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Lipschitz rigidity for scalar curvature

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Abstract. Let M be a closed smooth connected spin manifold of even dimension n , let g be a Riemannian metric of regularity $W^{1,p}$, $p > n$, on M whose distributional scalar curvature in the sense of Lee–LeFloch is bounded below by $n(n-1)$, and let $f: (M, g) \rightarrow \mathbb{S}^n$ be a 1-Lipschitz continuous (not necessarily smooth) map of nonzero degree to the unit n -sphere. Then f is a metric isometry. This generalizes a result of Llarull (1998) and answers in the affirmative a question of Gromov (2019) in his *Four lectures*.

Our proof is based on spectral properties of Dirac operators for low regularity Riemannian metrics and twisted with Lipschitz bundles. We argue that the existence of a nonzero harmonic spinor field forces f to be quasiregular in the sense of Reshetnyak, and in this way connect the powerful theory for quasiregular maps to the Atiyah–Singer index theorem.

Keywords: lower scalar curvature bounds, Lipschitz maps, twisted Dirac operators, low regularity metrics, scalar curvature distribution, quasiregular maps.

1. Introduction

Extremality and rigidity properties of Riemannian manifolds with lower scalar curvature bounds have played a major role in differential geometry for many years. We refer to Gromov’s *Four lectures on scalar curvature* [12] for a comprehensive overview of the subject. The following fundamental result of Llarull illustrates the striking interplay of metric, curvature and homological information in this context. Throughout this paper we denote by \mathbb{S}^n the unit n -sphere equipped with its standard Riemannian metric and induced Riemannian distance.

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Theorem 1.1 ([18, Theorem B]). *Let M be a closed smooth connected oriented manifold of dimension $n \geq 2$ which admits a spin structure and is equipped with a smooth Riemannian metric g of scalar curvature $\text{scal}_g \geq n(n - 1)$. Furthermore, let $f : (M, d_g) \rightarrow \mathbb{S}^n$ be a smooth 1-Lipschitz map of nonzero degree. Then f is a Riemannian isometry.*

Here d_g denotes the Riemannian distance on M induced by g . For M equal to the unit n -sphere and $f = \text{id}_M$, Theorem 1.1 says that the round metric on \mathbb{S}^n cannot be dominated by a smooth Riemannian metric g with $\text{scal}_g \geq n(n - 1)$, except by the round metric itself (see Llarull [18, Theorem A]).

The proof of Theorem 1.1 is based on spectral properties of twisted Dirac operators and emphasizes the fruitful interplay of spin and scalar curvature geometry. In fact, it is not known whether the spin condition is dispensable in any dimension greater than 4 (see [6] for recent progress in dimension 4).

Goette and Semmelmann proved in [9, Theorem 2.4] and [8, Theorem 0.1] generalizations of Theorem 1.1 where \mathbb{S}^n is replaced by closed Riemannian manifolds with nonnegative curvature operators and nonvanishing Euler characteristics, and by closed Kähler manifolds of positive Ricci curvature, respectively. Lott [19, Theorem 1.1] proved similar extremality and rigidity results for smooth Riemannian manifolds with boundary.

In the present work we will generalize Theorem 1.1 to Riemannian metrics g and comparison maps f with regularity less than C^1 , thus highlighting the metric content of Llarull’s theorem. This is close in spirit to other recent results, such as the conservation of lower scalar curvature bounds under C^0 -convergence of smooth Riemannian metrics by Gromov [10] and Bamler [1], the definition of lower scalar curvature bounds for C^0 -Riemannian metrics via regularizing Ricci flows by Burkhardt–Guim [4], and the positive mass theorem under low regularity assumptions by Lee–LeFloch [15].

Reminder 1.2. Let M be a connected smooth manifold and let g be a continuous Riemannian metric on M , that is, $g \in C^0(M, T^*M \otimes T^*M)$ such that g_x is a scalar product on T_xM for all $x \in M$. Given an absolutely continuous curve $\gamma : [0, 1] \rightarrow M$, the *length* of γ is then defined by the formula

$$\ell(\gamma) := \int_0^1 |\gamma'(t)|_g dt,$$

using the fact that $|\gamma'|_g \in L^1([0, 1], \mathbb{R})$ [5, Proposition 3.7].

Thus g induces a path metric $d_g : M \times M \rightarrow \mathbb{R}_{\geq 0}$, called the *Riemannian distance*,

$$d_g(x, y) = \inf \{ \ell(\gamma) \mid \gamma : [0, 1] \rightarrow M \text{ absolutely continuous, } \gamma(0) = x, \gamma(1) = y \}.$$

The metric d_g induces the manifold topology on M . One obtains the same distance function d_g if the infimum is taken only over all regular smooth curves $\gamma : [0, 1] \rightarrow M$ from x to y .

For more information about metric properties of smooth manifolds equipped with continuous Riemannian metrics, see Burtscher [5, Section 4].

Definition 1.3. Let M be a smooth manifold. A continuous Riemannian metric g on M is called *admissible* if there exists some $p > n$ with

$$g \in W_{\text{loc}}^{1,p}(M, T^*M \otimes T^*M).$$

Recall that, by the Sobolev embedding theorem, each section in $W_{\text{loc}}^{1,p}(M, T^*M \otimes T^*M)$, $p > n$, has a unique continuous representative. We remark that Lipschitz continuous Riemannian metrics, which often arise from smooth Riemannian metrics in geometric gluing constructions as in Theorem B below, are admissible.

For an admissible Riemannian metric g on M , the scalar curvature is defined as a distribution $\text{scal}_g: C_c^\infty(M) \rightarrow \mathbb{R}$; see Lee–LeFloch [15] and Section 3 below. In particular, we can define lower scalar curvature bounds in the distributional sense; see Definition 5.3.

Reminder 1.4. Let M and N be smooth manifolds with continuous Riemannian metrics g and h . Let $f: (M, d_g) \rightarrow (N, d_h)$ be Lipschitz continuous. If f is (totally) differentiable at $x \in M$, we denote by

$$d_x f: T_x M \rightarrow T_{f(x)} N$$

the differential of f at x . By Rademacher’s theorem, f is differentiable almost everywhere on M with differential of regularity L_{loc}^∞ .

For $n \geq 3$ it suffices to assume in Theorem 1.1 that the smooth map f is not 1-Lipschitz, but only *area-nonincreasing*, i.e., for all $x \in M$ the induced map $\Lambda^2 d_x f: \Lambda^2 T_x M \rightarrow \Lambda^2 T_{f(x)} \mathbb{S}^n$ is norm-bounded by 1 [18, Theorem C]. This notion is generalized to Lipschitz maps as follows.

Definition 1.5. In the situation of Reminder 1.4, we say that df is *area-nonincreasing a.e.* if for almost all $x \in M$ where f is differentiable, the operator norm of the induced map

$$\Lambda^2 d_x f: \Lambda^2 T_x M \rightarrow \Lambda^2 T_{f(x)} N$$

on the second exterior power of $T_x M$ satisfies $|\Lambda^2 d_x f| \leq 1$.

Each 1-Lipschitz map $f: (M, d_g) \rightarrow (N, d_h)$ satisfies $|d_x f| \leq 1$ for all $x \in M$ where f is differentiable; see Proposition 2.1. In particular, df is area-nonincreasing a.e.

We now state our main result. It provides an affirmative answer to a question posed by Gromov [12, Section 4.5, Question (b)].

Theorem A. *Let M be a closed smooth connected oriented manifold of even dimension n which admits a spin structure, let g be an admissible Riemannian metric with $\text{scal}_g \geq n(n - 1)$, and let $f: (M, d_g) \rightarrow \mathbb{S}^n$ be a Lipschitz continuous map of nonzero degree. Furthermore, if $n = 2$, assume that f is 1-Lipschitz, and if $n \geq 4$, then assume that df is area-nonincreasing a.e. Then f is a metric isometry.*

Remark 1.6. (a) If g is assumed to be smooth then each metric isometry $(M, d_g) \rightarrow \mathbb{S}^n$ is a smooth Riemannian isometry by the Myers–Steenrod theorem.

(b) By Proposition 3.6 each metric isometry between smooth manifolds equipped with admissible Riemannian metrics preserves the scalar curvature distributions. In particular, in Theorem A, we get $\text{scal}_g \equiv n(n-1)$. This will also follow from the proof of Theorem A in Section 6.

(c) In the first version of this paper, we conjectured that Theorem A also holds for all odd $n \geq 3$. In the meantime the preprint [16] by Lee-Tam appeared which develops an alternative approach to Theorem A based on the Ricci and harmonic map heat flows. In this way they generalize Theorem A to all $n \geq 2$ under the assumption that f is 1-Lipschitz. So far, however, the case of area-nonincreasing a.e. maps f seems to be accessible only via the Dirac operator method and remains open for odd $n \geq 3$. We are currently working on generalizing Theorem A to these cases.

Our approach provides a new perspective on scalar curvature rigidity results for smooth Riemannian manifolds with boundary, previously obtained by studying boundary value problems for Dirac-type operators as in Bär-Hanke [2, Theorem 2.19], Cecchini-Zeidler [7, Corollary 1.17], and Lott [19, Corollary 1.2]. This is illustrated by the following strong comparison principle “larger than hemispheres”. We denote by \mathbb{D}_\pm^n the closed upper, respectively lower hemispheres of \mathbb{S}^n , equipped with the Riemannian metrics induced from \mathbb{S}^n . Furthermore, we identify \mathbb{S}^{n-1} with the equator sphere of \mathbb{S}^n .

Theorem B. *Let (M, g) be a compact smooth connected oriented Riemannian manifold of even dimension n which admits a spin structure and has boundary ∂M , and let $f: (M, d_g) \rightarrow \mathbb{S}^n$ be a Lipschitz continuous map such that*

- ▷ if $n = 2$, then f is 1-Lipschitz,
- ▷ if $n \geq 4$, then df is area-nonincreasing a.e.,
- ▷ $\text{scal}_g \geq n(n-1)$,
- ▷ g has nonnegative mean curvature along ∂M with respect to the interior normal,¹
- ▷ $f(\partial M) \subset \mathbb{D}_-^n$ and $f: (M, \partial M) \rightarrow (\mathbb{S}^n, \mathbb{D}_-^n)$ is of nonzero degree.

Then $\text{im}(f) = \mathbb{D}_+^n$ and $f: (M, g) \rightarrow \mathbb{D}_+^n$ is a smooth Riemannian isometry.

We derive this result from Theorem A by a doubling procedure in Section 6. Unlike Gromov [11, 12], we work directly with the resulting nonsmooth metric on the double, thus dispensing with “smoothing the corners”.

If f is smooth, and $f(\partial M) \subset \mathbb{S}^{n-1}$ with $f|_{\partial M}$ 1-Lipschitz with respect to $d_g|_{\partial M}$, then Theorem B follows from Lott [19, Corollary 1.2].

Remark 1.7. Our methods can also be applied to the more general situation treated in [9]. This issue is not discussed further in our paper.

Our paper is structured as follows. In Section 2 we provide an infinitesimal characterization of metric isometries between smooth manifolds equipped with continuous

¹With this convention the unit ball in \mathbb{R}^n has mean curvature $n-1$.

Riemannian metrics. This discussion, which is of independent interest, makes essential use of Reshetnyak's theory of quasiregular maps; see Rickman [21] for a comprehensive introduction.

In Section 3 we introduce the scalar curvature distribution of an admissible Riemannian metric, following Lee–LeFloch [15], and derive some properties relevant for our discussion.

In Section 4 we construct the spinor Dirac operator twisted with Lipschitz bundles on a spin manifold equipped with an admissible Riemannian metric. Relying on previous work of Bartnik–Chruściel [3], we establish its main functional-analytic properties and prove an index formula in Theorem 4.8.

The main result of Section 5 is Theorem 5.1 which provides a Schrödinger–Lichnerowicz formula for the twisted Dirac operator considered in Section 4 if the twist bundle is the pullback of a smooth bundle along a Lipschitz map.

Theorems A and B are proved in Section 6. Similar to the original proof of Theorem 1.1, the Schrödinger–Lichnerowicz formula is applied to a nonzero harmonic spinor field on M – whose existence is guaranteed by the index formula – which implies that the differential of f is an isometry almost everywhere. However, in the absence of an inverse function theorem for Lipschitz maps, this no longer implies that f is bijective, let alone a metric isometry. In fact, f could have folds: see Example 2.6.

Here our new observation is that the existence of this spinor field further implies that the differential of $f: M \rightarrow \mathbb{S}^n$ is *orientation preserving almost everywhere*, possibly after reversing the orientation of M . As explained in Section 2, this implies that f is a quasiregular map without branch points, hence a homeomorphism which must be a metric isometry by a curve length comparison argument.

2. A characterization of metric isometries

Let M and N be connected smooth manifolds, equipped with continuous Riemannian metrics g and h . Throughout this section, let

$$f: (M, d_g) \rightarrow (N, d_h) \tag{2.1}$$

be a locally Lipschitz continuous map.

Proposition 2.1. *If f is L -Lipschitz, then $|d_x f| \leq L$ for all $x \in M$ where f is differentiable.*

Proof. Let $x \in M$ where f is differentiable and let $v \in T_x M$ with $|v|_{g_x} = 1$. Let $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ be a smooth curve with $\gamma(0) = x$ and $\gamma'(0) = v$. The curve $f \circ \gamma: (-\varepsilon, \varepsilon) \rightarrow N$ is Lipschitz continuous, hence absolutely continuous. Furthermore, it is differentiable at $t = 0$ by our choice of x .

Let $\eta > 0$. Since f is L -Lipschitz, and since $|v|_{g_x} = 1$ and both g and γ' are continuous maps, we can assume (at least for some smaller ε) that for $\delta \in (-\varepsilon, \varepsilon)$ we get

$$d_h(f \circ \gamma(\delta), f \circ \gamma(0)) \leq L \cdot d_g(\gamma(\delta), \gamma(0)) \leq L \left| \int_0^\delta |\gamma'(t)|_{g_{\gamma(t)}} dt \right| \leq (L + \eta) \cdot |\delta|. \tag{2.2}$$

By [5, Proposition 4.10] we have

$$\lim_{\delta \rightarrow 0} \frac{d_h(f \circ \gamma(\delta), f \circ \gamma(0))}{|\delta|} = |(f \circ \gamma)'(0)|_h$$

and together with (2.2) this shows

$$|(f \circ \gamma)'(0)|_h \leq L + \eta.$$

Letting η go to 0, we obtain $|(f \circ \gamma)'(0)|_h \leq L$, finishing the proof. ■

If g and h are smooth, f is proper (i.e., preimages of compact sets are compact), and f induces a surjective map $\pi_1(M) \rightarrow \pi_1(N)$, then f is a metric isometry if and only if it is a Riemannian isometry, i.e., the differential $d_x f: (T_x M, g_x) \rightarrow (T_{f(x)} N, h_{f(x)})$ is an isometry for all $x \in M$. In this section we provide a similar characterization for continuous g and h .

Definition 2.2. We say that df is

- ▷ an *isometry a.e.* if for almost all $x \in M$ where f is differentiable, the differential $d_x f: (T_x M, g_x) \rightarrow (T_{f(x)} N, h_{f(x)})$ is an isometry,
- ▷ a *locally orientation preserving isometry a.e.* if for each $p \in M$ there exist oriented neighborhoods $p \in U \subset M$ and $f(p) \in V \subset N$ such that for almost all $x \in U$ where f is differentiable, the differential $d_x f: (T_x M, g_x) \rightarrow (T_{f(x)} N, h_{f(x)})$ is an orientation preserving isometry.

Proposition 2.3. *If f is a metric isometry, then the differential df is a locally orientation preserving isometry a.e.*

Proof. It follows from Proposition 2.1 that df is an isometry a.e. Furthermore, the local homological mapping degree of the homeomorphism $f: M \rightarrow N$ is locally constant and hence the differential $d_x f$ is a locally orientation preserving isometry a.e. ■

The following result provides a converse.

Theorem 2.4. *Let f be proper, and suppose it induces a surjective map $\pi_1(M) \rightarrow \pi_1(N)$ and df is a locally orientation preserving isometry a.e. Then f is a metric isometry.*

Remark 2.5. For smooth g and h this is implied by [13, Theorem 1.1]. [13, Question 3 on p. 374] asks whether this result generalizes to metrics of regularity less than $C^{1,\alpha}$. Our Theorem 2.4 gives a positive answer for continuous g and h .

Example 2.6. The 1-Lipschitz map $\rho: \mathbb{S}^n \rightarrow \mathbb{S}^n$ with

$$\rho(x^0, x^1, \dots, x^n) = (-|x^0|, x^1, \dots, x^n)$$

which leaves \mathbb{D}^n pointwise fixed and reflects \mathbb{D}^n_+ onto \mathbb{D}^n_- is not a metric isometry. Indeed, the differential $d\rho$ is an isometry a.e., but not a locally orientation preserving isometry a.e.

The proof of Theorem 2.4 relies on the theory of quasiregular maps. We recall the basic definition from [21], restricted to our setting.

Definition 2.7. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be a domain, that is, a nonempty, open and connected subset of \mathbb{R}^n . Let $f \in \text{Lip}_{\text{loc}}(G, \mathbb{R}^n)$ be a locally Lipschitz map.

Let $K \geq 1$. The map f is called K -quasiregular if for almost all $x \in G$ where f is differentiable, we have

$$|d_x f|^n \leq K \det(d_x f).$$

The map f is called quasiregular if f is K -quasiregular for some $K \geq 1$.

Remark 2.8. Since locally Lipschitz maps are continuous and of regularity $W^{1,\infty}_{\text{loc}}$, it follows from [21, Proposition I.1.2] that K -quasiregular locally Lipschitz maps in the sense of Definition 2.7 are K^{n-1} -quasiregular in the sense of [21, Definition I.2.1 and the subsequent discussion].

Note that for quasiregular f , the Jacobian $\det(df)$ is nonnegative almost everywhere on G .

The following fact is implied by [21, Theorem VI.8.14].

Proposition 2.9. For all $n \geq 3$ there exists $\varepsilon = \varepsilon(n) > 0$ with the following property. Let $G \subset \mathbb{R}^n$ be a domain and let $f \in \text{Lip}_{\text{loc}}(G, \mathbb{R}^n)$ be $(1 + \varepsilon)$ -quasiregular and non-constant. Then f is a local homeomorphism, that is, for all $x \in G$ there exists an open neighborhood $x \in U \subset G$ such that $f(U) \subset \mathbb{R}^n$ is open and $f|_U: U \rightarrow f(U)$ is a homeomorphism.

Remark 2.10. Since each complex analytic map $\mathbb{R}^2 \supset G \rightarrow \mathbb{R}^2$ is 1-quasiregular, the requirement $n \geq 3$ is necessary.

Lemma 2.11. Let γ_1 and γ_2 be scalar products on \mathbb{R}^n . Assume that there exists $0 < \delta \leq 1/2$ such that for all $v \in \mathbb{R}^n$ and $i = 1, 2$ we have

$$(1 - \delta)|v| \leq |v|_{\gamma_i} \leq (1 + \delta)|v|.$$

Then, for all linear isometries $A: (\mathbb{R}^n, \gamma_1) \rightarrow (\mathbb{R}^n, \gamma_2)$, we have

$$|A|^n \leq (1 + 4 \cdot 3^{2n} \cdot \delta)|\det A|.$$

Proof. First,

$$|A| = \max_{|v|=1} |Av| \leq (1 + \delta) \max_{|v|_{\gamma_1}=1} |Av| \leq \frac{1 + \delta}{1 - \delta} \max_{|v|_{\gamma_1}=1} |Av|_{\gamma_2} = \frac{1 + \delta}{1 - \delta}.$$

Second, by a singular value decomposition of A , there exist orthonormal bases (e_1, \dots, e_n) and $(\varepsilon_1, \dots, \varepsilon_n)$ of \mathbb{R}^n together with real numbers $\mu_i > 0$ such that

$$Ae_i = \mu_i \varepsilon_i, \quad 1 \leq i \leq n.$$

The numbers μ_i satisfy

$$\mu_i = |Ae_i| = |e_i|_{\gamma_1} \cdot \left| A \frac{e_i}{|e_i|_{\gamma_1}} \right| \geq \frac{1-\delta}{1+\delta} \cdot \left| A \frac{e_i}{|e_i|_{\gamma_1}} \right|_{\gamma_2} = \frac{1-\delta}{1+\delta},$$

so that

$$|\det A| = \prod_{i=1}^n \mu_i \geq \left(\frac{1-\delta}{1+\delta} \right)^n.$$

Altogether, using $\delta \leq 1/2$, we get

$$\begin{aligned} |A|^n &\leq \left(\frac{1+\delta}{1-\delta} \right)^n \leq \left(\frac{1+\delta}{1-\delta} \right)^{2n} |\det A| \\ &= \left(1 + \frac{2\delta}{1-\delta} \right)^{2n} |\det A| \leq (1+4\delta)^{2n} |\det A|. \end{aligned}$$

The claim now follows from

$$(1+4\delta)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} (4\delta)^j = 1 + 4\delta \sum_{j=1}^{2n} \binom{2n}{j} \underbrace{(4\delta)^{j-1} \cdot 1^{2n-j}}_{\leq 2^j} \leq 1 + 4\delta(2+1)^{2n}. \blacksquare$$

The following fact is close to [17, Lemma 4.2].

Lemma 2.12. *Let $n = \dim M \geq 2$ and df be a locally orientation preserving isometry a.e. Then, for each $\varepsilon > 0$ and each $x \in M$, there exist local charts (U, ϕ) around x and (V, ψ) around $f(x) \in N$ such that $f(U) \subset V$ and the induced map*

$$\hat{f} = \psi \circ f \circ \phi^{-1}: \phi(U) \rightarrow \psi(V) \subset \mathbb{R}^n$$

is a $(1 + \varepsilon)$ -quasiregular Lipschitz map.

Proof. By assumption made in (2.1), we also have $\dim N = n$. Choose connected smooth local oriented charts (U, ϕ) around $x \in M$ and (V, ψ) around $y := f(x) \in N$ with $f(U) \subset V$, and let $\hat{f} = \psi \circ f \circ \phi^{-1}: \phi(U) \rightarrow \psi(V) \subset \mathbb{R}^n$ be the induced map which we can assume to be Lipschitz by convention (2.1).

Let $\hat{g} = (\phi^{-1})^*(g)$ and $\hat{h} = (\psi^{-1})^*(h)$ be the induced continuous Riemannian metrics on $\phi(U) \subset \mathbb{R}^n$ and $\psi(V) \subset \mathbb{R}^n$. We may assume that for almost all $\zeta \in \phi(U)$ where \hat{f} is differentiable, the differential $d_\zeta \hat{f}: (\mathbb{R}^n, \hat{g}_\zeta) \rightarrow (\mathbb{R}^n, \hat{h}_{\hat{f}(\zeta)})$ is an orientation preserving isometry.

By composing ϕ and ψ with linear orientation preserving isomorphisms of \mathbb{R}^n , we may assume that $\hat{g}_{\phi(x)}$ and $\hat{h}_{\psi(y)}$ are equal to the standard Euclidean scalar product. Since \hat{g} and \hat{h} are continuous, this implies, possibly after passing to smaller U and V , that with $\delta := \varepsilon/(4 \cdot 3^{2n})$ we get

$$\begin{aligned} (1-\delta)|v| &\leq |v|_{\hat{g}_\zeta} \leq (1+\delta)|v|, \quad \forall \zeta \in \phi(U), \forall v \in \mathbb{R}^n, \\ (1-\delta)|v| &\leq |v|_{\hat{h}_\xi} \leq (1+\delta)|v|, \quad \forall \xi \in \psi(V), \forall v \in \mathbb{R}^n. \end{aligned}$$

By Lemma 2.11 the assertion follows. ■

Proposition 2.13. *The map f in Theorem 2.4 is a homeomorphism.*

Proof. For $n = \dim M \geq 3$, Proposition 2.9 and Lemma 2.12 imply that f is a local homeomorphism and hence a covering map since f is proper. Since f induces a surjective map $\pi_1(M) \rightarrow \pi_1(N)$, this shows that f is a homeomorphism.

For $n = 2$ we consider the Riemannian products $(M \times \mathbb{R}, g + dt^2)$ and $(N \times \mathbb{R}, h + dt^2)$ where dt^2 is the standard Riemannian metric on \mathbb{R} and observe that the differential of the locally Lipschitz map

$$f \times \text{id}: (M \times \mathbb{R}, d_{g+dt^2}) \rightarrow (N \times \mathbb{R}, d_{h+dt^2})$$

is a locally orientation preserving isometry a.e. Hence $f \times \text{id}$ is a homeomorphism and the same then holds for f . For $n = 1$ a similar trick applies. ■

With Proposition 2.13, the proof of Theorem 2.4 is completed by the following.

Proposition 2.14. *Let f be a homeomorphism and let df be an isometry a.e. Then f is a metric isometry.*

Proof. We need to show that for all $x, y \in M$ we have

$$d_g(x, y) \leq d_h(f(x), f(y)), \tag{2.3}$$

$$d_g(x, y) \geq d_h(f(x), f(y)). \tag{2.4}$$

Since the local homological mapping degree of f is locally constant, we may assume that the differential df is an orientation preserving isometry a.e. Then

$$M_{\text{reg}} := \{x \in M \mid f \text{ is differentiable at } x \text{ and } d_x f \text{ is an orientation preserving isometry}\}$$

is a subset of M of full measure and contained in the subset of M where f is differentiable. In particular, since f is Lipschitz, the image $f(M \setminus M_{\text{reg}}) \subset N$ has measure zero [20, Lemma 4.1].

Let $p \in M_{\text{reg}}$ and set $q = f(p) \in N$. Then the map f^{-1} is differentiable at q with differential $(d_p f)^{-1}$. It follows that $f^{-1}: N \rightarrow M$ is almost everywhere differentiable and at almost all $q \in N$ where f^{-1} is differentiable its differential $d_q f^{-1}$ is an isometry. Note that we cannot assume at this point that f^{-1} is Lipschitz.

Given $\eta > 0$, there exists a regular smooth curve $\gamma: [0, 1] \rightarrow N$ joining $f(x)$ and $f(y)$ for which

$$\ell_h(\gamma) \leq d_h(f(x), f(y)) + \eta.$$

We claim that

$$d_g(x, y) \leq \ell_h(\gamma). \tag{2.5}$$

This implies (2.3) by letting η go to zero.

To show (2.5) we may (after subdividing γ) assume that $\gamma: [0, 1] \rightarrow N$ is an embedded smooth curve and that there exist local charts (U, ϕ) on M and (V, ψ) on N such that $\text{im}(\gamma) \subset V$ and

$$\psi \circ \gamma(t) = (t, 0, \dots, 0).$$

Furthermore, by Proposition 2.12 we may assume that $\hat{f}: \phi(U) \rightarrow \psi(V)$ is quasiregular. By [21, Corollary II.6.5] the inverse $\hat{f}^{-1}: \psi(V) \rightarrow \phi(U)$ is also quasiregular.

Choose $\delta > 0$ such that $[0, 1] \times (-\delta, \delta)^{n-1} \subset \psi(V)$ and for $\tau \in 0 \times (-\delta, \delta)^{n-1}$ and $t \in [0, 1]$ set

$$\gamma_\tau(t) = \psi^{-1}((t, 0, \dots, 0) + \tau) \in V.$$

Note that $\gamma_0 = \gamma$. Since $\hat{f}^{-1}: \psi(V) \rightarrow \phi(U)$ is quasiregular, it is absolutely continuous on lines [21, p. 5], and hence $f^{-1} \circ \gamma_\tau: [0, 1] \rightarrow U \subset M$ is absolutely continuous for almost all τ . Since $d_g: M \times M \rightarrow \mathbb{R}_{\geq 0}$ is continuous and $\ell_h(\gamma_\tau)$ depends continuously on τ , we may therefore assume in showing (2.5) that $f^{-1} \circ \gamma: [0, 1] \rightarrow U$ is absolutely continuous.

Furthermore, since $f(M \setminus M_{\text{reg}}) \subset N$ has measure zero, the intersection $\text{im}(\gamma_\tau) \cap f(M \setminus M_{\text{reg}})$ has measure zero for almost all τ by the Fubini theorem (for the 1-dimensional Lebesgue measure on the 1-dimensional submanifold $\text{im}(\gamma_\tau) \subset N$). By a similar continuity argument as before, we may therefore assume that $\text{im}(\gamma) \cap f(M \setminus M_{\text{reg}})$ has measure zero. In particular, $d_q f^{-1}$ exists and is an isometry for almost all $q \in \text{im}(\gamma)$.

Under these assumptions, we obtain

$$\begin{aligned} d_g(x, y) &\leq \ell_g(f^{-1} \circ \gamma) = \int_0^1 |(f^{-1} \circ \gamma)'(t)|_g dt \\ &= \int_0^1 |\gamma'(t)|_h dt = \ell_h(\gamma). \end{aligned}$$

This yields (2.5) and finishes the proof of (2.3).

The proof of (2.4) is analogous and in fact easier since f is Lipschitz by assumption. Assume that $\gamma: [0, 1] \rightarrow M$ is an embedded smooth curve joining x and y and there are local charts (U, ϕ) on M and (V, ψ) on N such that $\text{im}(\gamma) \subset U$ and

$$\phi \circ \gamma(t) = (t, 0, \dots, 0).$$

It is enough to show that under these circumstances we get

$$d_h(f(x), f(y)) \leq \ell_g(\gamma). \tag{2.6}$$

Since f is Lipschitz, $f \circ \gamma: [0, 1] \rightarrow V$ is absolutely continuous. By a similar argument to that for (2.3), considering a family of curves $\gamma_\tau(t) = \phi^{-1}((t, 0, \dots, 0) + \tau)$, we may assume that for almost all $p \in \text{im}(\gamma)$ the map f is differentiable at p and $d_p f$ is an isometry. This implies (2.6) and completes the proof of (2.4). ■

Remark 2.15. Assume that f is proper and a locally orientation preserving isometry a.e., but f does not necessarily induce a surjective map $\pi_1(M) \rightarrow \pi_1(N)$. Let $\pi: \tilde{N} \rightarrow N$ be the universal covering of N and let $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$ be the pullback of f along π , which is still proper. With the continuous Riemannian metric \tilde{g} and \tilde{h} induced from g and h , respectively, the map \tilde{f} is a locally orientation preserving isometry a.e.

Theorem 2.4 implies that the map $\tilde{f}: (\tilde{M}, d_{\tilde{g}}) \rightarrow (\tilde{N}, d_{\tilde{h}})$ is a metric isometry. Hence the original map $f: (M, d_g) \rightarrow (N, d_h)$ is a locally isometric covering map.

3. Distributional scalar curvature

Let M be a smooth manifold of dimension n . Given a real or complex Lipschitz vector bundle $E \rightarrow M$, we denote by $\text{Lip}_{\text{loc}}(M, E)$ the vector space of locally Lipschitz continuous sections of E and by $L^p_{\text{loc}}(M, E)$ the vector space of

- ▷ locally p -integrable sections of E if $1 \leq p < \infty$,
- ▷ locally essentially bounded integrable sections of E if $p = \infty$.

These spaces are defined with respect to some, hence any, bundle metric on E and continuous volume density on M .

Definition 3.1. An *affine connection* on E is a linear map

$$\nabla: \text{Lip}_{\text{loc}}(M, E) \rightarrow L^2_{\text{loc}}(M, T^*M \otimes E)$$

which for all scalar valued Lipschitz functions f on M satisfies the Leibniz rule

$$\nabla(f\eta) = df \otimes \eta + f \nabla\eta.$$

Let $\langle -, - \rangle$ be a Euclidean (if E is real) or Hermitian (if E is complex) bundle metric on E . We say that a connection ∇ on E is *metric* if, for all $\eta_1, \eta_2 \in \text{Lip}_{\text{loc}}(M, E)$,

$$d\langle \eta_1, \eta_2 \rangle = \langle \nabla\eta_1, \eta_2 \rangle + \langle \eta_1, \nabla\eta_2 \rangle. \tag{3.1}$$

Working in local Lipschitz frames of E , each affine connection ∇ uniquely extends to a linear map $\nabla: W^{1,2}_{\text{loc}}(M, E) \rightarrow L^2_{\text{loc}}(M, T^*M \otimes E)$ satisfying the Leibniz rule. If the given connection ∇ is metric, then (3.1) continues to hold for $\eta_1, \eta_2 \in W^{1,2}_{\text{loc}}(M, E)$. Furthermore, the usual partition of unity argument shows that on each Lipschitz bundle $E \rightarrow M$ with bundle metric there exists a metric connection.

Example 3.2. Let g be an admissible Riemannian metric of regularity $W^{1,p}_{\text{loc}}$, $p > n$, on M . With respect to C^1 -regular local coordinates (x^1, \dots, x^n) on $U \subset M$, the metric g has Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \in L^p_{\text{loc}}(U), \quad 1 \leq i, j, k \leq n.$$

Here, as usual, $(g^{k\ell})$ is the inverse of the metric tensor (g_{ij}) , and we sum over the index ℓ according to the Einstein summation convention.

Using the multiplication $\text{Lip}_{\text{loc}}(U, \mathbb{R}) \times L^p_{\text{loc}}(U) \rightarrow L^2_{\text{loc}}(U)$ and the inclusion $L^\infty_{\text{loc}}(U) \subset L^2_{\text{loc}}(U)$, we obtain the *Levi-Civita connection*

$$\nabla_g(f^j \partial_j) = dx^i \otimes (\partial_i f^k + f^j \Gamma^k_{ij}) \partial_k,$$

which is metric with respect to g .

The scalar curvature of metrics of regularity less than C^2 cannot be defined in the usual way. However, it can be defined as a distribution as follows (see [15, Section 2.1]).

Let (x^1, \dots, x^n) be smooth local coordinates on $U \subset M$. Recall that, for C^2 -regular g , the Riemannian curvature tensor on U is written in terms of Christoffel symbols as

$$R_{ijk}{}^\ell = \partial_j \Gamma_{ik}^\ell - \partial_i \Gamma_{jk}^\ell + \Gamma_{ik}^p \Gamma_{jp}^\ell - \Gamma_{jk}^p \Gamma_{ip}^\ell.$$

Hence the scalar curvature of g has the local expression

$$\text{scal}_g|_U = \partial_k V^k + F, \tag{3.2}$$

where V^k and F are the smooth functions on U defined by

$$V^k := g^{ij} \Gamma_{ij}^k - g^{ik} \Gamma_{ji}^j, \tag{3.3}$$

$$F := -(\partial_k g^{ij}) \Gamma_{ij}^k + (\partial_k g^{ik}) \Gamma_{ji}^j + g^{ij} (\Gamma_{kl}^k \Gamma_{ij}^\ell - \Gamma_{jl}^k \Gamma_{ik}^\ell). \tag{3.4}$$

Note that in (3.2) the second order derivatives of the metric tensor occur only linearly, while its first order derivatives occur quadratically.

Let $d\mu_g$ be the Riemannian volume element of g which, locally on U , we write in the form $d\mu_g = \sqrt{\det(g_{ij})} dx$. For $u \in C_c^\infty(U)$, we hence obtain, using integration by parts,

$$\int_M \text{scal}_g u \, d\mu_g = \int_U \left(-V^k \partial_k (u \sqrt{\det(g_{ij})}) + F u \sqrt{\det(g_{ij})} \right) dx. \tag{3.5}$$

In particular, the right hand integral is independent of the choice of smooth local coordinates on U .

Now let g be admissible of regularity $W_{\text{loc}}^{1,p}$, $p > n$. Then in (3.3) and (3.4) we have $V^k \in L_{\text{loc}}^2(U)$ and $F \in L_{\text{loc}}^1(U)$. Furthermore, since g is of regularity $W_{\text{loc}}^{1,p}$, $\det(g_{ij}) > 0$ and $\sqrt{\cdot}: (0, \infty) \rightarrow (0, \infty)$ is smooth with locally bounded derivative, we obtain $\sqrt{\det(g_{ij})} \in W_{\text{loc}}^{1,p}(U)$ by the chain rule for Sobolev functions. Hence the right hand side of (3.5) is still defined. Obviously, for fixed u it is continuous in g in the $W_{\text{loc}}^{1,p}$ -topology, which we will use frequently. For example, since any $W_{\text{loc}}^{1,p}$ -regular metric can be approximated by C^2 -regular metrics in the $W_{\text{loc}}^{1,p}$ -topology, the right hand side of (3.5) is independent of the choice of smooth local coordinates.

By a partition of unity on M , we hence obtain a linear functional

$$\text{scal}_g: C_c^\infty(M) \rightarrow \mathbb{R}$$

which is uniquely determined by

$$\langle\langle \text{scal}_g, u \rangle\rangle = \int_U \left(-V^k \partial_k (u \sqrt{\det(g_{ij})}) + F u \sqrt{\det(g_{ij})} \right) dx$$

if u is supported in a coordinate neighborhood $U \subset M$ with smooth coordinates (x^1, \dots, x^n) .

Definition 3.3. The functional $\text{scal}_g: C_c^\infty(M) \rightarrow \mathbb{R}$ is called the *scalar curvature distribution* of g .

Remark 3.4. Our definition of the distribution scal_g coincides with [15, Definition 2.1] which uses a smooth background metric on M instead of local coordinates.

The definition of $\langle\langle \text{scal}_g, u \rangle\rangle$ easily extends to Lipschitz functions $u \in \text{Lip}_c(M, \mathbb{C})$ and then defines a distribution with values in \mathbb{C} . This implies that, if $E \rightarrow M$ is a complex Lipschitz bundle with Hermitian inner product $\langle -, - \rangle$, we obtain a sesquilinear functional

$$\mathcal{S}_g : \text{Lip}_c(M, E) \times \text{Lip}_c(M, E) \rightarrow \mathbb{C}, \quad \mathcal{S}_g(\eta_1, \eta_2) := \langle\langle \text{scal}_g, \langle \eta_1, \eta_2 \rangle \rangle\rangle.$$

Proposition 3.5. *The scalar curvature distribution \mathcal{S}_g extends to a continuous sesquilinear functional*

$$\mathcal{S}_g : W_{\text{loc}}^{1,2}(M, E) \times W_{\text{loc}}^{1,2}(M, E) \rightarrow \mathbb{C}.$$

Proof. Let g be of regularity $W_{\text{loc}}^{1,p}$, $p > n$. Let $\eta, \theta \in W_{\text{loc}}^{1,2}(M, E)$. Then $\langle \eta, \theta \rangle \in L_{\text{loc}}^{\frac{n}{n-2}}(M)$ for $n \geq 3$ and $\langle \eta, \theta \rangle \in L_{\text{loc}}^{\frac{p}{p-2}}(M)$ for $n = 2$ by the Sobolev embedding theorem. Pick a metric connection ∇^E on E . We then obtain

$$d\langle \eta, \theta \rangle = \langle \nabla^E \eta, \theta \rangle + \langle \eta, \nabla^E \theta \rangle \in L_{\text{loc}}^{\frac{n}{n-1}}(M, T^*M)$$

by the Hölder inequality.

Moreover, if (η_i) and (θ_i) are sequences in $W_{\text{loc}}^{1,2}(M, E)$ converging to η and θ in $W_{\text{loc}}^{1,2}(M, E)$, then $\langle \eta_i, \theta_i \rangle \rightarrow \langle \eta, \theta \rangle$ in $L_{\text{loc}}^{\frac{n}{n-2}}(M)$ for $n \geq 3$, respectively in $L_{\text{loc}}^{\frac{p}{p-2}}(M)$ for $n = 2$, and $d\langle \eta_i, \theta_i \rangle \rightarrow d\langle \eta, \theta \rangle$ in $L_{\text{loc}}^{\frac{n}{n-1}}(M, T^*M)$ by a similar argument.

Writing the right hand integrand in (3.5) in the form

$$-V^k \cdot \partial_k u \cdot \sqrt{\det(g_{ij})} - V^k \cdot u \cdot \partial_k \sqrt{\det(g_{ij})} + F \cdot u \cdot \sqrt{\det(g_{ij})}$$

for $u := \langle \eta, \theta \rangle$, the claim now follows from the Hölder inequality. Indeed, since $\sqrt{\det(g_{ij})} \in W_{\text{loc}}^{1,p}(U)$, we can argue as follows:

- ▷ $V^k \in L_{\text{loc}}^n(U)$, $\partial_k u \in L_{\text{loc}}^{\frac{n}{n-1}}(U)$, $\sqrt{\det(g_{ij})} \in C^0(U)$ and $\frac{1}{n} + \frac{n-1}{n} = 1$,
- ▷ $V^k \in L_{\text{loc}}^n(U)$, $u \in L_{\text{loc}}^{\frac{n}{n-2}}(U)$, $\partial_k \sqrt{\det(g_{ij})} \in L_{\text{loc}}^n(U)$ and $\frac{1}{n} + \frac{n-2}{n} + \frac{1}{n} = 1$ for $n \geq 3$,
- ▷ $V^k \in L_{\text{loc}}^p(U)$, $u \in L_{\text{loc}}^{\frac{p}{p-2}}(U)$, $\partial_k \sqrt{\det(g_{ij})} \in L_{\text{loc}}^p(U)$ and $\frac{1}{p} + \frac{p-2}{p} + \frac{1}{p} = 1$ for $n = 2$,
- ▷ $F \in L_{\text{loc}}^{n/2}(U)$, $u \in L_{\text{loc}}^{\frac{n}{n-2}}(U)$, $\sqrt{\det(g_{ij})} \in C^0(U)$ and $\frac{2}{n} + \frac{n-2}{n} = 1$ for $n \geq 3$,
- ▷ $F \in L_{\text{loc}}^{p/2}(U)$, $u \in L_{\text{loc}}^{\frac{p}{p-2}}(U)$, $\sqrt{\det(g_{ij})} \in C^0(U)$ and $\frac{2}{p} + \frac{p-2}{p} = 1$ for $n = 2$. ■

We finally show that the scalar curvature distribution is invariant under metric isometries.

Proposition 3.6. *Let M and N be smooth manifolds of dimension n and equipped with admissible Riemannian metrics g and h of regularity $W_{\text{loc}}^{1,p}$, $p > n$. Let $f : (M, d_g) \rightarrow (N, d_h)$ be a metric isometry. Then $\text{scal}_g = f^* \text{scal}_h$ in the sense that, for all $u \in C_c^\infty(N)$,*

$$\langle\langle \text{scal}_g, u \circ f \rangle\rangle = \langle\langle \text{scal}_h, u \rangle\rangle.$$

Proof. Working in local coordinate neighborhoods we may assume that M and N are open subsets of \mathbb{R}^n . Let (u^1, \dots, u^n) be local harmonic coordinates on $\Omega' \subset N \subset \mathbb{R}^n$. Since h is $W_{\text{loc}}^{1,p}$ -regular, by [23, Chapter 3, (9.40)] we have $u^i \in W_{\text{loc}}^{2,p}(\Omega')$ for $1 \leq i \leq n$.

Let $\Omega := f^{-1}(\Omega') \subset M \subset \mathbb{R}^n$. In order to simplify the notation we replace M and N by Ω and Ω' .

Since $f: \Omega \rightarrow \Omega'$ is a metric isometry, the functions $v^i := u^i \circ f: \Omega \rightarrow \mathbb{R}$, $1 \leq i \leq n$, are weakly harmonic, hence harmonic, for the $W_{\text{loc}}^{1,p}$ -regular metric g , by an argument as in [22, p. 2417]. Again, this implies $v^i \in W_{\text{loc}}^{2,p}(\Omega)$ for $1 \leq i \leq n$.

The Sobolev embedding theorem implies that $u = (u^1, \dots, u^n): \Omega \rightarrow u(\Omega) \subset \mathbb{R}^n$ and $v = (v^1, \dots, v^n): \Omega' \rightarrow v(\Omega') \subset \mathbb{R}^n$ are C^1 -diffeomorphisms. Furthermore, a short calculation shows that their inverses are of regularity $W_{\text{loc}}^{2,p}$ and the same holds for

$$f = u^{-1} \circ v: \Omega \rightarrow \Omega'.$$

By Proposition 2.3 we have $f^*h = g$. It remains to show that this and the $W^{2,p}$ -regularity of f imply that

$$\langle\langle \text{scal}_g, u \circ f \rangle\rangle = \langle\langle \text{scal}_h, u \rangle\rangle, \quad u \in C_c^\infty(\Omega'). \tag{3.6}$$

To prove this claim, let (h_ν) be a sequence of smooth Riemannian metrics on Ω' converging to h in the $W_{\text{loc}}^{1,p}$ -topology. Since f is $W_{\text{loc}}^{2,p}$ -regular, this implies that $g_\nu := f^*h_\nu \in W_{\text{loc}}^{1,p}(\Omega, T^*\Omega \otimes T^*\Omega)$ converges to g in the $W_{\text{loc}}^{1,p}$ -topology. To prove (3.6) we may therefore assume that h is smooth. This implies that $g = f^*h$ is still of regularity $W_{\text{loc}}^{1,p}$.

Next let (f_ν) be a sequence of C^∞ -diffeomorphisms $\Omega \rightarrow \Omega'$ converging to f in the $W_{\text{loc}}^{2,p}$ -topology. This implies that $g_\nu := f_\nu^*h$ converges to g in the $W_{\text{loc}}^{1,p}$ -topology. To prove (3.6) we may hence assume that both h and f , and hence $g = f^*h$, are smooth.

But for smooth g , f and h , the scalar curvatures $\text{scal}_g: \Omega \rightarrow \mathbb{R}$ and $\text{scal}_h: \Omega' \rightarrow \mathbb{R}$ are defined by the classical formula (3.2), and furthermore $\text{scal}_g = \text{scal}_h \circ f$ by the invariance of the scalar curvature of smooth Riemannian metrics under smooth diffeomorphisms. From this, (3.6) follows. ■

4. Dirac operators twisted with Lipschitz bundles

Let M be a smooth oriented n -dimensional manifold which admits a spin structure, i.e., the second Stiefel–Whitney class of M vanishes. Let γ be some smooth Riemannian metric on M , which remains fixed throughout. Since M admits a spin structure, we may choose a smooth $\text{Spin}(n)$ -principal bundle $P_{\text{Spin}}(M, \gamma) \rightarrow M$ together with a 2-fold covering map to the $\text{SO}(n)$ -principal bundle $P_{\text{SO}}(M, \gamma) \rightarrow M$ of smooth positively oriented γ -orthonormal frames which is compatible with the canonical map $\text{Spin}(n) \rightarrow \text{SO}(n)$.

Let W be a complex left Cl_n -module with $\text{Spin}(n)$ -invariant Hermitian inner product. We obtain the associated *spinor bundle*

$$S = P_{\text{Spin}}(M, \gamma) \times_{\text{Spin}(n)} W \rightarrow M.$$

It carries a Hermitian inner product induced by the inner product on W and a metric connection ∇^γ induced by the Levi-Civita connection on (M, γ) . Furthermore, it carries

a skew-adjoint module structure over the Clifford algebra bundle $\text{Cl}(TM, \gamma) \rightarrow M$ with respect to which ∇^γ acts as a derivation. See [14, Chapter II] for details.

Let g be an admissible Riemannian metric on M of regularity $W_{\text{loc}}^{1,p}$, $p > n$. In order to avoid the discussion of nonsmooth spinor bundles, we develop the spin geometry for (M, g) entirely on the smooth bundle $S \rightarrow M$, similarly to [15, Section 3.1].

We recall that there exists a unique γ -self-adjoint and positive linear isomorphism $B_g: TM \rightarrow TM$ covering $\text{id}_M: M \rightarrow M$ and satisfying

$$\gamma(v, w) = g(B_g(v), w), \quad v, w \in T_x M, x \in M.$$

It has a unique positive square root $b_g: TM \rightarrow TM$ which is again γ -self-adjoint and satisfies

$$\gamma(v, w) = g(b_g v, b_g w), \quad v, w \in T_x M, x \in M.$$

Let $S^2(T^*M) \rightarrow M$ denote the vector bundle of symmetric $(0, 2)$ -tensors over M , and let $S_+^2(T^*M) \subset S^2(T^*M)$ be the open subset of positive definite symmetric $(0, 2)$ -tensors.

Lemma 4.1. *For all $p > n$, the assignment $g \mapsto b_g$ defines a continuous map*

$$b: W_{\text{loc}}^{1,p}(M, S_+^2(T^*M)) \rightarrow W_{\text{loc}}^{1,p}(M, \text{End}(TM)).$$

In particular, each b_g maps local smooth γ -orthonormal frames to local $W^{1,p}$ -regular g -orthonormal frames.

Proof. Let $\text{Sym} \subset \mathbb{R}^{n \times n}$ be the linear subspace of symmetric matrices and $\text{Sym}^+ \subset \text{Sym}$ be the open subset of positive definite symmetric matrices. The map $\text{Sym}^+ \rightarrow \text{Sym}^+$, $A \mapsto A^2$, is bijective and smooth with differential which is everywhere invertible. It is hence a diffeomorphism with smooth inverse which we denote by $A \mapsto \sqrt{A}$.

Denoting by $\text{Sym}^+(TM) \rightarrow M$ the smooth bundle of positive definite γ -self-adjoint endomorphisms of TM , we obtain a continuous map

$$\sqrt{\cdot}: W_{\text{loc}}^{1,p}(M, \text{Sym}^+(TM)) \rightarrow W_{\text{loc}}^{1,p}(M, \text{Sym}^+(TM)),$$

induced by the map $A \mapsto \sqrt{A}$ in each fiber.

The assignment $g \mapsto B_g$ induces a continuous map

$$B: W_{\text{loc}}^{1,p}(M, S_+^2(T^*M)) \rightarrow W_{\text{loc}}^{1,p}(M, \text{Sym}^+(TM)).$$

Since $b = \sqrt{B}$, this concludes the proof of Lemma 4.1. ■

Notation 4.2. For $v \in TM$ we set $v^g := b_g(v)$.

Let (e_1, \dots, e_n) be a smooth positively oriented γ -orthonormal frame of TM over $U \subset M$, that is, a section of the bundle $\text{P}_{\text{SO}}(M, \gamma)|_U \rightarrow U$, and choose a lift to a section of $\text{P}_{\text{Spin}}(M, \gamma)|_U \rightarrow U$. This induces a smooth trivialization $S|_U \cong U \times W$. In particular, each $\sigma \in W$ induces a smooth section of $S|_U = \text{P}_{\text{Spin}}(M, \gamma)|_U \times_{\text{Spin}(n)} W$ which is denoted again by σ and called a *constant local spinor field* with respect to the chosen trivialization of $\text{P}_{\text{Spin}}(M, \gamma)|_U$.

Let $\omega_{ji}^g \in L^2_{\text{loc}}(U, T^*U)$, $1 \leq i, j \leq n$, be the connection 1-forms of the Levi-Civita connection ∇_g on $TM \rightarrow M$ with respect to the g -orthonormal frame (e_1^g, \dots, e_n^g) of $TM|_U$, that is,

$$\nabla_g(e_i^g) = \sum_{j=1}^n \omega_{ji}^g(e_j^g).$$

We now define

$$\nabla_g^S \sigma := \frac{1}{2} \sum_{i < j} \omega_{ji}^g e_i \cdot e_j \cdot \sigma \in L^2_{\text{loc}}(U, S|_U). \tag{4.1}$$

Imposing the Leibniz rule, this extends to a metric connection ∇_g^S on $S|_U \rightarrow U$ in the sense of Definition 3.1. This definition is independent of the choice of (e_1, \dots, e_n) and its lift to $P_{\text{Spin}}(M, \gamma)|_U$ and can hence be used to define a global metric connection ∇_g^S on $S \rightarrow M$.

Definition 4.3. We call ∇_g^S the *spinor connection* on $S \rightarrow M$ with respect to g .

Remark 4.4. If g is smooth, then b_g induces a smooth map $\beta_g: P_{\text{SO}}(M, \gamma) \rightarrow P_{\text{SO}}(M, g)$ of $\text{SO}(n)$ -principal bundles (compare the proof of Lemma 4.1) and there exists a spin structure $P_{\text{Spin}}(M, g) \rightarrow P_{\text{SO}}(M, g)$ together with a commutative diagram of smooth fiber bundles over M ,

$$\begin{CD} P_{\text{Spin}}(M, \gamma) @>\tilde{\beta}_g>> P_{\text{Spin}}(M, g) \\ @VVV @VVV \\ P_{\text{SO}}(M, \gamma) @>\beta_g>> P_{\text{SO}}(M, g) \end{CD}$$

where $\tilde{\beta}_g$ is $\text{Spin}(n)$ -equivariant.

Let $S_g := P_{\text{Spin}}(M, g) \times_{\text{Spin}(n)} W \rightarrow M$ be the associated smooth spinor bundle which is usually considered on the spin Riemannian manifold (M, g) . We obtain a smooth unitary vector bundle isomorphism

$$\Xi_g: S = P_{\text{Spin}}(M, \gamma) \times_{\text{Spin}(n)} W \xrightarrow{\tilde{\beta}_g \times_{\text{Spin}(n)} \text{id}} P_{\text{Spin}}(M, g) \times_{\text{Spin}(n)} W = S_g$$

which is compatible with the $\text{Cl}(TM, \gamma)$ -module, respectively $\text{Cl}(TM, g)$ -module structure on S and S_g intertwined by β_g .

Hence the local expression of the spinor connection on S_g from [14, Theorem II.4.14] pulls back to our defining equation (4.1) under Ξ_g .

Let $E \rightarrow M$ be a Hermitian Lipschitz bundle together with a Lipschitz connection ∇^E which we do not assume to be metric. The tensor product bundle $S \otimes E \rightarrow M$ is a Lipschitz bundle with induced inner product and connection

$$\nabla^{S \otimes E} = \nabla_g^S \otimes 1 + 1 \otimes \nabla^E. \tag{4.2}$$

Definition 4.5. We define the *spin Dirac operator twisted with (E, ∇^E)* ,

$$D_E = D_{g,E}: \text{Lip}_{\text{loc}}(M, S \otimes E) \rightarrow L^2_{\text{loc}}(M, S \otimes E),$$

locally over an open subset $U \subset M$ equipped with a smooth γ -orthonormal frame (e_1, \dots, e_n) of TM and using Notation 4.2, by

$$D_{g,E}(\psi) := \sum_{i=1}^n e_i \cdot \nabla_{e_i}^{S \otimes E} \psi.$$

This expression is independent of the choice of (e_1, \dots, e_n) and hence gives a global definition of $D_{g,E}$.

From now on let M be closed. Recall from [15, Lemma 3.1] that the L^2 -norm and $W^{1,2}$ -norm on $\text{Lip}(M, S \otimes E)$,

$$\|\psi\|_{L^2} = \int_M |\psi|^2 d\mu_g, \quad \|\psi\|_{W^{1,2}} = \int_M (|\psi|^2 + |\nabla^{S \otimes E} \psi|^2) d\mu_g,$$

are equivalent to the corresponding norms defined for the metric γ (which induces a different spinor connection on S) and hence define the same L^2 - and $W^{1,2}$ -completions. If we wish to emphasize the precise norm, we add the relevant metric as a subscript.

We consider D_E as an unbounded operator

$$D_E: L^2_\gamma(M, S \otimes E) \overset{\text{dense}}{\supset} \text{Lip}(M, S \otimes E) \rightarrow L^2_\gamma(M, S \otimes E).$$

Since integration by parts holds for Lipschitz sections, the twisted Dirac operator D_E has a formal adjoint and is hence closable with closure

$$\bar{D}_E: \text{dom}(\bar{D}_E) \rightarrow L^2(M, S \otimes E).$$

Proposition 4.6. *We have*

$$\text{dom}(\bar{D}_E) = H^1(M, S \otimes E).$$

Proof. Fix local coordinates (x^1, \dots, x^n) on $U \subset M$, a smooth trivialization $S|_U \cong U \times \mathbb{C}^q$ and a Lipschitz trivialization $E|_U \cong U \times \mathbb{C}^r$. We obtain an induced local trivialization $(S \otimes E)|_U \cong U \times \mathbb{C}^{q+r}$ with respect to which we can write

$$D_E(u) = a^j \partial_j u + bu \tag{4.3}$$

where $a^j \in W^{1,p}_{\text{loc}}(U, \text{End}(\mathbb{C}^{q+r}))$ and $b \in L^p_{\text{loc}}(U, \text{End}(\mathbb{C}^{q+r}))$. Hence the operator D_E satisfies the regularity conditions [3, (3.4)].

Let $\sharp^{gt}: T^*M \cong TM$ be the musical isomorphism for g . By a standard computation, the principal symbol $\sigma_\xi(D_E)$ for $\xi \in T^*M$ is given by

$$\psi \mapsto i(\sharp^{gt}(\xi) \cdot \psi),$$

and hence the ellipticity condition [3, (3.5)] is satisfied with a constant η which can be chosen uniformly on M .

The assertion of Proposition 4.6 is hence implied by [3, Theorem 3.7]. ■

Proposition 4.7. *If the connection ∇^E is metric, then the twisted Dirac operator D_E is formally self-adjoint with respect to the L^2 -inner product on $\text{Lip}(M, S \otimes E)$ induced by g , that is,*

$$(D_E \psi_1, \psi_2)_g = (\psi_1, D_E \psi_2)_g, \quad \psi_1, \psi_2 \in \text{Lip}(M, S \otimes E).$$

In particular, the operator

$$\bar{D}_E: H^1(M, S \otimes E) \rightarrow L^2_g(M, S \otimes E)$$

is self-adjoint.

Proof. For smooth g this holds by the same argument as in [14, proof of Lemma 5.1], since integration by parts holds for Lipschitz sections. If g is of regularity $W^{1,p}$, $p > n$, we choose a sequence of smooth metrics g_ν on M converging to g in the $W^{1,p}$ -topology, hence in the L^∞ -topology. By Lemma 4.1 we obtain

$$\lim_{\nu \rightarrow \infty} (D_{g_\nu, E} \psi_1, \psi_2)_{g_\nu} = (D_{g, E} \psi_1, \psi_2)_g, \quad \lim_{\nu \rightarrow \infty} (\psi_1, D_{g_\nu, E} \psi_2)_{g_\nu} = (\psi_1, D_{g, E} \psi_2)_g. \quad \blacksquare$$

In the remainder of this section let

- ▷ M be of even dimension n ,
- ▷ W be the unique irreducible complex Cl_n -module with the usual $\mathbb{Z}/2$ -grading $W = W^+ \oplus W^-$ and a $\text{Spin}(n)$ -invariant Hermitian metric,
- ▷ $D_E = \bar{D}_{g, E}: H^1(M, S \otimes E) \rightarrow L^2(M, S \otimes E)$ be the closure of the twisted Dirac operator from Definition 4.5.

We obtain induced $\mathbb{Z}/2$ -gradings

$$S^\pm = P_{\text{Spin}(M, \gamma)} \times_{\text{Spin}(n)} W^\pm, \quad S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E),$$

and hence induced $\mathbb{Z}/2$ -gradings

$$\begin{aligned} L^2(M, S \otimes E) &= L^2(M, S^+ \otimes E) \oplus L^2(M, S^- \otimes E), \\ H^1(M, S \otimes E) &= H^1(M, S^+ \otimes E) \oplus H^1(M, S^- \otimes E) \end{aligned}$$

such that the operator D_E is odd with respect to these gradings, that is,

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}.$$

If the connection ∇^E on E is metric, then the operators $D_E^\pm: H^2(M, S^\pm \otimes E) \rightarrow L^2_g(M, S^\mp \otimes E)$ are adjoint to each other by Proposition 4.7.

Let $\hat{A}(M)$ denote the total \hat{A} -class of the tangent bundle of M .

Theorem 4.8. *The operator $D_E^+: H^1(M, S^+ \otimes E) \rightarrow L^2(M, S^- \otimes E)$ is Fredholm with index*

$$\text{index}(D_E^+) = \langle \hat{A}(M) \cup \text{ch}(E), [M] \rangle.$$

Theorem 4.8 is well known for smooth g and smooth twist bundles $E \rightarrow M$ by the Atiyah–Singer index theorem [14, Theorem III.13.10]. We will reduce the general case to this case using the invariance of the index in norm-continuous families of Fredholm operators.

In a first step, we put a smooth structure on the twist bundle $E \rightarrow M$.

Lemma 4.9. *There is a smooth structure on $E \rightarrow M$ which is compatible with the given Lipschitz structure on E .*

Proof. Let $\mu: M \rightarrow \mathbf{Gr}_r(\mathbb{C}^N)$ be a Lipschitz map which classifies the bundle E where $r = \text{rank } E$ and N is a large enough integer. Then μ is homotopic, through a Lipschitz homotopy, to a smooth map $\tilde{\mu}: M \rightarrow \mathbf{Gr}_r(\mathbb{C}^N)$. The map $\tilde{\mu}$ classifies a smooth bundle $\tilde{E} \rightarrow M$ which is Lipschitz isomorphic to $E \rightarrow M$, and we can use this Lipschitz isomorphism to pull back the smooth structure on \tilde{E} to a smooth structure on E . ■

Endow the smooth bundle $E \rightarrow M$ with a smooth inner product and a smooth connection $\tilde{\nabla}^E$ which remain fixed from now on. We obtain families which depend continuously on $t \in [0, 1]$ of

- ▷ $W^{1,p}$ -metrics $g_t := (1 - t)g + t\gamma$ on M ,
- ▷ (not necessarily metric) Lipschitz connections $\nabla_t^E = (1 - t) \cdot \nabla^E + t \cdot \tilde{\nabla}^E$ on E ,
- ▷ twisted Dirac operators $D_t = \tilde{D}_{g_t, (E, \nabla_t^E)}: H^1(M, S \otimes E) \rightarrow L^2(M, S \otimes E)$.

Furthermore, $a^j(t) \in W_{\text{loc}}^{1,p}(U, \text{End}(\mathbb{C}^{r+q}))$ and $b(t) \in L_{\text{loc}}^p(U, \text{End}(\mathbb{C}^{r+q}))$ in (4.3) depend continuously on t .

Proposition 4.10. *The following hold:*

- (a) $D_t^+: H^1(M, S^+ \otimes E) \rightarrow L^2(M, S^- \otimes E)$ is bounded for $t \in [0, 1]$ and the map $[0, 1] \rightarrow \mathcal{B}(H^1(M, S^+ \otimes E), L^2(M, S^- \otimes E)), t \mapsto D_t^+$, is continuous.
- (b) D_t^+ is Fredholm for $t \in [0, 1]$.

Proof. The first statement follows from [3, Theorem 3.7] and the local uniform boundedness and continuity of the coefficients $a^j(t)$ and $b(t)$ in t . The Fredholm property follows from [3, Corollary 4.5]. ■

Theorem 4.8 now follows from Proposition 4.10 and the invariance of the Fredholm index of norm-continuous families of Fredholm operators, which implies

$$\begin{aligned} \text{index}(D_{g, (E, \nabla^E)}^+) &= \text{index}(D_0^+) = \text{index}(D_1^+) \\ &= \text{index}(D_{\gamma, (E, \tilde{\nabla}^E)}) = \langle \hat{A}(M) \cup \text{ch}(E), [M] \rangle. \end{aligned}$$

5. An integral Schrödinger–Lichnerowicz formula

Let (M, γ) be a closed smooth n -dimensional Riemannian spin manifold and let $S \rightarrow M$ be a smooth spinor bundle as in Section 4 associated to γ and some Cl_n -representation W with $\text{Spin}(n)$ -invariant Hermitian inner product.

Let $E \rightarrow M$ be a smooth Hermitian vector bundle with metric connection ∇^E . If g is a smooth metric on M , then the twisted Dirac operator $D_E = D_{g,E}$ from Definition 4.5 satisfies the Schrödinger–Lichnerowicz formula [14, Theorem II.8.17],

$$D_E^2 = \nabla_g^* \circ \nabla_g + \frac{1}{4} \text{scal}_g + \mathcal{R}^E.$$

Here \mathcal{R}^E is a self-adjoint bundle endomorphism of $S \otimes E$ depending on the curvature R^E of E .

Now let g be an admissible Riemannian metric on M . In this case an integral form of the Schrödinger–Lichnerowicz formula still holds for the untwisted Dirac operator [15, Proposition 3.2]. Here we will generalize this formula to Dirac operators twisted with a Lipschitz bundle $E \rightarrow M$ and draw some conclusions. Since in general the curvature of Lipschitz bundles over M is not defined, we will work in the following setting.

Let (N, h) be a smooth Riemannian manifold, set $\ell := \dim N$ and let (E_0, ∇^{E_0}) be a smooth Hermitian vector bundle over N with smooth metric connection ∇^{E_0} . Take a Lipschitz map

$$f: (M, d_g) \rightarrow (N, d_h)$$

and let $E := f^*(E_0) \rightarrow M$ be the pullback bundle of E_0 under f . This is a Hermitian Lipschitz bundle with pullback metric connection $\nabla^E = f^*\nabla^{E_0}$.

Denote by $R^{E_0} \in \Omega^2(N, \text{End}(E_0))$ the curvature of the connection ∇^{E_0} and let

$$R^E = f^*(R^{E_0}) \in L^\infty \Omega^2(M, \text{End}(E))$$

be the pullback of R^{E_0} along f .

For $x \in M$ and $\sigma \otimes \eta \in (S \otimes E)_x$, we set (recall Notation 4.2)

$$\mathcal{R}_g^E(\sigma \otimes \eta) := \frac{1}{2} \sum_{i,j=1}^n (e_i \cdot e_j \cdot \sigma) \otimes (R_{e_i^g, e_j^g}^E \eta) \tag{5.1}$$

where (e_1, \dots, e_n) is some orthonormal basis of $(T_x M, \gamma_x)$. Observe that \mathcal{R}_g^E defines a section in $L^\infty(M, \text{End}(S \otimes E))$ and hence a bounded operator

$$\mathcal{R}_g^E: L^2(M, S \otimes E) \rightarrow L^2(M, S \otimes E).$$

Let $D = \bar{D}_{g,E}: H^1(M, S \otimes E) \rightarrow L^2(M, S \otimes E)$ be the self-adjoint spin Dirac operator on M twisted with (E, ∇^E) , see Proposition 4.7. We denote by ∇ the tensor product connection $\nabla^{S \otimes E}$ defined in (4.2).

Theorem 5.1 (Integral Schrödinger–Lichnerowicz formula). *For all ψ_1 and ψ_2 in $H^1(M, S \otimes E)$ we have*

$$(D \psi_1, D \psi_2)_g = (\nabla \psi_1, \nabla \psi_2)_g + \frac{1}{4} \langle\langle \text{scal}_g, \langle \psi_1, \psi_2 \rangle \rangle\rangle + (\mathcal{R}_g^E \psi_1, \psi_2)_g.$$

The proof of Theorem 5.1 is based on the following approximation result.

Lemma 5.2. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lipschitz function with compact support $K \subset \mathbb{R}^n$ and let $K \subset U \subset \mathbb{R}^n$ be an open neighborhood. Then there exists a sequence of smooth functions $f_\nu: \mathbb{R}^n \rightarrow \mathbb{R}$ with compact supports in U and satisfying*

$$\lim_{\nu \rightarrow \infty} (\|f_\nu - f\|_{L^\infty} + \|f_\nu - f\|_{H^1}) = 0, \quad \max_\nu \|df_\nu\|_{L^\infty} < \infty. \tag{5.2}$$

Proof. Let $\rho: \mathbb{R}^n \rightarrow [0, \infty)$ be a compactly supported smooth function such that $\int_{\mathbb{R}^n} \rho = 1$. For $\varepsilon > 0$ set $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(x/\varepsilon)$ and consider the convolution

$$\rho_\varepsilon * f(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(\tau) \cdot f(x - \tau) d\tau.$$

For small enough ε we have $\text{supp}(\rho_\varepsilon * f) \subset U$. Setting $f_\nu := \rho_{1/\nu} * f$ we have

$$\lim_{\nu \rightarrow \infty} \|f_\nu - f\|_{L^\infty} = 0, \quad \max_\nu \|df_\nu\|_{L^\infty} < \infty.$$

For a compactly supported function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ we denote by $\hat{u}: \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) dx,$$

its Fourier transform. We have $\hat{\rho}(0) = \int_{\mathbb{R}^n} \rho = 1$ and $\hat{\rho}_\varepsilon(\xi) = \hat{\rho}(\varepsilon\xi)$. As $\hat{\rho}$ is a Schwartz function, there is a uniform (in ε and ξ) bound C on $|\hat{\rho}_\varepsilon|$.

Let $\eta > 0$. As $\hat{f}(\xi)$ is a Schwartz function, $|\hat{f}(\xi)|^2(1 + |\xi|^2)$ is integrable. Then there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} |\hat{f}(\xi)|^2(1 + |\xi|^2) d\xi < \frac{\eta}{2(C + 1)^2}.$$

Set $F := \max \{ \max_{\xi \in B_R} |\hat{f}(\xi)|^2(1 + |\xi|^2), 1 \}$. There exists $\varepsilon_\eta > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_\eta$, we get

$$\sup_{\xi \in B_R} (\hat{\rho}_\varepsilon(\xi) - 1)^2 < \frac{\eta}{2F \text{vol}(B_R)}.$$

For $0 < \varepsilon \leq \varepsilon_\eta$ we hence obtain, denoting by μ the Lebesgue measure on \mathbb{R}^n ,

$$\begin{aligned} & \|\hat{\rho}_\varepsilon \hat{f} - \hat{f}\|_{L^2((1+|\xi|^2)\mu)}^2 \\ & \leq \int_{B_R} (\hat{\rho}_\varepsilon - 1)^2 |\hat{f}|^2(1 + |\xi|^2) d\xi + \int_{\mathbb{R}^n \setminus B_R} (|\hat{\rho}_\varepsilon| + 1)^2 |\hat{f}|^2(1 + |\xi|^2) d\xi \\ & \leq \frac{\eta}{2F \text{vol}(B_R)} \int_{B_R} F d\xi + (C + 1)^2 \int_{\mathbb{R}^n \setminus B_R} |\hat{f}|^2(1 + |\xi|^2) d\xi \\ & \leq \eta/2 + \eta/2 = \eta. \end{aligned}$$

Using the definition of the H^1 -norm in the Fourier picture, there is a constant $\Lambda > 0$ which only depends on n such that for $\varepsilon > 0$ we have

$$\|\rho_\varepsilon * f - f\|_{H^1}^2 \leq \Lambda \cdot \|\hat{\rho}_\varepsilon \hat{f} - \hat{f}\|_{L^2((1+|\xi|^2)\mu)}^2.$$

Letting η go to 0 in the previous estimates, this shows $\lim_{\nu \rightarrow \infty} \|f_\nu - f\|_{H^1} = 0$, as required. ■

Proof of Theorem 5.1. By Proposition 3.5, it is enough to show the assertion for $\psi_1, \psi_2 \in \text{Lip}(M, S \otimes E)$, which we assume from now on.

Step 1: Let (g_ν) be a sequence of smooth metrics on M with

$$\lim_{\nu \rightarrow \infty} \|g_\nu - g\|_{W^{1,p}} = 0.$$

We will show that if Theorem 5.1 holds for all g_ν , then it also holds for g .

Note that

$$\begin{aligned} (D_{g_\nu} \psi_1, D_{g_\nu} \psi_2)_{g_\nu} &\rightarrow (D_g \psi_1, D_g \psi_2)_g, \\ (\nabla_{g_\nu} \psi_1, \nabla_{g_\nu} \psi_2)_{g_\nu} &\rightarrow (\nabla_g \psi_1, \nabla_g \psi_2)_g, \\ \langle\langle \text{scal}_{g_\nu}, \langle \psi_1, \psi_2 \rangle \rangle\rangle &\rightarrow \langle\langle \text{scal}_g, \langle \psi_1, \psi_2 \rangle \rangle\rangle. \end{aligned}$$

Hence it remains to show that

$$(\mathcal{R}_{g_\nu}^E \psi_1, \psi_2)_{g_\nu} \rightarrow (\mathcal{R}_g^E \psi_1, \psi_2)_g.$$

Let (e_1, \dots, e_n) be a smooth γ -orthonormal frame of TM over some open subset $U \subset M$. Without loss of generality, we may assume that ψ_1 is supported in a compact subset $K \subset U$. By the defining equation (5.1) there is a constant C , not depending on ν , such that on U we have

$$|\mathcal{R}_{g_\nu}^E \psi_1 - \mathcal{R}_g^E \psi_1| \leq C \max_{i,j} |R_{e_i^{g_\nu}, e_j^{g_\nu}}^E - R_{e_i^g, e_j^g}^E| |\psi_1| \quad \text{a.e.}$$

As $p > n$, and hence $\lim_{\nu \rightarrow \infty} \|g_\nu - g\|_{L^\infty(K)} = 0$ and $\lim_{\nu \rightarrow \infty} \|e_i^{g_\nu} - e_i^g\|_{L^\infty(K)} = 0$ for $1 \leq i \leq n$ by Lemma 4.1, we obtain

$$\left\| \frac{d\mu_{g_\nu}}{d\mu_\gamma} - \frac{d\mu_g}{d\mu_\gamma} \right\|_{L^\infty(K)} \rightarrow 0, \quad \|R_{e_i^{g_\nu}, e_j^{g_\nu}}^E - R_{e_i^g, e_j^g}^E\|_{L^\infty(K)} \rightarrow 0, \quad 1 \leq i, j \leq n.$$

We conclude that

$$\begin{aligned} &|(\mathcal{R}_{g_\nu}^E \psi_1, \psi_2)_{g_\nu} - (\mathcal{R}_g^E \psi_1, \psi_2)_g| \\ &= \left| \int_M \langle \mathcal{R}_{g_\nu}^E \psi_1, \psi_2 \rangle \frac{d\mu_{g_\nu}}{d\mu_\gamma} d\mu_\gamma - \int_M \langle \mathcal{R}_g^E \psi_1, \psi_2 \rangle \frac{d\mu_g}{d\mu_\gamma} d\mu_\gamma \right| \\ &\leq \left(C_n \max_{i,j} \|R_{e_i^{g_\nu}, e_j^{g_\nu}}^E - R_{e_i^g, e_j^g}^E\|_{L^\infty(K)} \left\| \frac{d\mu_{g_\nu}}{d\mu_\gamma} \right\|_{L^\infty(K)} \right. \\ &\quad \left. + \|\mathcal{R}_g^E\|_{L^\infty(K)} \left\| \frac{d\mu_{g_\nu}}{d\mu_\gamma} - \frac{d\mu_g}{d\mu_\gamma} \right\|_{L^\infty(K)} \right) \int_M |\psi_1| |\psi_2| d\mu_\gamma. \end{aligned}$$

For $\nu \rightarrow \infty$ this tends to 0, concluding Step 1.

Step 2: We prove Theorem 5.1 for smooth g which together with Step 1 completes the proof of Theorem 5.1. Choose open subsets $U \subset M$ and $V \subset N$ with $f(U) \subset V$ and

\triangleright a smooth γ -orthonormal frame (e_1, \dots, e_n) of $TM|_U$,

- ▷ local coordinates (x^1, \dots, x^n) on U and (y^1, \dots, y^ℓ) on V ,
- ▷ a unitary trivialization $E_0|_V \cong V \times \mathbb{C}^r$.

Since both sides of (5.1) are sesquilinear forms, it suffices to show Theorem 5.1 for sections $\psi_1, \psi_2 \in \text{Lip}(M, S \otimes E)$ which are supported in a compact subset $K \subset U$. With the given trivialization of $E_0|_V$ and a continuity argument we can in fact assume that $\psi_1, \psi_2 \in C^\infty(U, S|_U \otimes \mathbb{C}^r)$, supported in K .

Let $\omega_0 \in \Omega^1(V, M_r(\mathbb{C}))$ be the smooth connection 1-form of ∇^{E_0} with respect to the local trivialization $E_0|_V \cong V \times \mathbb{C}^r$ and define

$$\omega := f^* \omega_0 \in L^\infty \Omega^1(U, M_r(\mathbb{C})).$$

Let

$$\hat{D}: \text{Lip}(U, S|_U \otimes \mathbb{C}^r) \rightarrow L^2(U, S|_U \otimes \mathbb{C}^r)$$

be the spin Dirac operator on U twisted with the trivial bundle $E|_U \cong U \times \mathbb{C}^r$ endowed with the metric Lipschitz connection $d + \omega$.

Let $\Omega_0 = d\omega_0 + \omega_0 \wedge \omega_0 \in \Omega^2(V, M_r(\mathbb{C}))$ be the smooth curvature form of the connection $d + \omega_0$ and set

$$\Omega := f^* \Omega_0 \in L^\infty \Omega^2(U, M_r(\mathbb{C})).$$

Let $\mathcal{R}: L^\infty(U, S|_U \otimes \mathbb{C}^r) \rightarrow L^\infty(U, S|_U \otimes \mathbb{C}^r)$ be defined by

$$\mathcal{R}(\sigma \otimes \eta) := \frac{1}{2} \sum_{i,j=1}^n (e_i \cdot e_j \cdot \sigma) \otimes (\Omega_{e_i^g, e_j^g} \eta)$$

on simple tensors $\sigma \otimes \eta \in S|_U \otimes \mathbb{C}^r$. Let $\hat{\nabla}$ be the tensor product connection $\nabla^S|_U \otimes 1 + 1 \otimes (d + \omega)$ on $S|_U \times \mathbb{C}^r$. It remains to show that

$$(\hat{D}\psi_1, \hat{D}\psi_2)_g = (\hat{\nabla}\psi_1, \hat{\nabla}\psi_2)_g + \langle\langle \text{scal}_g, \langle \psi_1, \psi_2 \rangle \rangle\rangle + (\mathcal{R}\psi_1, \psi_2)_g. \tag{5.3}$$

By Lemma 5.2 there is a sequence of smooth functions $f_\nu: U \rightarrow V$ such that, with respect to the chosen local coordinates on U and V ,

$$\lim_{\nu \rightarrow \infty} (\|f_\nu - f\|_{L^\infty(K)} + \|f_\nu - f\|_{H^1(K)}) = 0, \quad \max_\nu \|df_\nu\|_{L^\infty(K)} < \infty. \tag{5.4}$$

Set $\omega_\nu := f_\nu^* \omega_0 \in \Omega^1(U, M_r(\mathbb{C}))$ and let

$$\hat{D}_\nu: C_c^\infty(U, S|_U \otimes \mathbb{C}^r) \rightarrow C_c^\infty(U, S|_U \otimes \mathbb{C}^r)$$

be the formally self-adjoint spin Dirac operator on U twisted with the trivial bundle $E|_U \cong U \times \mathbb{C}^r$ endowed with the smooth metric connection $d + \omega_\nu$.

Set $\Omega_\nu := f_\nu^* \Omega_0 \in \Omega^2(U, M_n(\mathbb{C}))$ and let $\mathcal{R}_\nu: C^\infty(U, S|_U \otimes \mathbb{C}^r) \rightarrow C^\infty(U, S|_U \otimes \mathbb{C}^r)$ be defined on simple tensors $\sigma \otimes \eta \in S|_U \otimes \mathbb{C}^r$ by

$$\mathcal{R}_\nu(\sigma \otimes \eta) := \frac{1}{2} \sum_{i,j=1}^n (e_i \cdot e_j \cdot \sigma) \otimes ((\Omega_\nu)_{e_i^g, e_j^g} \eta).$$

Denote by $\hat{\nabla}_\nu$ the smooth tensor product connection $\nabla^S|_U \otimes 1 + 1 \otimes (d + \omega_\nu)$.

Using the classical Schrödinger–Lichnerowicz formula [14, Theorem II.8.17] and Remark 4.4, we find for all ν that

$$(\hat{D}_\nu \psi_1, \hat{D}_\nu \psi_2)_g = (\hat{\nabla}_\nu \psi_1, \hat{\nabla}_\nu \psi_2)_g + \langle\langle \text{scal}_g, \langle \psi_1, \psi_2 \rangle \rangle\rangle + (\mathcal{R}_\nu \psi_1, \psi_2)_g. \tag{5.5}$$

In order to establish (5.3), it hence remains to show that

$$\lim_{\nu \rightarrow \infty} \|\hat{\nabla} \psi_\kappa - \hat{\nabla}_\nu \psi_\kappa\|_{L^2(K)} = 0, \quad \kappa = 1, 2, \tag{5.6}$$

$$\lim_{\nu \rightarrow \infty} \|\mathcal{R} \psi_1 - \mathcal{R}_\nu \psi_1\|_{L^2(K)} = 0. \tag{5.7}$$

Since $\hat{\nabla} - \hat{\nabla}_\nu = 1 \otimes (\omega - \omega_\nu)$ we have

$$\|\hat{\nabla} \psi_\kappa - \hat{\nabla}_\nu \psi_\kappa\|_{L^2(K)} \leq \|\psi_\kappa\|_{L^\infty(K)} \|\omega - \omega_\nu\|_{L^2(K)}.$$

Write $\omega_0 = \sum_{\alpha=1}^\ell dy^\alpha \otimes \Gamma_\alpha$ with $\Gamma_\alpha \in C^\infty(V, M_r(\mathbb{C}))$. With $f = (f^1, \dots, f^\ell)$ and $f_\nu = (f_\nu^1, \dots, f_\nu^\ell)$ we obtain

$$\omega = \sum_{\alpha=1}^\ell \sum_{i=1}^n (\partial_i f^\alpha) dx^i \otimes (\Gamma_\alpha \circ f), \quad \omega_\nu = \sum_{\alpha=1}^\ell \sum_{i=1}^n (\partial_i f_\nu^\alpha) dx^i \otimes (\Gamma_\alpha \circ f_\nu).$$

Setting $\Gamma = (\Gamma_1, \dots, \Gamma_\ell)$ we hence obtain, a.e. over K ,

$$|\omega - \omega_\nu| \leq |df - df_\nu| \cdot \|\Gamma \circ f\|_{L^\infty(K)} + |df_\nu| \cdot \|\Gamma \circ f - \Gamma \circ f_\nu\|_{L^\infty(K)}.$$

By the triangle inequality for the L^2 -norm this implies

$$\begin{aligned} \|\omega - \omega_\nu\|_{L^2(K)} &\leq \|\Gamma \circ f\|_{L^\infty(K)} \|df - df_\nu\|_{L^2(K)} \\ &\quad + \|\Gamma \circ f - \Gamma \circ f_\nu\|_{L^\infty(K)} \|df_\nu\|_{L^2(K)}. \end{aligned}$$

With (5.4) the last expression tends to 0 as $\nu \rightarrow \infty$, proving (5.6).

For (5.7), we write

$$\Omega_0 = \sum_{1 \leq \alpha < \beta \leq \ell} (dy^\alpha \wedge dy^\beta) \otimes B_{\alpha,\beta}, \quad B_{\alpha,\beta} \in C^\infty(V, M_r(\mathbb{C})),$$

and obtain

$$\begin{aligned} \Omega &= \sum_{\alpha < \beta} \sum_{i < j} [(\partial_i f^\alpha)(\partial_j f^\beta) - (\partial_j f^\alpha)(\partial_i f^\beta)] dx^i \wedge dx^j \otimes (B_{\alpha,\beta} \circ f), \\ \Omega_\nu &= \sum_{\alpha < \beta} \sum_{i < j} [(\partial_i f_\nu^\alpha)(\partial_j f_\nu^\beta) - (\partial_j f_\nu^\alpha)(\partial_i f_\nu^\beta)] dx^i \wedge dx^j \otimes (B_{\alpha,\beta} \circ f_\nu). \end{aligned}$$

Therefore we get

$$(\mathcal{R} - \mathcal{R}_\nu)(\sigma \otimes \eta) = \frac{1}{2} \sum_{i,j=1}^n (e_i \cdot e_j \cdot \sigma) \otimes (\Delta_{e_i^g, e_j^g}^{(\nu)} \eta + \Lambda_{e_i^g, e_j^g}^{(\nu)} \eta),$$

where

$$\begin{aligned} \Delta^{(\nu)} &:= \sum_{\alpha < \beta} \sum_{i < j} [(\partial_i f_\nu^\alpha)(\partial_j f_\nu^\beta) - (\partial_j f_\nu^\alpha)(\partial_i f_\nu^\beta)] dx^i \wedge dx^j \\ &\quad \otimes [(B_{\alpha,\beta} \circ f) - (B_{\alpha,\beta} \circ f_\nu)], \\ \Lambda^{(\nu)} &:= \sum_{\alpha < \beta} \sum_{i < j} Z_{i,j}^{\alpha,\beta} dx^i \wedge dx^j \otimes (B_{\alpha,\beta} \circ f), \\ Z_{i,j}^{\alpha,\beta} &:= (\partial_i f^\alpha \partial_j f^\beta - \partial_j f^\alpha \partial_i f^\beta) - (\partial_i f_\nu^\alpha \partial_j f_\nu^\beta - \partial_j f_\nu^\alpha \partial_i f_\nu^\beta). \end{aligned}$$

It remains to show that

$$\lim_{\nu \rightarrow \infty} \|\Delta^{(\nu)}\|_{L^2(K)} = 0, \quad \lim_{\nu \rightarrow \infty} \|\Lambda^{(\nu)}\|_{L^2(K)} = 0. \tag{5.8}$$

Since $|(\partial_i f_\nu^\alpha)(\partial_j f_\nu^\beta) - (\partial_j f_\nu^\alpha)(\partial_i f_\nu^\beta)| \leq 2|df_\nu|^2$ a.e., there exists a constant C , not depending on ν , such that, with $B := (B_{\alpha,\beta})_{1 \leq \alpha,\beta \leq \ell}$, we obtain

$$\|\Delta^{(\nu)}\|_{L^2(K)} \leq C \|df_\nu\|_{L^\infty(K)}^2 \|(B \circ f) - (B \circ f_\nu)\|_{L^\infty(K)}.$$

From (5.4) it follows that the last expression tends to 0 as $\nu \rightarrow \infty$, proving the first part of (5.8).

For the second part in (5.8) we write

$$\begin{aligned} Z_{i,j}^{\alpha,\beta} &= \partial_i f^\alpha (\partial_j f^\beta - \partial_j f_\nu^\beta) - \partial_j f^\alpha (\partial_i f^\beta - \partial_i f_\nu^\beta) \\ &\quad + (\partial_i f^\alpha - \partial_i f_\nu^\alpha) \partial_j f_\nu^\beta - (\partial_j f^\alpha - \partial_j f_\nu^\alpha) \partial_i f_\nu^\beta, \end{aligned}$$

from which

$$|Z_{i,j}^{\alpha,\beta}| \leq 2(|df| + |df_\nu|) \cdot |df - df_\nu| \quad \text{a.e.}$$

We deduce that there exists a constant C , not depending on ν , such that

$$\|\Lambda^{(\nu)}\|_{L^2(K)} \leq C (\|df\|_{L^\infty(K)} + \|df_\nu\|_{L^\infty(K)}) \|df - df_\nu\|_{L^2(K)} \|B \circ f\|_{L^\infty(K)}.$$

Again using (5.4) it follows that the last expression tends to 0 as $\nu \rightarrow \infty$, so that the second equality in (5.8) holds as well.

Hence the proof of Theorem 5.1 is complete. ■

In the remainder of this section we apply the Schrödinger–Lichnerowicz formula to harmonic spinor fields under lower bounds of scal_g and \mathcal{R}^E . From now on let M be connected.

Definition 5.3. For a function $\vartheta \in L^\infty(M)$, we denote by

$$\mathcal{J}_\vartheta: L^1(M) \rightarrow \mathbb{R}, \quad u \mapsto \int_M \vartheta u \, d\mu_g,$$

the associated regular distribution.

▷ We say that $\text{scal}_g \geq \vartheta$ in the distributional sense if for all $u \in C_c^\infty(M)$ with $u \geq 0$,

$$\langle\langle \text{scal}_g, u \rangle\rangle \geq \mathcal{J}_\vartheta(u).$$

▷ We say that $\mathcal{R}_g^E \geq \vartheta$ fiberwise in the distributional sense if for all $\psi \in \text{Lip}(M, S \otimes E)$,

$$(\mathcal{R}_g^E \psi, \psi)_g \geq \mathcal{J}_\vartheta(|\psi|^2).$$

Proposition 5.4. *Suppose there exists $\vartheta \in L^\infty(M)$ with*

$$\frac{1}{4} \text{scal}_g \geq \vartheta, \tag{5.9}$$

$$\mathcal{R}_g^E \geq -\vartheta. \tag{5.10}$$

Then the following assertions hold:

(a) For each $\psi \in \ker D_{g,E}$ the norm $|\psi| \in H^1(M)$ is constant a.e.

(b) If $\ker D_{g,E} \neq 0$, then $\frac{1}{4} \text{scal}_g = \mathcal{J}_\vartheta$.

Proof. Let $\psi \in \ker D_{g,E} \subset H^1(M, S \otimes E)$. From Theorem 5.1 and our assumptions we obtain, using L^2 -norms with respect to the metric g on M ,

$$\begin{aligned} 0 &= \|\nabla_g \psi\|_{L^2}^2 + \frac{1}{4} \langle \text{scal}_g, |\psi|^2 \rangle + (\mathcal{R}_g^E \psi, \psi)_g \\ &\geq \|\nabla_g \psi\|_{L^2}^2 + \mathcal{J}_\vartheta(|\psi|^2) + \mathcal{J}_{-\vartheta}(|\psi|^2) \geq \|\nabla_g \psi\|_{L^2}^2. \end{aligned}$$

This implies $\|\nabla_g \psi\|_{L^2} = 0$ and hence $\nabla_g \psi = 0$ a.e.

Observe that $|\psi|^2 \in W^{1,1}(M)$. Since ∇_g is a metric connection, we furthermore have

$$d|\psi|^2 = 2\langle \nabla_g \psi, \psi \rangle_{S \otimes E} = 0 \quad \text{a.e.}$$

Since M is connected, we deduce that $|\psi|^2$ is constant a.e., finishing the proof of (a).

For (b) it remains to show that for all smooth $u: M \rightarrow [0, 1]$ we get $\frac{1}{4} \langle \text{scal}_g, u \rangle \leq \mathcal{J}_\vartheta(u)$. For a contradiction, assume that there exists $\varepsilon > 0$ and a smooth $u: M \rightarrow [0, 1]$ with

$$\frac{1}{4} \langle \text{scal}_g, u \rangle = \mathcal{J}_\vartheta(u) + \varepsilon.$$

Let $\psi \in \ker D_{g,E}$ be a nonzero H^1 -section. By (a) there exists a constant $C > 0$ with $|\psi|^2 \equiv C = Cu + C(1-u)$ a.e. Both u and $1-u$ are nonnegative and hence

$$\begin{aligned} 0 &= (D_{g,E} \psi, D_{g,E} \psi) \\ &\stackrel{\text{Th. 5.1}}{=} \underbrace{\|\nabla_g \psi\|_{L^2}^2}_{\geq 0} + \underbrace{\langle \mathcal{R}_g^E \psi, \psi \rangle + \mathcal{J}_\vartheta(|\psi|^2)}_{\geq 0 \text{ by (5.10)}} + \frac{1}{4} \langle \text{scal}_g, |\psi|^2 \rangle - \mathcal{J}_\vartheta(|\psi|^2) \\ &\geq C \left(\underbrace{\frac{1}{4} \langle \text{scal}_g, u \rangle - \mathcal{J}_\vartheta(u)}_{=\varepsilon} + \underbrace{\frac{1}{4} \langle \text{scal}_g, 1-u \rangle - \mathcal{J}_\vartheta(1-u)}_{\geq 0 \text{ by (5.9)}} \right) \geq C\varepsilon > 0, \end{aligned}$$

a contradiction. ■

6. Proofs of Theorem A and B

Proof of Theorem A. Pick a smooth Riemannian metric γ on M and let $S \rightarrow M$ be the smooth spinor bundle associated to γ , some spin structure on (M, γ) and the irreducible

complex Cl_n -module W as in Section 4. Furthermore, let $\Sigma \rightarrow \mathbb{S}^n$ be the smooth spinor bundle associated to the standard round metric g_0 on \mathbb{S}^n , the unique spin structure on (\mathbb{S}^n, g_0) and the Cl_n -module W . Both spinor bundles S and Σ are equipped with the respective metric spinor connections. Since n is even we have the $\mathbb{Z}/2$ -grading $\Sigma = \Sigma^+ \oplus \Sigma^-$. By pulling back (Σ, ∇^Σ) along f we obtain the Hermitian Lipschitz bundle $E \rightarrow M$ with induced metric connection ∇^E . Let

$$D_{g,E}: H^1(M, S \otimes E) \rightarrow L^2(M, S \otimes E)$$

be the self-adjoint spin Dirac operator on M for the twist bundle (E, ∇^E) . Since $E = E^+ \oplus E^-$ is also $\mathbb{Z}/2$ -graded, we obtain an induced operator

$$D_{g,E}^+: H^1(M, (S^+ \otimes E^+) \oplus (S^- \otimes E^-)) \rightarrow L^2(M, (S^- \otimes E^+) \oplus (S^+ \otimes E^-)).$$

According to Theorem 4.8, the index is equal to

$$\text{index}(D_{g,E}^+) = \langle \hat{A}(M) \cup f^*(\text{ch}(\Sigma^+) - \text{ch}(\Sigma^-)), [M] \rangle. \tag{6.1}$$

The Chern character difference is computed in [14, Proposition III.11.24]. This calculation shows

$$\text{ch}(\Sigma^+) - \text{ch}(\Sigma^-) = (-1)^{n/2} e(T\mathbb{S}^n),$$

where $e(T\mathbb{S}^n) \in H^n(\mathbb{S}^n; \mathbb{Q})$ is the Euler class of the tangent bundle $T\mathbb{S}^n \rightarrow \mathbb{S}^n$. Plugging this result into (6.1) and using the fact that the $H^0(M; \mathbb{Q})$ -component of $\hat{A}(M)$ is equal to 1, we obtain

$$\begin{aligned} (-1)^{n/2} \cdot \text{index}(D_{g,E}^+) &= \langle \hat{A}(M) \cup f^*(e(T\mathbb{S}^n)), [M] \rangle \\ &= \langle f^*(e(T\mathbb{S}^n)), [M] \rangle \\ &= \langle e(T\mathbb{S}^n), f_*([M]) \rangle \\ &= \text{deg}(f) \cdot \underbrace{\langle e(T\mathbb{S}^n), [\mathbb{S}^n] \rangle}_{=2}. \end{aligned}$$

Since $\text{deg}(f) \neq 0$ by assumption, the last expression is nonzero, and using the fact that $D_{g,E}$ is self-adjoint, we conclude that there exists $0 \neq \psi \in \ker D_{g,E}$.

We now analyze the term \mathcal{R}_g^E in the Schrödinger–Lichnerowicz formula of Theorem 5.1. For the next proposition, recall the shorthand $v^g = b_g(v)$ from Notation 4.2.

Proposition 6.1. *Let $x \in M$ be a point where f is differentiable. Assume that either $|d_x f| \leq 1$, or $n \geq 4$ and $|\Lambda^2 d_x f| \leq 1$. Then for all $\omega \in (S \otimes E)_x$ we have*

$$\langle \mathcal{R}_g^E \omega, \omega \rangle \geq -\frac{1}{4} n(n-1) |\omega|^2.$$

Furthermore, for $\omega \neq 0$ equality holds if and only if $d_x f: (T_x M, g_x) \rightarrow T_{f(x)} \mathbb{S}^n$ is an isometry and

$$(v \cdot w \otimes d_x f(v^g)) \cdot d_x f(w^g) \cdot \omega = \omega$$

for all γ -orthonormal vectors $v, w \in T_x M$.

Proof. By a singular value decomposition of $d_x f \circ b_g$, there exists a γ -orthonormal basis (e_1, \dots, e_n) of $T_x M$, a g_0 -orthonormal basis $(\varepsilon_1, \dots, \varepsilon_n)$ of $T_{f(x)}\mathbb{S}^n$ and real numbers $\mu_i \geq 0, 1 \leq i \leq n$, with $d_x f(e_i^g) = \mu_i \cdot \varepsilon_i$.

Since the curvature operator of (\mathbb{S}^n, g_0) acts as the identity on 2-forms, [14, Chapter II, (4.37)] (see also [18, Lemma 4.3]) implies

$$R_g^E \omega = \frac{1}{2} \sum_{i,j=1}^n \mu_i \mu_j (e_i \cdot e_j \otimes R_{\varepsilon_i, \varepsilon_j}^\Sigma) \omega = \frac{1}{4} \sum_{i \neq j} \mu_i \mu_j (e_i \cdot e_j \otimes \varepsilon_j \cdot \varepsilon_i) \cdot \omega.$$

Each Clifford multiplication operator $e_i \cdot e_j \otimes \varepsilon_j \cdot \varepsilon_i: (S \otimes E)_x \rightarrow (S \otimes E)_x$ is a self-adjoint involution, hence of norm 1, so that

$$\langle (e_i \cdot e_j \otimes \varepsilon_j \cdot \varepsilon_i) \cdot \omega, \omega \rangle \geq -|\omega|^2.$$

We conclude

$$\begin{aligned} \langle R_g^E \omega, \omega \rangle &= \frac{1}{4} \sum_{i \neq j} \mu_i \mu_j \langle (e_i \cdot e_j \otimes \varepsilon_j \cdot \varepsilon_i) \cdot \omega, \omega \rangle \\ &\geq -\frac{1}{4} \sum_{i \neq j} \mu_i \mu_j |\omega|^2 \geq -\frac{1}{4} n(n-1) |\omega|^2. \end{aligned}$$

Furthermore, for $\omega \neq 0$ equality holds if and only if

- (a) $\mu_i = 1$ for all $1 \leq i \leq n$, that is, $d_x f \circ b_g: (T_x M, \gamma_x) \rightarrow T_{f(x)}\mathbb{S}^n$ is an isometry,
- (b) $(e_i \cdot e_j \otimes \varepsilon_i \cdot \varepsilon_j) \cdot \omega = \omega$ for $1 \leq i \neq j \leq n$.

For (a) we use the fact that either $|d_x f| \leq 1$ (hence $\mu_i \leq 1$ for $1 \leq i \leq n$), or $n \geq 4$ and $|\Lambda^2 d_x f| \leq 1$ (hence $\mu_i \mu_j \leq 1$ for all $1 \leq i \neq j \leq n$).

Since $b_g: (T_x M, \gamma_x) \rightarrow (T_x M, g_x)$ is an isometry, assertions (a) and (b) are equivalent to the conditions stated in Proposition 6.1. ■

Since f is differentiable almost everywhere, Proposition 6.1 implies

$$\mathcal{R}^E \geq -\frac{1}{4} n(n-1)$$

fiberwise in the distributional sense. Together with our assumption $\text{scal}_g \geq n(n-1)$ in the distributional sense, Proposition 5.4 with $\vartheta := \frac{1}{4} n(n-1)$ implies that there exists $C > 0$ with $|\psi| = C$ a.e. and $\text{scal}_g = \mathcal{J}_{n(n-1)}$.

From Theorem 5.1 we hence obtain

$$\begin{aligned} 0 &= \|\mathbb{D}_{g,E} \psi\|_{L^2}^2 \\ &\geq \frac{1}{4} \langle \langle \text{scal}_g, |\psi|^2 \rangle \rangle + (\mathcal{R}^E \psi, \psi)_g = \frac{C^2}{4} n(n-1) \text{vol}(M, g) + (\mathcal{R}^E \psi, \psi)_g. \end{aligned}$$

Using the equality statement of Proposition 6.1, this implies that at almost all points x where f is differentiable, the map $d_x f: (T_x M, g_x) \rightarrow T_{f(x)}\mathbb{S}^n$ is an isometry and

$$(v \cdot w \otimes d_x f(v^g) \cdot d_x f(w^g)) \cdot \psi(x) = \psi(x) \quad \text{for all } \gamma\text{-orthonormal } v, w \in T_x M. \tag{6.2}$$

- Let $M_{\text{reg}} \subset M$ be the subset of full measure of all $x \in M$ where f is differentiable and
- ▷ $d_x f: (T_x M, g_x) \rightarrow T_{f(x)} \mathbb{S}^n$ is an isometry,
 - ▷ $(v \cdot w \otimes d_x f(v^g) \cdot d_x f(w^g)) \cdot \psi(x) = \psi(x)$ for all γ -orthonormal $v, w \in T_x M$,
 - ▷ $|\psi(x)| = C > 0$, where C is independent of x .

Proposition 6.2. *The differential $d_x f$ is either orientation preserving at almost all $x \in M_{\text{reg}}$ or orientation reversing at almost all $x \in M_{\text{reg}}$.*

Together with the previous discussion, this implies, possibly after reversing the orientation of M , that df is an orientation preserving isometry a.e. on M and the proof of Theorem A is completed by Theorem 2.4.

To prove Proposition 6.2, let

$$M_{\pm} := \{x \in M_{\text{reg}} \mid \det(d_x f) = \pm 1\} \subset M_{\text{reg}}$$

be the subset of M_{reg} where df is orientation preserving or orientation reversing, respectively.

Consider the Clifford algebra bundles $\text{Cl}(M) \rightarrow M$ for (M, γ) and $\text{Cl}(\mathbb{S}^n) \rightarrow \mathbb{S}^n$ for (\mathbb{S}^n, g_0) . The *oriented volume elements*

$$\text{vol}_{\gamma} = e_1 \cdot \dots \cdot e_n, \quad \text{vol}_{g_0} = \varepsilon_1 \cdot \dots \cdot \varepsilon_n,$$

where (e_1, \dots, e_n) is a local smooth positively oriented γ -orthonormal frame of TM and $(\varepsilon_1, \dots, \varepsilon_n)$ is a local smooth positively oriented g_0 -orthonormal frame of $T\mathbb{S}^n$, define global smooth sections of $\text{Cl}(M)$ and $\text{Cl}(\mathbb{S}^n)$. Hence we obtain a self-adjoint Lipschitz involution of the Lipschitz bundle $S \otimes E \rightarrow M$ which over $x \in M$ acts by left Clifford multiplication with $\text{vol}_{\gamma}(x) \otimes \text{vol}_{g_0}(f(x))$. Let $S \otimes E = W^+ \oplus W^-$ be the resulting orthogonal splitting into ± 1 -eigenbundles and let

$$\pi_+ \in \text{Lip}(M, \text{End}(S \otimes E))$$

be the orthogonal projection onto W^+ .

Now take $x \in M_{\text{reg}}$ and a positively oriented γ -orthonormal basis (e_1, \dots, e_n) of $T_x M$. Then $(d_x f(e_1^g), \dots, d_x f(e_n^g))$ is a g_0 -orthonormal basis of $T_{f(x)} \mathbb{S}^n$, which is positively oriented if and only if $d_x f \circ b_g: T_x M \rightarrow T_{f(x)} \mathbb{S}^n$ is orientation preserving. The last condition is equivalent to $x \in M_+$ as the γ -self-adjoint map $b_g: T_x M \rightarrow T_x M$ (defined in Lemma 4.1) is positive.

Applying (6.2) iteratively for $(v, w) = (e_1, e_2), (e_3, e_4), \dots, (e_{n-1}, e_n)$, using the fact that n is even, we get

$$((e_1 \cdot \dots \cdot e_n) \otimes (d_x f(e_1^g) \cdot \dots \cdot d_x f(e_n^g))) \cdot \psi(x) = \psi(x),$$

and together with the previous discussion, this shows

$$(\text{vol}_{\gamma}(x) \otimes \text{vol}_{g_0}(f(x))) \cdot \psi(x) = \pm \psi(x) \quad \text{if } x \in M_{\pm}.$$

Consequently,

$$\pi_+ \psi(x) = \begin{cases} \psi(x) & \text{for all } x \in M_+, \\ 0 & \text{for all } x \in M_-. \end{cases}$$

As $\psi \in W^{1,2}(M, S \otimes E)$ and π_+ is Lipschitz, we have $|\pi_+ \psi| \in W^{1,2}(M, \mathbb{R})$, and this function is identically 0 on M_- and constant with nonzero value C on M_+ . Because M is connected and the map is of Sobolev regularity $W^{1,2}$, either M_- or M_+ has measure zero, which finishes the proof of Proposition 6.2. ■

Proof of Theorem B. Choose a smooth collar neighborhood of $\partial M \subset M$ and let (\hat{M}, \hat{g}) be the smooth double of (M, g) with reflected metric. The metric \hat{g} on \hat{M} is Lipschitz and hence admissible. Since the boundary has positive mean curvature, [15, Proposition 5.1] implies that $\text{scal}_{\hat{g}} \geq n(n-1)$ in the distributional sense.

We define $\hat{f}: \hat{M} \rightarrow \mathbb{S}^n$ to be f on the first copy of M in \hat{M} , and $\rho \circ f$ on the second copy of M , where ρ was defined in Example 2.6. Since ρ is 1-Lipschitz and $f(\partial M) \subset \mathbb{D}_-^n$ by assumption, the map $\hat{f}: \hat{M} \rightarrow \mathbb{S}^n$ is Lipschitz. Furthermore, if $n \geq 4$, then $d\hat{f}$ is area-nonincreasing a.e., and if $n = 2$, then \hat{f} is 1-Lipschitz.

The map \hat{f} sends the second copy of M in \hat{M} to the lower hemisphere \mathbb{D}_- , hence

$$\deg \hat{f} = \deg(f: (M, \partial M) \rightarrow (\mathbb{S}^n, \mathbb{D}_-^n)) \neq 0.$$

Theorem A now implies that \hat{f} is a metric isometry. We conclude, using smoothness of g , that $\text{im}(f) = \mathbb{D}_+^n$ and that $f: (M, g) \rightarrow \mathbb{D}_+^n$ is a smooth Riemannian isometry. ■

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