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Pulsating waves in a multidimensional reaction-diffusion system of epidemic type

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Abstract. In this work we investigate the existence of front propagation for a two-component reaction -diffusion system of epidemic type posed in a multi-dimensional periodic medium. This system is a spatially heterogeneous version of the well-known Kermack–McKendrick epidemic model with Fickian diffusion. We derive sufficient conditions for the existence of pulsating travelling waves propagating in an arbitrarily given unit direction. Specifically, we prove that the set of admissible wave speeds contains a semi-infinite interval. Then, for each direction of propagation and each admissible speed, there exists a pulsating travelling wave solution of the system which is globally bounded.

Keywords: nonmonotone reaction-diffusion systems, pulsating travelling waves, uniform boundedness, epidemic models.

1. Introduction

This paper is concerned with the existence of pulsating travelling waves for the following two-component heterogeneous reaction-diffusion system:

$$\begin{cases} \partial_t S - \nabla \cdot (D_S(x)\nabla S) = -\beta(x)SI, \\ \partial_t I - \nabla \cdot (D_I(x)\nabla I) = \beta(x)SI - \gamma(x)I, \end{cases} \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where N is some given positive integer and without loss of generality, the spatial heterogeneities are assumed to be $\mathbb{Z}^N$-periodic. More precisely, we denote by $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ the N-dimensional torus and the coefficients arising in (1.1) satisfy the following conditions.

Assumption 1.1. (i) The diffusion matrix fields $D_S : \mathbb{T}^N \to \mathcal{S}_N(\mathbb{R})$ and $D_I : \mathbb{T}^N \to \mathcal{S}_N(\mathbb{R})$ are of class $C^{1+\alpha}$ for some exponent $\alpha \in (0,1)$, where $\mathcal{S}_N(\mathbb{R})$ denotes

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the space of real symmetric $N \times N$ matrices. Furthermore, D_S and D_I are uniformly elliptic in the sense that there exist positive constants $\theta \leq \Theta$ such that for all $(x, \xi) \in \mathbb{T}^N \times \mathbb{R}^N$,

$$\theta \|\xi\|^2 \leq \xi^T D_S(x)\xi \leq \Theta \|\xi\|^2 \quad \text{and} \quad \theta \|\xi\|^2 \leq \xi^T D_I(x)\xi \leq \Theta \|\xi\|^2. \tag{1.2}$$

(ii) The functions $\beta : \mathbb{T}^N \to \mathbb{R}$ and $\gamma : \mathbb{T}^N \to \mathbb{R}$ are positive and of class C^α.

System (1.1) of reaction-diffusion equations is a spatially heterogeneous version of the well-known Kermack–McKendrick SIR epidemic model with Fickian diffusion. In epidemiology, such a model describes the evolution of an infectious disease within a spatially distributed population of individuals. Here the functions $S(t, x)$ and $I(t, x)$ denote the densities of the susceptible and infected individuals at time t and at spatial position $x \in \mathbb{R}^N$, respectively. The contamination process follows the usual mass-action incidence with a contact rate function $\beta(x)$, while the function $\gamma(x)$ corresponds to the additional mortality (or recovered) rate due to the disease (or the treatment). We refer the reader to the original 1927 paper of Kermack and McKendrick [28] and to the monograph of Murray [32] for more details about this model mechanism.

In this work we shall focus on the propagation of fronts associated with system (1.1). Note that the model (1.1) does not involve the so-called vital dynamics (birth and death, sometimes called demography). Thus, one needs to prescribe the distribution of populations ahead of the travelling epidemic front. For that purpose, let S_0 be a positive constant. For a given population size S_0 in areas of the absence of disease, namely

$$(S_0, 0) \text{ is a disease-free equilibrium,} \tag{1.3}$$

we shall study the existence of a travelling epidemic wave that connects $(S_0, 0)$ and some unknown stationary state after the epidemic to be determined in the future.

In a spatially homogeneous medium, Hosono and Ilyas [27] proved the existence of travelling wave solutions via phase plane analysis. To be more specific, they studied the parabolic system

$$\begin{cases} \partial_t S - d_1 \partial_x^2 S = -\beta S I, \\ \partial_t I - d_2 \partial_x^2 I = \beta S I - \gamma I, \end{cases} \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.4}$$

where d_1, d_2, β and γ are positive constants. Assume that

$$\mathscr{R}_0 := \beta S_0 / \gamma > 1. \tag{1.5}$$

Then for each $c \geq c_\star := 2\sqrt{\beta S_0 d_2 (1 - 1/\mathscr{R}_0)}$, there exists a constant $S^- \in (0, S_0)$ such that problem (1.4) has a travelling wave solution of the form $(S, I)(t, x) = (U, V)(\xi)$ with $\xi := x - ct \in \mathbb{R}$ which exhibits the following behaviour at infinity:

$$U(+\infty) = S_0, \quad V(+\infty) = 0 \quad \text{and} \quad U(-\infty) = S^-, \ V(-\infty) = 0.$$

If $c < c_\star$ or $\mathscr{R}_0 \leq 1$, then there exists no travelling wave solution of (1.4). The quantity $\mathscr{R}_0$ defined in (1.5) is called the *basic reproduction number* of the Kermack–McKendrick model [28]. In epidemiology, the basic reproduction number plays the role of an epidemic threshold. If the basic reproduction number is greater than 1, then an epidemic propagates; otherwise, the disease dies out. This is often referred to as the *threshold phenomenon* (see [32]). Note that the above results show that $\mathscr{R}_0$ also gives a sharp threshold for the existence of travelling waves of problem (1.4).

Starting from the aforementioned pioneering works, growing attention has been paid to the threshold phenomenon for various epidemic models and to the travelling wave problem for some diffusive epidemic models. We refer to a nice survey by Ruan [36] for earlier works. As far as travelling waves for some problems similar to (1.4) in a homogeneous medium are concerned, we mention the works of Ducrot et al. [20, 21] for age-structured epidemic models and the work of Wang and Wu [38] for a diffusive epidemic model with nonlocal interaction and delay. A travelling wave solution in the aforementioned works is often referred to as a *travelling pulse* to emphasize that the profile of the infections behaves like a spike shape in the moving frame. Recently, travelling wave solutions for nonautonomous reaction-diffusion systems with a structure similar to the model (1.4) have been studied in [2, 39, 42].

Let us also mention that in a very recent work by Berestycki, Nordmann and Rossi [11], the authors point out that models similar to (1.4) are now also used to describe collective behaviours in various social contexts. In that setting, we refer to S as the level of social unrest, while I as a field of social tension. Based on the modelling of social phenomena, the propagation dynamics for system (1.4) in a more general form were investigated again in [11].

Structural analogues of system (1.1) also arise in combustion theory. As a typical example, the thermo-diffusive system (also called the KPP-type reaction-diffusion system) has been extensively studied. Let us describe such a system more precisely. In that setting, the function I represents the temperature of the reactant while S is its concentration. Set $D_S(x) \equiv \mathrm{Le}^{-1}\mathscr{I}$ and $D_I(x) \equiv \mathscr{I}$, where $\mathscr{I}$ denotes the identity matrix and the quantity $\mathrm{Le} > 0$ represents the ratio of thermal and material diffusivities, known as the *Lewis number*. For all $\mathrm{Le} > 0$, Giletti [22] proved the existence of multi-dimensional travelling waves inside a straight cylinder $\Omega = \mathbb{R}_x \times \omega_y$ for system (1.1) with Neumann boundary conditions in the presence of a shear flow, where ω_y is a smooth bounded domain of $\mathbb{R}^{N-1}$ and the parameters β and γ in (1.1) vary only with the cross variable y. In the context of combustion, one may also refer to [6, 25] and the references therein for earlier works on the existence of travelling waves for this kind of systems with $\gamma \equiv 0$ and equipped with various boundary conditions.

Coming back to the model (1.1), as mentioned above, the aim of the present paper is to investigate the propagation of fronts associated with the spatially heterogeneous reaction-diffusion system (1.1). Since we focus here on periodic environments, the so-called *pulsating travelling wave*, which is a generalization of the notion of *planar* travelling wave in homogeneous media, should be considered. A precise definition will be given below. The key feature of this kind of travelling wave is that its shape varies periodically as the wave moves in the medium. In particular, the profile of pulsating waves propagating in an arbitrarily given unit direction with rational coordinates exhibits periodicity in both time and space (see [13]). For scalar heterogeneous reaction-diffusion-advection equations, including periodic cases or more general ones, much more attention has been paid to propagation phenomena, for which we refer to [5, 10] for good overviews. As far as a periodic medium is concerned, let us also mention the work of Weinberger [40] on a general theory of spreading properties for order-preserving evolution problems, and the further generalization of this theory by Liang and Zhao [30].

Equipped with the prescribed values in (1.3), and following the notion of pulsating fronts for scalar spatially periodic equations introduced in [37] and further developed in [4, 41] (see also the references therein), we now introduce the definition of a pulsating travelling wave for problem (1.1):

Definition 1.1 (Pulsating travelling wave). Let $e \in \mathbb{S}^{N-1}$ be an arbitrarily given vector. An entire (classical) solution $(S, I)(t, x)$ of the parabolic system (1.1), that is, a solution defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, is said to be a *pulsating travelling wave* of problem (1.1) propagating in the direction e with effective speed $c \neq 0$ if the following conditions are satisfied:

(i) The function pair (S, I) is bounded:

$$0 < S(t, x) < S_0, \quad I(t, x) > 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

(ii) The function pair (S, I) satisfies the pulsating relations

$$(S, I)\left(t + \frac{k \cdot e}{c}, x\right) = (S, I)(t, x - k), \quad \forall (t, x, k) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{Z}^N,$$

(iii) The function pair (S, I) exhibits the following behaviour at $x \cdot e = +\infty$:

$$\lim_{x \cdot e \to +\infty} (S, I)(t, x) = (S_0, 0),$$

where the limit is understood to hold locally uniformly for $t \in \mathbb{R}$ and uniformly with respect to the directions of $e^{\perp} := \{\eta \in \mathbb{R}^N \mid \eta \cdot e = 0\}$.

Remark 1.1. The first and third conditions describe the basic characteristics of an epidemic wave associated with the model (1.1). The second condition is essential to the notion of pulsating travelling wave already mentioned above. Note also that we do not impose a prescribed value at $x \cdot e = -\infty$.

Ducrot and Giletti [18] considered a special case of model (1.1) when $D_S(x) \equiv \mathbf{0}$ and $D_I(x) \equiv \mathcal{I}$ (the identity matrix). In this situation, by setting $u(t, x) = \int_0^t I(s, x)\,\mathrm{d}s$, the system can be reduced to a single reaction-diffusion equation satisfied by u posed in a periodic medium with a KPP-type nonlinearity. Under suitable assumptions about the initial data, they studied the spreading speed and convergence to a one-dimensional pulsating wave by applying some results of Berestycki et al. [7–9]. See also [3, 31] for earlier works using such a change of variables.

However, there are very few results about the existence of pulsating waves for non-monotone systems in which all components involve the spatial diffusion. The results one may mention here are the work of Henderson [26] for a reactive Boussinesq system posed on a periodic domain which is unbounded in one direction, and the work of Constantin et al. [12] for a one-dimensional system describing the propagation of low Mach number flames in sprays. In addition, Griette and Matano [23] recently studied the propagation dynamics for a spatially periodic reaction-diffusion system whose reaction terms are partly competitive and partly cooperative depending on the value of the solution (which is motivated by an evolutionary-epidemic model in [1]). Let us emphasize that the aforementioned works all focus on one-dimensional pulsating waves.

To the best of our knowledge, the present paper is the first to prove the existence of multidimensional pulsating waves for nonmonotone reaction-diffusion systems posed in periodic environments, especially for the epidemic and KPP-type systems mentioned above. Finally, let us mention that Ducasse [15] has established threshold results associated with the heterogeneous epidemic model (1.1) posed on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with Neumann boundary conditions.

2. Main results

In this section, we state our main results and we outline the proof. To that end we need to introduce some notations that will be used throughout this paper.

A criterion for the existence of pulsating waves is formulated in terms of the periodic principal eigenvalue of an elliptic operator derived from the linearization of (1.1) around $(S_0, 0)$. More precisely, let us consider the unique $\mu_0 \in \mathbb{R}$ such that there exists a function $\phi \in C^2(\mathbb{T}^N)$ which satisfies

$$\begin{cases} -\nabla \cdot (D_I(x)\nabla\phi) - (\beta(x)S_0 - \gamma(x))\phi = \mu_0\phi \quad \text{in } \mathbb{T}^N, \\ \phi(x) > 0, \quad \forall x \in \mathbb{T}^N. \end{cases} \tag{2.1}$$

The existence of such a principal eigenvalue follows from the Krein–Rutman theorem. Note also that analogues are often used to describe the so-called monostable or bistable assumptions for scalar periodic reaction-diffusion equations (see [9, 16, 24] for instance). In what follows we shall see that the quantity μ_0 provides an almost sharp threshold for the existence of pulsating waves in any given unit direction for problem (1.1).

Next, to state the existence results, we need to introduce a few more notions to define a critical value of admissible wave speed. This relies on some heuristic arguments. According to Definition 1.1, the component S of the solution should be very close to S_0 far ahead of the propagating front. Similar to the scalar KPP equations, we expect that the component I of the solution would have exponential decay at the leading edge of the front. Furthermore, in the multidimensional framework, we need to analyze the dependence of spreading properties on the direction of propagation. As a consequence, for a given direction $e \in \mathbb{S}^{N-1}$ and speed $c > 0$, we shall use the following ansatz for some decay rate $\lambda > 0$ and for some positive function $\varphi \in C^2(\mathbb{T}^N)$:

$$\begin{cases} S(t,x) \approx S_0, \\ I(t,x) \approx e^{-\lambda(x\cdot e - ct)}\varphi(x), \end{cases} \quad \text{for } x\cdot e \gg ct \text{ and } (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Plugging the above ansatz into the I-equation of system (1.1), we obtain the equation satisfied by φ:

$$\begin{cases} L_{I,\lambda}\varphi = c\lambda\varphi & \text{in } \mathbb{T}^N, \\ \varphi(x) > 0, & \forall x \in \mathbb{T}^N, \end{cases}$$

where the elliptic operator $L_{I,\lambda} : C^2(\mathbb{T}^N) \to C^0(\mathbb{T}^N)$ is defined by

$$\begin{aligned} L_{I,\lambda}\psi &= \nabla \cdot (D_I(x)\nabla\psi) - 2\lambda e D_I(x)\nabla\psi \\ &\quad + [\lambda^2 e D_I(x) e - \lambda \nabla \cdot (D_I(x)e) + \beta(x)S_0 - \gamma(x)]\psi \quad \text{for } \lambda \in \mathbb{R}. \end{aligned}$$

For each $\lambda \in \mathbb{R}$ and each $e \in \mathbb{S}^{N-1}$, we denote by $k_\lambda(e)$ the periodic principal eigenvalue of the operator $L_{I,\lambda}$ with periodicity conditions. Then $k_\lambda(e)$ can be characterized by the existence of a positive function $\varphi_e \in C^2(\mathbb{T}^N)$ which is the principal eigenfunction of the operator $L_{I,\lambda}$ associated with $k_\lambda(e) = c\lambda$ (see [4, 34] for instance). Furthermore, according to the result of Nadin [34, Theorem 2.1], the periodic principal eigenvalue $k_\lambda(e)$ can be expressed by the following variational formula:

$$k_\lambda(e) = \max_{\psi \in \mathcal{F}} \left\{ -\int_{\mathbb{T}^N} \nabla\psi D_I(x)\nabla\psi \,\mathrm{d}x + \int_{\mathbb{T}^N} (\beta(x)S_0 - \gamma(x))\psi^2 \,\mathrm{d}x + \lambda^2 \mathcal{D}_e(\psi^2 D_I) \right\},$$

where

$$\mathcal{F} = \left\{ \psi \in C^1(\mathbb{T}^N) \;\middle|\; \psi > 0 \text{ and } \int_{\mathbb{T}^N} \psi^2 = 1 \right\},$$

and the functional $\mathcal{D}_e : C^0(\mathbb{T}^N, \mathcal{M}_N(\mathbb{R})) \to \mathbb{R}$ defined by

$$\mathcal{D}_e(A) = \min_{\Psi \in C^1(\mathbb{T}^N)} \int_{\mathbb{T}^N} (e + \nabla\Psi) A(x) (e + \nabla\Psi) \,\mathrm{d}x$$

gives the *effective diffusivity* of a diffusion matrix A in a given direction $e \in \mathbb{S}^{N-1}$.

Recalling the definition of μ_0 in (2.1), one has

$$\mu_0 = -k_0(e), \quad \forall e \in \mathbb{S}^{N-1}.$$

Furthermore, μ_0 can be directly expressed in terms of the Rayleigh quotient:

$$\mu_0 = \min_{\phi \in C^1(\mathbb{T}^N)\setminus\{0\}} \frac{\int_{\mathbb{T}^N} [\nabla\phi D_I(x)\nabla\phi - (\beta(x)S_0 - \gamma(x))\phi^2] \,\mathrm{d}x}{\int_{\mathbb{T}^N} \phi^2}.$$

In particular, if

$$\int_{\mathbb{T}^N} [\beta(x)S_0 - \gamma(x)] \,\mathrm{d}x > 0, \tag{2.2}$$

then it follows from the above formula that μ_0 is negative. Observe from (1.5) that in a homogeneous medium, the condition (2.2) amounts to the basic reproduction number $\mathcal{R}_0 > 1$. However, under a heterogeneous framework, it is not easy to get a sufficient and necessary condition for μ_0 to be negative.

Assume now that $\mu_0 < 0$. Then one may define a quantity $c^*(e)$ (see (3.8) for an explicit formula) with the help of the periodic principal eigenvalue $k_\lambda(e)$. This quantity turns out to be positive and depends continuously on the direction $e \in \mathbb{S}^{N-1}$. More detailed properties of the quantities introduced above will be stated in Section 3.2 below.

With the above notations, our main result is the following existence theorem.

Theorem 2.1. *Let Assumption 1.1 be satisfied and assume further that $\mu_0 < 0$. Then for each direction $e \in \mathbb{S}^{N-1}$, one has $c^*(e) > 0$, and for each wave speed $c > c^*(e)$ problem (1.1) admits a pulsating travelling wave solution (S, I, c) propagating in the direction e according to Definition 1.1.*

Remark 2.2. The condition $\mu_0 < 0$ is almost optimal for the existence of pulsating waves for problem (1.1). Indeed, if $\mu_0 > 0$, then any bounded entire solution $(S, I)(t, x)$ of (1.1) with $0 \le S \le S_0$ satisfies $I \equiv 0$ (see Section 3.3 below for more details), that is, without epidemic to occur. For the critical case $\mu_0 = 0$, the analogy with the scalar KPP equations suggests that it would also lead to the extinction of the disease, but such problems remain to be solved mathematically for systems as in (1.1) (see also [15]). One may mention that a special case has been treated in [17] which considered system (1.1) but posed on a straight cylinder with heterogeneity on the cross section.

Outline of the proof. The general idea of the proof of Theorem 2.1 follows the scheme of our previous paper [13]. However, generalization to nonmonotone reaction-diffusion systems, especially those with structures similar to the model (1.1), is not trivial.

In Section 3, we derive the equations satisfied by the profile of pulsating travelling waves of problem (1.1) by using the formulation of the wave profile proposed in [13]. As already seen in [13], such a system of equations is nondegenerate and parabolic. In addition, we state some technical results on the periodic principal eigenvalue $k_\lambda(e)$ and the quantity $c^*(e)$. The proof of the nonexistence results claimed in Remark 2.2 ends this section.

In Section 4, we first build an invariant domain of the solutions for any direction of propagation. Next, we prove the existence of pulsating waves propagating in any given unit direction with rational coordinates (see Theorem 4.6). A key lemma allows us to transform this problem into a space-time periodic problem. With the help of the Poincaré mapping and using suitable fixed-point arguments, we construct approximate solutions for this problem posed on a finite domain. Lastly, we pass to unbounded domains after deriving some uniform estimates of the solutions. Here we extend some arguments of Ducrot et al. [19]. Their idea comes from a very natural contradiction: in the epidemiological context, it can be explained that the susceptible population would become extinct if there are too many infectious individuals, which conversely reduces the infected one. However, unlike the Cauchy problem in [2, 19], the spatial heterogeneities coupled with some boundary value problems make this generalization more complicated and technical (see Lemma 4.10).

In Section 5, we complete the proof of Theorem 2.1 by a limiting argument for a general direction of propagation. The key point is to prove that the component I of the solutions remains uniformly bounded for any rational direction of propagation. The difficulty of the proof is due to the lack of periodicity. Nevertheless, once again, based on the very natural law of the epidemic already mentioned above, one can still reach some contradictions to obtain uniform estimates of the solutions.

3. Preliminaries

This section is devoted to the description of our methodology and to collecting some important properties of the periodic principal eigenvalue $k_\lambda(e)$, as well as to the proof of the nonexistence results claimed in Remark 2.2.

3.1. Description of the pulsating wave profile

Here we use the formulation of the wave profile proposed in our previous paper [13] to derive the equations satisfied by the profile of pulsating waves for problem (1.1). In [13], the pulsating travelling waves for a periodic Fisher-KPP reaction-diffusion equation were investigated.

Assume that an entire solution $(S, I) = (S, I)(t, x)$ of the parabolic system (1.1) is a pulsating travelling wave of (1.1) propagating in the direction $e \in \mathbb{S}^{N-1}$ with effective speed $c \neq 0$ according to Definition 1.1. We denote by $\{e_1, \dots, e_N\}$ the canonical basis of $\mathbb{R}^N$ and we consider a linear orthogonal transformation $\mathcal{R} \in \mathcal{O}(\mathbb{R}^N)$ (the corresponding matrix representation uses the same notation below) such that

$$\mathcal{R}e_1 = e,$$
$$e^\perp = \text{span}\{\mathcal{R}e_2, \dots, \mathcal{R}e_N\} = \{\alpha \in \mathbb{R}^N \mid \alpha = \mathcal{R}(0, y)^T,\ \forall y \in \mathbb{R}^{N-1}\}.$$

For any $k \in \mathbb{Z}^N$, we decompose k along e and $e^\perp$, using the following notations:

$$k = (k \cdot e)e + k_\perp \quad \text{with} \quad k_\perp = k - (k \cdot e)e \in e^\perp$$

so that

$$\mathcal{R}^{-1}k = \begin{pmatrix} k \cdot e \\ 0_{\mathbb{R}^{N-1}} \end{pmatrix} + \mathcal{R}^{-1}k_\perp$$

and

$$\mathcal{R}^{-1}k_\perp = \begin{pmatrix} 0 \\ \widehat{\mathcal{R}}k_\perp \end{pmatrix} \in \bigoplus_{i=2}^{N} \mathbb{R}e_i = \{0\} \times \mathbb{R}^{N-1}$$

for some linear mapping $\widehat{\mathcal{R}} : e^\perp \to \mathbb{R}^{N-1}$. Now we consider the frame moving with speed $c > 0$ and we define a function pair $(U, V) = (U, V)(\xi, t, y)$ for $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$ by

$$(U, V)(\xi, t, y) = (S, I)(t, x) \quad \text{with} \quad x = \mathcal{R}\begin{pmatrix} \xi + ct \\ y \end{pmatrix}. \tag{3.1}$$

From the pulsating relations of (S, I) we have, for all $k \in \mathbb{Z}^N$,

$$\begin{aligned}(U, V)(\xi, t, y) &= (S, I)\left(t + \frac{k \cdot e}{c}, \mathcal{R}(\xi + ct, y)^T + k\right)\\ &= (S, I)\left(t + \frac{k \cdot e}{c}, \mathcal{R}((\xi + ct, y)^T + \mathcal{R}^{-1}k)\right)\\ &= (S, I)\left(t + \frac{k \cdot e}{c}, \mathcal{R}\left(\left(\xi + c\left(t + \frac{k \cdot e}{c}\right), y\right)^T + \mathcal{R}^{-1}k_\perp\right)\right)\\ &= (U, V)\left(\xi, t + \frac{k \cdot e}{c}, y + \widehat{\mathcal{R}}k_\perp\right).\end{aligned}$$

Moreover, the 3-tuple (U, V, c) satisfies the parabolic system

$$\begin{cases}\partial_t U - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi, t, y)\nabla_{\xi,y} U) - cU_\xi = -\widetilde{\beta}(\xi, t, y)UV,\\ \partial_t V - \nabla_{\xi,y} \cdot (\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y} V) - cV_\xi = \widetilde{\beta}(\xi, t, y)UV - \widetilde{\gamma}(\xi, t, y)V\end{cases} \tag{3.2}$$

posed for $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$, where the matrix-valued functions $\widetilde{D}_S : \mathbb{R}^{N+1} \to \mathcal{M}_N(\mathbb{R})$ and $\widetilde{D}_I : \mathbb{R}^{N+1} \to \mathcal{M}_N(\mathbb{R})$ are defined by

$$\widetilde{D}_S(\xi, t, y) = \mathcal{R}D_S\left(\mathcal{R}\begin{pmatrix}\xi + ct\\ y\end{pmatrix}\right)\mathcal{R}^T \quad \text{and} \quad \widetilde{D}_I(\xi, t, y) = \mathcal{R}D_I\left(\mathcal{R}\begin{pmatrix}\xi + ct\\ y\end{pmatrix}\right)\mathcal{R}^T,$$

and the functions $\widetilde{\beta} : \mathbb{R}^{N+1} \to (0, \infty)$ and $\widetilde{\gamma} : \mathbb{R}^{N+1} \to (0, \infty)$ are defined by

$$\widetilde{\beta}(\xi, t, y) = \beta(\mathcal{R}(\xi + ct, y)^T) \quad \text{and} \quad \widetilde{\gamma}(\xi, t, y) = \gamma(\mathcal{R}(\xi + ct, y)^T).$$

Here the vector differential operator $\nabla_{\xi,y} = (\frac{\partial}{\partial\xi}, \nabla_y)^T$ stands for the gradient of the unknown functions U and V with respect to $(\xi, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, while the operator

$$u = (u_1, \ldots, u_N) \mapsto \nabla_{\xi,y} \cdot u = \frac{\partial u_1}{\partial \xi} + \sum_{i=1}^{N-1} \frac{\partial u_{i+1}}{\partial y_i}$$

denotes the divergence of a vector-valued function $u : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}^N$ with respect to (ξ, y) (see [13] for more details on the above computations).

Consequently, when $(S, I) = (S, I)(t, x)$ is a pulsating travelling wave of problem (1.1) in the direction $e \in \mathbb{S}^{N-1}$ with speed $c > 0$, then the function pair given by

$$(U, V)(\xi, t, y) = (S, I)(t, \mathcal{R}(\xi + ct, y)^T)$$

becomes a bounded entire solution of (3.2) and the pulsating relations for (S, I) can be rewritten as

$$(U, V)(\xi, t, y) = (U, V)\left(\xi, t + \frac{k \cdot e}{c}, y + \widehat{\mathcal{R}}k_\perp\right),\ \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1},\ \forall k \in \mathbb{Z}^N. \tag{3.3}$$

As in [13], we shall call (3.3) the *$\mathcal{R}$-pulsating condition*. Furthermore, the function pair (U, V) satisfies the asymptotic condition

$$\lim_{\xi \to +\infty} (U, V)(\xi, t, y) = (S_0, 0) \quad \text{uniformly for } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}. \tag{3.4}$$

Note also that since the functions β and γ are both $\mathbb{Z}^N$-periodic in x, they also satisfy the $\mathcal{R}$-pulsating condition, that is,

$$(\widetilde{\beta}, \widetilde{\gamma})\left(\xi, t + \frac{k \cdot e}{c}, y + \widehat{\mathcal{R}} k_\perp\right) = (\widetilde{\beta}, \widetilde{\gamma})(\xi, t, y),\ \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1},\ \forall k \in \mathbb{Z}^N.$$

Likewise, all entries of the matrix-valued functions $\widetilde{D}_S$ and $\widetilde{D}_I$ also satisfy the $\mathcal{R}$-pulsating condition. Furthermore, since $\mathcal{R} \in \mathcal{O}(\mathbb{R}^N)$ and the matrix fields D_S and D_I are both symmetric, it follows that $\widetilde{D}_S$ and $\widetilde{D}_I$ are also symmetric and satisfy the same uniform elliptic conditions (1.2) as D_S and D_I.

We will work with the uniformly parabolic system (3.2) in almost all the proofs. In particular, the wave profile used in [12] is actually the one-dimensional case presented here (namely without the variable y).

3.2. The corresponding periodic eigenvalue problem

In order to build an invariant domain and to perform an approximation argument via a dense set of propagation directions, we recall some important properties of the periodic principal eigenelements for an elliptic eigenvalue problem.

Consider the V-equation of system (3.2) linearized around $(S_0, 0)$:

$$\partial_t V - \nabla_{\xi,y} \cdot (\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y} V) - cV_\xi = \big(\widetilde{\beta}(\xi, t, y)S_0 - \widetilde{\gamma}(\xi, t, y)\big)V$$

posed for $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$. For clarity, we denote by

$$X := \mathcal{R}(\xi + ct, y)^T$$

the generic element of $\mathbb{T}^N$. We look for a solution to the above equation of the form

$$\varphi(\xi, t, y) = e^{-\lambda\xi}\Phi(X), \quad X \in \mathbb{T}^N.$$

Then the pair $(\lambda, \Phi) \in \mathbb{R} \times C^2(\mathbb{T}^N)$ has to solve the elliptic equation

$$L_{I,\lambda}\psi = c\lambda\psi \quad \text{in } \mathbb{T}^N, \tag{3.5}$$

where $L_{I,\lambda}$ is the elliptic operator

$$\begin{aligned} L_{I,\lambda}\psi := &\ \nabla \cdot (D_I(X)\nabla\psi) - 2\lambda e D_I(X)\nabla\psi \\ &+ [\lambda^2 e D_I(X)e - \lambda\nabla \cdot (D_I(X)e) + a(X)]\psi \end{aligned}$$

with $a(X) := \beta(X)S_0 - \gamma(X)$ for all $X \in \mathbb{T}^N$. To solve (3.5), let us consider, for each $\lambda \in \mathbb{R}$ and each $e \in \mathbb{S}^{N-1}$, the periodic principal eigenvalue $k_\lambda(e) \in \mathbb{R}$ of the problem

$$\begin{cases} L_{I,\lambda}\Phi_{\lambda,e} = k_\lambda(e)\Phi_{\lambda,e} & \text{in } \mathbb{T}^N, \\ \Phi_{\lambda,e} \in C^2(\mathbb{T}^N), \quad \Phi_{\lambda,e} > 0. \end{cases} \tag{3.6}$$

Recalling the definition of μ_0 in (2.1), one has $\mu_0 = -k_0(e)$ for any $e \in \mathbb{S}^{N-1}$. This means that the quantity μ_0 does not depend on the direction of propagation. Let $\Phi_{\lambda,e}$ be the unique principal eigenfunction of (3.6) such that

$$\Phi_{\lambda,e} > 0 \quad \text{and} \quad \|\Phi_{\lambda,e}\|_\infty = 1. \tag{3.7}$$

A few properties of the eigenelements $(k_\lambda(e), \Phi_{\lambda,e})$ are collected in the following proposition.

Proposition 3.1. *The following properties hold:*

(i) *For each $e \in \mathbb{S}^{N-1}$, $k_\lambda(e)$ is analytic and convex with respect to $\lambda \in \mathbb{R}$.*

(ii) *For each $\lambda \in \mathbb{R}$, the function $\mathbb{S}^{N-1} \ni e \mapsto k_\lambda(e)$ is continuous. Moreover, the principal eigenfunction $\Phi_{\lambda,e}(X)$ depends continuously on both $\lambda \in \mathbb{R}$ and $e \in \mathbb{S}^{N-1}$ with respect to the uniform topology.*

Proposition 3.2. *Assume that $\mu_0 < 0$. Then*

$$c^*(e) := \inf_{\lambda>0} \frac{k_\lambda(e)}{\lambda} \in \mathbb{R}, \tag{3.8}$$

and for each $c > c^*(e)$, the positive real number

$$\lambda_c(e) := \min\{\lambda > 0 \mid k_\lambda(e) - c\lambda = 0\}$$

is well-defined and the set

$$F_c = \{\lambda \in (0,\infty) \mid k_\lambda(e) - c\lambda = 0\}$$

is either the singleton $\{\lambda_c\}$, or equal to $\{\lambda_c, \lambda_c^+\}$ with $\lambda_c < \lambda_c^+$. The set $F_{c^*(e)}$ is either empty or a singleton $\{\lambda^*\}$, and if it is $\{\lambda^*\}$, then the multiplicity of $\{\lambda^*\}$ as a root of $k_\lambda(e) - c\lambda = 0$ is $2m + 2$ for some $m \in \mathbb{N}$.

These results can be found in [24, 34]. As a direct consequence of Proposition 3.1 (ii) and the formula (3.8) for $c^*(e)$, we have the following results.

Proposition 3.3. *Assume that $\mu_0 < 0$. Then the following statements hold*:

(i) *The mapping $\mathbb{S}^{N-1} \ni e \mapsto c^*(e)$ is continuous.*

(ii) *$\lambda_c(e)$ is continuous in both $e \in \mathbb{S}^{N-1}$ and $c \in (c^*(e), \infty)$.*

3.3. Nonexistence of solutions

This subsection is concerned with the nonexistence results claimed in Remark 2.2. More precisely, we have the following proposition.

Proposition 3.4. *Let Assumption 1.1 be satisfied. If $\mu_0 > 0$, then any bounded entire solution $(S, I)(t, x)$ of the parabolic system (1.1) with*

$$0 \leq S(t,x) \leq S_0, \quad I(t,x) \geq 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{3.9}$$

satisfies $I \equiv 0$. In particular, problem (1.1) does not admit any pulsating travelling wave in any given direction $e \in \mathbb{S}^{N-1}$ according to Definition 1.1.

Proof. Assume that $(S, I) = (S, I)(t, x)$ is a bounded entire solution of (1.1) satisfying (3.9) and assume that $\mu_0 > 0$. Recall that μ_0 is the principal eigenvalue of the problem

$$\begin{cases} -\nabla \cdot (D_I(x)\nabla\phi) - (\beta(x)S_0 - \gamma(x))\phi = \mu_0\phi \quad \text{in } \mathbb{T}^N, \\ \phi \in C^2(\mathbb{T}^N), \quad \phi(x) > 0, \quad \forall x \in \mathbb{T}^N. \end{cases}$$

Let ϕ_0 be the unique principal eigenfunction associated with μ_0 such that $\min_{\mathbb{T}^N} \phi_0 = 1$. Since $0 \leq S(t,x) \leq S_0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, the function $\widehat{I}(t,x) := e^{-\mu_0 t}\phi_0(x)$ is a supersolution for the I-equation of the parabolic system (1.1), that is,

$$\begin{aligned} \partial_t \widehat{I}(t,x) - \nabla \cdot (D_I(x)\nabla \widehat{I}(t,x)) &= \big(\beta(x)S_0 - \gamma(x)\big)\widehat{I}(t,x) \\ &\geq \beta(x)S(t,x)\widehat{I}(t,x) - \gamma(x)\widehat{I}(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \end{aligned}$$

Since I is bounded, there exists some constant $M > 0$ such that

$$0 \leq I(t,x) \leq M, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

It then follows from the parabolic comparison principle that

$$0 \leq I(t+s,x) \leq M\widehat{I}(s,x) = Me^{-\mu_0 s}\phi_0(x), \quad \forall s \geq 0, \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Since $\mu_0 > 0$, one deduces that $I(t+s,x)$ tends to 0 as $s \to \infty$ uniformly on any compact subset of $\mathbb{R} \times \mathbb{R}^N$. This means that the bounded entire solution of (1.1) satisfying (3.9) can only be such that $I \equiv 0$. This completes the proof of Proposition 3.4. ∎

4. Existence of solutions

Throughout this section we assume that

$$\mu_0 < 0.$$

4.1. Construction of sub- and supersolutions

In this subsection, we construct suitable sub- and supersolutions of system (3.2) with any direction $e \in \mathbb{S}^{N-1}$. In particular, these subsolutions ensure that the solution obtained in both limiting procedures is nontrivial and also yield the positivity of pulsating travelling wave solutions.

To do this we first prove that the function $(0,\infty) \ni \lambda \mapsto k_\lambda(e)/\lambda$ can reach the minimum and the quantity $c^*(e)$ defined by (3.8) is positive for any $e \in \mathbb{S}^{N-1}$ under our hypothesis $\mu_0 < 0$.

Lemma 4.1. *Assume that $\mu_0 < 0$. Then for each $e \in \mathbb{S}^{N-1}$, one has*

$$c^*(e) = \min_{\lambda>0} \frac{k_\lambda(e)}{\lambda} > 0.$$

Proof. Consider the periodic Fisher-KPP equation

$$u_t - \nabla \cdot (D_I(x)\nabla u) = u(a(x) - u), \quad t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{4.1}$$

with $a(x) := \beta(x)S_0 - \gamma(x)$. Here the function a is $\mathbb{Z}^N$-periodic in x, of class C^α but may change sign. According to the results of Berestycki et al. [8, 9], if $\mu_0 < 0$, then there exists a unique positive stationary state $p(x)$ of problem (4.1) which turns out to be $\mathbb{Z}^N$-periodic. Furthermore, there exists $c_0^* > 0$ such that problem (4.1) has a pulsating front solution (u, c) propagating in the direction $e \in \mathbb{S}^{N-1}$ which satisfies

$$\begin{cases} u\left(t + \dfrac{k\cdot e}{c}, x\right) = u(t, x-k), \quad \forall (t,x,k) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{Z}^N, \\ \lim\limits_{x\cdot e\to+\infty} u(t,x) = 0,\ \lim\limits_{x\cdot e\to-\infty} u(t,x) = p(x), \quad \forall t \in \mathbb{R}, \end{cases}$$

if and only if $c \geq c_0^*$, and the minimal speed is given by the variational representation

$$c_0^* = \min\{c \in \mathbb{R} \mid \exists \lambda > 0 \text{ such that } \mu_c(\lambda) = 0\},$$

where $\mu_c(\lambda)$ denotes the periodic principal eigenvalue of the elliptic operator

$$\psi \mapsto \nabla\cdot(D_I(x)\nabla\psi) - 2\lambda e D_I(x)\nabla\psi + [\lambda^2 e D_I(x) e - \lambda\nabla\cdot(D_I(x)e) - c\lambda + a(x)]\psi$$

posed for $\psi \in C^2(\mathbb{T}^N)$ with periodicity conditions. These conclusions yield

$$c^*(e) = \inf_{\lambda>0}\frac{k_\lambda(e)}{\lambda} = \min_{\lambda>0}\frac{k_\lambda(e)}{\lambda} = c_0^* > 0, \quad \forall e \in \mathbb{S}^{N-1}. \qquad \blacksquare$$

For notational simplicity, we temporarily forget the dependence of $k_\lambda(e)$, $\Phi_{\lambda,e}$ and $\lambda_c(e)$ on the direction $e \in \mathbb{S}^{N-1}$ and we still denote by

$$X = \mathcal{R}(\xi + ct, y)^T$$

the generic element of $\mathbb{T}^N$. From Lemma 4.1, we see that the set F_c defined in Proposition 3.2 equals $\{\lambda_c, \lambda_c^+\}$ when $c > c^*(e)$. One can now construct a suitable sub- and supersolution pair of system (3.2).

Note first that the constant S_0 is a supersolution for the U-equation of (3.2).

Lemma 4.2. *Assume that $\mu_0 < 0$. Then for each $c > c^*(e)$, the function*

$$\overline{V}(\xi, t, y) = e^{-\lambda_c \xi} \Phi_{\lambda_c}(X)$$

is a solution for the V-equation of (3.2) with $U(\xi, t, y) \equiv S_0$, where the function Φ_{λ_c} is the unique principal eigenfunction of problem (3.6) in the sense of (3.7) with $\lambda = \lambda_c$.

Proof. To prove the lemma we compute, for all $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$,

$$\begin{aligned}
&\partial_t \overline{V} - \nabla_{\xi,y} \cdot \big(\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y}\overline{V}\big) - c\partial_\xi \overline{V} \\
&\qquad = \big[c\lambda_c \Phi_{\lambda_c} - \nabla \cdot (D_I(X)\nabla\Phi_{\lambda_c}) + 2\lambda_c e D_I(X)\Phi_{\lambda_c} \\
&\qquad\quad - \big(\lambda_c^2 e D_I(X)e - \lambda_c \nabla \cdot (D_I(X)e)\big)\Phi_{\lambda_c}\big]e^{-\lambda_c \xi} \\
&\qquad = [c\lambda_c - (k_{\lambda_c} - a(X))]e^{-\lambda_c \xi}\Phi_{\lambda_c}(X) \\
&\qquad = \big(\widetilde{\beta}(\xi, t, y)S_0 - \widetilde{\gamma}(\xi, t, y)\big)\overline{V}.
\end{aligned}$$

This gives the conclusion. ∎

In order to construct a subsolution for the U-equation of (3.2), let us introduce for each $\delta \in \mathbb{R}$ the principal eigenvalue problem associated with the S-equation of (1.1):

$$\begin{cases} L_{S,\delta}\Psi_{\delta,e} = \nu(\delta, e)\Psi_{\delta,e} & \text{in } \mathbb{T}^N, \\ \Psi_{\delta,e} \in C^2(\mathbb{T}^N), \quad \Psi_{\delta,e} > 0, \end{cases} \tag{4.2}$$

where $L_{S,\delta}$ is the elliptic operator

$$L_{S,\delta}\psi := \nabla \cdot (D_S(X)\nabla\psi) - 2\delta e D_S(X)\nabla\psi + [\delta^2 e D_S(X)e - \delta\nabla \cdot (D_S(X)e)]\psi$$

with periodicity conditions. As with $k_\lambda(e)$ in Section 3.2, here the principal eigenvalue $\nu(\delta, e)$ depends continuously on both $\delta \in \mathbb{R}$ and $e \in \mathbb{S}^{N-1}$. Moreover, the principal eigenfunction $\Psi_{\delta,e}(X)$, up to some normalization, also depends continuously on both δ and e with respect to the uniform topology. For notational simplicity, we also temporarily forget the dependence of $\nu(\delta, e)$ and $\Psi_{\delta,e}$ on e.

Lemma 4.3. *Assume that $\mu_0 < 0$. For any $c > c^*(e)$, let $\underline{U}$ be the function defined by*

$$\underline{U}(\xi, t, y) = S_0(1 - K_1 e^{-\delta_1 \xi}\Psi_{\delta_1}(X)), \tag{4.3}$$

where $\Psi_{\delta_1} > 0$ is the unique principal eigenfunction of problem (4.2) in the sense of $\|\Psi_{\delta_1}\|_\infty = 1$ with $\delta = \delta_1$. Then there exist $\delta_1 \in (0, \lambda_c)$ small enough and $K_1 > 1$ large enough such that the function $\underline{U}$ satisfies the differential inequality

$$\partial_t \underline{U} - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi, t, y)\nabla_{\xi,y}\underline{U}) - c\partial_\xi \underline{U} \leq -\widetilde{\beta}(\xi, t, y)\underline{U}\,\overline{V} \tag{4.4}$$

for any $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$.

Proof. Substituting $\underline{U}$ and $\overline{V}$ into (4.4), inequality (4.4) is equivalent to

$$K_1(c\delta_1 - \nu(\delta_1)) \geq \beta(X)\left(\frac{1}{\Psi_{\delta_1}(X)} - K_1 e^{-\delta_1\xi}\right)e^{(\delta_1-\lambda_c)\xi}\Phi_{\lambda_c}(X),\ \forall(\xi, X) \in \mathbb{R} \times \mathbb{T}^N. \tag{4.5}$$

Since $0 < \Phi_{\lambda_c} \leq 1$, this inequality is satisfied for any $\xi > 0$ as long as $0 < \delta_1 \leq \lambda_c$ and

$$K_1(c\delta_1 - \nu(\delta_1)) \geq \beta(X)\Psi_{\delta_1}^{-1}(X), \quad \forall X \in \mathbb{T}^N.$$

From [4, Proposition 5.7], we know that the principal eigenvalue $\nu(\delta)$ of problem (4.2) is convex with respect to $\delta \in \mathbb{R}$ and $\nu(0) = \nu'(0) = 0$, whence $\nu(\delta)$ is nonnegative. Since $c > c^*(e) > 0$, we can choose $\delta_1 > 0$ small enough such that

$$0 < \delta_1 < \lambda_c \quad \text{and} \quad 0 \leq \nu(\delta_1) < c\delta_1.$$

Therefore, it is sufficient to choose K_1 so that

$$K_1 \geq \frac{\overline{\beta}}{m_{\delta_1}(c\delta_1 - \nu(\delta_1))} > 0 \quad \text{with} \quad m_{\delta_1} := \min_{X\in\mathbb{T}^N} \Psi_{\delta_1}(X) > 0 \quad \text{and} \quad \overline{\beta} := \max_{X\in\mathbb{T}^N} \beta(X).$$

When $\xi \leq 0$, with the choice above for the parameters K_1 and δ_1, inequality (4.5) is satisfied provided that

$$1 \geq (1 - m_{\delta_1} K_1 e^{-\delta_1\xi})e^{(\delta_1-\lambda_c)\xi},$$

which is true if we fix $K_1 > 1$ so that

$$K_1 = \max\left\{\frac{1}{m_{\delta_1}}, \frac{\overline{\beta}}{m_{\delta_1}(c\delta_1 - \nu(\delta_1))}\right\}.$$

This ends the proof of Lemma 4.3. ∎

Finally, we prove the following.

Lemma 4.4. *Assume that $\mu_0 < 0$. For any $c > c^*(e)$, let $\underline{V}$ be the function defined by*

$$\underline{V}(\xi, t, y) = \Phi_{\lambda_c}(X)e^{-\lambda_c\xi} - K_2\Phi_{\lambda_c+\delta_2}(X)e^{-(\lambda_c+\delta_2)\xi}, \tag{4.6}$$

where the functions Φ_{λ_c} and $\Phi_{\lambda_c+\delta_2}$ are the unique principal eigenfunctions of problem (3.6) in the sense of (3.7) corresponding to λ_c and $\lambda_c + \delta_2$, respectively. Then there exist $\delta_2 \in (0, \min\{\delta_1, \lambda_c^+ - \lambda_c\})$ small enough and $K_2 > 0$ large enough such that $\underline{V}$ satisfies the differential inequality

$$\partial_t\underline{V} - \nabla_{\xi,y}\cdot(\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y}\underline{V}) - c\partial_\xi\underline{V} \leq \widetilde{\beta}(\xi, t, y)\underline{U}^+\underline{V} - \widetilde{\gamma}(\xi, t, y)\underline{V} \tag{4.7}$$

on the set

$$\Omega^+ := \{(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \mid \underline{V}(\xi, t, y) \geq 0\}.$$

Proof. By Propositions 3.1–3.2, one has $k'_{\lambda_c} < c$ for any $c > c^*(e)$ (the prime denotes the derivative with respect to λ) since λ_c is the smallest positive solution such that $k_\lambda - c\lambda = 0$, and since k_λ is convex in λ, also $k_0 = -\mu_0 > 0$. Set $r_\delta := c(\lambda_c + \delta) - k_{\lambda_c+\delta}$. It is not hard to check that the function $\delta \mapsto r_\delta$ satisfies the following properties:

$$r_0 = c\lambda_c - k_{\lambda_c} = 0, \qquad \left.\frac{\mathrm{d}r_\delta}{\mathrm{d}\delta}\right|_{\delta=0} = c - k'_{\lambda_c} > 0, \quad \forall c > c^*(e).$$

Therefore, we can choose $\delta_2 > 0$ small enough such that

$$\begin{cases} 0 < \delta_2 < \min\{\delta_1, \lambda_c^+ - \lambda_c\}, \\ r_{\delta_2} = c(\lambda_c + \delta_2) - k_{\lambda_c+\delta_2} > 0. \end{cases} \tag{4.8}$$

Since $0 < \Psi_{\delta_1} \leq 1$, we have

$$\underline{U}(\xi, t, y) = S_0(1 - K_1 e^{-\delta_1\xi}\Psi_{\delta_1}(X)) \geq S_0(1 - K_1 e^{-\delta_1\xi}),\ \forall(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1},$$

which implies that

$$\underline{U}^+(\xi, t, y) = \underline{U}(\xi, t, y), \quad \forall\xi > \xi_0 := \frac{1}{\delta_1}\ln K_1,\ \forall(t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}.$$

Next we fix $K_2 > 0$ so that

$$K_2 = \max\left\{K_1 \sup_{\mathbb{T}^N} \frac{\Phi_{\lambda_c}}{\Phi_{\lambda_c+\delta_2}}, \frac{K_1 S_0 \overline{\beta}}{r_{\delta_2}} \sup_{\mathbb{T}^N} \frac{\Phi_{\lambda_c}}{\Phi_{\lambda_c+\delta_2}}\right\} \quad \text{with} \quad \overline{\beta} = \max_{X\in\mathbb{T}^N} \beta(X). \tag{4.9}$$

With such a choice for the parameters, one has $K_1 > K_1^{\delta_2/\delta_1} \geq e^{\delta_2\xi}$ for any $\xi \leq \xi_0$. Therefore,

$$\sup_{(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}} \underline{V}(\xi, t, y) < 0, \quad \forall\xi \leq \xi_0,$$
$$\Omega^+ := \{(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \mid \underline{V}(\xi, t, y) \geq 0\} \subset (\xi_0, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1}.$$

Now, in order to prove the differential inequality (4.7) we compute, for all $(\xi, t, y) \in \Omega^+$,

$$\begin{aligned}
&\partial_t\underline{V} - \nabla_{\xi,y} \cdot (\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y}\underline{V}) - c\partial_\xi\underline{V} - \widetilde{\beta}(\xi, t, y)\underline{U}^+\underline{V} + \widetilde{\gamma}(\xi, t, y)\underline{V} \\
&\quad = (\beta(X)S_0 - \gamma(X))\underline{V} - K_2[c(\lambda_c + \delta_2) - k_{\lambda_c+\delta_2}]\Phi_{\lambda_c+\delta_2}(X)e^{-(\lambda_c+\delta_2)\xi} \\
&\qquad - [\beta(X)S_0(1 - K_1e^{-\delta_1\xi}\Psi_{\delta_1}(X)) - \gamma(X)]\underline{V} \\
&\quad \leq K_1S_0\beta(X)\underline{V}e^{-\delta_1\xi} - K_2[c(\lambda_c + \delta_2) - k_{\lambda_c+\delta_2}]\Phi_{\lambda_c+\delta_2}(X)e^{-(\lambda_c+\delta_2)\xi} \\
&\quad = \left[K_1S_0\beta(X)\left(\frac{\Phi_{\lambda_c}(X)}{\Phi_{\lambda_c+\delta_2}(X)}e^{(\delta_2-\delta_1)\xi} - K_2e^{-\delta_1\xi}\right) - K_2r_{\delta_2}\right]\Phi_{\lambda_c+\delta_2}(X)e^{-(\lambda_c+\delta_2)\xi} \\
&\quad \leq 0
\end{aligned}$$

due to $0 < \Psi_{\delta_1} \leq 1$ and with the choices (4.8)–(4.9) for the parameters K_2 and δ_2.

This ends the proof of Lemma 4.7. ∎

4.2. *Pulsating waves in each rational direction*

In this subsection, we prove the existence of pulsating travelling waves of problem (1.1) propagating along each direction with rational coordinates in $\mathbb{S}^{N-1}$ under the framework of Section 3.1.

4.2.1. Derivation of a space-time periodic problem. The aim of this part is to transform problem (3.2)–(3.3) into a space-time periodic problem, and then to state the results that will be proved in the next two parts. To do so let us first recall the following result, which has been proved in our previous paper [13]:

Lemma 4.5. *Assume that $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ for some $N \geq 2$. Then there exist an orthogonal basis $\{\zeta_i\}_{i=1}^N$ of $\mathbb{R}^N$ and N constants $\tau_i > 0$ such that $\zeta_1 = \zeta$ and*

$$\mathbb{Z}^N = \bigoplus_{i=1}^{N} \tau_i \mathbb{Z}\zeta_i, \quad \zeta_i \perp \zeta_j \ (i \neq j), \quad i, j = 1, \ldots, N.$$

In particular, if $\zeta = e_\ell$ for some $\ell \in \{1, \ldots, N\}$, a standard coordinate vector of $\mathbb{R}^N$, then $\tau_i = 1$ for all $1 \leq i \leq N$. Now we apply this key lemma to derive a space-time periodic problem from (3.2)–(3.3). For any $k \in \mathbb{Z}^N$ and for some given vector $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$, we decompose k along ζ and $\zeta^\perp$, that is,

$$k = (k \cdot \zeta)\zeta + k_\perp \quad \text{with } k_\perp \in \zeta^\perp.$$

By Lemma 4.5, we have

$$k \cdot \zeta = \tau_1 p_1 \quad \text{and} \quad k_\perp = k - (k \cdot \zeta)\zeta = \sum_{i=2}^{N} \tau_i p_i \zeta_i,$$

where $p_i \in \mathbb{Z}$ for all $1 \leq i \leq N$. Recalling the moving frame of Section 3.1, we find that the $\mathcal{R}$-pulsating condition (3.3) in a given rational direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ actually means that

$$\begin{aligned} &\forall p_1 \in \mathbb{Z},\ \forall \tau \in \tau_2\mathbb{Z} \times \cdots \times \tau_N\mathbb{Z},\ \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}, \\ &(U, V)\left(\xi, t + \frac{k \cdot \zeta}{c}, y + \hat{\mathcal{R}} k_\perp\right) = (U, V)\left(\xi, t + \frac{p_1 \tau_1}{c}, y + \tau\right) = (U, V)(\xi, t, y). \end{aligned} \tag{4.10}$$

In other words, under the frame moving with a speed $c > 0$, the profile $(U, V)(\xi, t, y)$ of pulsating waves in a unit rational direction is periodic with respect to the last two variables (t, y). Furthermore, when we fix a direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ and a wave speed $c > 0$, then the corresponding periods τ_1/c and $\tau_2, \ldots, \tau_N$ are also automatically determined.

Note that all the parameters arising in system (3.2) satisfy the $\mathcal{R}$-pulsating condition (3.3). Hence they also share the periodicity conditions as in (4.10) in the direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$. In the following, we always say that a function is τ_1/c-periodic in t and τ-periodic in y if it satisfies the periodicity conditions of (4.10).

Recalling system (3.2) satisfied by the profile function pair (U, V), we conclude that the existence of pulsating travelling waves for problem (1.1) in the direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ is equivalent to solving the following space-time periodic problem posed for $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$:

$$\begin{cases} \partial_t U - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi, t, y)\nabla_{\xi,y} U) - cU_\xi = -\widetilde{\beta}(\xi, t, y)UV, \\ \partial_t V - \nabla_{\xi,y} \cdot (\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y} V) - cV_\xi = \widetilde{\beta}(\xi, t, y)UV - \widetilde{\gamma}(\xi, t, y)V, \\ (U, V) \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y. \end{cases} \tag{4.11}$$

To be more precise we will prove the following existence theorem.

Theorem 4.6 (Existence of bounded entire solutions for (4.11)). *Let Assumption 1.1 be satisfied, and assume further that $\mu_0 < 0$. Then for each direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ and for each speed $c > c^*(\zeta)$ where $c^*(\zeta)$ is given by (3.8) with $e = \zeta$, problem (4.11) admits a bounded and entire (classical) solution $(U, V) = (U, V)(\xi, t, y)$ and furthermore the following properties hold:*

(i) *The function pair (U, V) satisfies*

$$0 < \min_{(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}} U(\xi, t, y) \le U < S_0, \qquad \min_{(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}} V(\xi, t, y) > 0, \quad \forall \xi \in \mathbb{R}.$$

(ii) *The function pair (U, V) satisfies the following asymptotics at $\xi = +\infty$:*

$$\lim_{\xi\to+\infty} (U, V)(\xi, t, y) = (S_0, 0) \quad \textit{uniformly for } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}.$$

(iii) *The function V has the following exponential decay:*

$$V(\xi, t, y) \sim \Phi_{\lambda_c}(X)e^{-\lambda_c \xi} \quad \textit{with} \quad X = \mathcal{R}(\xi + ct, y)^T \textit{ as } \xi \to +\infty$$

uniformly for $(t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, where Φ_{λ_c} is the unique principal eigenfunction of problem (3.6) in the sense of (3.7) with $\lambda = \lambda_c$.

Roughly speaking, we split the proof of Theorem 4.6 into two steps: the first one considers a similar space-time periodic problem posed on a finite domain and the second one is to pass to the limit in the unbounded domains. Such an approach has been widely used to show the existence of travelling waves for reaction-diffusion systems posed on a straight cylinder (see [6, 22, 25]) and for some age-structured epidemic models (see [20, 21]), or the existence of pulsating fronts for scalar periodic reaction-diffusion equations (see [4, 9, 33]). Let us mention that instead of dealing with a boundary value problem for the elliptic system satisfied by the wave profile in [6, 22, 25], we need to consider a boundary value problem for the parabolic system in the first step. As already underlined, the main difficulty in the second step is to derive some uniform estimates of the solutions.

4.2.2. Existence result in a finite cylinder. Here we construct a solution of problem (4.11) on a bounded domain with respect to ξ. In all this subsection and the next subsection, we fix a vector $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ and a speed $c > c^*(\zeta)$. To proceed we introduce some notations. Let C be the periodicity cell defined by

$$C = (0, \tau_2) \times \cdots \times (0, \tau_N)$$

and we denote by

$$\mathbb{T}_\tau^{N-1} = \mathbb{R}/\tau_2\mathbb{Z} \times \cdots \times \mathbb{R}/\tau_N\mathbb{Z}$$

the $(N-1)$-dimensional torus with an $(N-1)$-tuple $(\tau_2, \ldots, \tau_N)$ given in Lemma 4.5 (here we use the torus just to simplify some notions and function spaces that will be introduced below). Now we consider the following two functions:

$$\underline{U}^+(\xi, t, y) := \max(0, \underline{U}(\xi, t, y)), \quad \underline{V}^+(\xi, t, y) := \max(0, \underline{V}(\xi, t, y)),$$

where $\underline{U}$ and $\underline{V}$ are defined by (4.3) and (4.6), respectively. Then the functions $\underline{U}^+$ and $\underline{V}^+$ are continuous and satisfy

$$\underline{U}^+(\xi, t, y) < S_0, \quad \underline{V}^+(\xi, t, y) < \overline{V}(\xi, t, y), \quad \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1},$$

where the function $\overline{V}$ is defined in Lemma 4.2. Furthermore, it is easy to check that in the direction ζ, these functions are also τ_1/c-periodic in t and τ-periodic in y.

Let $a > 0$ be given and set $\Sigma_a = (-a, a) \times \mathbb{R} \times \mathbb{T}_\tau^{N-1}$. Consider the following boundary value problem associated with (4.11):

$$\begin{cases} \partial_t U - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi, t, y)\nabla_{\xi,y} U) - cU_\xi = -\widetilde{\beta}(\xi, t, y)UV & \text{in } \Sigma_a, \\ \partial_t V - \nabla_{\xi,y} \cdot (\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y} V) - cV_\xi = \widetilde{\beta}(\xi, t, y)UV - \widetilde{\gamma}(\xi, t, y)V & \text{in } \Sigma_a, \\ (U, V) \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \end{cases} \tag{4.12}$$

together with the boundary conditions

$$\begin{cases} U(-a, t, y) = \underline{U}^+(-a, t, y), \quad U(a, t, y) = \underline{U}^+(a, t, y) \\ V(-a, t, y) = \underline{V}^+(-a, t, y), \quad V(a, t, y) = \underline{V}^+(a, t, y) \end{cases} \text{for } (t, y) \in \mathbb{R} \times \mathbb{T}_\tau^{N-1}. \tag{4.13}$$

Note first that from the proof of Lemma 4.4, one may introduce a positive constant

$$a_0 \geq \frac{1}{\delta_2} \ln\left(K_2 \sup_{\mathbb{T}^N} \frac{\Phi_{\lambda_c+\delta_2}}{\Phi_{\lambda_c}}\right) \geq \frac{1}{\delta_2} \ln K_1 > \frac{1}{\delta_1} \ln K_1$$

such that for any $a > a_0$, $\underline{U}^+(-a, t, y) \equiv 0$ and $\underline{V}^+(-a, t, y) \equiv 0$, while $\underline{U}^+(a, \cdot, \cdot)$ and $\underline{V}^+(a, \cdot, \cdot)$ are strictly positive. Then we will prove the following result.

Proposition 4.7. *Let Assumption 1.1 be satisfied, and assume further that $\mu_0 < 0$. Then for any $a > a_0$, problem (4.12)–(4.13) has a classical solution (U_a, V_a) which satisfies*

$$\begin{aligned} &\underline{U}^+(\xi, t, y) \leq U_a(\xi, t, y) \leq S_0, && \forall (\xi, t, y) \in \overline{\Sigma_a}, \\ &\underline{V}^+(\xi, t, y) \leq V_a(\xi, t, y) \leq \overline{V}(\xi, t, y), && \forall (\xi, t, y) \in \overline{\Sigma_a}. \end{aligned}$$

Proof. The proof is divided into four steps with several lemmas.

Step 1. *A fixed-point problem and its well-posedness.*

In order to solve the boundary value problem (4.12)–(4.13), we first reformulate it as a fixed-point problem. For that purpose we begin with some notations. Fix $a > a_0$ and let

$$C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2)$$

be the space of continuous vector-valued functions from $\overline{\Sigma_a}$ to $\mathbb{R}^2$ that are τ_1/c-periodic in t. Recalling $\Sigma_a = (-a, a) \times \mathbb{R} \times \mathbb{T}^{N-1}_\tau$, any function in $C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2)$ is also τ-periodic in y. We endow this function space with the usual sup-norm. Then it is a Banach space. To apply the Schauder fixed-point theorem, let us now define a bounded closed and convex subset E_a of $C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2)$ by

$$E_a = \{(U, V) \in C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2) \mid \underline{U}^+ \le U \le S_0 \text{ and } \underline{V}^+ \le V \le \overline{V} \ \text{ in } \overline{\Sigma_a}\},$$

and furthermore we define a mapping $\mathcal{T}_a : E_a \to C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2)$ by

$$\mathcal{T}_a(U_\star, V_\star) = (U, V), \quad \forall (U_\star, V_\star) \in E_a,$$

where $(U, V) \in C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2)$ is the solution of the linear problem

$$\begin{cases} \partial_t U - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi, t, y)\nabla_{\xi,y} U) - cU_\xi + \widetilde{\beta}(\xi, t, y)UV_\star = 0 & \text{in } \Sigma_a, \\ \partial_t V - \nabla_{\xi,y} \cdot (\widetilde{D}_I(\xi, t, y)\nabla_{\xi,y} V) - cV_\xi + \widetilde{\gamma}(\xi, t, y)V = \widetilde{\beta}(\xi, t, y)U_\star V_\star & \text{in } \Sigma_a, \\ (U, V) \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \end{cases} \tag{4.14}$$

together with the boundary conditions

$$\begin{cases} U(-a, t, y) = \underline{U}^+(-a, t, y), \quad U(a, t, y) = \underline{U}^+(a, t, y) \\ V(-a, t, y) = \underline{V}^+(-a, t, y), \quad V(a, t, y) = \underline{V}^+(a, t, y) \end{cases} \text{for } (t, y) \in \mathbb{R} \times \mathbb{T}^{N-1}_\tau. \tag{4.15}$$

Next we consider this fixed-point problem on the set E_a and we first investigate its solvability.

Lemma 4.8. *For any given pair $(U_\star, V_\star) \in E_a$, problem (4.14)–(4.15) admits a unique strong solution $(U, V) = (U, V)(\xi, t, y)$ in $C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2)$.*

Proof. The proof is inspired by some ideas of Nadin [33, Lemma 4.3]. Let ϕ be the function defined by

$$\phi(\xi, t, y) = \frac{\xi + a}{2a} \underline{U}(a, t, y). \tag{4.16}$$

This function is τ_1/c-periodic in t and τ-periodic in y. We make the change of variable $u = U - \phi$. Then the function u satisfies

$$\begin{cases} \mathcal{L}_S u + (\widetilde{\beta} V_\star)(\xi, t, y)u = g(\xi, t, y) \quad \text{in } \Sigma_a, \\ u \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \\ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}, \quad u(\pm a, t, y) = 0, \end{cases} \tag{4.17}$$

where $\mathcal{L}_S$ is the parabolic operator given by

$$\mathcal{L}_S u = \partial_t u - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi,t,y)\nabla_{\xi,y} u) - c\partial_\xi u$$

and we have set

$$g(\xi,t,y) := -(\widetilde{\beta} V_\star)(\xi,t,y)\phi - \mathcal{L}_S \phi.$$

Note that the function g is continuous in $\overline{\Sigma_a}$ and also periodic with respect to the last two variables (t, y). We are looking for a unique solution of problem (4.17) which is continuous in $\overline{\Sigma_a}$. To that end we define the set

$$\Gamma^0_{\text{per}} = \Big\{ v \in L^2_{\text{loc}}([-a,a] \times \mathbb{R}^{N-1}) \;\Big|\; \text{for any } \tau \in \prod_{i=2}^{N} \tau_i \mathbb{Z}, \\ v(\xi, y + \tau) = v(\xi, y) \text{ for a.e. } (\xi, y) \in [-a,a] \times \mathbb{R}^{N-1} \Big\}$$

and endow it with the L^2-norm on $(-a,a) \times \mathbb{T}^{N-1}_\tau$. Then it is a Banach space. Now, for $\epsilon > 0$ small we consider the following initial boundary value problem:

$$\begin{cases} \mathcal{L}_S u + (b(\xi,t,y) + \epsilon)u = g(\xi,t,y) & \text{for } t > 0 \text{ and } (\xi, y) \in (-a,a) \times \mathbb{T}^{N-1}_\tau, \\ u(\pm a, t, y) = 0 & \text{for } t > 0 \text{ and } y \in \mathbb{R}^{N-1}, \\ u(\xi, 0, y) = u_0(\xi, y) & \text{for } (\xi, y) \in (-a,a) \times \mathbb{R}^{N-1}, \end{cases} \tag{4.18}$$

where we have set $b(\xi,t,y) := (\widetilde{\beta} V_\star)(\xi,t,y)$. For any $u_0 \in \Gamma^0_{\text{per}}$, let $u = u(\xi,t,y)$ be the solution of (4.18) with an initial datum $u(\xi, 0, y) = u_0(\xi, y)$. So necessarily u is τ-periodic with respect to y due to the periodicity conditions (in which sense we shall elaborate below). From the classical parabolic theory (see [29, Theorem 3.4.2] for instance), there exists a unique weak solution of (4.18) in the class

$$u \in C^0\big([0,\infty), L^2((-a,a) \times \mathbb{T}^{N-1}_\tau)\big) \cap L^2\big(0,\infty; H^1_0((-a,a) \times \mathbb{T}^{N-1}_\tau)\big).$$

Since the functions b and g are bounded, it further follows from [29, Theorem 3.7.1] that

$$u \in L^\infty((-a,a) \times (0,\infty) \times \mathbb{T}^{N-1}_\tau). \tag{4.19}$$

In order to look for a periodic (in time) solution of (4.18), for all $u_0 \in \Gamma^0_{\text{per}}$ we investigate the Poincaré mapping

$$\mathcal{T}_\epsilon : \Gamma^0_{\text{per}} \to \Gamma^0_{\text{per}}, \quad u_0 \mapsto u(\tau_1/c).$$

Since $\min_{(\xi,t,y)\in\overline{\Sigma_a}} b(\xi,t,y) \geq 0$, we can choose some η such that $0 < \eta < \epsilon + \min b$. Next, take $u_{01}, u_{02} \in \Gamma^0_{\text{per}}$ and set

$$W(\xi,t,y) := (u_1(\xi,t,y) - u_2(\xi,t,y))e^{\eta t}.$$

Then the function W satisfies

$$W_t - \nabla_{\xi,y} \cdot (\widetilde{D}_S(\xi,t,y)\nabla_{\xi,y} W) - cW_\xi + (b(\xi,t,y) + \epsilon - \eta)W = 0.$$

Multiplying this equation by W and integrating by parts over $(-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^{N-1}$, we obtain

$$\frac{1}{2}\int_{-a}^{a}\int_{\mathbb{T}_\tau^{N-1}}\left(W^2(\xi,\tau_1/c,y)-W^2(\xi,0,y)\right)\mathrm{d}\xi\,\mathrm{d}y$$
$$=-\int_{-a}^{a}\int_{0}^{\tau_1/c}\int_{\mathbb{T}_\tau^{N-1}}[\nabla_{\xi,y}W\widetilde{D}_S(\xi,t,y)\nabla_{\xi,y}W+(b(\xi,t,y)+\epsilon-\eta)W^2]\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}y.$$

From the uniform ellipticity of $\widetilde{D}_S$ and with the above choice for η, one gets

$$\|u_2(\tau_1/c)-u_1(\tau_1/c)\|_{\Gamma^0_{\text{per}}}\le e^{-\eta\tau_1/c}\|u_2(0)-u_1(0)\|_{\Gamma^0_{\text{per}}}.$$

Since $\eta\tau_1/c>0$, the mapping $\mathcal{T}_\epsilon$ is a contraction from Γ^0_{per} into itself. It then follows from the Banach fixed-point theorem that $\mathcal{T}_\epsilon$ admits a unique fixed point in Γ^0_{per}, denoted by u_0^ϵ. In other words, there exists a unique space-time periodic solution u^ϵ of problem (4.18) which satisfies

$$u^\epsilon\in C^0\big([0,\tau_1/c],L^2((-a,a)\times\mathbb{T}_\tau^{N-1})\big)\cap L^2\big(0,\tau_1/c;H_0^1((-a,a)\times\mathbb{T}_\tau^{N-1})\big)$$

and

$$\begin{cases}\mathcal{L}_S u^\epsilon+(b(\xi,t,y)+\epsilon)u^\epsilon=g(\xi,t,y) & \text{in }\Sigma_a,\\ u^\epsilon(\cdot,0,\cdot)=u^\epsilon(\cdot,\tau_1/c,\cdot) & \text{in }\Gamma^0_{\text{per}}.\end{cases}\tag{4.20}$$

Next, by letting $\epsilon\to0$, we expect to obtain a limit function that is at least continuous in $\overline{\Sigma_a}$, and that is the desired solution for problem (4.17). To that end we fix some $\epsilon_0>0$ and we consider a solution family $\{u^\epsilon\}$ of problem (4.20) with any $\epsilon\in(0,\epsilon_0)$. Furthermore, we first use the classical bootstrap arguments to improve the regularity of the solution. Recall that the diffusion matrix field D_S is assumed to be of class $C^{1+\alpha}(\mathbb{T}^N,\mathcal{S}_N)$ and $\widetilde{D}_S(\xi,t,y)=\mathcal{R}D_S(X)\mathcal{R}^T$ with $X\in\mathbb{T}^N$. Note also that the coefficient b and the free term g of equation (4.20) are continuous and bounded in $\overline{\Sigma_a}$. By (4.19), we know that the space-time periodic solution u^ϵ of (4.20) is actually bounded in $L^\infty(\Sigma_a)$ for any $\epsilon\in(0,\epsilon_0)$. Thus, the fixed point $u_0^\epsilon=\mathcal{T}_\epsilon(u_0^\epsilon)$ belongs to $L^\infty((-a,a)\times\mathbb{T}_\tau^{N-1})$. By parabolic regularity, we obtain

$$u^\epsilon\in C^0\big([0,\tau_1/c],L^p((-a,a)\times\mathbb{T}_\tau^{N-1})\big)\cap L^\infty\big(0,\tau_1/c\,;W_0^{1,p}((-a,a)\times\mathbb{T}_\tau^{N-1})\big)$$

for any $p\in(1,\infty)$ and any $\epsilon\in(0,\epsilon_0)$. Since u^ϵ is τ_1/c-periodic in t, it follows from the fixed-point argument that $u_0^\epsilon=\mathcal{T}_\epsilon(u_0^\epsilon)\in W_0^{1,p}((-a,a)\times\mathbb{T}_\tau^{N-1})$ and furthermore parabolic regularity shows that

$$u^\epsilon\in C^0\big([0,\tau_1/c],W_0^{1,p}((-a,a)\times\mathbb{T}_\tau^{N-1})\big)$$
$$\cap L^\infty\big(0,\tau_1/c;W_0^{1,p}\cap W^{2,p}((-a,a)\times\mathbb{T}_\tau^{N-1})\big)$$

for any $p\in(1,\infty)$ and any $\epsilon\in(0,\epsilon_0)$. Again the same argument as above implies that for any $\epsilon\in(0,\epsilon_0)$ one has

$$u_0^\epsilon=\mathcal{T}_\epsilon(u_0^\epsilon)\in W_0^{1,p}((-a,a)\times\mathbb{T}_\tau^{N-1})\cap W^{2,p}((-a,a)\times\mathbb{T}_\tau^{N-1}).$$

Finally, from [29, Theorem 4.9.1] and using the periodicity of u^ϵ in t, we obtain

$$u^\epsilon \in W_p^{2,1,2}((-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^{N-1}) \quad \text{for any } p\in(1,\infty),$$

and

$$u^\epsilon(\pm a,t,y)=0 \quad \text{for a.e. } (t,y)\in\mathbb{R}\times\mathbb{T}_\tau^{N-1}.$$

Moreover, there exists a constant $M>0$ independent of ϵ such that

$$\|u^\epsilon\|_{W_p^{2,1,2}((-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^{N-1})} \le M(\|g\|_\infty + \|u_0^\epsilon\|_{W^{2,p}((-a,a)\times\mathbb{T}_\tau^{N-1})}),$$

and further the periodicity in t and the boundedness of g yield

$$\|u^\epsilon\|_{W_p^{2,1,2}((-a,a)\times(-T,T)\times\mathbb{T}_\tau^{N-1})} \le M, \quad \forall\epsilon\in(0,\epsilon_0),\ \forall\, T>0,\ \forall p\in(1,\infty).$$

Therefore, we conclude that problem (4.20) admits a unique strong solution u^ϵ and the solution family $\{u^\epsilon\}$ is uniformly bounded in $W_{p,\mathrm{loc}}^{2,1,2}(\overline{\Sigma_a})$ for all $p\in(1,\infty)$.

One can now pass to the limit $\epsilon\to0$. To do so we consider a sequence $\{\epsilon_n\}\subset(0,\epsilon_0)$ such that $\epsilon_n\to0$ as $n\to\infty$. The above conclusions ensure that the sequence u^{ϵ_n} converges (up to extracting a subsequence) weakly in $W_{p,\mathrm{loc}}^{2,1,2}(\overline{\Sigma_a})$ for all $p\in(1,\infty)$ and strongly in $C_{\mathrm{loc}}^{1,0,1}(\overline{\Sigma_a})$ to a space-time periodic function u such that

$$\mathcal{L}_S u + (\widetilde{\beta}V_\star)(\xi,t,y)u = g(\xi,t,y) \quad \text{in } \Sigma_a.$$

Furthermore, the above convergence and the periodicity conditions also imply that the limit function u is at least continuous in $\overline{\Sigma_a}$ and satisfies the pointwise boundary conditions:

$$u(-a,t,y)=u(a,t,y)=0, \quad \forall(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}.$$

Consequently, the function $U=u+\phi$ where ϕ is given by (4.16) is a solution of the U-equation for the linear boundary problem (4.14)–(4.15). It remains to prove uniqueness. To this end, let U_1 and U_2 be two solutions of problem (4.14)–(4.15) associated with the U-equation. Then

$$(U_1-U_2)_t - \nabla_{\xi,y}\cdot(\widetilde{D}_S\nabla_{\xi,y}(U_1-U_2)) - c(U_1-U_2)_\xi + \widetilde{\beta}V_\star(U_1-U_2)=0 \ \text{ in } \Sigma_a.$$

Since U_1 and U_2 are periodic in t and satisfy the same boundary conditions, multiplying the above equation by U_1-U_2 and integrating by parts over Σ_a, one gets

$$\int_{-a}^{a}\int_0^{\tau_1/c}\int_{\mathbb{T}_\tau^{N-1}} [\nabla_{\xi,y}(U_1-U_2)\widetilde{D}_S\nabla_{\xi,y}(U_1-U_2) + \widetilde{\beta}V_\star(U_1-U_2)^2]\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}y = 0.$$

Recall that the function $\widetilde{\beta}(\xi,t,y)=\beta(X)$ with $X=\mathcal{R}(\xi+ct,y)^T\in\mathbb{T}^N$ is strictly positive everywhere. Moreover, $V_\star\ge \underline{V}^+$ in Σ_a is nonnegative and nonzero for any $a>a_0$. From the uniform ellipticity of $\widetilde{D}_S$, we conclude that the above equality holds if and only if $U_1\equiv U_2$ in $\overline{\Sigma_a}$.

In the same way let ψ be the function defined by

$$\psi(\xi,t,y) = \frac{\xi+a}{2a}\underline{V}(a,t,y).$$

We make the change of variable $v = V - \psi$. Then v satisfies

$$\begin{cases} \mathcal{L}_I v + \widetilde{\gamma}(\xi,t,y)v = h(\xi,t,y) & \text{in } \Sigma_a, \\ v \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \\ \forall (t,y) \in \mathbb{R}\times\mathbb{R}^{N-1}, \quad v(\pm a,t,y) = 0, \end{cases}$$

where $\mathcal{L}_I$ is the parabolic operator given by

$$\mathcal{L}_I v = \partial_t v - \nabla_{\xi,y}\cdot(\widetilde{D}_I(\xi,t,y)\nabla_{\xi,y}v) - c\partial_\xi v$$

and we have set

$$h(\xi,t,y) := \widetilde{\beta}(\xi,t,y)U_\star V_\star - (\mathcal{L}_I + \widetilde{\gamma}(\xi,t,y))\psi.$$

The same argument as above enables us to obtain the existence and uniqueness of a space-time periodic strong solution v for this problem. Here we do not perform an approximation procedure since the function $\widetilde{\gamma}$ is strictly positive. Thus, it is sufficient to set $\eta = \min_{\overline{\Sigma_a}}\widetilde{\gamma}$ when we construct a contractive Poincaré mapping.

As a consequence, $(U,V) = (u+\phi, v+\psi) \in C^0_{t\text{-per}}(\overline{\Sigma_a},\mathbb{R}^2)$ is the unique strong solution of problem (4.14)–(4.15).

This ends the proof of Lemma 4.8. ∎

Step 2. *The set E_a is invariant by $\mathcal{T}_a$, that is, $\mathcal{T}_a(E_a) \subset E_a$.*

This result follows from successive applications of the weak parabolic maximum principle for a space-time periodic problem. Note that here we cannot directly apply the classical weak maximum principle posed for a finite time interval. Therefore, for the sake of completeness, we prove this extension on the domain $\Sigma_a = (-a,a)\times\mathbb{R}\times\mathbb{T}_\tau^{N-1}$. In particular, the coefficient of the zero-order term of equations is not strictly positive. We denote by $A = A(\xi,t,y)$ the diffusion matrix fields $\widetilde{D}_S$ or $\widetilde{D}_I$ and then we prove the following auxiliary lemma.

Lemma 4.9. *Assume that $u \in W_p^{2,1,2}((-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^{N-1})$ (for some $p \geq N+2$) satisfies*

$$\begin{cases} \partial_t u - \nabla_{\xi,y}\cdot(A(\xi,t,y)\nabla_{\xi,y}u) - c\partial_\xi u + h(\xi,t,y)u \geq 0 & \text{in } \Sigma_a, \\ u \text{ is } \tau_1/c\text{-periodic in } t, \\ u \geq 0 \quad \text{on } \partial\Sigma_a, \end{cases}$$

where $h \in C^0(\overline{\Sigma_a})$ is τ_1/c-periodic in t and nonnegative. Then $u \geq 0$ in $\overline{\Sigma_a}$.

Proof. Firstly, as $p \geq N+2$, from Sobolev's embeddings [29, Lemma 2.3.3] and the periodicity in t, we know that the function u belongs to $C^{1,0,1}(\overline{\Sigma_a})$ at least. Now set

$u^-(\xi,t,y) := \max(-u(\xi,t,y),0)$. Since $u \geq 0$ on $\partial\Sigma_a$, one has $u^-(\pm a,\cdot,\cdot) = 0$. Multiply the equation by u^- and integrate by parts over Σ_a. Since u is periodic in t, we obtain

$$-\int_{-a}^{a}\int_{0}^{\tau_1/c}\int_{\mathbb{T}_\tau^{N-1}} \nabla_{\xi,y}u^- A\nabla_{\xi,y}u^- \,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}y \geq \int_{-a}^{a}\int_{0}^{\tau_1/c}\int_{\mathbb{T}_\tau^{N-1}} h(\xi,t,y)(u^-)^2\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}y.$$

Since h is nonnegative and A is uniformly elliptic, the left-hand integral is nonpositive while the right-hand one is nonnegative. Therefore, $u^-(\xi,t,y) \equiv 0$, that is, $u \geq 0$ in $\overline{\Sigma_a}$. ∎

One can now complete the proof of $\mathcal{T}_a(E_a) \subset E_a$. For any given pair $(U_\star, V_\star) \in E_a$, let $(U,V) = \mathcal{T}_a(U_\star,V_\star)$ be the unique solution of (4.14)–(4.15). By Lemma 4.8, we know that $(U,V) \in C^0_{t\text{-per}}(\overline{\Sigma_a},\mathbb{R}^2)$ is a strong solution in the sense of $W_p^{2,1,2}$ for any $p \in (1,\infty)$. We apply Lemma 4.9 to the following problem:

$$\begin{cases} \partial_t U - \nabla_{\xi,y}\cdot(\widetilde{D}_S(\xi,t,y)\nabla_{\xi,y}U) - cU_\xi + (\widetilde{\beta}V_\star)(\xi,t,y)U = 0 \quad \text{in } \Sigma_a,\\ U \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y,\\ \forall (t,y) \in \mathbb{R}\times\mathbb{T}_\tau^{N-1}, \quad U(-a,t,y) = 0,\ U(a,t,y) = \underline{U}(a,t,y) > 0. \end{cases}$$

Then we obtain $U \geq 0$ in $\overline{\Sigma_a}$. Likewise, since $U(\pm a,\cdot,\cdot) \leq S_0$, applying Lemma 4.9 to the function $W(\xi,t,y) := S_0 - U(\xi,t,y)$ yields $U \leq S_0$ in $\overline{\Sigma_a}$. We conclude that

$$0 \leq U(\xi,t,y) \leq S_0, \quad \forall (\xi,t,y) \in \overline{\Sigma_a}. \tag{4.21}$$

From Lemma 4.2, the function $W := \overline{V} - V$ satisfies

$$\begin{cases} \partial_t W - \nabla_{\xi,y}\cdot(\widetilde{D}_I(\xi,t,y)\nabla_{\xi,y}W) - cW_\xi + \widetilde{\gamma}(\xi,t,y)W \\ \qquad = \widetilde{\beta}S_0\overline{V} - \widetilde{\beta}U_\star V_\star \geq 0 \quad \text{in } \Sigma_a,\\ W \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y,\\ \forall (t,y) \in \mathbb{R}\times\mathbb{T}_\tau^{N-1}, \quad W(\pm a,t,y) = (\overline{V} - \underline{V}^+)(\pm a,t,y) \geq 0. \end{cases}$$

Therefore, $V \leq \overline{V}$ in $\overline{\Sigma_a}$. Further, since $U \geq 0$ and $V \leq \overline{V}$ in $\overline{\Sigma_a}$, it follows from Lemma 4.3 that for any $(\xi,t,y) \in \Sigma_a$ we have

$$\begin{cases} \partial_t(U - \underline{U}) - \nabla_{\xi,y}\cdot\big(\widetilde{D}_S(\xi,t,y)\nabla_{\xi,y}(U - \underline{U})\big) - c(U - \underline{U})_\xi + (\widetilde{\beta}\overline{V})(U - \underline{U}) \geq 0,\\ U - \underline{U} \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y,\\ \forall (t,y) \in \mathbb{R}\times\mathbb{T}_\tau^{N-1}, \quad (U - \underline{U})(\pm a,t,y) \geq 0. \end{cases}$$

Thus, using Lemma 4.9 and (4.21), we conclude that

$$\underline{U}^+(\xi,t,y) = \max(0,\underline{U}(\xi,t,y)) \leq U(\xi,t,y) \leq S_0, \quad \forall (\xi,t,y) \in \overline{\Sigma_a}.$$

Let us now verify that $V \geq \underline{V}^+$ in $\overline{\Sigma_a}$. Firstly, since $V(\pm a, \cdot, \cdot) \geq 0$, the same argument as above easily yields $V \geq 0$ in $\overline{\Sigma_a}$. Next, from Lemma 4.4, we obtain

$$\partial_t(V - \underline{V}) - \nabla_{\xi,y} \cdot \big(\widetilde{D}_I(\xi,t,y)\nabla_{\xi,y}(V - \underline{V})\big) - c(V - \underline{V})_\xi + \widetilde{\gamma}(\xi,t,y)(V - \underline{V})$$
$$= \widetilde{\beta}(\xi,t,y)U_\star V_\star - \widetilde{\beta}(\xi,t,y)\underline{U}^+\underline{V} \geq 0 \quad \text{in } \Sigma_a \cap \Omega^+, \tag{4.22}$$

and

$$\begin{cases} (V - \underline{V}) \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \\ (V - \underline{V})(\xi,t,y) \geq 0 \quad \text{on } \partial(\Sigma_a \cap \Omega^+), \end{cases}$$

where

$$\partial(\Sigma_a \cap \Omega^+) = (\overline{\Sigma_a} \cap \partial\Omega^+) \cup (\partial\Sigma_a \cap \mathring{\Omega}^+).$$

Now we claim that $V \geq \underline{V}$ in $\overline{\Sigma_a} \cap \Omega^+$. Without loss of generality, one may take a sequence $\{(U_\star^n, V_\star^n)(\xi,t,y)\}$ of smooth functions such that the corresponding limit solution V is of class $C^{2,1,2}$. Indeed, if

$$\min_{(\xi,t,y)\in \overline{\Sigma_a} \cap \Omega^+} (V - \underline{V})(\xi,t,y) < 0,$$

then the point $P_0 := (\xi_0, t_0, y_0)$ where $V - \underline{V}$ attains its minimum lies in $\Sigma_a \cap \mathring{\Omega}^+$ because $(V - \underline{V})(\pm a, \cdot, \cdot) = 0$ and $V \geq \underline{V}$ on $\partial\Omega^+$. Since $V - \underline{V}$ is periodic in t, one has

$$(V - \underline{V})_t(P_0) = 0,\ \nabla_{\xi,y}(V - \underline{V})(P_0) = 0_{\mathbb{R}^N},\ D^2_{\xi,y}(V - \underline{V})(P_0) \text{ is nonnegative-definite.}$$

Using the uniform ellipticity and symmetry of $\widetilde{D}_I$, one can further verify that

$$\nabla_{\xi,y} \cdot (\widetilde{D}_I \nabla_{\xi,y}(V - \underline{V}))(P_0) \geq 0.$$

Since $\widetilde{\gamma}$ is strictly positive, we obtain

$$\big(\partial_t(V - \underline{V}) - \nabla_{\xi,y} \cdot (\widetilde{D}_I \nabla_{\xi,y}(V - \underline{V})) - c(V - \underline{V})_\xi + \widetilde{\gamma}(V - \underline{V})\big)(P_0) < 0,$$

which contradicts the differential inequality (4.22). Finally, as $V \geq 0$ in $\overline{\Sigma_a}$, we conclude that

$$\underline{V}^+(\xi,t,y) = \max(0, \underline{V}(\xi,t,y)) \leq V(\xi,t,y) \leq \overline{V}(\xi,t,y), \quad \forall (\xi,t,y) \in \overline{\Sigma_a}.$$

This completes the proof of $\mathcal{T}_a(E_a) \subset E_a$.

Step 3. *The mapping $\mathcal{T}_a : E_a \to E_a$ is compact.*

We need to prove that $\overline{\mathcal{T}_a(E_a)}$ is compact in E_a. The main arguments are based on parabolic estimates and Sobolev embedding theorems. Let $\{(U_\star^n, V_\star^n)\}_{n\in\mathbb{N}}$ be a sequence in E_a and let

$$(U^n, V^n) = \mathcal{T}_a(U_\star^n, V_\star^n) \in C^0_{t\text{-per}}(\overline{\Sigma_a}, \mathbb{R}^2), \quad n \in \mathbb{N},$$

be the solutions associated with problem (4.14)–(4.15). Recalling Assumption 1.1 about the diffusion matrix fields and since the functions $\underline{U}^+$ and $\underline{V}^+$ are continuous and bounded, the standard parabolic L^p-estimates up to the boundary give a constant M independent of n such that

$$\|U^n\|_{W_p^{2,1,2}((-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^N)} \le M \quad\text{and}\quad \|V^n\|_{W_p^{2,1,2}((-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^N)} \le M$$

for any $p \in (1,\infty)$ (see [29, Chapter IV] or [35, Theorem 48.1] for a collection of those estimates). Now, using Sobolev's embeddings [29, Lemma 2.3.3] for p large enough, we see that the embedding

$$W_p^{2,1,2}((-a,a)\times(0,\tau_1/c)\times\mathbb{T}_\tau^N) \hookrightarrow C^{\nu,\nu/2,\nu}([-a,a]\times[0,\tau_1/c]\times\mathbb{T}_\tau^N)$$

with $\nu := 2-(N+2)/p > 0$ is compact. Thus, from the periodicity of (U^n,V^n) in t, one may extract a subsequence of $\{(U^n,V^n)\}_{n\in\mathbb{N}}$, not relabelled, such that $\{(U^n,V^n)\}$ converges uniformly (by periodicity) in $\overline{\Sigma_a}$ to a function pair $(U,V) \in C^0_{t\text{-per}}(\overline{\Sigma_a},\mathbb{R}^2)$. Since (U^n,V^n) satisfies system (4.14) and the boundary condition (4.15) for each $n \in \mathbb{N}$, the limit function pair (U,V) also satisfies (4.14) and (4.15) in $\overline{\Sigma_a}$. Furthermore, the invariance in Step 2 implies that (U,V) belongs to E_a. Therefore, the operator $\mathcal{T}_a$ is compact from E_a into itself.

Step 4. *A fixed point of $\mathcal{T}_a$.*

We conclude from the previous steps and the Schauder fixed-point theorem that the mapping $\mathcal{T}_a$ has a fixed point in E_a, that is, $(U_\star,V_\star) \in E_a$ satisfying $\mathcal{T}_a(U_\star,V_\star) = (U_\star,V_\star)$. As a consequence, problem (4.12)–(4.13) has a solution $(U_a,V_a) \in E_a$. Furthermore, by parabolic regularity up to the boundary, (U_a,V_a) is a classical solution of (4.12)–(4.13) in the sense of $C^{2+\alpha,1+\alpha/2,2+\alpha}(\overline{\Sigma_a})^2$ (recalling Assumption 1.1 about the coefficients).

This ends the proof of Proposition 4.7. ∎

4.2.3. Passage to the limit in unbounded domains. In this part, we are going to pass to the limit $a \to \infty$ in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}$ for the solution $(U_a,V_a)(\xi,t,y)$ in Proposition 4.7. As already underlined, the main difficulty is to obtain estimates of solutions independent of the length of the finite interval $[-a,a]$.

Let $\{a_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers such that $a_n \to +\infty$ as $n \to \infty$ and let (U_{a_n},V_{a_n}) be a solution of problem (4.12)–(4.13) in Σ_{a_n} with $a = a_n$. Note first that the sequence $\{(U_{a_n},V_{a_n})\}$ satisfies the upper and lower estimates

$$\max(0,\underline{U}) \le U_{a_n} \le S_0 \quad\text{and}\quad \max(0,\underline{V}) \le V_{a_n} \le \overline{V} \quad\text{in } \overline{\Sigma_{a_n}}.$$

Therefore, the component U_{a_n} of the solution is bounded for any $n \in \mathbb{N}$. For notational concision, we denote by $(U_n,V_n) = (U_{a_n},V_{a_n})$ the solution of (4.12)–(4.13) in Σ_{a_n}. Next in order to perform a limit procedure as $n \to \infty$, we need to derive new estimates for the sequence of functions $\{V_n(\xi,t,y)\}$ to ensure that it is uniformly bounded. To that end we extend some ideas of Ducrot et al. [19], where the authors proved the weak dissipativity of the solution semiflow for a prey-predator system (which actually covers a large class of models) in a homogeneous medium.

Lemma 4.10 (Uniform boundedness). *The function $V_n(\xi, t, y)$ is uniformly bounded on $\overline{\Sigma_{a_n}}$, that is, there exists a constant $M > 0$ such that for any n large enough,*

$$V_n(\xi, t, y) \leq M, \quad \forall (\xi, t, y) \in [-a_n, a_n] \times \mathbb{R} \times \mathbb{R}^{N-1},$$

where the bound $M = M(\zeta)$ depends only on the direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$.

Proof. Let us first recall that for each n large enough, (U_n, V_n) satisfies the following boundary conditions:

$$\begin{cases} U(-a_n, t, y) = 0, & U(a_n, t, y) = \underline{U}(a_n, t, y) \\ V(-a_n, t, y) = 0, & V(a_n, t, y) = \underline{V}(a_n, t, y) \end{cases} \quad \text{for } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}.$$

Thus, the sequences $\{U_n(\pm a_n, \cdot, \cdot)\}$ and $\{V_n(\pm a_n, \cdot, \cdot)\}$ are uniformly bounded. Moreover, the component $U_n(\xi, t, y) \leq S_0$ is uniformly bounded with respect to $(\xi, t, y) \in \overline{\Sigma_{a_n}}$ and $n \in \mathbb{N}$. Next we prove that the other sequence $\{V_n(\xi, t, y)\}$ is also uniformly bounded in Σ_{a_n}.

Assume by contradiction that there is a sequence $\{(\xi_n, t_n, y_n)\}$ with $\xi_n \in (-a_n, a_n)$, $t_n \in \mathbb{R}$ and $y_n \in \mathbb{R}^{N-1}$ such that

$$V_n(\xi_n, t_n, y_n) \to \infty \quad \text{as } n \to \infty.$$

One may suppose, without loss of generality, that

$$\sup_{(\xi,y)\in(-a_n,a_n)\times\mathbb{R}^{N-1}} V_n(\xi, t_n, y) = n \quad \text{and} \quad V_n(\xi_n, t_n, y_n) \in [n/2, n]. \tag{4.23}$$

For all n, there exist $\bar{t}_n \in \frac{\tau_1}{c}\mathbb{Z}$ and $\bar{y}_n \in \prod_{i=2}^N \tau_i \mathbb{Z}$ such that $s_n = t_n - \bar{t}_n \in [0, \tau_1/c]$ and $z_n = y_n - \bar{y}_n \in \overline{C} = \prod_{i=2}^N [0, \tau_i]$. One may assume that, up to extracting a subsequence, $s_n \to s_\infty$ and $z_n \to z_\infty$ as $n \to \infty$. Note that

$$V_n(\xi, t, y) \leq \overline{V}(\xi, t, y) = e^{-\lambda_c \xi} \Phi_{\lambda_c}(X), \quad \forall (\xi, t, y) \in \overline{\Sigma_{a_n}}, \ \forall n \in \mathbb{N}.$$

Since Φ_{λ_c} is strictly positive and bounded for all $X = \mathcal{R}(\xi + ct, y)^T \in \mathbb{T}^N$, one has $\xi_n \to -\infty$ as $n \to \infty$. Clearly, $a_n - \xi_n \to \infty$ as $n \to \infty$ and up to taking a subsequence, one may assume that $-a_n - \xi_n \to -\kappa$ for some $\kappa \in [0, \infty]$ as $n \to \infty$. Two cases may occur: either $\kappa < \infty$ or $\kappa = \infty$. Consider now the sequence of functions

$$(U_n, V_n)(\xi + \xi_n, t + t_n, y), \quad (\xi, t, y) \in [-a_n - \xi_n, a_n - \xi_n] \times \mathbb{R} \times \mathbb{R}^{N-1}.$$

Define

$$\widetilde{V}_n(\xi, t, y) = \frac{V_n(\xi + \xi_n, t + t_n, y)}{V_n(\xi_n, t_n, y_n)}. \tag{4.24}$$

For each n and all $(\xi, t, y) \in [-a_n - \xi_n, a_n - \xi_n) \times \mathbb{R} \times \mathbb{R}^{N-1}$, the function $\widetilde{V}_n$ satisfies

$$\partial_t \widetilde{V}_n - \nabla_{\xi,y} \cdot \left(\widetilde{D}_I(\xi + \xi_n, t + s_n, y)\nabla_{\xi,y}\widetilde{V}_n\right) - c\partial_\xi \widetilde{V}_n - h_n(\xi, t, y)\widetilde{V}_n = 0.$$

where

$$h_n(\xi,t,y) = (\widetilde{\beta}U_n - \widetilde{\gamma})(\xi+\xi_n, t+s_n, y),\ (\xi,t,y) \in [-a_n-\xi_n, a_n-\xi_n]\times\mathbb{R}\times\mathbb{R}^{N-1}. \tag{4.25}$$

Since $\widetilde{\beta}$ and $\widetilde{\gamma}$ are globally bounded and $\{U_n(\xi,t,y)\}$ is uniformly bounded, the sequence of functions $\{h_n(\xi,t,y)\}$ is also uniformly bounded with respect to $n \in \mathbb{N}$ on the domain $[-a_n-\xi_n, a_n-\xi_n]\times\mathbb{R}\times\mathbb{R}^{N-1}$. From our construction (4.23), one may observe that

$$V_n(\xi+\xi_n, t_n, y) \le n \le 2V_n(\xi_n, t_n, y_n), \quad \forall \xi \in (-a_n-\xi_n, a_n-\xi_n),$$

which implies that $\widetilde{V}_n(\xi,0,y) \le 2$ for all $(\xi,y) \in (-a_n-\xi_n, a_n-\xi_n)\times\mathbb{R}^{N-1}$. Furthermore, for all $(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}$,

$$\widetilde{V}_n(-a_n-\xi_n, t, y) = 0 \quad \text{and} \quad \widetilde{V}_n(a_n-\xi_n, t, y) = \frac{V_n(a_n, t+t_n, y)}{V_n(\xi_n, t_n, y_n)} \le 1.$$

It then follows from the parabolic comparison principle that

$$\widetilde{V}_n(\xi,t,y) \le 2e^{\|h_n\|_\infty t}, \quad \forall (\xi,t,y) \in [-a_n-\xi_n, a_n-\xi_n]\times[0,\tau_1/c]\times\mathbb{R}^{N-1}.$$

Since h_n is globally bounded and $\widetilde{V}_n$ is periodic in t for each n, there exists some constant $K > 2$ such that

$$\widetilde{V}_n(\xi,t,y) \le K, \quad \forall (\xi,t,y) \in [-a_n-\xi_n, a_n-\xi_n]\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

Consequently, $\{\widetilde{V}_n(\xi,t,y)\}$ is uniformly bounded with respect to $n\in\mathbb{N}$ on each domain of the form $[-a_n-\xi_n, a_n-\xi_n]\times\mathbb{R}\times\mathbb{R}^{N-1}$. Furthermore, since $\widetilde{D}_I(\xi,t,y) = \mathcal{R}D_I(X)\mathcal{R}^T$ with $\mathcal{R}\in\mathcal{O}(\mathbb{R}^N)$ and D_I is of class $C^{1+\alpha}$ with respect to $X = \mathcal{R}(\xi+ct, y)^T \in \mathbb{T}^N$, one may assume that up to a subsequence,

$$\widetilde{D}_I(\xi+\xi_n, t+s_n, y) \to \widetilde{D}_{I,\infty}(\xi, t+s_\infty, y) \text{ in } C^{1,0,1}_{\text{loc}}((-\kappa,\infty)\times\mathbb{R}\times\mathbb{R}^{N-1}) \text{ as } n\to\infty$$

locally uniformly with respect to $\xi\in(-\kappa,\infty)$ and uniformly for $(t,y)\in\mathbb{R}^N$ because of the periodicity conditions.

Now if $\kappa=\infty$, then by parabolic regularity, one may assume that, possibly along a subsequence still denoted by n,

$$\widetilde{V}_n(\xi,t,y) \to \widetilde{V}_\infty(\xi,t,y) \text{ locally uniformly for } \xi\in\mathbb{R} \text{ and uniformly for } (t,y)\in\mathbb{R}^N$$

as $n\to\infty$, where the convergence is weak in $W^{2,1,2}_{p,\text{loc}}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1})$ for all $p\in(1,\infty)$ and strong in $C^{1,0,1}_{\text{loc}}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1})$ and where the limit function $\widetilde{V}_\infty$ satisfies

$$\begin{cases} \partial_t\widetilde{V}_\infty - \nabla_{\xi,y}\cdot\big(\widetilde{D}_{I,\infty}(\xi,t+s_\infty,y)\nabla_{\xi,y}\widetilde{V}_\infty\big) - c\partial_\xi\widetilde{V}_\infty - h_\infty(\xi,t,y)\widetilde{V}_\infty = 0, \\ \widetilde{V}_\infty \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \\ 0 \le \widetilde{V}_\infty(\xi,t,y) \le K, \quad \forall(\xi,t,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}, \\ \widetilde{V}_\infty(0,0,z_\infty) = 1. \end{cases}$$

Here the bounded function h_∞ is the L^∞_{loc}-weak star limit of the bounded sequence $\{h_n(\xi,t,y)\}$ defined in (4.25). Thanks to $\widetilde{V}_\infty(0,0,z_\infty)=1$ and since $\widetilde{V}_\infty$ is nonnegative, the strong maximum principle shows that $\widetilde{V}_\infty$ is positive everywhere. Thus, from the definition of $\widetilde{V}_n(\xi,t,y)$, we conclude that

$$V_n(\xi+\xi_n,t+t_n,y)\to\infty \quad \text{locally uniformly for } (\xi,t,y)\in\mathbb{R}^{N+1} \text{ as } n\to\infty. \tag{4.26}$$

If $\kappa<\infty$, we define $\widehat{V}_n(\xi,t,y)=\widetilde{V}_n(\xi-a_n-\xi_n,t,y)$ for $(\xi,t,y)\in[0,2a_n]\times\mathbb{R}\times\mathbb{R}^{N-1}$, where $\widetilde{V}_n$ is given by (4.24). Then $\{\widehat{V}_n(\xi,t,y)\}$ is uniformly bounded with respect to $n\in\mathbb{N}$ on each domain of the form $[0,2a_n]\times\mathbb{R}\times\mathbb{R}^{N-1}$. By parabolic regularity, one may assume that, possibly along a subsequence still denoted by n,

$$\widehat{V}_n(\xi,t,y)\to\widehat{V}_\infty(\xi,t,y) \text{ locally uniformly for } \xi\in\mathbb{R}_+ \text{ and uniformly for } (t,y)\in\mathbb{R}^N$$

as $n\to\infty$, where the convergence is weak in $W^{2,1,2}_{p,loc}(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^{N-1})$ for any $p\in(1,\infty)$ and strong in $C^{1,0,1}_{loc}(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^{N-1})$. Notice that the above convergence also implies that

$$\widehat{V}_n(0,t,y)\to 0 \quad \text{uniformly for } (t,y)\in\mathbb{R}\times\mathbb{R}^{N-1} \text{ as } n\to\infty.$$

As a consequence, the limit function $\widehat{V}_\infty$ satisfies

$$\begin{cases}\partial_t\widehat{V}_\infty-\nabla_{\xi,y}\cdot\left(\widetilde{D}_{I,\infty}(\xi-\kappa,t+s_\infty,y)\nabla_{\xi,y}\widehat{V}_\infty\right)-c\partial_\xi\widehat{V}_\infty-h_\infty(\xi-\kappa,t,y)\widehat{V}_\infty=0,\\ \widehat{V}_\infty \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y,\\ 0\le\widehat{V}_\infty(\xi,t,y)\le K, \quad \forall(\xi,t,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^{N-1},\\ \widehat{V}_\infty(0,t,y)=0, \quad \forall(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1},\end{cases}$$

Furthermore, since $\widehat{V}_n(a_n+\xi_n,0,y_n)=1$, one has $\widehat{V}_\infty(\kappa,0,z_\infty)=1$, which also implies that $\kappa>0$. Therefore, the strong maximum principle yields

$$\widehat{V}_\infty(\xi,t,y)>0, \quad \forall\xi\in(0,\infty),\ \forall(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}.$$

From the definition of $\{\widehat{V}_n(\xi,t,y)\}$ and (4.24), we conclude that

$$V_n(\xi+\xi_n,t+t_n,y)\to\infty \text{ locally uniformly for } (\xi,t,y)\in(-\kappa,\infty)\times\mathbb{R}^N \text{ as } n\to\infty. \tag{4.27}$$

Now we make the following claim.

Claim 1. *In both cases, that is, for any $\kappa\in(0,\infty]$, the sequence $\{U_n(\xi,t,y)\}$ satisfies*

$$\lim_{n\to\infty}U_n(\xi+\xi_n,t+t_n,y)=0, \quad \forall(\xi,t,y)\in(-\kappa,\infty)\times\mathbb{R}\times\mathbb{R}^{N-1}. \tag{4.28}$$

Before proving the claim, let us first complete the proof of Lemma 4.10. Recalling the definition of $\{h_n(\xi,t,y)\}$ in (4.25) and $\widetilde{\gamma}(\xi,t,y)=\gamma(X)>0$ with

$X = \mathcal{R}(\xi + ct, y)^T \in \mathbb{T}^N$, Claim 1 implies that the L^∞_{loc}-weak star limit h_∞ satisfies

$$h_\infty(\xi, t, y) \le -\underline{\gamma}, \quad \forall (\xi, t, y) \in (-\kappa, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1},$$

where we have set $\underline{\gamma} = \min_{\mathbb{T}^N} \gamma > 0$. As a consequence, the limit function $(-\kappa, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1} \ni (\xi, t, y) \mapsto W_\infty(\xi, t, y)$ given by

$$W_\infty(\xi, t, y) = \widetilde{V}_\infty(\xi, t, y) \text{ if } \kappa = \infty, \quad W_\infty(\xi, t, y) = \widehat{V}_\infty(\xi + \kappa, t, y) \text{ if } \kappa < \infty$$

satisfies the problem

$$\begin{cases} \partial_t W_\infty \le \nabla_{\xi,y} \cdot \big(\widetilde{D}_{I,\infty}(\xi, t + s_\infty, y) \nabla_{\xi,y} W_\infty\big) + c\partial_\xi W_\infty - \underline{\gamma} W_\infty, \\ W_\infty \text{ is } \tau_1/c\text{-periodic in } t \text{ and } \tau\text{-periodic in } y, \\ 0 < W_\infty(\xi, t, y) \le K, \quad \forall (\xi, t, y) \in (-\kappa, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1}, \\ W_\infty(0, 0, z_\infty) = 1, \end{cases}$$

together with the boundary condition

$$W_\infty(-\kappa, t, y) = 0 \quad \text{for all } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \text{ when } \kappa < \infty.$$

Notice that the function $e^{-\underline{\gamma} t}$ is a supersolution of the above equation. Since $K > 2$, it follows from the parabolic comparison principle that

$$W_\infty(\xi, t, y) \le K e^{-\underline{\gamma} t}, \quad \forall (\xi, t, y) \in (-\kappa, \infty) \times \mathbb{R}_+ \times \mathbb{R}^{N-1}.$$

Since W_∞ is τ_1/c-periodic in t, we have

$$W_\infty(0, 0, z_\infty) = W_\infty(0, n\tau_1/c, z_\infty) \le K e^{-\frac{n\tau_1}{c}\underline{\gamma}}, \quad \forall n \ge 1.$$

Finally, as $\underline{\gamma} > 0$ and $\tau_1/c > 0$, letting $n \to \infty$ one gets a contradiction with the normalization $W_\infty(0, 0, z_\infty) = 1$.

This completes the proof of Lemma 4.10. ∎

Proof of Claim 1. Let $M, R, T > 0$ be given. Choose $\varepsilon \in (-\kappa, 0)$ and let $U = U^{\varepsilon,n}_{R,T}(\xi, t, y)$ be the solution of the following problem:

$$\begin{cases} \partial_t U - \nabla_{\xi,y} \cdot \big(\widetilde{D}_S(\xi + \xi_n, t + s_n, y) \nabla_{\xi,y} U\big) - cU_\xi + \underline{\beta} M U = 0 & \text{in } \Omega^\varepsilon_{R,T}, \\ U(\varepsilon, t, y) = U(R, t, y) = S_0 & \text{for } |t| < T \text{ and } y \in \mathbb{R}^{N-1}, \\ U(\xi, -T, y) = S_0 & \text{for } \varepsilon \le \xi \le R \text{ and } y \in \mathbb{R}^{N-1}, \end{cases}$$

where

$$\Omega^\varepsilon_{R,T} := (\varepsilon, R) \times (-T, T) \times \mathbb{T}^{N-1}_\tau.$$

Note first that $U^{\varepsilon,n}_{R,T}$ is still τ-periodic in y since $\widetilde{D}_S$ is τ-periodic in y and the constant S_0 is spatially periodic. Furthermore, by (4.26)–(4.27), one can choose $N_0 > 0$ large enough (depending on M, ε, R and T) such that

$$V_n(\xi + \xi_n, t + t_n, y) \ge M, \quad \forall n \ge N_0, \ \forall (\xi, t, y) \in \overline{\Omega^\varepsilon_{R,T}}.$$

Therefore, the parabolic comparison principle ensures that

$$U_n(\xi+\xi_n, t+t_n, y) \le U^{\varepsilon,n}_{R,T}(\xi,t,y), \quad \forall n \ge N_0,\ \forall (\xi,t,y) \in \overline{\Omega^{\varepsilon}_{R,T}}. \tag{4.29}$$

Observe further that

$$0 \le U^{\varepsilon,n}_{R,T}(\xi,t,y) \le S_0, \quad \forall n \ge N_0,\ \forall (\xi,t,y) \in \overline{\Omega^{\varepsilon}_{R,T}}.$$

Recall that $\widetilde{D}_S(\xi,t,y) = \mathcal{R} D_S(X)\mathcal{R}^T$ with $\mathcal{R} \in \mathcal{O}(\mathbb{R}^N)$ and D_S is a matrix-valued function of class $C^{1+\alpha}$ with respect to $X = \mathcal{R}(\xi+ct, y)^T \in \mathbb{T}^N$. As a consequence of the above uniform boundedness of $U^{\varepsilon,n}_{R,T}$ and Schauder parabolic regularity, for any positive constants R, T and any sequence $\{n_k\}_{k\in\mathbb{N}}$ tending to infinity as $k\to\infty$, one may extract a subsequence (not relabelled) such that for some $\alpha' \in (0,\alpha)$,

$$\widetilde{D}_S(\xi+\xi_{n_k}, t+s_{n_k}, y) \to \widetilde{D}_{S,\infty}(\xi, t+s_\infty, y) \quad \text{in } C^{1+\alpha',\alpha/2,1+\alpha}(\overline{\Omega^{\varepsilon}_{R,T}})$$

and

$$U^{\varepsilon,n_k}_{R,T}(\xi,t,y) \to U^{\varepsilon}_{R,T}(\xi,t,y) \quad \text{uniformly for } (\xi,t,y) \in \overline{\Omega^{\varepsilon}_{R/2,T/2}}.$$

Thanks to the uniform boundedness of $U^{\varepsilon}_{R,T}$ with respect to ε, R and T, using parabolic regularity again, for any sequences $\{R_k\}$ and $\{T_k\}$ tending to infinity, one may extract subsequences (not relabelled) such that

$$U^{\varepsilon}_{R_k,T_k} \to U^{\varepsilon} \text{ locally uniformly for } (\xi,t) \in [\varepsilon,\infty)\times\mathbb{R} \text{ and uniformly for } y \in \mathbb{R}^{N-1},$$

where the limit function U^{ε} becomes a bounded ($0 \le U^{\varepsilon} \le S_0$) classical solution of the following boundary value problem:

$$\begin{cases} \partial_t U - \nabla_{\xi,y}\cdot\big(\widetilde{D}_{S,\infty}(\xi,t+s_\infty,y)\nabla_{\xi,y}U\big) - cU_\xi + \underline{\beta} M U = 0 \ \text{ in } [\varepsilon,\infty)\times\mathbb{R}^N, \\ U \text{ is } \tau\text{-periodic in } y, \\ U(\varepsilon,t,y) = S_0 \quad \text{for } (t,y) \in \mathbb{R}\times\mathbb{R}^{N-1}. \end{cases} \tag{4.30}$$

Next we prove that the bounded solution U^{ε} of problem (4.30) tends to 0 as $M\to\infty$. To that end we look for a supersolution of the form

$$\overline{U^{\varepsilon}}(\xi,t,y) = S_0 e^{-\lambda_M(\xi-\varepsilon)},$$

where the parameter $\lambda_M > 0$ will be chosen later so that $\lambda_M \to \infty$ as $M\to\infty$. In order for $e^{-\lambda\xi}$ to be a supersolution of (4.30), it is sufficient that

$$[\lambda\nabla_{\xi,y}\cdot(\widetilde{D}_{S,\infty}(\xi,t+s_\infty,y)e_1) - \lambda^2 e_1 \widetilde{D}_{S,\infty}(\xi,t+s_\infty,y)e_1 + c\lambda + \underline{\beta} M]e^{-\lambda\xi} \ge 0. \tag{4.31}$$

Coming back to the original variable, let us recall the notations of Section 3.1:

$$\widetilde{D}_S(\xi+\xi_n, t+s_n, y) = \mathcal{R} D_S(X+X_n)\mathcal{R}^T, \quad X = \mathcal{R}\begin{pmatrix}\xi+ct\\ y\end{pmatrix}, X_n = \mathcal{R}\begin{pmatrix}\xi_n + cs_n\\ 0_{\mathbb{R}^{N-1}}\end{pmatrix},$$

where $X \in \mathbb{T}^N$, $\{X_n\} \subset \mathbb{T}^N$ and $\mathcal{R} \in \mathcal{O}(\mathbb{R}^N)$ is such that $\mathcal{R}e_1 = \zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$. By compactness of $\mathbb{T}^N$ and since $D_S \in C^{1+\alpha}(\mathbb{T}^N, \mathcal{S}_N)$, one may assume that, up to extracting a subsequence,

$$D_S(X + X_n) \to D_S(X + X_\infty) \quad \text{in } C^{1+\alpha}(\mathbb{T}^N) \text{ as } n \to \infty.$$

In other words, the limit matrix-valued function $\widetilde{D}_{S,\infty}$ can be equivalently written as

$$\widetilde{D}_{S,\infty}(\xi, t + s_\infty, y) = \mathcal{R} D_S(X + X_\infty)\mathcal{R}^T,$$

and since the vector-valued function $\nabla_{\xi,y}(\widetilde{D}_{S,\infty}e_1)$ is bounded, so is $\nabla_{\xi,y} \cdot (\widetilde{D}_{S,\infty}e_1)$. Therefore, using the uniform ellipticity (1.2) of D_S, we find that inequality (4.31) holds true as long as

$$\begin{aligned}
&\lambda \nabla_{\xi,y} \cdot (\widetilde{D}_{S,\infty}(\xi, t + s_\infty, y)e_1) - \lambda^2 e_1 \widetilde{D}_{S,\infty}(\xi, t + s_\infty, y)e_1 + c\lambda + \underline{\beta} M \\
&\quad = \lambda \nabla_{\xi,y} \cdot (\widetilde{D}_{S,\infty}(\xi, t + s_\infty, y)e_1) - \lambda^2 \zeta D_S(X + X_\infty)\zeta + c\lambda + \underline{\beta} M \\
&\quad \geq -\Theta\lambda^2 + \big(c - \|\nabla_{\xi,y} \cdot (\widetilde{D}_{S,\infty}e_1)\|_\infty\big)\lambda + \underline{\beta} M \geq 0.
\end{aligned}$$

Set

$$\lambda_M := \sqrt{\frac{\underline{\beta} M}{2\Theta}}. \tag{4.32}$$

Then for any $M > 0$, λ_M satisfies the above inequality, whence $e^{-\lambda_M \xi}$ is a supersolution of (4.30). Further, applying the weak maximum principle to problem (4.30), one can obtain

$$U^\varepsilon(\xi, t, y) \leq S_0 e^{-\lambda_M(\xi - \varepsilon)}, \quad \forall \xi \geq \varepsilon,\ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}. \tag{4.33}$$

Indeed, assume by contradiction (recall that the solution U^ε of problem (4.30) is no longer periodic with respect to t) that

$$\eta := \sup_{(\xi,t,y) \in (\varepsilon,\infty) \times \mathbb{R}^N} (U^\varepsilon(\xi, t, y) - S_0 e^{-\lambda_M(\xi - \varepsilon)}) > 0.$$

Then there exists a sequence $\{(\xi_n', t_n', y_n')\} \subset (\varepsilon, \infty) \times \mathbb{R} \times \mathbb{T}_\tau^{N-1}$ with $y_n' \to y_\infty' \in \mathbb{T}_\tau^{N-1}$ as $n \to \infty$ such that

$$U^\varepsilon(\xi_n', t_n', y_n') - S_0 e^{-\lambda_M(\xi_n' - \varepsilon)} \to \eta \quad \text{as } n \to \infty. \tag{4.34}$$

Note that the diffusion matrix field $\widehat{D}_S$ is still τ_1/c-periodic in t. For all n, there exists some $\bar{t_n'} \in \frac{\tau_1}{c}\mathbb{Z}$ such that $s_n' = t_n' - \bar{t_n'} \in [0, \tau_1/c]$. One may assume that, up to taking a subsequence, $s_n' \to s_\infty'$ as $n \to \infty$. As for the limit of the sequence $\{\xi_n'\}$, we need to distinguish two cases:

Case 1: $\xi_n' \to \xi_\infty' \in [\varepsilon, \infty)$ as $n \to \infty$. We define

$$W_n(\xi, t, y) = U^\varepsilon(\xi, t + t_n', y) - S_0 e^{-\lambda_M(\xi - \varepsilon)}, \quad (\xi, t, y) \in [\varepsilon, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1}.$$

As $0 \le U^\varepsilon \le S_0$ and $\lambda_M > 0$, the sequence $\{W_n\}$ is uniformly bounded with respect to n and ε. Moreover, for each n and any $(\xi, t, y) \in [\varepsilon, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1}$, the function W_n satisfies

$$\begin{cases} \partial_t W_n - \nabla_{\xi,y} \cdot \big(\widetilde{D}_{S,\infty}(\xi, t + s'_n + s_\infty, y)\nabla_{\xi,y} W_n\big) - c\partial_\xi W_n + \underline{\beta} M W_n \le 0, \\ W_n \text{ is } \tau\text{-periodic in } y \text{ and } W_n(\varepsilon, t, y) = 0 \text{ for all } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}. \end{cases}$$

From standard parabolic estimates, the sequence $\{W_n\}$ converges (up to taking a subsequence) in $C^{2,1,2}_{\text{loc}}([\varepsilon, \infty) \times \mathbb{R} \times \mathbb{R}^{N-1})$ to a bounded function W_∞ which satisfies

$$\begin{cases} \partial_t W_\infty - \nabla_{\xi,y} \cdot \big(\widetilde{D}_{S,\infty}(\xi, t + s'_\infty + s_\infty, y)\nabla_{\xi,y} W_\infty\big) - c\partial_\xi W_\infty + \underline{\beta} M W_\infty \le 0, \\ W_\infty \text{ is } \tau\text{-periodic in } y \text{ and } W_\infty(\varepsilon, t, y) = 0 \text{ for all } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}. \end{cases} \tag{4.35}$$

By (4.34), $W_\infty(\xi'_\infty, 0, y'_\infty) = \eta > 0$, which implies that $\xi'_\infty > \varepsilon$. Therefore, from (4.34) and using the uniform ellipticity and symmetry of $\widetilde{D}_{S,\infty}$, one gets

$$\partial_t W_\infty(\xi'_\infty, 0, y'_\infty) = \partial_\xi W_\infty(\xi'_\infty, 0, y'_\infty) = 0, \quad \nabla_{\xi,y} \cdot (\widetilde{D}_{S,\infty} \nabla_{\xi,y} W_\infty)(\xi'_\infty, 0, y'_\infty) \le 0.$$

Thanks to $\underline{\beta} M > 0$, a contradiction with (4.35) has been achieved.

Case 2: $\xi'_n \to \infty$ as $n \to \infty$. We define

$$\widehat{W}_n(\xi, t, y) = U^\varepsilon(\xi + \xi'_n, t + t'_n, y) - S_0 e^{-\lambda_M(\xi + \xi'_n - \varepsilon)}, \quad (\xi, t, y) \in [\varepsilon - \xi'_n, \infty) \times \mathbb{R}^N.$$

For each n, the function $\widehat{W}_n$ satisfies

$$\begin{cases} \partial_t \widehat{W}_n - \nabla_{\xi,y} \cdot (\widehat{D}_{S,n}(\xi, t, y)\nabla_{\xi,y} \widehat{W}_n) - c\partial_\xi \widehat{W}_n + \underline{\beta} M \widehat{W}_n \le 0, \\ \widehat{W}_n \text{ is } \tau\text{-periodic in } y \text{ and } \widehat{W}_n(\varepsilon - \xi'_n, t, y) = 0 \text{ for all } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}, \end{cases}$$

where the matrix-valued function $\widehat{D}_{S,n}$ is given by

$$\widehat{D}_{S,n}(\xi, t, y) = \widetilde{D}_{S,\infty}(\xi + \xi'_n, t + s'_n + s_\infty, y), \quad (\xi, t, y) \in [\varepsilon - \xi'_n, \infty) \times \mathbb{R}^N. \tag{4.36}$$

Because of the uniform boundedness of $\{\widehat{W}_n\}$ with respect to n and ε, using parabolic regularity, one may assume that, possibly along a subsequence still denoted by n,

$$\widehat{W}_n(\xi, t, y) \to \widehat{W}_\infty(\xi, t, y) \text{ locally uniformly for } (\xi, t) \in \mathbb{R}^2, \text{ uniformly for } y \in \mathbb{R}^{N-1}$$

as $n \to \infty$, where the limit function $\widehat{W}_\infty$ becomes an entire solution of the problem

$$\begin{cases} \partial_t W - \nabla_{\xi,y} \cdot (\widehat{D}_{S,\infty}(\xi, t, y)\nabla_{\xi,y} W) - c\partial_\xi W + \underline{\beta} M W = 0 \quad \text{in } \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}, \\ W \text{ is } \tau\text{-periodic in } y, \\ 0 \le W(\xi, t, y) \le S_0, \quad \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}. \end{cases} \tag{4.37}$$

Here the matrix-valued function $\widehat{D}_{S,\infty}$ is the limit of the sequence $\{\widehat{D}_{S,n}\}$ defined by (4.36) in the topology of $C^{1+\alpha',\alpha/2,1+\alpha}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1})$. Furthermore, from the definition of $\{\widehat{W}_n(\xi,t,y)\}$ and (4.34), the above convergence also yields

$$\widehat{W}_\infty(0,0,y'_\infty)=\eta>0. \tag{4.38}$$

However, as $\underline{\beta}M>0$, it is easy to verify that a bounded entire solution of (4.37) can only be 0, so $\widehat{W}_\infty(\xi,t,y)\equiv 0$, contrary to (4.38).

Eventually, as a consequence of (4.33) we obtain

$$\limsup_{\substack{R\to\infty\\T\to\infty}}\limsup_{n\to\infty}U^{\varepsilon,n}_{R,T}(\xi,t,y)\le S_0e^{-\lambda_M(\xi-\varepsilon)},\quad \forall(\xi,t,y)\in[\varepsilon,\infty)\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

Further, we deduce from (4.29) that for all $(\xi,t,y)\in[\varepsilon,\infty)\times\mathbb{R}\times\mathbb{R}^{N-1}$,

$$\limsup_{n\to\infty}U_n(\xi+\xi_n,t+t_n,y)\le S_0e^{-\lambda_M(\xi-\varepsilon)},\quad \forall M>0.$$

Recalling the definition of λ_M in (4.32) and by letting $M\to\infty$, one gets

$$\lim_{n\to\infty}U_n(\xi+\xi_n,t+t_n,y)=0,\quad \forall(\xi,t,y)\in(\varepsilon,\infty)\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

Since $\varepsilon\in(-\kappa,0)$ can be chosen arbitrarily, Claim 1 follows. ■

In the remaining part of this subsection, we pass to the limit $n\to\infty$ and complete the proof of Theorem 4.6.

Proof of Theorem 4.6. Using parabolic regularity coupled with the uniform estimates in Lemma 4.10, there exist a subsequence of $\{(U_n,V_n)\}$ (not relabelled) and a bounded limit function pair (U,V) such that

$$(U_n,V_n)(\xi,t,y)\to(U,V)(\xi,t,y)\text{ locally uniformly for }\xi\in\mathbb{R},\text{ uniformly for }(t,y)\in\mathbb{R}^N,$$

where the convergence holds in the topology of $C^{2+\alpha',1+\alpha'/2,2+\alpha'}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1})^2$ for all $\alpha'\in[0,\alpha)$ due to Schauder parabolic estimates (recalling Assumption 1.1 about the coefficients). Furthermore, the limit function pair $(U,V)\in C^{2,1,2}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1})^2$ satisfies the problem

$$\begin{cases}\partial_tU-\nabla_{\xi,y}\cdot(\widetilde{D}_S(\xi,t,y)\nabla_{\xi,y}U)-cU_\xi=-\widetilde{\beta}(\xi,t,y)UV & \text{in }\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1},\\ \partial_tV-\nabla_{\xi,y}\cdot(\widetilde{D}_I(\xi,t,y)\nabla_{\xi,y}V)-cV_\xi=\widetilde{\beta}(\xi,t,y)UV-\widetilde{\gamma}(\xi,t,y)V & \text{in }\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1},\\ (U,V)\text{ is }\tau_1/c\text{-periodic in }t\text{ and }\tau\text{-periodic in }y. \end{cases}$$

Since the functions U and V are bounded, they also enjoy the following estimates:

$$\max(0,\underline{U})\le U\le S_0\quad\text{and}\quad\max(0,\underline{V})\le V\le\overline{V}\quad\text{in }\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}. \tag{4.39}$$

In particular, the lower estimates for (U,V) ensure that for some ξ_0 large enough,

$$\min_{(t,y)\in[0,\tau_1/c]\times\mathbb{T}^{N-1}_\tau}U(\xi,t,y)>0\quad\text{and}\quad\min_{(t,y)\in[0,\tau_1/c]\times\mathbb{T}^{N-1}_\tau}V(\xi,t,y)>0,\quad\forall\xi>\xi_0.$$

Thus the strong maximum principle and the periodicity conditions yield $U > 0$ and $V > 0$ in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$ and furthermore one also deduces that

$$\min_{(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}} U(\xi, t, y) > 0 \quad \text{and} \quad \min_{(t,y)\in\mathbb{R}\times\mathbb{R}^{N-1}} V(\xi, t, y) > 0, \quad \forall \xi \in \mathbb{R}.$$

Next, as $\widetilde{\beta} > 0$, it follows from the positivity of V and the strong maximum principle that

$$U(\xi, t, y) < S_0, \quad \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}.$$

Observe that the estimates (4.39) also ensure that

$$\lim_{\xi\to+\infty} (U, V)(\xi, t, y) = (S_0, 0) \quad \text{uniformly for } (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}.$$

We have completed the proof of the first two properties in Theorem 4.6. The third property follows from the estimates for V in (4.39) since the functions $\underline{V}$ and $\overline{V}$ have the same exponential decay as $\xi \to +\infty$.

This ends the proof of Theorem 4.6. ∎

5. Proof of Theorem 2.1

In this section, we complete the proof of Theorem 2.1. Throughout this section we again assume that

$$\mu_0 < 0.$$

Before going any further, let us make a brief summary. Coming back to the original variables (t, x), Theorem 4.6 equivalently implies that for each $c > c^*(\zeta)$, problem (1.1) admits a pulsating travelling wave $(S, I) = (S, I)(t, x; \zeta)$ propagating along the direction $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ with speed c according to Definition 1.1. Furthermore, from Lemmas 4.3–4.4 and Theorem 4.6, the following properties hold:

(i) For each $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$, the function pair (S, I) is positive and satisfies the following lower and upper estimates:

$$\begin{aligned} &\max(0, \underline{S}(t, x; \zeta)) \le S(t, x; \zeta) < S_0, && \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\ &\max(0, \underline{I}(t, x; \zeta)) \le I(t, x; \zeta) \le M(\zeta), && \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \end{aligned} \tag{5.1}$$

where the functions $\underline{S}$ and $\underline{I}$ are defined as

$$\begin{aligned} &\underline{S}(t, x; \zeta) = S_0\big(1 - K_1 e^{-\delta_1(x\cdot\zeta - ct)} \Psi_{\delta_1}(x; \zeta)\big), \\ &\underline{I}(t, x; \zeta) = e^{-\lambda_c(\zeta)(x\cdot\zeta - ct)} \Phi_{\lambda_c(\zeta)}(x; \zeta) - K_2 e^{-(\lambda_c(\zeta)+\delta_2)(x\cdot\zeta - ct)} \Phi_{\lambda_c(\zeta)+\delta_2}(x; \zeta). \end{aligned}$$

(ii) For each $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$, the function pair (S, I) satisfies the pulsating relations

$$(S, I)(t + k \cdot \zeta/c, x) = (S, I)(t, x - k), \quad \forall (t, x, k) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{Z}^N.$$

(iii) For each $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$, the function pair (S, I) exhibits the following behaviour at $x \cdot \zeta = +\infty$:

$$\lim_{x\cdot\zeta \to +\infty} (S, I)(t, x) = (S_0, 0)$$

locally uniformly for $t \in \mathbb{R}$ and uniformly in the directions of $\zeta^\perp$.

In order to obtain the existence of pulsating travelling waves for problem (1.1) propagating along any given direction in $\mathbb{S}^{N-1}$, we will perform an approximation argument with the help of a dense subset of $\mathbb{S}^{N-1}$. This requires uniform estimates of the solutions with respect to the direction of propagation. Unlike the observations in Section 4.2, the profile of pulsating waves propagating in a unit direction with some irrational coordinates no longer exhibits periodicity in space and time (see below for a more precise explanation).

5.1. Uniform boundedness with respect to direction

Note first that the component S of the solution is globally bounded for any $\zeta \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$. We now prove that the other component $I(t, x; \zeta)$ is also bounded on any subset of $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$ and for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, namely the upper bound in (5.1) is actually independent of the direction of propagation. To this end, we further extend some arguments developed in Lemma 4.10 to more general heterogeneous media.

Lemma 5.1. *Let $\mathcal{K}$ be any subset of $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$. If a function pair (S, I) defined by*

$$(S, I) : \mathbb{R} \times \mathbb{R}^N \times \mathcal{K} \to (0, \infty) \times (0, \infty),$$
$$(t, x, e) \mapsto (S, I)(t, x; e),$$

is a pulsating travelling wave of problem (1.1) propagating along the direction e with the direction varying speed $c(e)$ defined by

$$c : \mathcal{K} \to (c^*(e), \infty), \quad e \mapsto c(e),$$

where the formula of $c^*(e)$ for each $e \in \mathcal{K}$ is given by (3.8), then there exists a constant $M = M(\mathcal{K}) > 0$ such that

$$I(t, x; e) \le M, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N,\ \forall e \in \mathcal{K}.$$

Proof. We still work with the moving frame of Section 3.1. Recall that (S, I) is a pulsating travelling wave of problem (1.1) propagating in the direction $e \in \mathbb{S}^{N-1}$ with speed $c > 0$ if and only if the function pair (U, V) defined by

$$(U, V)(\xi, t, y) = (S, I)(t, \mathcal{R}(\xi + ct, y)^T)$$

satisfies the parabolic system (3.2) and the $\mathcal{R}$-pulsating condition

$$(U, V)\left(\xi, t + \frac{k \cdot e}{c}, y + \widehat{\mathcal{R}} k_\perp\right) = (U, V)(\xi, t, y), \quad \forall (\xi, t, y) \in \mathbb{R}^{N+1},\ \forall k \in \mathbb{Z}^N. \tag{5.2}$$

Here $\mathcal{R} \in \mathcal{O}(\mathbb{R}^N)$ is such that $\mathcal{R}e_1 = e \in \mathbb{S}^{N-1}$. By estimate (5.1), for any $e \in \mathcal{K}$, the function V is bounded, that is,

$$V(\xi, t, y) \le M(e), \quad \forall (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}.$$

Next we prove that the bound $M(e)$ is actually independent of the direction. To that end we still argue by contradiction and use the following notation:

$$(U^n, V^n)(\xi, t, y) := (U, V)(\xi, t, y; e^n) = (S, I)(t, \mathcal{R}_n(\xi + c^n t, y)^T),$$

which denotes the solution of (3.2) in the direction e^n with $c^n := c(e^n)$ for each $n \in \mathbb{N}$.

Assume by contradiction that there exist a sequence $\{(\xi_n, t_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$ of points and a sequence $\{e^n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ of directions such that

$$V^n(\xi_n, t_n, y_n) \to \infty \quad \text{as } n \to \infty.$$

One may assume, without loss of generality, that

$$\sup_{(\xi,t,y)\in\mathbb{R}^{N+1}} V^n(\xi, t, y) = \|V^n\|_\infty \quad \text{and} \quad V^n(\xi_n, t_n, y_n) \in [\|V^n\|_\infty/2, \|V^n\|_\infty]. \tag{5.3}$$

Note that the component U^n of the solution satisfies the lower and upper estimates

$$\max(0, \underline{U}(\xi, t, y; e^n)) \le U^n(\xi, t, y) < S_0, \quad \forall (\xi, t, y) \in \mathbb{R}^{N+1}, \ \forall n \in \mathbb{N}.$$

Hence $\{U^n\}$ is uniformly bounded. By compactness of $\mathbb{S}^{N-1}$, one may assume that $e^n \to e \in \mathbb{S}^{N-1}$ as $n \to \infty$. Equivalently, $\|\mathcal{R}_n - \mathcal{R}\|_{\mathcal{O}(\mathbb{R}^N)} \to 0$ and $\mathcal{R}_n e_1 \to \mathcal{R}e_1$ as $n \to \infty$. On the other hand, as $0 < c^*(e^n) < c^n < \infty$ for each $n \in \mathbb{N}$, the sequence $\{c_n\}_{n\in\mathbb{N}}$ of speeds is bounded. Thus, one may assume again that $c^n \to c^\infty \ge c^*(e)$ as $n \to \infty$ since the mapping $\mathbb{S}^{N-1} \ni e \mapsto c^*(e)$ is continuous.

To proceed let us first point out that for a vector in $\mathbb{S}^{N-1}$ with some irrational coordinates, the $\mathcal{R}$-pulsating condition (5.2) no longer yields periodicity with respect to (t, y). Next, we define

$$W_n(\xi, t, y) = \frac{V^n(\xi + \xi_n, t + t_n, y + y_n)}{V^n(\xi_n, t_n, y_n)}.$$

For each n, the function W_n satisfies

$$\partial_t W_n - \nabla_{\xi,y} \cdot (\widetilde{\mathfrak{D}}_{I,n}(\xi, t, y)\nabla_{\xi,y} W_n) - c^n \partial_\xi W_n - b^n(\xi, t, y) W_n = 0 \ \text{ in } \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1},$$

where the sequence $\{\widetilde{\mathfrak{D}}_{I,n}(\xi, t, y)\}$ of matrix-valued functions is defined by

$$\widetilde{\mathfrak{D}}_{I,n}(\xi, t, y) = \widetilde{D}_{I,n}(\xi + \xi_n, t + t_n, y + y_n) = \mathcal{R}_n D_I\left(\mathcal{R}_n \begin{pmatrix} \xi + \xi_n + c^n(t + t_n) \\ y + y_n \end{pmatrix}\right)\mathcal{R}_n^T$$

with

$$\widetilde{D}_{I,n}(\xi, t, y) := \widetilde{D}_I(\xi, t, y; e^n) = \mathcal{R}_n D_I\left(\mathcal{R}_n \begin{pmatrix} \xi + c^n t \\ y \end{pmatrix}\right)\mathcal{R}_n^T,$$

while the sequence $\{b^n(\xi, t, y)\}$ of functions is defined by

$$b^n(\xi, t, y) = (\widetilde{\beta}_n U^n - \widetilde{\gamma}_n)(\xi + \xi_n, t + t_n, y + y_n) \tag{5.4}$$

with

$$(\widetilde{\beta}_n, \widetilde{\gamma}_n)(\xi + \xi_n, t + t_n, y + y_n) = (\beta, \gamma)\left(\mathcal{R}_n \begin{pmatrix} \xi + c^n t \\ y \end{pmatrix} + \mathcal{R}_n \begin{pmatrix} \xi_n + c^n t_n \\ y_n \end{pmatrix}\right),$$
$$(\widetilde{\beta}_n, \widetilde{\gamma}_n)(\xi, t, y) := (\widetilde{\beta}, \widetilde{\gamma})(\xi, t, y; e^n) = (\beta, \gamma)\left(\mathcal{R}_n \begin{pmatrix} \xi + c^n t \\ y \end{pmatrix}\right).$$

Note first the following convergence for the coefficient sequences as $n \to \infty$:

$$\widetilde{D}_{I,n}(\xi, t, y) \to \widetilde{D}_I(\xi, t, y; e) = \mathcal{R} D_I\left(\mathcal{R} \begin{pmatrix} \xi + c^\infty t \\ y \end{pmatrix}\right)\mathcal{R}^T,$$
$$(\widetilde{\beta}_n, \widetilde{\gamma}_n)(\xi, t, y) \to (\widetilde{\beta}, \widetilde{\gamma})(\xi, t, y; e) = (\beta, \gamma)\left(\mathcal{R} \begin{pmatrix} \xi + c^\infty t \\ y \end{pmatrix}\right)$$

with respect to the uniform topology. Furthermore, since $\{U^n\}$ is uniformly bounded, the sequence $\{b^n(\xi, t, y)\}$ of functions defined in (5.4) is also uniformly bounded for all $(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$ and $n \in \mathbb{N}$. On the other hand, since we can rewrite

$$D_I\left(\mathcal{R}_n \begin{pmatrix} \xi + \xi_n + c^n(t + t_n) \\ y + y_n \end{pmatrix}\right) = D_I\left(\mathcal{R}_n \begin{pmatrix} \xi + c^n t \\ y \end{pmatrix} + \mathcal{R}_n \begin{pmatrix} \xi_n + c^n t_n \\ y_n \end{pmatrix}\right)$$
$$= D_I(X + X_n)$$

with $X = \mathcal{R}_n(\xi + c^n t, y)^T \in \mathbb{T}^N$ and $\{X_n\}_{n\in\mathbb{N}} \subset \mathbb{T}^N$, by the compactness of $\mathbb{T}^N$ and since $D_I \in C^{1+\alpha}(\mathbb{T}^N, \mathcal{S}_N)$, one has

$$\mathcal{R}_n D_I\left(\mathcal{R}_n \begin{pmatrix} \xi + \xi_n + c^n(t + t_n) \\ y + y_n \end{pmatrix}\right)\mathcal{R}_n^T \to \mathcal{R} D_I(X + X_\infty)\mathcal{R}^T \quad \text{in } C^{1+\alpha}(\mathbb{T}^N)$$

as $n \to \infty$. Thus, one may assume at least that $\widetilde{\mathfrak{D}}_{I,n}(\xi, t, y) \to \widetilde{\mathfrak{D}}_{I,\infty}(\xi, t, y)$ in $C^{1,0,1}_{\text{loc}}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1})$ as $n \to \infty$. From our construction (5.3), one may observe that

$$W_n(0, 0, 0) = 1, \quad 0 \le W_n(\xi, t, y) \le 2, \quad \forall(\xi, t, y) \in \mathbb{R}^{N+1}, \ \forall n \in \mathbb{N}.$$

As a consequence of this uniform boundedness and parabolic regularity, one may assume that, possibly along a subsequence still denoted by n,

$$W_n(\xi, t, y) \to W_\infty(\xi, t, y) \text{ locally uniformly for } (\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \text{ as } n \to \infty,$$

where the convergence is weak in $W^{2,1,2}_{p,\text{loc}}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1})$ for any $p \in (1, \infty)$ and strong in $C^{1,0,1}_{\text{loc}}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1})$. Furthermore, the limit function W_∞ satisfies

$$\begin{cases} \partial_t W_\infty - \nabla_{\xi,y} \cdot (\widetilde{\mathfrak{D}}_{I,\infty}(\xi, t, y)\nabla_{\xi,y} W_\infty) - c^\infty \partial_\xi W_\infty - b^\infty(\xi, t, y) W_\infty = 0, \\ W_\infty(0, 0, 0) = 1, \quad 0 \le W_\infty(\xi, t, y) \le 2, \quad \forall(\xi, t, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}, \end{cases}$$

where the bounded function b^∞ is the L^∞-weak star limit of $\{b^n(\xi,t,y)\}$ defined in (5.4). Since W_∞ is nonnegative and $W_\infty(0,0,0)=1$, it follows from the strong maximum principle that W_∞ is positive everywhere. Thus, from the definition of $\{W_n(\xi,t,y)\}$, we conclude that

$$V^n(\xi+\xi_n, t+t_n, y+y_n) \to \infty \text{ locally uniformly for } (\xi,t,y)\in\mathbb{R}^{N+1} \text{ as } n\to\infty. \tag{5.5}$$

Next, following the proof of Lemma 4.10, we make the following claim.

Claim 2. *The sequence* $\{U^n(\xi,t,y)\}$ *of functions satisfies*

$$\lim_{n\to\infty} U^n(\xi+\xi_n, t+t_n, y+y_n)=0, \quad \forall(\xi,t,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

The proof of this claim is also postponed and let us first complete the proof of Lemma 5.1. Using Claim 2, since $\widetilde{\gamma}_n(\xi,t,y):=\widetilde{\gamma}(\xi,t,y;e^n)>0$ for any $(\xi,t,y)\in\mathbb{R}^{N+1}$ and $n\in\mathbb{N}$, we find that the L^∞-weak star limit b^∞ of $\{b_n(\xi,t,y)\}$ defined in (5.4) satisfies

$$b^\infty(\xi,t,y)\le -\underline{\widetilde{\gamma}}, \quad \forall(\xi,t,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1},$$

where we have set

$$\underline{\widetilde{\gamma}} := \inf_{n\in\mathbb{N}} \inf_{(\xi,t,y)\in\mathbb{R}^{N+1}} \widetilde{\gamma}(\xi,t,y;e^n) = \min_{X\in\mathbb{T}^N}\gamma(X)>0.$$

Thus, the limit function W_∞ has the following properties:

$$\begin{aligned}
&\partial_t W_\infty \le \nabla_{\xi,y}\cdot(\widetilde{\mathcal{D}}_{I,\infty}(\xi,t,y)\nabla_{\xi,y}W_\infty) + c^\infty\partial_\xi W_\infty - \underline{\widetilde{\gamma}}W_\infty,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall(\xi,t,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1},\\
&W_\infty(0,0,0)=1, \quad 0<W_\infty(\xi,t,y)\le 2, \quad \forall(\xi,t,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}.
\end{aligned}$$

Notice that $e^{-\underline{\widetilde{\gamma}}t}$ is a supersolution of the above equation. From the parabolic comparison principle, we obtain

$$W_\infty(\xi,t+s,y)\le 2e^{-\underline{\widetilde{\gamma}}s}, \quad \forall s\ge 0,\ \forall(\xi,t,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

This implies, with $(\xi,t,y)=(0,-s,0)$, that

$$W_\infty(0,0,0)\le 2e^{-\underline{\widetilde{\gamma}}s}, \quad \forall s>0.$$

Finally, as $\underline{\widetilde{\gamma}}>0$, this contradicts the normalization $W_\infty(0,0,0)=1$.

This ends the proof of Lemma 5.1. ∎

Proof of Claim 2. Let $R,T>0$ be given and consider for each $n\in\mathbb{N}$ the solution

$$U=U^n_{R,T}(\xi,t,y):=U_{R,T}(\xi,t,y;e^n) \quad \text{with } e^n\in\mathcal{K}$$

of the following problem:

$$\begin{cases} \partial_t U - \nabla_{\xi,y} \cdot (\widetilde{\mathcal{D}}_{S,n}(\xi,t,y)\nabla_{\xi,y} U) - c^n \partial_\xi U + \underline{\widetilde{\beta}} U = 0 & \text{for } |t| < T,\ (\xi,y) \in B_R, \\ U(\xi,t,y;e^n) = S_0 & \text{for } |t| < T,\ (\xi,y) \in \partial B_R, \\ U(\xi,-T,y;e^n) = S_0 & \text{for } (\xi,y) \in \overline{B_R}, \end{cases}$$

where B_R denotes the open ball of $\mathbb{R}^N$ with radius R and centre $\mathbf{0} = (0, 0_{\mathbb{R}^{N-1}})$, and the sequence $\{\widetilde{\mathcal{D}}_{S,n}(\xi,t,y)\}$ of matrix-valued functions is defined by

$$\widetilde{\mathcal{D}}_{S,n}(\xi,t,y) = \widetilde{D}_{S,n}(\xi+\xi_n, t+t_n, y+y_n) = \mathcal{R}_n D_S\left(\mathcal{R}_n \begin{pmatrix} \xi+\xi_n+c^n(t+t_n) \\ y+y_n \end{pmatrix}\right)\mathcal{R}_n^T$$

with

$$\widetilde{D}_{S,n}(\xi,t,y) = \widetilde{D}_S(\xi,t,y;e^n) = \mathcal{R}_n D_S\left(\mathcal{R}_n \begin{pmatrix} \xi+c^n t \\ y \end{pmatrix}\right)\mathcal{R}_n^T,$$

Here we have set

$$\underline{\widetilde{\beta}} := \inf_{n\in\mathbb{N}} \inf_{(\xi,t,y)\in\mathbb{R}^{N+1}} \widetilde{\beta}(\xi,t,y;e^n) = \min_{X\in\mathbb{T}^N} \beta(X) > 0,$$

where the equality holds true since $\widetilde{\beta}$ is independent of the direction. By (5.5), we can choose $N_0 > 0$ large enough (depending on R and T) such that

$$V^n(\xi+\xi_n, t+t_n, y+y_n) \geq 1, \quad \forall n \geq N_0,\ \forall t \in [-T,T],\ \forall (\xi,y) \in \overline{B_R}.$$

Thus, the parabolic comparison principle ensures that

$$U^n(\xi+\xi_n, t+t_n, y+y_n) \leq U^n_{R,T}(\xi,t,y), \quad \forall n \geq N_0,\ \forall t \in [-T,T],\ \forall (\xi,y) \in \overline{B_R}.$$

Observe further that

$$0 \leq U^n_{R,T}(\xi,t,y) \leq S_0, \quad \forall n \geq N_0,\ \forall t \in [-T,T],\ \forall (\xi,y) \in \overline{B_R}.$$

As a consequence of this uniform boundedness and Schauder parabolic regularity, for any positive constants R, T and any sequence $\{n_k\}_{k\in\mathbb{N}}$ tending to infinity as $k \to \infty$, one may extract a subsequence, still denoted by $\{n_k\}$, such that

$$U^{n_k}_{R,T}(\xi,t,y) \to \widehat{U}_{R,T}(\xi,t,y;e) \quad \text{uniformly for } |t| \leq T/2 \text{ and } (\xi,y) \in \overline{B_{R/2}},$$

where the limit function $\widehat{U}_{R,T}$ satisfies

$$\partial_t U - \nabla_{\xi,y} \cdot (\widetilde{\mathcal{D}}_{S,\infty}(\xi,t,y)\nabla_{\xi,y} U) - c^\infty U_\xi + \underline{\widetilde{\beta}} U = 0 \quad \text{for } |t| \leq T/2 \text{ and } (\xi,y) \in \overline{B_{R/2}}.$$

Here the matrix-valued function $\widetilde{\mathcal{D}}_{S,\infty}$ is a limit of the sequence $\{\widetilde{\mathcal{D}}_{S,n_k}(\xi,t,y)\}_{k\geq 0}$ in the topology of $C^{1+\alpha',\alpha'/2,1+\alpha'}$ for some $\alpha' \in (0,\alpha)$. Again, thanks to the uniform boundedness of $\widehat{U}_{R,T}$ with respect to R and T, using parabolic regularity, for any sequences $\{R_k\}$ and $\{T_k\}$ tending to infinity, one may extract subsequences (not relabelled) such that

$$\widehat{U}_{R_k,T_k}(\xi,t,y;e) \to \widehat{U}(\xi,t,y;e) \quad \text{locally uniformly for } (\xi,t,y) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1},$$

where the limit function $\widehat{U}$ becomes a bounded entire solution of the problem

$$\partial_t U - \nabla_{\xi,y} \cdot (\widetilde{\mathscr{D}}_{S,\infty}(\xi,t,y)\nabla_{\xi,y} U) - c^{\infty} U_{\xi} + \underline{\widetilde{\beta}} U = 0, \quad (\xi,t,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}.$$

As $\underline{\widetilde{\beta}} > 0$, one can readily verify that $\widehat{U}(\xi,t,y;e) \equiv 0$.

As a consequence, for any $(\xi,t,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$,

$$\limsup_{n\to\infty} U^n(\xi+\xi_n, t+t_n, y+y_n) \le \limsup_{\substack{R\to\infty \\ T\to\infty}} \limsup_{n\to\infty} U^n_{R,T}(\xi,t,y) = 0.$$

This ends the proof of Claim 2. ∎

5.2. Rational approximation to any direction of propagation

In this subsection, we prove the existence of pulsating travelling waves for problem (1.1) propagating in any given direction $e \in \mathbb{S}^{N-1}$. To do so we first recall the following well-known result (see [7, Proposition 4.1] or [14, Lemma 4.1]):

Proposition 5.2. *The set $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$ is dense in $\mathbb{S}^{N-1}$.*

Equipped with Lemma 5.1 and the above proposition, we are now able to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Fix any $e \in \mathbb{S}^{N-1}$. Then there exists a sequence $\{\zeta_m\}_{m\in\mathbb{N}} \subset \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ such that $\zeta_m \to e$ as $m \to \infty$. Since the mapping $\mathbb{S}^{N-1} \ni e \mapsto c^*(e)$ is continuous, for each speed $c_e \in (c^*(e), \infty)$, we can choose a sequence $\{c_m\}_{m\in\mathbb{N}} \subset (c^*(\zeta_m), \infty)$ such that $c_m \to c_e$ as $m \to \infty$. Now we consider the following sequences of solutions corresponding to problem (1.1):

$$c_m > c^*(\zeta_m), \quad (S_m, I_m)(t,x) := (S,I)(t,x;\zeta_m), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ m \in \mathbb{N}.$$

For each $m \in \mathbb{N}$, the function pair (S_m, I_m) satisfies the pulsating relations

$$(S_m, I_m)\left(t + \frac{k\cdot\zeta_m}{c_m}, x\right) = (S_m, I_m)(t, x-k), \quad \forall (t,x,k) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{Z}^N,$$

as well as the lower and upper estimates

$$\max(0, \underline{S}_m) \le S_m < S_0 \quad \text{and} \quad \max(0, \underline{I}_m) \le I_m \le \overline{I}_m \quad \text{in } \mathbb{R} \times \mathbb{R}^N,$$

where the functions $\underline{S}_m := \underline{S}(\cdot,\cdot;\zeta_m)$, $\underline{I}_m := \underline{I}(\cdot,\cdot;\zeta_m)$ and $\overline{I}_m := \overline{I}(\cdot,\cdot;\zeta_m)$ are of the following forms:

$$\begin{aligned}
&\underline{S}(t,x;\zeta_m) = S_0[1 - K_1 e^{-\delta_1(x\cdot\zeta_m - c_m t)}\Psi_{\delta_1}(x;\zeta_m)],\\
&\overline{I}(t,x;\zeta_m) = e^{-\lambda_{c_m}(\zeta_m)(x\cdot\zeta_m - c_m t)}\Phi_{\lambda_{c_m}(\zeta_m)}(x;\zeta_m),\\
&\underline{I}(t,x;\zeta_m) = \overline{I}(t,x;\zeta_m) - K_2 e^{-(\lambda_{c_m}(\zeta_m)+\delta_2)(x\cdot\zeta_m - c_m t)}\Phi_{\lambda_{c_m}(\zeta_m)+\delta_2}(x;\zeta_m).
\end{aligned}$$

Applying Lemma 5.1 to $\mathcal{K} = \{\zeta_m\}_{m\in\mathbb{N}}$, we find that the sequence $\{(S_m, I_m)(t,x)\}$ is uniformly bounded for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $m \in \mathbb{N}$. By parabolic regularity, there exist a subsequence of $\{(S_m, I_m)\}$ (not relabelled) and a bounded limit function pair (S, I) such that

$$(S_m, I_m)(t,x) \to (S, I)(t,x;e) \quad \text{in } C^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N)^2 \text{ as } m \to \infty.$$

Moreover, $c^*(\zeta_m) \to c^*(e)$ as $m \to \infty$ since the mapping $\mathbb{S}^{N-1} \ni e \mapsto c^*(e)$ is continuous. Consequently, the 3-tuple limit (c_e, S, I) satisfies $c_e > c^*(e)$ and the parabolic system

$$\begin{cases} \partial_t S - \nabla\cdot(D_S(x)\nabla S) = -\beta(x)SI, \\ \partial_t I - \nabla\cdot(D_I(x)\nabla I) = \beta(x)SI - \gamma(x)I, \end{cases} \quad \text{for } (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Furthermore, it is not hard to verify that the above convergence also yields

$$(S, I)\left(t + \frac{k\cdot e}{c_e}, x\right) = (S, I)(t, x-k), \quad \forall (t,x,k) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{Z}^N. \tag{5.6}$$

From Propositions 3.1 and 3.3, we know that the mapping $\mathbb{S}^{N-1} \ni e \mapsto \lambda_c(e)$ is continuous in both $r \in \mathbb{S}^{N-1}$ and $c \in (c^*(e), \infty)$, and that the principal eigenfunction $\Phi_\lambda(\cdot; e)$ depends continuously on both $\lambda \in \mathbb{R}$ and $e \in \mathbb{S}^{N-1}$ with respect to the uniform topology. The same conclusion holds true for the principal eigenfunction $\Psi_\delta(\cdot; e)$. Since (S, I) are bounded and $c_e > c^*(e)$, they also enjoy the following lower and upper estimates:

$$\begin{aligned} &\max(0, \underline{S}(t,x;e)) \le S(t,x;e) \le S_0, && \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ &\max(0, \underline{I}(t,x;e)) \le I(t,x;e) \le \overline{I}(t,x;e), && \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, \end{aligned} \tag{5.7}$$

where the functions $\underline{S}$, $\underline{I}$ and $\overline{I}$ are the respective limits of $\{\underline{S}_m\}$, $\{\underline{I}_m\}$ and $\{\overline{I}_m\}$ in the uniform topology, and they are of the following form:

$$\begin{aligned} &\underline{S}(t,x;e) = S_0\big(1 - K_1 e^{-\delta_1(x\cdot e - c_e t)}\Psi_{\delta_1}(x;e)\big), \\ &\overline{I}(t,x;e) = e^{-\lambda_{c_e}(e)(x\cdot e - c_e t)}\Phi_{\lambda_{c_e}(e)}(x;e), \\ &\underline{I}(t,x;e) = \overline{I}(t,x;e) - K_2 e^{-(\lambda_{c_e}(e)+\delta_2)(x\cdot e - c_e t)}\Phi_{\lambda_{c_e}(e)+\delta_2}(x;e). \end{aligned}$$

In order to obtain our result, we also need to prove that

$$\lim_{x\cdot e\to+\infty}(S, I)(t,x) = (S_0, 0)$$

locally uniformly for $t \in \mathbb{R}$ and uniformly in the directions of $e^\perp$, as well as

$$0 < S(t,x;e) < S_0, \quad I(t,x;e) > 0, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Using the pulsating relations (5.6) for (S, I) and the positivity of c_e, the former is equivalent to proving that

$$\lim_{t\to-\infty}(S, I)(t,x;e) = (S_0, 0) \quad \text{locally uniformly for } x \in \mathbb{R}^N,$$

which follows immediately from the estimates (5.7) for (S, I) since $\underline{S}(t,x;e)$ is decreasing in t while $\overline{I}(t,x;e)$ is increasing in t. The latter inequality follows from the strong maximum principle. Indeed, the lower estimates for (S, I) in (5.7) ensure that there exists t_0 large enough such that

$$\inf_{x\in\mathbb{R}^N} S(t,x;e) > 0, \qquad \inf_{x\in\mathbb{R}^N} I(t,x;e) > 0, \qquad \forall t < -t_0.$$

Therefore, $S(t,x;e) > 0$ and $I(t,x;e) > 0$ for any $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Furthermore, as $\beta > 0$, the positivity of I and the strong maximum principle yield $S(t,x;e) < S_0$ for any $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

This ends the proof of Theorem 2.1. ■

6. Some comments and future work

Let us mention some issues that will be considered in the future, and comment on the difficulties that arise.

Firstly, in a forthcoming work, based on the periodicity exhibited by the profile in each unit rational direction, we will investigate the asymptotic behaviour of pulsating waves for problem (1.1) far behind the propagating front. More precisely, the pulsating travelling wave we investigate here would have a travelling pulse of the infected populations. This is a reasonable and natural expectation because no flux of susceptible individuals is involved in the model (1.1). Furthermore, there exists a constant $S_\infty \in [0, S_0)$ that is the desired limit of the component S. This quantity describes the severity of the epidemic. In general, it depends on the wave speed c and the population size S_0 before the introduction of the disease. For a general unit direction, this problem remains open for the moment.

Secondly, we expect that the critical value c^* corresponds to the minimal wave speed. A general methodology is to use a limiting argument $c_n \to c^*$ with larger speeds $c_n > c^*$. The main difficulty is proving that the resulting solution is nontrivial. On the one hand, as underlined above, the state $S_\infty = S_\infty^n$ after the epidemic does depend on the wave speeds c_n and the pulsating wave exhibits a pulse-like shape of the infections. On the other hand, for scalar periodic reaction-diffusion equations, the pulsating front of monostable or bistable type is monotonic in time (see [5, 24]), but this is not the desired property of the pulsating wave for problem (1.1). We leave this issue for future work.

Lastly, we have not focused in this work on the nonexistence of pulsating waves for problem (1.1) with small speeds $c < c^*$. This is also left for future work.

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