Homology growth of polynomially growing mapping tori

Naomi Andrew, Yassine Guerch, Sam Hughes, and Monika Kudlinska

Abstract. We prove that residually finite mapping tori of polynomially growing automorphisms of hyperbolic groups, groups hyperbolic relative to finitely many virtually polycyclic groups, right-angled Artin groups (when the automorphism is untwisted), and right-angled Coxeter groups have the cheap rebuilding property of Abert, Bergeron, Fraczyk, and Gaboriau. In particular, their torsion homology growth vanishes for every Farber sequence in every degree.

1. Introduction

Let Γ be a residually finite group of type F. By Lück's celebrated approximation theorem, the ith ℓ^2-Betti number $b_i^{(2)}(\Gamma)$ of Γ is a measure of the growth of the ith homology of Γ with rational coefficients [29]. More precisely, if $(\Gamma_k)_{k\in\mathbb{N}}$ is a descending sequence of finite index normal subgroups of Γ such that $\bigcap_{k\in\mathbb{N}} \Gamma_k = 1$, then

$$b_i^{(2)}(\Gamma) = \lim_{k\to\infty} \frac{\dim_{\mathbb{Q}} H_i(\Gamma_k, \mathbb{Q})}{[\Gamma : \Gamma_k]}.$$

The ℓ^2-Betti numbers are important group invariants which have found many applications in topology and group theory (see [30] and the references therein for a comprehensive account). It is thus natural to study the growth of other homology groups associated to Γ, as well as the growth of the torsion part $|H_i(\Gamma_k, \mathbb{Z})_{\text{tors}}|$ of the homology.

This paper is concerned with the growth of the mod-p Betti numbers of certain groups Γ, as well as the *homology torsion growth*, which is defined to be

$$t_j(\Gamma; \Gamma_k) = \limsup_{k\to\infty} \frac{\log|H_j(\Gamma_k, \mathbb{Z})_{\text{tors}}|}{[\Gamma : \Gamma_k]},$$

where $(\Gamma_k)_{k\in\mathbb{N}}$ is a Farber sequence of Γ.

These invariants have been computed in right-angled Artin groups and certain graph products [4, 34] (see also [14] where non-vanishing is proven for certain Bestvina–Brady groups). Moreover, there exist results proving vanishing of homology growth in various instances [1, 7, 25, 36]. We also mention the work of Bader–Gelander–Sauer which gives

an upper bound for the homology torsion of negatively curved Riemannian manifolds of dimension at least four in terms of their volume [5].

The main aim of this paper is to calculate homology growth for certain classes of group extensions of the form $G \rtimes_\phi \mathbb{Z}$, where $\phi \in \mathrm{Aut}(G)$ is *polynomially growing* (see Section 2.1 for the precise definition). We prove the following theorem.

Theorem A. *Let Γ be a group isomorphic to one of the following:*

- *$G \rtimes_\phi \mathbb{Z}$ with G residually finite and hyperbolic;*
- *$G \rtimes_\phi \mathbb{Z}$ with G residually finite and hyperbolic relative to a finite collection of virtually polycyclic groups;*
- *$A_L \rtimes_\phi \mathbb{Z}$ where A_L is a right-angled Artin group and $\phi \in \mathrm{Aut}(A_L)$ is untwisted (see Section 5); or*
- *$W_L \rtimes_\phi \mathbb{Z}$ where W_L is a right-angled Coxeter group.*

If ϕ is polynomially growing, then for every Farber sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of Γ, every $j \geq 0$ and every field $\mathbb{K}$, we have

$$\lim_{k\to\infty} \frac{\dim_\mathbb{K} H_j(\Gamma_k, \mathbb{K})}{[\Gamma : \Gamma_k]} = 0 \quad and \quad \lim_{k\to\infty} \frac{\log|H_j(\Gamma_k, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]} = 0.$$

We note that in the case where G is a finite rank free group and $\phi \in \mathrm{Aut}(G)$ a polynomially growing automorphism, the result was already known by previous work of the first, third and fourth authors [3].

The key tool in proving Theorem A is the *cheap rebuilding property* developed by Abert–Bergeron–Fraczyk–Gaboriau in [1], which implies vanishing of homology torsion growth in residually finite groups. Crucially for us, a group Γ has the cheap rebuilding property whenever it acts co-compactly on a contractible CW-complex with stabilisers that satisfy the cheap rebuilding property (see Section 2.3 for more details, and specifically Theorem 2.14).

We prove that the groups considered in Theorem A all satisfy the cheap rebuilding property. We do so by finding ϕ-invariant splittings of G, which induce splittings of the extension $G \rtimes_\phi \mathbb{Z}$. In the one-ended relatively hyperbolic case, we deploy the theory of JSJ decompositions [23] following Guirardel–Levitt. In the case of right-angled Artin and Coxeter groups, we use the work of Fioravanti on coarse median preserving automorphisms of those groups [13].

For the case when the group G is infinitely ended, we prove a combination theorem for the cheap rebuilding property of the mapping torus $G \rtimes_\phi \mathbb{Z}$ of a polynomially growing automorphism $\phi \in \mathrm{Aut}(G)$.

Theorem 3.1. *Let $G = G_1 * \cdots * G_k * F_N$ be a free product of residually finite groups. Fix $\alpha \in \mathbb{N}$. Let ϕ be a polynomially growing automorphism of G which preserves the conjugacy classes of the factors G_i. Suppose that for every $i \in \{1, \ldots, k\}$, the group*

$G_i \rtimes_{\phi|_{G_i}} \mathbb{Z}$ *has the cheap α-rebuilding property. Then the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.*

We note that Theorem 3.1 recovers the main theorem from [3] on the cheap rebuilding property for mapping tori of polynomially growing outer automorphisms of finite rank free groups.

The key observation in the proof of Theorem 3.1 is that if a polynomially growing automorphism preserves a free product decomposition, then after possibly passing to a power it preserves a *sporadic* free factor system (see Proposition 2.11). Hence, it preserves a G-tree T which is the Bass–Serre tree corresponding to the sporadic free factor system.

1.1. The ℓ^2-torsion conjecture

The *integral torsion* $\rho^{\mathbb{Z}}(\Gamma)$ of a group Γ is defined to be the sum of the homology torsion gradients:

$$\rho^{\mathbb{Z}}(\Gamma) := \sum_{j \geq 0} t_j(\Gamma; \Gamma_k).$$

An important conjecture of Lück relates the integral torsion $\rho^{\mathbb{Z}}(\Gamma)$ with its ℓ^2-torsion $\rho^{(2)}(\Gamma)$.

Conjecture 1.1 ([31, Conjecture 1.11 (3)]). *Let Γ be an infinite residually finite ℓ^2-acyclic group of type* VF. *Then, $\rho^{(2)}(\Gamma) = \rho^{\mathbb{Z}}(\Gamma)$.*

The conjecture is known to hold for some classes of groups, including amenable groups by the work of Kar–Kropholler–Nikolov [25] and Li–Thom [28], and fundamental groups of closed aspherical manifolds which admit a non-trivial S^1-action, or contain a non-trivial elementary amenable normal subgroup, by the work of Lück [31]. Okun–Schreve showed that the conjecture is also true in the case of right-angled Artin groups, where the torsion does not vanish in general [34].

The work of Clay [9], combined with the results mentioned above, confirms the conjecture in the case of mapping tori of polynomially growing outer automorphisms of finite rank free groups. Hence, our work leads to the following natural question.

Question 1.2. Let Γ be the mapping torus of a polynomially growing automorphism $\phi \in \text{Aut}(G)$ as in Theorem A. Is it true that the ℓ^2-torsion $\rho^{(2)}(\Gamma)$ of Γ vanishes?

2. Background

2.1. Growth of automorphisms

Fix a finite generating set S of a group G. For any $g \in G$, let $\|[g]\|_S$ denote the length of the shortest word in the conjugacy class $[g]$ of g, and $|g|_S$ the length of the shortest word

in the generators S representing the element g in G. An outer automorphism $\Phi \in \mathrm{Out}(G)$ is said to *grow polynomially*, if for every conjugacy class c in G, there exist some integer $d \geq 0$ and real number $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|\Phi^n(c)\|_S \leq Cn^d + C.$$

An automorphism $\phi \in \mathrm{Aut}(G)$ is said to *grow polynomially*, if for every element $g \in G$, there exist some integer $d \geq 0$ and $C > 0$ such that for all $n \in \mathbb{N}$,

$$|\phi^n(g)|_S \leq Cn^d + C.$$

Note that for any two finite generating sets S and S', the corresponding word metrics on G are bi-Lipschitz equivalent. In particular, the definitions of growth are independent of the choice of a finite generating set.

Remark 2.1. Many naturally occurring groups, including torsion-free hyperbolic groups and abelian groups, experience a growth rate dichotomy: for every $\phi \in \mathrm{Aut}(G)$, each $g \in G$ admits either polynomial or exponential growth under the iterations of ϕ (see [10, Theorem 1.1]). In the case of free groups, Levitt [27] gave complete classification of all possible types of growth for automorphisms of free groups. However, as shown by Coulon [10], there exist groups with automorphisms which exhibit more exotic types of growth.

2.2. Free products, free factor systems and the graph of free factors

Let $G_1, \dots, G_k$ be a finite collection of non-trivial finitely generated groups, let F_N be a free group of rank N and let

$$G = G_1 * \cdots * G_k * F_N.$$

We denote by $\mathcal{F}$ the set of conjugacy classes of the groups G_i with $i \in \{1, \dots, k\}$. We refer to the pair $(G, \mathcal{F})$ as a *free product*.

Definition 2.2. The pair $(G, \mathcal{F})$ is a *sporadic free product* if one of the following holds:

  (1) We have $k = 0$ and $G = \mathbb{Z}$.
  (2) We have $k = 1$ and $G = G_1$ or $G = G_1 * \mathbb{Z}$.
  (3) We have $k = 2$ and $G = G_1 * G_2$.

Otherwise, the pair $(G, \mathcal{F})$ is a *nonsporadic free product*.

Given a free product $(G, \mathcal{F})$, an element $g \in G$ is *peripheral* if there exists $[A] \in \mathcal{F}$ with $g \in A$. Otherwise, we say that g is *nonperipheral*.

A *free factor system* of $(G, \mathcal{F})$ is a finite collection $\mathcal{F}' = \{[A_1], \dots, [A_\ell]\}$ of conjugacy classes of non-trivial finitely generated subgroups of G such that:

(1) for every $i \in \{1, \ldots, k\}$, the group G_i is contained in some subgroup A of G with $[A] \in \mathcal{F}'$;

(2) there exists a subgroup B of G such that $G = A_1 * \cdots * A_\ell * B$.

A free factor system of $(G, \mathcal{F})$ is *proper* if it is distinct from $\mathcal{F}$ and $\{[G]\}$.

There is a natural partial order on the set of free factor systems of $(G, \mathcal{F})$, where $\mathcal{F}_1 \leq \mathcal{F}_2$ if, for every $[A] \in \mathcal{F}_1$, there exists $[B] \in \mathcal{F}_2$ with $A \subseteq B$. Note that $\mathcal{F}$ is minimal for this partial order. A *free factor* of $(G, \mathcal{F})$ is an element of a free factor system.

Definition 2.3. Let $(G, \mathcal{F})$ be a free product. The *free factor graph of* $(G, \mathcal{F})$, denoted by $\mathrm{FF}(G, \mathcal{F})$, is the graph whose vertices are the proper free factors of $(G, \mathcal{F})$, two free factors $\mathcal{F}_1$ and $\mathcal{F}_2$ being adjacent if $\mathcal{F}_1 < \mathcal{F}_2$ or $\mathcal{F}_2 < \mathcal{F}_1$.

By results of Guirardel and Horbez [20, Proposition 2.11] (see also the work of Bestvina and Feighn [8] and Handel and Mosher [24] for the case $G = F_N$), the graph $\mathrm{FF}(G, \mathcal{F})$ is Gromov hyperbolic.

We say that an outer automorphism $\Phi \in \mathrm{Out}(G)$ *preserves* a free factor system $\mathcal{F}$ if Φ fixes each conjugacy class in $\mathcal{F}$. We write $\mathrm{Out}(G, \mathcal{F})$ to denote the subgroup of $\mathrm{Out}(G)$ which consists of the outer automorphisms which preserve $\mathcal{F}$. The group $\mathrm{Out}(G, \mathcal{F})$ has a natural action by isometries on the graph $\mathrm{FF}(G, \mathcal{F})$ induced by its action on the set of free factors of $(G, \mathcal{F})$. The next result describes the loxodromic elements of $\mathrm{FF}(G, \mathcal{F})$. We say that an element $\Phi \in \mathrm{Out}(G, \mathcal{F})$ is *fully irreducible* if no positive power of Φ preserves a proper free factor system of $(G, \mathcal{F})$. When $(G, \mathcal{F})$ is a free product and $\mathcal{F} = \emptyset$, this reduces to the usual notion of being fully irreducible for elements of $\mathrm{Out}(F_N)$, where F_N is a free group of rank N.

Theorem 2.4 ([20, Theorem 4.1]). *Let* $(G, \mathcal{F})$ *be a nonsporadic free product. An element* $\Phi \in \mathrm{Out}(G, \mathcal{F})$ *is a loxodromic element of* $\mathrm{FF}(G, \mathcal{F})$ *if and only if* Φ *is fully irreducible.*

The next theorem gives an existence condition of fully irreducible elements in subgroups of $\mathrm{Out}(G, \mathcal{F})$.

Theorem 2.5 ([20, Theorem 7.1]). *Let* $(G, \mathcal{F})$ *be a nonsporadic free product and let* H *be a finitely generated subgroup of* $\mathrm{Out}(G, \mathcal{F})$. *If* H *does not virtually preserve a proper* $(G, \mathcal{F})$*-free factor system, then* H *contains a fully irreducible outer automorphism.*

We now describe the Gromov boundary of $\mathrm{FF}(G, \mathcal{F})$. A $(G, \mathcal{F})$*-tree* is an $\mathbb{R}$-tree T equipped with an action of G by isometries such that for every $i \in \{1, \ldots, k\}$, the group G_i fixes a point in T. Given a $(G, \mathcal{F})$-tree T and a point $x \in T$, we denote by $\mathrm{Stab}(x)$ the stabiliser of x.

A $(G, \mathcal{F})$-tree is *very small* if tripod stabilisers are trivial and arc stabilisers are cyclic (maybe trivial), nonperipheral and root-closed.

A *Grushko* $(G, \mathcal{F})$-*tree* is a $(G, \mathcal{F})$-tree T such that T is simplicial, the action of G is minimal, edge stabilisers are trivial and, for every vertex v, either Stab(v) is trivial or $[\text{Stab}(v)] \in \mathcal{F}$. Recall that minimal means that G does not preserve a proper subtree of T.

Note that, if $[A]$ is a free factor of $(G, \mathcal{F})$, the free factor system $\mathcal{F}$ induces a free factor system $\mathcal{F}|_A$ of A. A $(G, \mathcal{F})$-*arational tree* is a very small $(G, \mathcal{F})$-tree T which is not a Grushko tree and such that for every free factor $[A]$ of $(G, \mathcal{F})$, the action of $(A, \mathcal{F}|_A)$ on its minimal tree in T induces a Grushko $(A, \mathcal{F}|_A)$-tree.

The following theorem relates the Gromov boundary of FF$(G, \mathcal{F})$ with the $(G, \mathcal{F})$-arational trees.

Proposition 2.6 ([20, Theorem 3.4]). *Let $(G, \mathcal{F})$ be a nonsporadic free product and let H be a finitely generated subgroup of* Out$(G, \mathcal{F})$. *If H has a finite orbit in ∂_∞FF$(G, \mathcal{F})$, then H has a finite index subgroup which fixes the homothety class of a $(G, \mathcal{F})$-arational tree.*

Remark 2.7. Note that [20, Theorem 3.4] only shows that H has a finite index subgroup preserving the G-equivariant homeomorphism class (for the observers' topology) of a $(G, \mathcal{F})$-arational tree T. However, by [19, Proposition 13.5], the set of projective classes of $(G, \mathcal{F})$-arational trees which are equivalent to T is a finite-dimensional simplex. Therefore, the group H has a finite index subgroup preserving the homothety class of a $(G, \mathcal{F})$-arational tree equivalent to T corresponding to an extremal point of the simplex.

If T is a $(G, \mathcal{F})$-arational tree, we denote by $[T]$ its homothety class and by SF: Stab$([T]) \to \mathbb{R}_+^*$ the stretching factor homomorphism.

Lemma 2.8 ([20, Proposition 6.3, Corollary 6.12]). *Let T be a $(G, \mathcal{F})$-arational tree. For every $\Phi \in$ Stab$([T])$, we have SF$(\Phi) \neq 1$ if and only if Φ is fully irreducible.*

Lemma 2.9. *Let $(G, \mathcal{F})$ be a nonsporadic free product and let $\Phi \in$ Out$(G, \mathcal{F})$ be polynomially growing. Then Φ is not fully irreducible.*

Proof. Suppose towards a contradiction that Φ is fully irreducible. By Theorem 2.4, the element Φ is a loxodromic element of FF$(G, \mathcal{F})$. In particular, Φ acts on FF$(G, \mathcal{F})$ with North-South dynamics and has exactly two finite orbits in ∂_∞FF$(G, \mathcal{F})$ consisting of its attracting and repelling fixed points. Let ξ_+ be the attracting fixed point of Φ. By Proposition 2.6, up to taking a power of Φ, we may suppose that Φ fixes the homothety class of a $(G, \mathcal{F})$-arational tree T associated with ξ_+. By Lemma 2.8, the stretching factor SF$_T(\Phi)$ of Φ is distinct from 1. Since SF$_T : \langle \Phi \rangle \to \mathbb{R}_+^*$ is a homomorphism, either SF$_T(\Phi) > 1$ or SF$_T(\Phi^{-1}) > 1$. Up to replacing Φ by Φ^{-1}, we may assume that $\lambda = $ SF$_T(\Phi) > 1$. Note that we have only possibly replaced Φ by Φ^{-1} without changing the tree T. In particular, we are not considering the repelling fixed point ξ_- of Φ. This is why SF$_T(\Phi) \circ$ SF$_T(\Phi^{-1}) = 1$.

Let $g \in G$ and let $\ell([g])$ be the translation length of the conjugacy class of g in T. Since Φ preserves the homothety class of T, for every $m \in \mathbb{N}$, we have $\ell(\Phi^m([g])) = \lambda^m \ell([g])$. But $\ell(\Phi^m([g]))$ is bounded from above by a multiple of $\|\Phi^m([g])\|$ (see, for instance, [11, Propositions 1.5 and 1.8]). As Φ is polynomially growing, this implies that, for every $g \in G$, we have $\ell([g]) = 0$. Since G is finitely generated, the group G fixes a point in T (see, for instance, [11, Section 3]), a contradiction. Thus, Φ is not fully irreducible. ∎

Remark 2.10. Another proof of the fact that $\mathrm{SF}_T(\Phi) > 1$ is the following. We follow the notations of the above proof.

Suppose that Φ is fully irreducible and let U be a relative train track associated with Φ, which exists by [16, Theorem 8.24] and [32, Theorem A]. Let $\lambda(\Phi)$ be the Perron–Frobenius eigenvalue of the associated transition matrix. A consequence of [17, Theorem 3.9] is that since $(G, \mathcal{F})$ is nonsporadic and Φ is fully irreducible, we have $\lambda(\Phi) > 1$.

Let $T_+ = \lim_{n \to \infty} \frac{T\Phi^n}{\lambda(\Phi)^n}$. By [18, Lemma 2.14.1], the space T^+ is an $\mathbb{R}$-tree in the boundary of the outer space $\mathbb{P}\mathcal{O}$ of $(G, \mathcal{F})$. Thus, $T_+\phi = \lambda(\Phi)T_+$.

Since Φ is fully irreducible, by [18, Theorem 5.1.2], it acts with a North-South dynamics on the outer space of $(G, \mathcal{F})$. By [20, Theorem 3], there exists a Φ-equivariant map $\psi \colon \mathbb{P}\mathcal{O} \to \mathrm{FF}(G, \mathcal{F})$ with a Φ-equivariant extension $\partial \psi$ from the set of $(G, \mathcal{F})$-arational trees to $\partial_\infty \mathrm{FF}(G, \mathcal{F})$. Since Φ also acts with a North-South dynamics on $\mathrm{FF}(G, \mathcal{F})$, we see that $\partial \psi$ sends T_+ to ξ_+, so that we can choose T in the proof of Lemma 2.9 to be T_+. In particular, we have $\mathrm{SF}_T(\Phi) = \lambda(\Phi) > 1$.

Bass–Serre theory implies that for any free product $(G, \mathcal{F})$, Grushko $(G, \mathcal{F})$-trees exist. In general, there are many of these trees, even working up to equivariant homothety: they form a *deformation space* which is invariant under an action of $\mathrm{Aut}(G, \mathcal{F})$, as studied in [21]. However, if $(G, \mathcal{F})$ is sporadic, then by [15, 26], there is a fixed point for this action – a $(G, \mathcal{F})$-tree preserved by all of $\mathrm{Aut}(G, \mathcal{F})$. The tree in question has a single orbit of edges and either one or two orbits of vertices (depending on the size of $\mathcal{F}$). A more elementary proof of the same fact using translation length functions is in [2, Proposition 4.8]: while the statement concerns indecomposable factors and the whole automorphism group, the argument applies just as well restricted to any sporadic free factor system and the subgroup $\mathrm{Aut}(G, \mathcal{F})$.

The sporadic splittings obtained in the following proposition will be of the form $G_1 * \mathbb{Z}$ or $G_1 * G_2$ rather than the degenerate case $G = \mathbb{Z}$. However, note that the free $\mathbb{Z}$ action on a line is invariant under the (finite) automorphism group, so the statement holds even in this case.

Proposition 2.11. *Let $(G, \mathcal{F})$ be a free product and let $\Phi \in \mathrm{Out}(G, \mathcal{F})$ be polynomially growing. There is a free factor system $\mathcal{F}'$ of $(G, \mathcal{F})$ and $k \in \mathbb{N}$ such that Φ^k preserves $\mathcal{F}'$ and $(G, \mathcal{F}')$ is sporadic. In particular, Φ^k preserves a Bass–Serre tree associated to $\mathcal{F}'$.*

Proof. If $(G, \mathcal{F})$ is sporadic, then the first claim is immediate. Otherwise, let $\mathcal{F}'$ be a maximal Φ-periodic proper free factor system. It suffices to prove that $(G, \mathcal{F}')$ is sporadic. Up to taking a power of Φ, we may suppose that $\mathcal{F}'$ is Φ-invariant. Thus, we may view Φ as an element of $\mathrm{Out}(G, \mathcal{F}')$. By maximality of $\mathcal{F}'$ and Theorem 2.5, the element Φ is fully irreducible. Then, since Φ is polynomially growing, by Lemma 2.9, it must be the case that $(G, \mathcal{F}')$ is sporadic.

Now suppose that $(G, \mathcal{F})$ is sporadic and that Φ preserves $\mathcal{F}$. Since there is a fixed point for the action of $\mathrm{Aut}(G, \mathcal{F})$ on its deformation space, this $(G, \mathcal{F})$-tree is preserved by Φ. ∎

2.3. The cheap rebuilding property

Let $\alpha \in \mathbb{N}$. In this section, we give the relevant background regarding the *cheap α-rebuilding property*, which was introduced by Abert, Bergeron, Fraczyk and Gaboriau [1] to prove that certain groups have vanishing (torsion) homology growth. Although we will not state the complete definition of this property, we list in the following propositions the properties which we will use in the rest of the paper. This property is relevant for our considerations by the following theorem.

Theorem 2.12 ([1, Theorem 10.20]). *Let $\alpha \in \mathbb{N}$ and let Γ be a residually finite group of type $F_{\alpha+1}$. Suppose that Γ has the cheap α-rebuilding property. For every Farber sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of Γ, each $j \leq \alpha$ and every coefficient field $\mathbb{K}$, we have*

$$\lim_{k \to \infty} \frac{\dim_{\mathbb{K}} H_j(\Gamma_k, \mathbb{K})}{[\Gamma : \Gamma_k]} = 0 \quad and \quad \lim_{k \to \infty} \frac{\log |H_j(\Gamma_k, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]} = 0.$$

We refer to [1] for the definition of a Farber sequence. Examples include decreasing sequences of finite index normal subgroups with trivial intersection.

Proposition 2.13 ([1, Corollary 10.13]). *Let Γ be a residually finite countable group and let $\alpha \in \mathbb{N}$. The following statements hold:*

(1) *Let $\Gamma' \subseteq \Gamma$ be a finite index subgroup. Then Γ has the cheap α-rebuilding property if and only if Γ' does.*

(2) *If Γ has an infinite normal subgroup N such that Γ/N is of type F_α and N has the cheap α-rebuilding property, then Γ has the cheap α-rebuilding property.*

(3) *For every $m \in \mathbb{N}^*$ and every $\alpha \in \mathbb{N}$, the group $\mathbb{Z}^m$ has the cheap α-rebuilding property.*

(4) *If Γ is infinite and virtually polycyclic, then it has the cheap α-rebuilding property.*

Note that the "infinite" assumption at various points in this proposition was necessary: finite groups do not have the cheap α-rebuilding property for any α.

Theorem 2.14 ([1, Theorem 10.9]). *Let Γ be a residually finite group acting on a CW-complex Ω in such a way that any element stabilising a cell fixes it pointwise. Let $\alpha \in \mathbb{N}$. Suppose that the following conditions hold:*

(1) *$\Gamma \backslash \Omega$ has finite α-skeleton.*

(2) *Ω is $(\alpha - 1)$-connected.*

(3) *For each cell $\omega \in \Omega$ of dimension $j \leq \alpha$, the stabiliser $\mathrm{Stab}_\Gamma(\omega)$ has the cheap $(\alpha - j)$-rebuilding property.*

Then Γ itself has the cheap α-rebuilding property.

3. A combination theorem for the cheap rebuilding property of mapping tori

Let $(G, \mathcal{F})$ be a free product. Recall that each element $[G_i] \in \mathcal{F}$ corresponds to the conjugacy class of a non-trivial finitely generated subgroup G_i of G, with the possibility that $\mathcal{F} = \emptyset$. The main result of this section, Theorem 3.1, is a combination theorem which allows us to deduce the cheap rebuilding property for some mapping tori of G, assuming that it holds for the mapping tori of the factors $\mathcal{F}$. We will use the combination theorem in subsequent sections to prove the cheap α-rebuilding property for a large family of mapping tori with polynomially growing monodromy.

Theorem 3.1. *Let $G = G_1 * \cdots * G_k * F_N$ be a free product of residually finite groups. Fix $\alpha \in \mathbb{N}$. Let ϕ be a polynomially growing automorphism of G which preserves the conjugacy classes of the factors G_i. Suppose that for every $i \in \{1, \ldots, k\}$, the group $G_i \rtimes_{\phi|_{G_i}} \mathbb{Z}$ has the cheap α-rebuilding property. Then the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.*

We note that the groups G_i in the free product are not required to be freely irreducible.

Proof. The proof is by induction on the Grushko rank $k + N$ of G. If $k = 1$ and $N = 0$, then the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property by the hypothesis. If $k = 0$ and $N = 1$, then $G \rtimes_\phi \mathbb{Z}$ is virtually $\mathbb{Z}^2$. By Proposition 2.13 (1) and (3), for every $\alpha \in \mathbb{N}$, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.

Suppose now that $k + N \geq 2$. Let Φ be the outer class of ϕ and let $\mathcal{F}$ be a sporadic free factor system given by Proposition 2.11. Let T be the canonical Bass–Serre tree of G associated to $\mathcal{F}$. The tree T has a unique orbit of edges and its edge stabilisers in G are trivial. In particular, vertex stabilisers in G of T are proper free factors of G, hence have a smaller Grushko rank than the one of G. Since T is canonical, it is preserved by Φ. Up to taking a power of Φ, we may suppose that Φ acts trivially on the underlying graph of $G \backslash T$.

The actions of G and Φ on T induce an action of $G \rtimes_\phi \mathbb{Z}$ on T. Edge stabilisers in $G \rtimes_\phi \mathbb{Z}$ are infinite cyclic and the stabiliser of a vertex v of T is isomorphic to

Stab$(v) \rtimes_{\phi_v} \mathbb{Z}$, where Stab$(v)$ is the vertex stabiliser of v in G and $\phi_v := \phi|_{\mathrm{Stab}(v)}$ is the automorphism of Stab(v) induced by a representative of Φ preserving Stab(v).

We now prove that $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property by applying Theorem 2.14 to the action of $G \rtimes_\phi \mathbb{Z}$ on T. Since the action is co-compact and since T is a tree, it suffices to prove that the stabiliser of any vertex of T has the cheap α-rebuilding property and that the stabiliser of any edge of T has the cheap $(\alpha - 1)$-rebuilding property.

Since edge stabilisers in $G \rtimes_\phi \mathbb{Z}$ are infinite cyclic, they have the cheap $(\alpha - 1)$-rebuilding property by Proposition 2.13 (3). Let v be a vertex of T. Note that since ϕ is a polynomially growing automorphism, so is $\phi_v \in \mathrm{Aut}(\mathrm{Stab}(v))$. Since the Grushko rank of Stab(v) is smaller than the one of G, by induction hypothesis, the group Stab$(v) \rtimes_{\phi_v} \mathbb{Z}$ has the cheap α-rebuilding property. Thus, by Theorem 2.14, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property, which concludes the proof. ∎

Remark 3.2. Theorem 3.1 recovers the main result of [3] which states that any free-by-cyclic group $F_n \rtimes_\phi \mathbb{Z}$ with polynomially growing monodromy $\phi \in \mathrm{Aut}(F_n)$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.

Remark 3.3. If G is infinitely ended, residually finite and accessible, then G admits a finite index subgroup H which is a free product of one-ended groups and a finitely generated free group, and the suspension $G \rtimes_\phi \mathbb{Z}$ has a finite index subgroup $H \rtimes_{\phi'} \mathbb{Z}$. Hence, by Proposition 2.13 (1), one can apply Theorem 3.1 in the setting of infinitely ended, accessible groups which perhaps have torsion.

4. The cheap rebuilding property for residually finite (relatively) hyperbolic groups

In this section, we prove the cheap rebuilding property for mapping tori of residually finite (relatively) hyperbolic groups. Theorem 3.1 is the main step in order to prove the cheap rebuilding property for infinitely ended hyperbolic groups. The one-ended case requires the use of the JSJ decomposition of the group, whose properties are presented after Lemma 4.1, following Guirardel–Levitt [23].

Lemma 4.1. *Let S be a compact, connected hyperbolic surface and let $G = \pi_1(S)$. Let $D \in \mathrm{MCG}(S)$ be a Dehn multi-twist. For every $\alpha \in \mathbb{N}$, the group $G \rtimes_D \mathbb{Z}$ has the cheap α-rebuilding property.*

Proof. Let $\beta = (\beta_1, \ldots, \beta_k)$ be the collection of pairwise non-isotopic simple closed curves in S associated to the multi-twist D. Then D preserves each connected component of the complement $S - \bigcup_{i=1}^{k} \beta_i$. Hence, the natural action of G on the simplicial tree T dual to the collection of lifts of the loops β_i in the universal cover of S extends to an action of the extension $G \rtimes_D \mathbb{Z}$. The edge stabilisers of the action are isomorphic to $\mathbb{Z}^2$ and the

vertex stabilisers are isomorphic to $\pi_1(S_0) \times \mathbb{Z}$, where S_0 is a connected component of the closure of $S - \bigcup_{i=1}^{k} \beta_i$. In particular, by Proposition 2.13 (2) and (3), the edge and vertex stabilisers satisfy the cheap α-rebuilding property for every α. Thus, by Theorem 2.14, the group $G \rtimes_D \mathbb{Z}$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$. ∎

Let $\mathscr{G}$ be the family of groups which are hyperbolic relative to a finite family of virtually polycyclic groups. For each $G \in \mathscr{G}$, we may assume that the peripheral family $\mathscr{P}$ does not contain virtually cyclic groups, since removing them does not destroy relative hyperbolicity (see [23, Section 9.3]). The family $\mathscr{G}$ includes the class of toral relatively hyperbolic groups, which are torsion-free groups hyperbolic relative to a finite collection $\mathscr{P}$ of finitely generated abelian subgroups.

We say a group G is *one-ended relative to a collection of subgroups* $\mathscr{P}$ if there does not exist a splitting of G over finite subgroups such that each group in $\mathscr{P}$ is conjugate into some vertex group. Note that a one-ended group is automatically one-ended relative to every collection of subgroups. We write $\text{Aut}(G; \mathscr{P})$ to denote the group of automorphisms of G which preserve $\mathscr{P}$.

Let $G \in \mathscr{G}$ and suppose that G is one-ended relative to $\mathscr{P}$. By the work of Guirardel and Levitt [23, Corollary 9.20] (see also [22, Section 3.3]), there is a canonical JSJ tree T_G for G, that is, a simplicial tree equipped with an action of G which is preserved by the elements of $\text{Aut}(G; \mathscr{P})$. The group $\text{Aut}(G; \mathscr{P})$ has a finite index subgroup $\mathcal{K}(T_G)$ which acts as the identity on the underlying graph of $G \backslash T_G$. Edge stabilisers of T_G in G are virtually polycyclic. If we further assume that G is torsion free, then the stabiliser $\text{Stab}_G(v)$ of a vertex v in T_G satisfies one of the following:

(1) The group $\text{Stab}_G(v)$ is isomorphic to the fundamental group of a compact hyperbolic surface S and the image of the natural homomorphism $\mathcal{K}(T_G) \to \text{Out}(\text{Stab}_G(v))$ is contained in the mapping class group $\text{MCG}(S)$ of S.

(2) There exists $[P] \in \mathscr{P}$ with $\text{Stab}_G(v) = P$. In particular, the group $\text{Stab}_G(v)$ is virtually polycyclic.

(3) The image of the natural homomorphism $\mathcal{K}(T_G) \to \text{Out}(\text{Stab}_G(v))$ is finite.

Recall that $\mathscr{G}$ denotes the family of groups which are hyperbolic relative to a finite family of virtually polycyclic groups, and we assume with no loss of generality that the peripheral subgroup is not virtually cyclic. Let $\mathscr{G}_{\text{rf}}$ denote the subset of the groups in $\mathscr{G}$ which are residually finite.

Lemma 4.2. *For any group $G \in \mathscr{G}_{\text{rf}}$, there exists a finite index subgroup $G' \leq G$ such that $G' \in \mathscr{G}_{\text{rf}}$ and G' is torsion free.*

Proof. It is well known, using, for instance, the action of G on a relative Rips complex (see [12]), that for a relatively hyperbolic group G, there exists a finite number of finite subgroups $F_1, \ldots, F_k$ of G, such that any finite order element $g \in G$ can be conjugated into some F_i or into a peripheral subgroup of G. Since polycyclic groups contain finitely many conjugacy classes of finite order elements by [6, Theorem 7.1], it follows

that G also has finitely many conjugacy classes of finite order elements. Now since G is residually finite, there exists a finite index normal subgroup G' of G which does not contain any of the finite conjugacy classes and thus is torsion free. Moreover, G' is clearly also residually finite and hyperbolic relative to a finite collection of virtually polycyclic subgroups.   ∎

Theorem 4.3. *Let $G \in \mathcal{G}_{\mathrm{rf}}$ and let $\Phi = [\phi] \in \mathrm{Out}(G)$ be a polynomially growing outer automorphism. Then $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for all α.*

Proof. By Lemma 4.2, we may assume that G is torsion free. By Theorem 3.1, it suffices to prove the result when G is one ended. Let T_G be the JSJ tree associated to G described above. By [33, Lemma 3.2], we may pass to a power of Φ which fixes the conjugacy class of each group in the peripheral system $\mathcal{P}$. That is, up to finite index (applying Proposition 2.13 (1)), we can assume ϕ is an element of $\mathrm{Aut}(G; \mathcal{P})$.

Since T_G is preserved by $\mathrm{Aut}(G; \mathcal{P})$, the action of G and ϕ on T_G induces an action of $G \rtimes_\phi \mathbb{Z}$ on T_G. Up to taking a power of ϕ, we may suppose that $\phi \in \mathcal{K}(T_G)$.

We prove Theorem 4.3 by applying Theorem 2.14 to the action of $G \rtimes_\phi \mathbb{Z}$ on T_G. As in the proof of Theorem 3.1, it suffices to prove that, for every cell $\omega \in T_G$, the group $\mathrm{Stab}(\omega)$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.

Edge stabilisers in $G \rtimes_\phi \mathbb{Z}$ are virtually polycyclic and infinite since $G \rtimes_\phi \mathbb{Z}$ is one ended. Thus, by Proposition 2.13 (4), they have the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.

Let $v \in T_G$. There are three cases for the vertex stabilisers.

Case 1. The stabiliser $\mathrm{Stab}_G(v)$ is the fundamental group of a compact hyperbolic surface S and $\mathcal{K}(T_G) \to \mathrm{Out}(\mathrm{Stab}_G(v))$ has image contained in $\mathrm{MCG}(S)$.

Since ϕ is polynomially growing, its image in $\mathrm{MCG}(S)$ is, in fact, a Dehn multitwist D. Thus, the stabiliser in $G \rtimes_\phi \mathbb{Z}$ of v is isomorphic to $\pi_1(S) \rtimes_D \mathbb{Z}$. By Lemma 4.1, it has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.   ◆

Case 2. The stabiliser $\mathrm{Stab}_G(v)$ is virtually polycyclic.

Here the stabiliser of v in $G \rtimes_\phi \mathbb{Z}$ is virtually polycyclic-by-$\mathbb{Z}$ (hence, polycyclic). Thus, it has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$ by Proposition 2.13 (4).   ◆

Case 3. The image of the natural homomorphism $\mathcal{K}(T_G) \to \mathrm{Out}(\mathrm{Stab}_G(v))$ is finite.

Up to taking a power of ϕ, we may suppose that the stabiliser of v in $G \rtimes_\phi \mathbb{Z}$ is isomorphic to $\mathrm{Stab}_G(v) \times \mathbb{Z}$. By, for instance, [22, Lemma 3.8], the group $\mathrm{Stab}_G(v)$ belongs to $\mathcal{G}$. By results of Dahmani [12, Theorem 0.1], the group $\mathrm{Stab}_G(v)$ is of type F_∞. By Proposition 2.13 (2), for every $\alpha \in \mathbb{N}$, the group $\mathrm{Stab}_G(v) \times \mathbb{Z}$ has the cheap α-rebuilding property.   ◆

Thus, Theorem 2.14 implies that $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.   ∎

Remark 4.4. The restriction to virtually polycyclic subgroups is also used in Case 3; to deduce that $\mathrm{Stab}_G(v)$ is of type F_∞ requires that arbitrary subgroups in $\mathcal{P}$ themselves are of finite type.

5. Right-angled Artin and Coxeter groups

Let L be a flag complex and recall that A_L and W_L, respectively, denote the *right-angled Artin* and *right-angled Coxeter groups* on L. The following maps induce automorphisms of both the right-angled Artin group $A(L)$ and the right-angled Coxeter group $W(L)$:

(1) *graph automorphisms*, which are the automorphisms induced by automorphisms of L;

(2) *inversions*, which send $v \mapsto v^{-1}$ and $u \mapsto u$ for $u \neq v$ and $u, v \in L^{(0)}$;

(3) *partial conjugations*, labelled by $k_{w,C}$ for $w \in L^{(0)}$ and a connected component C of $L \backslash \mathrm{st}(w)$, and defined as $k_{w,C}(u) = w^{-1}uw$ if $u \in C^{(0)}$ and $k_{w,C}(u) = u$ if $u \in L^{(0)} \backslash C$; and

(4) *folds*, labelled by $\tau_{v,w}$ for any $v, w \in L^{(0)}$ with $\mathrm{lk}\, v \subseteq \mathrm{lk}\, w$, and defined by $\tau_{v,w}(v) = vw$ and $\tau_{v,w}(u) = u$ for all $u \in L^{(0)} \backslash \{v\}$.

We say an automorphism of A_L is *untwisted* if it is contained in the subgroup $U(L) \leq \mathrm{Aut}(A_L)$ which is generated by graph automorphisms, inversions, partial conjugations and folds. We define the subgroup of untwisted automorphisms of the right-angled Coxeter groups analogously. Note that by [13, Proposition A (3)], untwisted automorphisms of A_L are exactly the automorphisms which preserve the standard coarse median structure on A_L. We will not use this fact or any results about coarse medians explicitly, but we note that it underpins much of our work in this section.

Theorem 5.1. *Let L be a flag complex on $[m]$ and let $\Gamma = A_L \rtimes_\phi \mathbb{Z}$. If ϕ is an untwisted and polynomially growing automorphism of A_L, then Γ has the cheap α-rebuilding property for all α.*

Proof. We proceed by induction on m, the number of vertices of L. When $m = 1$, we have that A_L is isomorphic to $\mathbb{Z}$. In this case, Γ is virtually $\mathbb{Z}^2$ and so the result follows from Proposition 2.13. We now suppose $m > 1$. Note that if $K \subset L$ is a full subcomplex, then any untwisted automorphism of A_L preserving A_K restricts to an untwisted automorphism of A_K. To prove the inductive step, there are three cases to consider.

Case 1. The factor A_L is freely reducible. In this case, A_L admits a Grushko splitting $A_{K_1} * \cdots * A_{K_k} * F_n$, where each K_i and $[n]$ is a full subcomplex of L. In particular, each subcomplex K_i contains at least one vertex but strictly less than m vertices. We replace ϕ by a sufficiently high power which preserves the conjugacy class of every factor A_{K_i}. By the inductive hypothesis, each $A_{K_i} \rtimes_{\phi|_{A_{K_i}}} \mathbb{Z}$ has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$. The conclusion follows from Theorem 3.1. ◆

Case 2. The factor A_L is freely and directly irreducible. By [13, Proposition D], the group A_L splits as an amalgamated free product $A_J *_{A_{J \cap K}} A_K$ with each $J, K, J \cap K \subset L$ non-empty and such that, possibly after replacing ϕ by a high enough power, the corresponding Bass–Serre tree T is ϕ-invariant. Moreover, for $X \in \{J, K, J \cap K\}$, we see $\phi(A_X) = A_X$ (see [13, Lemma 5.3]). It follows that Γ acts on T with stabilisers conjugate to $A_X \rtimes_{\phi|_{A_X}} \mathbb{Z}$. By the inductive hypothesis, the subgroups $A_X \rtimes_{\phi|_{A_X}} \mathbb{Z}$ satisfy the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$. The conclusion follows from Theorem 2.14.   ◆

Case 3. The factor A_L is directly reducible.

Now, A_L splits as a direct product $\prod_i A_{K_i} \times \mathbb{Z}^k$ for some $k \geq 0$, where each A_{K_i} is non-cyclic and directly irreducible. If $k > 0$, then since $\prod_i A_{K_i}$ is of type F_α for every $\alpha \in \mathbb{N}$, we have that A_L has the cheap α-rebuilding property for every α by Proposition 2.13 (2). Another application of Proposition 2.13 (2) gives us that the mapping torus $A_L \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$ and $\phi \in \mathrm{Aut}(A_L)$.

Hence, we may assume that $k = 0$. Thus, A_L acts on the product $X = \prod_i T_i$ where each T_i is a tree which arises from Proposition 2.11 if A_{K_i} is freely reducible, and from the amalgamated product splitting given by [13, Proposition D] as in Case 2 when A_{K_i} is both directly and freely irreducible.

As in the previous two cases, up to passing to a power, we may assume that ϕ fixes the quotient $A_L \backslash X$ pointwise, and that $\phi|_{A_{K_i}}$ preserves the stabilisers of A_{K_i} acting on T_i for each i. Since the stabilisers of A_L acting on X are products of the stabilisers of the A_{K_i} acting on T_i, we see that $\phi(\mathrm{Stab}_{A_L}(\sigma)) = \mathrm{Stab}_{A_L}(\sigma)$ for each cell $\sigma \in X$. Thus, $A_L \rtimes_\phi \mathbb{Z}$ acts on X with the stabiliser of a cell σ isomorphic to $A_{J_\sigma} \rtimes_{\phi|_{A_{J_\sigma}}} \mathbb{Z}$ for some RAAG A_{J_σ} where J_σ is non-empty and has strictly less vertices than L. The conclusion follows from Theorem 2.14.   ◆

This completes the proof of the inductive step and the theorem.   ∎

Theorem 5.2. *Let L be a flag complex on $[m]$ and let $\Gamma = W_L \rtimes_\phi \mathbb{Z}$. If ϕ is polynomially growing, then Γ has the cheap α-rebuilding property for all α.*

Proof. The proof is entirely analogous to Theorem 5.1 with the following modifications. First, we note that the subgroup of untwisted automorphisms of W_L has finite index in the automorphism group $\mathrm{Aut}(W_L)$ by [35]. Thus, we may pass to a power of ϕ which is untwisted.

In the case where $m = 1$, we have that $\Gamma = \mathbb{Z}/2 \times \mathbb{Z}$ which has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$ by Proposition 2.13. The three cases are now identical, taking into account the remarks after Theorem E and at the start of Section 5 of [13], since the results we use for A_L also hold for W_L.   ∎

Remark 5.3. We can actually say more regarding the cheap α-rebuilding property for mapping tori of automorphisms of RAAGs. Indeed, if L is $(\alpha - 1)$-connected, then A_L has the cheap α-rebuilding property [1, Theorem I]. In particular, if L is contractible, then

for any automorphism ϕ of A_L, the group $A_L \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for all α.

References

[1] M. Abert, N. Bergeron, M. Frączyk, and D. Gaboriau, On homology torsion growth. *J. Eur. Math. Soc. (JEMS)* **27** (2025), no. 6, 2293–2357 MR 4889244

[2] N. Andrew, Serre's property (FA) for automorphism groups of free products. *J. Group Theory* **24** (2021), no. 2, 385–414 Zbl 1498.20081 MR 4223849

[3] N. Andrew, S. Hughes, and M. Kudlinska, Torsion homology growth of polynomially growing free-by-cyclic groups. *Rocky Mountain J. Math.* **54** (2024), no. 4, 933–941 Zbl 07941281 MR 4790461

[4] G. Avramidi, B. Okun, and K. Schreve, Mod p and torsion homology growth in nonpositive curvature. *Invent. Math.* **226** (2021), no. 3, 711–723 Zbl 1547.20136 MR 4337971

[5] U. Bader, T. Gelander, and R. Sauer, Homology and homotopy complexity in negative curvature. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 8, 2537–2571 Zbl 1448.57032 MR 4118615

[6] G. Baumslag, F. B. Cannonito, D. J. S. Robinson, and D. Segal, The algorithmic theory of polycyclic-by-finite groups. *J. Algebra* **142** (1991), no. 1, 118–149 Zbl 0774.20019 MR 1125209

[7] N. Bergeron and A. Venkatesh, The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu* **12** (2013), no. 2, 391–447 Zbl 1266.22013 MR 3028790

[8] M. Bestvina and M. Feighn, Hyperbolicity of the complex of free factors. *Adv. Math.* **256** (2014), 104–155 Zbl 1348.20028 MR 3177291

[9] M. Clay, ℓ^2-torsion of free-by-cyclic groups. *Q. J. Math.* **68** (2017), no. 2, 617–634 Zbl 1394.20021 MR 3667215

[10] R. Coulon, Examples of groups whose automorphisms have exotic growth. *Algebr. Geom. Topol.* **22** (2022), no. 4, 1497–1510 Zbl 07611763 MR 4495664

[11] M. Culler and J. W. Morgan, Group actions on $\mathbb{R}$-trees. *Proc. Lond. Math. Soc. (3)* **55** (1987), no. 3, 571–604 Zbl 0658.20021 MR 0907233

[12] F. Dahmani, Classifying spaces and boundaries for relatively hyperbolic groups. *Proc. Lond. Math. Soc. (3)* **86** (2003), no. 3, 666–684 Zbl 1031.20039 MR 1974394

[13] E. Fioravanti, Coarse-median preserving automorphisms. *Geom. Topol.* **28** (2024), no. 1, 161–266 Zbl 1535.20211 MR 4711836

[14] S. P. Fisher, S. Hughes, and I. J. Leary, Homological growth of Artin kernels in positive characteristic. *Math. Ann.* **389** (2024), no. 1, 819–843 Zbl 1540.20101 MR 4735963

[15] M. Forester, Deformation and rigidity of simplicial group actions on trees. *Geom. Topol.* **6** (2002), no. 1, 219–267 Zbl 1118.20028 MR 1914569

[16] S. Francaviglia and A. Martino, Stretching factors, metrics and train tracks for free products. *Illinois J. Math.* **59** (2015), no. 4, 859–899 Zbl 1382.20031 MR 3628293

[17] S. Francaviglia, A. Martino, and D. Syrigos, The minimally displaced set of an irreducible automorphism of F_N is co-compact. *Arch. Math. (Basel)* **116** (2021), no. 4, 369–383 Zbl 1482.20015 MR 4244237

[18] S. Francaviglia, A. Martino, and D. Syrigos, On the action of relatively irreducible automorphisms on their train tracks. [v1] 2021, [v2] 2024, arXiv:2108.01680v2, to appear in *Ann. Inst. Fourier (Grenoble)*

[19] V. Guirardel and C. Horbez, Algebraic laminations for free products and arational trees. *Algebr. Geom. Topol.* **19** (2019), no. 5, 2283–2400 Zbl 1456.20020 MR 4023318

[20] V. Guirardel and C. Horbez, Boundaries of relative factor graphs and subgroup classification for automorphisms of free products. *Geom. Topol.* **26** (2022), no. 1, 71–126 Zbl 07525898 MR 4404875

[21] V. Guirardel and G. Levitt, The outer space of a free product. *Proc. Lond. Math. Soc. (3)* **94** (2007), no. 3, 695–714 Zbl 1168.20011 MR 2325317

[22] V. Guirardel and G. Levitt, Splittings and automorphisms of relatively hyperbolic groups. *Groups Geom. Dyn.* **9** (2015), no. 2, 599–663 Zbl 1368.20037 MR 3356977

[23] V. Guirardel and G. Levitt, *JSJ decompositions of groups*. Astérisque 395, Société Mathématique de France (SMF), Paris, 2017, 165 pp. Zbl 1391.20002 MR 3758992

[24] M. Handel and L. Mosher, Relative free splitting and free factor complexes I: hyperbolicity. [v1] 2014, [v3] 2025, arXiv:1407.3508v3

[25] A. Kar, P. Kropholler, and N. Nikolov, On growth of homology torsion in amenable groups. *Math. Proc. Cambridge Philos. Soc.* **162** (2017), no. 2, 337–351 Zbl 1383.20035 MR 3604918

[26] G. Levitt, Characterizing rigid simplicial actions on trees. In J. Burillo, S. Cleary, M. Elder, J. Taback, and E. Ventura (eds.), *Geometric methods in group theory*, pp. 27–33, Contemp. Math. 372, American Mathematical Society, Providence, RI, 2005 Zbl 1079.20040 MR 2139674

[27] G. Levitt, Counting growth types of automorphisms of free groups. *Geom. Funct. Anal.* **19** (2009), no. 4, 1119–1146 Zbl 1196.20038 MR 2570318

[28] H. Li and A. Thom, Entropy, determinants, and L^2-torsion. *J. Amer. Math. Soc.* **27** (2014), no. 1, 239–292 Zbl 1283.37031 MR 3110799

[29] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* **4** (1994), no. 4, 455–481 Zbl 0853.57021 MR 1280122

[30] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*. Ergeb. Math. Grenzgeb. (3) 44, Springer, Heidelberg, 2002 Zbl 1009.55001 MR 1926649

[31] W. Lück, Approximating L^2-invariants and homology growth. *Geom. Funct. Anal.* **23** (2013), no. 2, 622–663 Zbl 1273.22009 MR 3053758

[32] R. A. Lyman, Train track maps on graphs of groups. *Groups Geom. Dyn.* **16** (2022), no. 4, 1389–1422 Zbl 1527.20035 MR 4536434

[33] A. Minasyan and D. Osin, Fixed subgroups of automorphisms of relatively hyperbolic groups. *Q. J. Math.* **63** (2012), no. 3, 695–712 Zbl 1268.20046 MR 2967171

[34] B. Okun and K. Schreve, Torsion invariants of complexes of groups. *Duke Math. J.* **173** (2024), no. 2, 391–418 Zbl 1537.20098 MR 4728693

[35] A. Sale and T. Susse, Outer automorphism groups of right-angled Coxeter groups are either large or virtually abelian. *Trans. Amer. Math. Soc.* **372** (2019), no. 11, 7785–7803 Zbl 1442.20025 MR 4029681

[36] R. Sauer, Volume and homology growth of aspherical manifolds. *Geom. Topol.* **20** (2016), no. 2, 1035–1059 Zbl 1338.53067 MR 3493098

Naomi Andrew
Mathematical Institute, University of Oxford, Andrew Wiles Building, Observatory Quarter, Oxford OX2 6GG, UK; naomi.andrew@maths.ox.ac.uk

Yassine Guerch
Laboratoire de Mathématiques Nicolas Oresme, CNRS, UMR 6139, University of Caen Normandy, Esplanade de la Paix, CS 14032 Caen Cedex 5, France; yassine.guerch@unicaen.fr

Sam Hughes
Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany; hughes@math.uni-bonn.de, sam.hughes.maths@gmail.com

Monika Kudlinska
Emmanuel College, St Andrew's Street, Cambridge CB2 3AP, UK;
m.kudlinska@dpmms.cam.ac.uk