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Groups with finitely many Busemann points

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Abstract. We show that an infinite finitely generated group G is virtually $\mathbb{Z}$ if and only if every Cayley graph of G contains only finitely many Busemann points in its horofunction boundary. This complements a previous result of the second named author and M. Tointon.

1. Introduction

In a few important works it has been suggested to consider horofunctions as an analog to linear functionals, in (non-linear) general metric spaces (see, e.g., [3, 5, 6, 14] and references therein). The space of horofunctions (which will be precisely defined below) has been fruitfully used in the study of the geometry of metric spaces. Specifically in the case of hyperbolic spaces and hyperbolic groups, these have been used quite successfully. Less research has been devoted to the study of horofunction boundaries in the case of "small" groups. This is perhaps due to the fact that the space of horofunctions itself is not invariant when switching between different Cayley graphs of the same group. However, it seems to be true that some properties are shared by all Cayley graphs of a given group, and it is very interesting to determine such invariants.

Horofunctions are very well suited to finding *virtual characters*, that is, functions whose restriction to a finite index subgroup is a non-trivial homomorphism into an Abelian group. In fact, as will be explained below, if a horofunction has a finite orbit under the canonical group action, it is such a virtual character (with the implicit finite index subgroup being the stabilizer of the action). Finding such virtual characters is useful in many situations, because it provides a way of splitting the group into two parts. Specifically, if $\psi : G \to \mathbb{Z}$ is a surjective homomorphism, then $G \cong \mathbb{Z} \ltimes K$ for $K = \mathsf{Ker}\,\psi$. Even though such a decomposition is a semi-direct product, in many cases it is still useful in the geometric or algebraic analysis. For example, the main part in the proof of Gromov's theorem regarding groups of polynomial growth [2] is to show that every group of polynomial growth admits a virtual character. Most of the new proofs of this theorem are actually proofs of this result, including Kleiner's proof [8] and Ozawa's proof [9]. Even before these new proofs appeared, it was suggested by Karlsson [4] to use the horofunction boundary to provide the required virtual character in the polynomial growth case.

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Conjecture 1.1. *Let G be a finitely generated group of polynomial growth. Then, for any Cayley graph of G the horofunction boundary contains a finite orbit (for the canonical group action).*

See also [11, Conjecture 1.3].

Grigorchuk has conjectured that there is a "gap" in the possible growth functions of Cayley graphs (see [1]). Specifically, if the growth is small enough it is conjectured that the group actually has polynomial growth. Since horofunctions, and specifically Busemann points (see below), are related to geodesics in the graph, these may very well be good tools to study the growth. By Gromov's theorem mentioned, every polynomial growth group is virtually nilpotent and thus admits a virtual character. Walsh [12] has shown that the action of nilpotent groups on their horofunction boundary has a finite orbit. So we have the following (logically weaker) conjecture.

Conjecture 1.2. *Let Γ be a Cayley graph. Assume that the size of the ball of radius r in Γ is bounded by $C\exp(Cr^{\alpha})$ for some $C > 0$, $\alpha < \frac{1}{2}$, and all $r \in \mathbb{N}$.*

Then, the horofunction boundary of Γ contains a finite orbit for the underlying group action.

Let us remark that even smaller growth functions in Conjecture 1.2 are of interest. The state of the art follows from the work of Shalom and Tao [10] proving that any group of growth bounded by $C\exp(C(\log r)(\log\log r)^{\varepsilon})$ for some $\varepsilon > 0$ is virtually nilpotent (and so actually of polynomial growth). In a forthcoming work by Szusterman and the second named author, it is shown that if every Cayley graph of small enough growth has a finite orbit in its horofunction boundary, then the underlying group is actually virtually nilpotent.

Part of the difficulty of understanding the horofunction boundary of a Cayley graph, and especially connecting it to the underlying algebraic structure of the group, is that the boundary changes when changing the specific choice of Cayley graph. The properties of the boundary which are invariant to changing Cayley graphs are still somewhat mysterious. As a first step in considering such invariants, in [11] it was shown that any graph that has linear volume growth must have finitely many Busemann points in its boundary (see below for a precise definition of Busemann points). The other direction is not true for general graphs; we provide an example of a graph with arbitrary growth and a boundary consisting of just one point (see Example 4.2).

Our main result, Theorem 1.3, is a first step in understanding the relationship between algebraic properties of a group and the horofunction boundaries of its Cayley graphs. Theorem 1.3 states that an infinite finitely generated group is virtually $\mathbb{Z}$ if and only if for every Cayley graph there are finitely many Busemann points in its horofunction boundary. It is also shown that these conditions are equivalent to every Cayley graph having a finite horofunction boundary, and that when these equivalent conditions hold, every horofunction is a Busemann point. See below for a precise statement and definitions.

1.1. Notation and precise statement of results

Let G be a finitely generated group. Fix a finite symmetric generating set S for G; that is, S generates G as a group, $|S| < \infty$, and $S = S^{-1} = \{x^{-1} : x \in S\}$. The *Cayley graph* of G with respect to S, denoted $\Gamma(G,S)$, is the graph whose vertices are elements of G and edges are given by the relation: $\{x, xs\}$ is an edge for any $x \in G$ and $s \in S$. The graph metric of $\Gamma(G,S)$ is denoted by d_S. It is easy to check that d_S is left invariant in the sense that $d_S(zx, zy) = d_S(x,y)$ for all $x,y,z \in G$. Thus it is useful to use the notation $|x|_S = d_S(x,1)$, where $1 = 1_G$ denote the identity element in the group G. When the metric is clear from the context, we will omit the subscript $d = d_S$ and $|x| = |x|_S$.

Fix some Cayley graph $\Gamma(G,S)$. Let $L(G,S)$ be the set of all functions $h : G \to \mathbb{R}$ such that h is 1-Lipschitz (i.e., $|h(x) - h(y)| \le d(x,y)$ for all $x,y \in G$) and $h(1) = 0$. Equip $L(G,S)$ with the topology of pointwise convergence, and note that $L(G,S)$ is compact. The set G embeds into $L(G,S)$ by identifying $x \in G$ with the *Busemann function* $b_x : G \to \mathbb{R}$ given by $b_x(y) = d(x,y) - d(x,1)$. We denote closure of $\{b_x : x \in G\}$ in $L(G,S)$ by $\overline{\Gamma(G,S)}$ and define the *horofunction boundary*, or *horoboundary*, of $\Gamma(G,S)$ to be

$$\partial\Gamma(G,S) := \overline{\Gamma(G,S)} \setminus \{b_x : x \in G\}.$$

Elements of $\partial\Gamma(G,S)$ are called *horofunctions*. (The term *metric-functional* is also used, with good reason, see [7].) One notes that all horofunctions are integer valued (because the Busemann functions are, and we are dealing with pointwise convergence).

A *finite geodesic* in $\Gamma(G,S)$ is a finite sequence $(\gamma_k)_{k=0}^{n}$ such that

$$\forall\, 0 \le m \le k \le n \quad d(\gamma_k, \gamma_m) = k - m. \tag{1.1}$$

An *infinite geodesic* in $\Gamma(G,S)$ is an infinite sequence $(\gamma_k)_{k=0}^{\infty}$ such that (1.1) holds for all $0 \le m \le k$. For simplicity, when we write *geodesic* we refer to an infinite geodesic, and if we require finite geodesics, we will explicitly state *finite*.

It is shown in Lemma 3.2 that if $(\gamma_n)_n$ is an infinite geodesic, then $\gamma_\infty := \lim_{n\to\infty} b_{\gamma_n}$ exists (as a limit in $\overline{\Gamma(G,S)}$), and that $\gamma_\infty \in \partial\Gamma(G,S)$. Such a horofunction $\gamma_\infty \in \partial\Gamma(G,S)$ that is a limit of a geodesic is called a *Busemann point*. We denote by $\partial_b\Gamma(G,S)$ the closure (in $\overline{\Gamma(G,S)}$) of the set of all Busemann points. For a more in-depth discussion of Busemann points on Cayley graphs, see [13] and references therein.

The fact that any Cayley graph contains an infinite geodesic is classical, and can be proven using König's lemma. (In fact, that proof shows the existence of a *bi-infinite* geodesic, i.e., a sequence $(\gamma_t)_{t\in\mathbb{Z}}$ such that $\gamma_0 = 1$ and such that any finite subpath is a finite geodesic. This implies that there are always at least two Busemann points in the horoboundary. See also Karlsson's proof (by another method) in [6] that the horoboundary (there called metric boundary) of a finitely generated infinite group contains at least two elements.)

The horoboundary construction also provides an equivalence relation on geodesics: two geodesics α, β are equivalent if $\alpha_\infty = \beta_\infty$. Note that if $\alpha \cap \beta := \{x : \exists\, t,k \in \mathbb{N}, \alpha_t = \beta_k = x\}$ is infinite, then $\alpha_\infty = \beta_\infty$. However, the opposite is not guarantied.

One may have equivalent geodesics which have an empty intersection. (It is however true that geodesics α, β are equivalent if and only if there exists some geodesic γ such that $|\gamma \cap \alpha| = |\gamma \cap \beta| = \infty$, see [12].)

Our main result is the following.

Theorem 1.3. *Let G be an infinite finitely generated group. The following are equivalent:*

(1) *G is virtually $\mathbb{Z}$.*

(2) *For every Cayley graph of G, the Busemann boundary is finite.*

(3) *For every Cayley graph of G, the horofunction boundary is finite.*

Moreover, when G is virtually $\mathbb{Z}$ any horofunction is a Busemann point.

Recall that for a group property $\mathcal{P}$, the property of being virtually $\mathcal{P}$ is containing a finite index subgroup with the property $\mathcal{P}$. Thus, virtually $\mathbb{Z}$ means containing a finite index infinite cyclic subgroup.

Proof. (1) $\Rightarrow$ (2) was first shown in [11], and holds for general graphs of linear growth. In Proposition 4.1 we provide a very elegant and short proof by Sam Shepperd (communicated to us by M. Tointon).

Our main two new contributions are (2) $\Rightarrow$ (1) and the assertion that if G is virtually $\mathbb{Z}$, then every horofunction is a Busemann point.

The implication (2) $\Rightarrow$ (1) is shown in Lemma 3.1.

Theorem 4.3 shows that when G is virtually $\mathbb{Z}$ then in any Cayley graph, every horofunction is a Busemann point. This gives (1) $\Rightarrow$ (3) by combining the implication (1) $\Rightarrow$ (2) with Theorem 4.3.

(3) $\Rightarrow$ (2) is trivial since the Busemann boundary is contained in the horoboundary. ∎

Somewhat frustratingly, we have not proved the following.

Conjecture 1.4. *Let G be an infinite finitely generated group and assume that there exists some Cayley graph of G for which $\partial_b \Gamma(G, S)$ is finite.*

Then, G is virtually $\mathbb{Z}$.

2. Group action

There is a natural action of G on $\overline{\Gamma(G, S)}$, given by $x.h(y) = h(x^{-1}y) - h(x^{-1})$ for $x \in G$ and $h \in \overline{\Gamma(G, S)}$. The following basic properties are very easy to verify, and we leave the proof to the reader.

Proposition 2.1. *The action of G on $\overline{\Gamma(G, S)}$ has the following properties:*

(1) *The action is continuous, that is, if $h_n \to h$ then $x.h_n \to x.h$.*

(2) *The map $x \mapsto b_x$ is equivariant, that is, $x.b_y = b_{xy}$.*

(3) *The horoboundary* $\partial\Gamma(G,S)$ *and the Busemann boundary* $\partial_b\Gamma(G,S)$ *are invariant subsets.*

Two basic properties of horofunctions we require are the following. The proofs are included for completeness.

Proposition 2.2. *Let* $\Gamma(G,S)$ *be a Cayley graph.*

If $b_{x_n} \to h$ *and* $h \in \partial\Gamma(G,S)$*, then* $|x_n| \to \infty$.

Also, if $b_{x_n} \to h$ *and* $|x_n| \to \infty$*, then for every* $r \geq 0$ *there exists* $x \in G$ *with* $h(x) = -|x| = -r$.

In particular, horofunctions are not bounded from below.

Proof. Assume that $b_{x_n} \to h \in \overline{\Gamma(G,S)}$ and $|x_n| \leq r$ for all n. Then, since the ball of radius r is finite, it must be that there exists some $|x| \leq r$ such that $x_n = x$ for infinitely many n. Thus, $h = b_x \notin \partial\Gamma(G,S)$. This shows that if $b_{x_n} \to h \in \partial\Gamma(G,S)$, then $|x_n| \to \infty$.

Now, assume that $b_{x_n} \to h$ with $|x_n| \to \infty$. Fix some $r \geq 0$, let $z = x_n \in G$ be such that $b_z(y) = h(y)$ for all $|y| \leq r$. Since $|x_n| \to \infty$ we can choose z so that $|z| > r$. Let $\gamma = (1 = \gamma_0, \dots, \gamma_{|z|} = z)$ be a finite geodesic and let $x = \gamma_r$. Note that $|x| = r$ and that

$$h(x) = d(\gamma_{|z|}, \gamma_r) - d(\gamma_{|z|}, \gamma_0) = -r = -|x|. \qquad \blacksquare$$

Proposition 2.3. *Let* $h \in \partial\Gamma(G,S)$ *be a horofunction. Let*

$$K = \mathsf{stab}(h) = \{x \in G : x.h = h\}$$

be the stabilizer of h.

Then, the restriction $h|_K$ *is a homomorphism from* K *into* $\mathbb{Z}$ *(the additive group of integers).*

Proof. This follows immediately since for any $x \in K$ and any $y \in G$ we have

$$h(xy) = x^{-1}.h(y) + h(x) = h(y) + h(x). \qquad \blacksquare$$

We have seen that restricting a horofunction to the stabilizer subgroup (for the canonical group action) results in a homomorphism into $\mathbb{Z}$. One naive approach to attempt to provide a simple proof that a finite horoboundary implies that the group is virtually $\mathbb{Z}$ (similar to Conjecture 1.4) would be as follows.

G acts on $\partial\Gamma(G,S)$. Define

$$K = \{x \in G : \forall\, h \in \partial\Gamma(G,S), x.h = h\}$$
$$F = \{x \in K : \forall\, h \in \partial\Gamma(G,S), h(x) = 0\}.$$

One can show that F is a subgroup. In fact, it is possible to show that if $x_1, \dots, x_n \in F$ then the group they generate, $\langle x_1, \dots, x_n \rangle$, is contained in a ball of finite radius r, which depends only on the generators $x_1, \dots, x_n$.

At the moment we do not know an answer to the following.

Question 2.4. Is the subgroup F finitely generated when the horoboundary is finite? Is F always finitely generated (in any Cayley graph)?

Because of the properties above, if F is finitely generated, then it is finite.

When the horoboundary is finite, we have that $[G : K] < \infty$ and we can also write $\partial\Gamma(G, S) = \{h_1, \dots, h_d\}$. The map $\Psi(x) = (h_1(x), \dots, h_d(x))$ is then a map into $\mathbb{Z}^d$ whose restriction to K is a homomorphism. The kernel of $\Psi|_K$ is exactly F. So if F is finite, then one could show that K is virtually $\mathbb{Z}^n$ for some $n \leq d$, implying that also G is. Since G has a finite horoboundary, one can then prove that actually $n = 1$, along the lines of our proof of Theorem 1.3.

3. Finite Busemann boundaries

In this section we prove the main new part of Theorem 1.3, stated as the following lemma.

Lemma 3.1. *Let G be a finitely generated infinite group. Assume that S is a finite symmetric generating set such that for any $x \in G$ we have that the Busemann boundary $\partial_b\Gamma(G, U)$ is finite, where $U = S \cup \{x, x^{-1}\}$.*

Then, G is virtually $\mathbb{Z}$.

The proof of Lemma 3.1 is at the end of the section, and will use the following lemmas.

Lemma 3.2. *Let $\Gamma(G, S)$ be a Cayley graph. Let $(\gamma_t)_t$ be a geodesic.*

Then, the pointwise limit

$$\gamma_\infty = \lim_{t \to \infty} b_{\gamma_t}$$

exists, and $\gamma_\infty \in \partial\Gamma(G, S)$.

Also, there exists t_0 such that for all $t \geq t_0$ we have that $\gamma_\infty(\gamma_t) = -|\gamma_t|$.

If $\gamma_0 = 1$, then $\gamma_\infty(\gamma_t) = -t$ for all $t \in \mathbb{N}$.

Proof. Let $(\gamma_t)_t$ be a geodesic with $\gamma_0 = 1$. Set $b_t = b_{\gamma_t}$. For any $x \in G$, by the triangle inequality $b_x(y) = d(y, x) - |x| \geq -|y|$. Thus, for fixed $y \in G$ the sequence $(b_t(y))_t$ is bounded below by $-|y|$. Because γ is a geodesic with $\gamma_0 = 1$, we have that $|\gamma_{t+1}| - |\gamma_t| = d(\gamma_{t+1}, \gamma_t)$. So another use of the triangle inequality gives

$$b_{t+1}(y) - b_t(y) = d(\gamma_{t+1}, y) - d(\gamma_t, y) - d(\gamma_{t+1}, \gamma_t) \leq 0.$$

We conclude that $(b_t(y))_t$ is a non-increasing sequence of integers, bounded from below, and hence must converge to some integer which we denote by $\gamma_\infty(y)$.

Also, for every $s \geq t$ we have that

$$b_s(\gamma_t) = d(\gamma_t, \gamma_s) - |\gamma_s| = -|\gamma_t|.$$

Taking $s \to \infty$ we have that $\gamma_\infty(\gamma_t) = -|\gamma_t| = -t$ for all t.

In particular, γ_∞ is unbounded from below and, therefore, cannot be equal to b_x for any $x \in G$, that is, $\gamma_\infty \in \partial\Gamma(G,S)$.

Now, let β be a geodesic, with $\beta_0 = x$. Then, $(\gamma_t = x^{-1}\beta_t)_t$ is a geodesic with $\gamma_0 = 1$, so the limit $\beta_\infty = x.\gamma_\infty$ exists.

Let t_0 be such that for all $t \geq t_0$ we have that $\gamma_\infty(x^{-1}) = d(\gamma_t, x^{-1}) - |\gamma_t|$. Then, for all $s \geq t \geq t_0$ we have that

$$d(\gamma_s,\gamma_t) - (d(\gamma_s,x^{-1}) - d(\gamma_t,x^{-1})) = d(\gamma_t,x^{-1}) - |\gamma_t| - (d(\gamma_s,x^{-1}) - |\gamma_s|) = 0,$$

so that for all $t \geq t_0$ and all large enough s,

$$\begin{aligned}\beta_\infty(\beta_t) &= d(\beta_s,\beta_t) - |\beta_s| \\ &= d(\gamma_s,\gamma_t) - (d(\gamma_s,x^{-1}) - d(\gamma_t,x^{-1})) - |\beta_t| = -|\beta_t|.\end{aligned}$$ ■

Lemma 3.3. *Let $\Gamma(G,S)$ be a Cayley graph. Let $K \leq G$ be a finite index subgroup $[G:K] < \infty$.*

Then, there exists $h \in \partial_b\Gamma(G,S)$ and $1 \neq y \in K$ such that $h(y) = -|y|$.

Proof. Let $R \subset G$ be a finite set of representatives for the cosets of K, that is, $G = \biguplus_{r\in R} rK$, and suppose that $1 \in R$. Let $(\gamma_t)_t$ be some geodesic in $\Gamma(G,S)$ such that $\gamma_0 = 1$ and let $\gamma_\infty \in \partial_b\Gamma(G,S)$. Since R is finite, there exists $r \in R$ such that $\gamma_t \in rK$ for infinitely many t. Denote $\beta_t = r^{-1}\gamma_t$. Note that β_t is a geodesic and $\beta_\infty = r^{-1}.\gamma_\infty$. By Lemma 3.2, there exists t_0 such that for every $t \geq t_0$ we have $\beta_\infty(\beta_t) = -|\beta_t|$. Since by definition $\beta_t \in K$ for infinitely many t, there exists t such that $1 \neq \beta_t \in K$ and $\beta_\infty(\beta_t) = -|\beta_t|$. We are done by taking $h = \beta_\infty$ and $y = \beta_t$. ■

Lemma 3.4. *Let $\Gamma(G,S)$ be a Cayley graph. Let $K \leq G$ be a subgroup.*

Suppose there exists a 1-Lipschitz map $h : G \to \mathbb{R}$ such that $h|_K$ is a group homomorphism, and suppose also that there is $1 \neq x \in K$ such that $h(x) = |x|_S$.

Then $|x^t|_S = t \cdot |x|_S$ for every $t \in \mathbb{N}$. If we denote $U = S \cup \{x, x^{-1}\}$ then for any $g \in G$ we have that $(gx^t)_t$ is a geodesic in $\Gamma(G,U)$.

Moreover, there exists a geodesic γ in $\Gamma(G,S)$ such that $\gamma_{t|x|} = x^t$ for all $t \in \mathbb{N}$.

Proof. By the triangle inequality, $|x^t|_S \leq t|x|_S$. By our assumptions on h, since $x \in K$ we have that

$$t|x|_S = th(x) = h(x^t) \leq |x^t|_S,$$

so $|x^t|_S = t|x|_S = h(x^t)$ for all $t \in \mathbb{N}$.

Next we show that $|x^t|_U = t$ for every $t \in \mathbb{N}$. Write $x^t = u_1 \cdots u_m$, where $u_j \in U$ for all $1 \leq j \leq m = |x^t|_U$. Let $J = \{j \mid u_j \in S\}$, so that if $j \notin J$ then $u_j \in \{x, x^{-1}\}$. Since $x \in U$, $m \leq t$. Also,

$$t \cdot |x|_S = |x^t|_S = |u_1 \cdots u_m|_S \leq \sum_{j=1}^m |u_j|_S \leq |J| + (m - |J|) \cdot |x|_S \leq m \cdot |x|_S,$$

where we have used that $|x|_S \geq 1$ because $x \neq 1$. Thus, $t \leq m$, implying that $|x^t|_U = m = t$.

Now, let $g \in G$. Since $(x^t)_t$ is a geodesic in $\Gamma(G,U)$ and the graph metric is left invariant, $d_U(gx^n, gx^m) = d_U(x^n, x^m) = |n-m|$, so $(gx^t)_t$ is also a geodesic in $\Gamma(G,U)$.

Finally, for the last assertion, we want to construct a geodesic γ in $\Gamma(G,S)$ such that $\gamma_{t|x|} = x^t$ for all $t \in \mathbb{N}$. To this end, set $m = |x|$ and let $(1 = x_0, x_1, \ldots, x_m = x)$ be a finite geodesic in $\Gamma(G,S)$. Define $\gamma_0 = 1$, and for $0 \leq j \leq m$, define $\gamma_{nm+j} = x^n x_j$.

We will show that γ is a geodesic in $\Gamma(G,S)$. Indeed, since γ is a path in the graph $\Gamma(G,S)$, it is immediate that $d(\gamma_t, \gamma_s) \leq s-t$ for all $s \geq t$, so we only need to prove a matching lower bound.

Let $k \geq n$ and $0 \leq i, j \leq m$ be such that $km + i \geq nm + j$. Then,

$$d(\gamma_{nm+j}, \gamma_{km+i}) = d(x^n x_j, x^k x_i) = d(x_j, x^{k-n} x_i).$$

If $k = n$, then $i \geq j$ and

$$d(\gamma_{nm+j}, \gamma_{km+i}) = d(x_j, x_i) = i - j = km + i - (nm + j)$$

because $(x_0, \ldots, x_m)$ is a finite geodesic. If $k > n$, then

$$\begin{aligned}(k-n+1)|x|_S = |x^{k-n+1}|_S &\leq d(x^{k-n+1}, x^{k-n}x_i) + d(x^{k-n}x_i, x_j) + d(x_j, 1)\\ &= d(x_m, x_i) + d(\gamma_{nm+j}, \gamma_{km+i}) + d(x_j, x_0)\\ &= d(\gamma_{nm+j}, \gamma_{km+i}) + |x|_S + j - i,\end{aligned}$$

where we have used again that $(x_0, \ldots, x_m)$ is a finite geodesic. Thus, in all cases we get that

$$d(\gamma_{nm+j}, \gamma_{km+i}) \geq km + i - (nm + j),$$

providing a matching lower bound to the upper bound and proving that γ is indeed a geodesic in $\Gamma(G,S)$. ■

Lemma 3.5. *Let $\Gamma(G,S)$ be a Cayley graph. Let $K \leq G$ be a subgroup.*

Suppose there exists a 1-Lipschtiz map $h : G \to \mathbb{R}$ such that $h|_K$ is a group homomorphism, and suppose also that there is $1 \neq x \in K$ such that $h(x) = |x|_S$. Let $U = S \cup \{x, x^{-1}\}$.

Then,

$$|\mathsf{Ker}\,(h|_K)| \leq |\partial_b \Gamma(G,U)|.$$

Proof. Let $g \in \mathsf{Ker}(h|_K)$. Let $\gamma_t = x^t$ and $\beta_t = gx^t$. Note that both $(\gamma_t)_t$ and $(\beta_t)_t$ are geodesics in $\Gamma(G,U)$ by Lemma 3.4.

Define $\psi(y) = \frac{1}{|x|_S} \cdot h(y)$. Notice that ψ is a 1-Lipschitz map on $\Gamma(G, U)$, since $|\psi(s)| \leq 1$ for every $s \in S$ and $|\psi(x)| = |\psi(x^{-1})| = 1$. Note also that $\psi|_K$ is a homomorphism to $\mathbb{R}$ and satisfies $\psi(\gamma_t) = \psi(\beta_t) = t$ for every t. Thus for every t,

$$d_U(\gamma_t, \gamma_0) = t = |\psi(\gamma_t) - \psi(\beta_0)| \leq d_U(\gamma_t, \beta_0)$$
$$d_U(\beta_t, \beta_0) = t = |\psi(\beta_t) - \psi(\gamma_0)| \leq d_U(\beta_t, \gamma_0)$$

Consider $\overline{\Gamma(G, U)}$. Note that

$$\begin{aligned} b_{\gamma_t}(\gamma_0) - b_{\gamma_t}(\beta_0) &= d_U(\gamma_t, \gamma_0) - d_U(\gamma_t, \beta_0) \leq 0 \\ &\leq d_U(\beta_t, \gamma_0) - d_U(\beta_t, \beta_0) = b_{\beta_t}(\gamma_0) - b_{\beta_t}(\beta_0). \end{aligned} \tag{3.1}$$

If $\gamma_\infty = \beta_\infty$, there exists t_0 such that for every $t \geq t_0$ we have that $b_{\gamma_t}(\gamma_0) - b_{\gamma_t}(\beta_0) = b_{\beta_t}(\gamma_0) - b_{\beta_t}(\beta_0)$, implying equality throughout (3.1). Thus, $d_U(\gamma_t, \beta_0) = d_U(\gamma_t, \gamma_0) = t$ for every $t \geq t_0$.

Now we shall prove that $g = 1$. Since $h(g) = 0$, $h(x^n) = n \cdot |x|_S$ and h is 1-Lipschitz, we have

$$t \cdot |x|_S = h(\gamma_t) = |h(\gamma_t) - h(g)| \leq d_S(\gamma_t, g),$$

so that $t \cdot |x|_S \leq d_S(\gamma_t, g)$ for every t.

Fix some $t \geq t_0$. Since $d_U(\gamma_t, \beta_0) = t$, there exists a finite geodesic $(z_j)_{j=0}^t$ in $\Gamma(G, U)$ such that $z_0 = \beta_0 = g$ and $z_t = \gamma_t = x^t$. Let $u_j = z_{j-1}^{-1} z_j$ for every $1 \leq j \leq t$, and let $J = \{j \mid u_j \in S\}$. We get

$$t \cdot |x|_S \leq d_S(\gamma_t, g) \leq \sum_{j=1}^{t} |u_j|_S = |J| + (t - |J|) \cdot |x|_S = t \cdot |x|_S - |J| \cdot (|x|_S - 1).$$

Because $x \neq 1$, it follows that $|J| = 0$, so $u_j \in \{x, x^{-1}\}$ for every $1 \leq j \leq t$. Thus $x^t = z_t = g x^m$ for some $m \in \mathbb{Z}$. But that means that $g = x^{t-m}$ and so $0 = h(g) = h(x^{t-m}) = (t - m) \cdot |x|_S$. This implies that $t - m = 0$, that is, $g = 1$.

Thus we have shown that the only $g \in \mathsf{Ker}(h|_K)$ such that $\gamma_\infty = g.\gamma_\infty$ is $g = 1$. So if $g, g' \in \mathsf{Ker}(h|_K)$ are such that $g.\gamma_\infty = g'.\gamma_\infty$, then $g = g'$. That is, the map $g \mapsto g.\gamma_\infty$ from $\mathsf{Ker}(h|_K)$ to $\partial_b \Gamma(G, U)$ is injective, completing the proof. ■

We now complete this section by proving Lemma 3.1.

Proof of Lemma 3.1. Let S be a finite symmetric generating set for an infinite group G such that for any $x \in G$, setting $U = S \cup \{x, x^{-1}\}$ gives a Cayley graph $\Gamma(G, U)$ with a finite Busemann boundary. Specifically, taking $x \in S$ tells us that $|\partial_b \Gamma(G, S)| < \infty$. Let

$$K = \{x \in G : \forall\, h \in \partial_b \Gamma(G, S), x.h = h\}$$

be the kernel of the action. Since $\partial_b \Gamma(G, S)$ is finite, $[G : K] < \infty$. By Lemma 3.3, there exist $h \in \partial_b \Gamma(G, S)$ and $1 \neq y \in K$ such that $h(y) = -|y|_S$. By Proposition 2.3, $h|_K$

is a homomorphism into $\mathbb{R}$, so $h(x) = |x|_S$ for $x = y^{-1}$. By Lemma 3.5, we have that $|\mathsf{Ker}(h|_K)| \leq |\partial_b \Gamma(G,U)|$, where $U = S \cup \{x,x^{-1}\}$. This is finite by our assumptions on S.

Since h is a non-trivial integer-valued homomorphism on the infinite group K, it must be that $h(K) \cong \mathbb{Z}$. Thus, $K/\mathsf{Ker}(h|_K) \cong \mathbb{Z}$, and as the kernel is finite, this implies that K is virtually $\mathbb{Z}$. As $[G : K] < \infty$, this completes the proof. ■

4. Graphs of linear growth

The "only if" direction of Theorem 1.3 was originally proven in [11] (in that paper horoboundary refers to what we call Busemann boundary). Sam Shepperd gave the following short and elegant proof.

By a graph of linear growth, we mean that the number of vertices in the ball of radius r in the graph grows at most linearly in the radius r.

Proposition 4.1. *Let Γ be a graph, with d denoting the graph distance. Let $o \in \Gamma$ be some vertex, and consider the Busemann boundary $\partial_b \Gamma$ with respect to this base point. Let $S_r = \{x \in \Gamma : d(x,o) = r\}$ be the sphere of radius r around o.*

Then,

$$|\partial_b \Gamma| \leq \liminf_{r\to\infty} |S_r|.$$

Specifically, if Γ has linear growth, then the Busemann boundary is finite.

Proof. For a geodesic γ in Γ, denote

$$S_r \cap \gamma = \{x \in S_r : \exists t, \gamma_t = x\}.$$

For any geodesic γ, there exists some $r(\gamma)$ such that for all $r \geq r(\gamma)$ we have $|S_r \cap \gamma| = 1$.

Now, let $\gamma^{(1)}, \ldots, \gamma^{(n)}$ be n geodesics, such that $\gamma_\infty^{(1)}, \ldots, \gamma_\infty^{(n)}$ are all distinct Busemann points. Since these geodesics are all pairwise non-equivalent, no two of them can intersect infinitely many times. Thus, there exists some r_0 such that for all $r \geq r_0$, we have

$$\forall\, i \neq j, \quad (S_r \cap \gamma^{(i)}) \cap (S_r \cap \gamma^{(j)}) = \emptyset.$$

By making sure that $r_0 \geq \max\{r(\gamma^{(1)}), \ldots, r(\gamma^{(n)})\}$, we get that

$$n = \left|\bigcup_{j=1}^{n} (S_r \cap \gamma^{(j)})\right| \leq |S_r|$$

for all $r \geq r_0$. Taking lim inf on the right-hand side completes the proof of the first assertion.

Now, if Γ has linear growth, then there exists a constant $C > 0$ such that for any $r \in \mathbb{N}$ we have

$$|\{x \in \Gamma : d(x,o) \leq r\}| = \sum_{k=0}^{r} |S_k| \leq C(r+1).$$

This is easily seen to imply that $\liminf_{r\to\infty} |S_r| < \infty$, implying the second assertion. ■

Note also that one can replace the geodesics in the proof above with infinite simple paths which have distinct limits in the horoboundary. It is not true, however, that every horofunction is the limit of a sequence forming a simple infinite path.

As mentioned, the converse statement to Proposition 4.1 for general graphs, that is, an analog of Theorem 1.3, is not true in general. Consider the following example with a graph of arbitrary growth, but only one point in the horofunction boundary.

Example 4.2. Consider the following graph: Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs, and fix some vertex $x_n \in \Gamma_n$ in each one.

The vertex set of our graph Γ is then defined to be $\mathbb{N} \cup \bigcup_n \Gamma_n$. Edges in Γ are given by:

- the original edges in $\mathbb{N}$ ($\{x, y\}$ is an edge if $|x - y| = 1$ for $x, y \in \mathbb{N}$),
- the original edges in each Γ_n, and
- additional edges $\{n, x_n\}$ for each $n \in \mathbb{N}$.

See Figure 1.

It is not difficult to compute that if $x \in \Gamma_n \cup \{n\}$ and if $y \in \Gamma_m \cup \{m\}$ for $m > n$, then

$$d(y,x) = d(y,m) + d(m,n) + d(n,x),$$

so that the only possible horofunction in this graph is given by

$$h(x) = \lim_{m\to\infty} (d(m,x) - m).$$

Theorem 4.3. *Let G be a group that is virtually $\mathbb{Z}$, and let $\Gamma(G, S)$ be any Cayley graph of G.*

Then, any horofunction in $\partial\Gamma(G, S)$ is a Busemann point; that is, $\partial\Gamma(G, S) = \partial_b\Gamma(G, S)$.

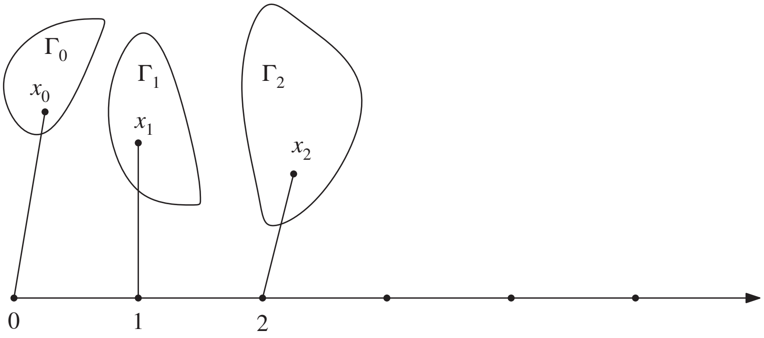

Figure 1. The graph Γ from Example 4.2.

Proof. Since G is virtually $\mathbb{Z}$ it has linear growth, so by Proposition 4.1 we know that the Busemann boundary $\partial_b\Gamma(G,S)$ is finite. G acts on $\partial_b\Gamma(G,S)$. Let

$$K = \{x \in G : \forall\, h \in \partial_b\Gamma(G,S), x.h = h\}$$

be the kernel of the action. Since $\partial_b\Gamma(G,S)$ is finite, $[G:K] < \infty$. By Lemma 3.3, there exist $\varphi \in \partial_b\Gamma(G,S)$ and $1 \neq y \in K$ such that $\varphi(y) = -|y|_S$. By Proposition 2.3, $\varphi|_K$ is a homomorphism into $\mathbb{R}$, so $\varphi(x) = |x|_S$ for $x = y^{-1}$. Since φ is 1-Lipschitz, by Lemma 3.4 there exists a geodesic α in $\Gamma(G,S)$ such that $\alpha_{n|x|} = x^n$ for all $n \in \mathbb{N}$. Applying the same logic to $y = x^{-1}$ but with the 1-Lipschitz function $-\varphi$, we can also define a geodesic β so that $\beta_{n|x|} = x^{-n}$ for all $n \in \mathbb{N}$.

Now, consider the subgroup $N = \langle x \rangle \cong \mathbb{Z}$. Since G is virtually $\mathbb{Z}$, it is impossible that $[G:N] = \infty$. Let R be a set of representatives for the cosets of N; that is, $G = \biguplus_{r\in R} rN$, with $|R| = [G:N] < \infty$. Assume that $1 \in R$.

Let $h \in \partial\Gamma(G,S)$ be an arbitrary horofunction. Let $(y_n)_n$ be a sequence in G such that $h = \lim_{n\to\infty} b_{y_n}$. By Proposition 2.2, the set $\{y_n\}_n$ is infinite, and thus, there must exist some $r \in R$ such that infinitely many y_n are in the coset rN. Thus, one of the two sets

$$\{y_n : \exists\, k, y_n = r\alpha_k\} \quad \text{or} \quad \{y_n : \exists\, k, y_n = r\beta_k\}$$

is infinite. That is, the sequence $(y_n)_n$ has infinitely many common points with one of the geodesics $r\alpha$ or $r\beta$. This implies that

$$h = \lim_{n\to\infty} b_{y_n} = \lim_{k\to\infty} b_{rx^k} \in \{r.\alpha_\infty, r.\beta_\infty\}.$$

As this was for an arbitrary horofunction $h \in \partial\Gamma(G,S)$, we have that

$$\partial\Gamma(G,S) \subset \{r.\alpha_\infty, r.\beta_\infty : r \in R\} \subset \partial_b\Gamma(G,S). \qquad \blacksquare$$

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