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Free metabelian groups are permutation stable

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Abstract. We prove that all finitely generated free metabelian groups are permutation stable by verifying that all invariant random subgroups of them are co-sofic. This partially answers to the question asked by Levit and Lubotzky whether all finitely generated metabelian groups are permutation stable. Our proof extends the range of application of Levit and Lubotzky's method, which is used to show permutation stability of permutational wreath products of finitely generated abelian groups, to non-split and non-permutational metabelian groups.

1. Introduction

Permutation stability of groups gets attention in recent years, and many groups have been found to be permutation stable. The aim of this paper is to give a new example of such groups.

Definition 1.1. Let $\mathrm{Sym}(n)$ be the symmetric group of degree n. The *Hamming metric* d_{H} on $\mathrm{Sym}(n)$ is defined by $d_{\mathrm{H}}(\sigma,\tau) = \frac{1}{n}|\{k \in \{1,\dots,n\} \mid \sigma(k) \neq \tau(k)\}|$.

Definition 1.2. Let G be a countable group. An *almost homomorphism* is a sequence $(\phi_n : G \to \mathrm{Sym}(n))_n$ of maps such that $d_{\mathrm{H}}(\phi_n(gh), \phi_n(g)\phi_n(h)) \to 0$ $(n \to \infty)$ for all $g, h \in G$.

We call G *permutation stable* (or *P-stable* for short) if for every almost homomorphism $(\phi_n : G \to \mathrm{Sym}(n))_n$, there exists a sequence of homomorphisms $(\psi_n : G \to \mathrm{Sym}(n))_n$ such that $d_{\mathrm{H}}(\phi_n(g), \psi_n(g)) \to 0$ $(n \to \infty)$ for every $g \in G$.

Let F be a free group and let F'' be its second derived subgroup. The group F/F'' is called a *free metabelian group*. Our main theorem is the following.

Theorem 1.3. *All finitely generated free metabelian groups are P-stable.*

By viewing a countably generated free metabelian group as a suitable limit of finitely generated metabelian groups, one can remove the assumption of finite generation (see Section 6).

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Corollary 1.4. *All countable free metabelian groups are P-stable.*

Historically, it is a classical question in operator theory whether "almost commuting" matrices are "near" commuting matrices. For example, let $U(n)$ be the unitary group of degree n and $\|\cdot\|$ a norm on the set of $n \times n$ matrices. If $A_n, B_n \in U(n)$ with $\|A_n B_n - B_n A_n\| \to 0$ $(n \to \infty)$, then do there exist $A'_n, B'_n \in U(n)$ such that $A'_n B'_n = B'_n A'_n$ and $\|A_n - A'_n\| + \|B_n - B'_n\| \to 0$ $(n \to \infty)$? This asks the stability of the relation $xy = yx$ with respect to the unitary matrices and the norm. The answer to this question is "No" for the operator norm [13], but "Yes" for the normalized Hilbert–Schmidt norm [7]. P-stability is a discrete version of this stability, that is, it considers permutations instead of matrices. Indeed, if $(U(n), \|\cdot\|)$ is replaced by $(\mathrm{Sym}(n), d_{\mathrm{H}})$, then the question asks whether the group $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$ is P-stable.

The first examples of P-stable groups are found in [8], where all finite groups are shown to be P-stable. In [3], all finitely generated abelian groups (e.g., $\mathbb{Z}^2$) are shown to be P-stable.

Recently, great progress was made by [5], which relates P-stability with invariant random subgroups.

Theorem 1.5 ([5, Theorem 1.3]). *Let G be a finitely generated amenable group. Then G is P-stable if and only if every invariant random subgroup of G is co-sofic.*

We refer to Definition 2.1 for the terminology. Invariant random subgroups are a notion generalizing normal subgroups and lattices simultaneously, introduced by [1]. Co-soficity is introduced by [6]. Theorem 1.5 is applied to finding a lot of P-stable groups, such as virtually polycyclic groups, the Baumslag–Solitar groups $\mathrm{BS}(1, n)$ for all $n \in \mathbb{Z}$ [5], the Grigorchuk group [14], and other uncountably many groups [10].

However, characterizing co-soficity of arbitrary invariant random subgroups is still difficult even for solvable groups. Levit and Lubotzky [9] give the following P-stable examples of finitely generated metabelian groups by combining Theorem 1.5 with the pointwise ergodic theorem.

Theorem 1.6 ([9, Theorem 1.4]). *Let A and Q be finitely generated abelian groups, and X a set on which Q acts with only finitely many orbits. Then the semidirect product $Q \ltimes \bigoplus_X A$ induced from the action of Q on X is P-stable.*

In [9, Section 1.5], it is asked whether all finitely generated metabelian groups are P-stable. Our Theorem 1.3 answers to this question affirmatively for all free metabelian groups that are basic examples of non-split metabelian groups.

Remark 1.7. Let A, Q, X be as in Theorem 1.6. We regard $N = \bigoplus_X A$ as a $\mathbb{Z}[Q]$-module. An extension of Q by N such that the conjugation on N induces the given $\mathbb{Z}[Q]$-structure is called a *permutational metabelian group* in [9]. They conjecture that all permutational metabelian groups are P-stable. The group F_2/F_2'' is permutational metabelian, but F_d/F_d'' is not if $d \geq 3$ (see Theorem 4.4).

Remark 1.8. Finitely generated free metabelian groups are already shown to be *Hilbert–Schmidt stable*, that is, they are stable with respect to the unitary groups and the normalized Hilbert–Schmidt norm [11].

Our proof of Theorem 1.3 basically follows the method of [9]. The main part is the construction of a Følner sequence with special properties (see Section 2.1). However, there are two differences between the setting of [9] and ours. First free metabelian groups are not split metabelian, that is, there is no natural lift of the quotient group $Q = F_d/F_d'$ to F_d/F_d''. Hence, we have to choose a lift and work with it. Lemma 5.1 is an elementary but useful observation. The second difference is that if $d \geq 3$, then F_d/F_d'' is not permutational metabelian, and this makes the $\mathbb{Z}[Q]$-structure on the commutator subgroup $N = F_d'/F_d''$ complicated. To apply the method of [9], we need to find a sequence of subgroups $M_n \leq N$ indicated in Theorem 2.16, and this is the most non-trivial part. Proposition 4.14 is important to find such subgroups.

As a final remark, P-stability of free solvable groups of derived length greater than 2 (i.e., $F/F^{(k)}$, where F is a free group and $F^{(k)}$ is its k-th derived subgroup with $k \geq 3$) is not known. The proof for free metabelian groups essentially relies on Hall's theorem that all finitely generated metabelian groups are residually finite. Indeed, this is used to show [9, Proposition 5.2] (Proposition 2.11). This does not hold for free solvable groups of derived length greater than 2, so we do not know even whether any Dirac IRS on a normal subgroup is co-sofic. These groups are known to be not locally extended residually finite (LERF) [2], that is, they have a subgroup which is not a limit of finite index subgroups (in the sense of Definition 2.1), but the subgroup constructed in [2] is not normal.

Organization of the paper. In Section 2, we review the terminology and tools established by [9]. In Section 3, we consider the division in the ring of Laurent polynomials with finitely many variables and coefficients in $\mathbb{Z}$. This is preparation for Section 4, in which we investigate the commutator subgroup of free metabelian groups. Then we prove Theorem 1.3 in Section 5, and prove Corollary 1.4 in Section 6.

2. Levit and Lubotzky's method

2.1. Weiss approximation

Let G be a countable group in this subsection.

Definition 2.1 ([6, Definition 15]). Let $\mathrm{Sub}(G)$ be the set of subgroups of G, and regard it as a closed subspace of 2^G with respect to the product topology. We endow the set $C(\mathrm{Sub}(G))^*$ with the weak* topology. An *invariant random subgroup* (or *IRS* for short) of G is a Borel probability measure on $\mathrm{Sub}(G)$ invariant under the conjugation by G. A *finite index* IRS of G is an IRS of G supported on the set of finite index subgroups of G. A *co-sofic* IRS of G is a weak*-limit of finite index IRSs of G.

Definition 2.2 ([9, Section 3]). Let $H \le G$ be a subgroup.

(i) Let $\delta_H \in C(\mathrm{Sub}(G))^*$ denote the Dirac measure on H. For a non-empty finite subset $F \subset G$, set $F * H = \frac{1}{|F|}\sum_{g\in F}\delta_{gHg^{-1}} \in C(\mathrm{Sub}(G))^*$.

(ii) A subset $F \subset G$ is a *transversal* for H in G if $G = \bigsqcup_{g\in F} gH$. Also, F is a *finite-to-one transversal* for H in G if F is a disjoint union of finitely many transversals for H in G.

Notation 2.3. Let $H, K \le G$ be a subgroup. Let $\mathrm{N}_G(H)$ denote the normalizer of H, that is, $\mathrm{N}_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Also set $\mathrm{N}_K(H) = \mathrm{N}_G(H) \cap K$.

Definition 2.4 ([9, Section 3]). Let $H \le G$ be a subgroup. Let $F_n \subset G$ be a sequence of non-empty finite subsets and $K_n \le G$ a sequence of finite index subgroups. The sequence of pairs (K_n, F_n) is a *Weiss approximation* of H if F_n is a finite-to-one transversal for $\mathrm{N}_G(K_n)$ in G for every n and $F_n * H - F_n * K_n \to 0$ in $C(\mathrm{Sub}(G))^*$.

Theorem 2.5 ([9, Theorem 3.10]). *Let μ be an IRS of an amenable group G and (F_n) a Følner sequence of G. Suppose for μ-a.e. $H \in \mathrm{Sub}(G)$, there exists a sequence of finite index subgroups $K_n \le G$ such that the sequence of pairs (K_n, F_n) is a Weiss approximation of H. Then μ is a co-sofic IRS of G.*

Proposition 2.6 ([9, Proposition 5.3]). *Let G be a finitely generated group. Suppose a normal subgroup $N \trianglelefteq G$ and the quotient group G/N are abelian. Let μ be an IRS of G. Then for μ-a.e. $H \in \mathrm{Sub}(G)$, $[N : \mathrm{N}_N(H)] < \infty$ holds.*

The following corollary is used in [9, Section 11] to prove their main theorem. We state it explicitly to make our argument simple, and give its proof for the reader's convenience.

Corollary 2.7 ([9]). *Let G be a finitely generated group. Suppose $N \trianglelefteq G$ and $Q = G/N$ are abelian groups. Also suppose that every subgroup $R \le Q$ admits a Følner sequence (F_n) of G satisfying the following property $(\#_R)$:*

($\#_R$) *If $H \le G$ is a subgroup with $HN/N = R$ and $[N : \mathrm{N}_N(H)] < \infty$, then there exists a sequence of finite index subgroups $K_n \le G$ such that the sequence of pairs (K_n, F_n) is a Weiss approximation of H.*

Then G is P-stable.

Proof. By Theorem 1.5, it suffices to show that every IRS of G is co-sofic. Let μ be an IRS of G. By the Krein–Milman theorem, μ is a weak*-limit of convex combinations of ergodic IRSs of G. Thus, we may assume that μ is ergodic. Since the map $H \in \mathrm{Sub}(G) \mapsto HN/N \in \mathrm{Sub}(Q)$ is G-invariant, it is constant on a μ-conull set. Let $R \in \mathrm{Sub}(Q)$ be the constant. By Proposition 2.6, the set

$$A := \{H \in \mathrm{Sub}(G) \mid HN/N = R \text{ and } [N : \mathrm{N}_N(H)] < \infty\}$$

is μ-conull. By property ($\#_R$), for every $H \in A$, there exists a sequence of finite index subgroups $K_n \leq G$ such that (K_n, F_n) is a Weiss approximation of H. By Theorem 2.5, μ is co-sofic. ■

2.2. A sufficient condition to be a Weiss approximation

Let G be a group and $N \trianglelefteq G$ a normal subgroup. Set $Q = G/N$.

Notation 2.8 ([9, Section 4.1]). For a subgroup $H \leq G$, we set

- $Q_H = HN/N \leq Q$,
- $N_H = N \cap H$, and
- let $\alpha_H : Q_H \to H/N_H$ be the natural isomorphism.

We denote $[H] = [Q_H, N_H, \alpha_H]$. Note that $\overline{\alpha_H(q)} = q$ for every $q \in Q_H$, where $\xi \in H/N_H \mapsto \overline{\xi} \in G/N$ denotes the natural quotient.

Proposition 2.9 ([9, Proposition 4.1]). *Let $R \leq Q$, $M \leq N$ be subgroups and $\alpha : R \to \mathrm{N}_G(M)/M$ a homomorphism satisfying $\overline{\alpha(q)} = q$ for every $q \in R$. Then there exists a unique subgroup $H \leq G$ such that $[H] = [R, M, \alpha]$.*

Let $q \in Q \mapsto \widehat{q} \in G$ be a (set-theoretic) section of the quotient map. For a subset $I \subset Q$, we set $\widehat{I} = \{\widehat{q} \in G \mid q \in I\}$.

Proposition 2.10 ([9, Proposition 4.3]). *Let $H \leq G$ be a subgroup and let $I \subset Q$ and $T \subset N$ be transversals for Q_H in Q and N_H in N, respectively. Then $\widehat{I}T$ is a transversal for H in G.*

Assume that G is finitely generated, and Q and N are abelian in the rest of this section. We use additive notation for the group operation in N (and use multiplicative notation for Q). The action of Q on N is induced from the conjugation by G.

Proposition 2.11 ([9, Proposition 5.2]). *Let $H \leq G$ be a subgroup such that HN is finitely generated. Then for any finite subset $T \subset N$, there exists a finite index subgroup $M \leq N$ such that $N_H \cap T = M \cap T$, $N_H \leq M$, and M is Q_H-invariant.*

Definition 2.12. Let Γ and Λ be groups and $\Delta, \Delta_n \leq \Lambda$ subgroups for $n \in \mathbb{N}$. Following [9, Definition 6.1], we say that a sequence of homomorphisms $\phi_n : \Gamma \to \mathrm{N}_\Lambda(\Delta_n)/\Delta_n$ is *consistent* with a homomorphism $\phi : \Gamma \to \mathrm{N}_\Lambda(\Delta)/\Delta$ if for every $\gamma \in \Gamma$, there exists $\lambda \in \Lambda$ such that $\phi_n(\gamma) = \lambda\Delta_n$ and $\phi(\gamma) = \lambda\Delta$ for every n.

Proposition 2.13 ([9, Corollary 6.3]). *Let $H \leq G$ be a subgroup and $N_n \leq N$ be a sequence of Q_H-invariant subgroups such that $N_n \to N_H$ in* Sub(N). *Then there exist homomorphisms $\alpha_n : Q_H \to \mathrm{N}_G(N_n)/N_n$ defined for all n sufficiently large such that the sequence α_n is consistent with α_H.*

Definition 2.14 ([9, Definition 7.2]). Let $T_n \subset N$ and $F_n \subset G$ be sequences of non-empty finite subsets. The sequence T_n is *adapted* to the sequence F_n if for every $g \in G$ and every finite subset $\Phi \subset N$,

$$\frac{|\{h \in F_n \mid [g,h] + \Phi \subset T_n\}|}{|F_n|} \to 1 \quad (n \to \infty). \tag{2.1}$$

Remark 2.15. By taking g to be the identity of G, convergence (2.1) implies $\Phi \subset T_n$ for every n sufficiently large. Hence, if T_n is adapted to some F_n, then $\bigcup_n \bigcap_{k \geq n} T_k = N$ holds.

Theorem 2.16 ([9, Theorem 7.5]). *Let $H \leq G$ be a subgroup with $[N : \mathrm{N}_N(H)] < \infty$. For every $n \in \mathbb{N}$, let $K_n \leq G$ be a finite index subgroup with $[K_n] = [Q_n, N_n, \alpha_n]$, $M_n \leq N$ a subgroup, and $I_n \subset Q$ and $T_n \subset M_n$ non-empty finite subsets. Assume that they satisfy the following conditions:*

- (i) *$Q_H \leq Q_n$ and $Q_n \to Q_H$ in* $\mathrm{Sub}(Q)$.
- (ii) *M_n is Q_H-invariant.*
- (iii) *The sequence $\alpha_n|_{Q_H}$ is consistent with α_H.*
- (iv) *$N_H \cap T_n = N_n \cap T_n$ and $N_H \cap M_n \leq N_n$.*
- (v) *The sequence T_n is adapted to the sequence $\widehat{I_n}$.*

Then for any sequence $P_n \subset M_n$ of non-empty finite subsets, we have

$$(\widehat{I_n} P_n) * H - (\widehat{I_n} P_n) * K_n \to 0 \quad \text{in } C(\mathrm{Sub}(G))^*.$$

Remark 2.17. The original statement of [9, Theorem 7.5] is different from the above. They assume the following two conditions (see [9, Definitions 7.1 and 7.3]):

- (a) (K_n, M_n, T_n) is a *controlled approximation* of H.
- (b) $\widehat{I_n} P_n$ is *adapted* to (K_n, M_n, T_n).

Condition (a) claims conditions (i)–(iv) and $\bigcup_n \bigcap_{k \geq n} T_k = N$ holds. The last condition follows from condition (v) by Remark 2.15. Condition (b) is stronger than condition (v), but they only use condition (v) for the proof of this theorem.

3. Division of Laurent polynomials

Throughout the paper, we mean by an interval $I \subset \mathbb{R}$ the intersection $I \cap \mathbb{Z}$ if there is no cause of confusion. Let $\mathbb{N}$ be the set of positive integers. Let $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ be the ceiling function, and $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ the floor function.

Definition 3.1. Let $\phi \in \mathbb{Z}[s^{\pm 1}]$ be a nonzero polynomial. Let m be the smallest degree of s in ϕ and let n be the non-negative integer such that $m + n$ is the largest degree of s

in ϕ. We call the integer n the *degree* of ϕ. We say that ϕ is *monic* if each of the coefficients of s^m and s^{m+n} belongs to $\{\pm 1\}$.

For $n \in \mathbb{N}$, set

$$M(n; s) = \operatorname{span}_{\mathbb{Z}}\{s^m \mid -\lceil n/2 \rceil < m \le \lfloor n/2 \rfloor\} \subset \mathbb{Z}[s^{\pm 1}].$$

Lemma 3.2. *Let $\phi \in \mathbb{Z}[s^{\pm 1}]$ be a monic polynomial with degree $n > 0$. Then for every $\psi \in \mathbb{Z}[s^{\pm 1}]$, there exists $\theta \in \mathbb{Z}[s^{\pm 1}]$ such that $\psi - \theta\phi \in M(n; s)$.*

Proof. Let m be the smallest degree of s in ϕ. Since $M(n; s)$ is closed under linear combination with coefficients in $\mathbb{Z}$, we may assume $\psi = s^k$ with $k \in \mathbb{Z}$. Suppose $k > 0$. We may assume that the coefficient of s^{n+m} in ϕ is 1. Since $s^{-m}\phi \in \mathbb{Z}[s]$ is a monic polynomial of s with degree n in the usual sense, there exists $\theta \in \mathbb{Z}[s]$ such that the polynomial

$$s^{k+\lceil n/2 \rceil - 1} - \theta \cdot s^{-m}\phi \in \mathbb{Z}[s]$$

of s has degree less than n. Then by multiplying $s^{1-\lceil n/2 \rceil}$ to this polynomial, we have

$$s^k - s^{-m+1-\lceil n/2 \rceil}\theta \cdot \phi \in M(n; s)$$

as desired. Next suppose $k \le 0$. We may assume that the coefficient of s^m in ϕ is 1. Since $s^{-n-m}\phi \in \mathbb{Z}[s^{-1}]$ is a monic polynomial of s^{-1} with degree n in the usual sense, there exists $\theta \in \mathbb{Z}[s^{-1}]$ such that the polynomial

$$s^{k-\lfloor n/2 \rfloor} - \theta \cdot s^{-n-m}\phi \in \mathbb{Z}[s^{-1}]$$

of s^{-1} has degree less than n. Then by multiplying $s^{\lfloor n/2 \rfloor}$ to this polynomial, we have

$$s^k - s^{-n-m+\lfloor n/2 \rfloor}\theta \cdot \phi \in M(n; s)$$

as desired. ∎

Let $d \in \mathbb{N}$. Now we consider the ring $\mathbb{Z}[s_1^{\pm 1}, \dots, s_d^{\pm 1}]$. For $n_1, \dots, n_d \in \mathbb{N} \cup \{\infty\}$, set

$$\begin{aligned} M(n_1, \dots, n_d; s_1, \dots, s_d) = \operatorname{span}_{\mathbb{Z}} &\left\{s_1^{m_1} \cdots s_d^{m_d} \mid m_i \in (-\lceil n_i/2 \rceil, \lfloor n_i/2 \rfloor] \text{ for every } i\right\} \\ &\subset \mathbb{Z}\big[s_1^{\pm 1}, \dots, s_d^{\pm 1}\big], \end{aligned}$$

where $(-\lceil \infty/2 \rceil, \lfloor \infty/2 \rfloor]$ means $(-\infty, \infty)$.

Proposition 3.3. *Let $1 \le c \le d$. For every $1 \le i \le c$, let $\phi_i \in \mathbb{Z}[s_i^{\pm 1}]$ be a monic polynomial with degree $n_i > 0$. Then for every $\psi \in \mathbb{Z}[s_1^{\pm 1}, \dots, s_d^{\pm 1}]$, there exist $\theta_i \in \mathbb{Z}[s_1^{\pm 1}, \dots, s_d^{\pm 1}]$ for $1 \le i \le c$ and $\lambda \in M(n_1, \dots, n_c, \infty, \dots, \infty; s_1, \dots, s_d)$ such that*

$$\psi = \sum_{i=1}^{c} \theta_i \phi_i + \lambda. \tag{3.1}$$

Moreover, such $\lambda \in M(n_1,\dots,n_c,\infty,\dots,\infty;s_1,\dots,s_d)$ *is unique, that is, if*

$$\psi = \sum_{i=1}^{c} \theta_i'\phi_i + \lambda'$$

with $\theta_i' \in \mathbb{Z}[s_1^{\pm1},\dots,s_d^{\pm1}]$ *and* $\lambda' \in M(n_1,\dots,n_c,\infty,\dots,\infty;s_1,\dots,s_d)$*, then* $\lambda = \lambda'$*.*

We call equation (3.1) *"division of* ψ *by* $\{\phi_i\}$*" and* λ *the remainder of the division.*

Proof. First we prove the existence by the induction on c. In the base case $c = 0$, there is nothing to prove. Now let $c \geq 1$. By the induction hypothesis, there exists $\theta_i \in \mathbb{Z}[s_1^{\pm1},\dots,s_d^{\pm1}]$ such that

$$\psi - \sum_{i=1}^{c-1} \theta_i\phi_i \in M(n_1,\dots,n_{c-1},\infty,\dots,\infty;s_1,\dots,s_d).$$

Thus, we may assume $\psi \in M(n_1,\dots,n_{c-1},\infty,\dots,\infty;s_1,\dots,s_d)$, and further $\psi = s_1^{m_1}\dots s_{c-1}^{m_{c-1}}\psi'$ with $m_i \in (-\lceil n_i/2\rceil, \lfloor n_i/2\rfloor]$ and $\psi' \in \mathbb{Z}[s_c^{\pm1},\dots,s_d^{\pm1}]$ since every element of $M(n_1,\dots,n_{c-1},\infty,\dots,\infty;s_1,\dots,s_d)$ is a linear combination with coefficients in $\mathbb{Z}$ of polynomials of this form. Then ψ' is represented as $\psi' = \sum_k s_c^k\psi_k'$ for some $\psi_k' \in \mathbb{Z}[s_{c+1}^{\pm1},\dots,s_d^{\pm1}]$. Since $M(n_1,\dots,n_c,\infty,\dots,\infty;s_1,\dots,s_d)$ is closed under multiplying elements of $\mathbb{Z}[s_{c+1}^{\pm1},\dots,s_d^{\pm1}]$, we may assume $\psi' = s_c^k$ with $k \in \mathbb{Z}$. Then by Lemma 3.2, there exists $\theta \in \mathbb{Z}[s_c^{\pm1}]$ such that $\psi' - \theta\phi_c \in M(n_c;s_c)$. Then we have

$$\psi - s_1^{m_1}\cdots s_{c-1}^{m_{c-1}}\theta\cdot\phi_c = s_1^{m_1}\cdots s_{c-1}^{m_{c-1}}(\psi' - \theta\phi_c) \in M(n_1,\dots,n_c,\infty,\dots,\infty;s_1,\dots,s_d).$$

The existence is proved.

Next we will prove the uniqueness. It suffices to show that if $\psi = \sum_{i=1}^{c}\theta_i\phi_i \in M(n_1,\dots,n_c,\infty,\dots,\infty;s_1,\dots,s_d)$ with $\theta_i \in \mathbb{Z}[s_1^{\pm1},\dots,s_d^{\pm1}]$, then $\psi = 0$. We call such $\Theta := (\theta_i)_{i=1}^{c}$ a representation of ψ. Suppose $\psi \neq 0$. Let $\sigma(\Theta)$ be the smallest integer $i \in [1,c]$ such that $\theta_i \neq 0$. We may assume that $\sigma(\Theta)$ is the largest among all the representations of ψ. Set $j = \sigma(\Theta)$. Let $\tau(\Theta) \geq 0$ be the difference between the largest and the smallest degrees of s_j in θ_j. We may assume that $\tau(\Theta)$ is the smallest among all the representations of ψ satisfying $\sigma(\Theta) = j$. Let l and m be the smallest degrees of s_j in θ_j and ϕ_j, respectively. For $j \leq i \leq c$, set $\theta_i = \sum_k s_j^k\mu_{i,k}$ with $\mu_{i,k} \in \mathbb{Z}[s_1^{\pm1},\dots,s_{j-1}^{\pm1},s_{j+1}^{\pm1},\dots,s_d^{\pm1}]$. Note that

$$\theta_j = \sum_{k=m}^{m+\tau(\Theta)} s_j^k\mu_{j,k}$$

holds. Also let the polynomial $\phi_j \in \mathbb{Z}[s_j^{\pm1}]$ be represented as

$$\phi_j = \sum_{k=l}^{l+n_j} a_k s_j^k$$

with $a_k \in \mathbb{Z}$.

Toward deducing a contradiction, we first assume $m + l \le -\lceil n_j/2 \rceil$. Then the sum of the terms of $\theta_j \phi_j$ whose degree of s_j is $m + l$ is $a_l s_j^{m+l} \mu_{j,m}$. For $i > j$, the sum of the terms of $\theta_i \phi_i$ whose degree of s_j is $m + l$ is $s_j^{m+l} \mu_{i,m+l} \phi_i$. Since $\psi = \sum_{i=j}^{c} \theta_i \phi_i \in M(n_1, \dots, n_c, \infty, \dots, \infty; s_1, \dots, s_d)$, the sum of the terms of ψ whose degree of s_j is $m + l$ must be 0. Thus, we have

$$a_l \mu_{j,m} + \sum_{i=j+1}^{c} \mu_{i,m+l} \phi_i = 0.$$

Note that $a_l \in \{\pm 1\}$ since ϕ_j is monic. Hence, we have

$$\begin{aligned} \psi &= \psi - \Bigg(\mu_{j,m} + a_l^{-1} \sum_{i=j+1}^{c} \mu_{i,m+l} \phi_i \Bigg) s_j^m \phi_j \\ &= \big(\theta_j - s_j^m \mu_{j,m}\big) \phi_j + \sum_{i=j+1}^{c} \big(\theta_i - a_l^{-1} s_j^m \mu_{i,m+l} \phi_j\big) \phi_i. \end{aligned}$$

This gives a new representation $\Theta' = (\theta'_i)_{i=1}^c$ of ψ such that $\theta'_i = 0$ if $i < j$, and $\theta'_j = \theta_j - s_j^m \mu_{j,m} = \sum_{k=m+1}^{m+\tau(\Theta)} s_j^k \mu_{i,k}$. Since $\sigma(\Theta) = j$ is the largest, we have $\sigma(\Theta') = j$. Then, however, $\tau(\Theta') < \tau(\Theta)$ holds, which contradicts the assumption of $\tau(\Theta)$ being the smallest.

We next assume $m + l > -\lceil n_j/2 \rceil$. The argument is analogous. The largest degrees of s_j in θ_j and ϕ_j are $m' := m + \tau(\Theta)$ and $l' := l + n_j$, respectively. Then

$$m' + l' \ge m + l + n_j > -\lceil n_j/2 \rceil + n_j = \lfloor n_j/2 \rfloor$$

holds. Since $\psi \in M(n_1, \dots, n_c, \infty, \dots, \infty; s_1, \dots, s_d)$, the sum of the terms of ψ whose degree of s_j is $m' + l'$ must be 0. Hence, we have

$$a_{l'} \mu_{j,m'} + \sum_{i=j+1}^{c} \mu_{i,m'+l'} \phi_i = 0$$

and

$$\psi = (\theta_j - s_j^{m'} \mu_{j,m'}) \phi_j + \sum_{i=j+1}^{c} (\theta_i - a_{l'}^{-1} s_j^{m'} \mu_{i,m'+l'} \phi_j) \phi_i$$

in the same way as above. This gives a new representation $\Theta' = (\theta'_i)_{i=1}^c$ of ψ such that $\theta'_i = 0$ if $i < j$, and $\theta'_j = \theta_j - s_j^{m'} \mu_{j,m'} = \sum_{k=m}^{m+\tau(\Theta)-1} s_j^k \mu_{i,k}$. Again we have $\sigma(\Theta') = j$ and $\tau(\Theta') < \tau(\Theta)$, which is a contradiction. ■

4. The commutator subgroup of free metabelian groups

Let G be a group and $N \trianglelefteq G$ a normal abelian subgroup. Set $Q = G/N$.

Notation 4.1. Let $g \in G \mapsto \overline{g} \in Q$ be the quotient map. We use additive notation for the group operation in N. The right action $N \curvearrowleft Q$ is induced from the conjugation by G, and it is denoted by $f^{\overline{g}} := g^{-1}fg$ for $f \in N$ and $g \in G$. We regard N as a right $\mathbb{Z}[Q]$-module with respect to this action, and the multiplication of $f \in N$ by $\phi \in \mathbb{Z}[Q]$ is denoted by f^{ϕ}. That is, if $\phi = \sum_{q \in Q} n_q \cdot q \in \mathbb{Z}[Q]$ with $n_q \in \mathbb{Z}$, then $f^{\phi} = \sum_{q \in Q} n_q f^q$ holds.

Remark 4.2. We keep using multiplicative notation for the group operation in Q even when Q is abelian. Then for $q_1, q_2 \in Q$, the elements $q_1 + q_2$ and $q_1 q_2$ of $\mathbb{Z}[Q]$ are distinguished. For $f \in N$, note that $f^{q_1+q_2} = g_1^{-1} f g_1 + g_2^{-1} f g_2$ and $f^{q_1 q_2} = g_2^{-1} g_1^{-1} f g_1 g_2$, where $g_1, g_2 \in G$ with $\overline{g_1} = q_1$ and $\overline{g_2} = q_2$.

Lemma 4.3. *For all $g, h, k \in G$, we have*

$$[g, hk] = [g, k] + [g, h]^{\overline{k}}. \tag{4.1}$$

Also for all $g, h \in G$ and $n \in \mathbb{N}$, we have

$$[g, h^n] = [g, h]^{1+\overline{h}+\cdots+\overline{h}^{n-1}}. \tag{4.2}$$

Proof. In general,

$$[g, hk] = g^{-1}k^{-1}h^{-1}ghk = g^{-1}k^{-1}gk \cdot k^{-1}g^{-1}h^{-1}ghk = [g, k] \cdot k^{-1}[g, h]k$$

holds, and we have equation (4.1). Then we have

$$\begin{aligned} [g, h^n] &= [g, h^{n-1}] + [g, h]^{\overline{h}^{n-1}} \\ &= [g, h^{n-2}] + [g, h]^{\overline{h}^{n-2}} + [g, h]^{\overline{h}^{n-1}} \\ &= \cdots \\ &= [g, h]^{1+\overline{h}+\cdots+\overline{h}^{n-1}}. \end{aligned}$$

∎

Let $G = F_d/F_d''$ with $d \geq 2$ and $N = G' = F_d'/F_d''$ in the rest of this section. Let $\{a_1, \ldots, a_d\}$ be a free generator of G and set $s_i = \overline{a_i} \in Q$ for $1 \leq i \leq d$. Then Q is the free abelian group over the basis $\Sigma_Q := \{s_1, \ldots, s_d\}$, and the group ring $\mathbb{Z}[Q]$ is identified with $\mathbb{Z}[s_1^{\pm 1}, \ldots, s_d^{\pm 1}]$. Since N is the commutator subgroup of G, it is generated by the set

$$X := \{[a_i, a_j] \mid 1 \leq i < j \leq d\}$$

as a $\mathbb{Z}[Q]$-module.

Theorem 4.4 ([4, Theorem 3]). *As $\mathbb{Z}[Q]$-modules, the subgroup N can be identified with $(\bigoplus_X \mathbb{Z}[Q])/\mathfrak{I}$, where $\mathfrak{I}$ is the $\mathbb{Z}[Q]$-submodule of $\bigoplus_X \mathbb{Z}[Q]$ generated by the subset*

$$\{[a_i, a_j]^{1-s_k} + [a_i, a_k]^{s_j-1} + [a_j, a_k]^{1-s_i} \mid 1 \leq i < j < k \leq d\}.$$

In particular, if $d = 2$, then $\mathfrak{I} = 0$ and N is isomorphic to $\mathbb{Z}[Q]$.

Corollary 4.5. *For all* $1 \le i < j < k \le d$, *we have*

$$[a_i, a_j]^{1-s_k} + [a_i, a_k]^{s_j - 1} + [a_j, a_k]^{1-s_i} = 0 \quad \text{in } N.$$

Lemma 4.6. *For all* $1 \le i < j < k \le d$ *and* $n \in \mathbb{N}$, *we have*

$$[a_i, a_j]^{s_k^n} = [a_i, a_j] + [a_i, a_k]^{(s_j-1)(1+s_k+\cdots+s_k^{n-1})} + [a_j, a_k]^{(1-s_i)(1+s_k+\cdots+s_k^{n-1})} \tag{4.3}$$

and

$$[a_i, a_j]^{s_k^{-n}} = [a_i, a_j] + [a_i, a_k]^{(1-s_j)(s_k^{-1}+s_k^{-2}+\cdots+s_k^{-n})} + [a_j, a_k]^{(s_i-1)(s_k^{-1}+s_k^{-2}+\cdots+s_k^{-n})} \tag{4.4}$$

in N.

Proof. By Corollary 4.5, we have

$$\begin{aligned}[a_i, a_j]^{s_k^n - 1} &= [a_i, a_j]^{(s_k-1)(1+s_k+\cdots+s_k^{n-1})} \\ &= [a_i, a_k]^{(s_j-1)(1+s_k+\cdots+s_k^{n-1})} + [a_j, a_k]^{(1-s_i)(1+s_k+\cdots+s_k^{n-1})}\end{aligned}$$

and obtain equation (4.3). By acting with s_k^{-n} to equation (4.3), we obtain equation (4.4). ■

For each $1 \le j \le d$, let $Q^{(j)}$ be the subgroup of Q generated by $\{s_i\}_{i=1}^{j}$.

Proposition 4.7. *For every* $f \in N$, *there exists a unique family* $(\phi_{i,j})_{1 \le i < j \le d}$ *of elements of* $\mathbb{Z}[Q]$ *satisfying*

(i) $\phi_{i,j} \in \mathbb{Z}[Q^{(j)}]$ *for all* i *and* j, *and*

(ii) $f = \sum_{1 \le i < j \le d} [a_i, a_j]^{\phi_{i,j}}$.

Proof. First, we prove the existence. Set

$$\tilde{N} = \sum_{1 \le i < j \le d} [a_i, a_j]^{\mathbb{Z}[Q^{(j)}]}.$$

Clearly $X \subset \tilde{N}$. Since N is generated by X as a $\mathbb{Z}[Q]$-module, it suffices to show that $\tilde{N}$ is Q-invariant. Let $1 \le i < j \le d$ and $\phi \in \mathbb{Z}[Q^{(j)}]$. We show $[a_i, a_j]^{\phi s_k^{\pm 1}} \in \tilde{N}$ for every $1 \le k \le d$. If $k \le j$, then $\phi s_k^{\pm 1} \in \mathbb{Z}[Q^{(j)}]$ and it is done. Suppose $k > j$. Then by equations (4.3) and (4.4), we have

$$[a_i, a_j]^{\phi s_k} = [a_i, a_j]^{\phi} + [a_i, a_k]^{\phi(s_j-1)} + [a_j, a_k]^{\phi(1-s_i)}$$

and

$$[a_i, a_j]^{\phi s_k^{-1}} = [a_i, a_j]^{\phi} + [a_i, a_k]^{\phi(1-s_j)s_k^{-1}} + [a_j, a_k]^{\phi(s_i-1)s_k^{-1}}.$$

By looking at the right-hand side, these are elements of $\tilde{N}$ as desired.

Next we will prove the uniqueness. By Theorem 4.4, it suffices to show the equation $\bigoplus_{1\leq i<j\leq d}[a_i,a_j]^{\mathbb{Z}[Q^{(j)}]}\cap\mathfrak{I}=0$. We may assume $d\geq 3$ since if $d=2$, then $\mathfrak{I}=0$. Suppose $0\neq f\in\bigoplus_{1\leq i<j\leq d}[a_i,a_j]^{\mathbb{Z}[Q^{(j)}]}\cap\mathfrak{I}$. Since $f\in\mathfrak{I}$, there exists a family $\Phi=(\phi_{i,j,k})_{1\leq i<j<k\leq d}$ of elements of $\mathbb{Z}[Q]$ such that

$$f=\sum_{1\leq i<j<k\leq d}\left([a_i,a_j]^{1-s_k}+[a_i,a_k]^{s_j-1}+[a_j,a_k]^{1-s_i}\right)^{\phi_{i,j,k}}. \tag{4.5}$$

We call such Φ a representation of f. Now we consider the lexicographic order on $\mathbb{N}^3$, that is,

$$(i,j,k)\leq(i',j',k')\Leftrightarrow i<i'\vee(i=i'\wedge j<j')\vee(i=i'\wedge j=j'\wedge k\leq k').$$

Let $\sigma(\Phi)$ be the smallest (i,j,k) such that $\phi_{i,j,k}\neq 0$. We may assume that $\sigma(\Phi)$ is the largest among all the representations of f. Set $\sigma(\Phi)=(i_0,j_0,k_0)$. By equation (4.5), the $[a_{i_0},a_{j_0}]$-th coordinate of f is

$$\sum_{l<i_0}(1-s_l)\phi_{l,i_0,j_0}+\sum_{i_0<l<j_0}(s_l-1)\phi_{i_0,l,j_0}+\sum_{l>j_0}(1-s_l)\phi_{i_0,j_0,l}. \tag{4.6}$$

However, by the definition of $\sigma(\Phi)$, we have $\phi_{l,i_0,j_0}=0$ if $l<i_0$, $\phi_{i_0,l,j_0}=0$ if $i_0<l<j_0$, and $\phi_{i_0,j_0,l}=0$ if $j_0<l<k_0$. Thus, polynomial (4.6) is equal to

$$\sum_{l\geq k_0}(1-s_l)\phi_{i_0,j_0,l}.$$

Since $f\in\bigoplus_{1\leq i<j\leq d}[a_i,a_j]^{\mathbb{Z}[Q^{(j)}]}$, we have $\sum_{l\geq k_0}(1-s_l)\phi_{i_0,j_0,l}\in\mathbb{Z}[Q^{(j_0)}]$, and this must be 0, which is verified by substituting $s_l=1$ for $l\geq k_0$ (note that $s_l\notin Q^{(j_0)}$). Hence, we have

$$(1-s_{k_0})\phi_{i_0,j_0,k_0}=-\sum_{l>k_0}(1-s_l)\phi_{i_0,j_0,l}. \tag{4.7}$$

By Proposition 3.3 (consider division of ϕ_{i_0,j_0,k_0} by $\{1-s_l\}_{l>k_0}$), we have

$$\phi_{i_0,j_0,k_0}=\sum_{l>k_0}(1-s_l)\psi_l+\lambda$$

for some $\psi_l\in\mathbb{Z}[Q]$ and $\lambda\in\mathbb{Z}[Q^{(k_0)}]$. Then substitute this into equation (4.7) and we have

$$(1-s_{k_0})\sum_{l>k_0}(1-s_l)\psi_l+(1-s_{k_0})\lambda=-\sum_{l>k_0}(1-s_l)\phi_{i_0,j_0,l}.$$

Since $(1-s_{k_0})\lambda\in\mathbb{Z}[Q^{(k_0)}]$, we have $\lambda=0$ by Proposition 3.3 (uniqueness of the remainder of division of both sides by $\{1-s_l\}_{l>k_0}$). Hence, we have

$$\phi_{i_0,j_0,k_0}=\sum_{l>k_0}(1-s_l)\psi_l. \tag{4.8}$$

Now we set $\alpha_{i,j,k} = [a_i, a_j]^{1-s_k} + [a_i, a_k]^{s_j-1} + [a_j, a_k]^{1-s_i}$ for $1 \le i < j < k \le d$. Note that

$$\alpha_{i,j,k}^{1-s_l} + \alpha_{i,j,l}^{s_k-1} + \alpha_{i,k,l}^{1-s_j} + \alpha_{j,k,l}^{s_i-1} = 0$$

for any $1 \le i < j < k < l \le d$. Then by equation (4.8), we have

$$\alpha_{i_0,j_0,k_0}^{\phi_{i_0,j_0,k_0}} = \sum_{l>k_0} \alpha_{i_0,j_0,k_0}^{(1-s_l)\psi_l} = \sum_{l>k_0} \left(\alpha_{i_0,j_0,l}^{1-s_{k_0}} + \alpha_{i_0,k_0,l}^{s_{j_0}-1} + \alpha_{j_0,k_0,l}^{1-s_{i_0}}\right)^{\psi_l}.$$

By adding $0 = -\alpha_{i_0,j_0,k_0}^{\phi_{i_0,j_0,k_0}} + \sum_{l>k_0} \left(\alpha_{i_0,j_0,l}^{1-s_{k_0}} + \alpha_{i_0,k_0,l}^{s_{j_0}-1} + \alpha_{j_0,k_0,l}^{1-s_{i_0}}\right)^{\psi_l}$ to equation (4.5), we have

$$f = \sum_{\substack{1\le i<j<k\le d \\ (i,j,k)\ne(i_0,j_0,k_0)}} \alpha_{i,j,k}^{\phi_{i,j,k}} + \sum_{l>k_0} \left(\alpha_{i_0,j_0,l}^{1-s_{k_0}} + \alpha_{i_0,k_0,l}^{s_{j_0}-1} + \alpha_{j_0,k_0,l}^{1-s_{i_0}}\right)^{\psi_l}.$$

This gives a new representation $\Phi' = (\phi'_{i,j,k})$ of f such that $\phi'_{i,j,k} = \phi_{i,j,k} = 0$ if $(i, j, k) < (i_0, j_0, k_0)$, and $\phi'_{i_0,j_0,k_0} = 0$. Then we have $\sigma(\Phi') > (i_0, j_0, k_0) = \sigma(\Phi)$, which contradicts the assumption of $\sigma(\Phi)$ being the largest. ■

Let $1 \le c \le d$. Set $U = Q^{(c)}$ and let V be the subgroup of Q generated by $\Sigma_V := \{s_i\}_{i=c+1}^{d}$. Fix $n \in \mathbb{N}$. Let O be the $\mathbb{Z}[Q]$-submodule of N generated by the set

$$\left\{[g, a_i^{2n}] \mid g \in G,\ c+1 \le i \le d\right\}.$$

Lemma 4.8. *If $f \in N$ and $c + 1 \le i \le d$, then $f^{1-s_i^{2n}} = -[f, a_i^{2n}] \in O$ holds.*

Proof. In G (using multiplicative notation), we have $[f, a_i^{2n}] = a_i^{-2n} f a_i^{2n} f^{-1}$. Recall from Notation 4.1 that $a_i^{-2n} f a_i = f^{s_i^{2n}}$. Hence in N (using additive notation), we have

$$-\left[f, a_i^{2n}\right] = -\left(f^{s_i^{2n}} + f^{-1}\right) = -f^{s_i^{2n}-1} = f^{1-s_i^{2n}}.$$ ■

Lemma 4.9. *For $1 \le i < j \le d$, let $O_{i,j}$ be the ideal of $\mathbb{Z}[Q^{(j)}]$ generated by*

$$\left\{1 - s_k^{2n} \mid k \in [c+1, j]\backslash\{i, j\}\right\} \cup \left\{1 + s_k + \cdots + s_k^{2n-1} \mid k \in [c+1, j] \cap \{i, j\}\right\}.$$

Then we have

$$O = \sum_{1\le i<j\le d} [a_i, a_j]^{O_{i,j}}. \tag{4.9}$$

Proof. Let $\widetilde{O}$ be the right-hand side of equation (4.9). Let $1 \le i < j \le d$ and $\phi \in O_{i,j}$. We show $[a_i, a_j]^{\phi} \in O$. We may assume that ϕ is in the generator of the ideal $O_{i,j}$, that is, $\phi = 1 - s_k^{2n}$ with $k \in [c+1, j]\backslash\{i, j\}$, or $\phi = 1 + s_k + \cdots + s_k^{2n-1}$ with

$k \in [c+1, j] \cap \{i, j\}$. In the first case, $[a_i, a_j]^{(1-s_k^{2n})} \in O$ by the previous lemma. In the second case, by equation (4.2) we have

$$[a_i, a_j]^{1+s_j+\cdots+s_j^{2n-1}} = [a_i, a_j^{2n}] \in O \quad \text{if } k = j,$$

and

$$[a_i, a_j]^{1+s_i+\cdots+s_i^{2n-1}} = -[a_j, a_i^{2n}] \in O \quad \text{if } k = i.$$

Hence, $O \supset \widetilde{O}$ holds.

Now we show the converse. Note that for all $g, h \in G$, and $c+1 \leq k \leq d$,

$$\left[gh, a_k^{2n}\right] = \left[g, a_k^{2n}\right]^{\bar{h}} + \left[h, a_k^{2n}\right]$$

holds by equation (4.1), and

$$[g^{-1}, a_k^{2n}] = -[g, a_k^{2n}]^{\bar{g}^{-1}}$$

holds. Since G is generated by $\{a_i\}_{i=1}^{d}$, O is generated by

$$A := \{[a_i, a_k^{2n}] = [a_i, a_k]^{1+s_k+\cdots+s_k^{2n-1}} \mid 1 \leq i \leq d,\, c+1 \leq k \leq d\}$$

as a $\mathbb{Z}[Q]$-module. Let $1 \leq i \leq d$ and $c+1 \leq k \leq d$. Then we have

$$1 + s_k + \cdots + s_k^{2n-1} \in O_{i,k} \quad \text{if } i < k,$$

and

$$1 + s_k + \cdots + s_k^{2n-1} \in O_{k,i} \quad \text{if } i > k.$$

If $i = k$, then $[a_i, a_k] = 0$. It follows that $A \subset \widetilde{O}$. Thus, it suffices to show that $\widetilde{O}$ is Q-invariant. Let $1 \leq i < j \leq d$ and $\phi \in O_{i,j}$. We show that $[a_i, a_j]^{s_k^{\pm 1}\phi} \in \widetilde{O}$ for every $1 \leq k \leq d$. Since $O_{i,j}$ is $Q^{(j)}$-invariant, we may assume $k > j$. Then by equations (4.3) and (4.4), we have

$$[a_i, a_j]^{s_k\phi} = [a_i, a_j]^{\phi} + [a_i, a_k]^{(s_j-1)\phi} + [a_j, a_k]^{(1-s_i)\phi}$$

and

$$[a_i, a_j]^{s_k^{-1}\phi} = [a_i, a_j]^{\phi} + [a_i, a_k]^{(1-s_j)s_k^{-1}\phi} + [a_j, a_k]^{(s_i-1)s_k^{-1}\phi}.$$

It suffices to show that $(1-s_j)\phi \in O_{i,k}$ and $(1-s_i)\phi \in O_{j,k}$. Since $\phi \in O_{i,j}$, there exists $\psi_l \in \mathbb{Z}[Q^{(j)}]$ for $c+1 \leq l \leq d$ such that

$$\phi = \sum_{l \in [c+1,j] \setminus \{i,j\}} \left(1 - s_l^{2n}\right)\psi_l + \sum_{l \in [c+1,j] \cap \{i,j\}} \left(1 + s_l + \cdots + s_l^{2n-1}\right)\psi_l.$$

Then we have

$$\begin{aligned}(1-s_j)\phi = &\sum_{l \in [c+1,j] \setminus \{i,j\}} (1-s_j)\left(1-s_l^{2n}\right)\psi_l \\ &+ 1_{[c+1,d]}(i) \cdot (1-s_j)\left(1 + s_i + \cdots + s_i^{2n-1}\right)\psi_i \\ &+ 1_{[c+1,d]}(j) \cdot \left(1 - s_j^{2n}\right)\psi_j,\end{aligned}$$

and thus $(1-s_j)\phi \in O_{i,k}$. In the same way, $(1-s_i)\phi \in O_{j,k}$ holds. Hence, $[a_i,a_j]^{s_k^{\pm 1}\phi} \in \widetilde{O}$ holds. ∎

Definition 4.10. For $1 \le i < j \le d$, define the subgroup $M_{i,j} \le \mathbb{Z}[Q^{(j)}]$ as follows:

(i) If $j \le c$, then let $M_{i,j} = \mathbb{Z}[Q^{(j)}]$.

(ii) If $j \ge c+1$, then let $M_{i,j}$ be the $\mathbb{Z}[U]$-submodule of $\mathbb{Z}[Q^{(j)}]$ generated by

$$\left\{ s_{c+1}^{m_{c+1}} \cdots s_j^{m_j} \;\middle|\; \begin{array}{l} m_k \in [-n+1, n] \text{ for } k \in [c+1, j]\setminus\{i,j\}, \text{ and} \\ m_k \in [-n+1, n-1] \text{ for } k \in [c+1, j] \cap \{i,j\} \end{array} \right\}.$$

Then we set

$$M = \sum_{1 \le i < j \le d} [a_i, a_j]^{M_{i,j}}. \tag{4.10}$$

We call M the *U-residue of O*.

Remark 4.11. Let $1 \le i < j \le d$. For $1 \le k \le j$, define $n_k \in \mathbb{N} \cup \{\infty\}$ by

$$\begin{aligned} n_k &= \infty \quad \text{if } k \in [1, c], \\ n_k &= 2n \quad \text{if } k \in [c+1, j]\setminus\{i,j\}, \end{aligned}$$

and

$$n_k = 2n-1 \quad \text{if } k \in [c+1, j] \cap \{i,j\}.$$

Then $M_{i,j} = M(n_1, \dots, n_j; s_1, \dots, s_j)$ holds.

Notation 4.12. Let A be a finitely generated free abelian group over the basis $\Sigma_A = \{s_1, \dots, s_d\}$. For $m \in \mathbb{N}$, we set

$$\mathrm{B}_A(m, \Sigma_A) = \left\{ s_1^{k_1} \cdots s_d^{k_d} \in A \mid -\lceil m/2 \rceil < k_i \le \lfloor m/2 \rfloor \text{ for every } i \right\}.$$

Lemma 4.13. *The subgroup M is U-invariant and contains*

$$\{x^v \mid x \in X,\ v \in \mathrm{B}_Q(2n-1, \Sigma_Q)\}.$$

Proof. Let $1 \le i < j \le d$ and $\phi \in M_{i,j}$. We show that $[a_i,a_j]^{s_k^{\pm 1}\phi} \in M$ for every $1 \le k \le c$. If $j \ge c+1$, then $s_k^{\pm 1}\phi \in M_{i,j}$ since $M_{i,j}$ is U-invariant, and it is done. Now suppose $j \le c$. If $k \le j \le c$, then $s_k^{\pm 1}\phi \in \mathbb{Z}[Q^{(j)}] = M_{i,j}$, and it is done. If $j < k \le c$, then by equations (4.3) and (4.4) we have

$$[a_i,a_j]^{s_k\phi} = [a_i,a_j]^{\phi} + [a_i,a_k]^{(s_j-1)\phi} + [a_j,a_k]^{(1-s_i)\phi}$$

and

$$[a_i,a_j]^{s_k^{-1}\phi} = [a_i,a_j]^{\phi} + [a_i,a_k]^{(1-s_j)s_k^{-1}\phi} + [a_j,a_k]^{(s_i-1)s_k^{-1}\phi}.$$

Since $\phi, s_i, s_j, s_k \in \mathbb{Z}[Q^{(k)}] = M_{i,k} = M_{j,k}$, we have $[a_i, a_j]^{s_k^{\pm 1}\phi} \in M$. Hence, M is U-invariant.

For the remaining claim, it suffices to show that

$$\{x^v \mid x \in X,\ v \in \mathrm{B}_V(2n-1, \Sigma_V)\} \subset M$$

since M is U-invariant. Let $1 \le i < j \le d$ and $v = s_{c+1}^{m_{c+1}} \cdots s_d^{m_d} \in \mathrm{B}_V(\Sigma_V, 2n-1)$. We show that $[a_i, a_j]^v \in M$. If v is the identity of Q, it is clear. Otherwise, let $k \in [c+1, d]$ be the largest integer such that $m_k \neq 0$. We prove by the induction on k. If $k \le j$, then $v \in M_{i,j}$ since $m_{c+1}, \ldots, m_k \in [-n+1, n-1]$, and thus $[a_i, a_j]^v \in M$ holds. Suppose $k > j$ and set $w = s_k^{-m_k} v = s_{c+1}^{m_{c+1}} \cdots s_{k-1}^{m_{k-1}}$. If $m_k > 0$, then by equation (4.3) we have

$$\begin{aligned}[a_i, a_j]^v = [a_i, a_j]^{s_k^{m_k} w} = [a_i, a_j]^w &+ [a_i, a_k]^{(s_j-1)(1+s_k+\cdots+s_k^{m_k-1})w} \\ &+ [a_j, a_k]^{(1-s_i)(1+s_k+\cdots+s_k^{m_k-1})w}.\end{aligned}$$

If $m_k < 0$, then by equation (4.4) we have

$$\begin{aligned}[a_i, a_j]^v = [a_i, a_j]^{s_k^{m_k} w} = [a_i, a_j]^w &+ [a_i, a_k]^{(1-s_j)(s_k^{-1}+s_k^{-2}+\cdots+s_k^{m_k})w} \\ &+ [a_j, a_k]^{(s_i-1)(s_k^{-1}+s_k^{-2}+\cdots+s_k^{m_k})w}.\end{aligned}$$

By the induction hypothesis, $[a_i, a_j]^w \in M$. It suffices to show that

$$(1-s_j)s_k^m w \in M_{i,k} \quad \text{and} \quad (1-s_i)s_k^m w \in M_{j,k}$$

for every $m \in [-n+1, n-1]$. This follows from

$$s_k^m w = s_{c+1}^{m_{c+1}} \cdots s_{k-1}^{m_{k-1}} s_k^m \in M_{i,k} \cap M_{j,k}, \tag{4.11}$$

$$s_j s_k^m w = s_j s_{c+1}^{m_{c+1}} \cdots s_{k-1}^{m_{k-1}} s_k^m \in M_{i,k} \tag{4.12}$$

and

$$s_i s_k^m w = s_i s_{c+1}^{m_{c+1}} \cdots s_{k-1}^{m_{k-1}} s_k^m \in M_{j,k}. \tag{4.13}$$

Claim (4.11) holds since $m_{c+1}, \ldots, m_{k-1} \in [-n+1, n-1]$. If $j \le c$, then $s_j s_k^m w \in M_{i,k}$ since $M_{i,k}$ is U-invariant. If $j \ge c+1$, then $s_j s_k^m w \in M_{i,k}$ since $m_j + 1 \in [-n+1, n]$. Thus claim (4.12) holds. Claim (4.13) also holds in the same way. Hence, $[a_i, a_j]^v \in M$ is proved. ■

Proposition 4.14. *Let $\pi : N \to N/O$ be the quotient map. Then the restriction $\pi|_M : M \to N/O$ is a bijection.*

Proof. Let $1 \le i < j \le d$. For $1 \le k \le j$, we define $\phi_k \in \mathbb{Z}[s_k^{\pm 1}]$ and $n_k \in \mathbb{N} \cup \{\infty\}$ by

$$\begin{aligned} \phi_k &= 0, & n_k &= \infty & &\text{if } k \in [1, c], \\ \phi_k &= 1 - s_k^{2n}, & n_k &= 2n & &\text{if } k \in [c+1, j] \backslash \{i, j\}, \end{aligned}$$

and

$$\phi_k = 1 + s_k + \cdots + s_k^{2n-1}, \quad n_k = 2n - 1 \quad \text{if } k \in [c+1, j] \cap \{i, j\}.$$

Then $O_{i,j}$ is the ideal of $\mathbb{Z}[Q^{(j)}]$ generated by $\{\phi_k\}_{k=1}^{j}$, and the equation

$$M_{i,j} = M(n_1, \dots, n_j; s_1, \dots, s_j)$$

holds. By Proposition 3.3, we have $M_{i,j} + O_{i,j} = \mathbb{Z}[Q^{(j)}]$ and $M_{i,j} \cap O_{i,j} = 0$.

Now we show the proposition. By Proposition 4.7, we have

$$N = \sum_{1 \le i < j \le d} [a_i, a_j]^{\mathbb{Z}[Q^{(j)}]}.$$

Also, by equations (4.10) and (4.9), we have

$$M = \sum_{1 \le i < j \le d} [a_i, a_j]^{M_{i,j}} \quad \text{and} \quad O = \sum_{1 \le i < j \le d} [a_i, a_j]^{O_{i,j}}.$$

Hence, we have

$$M + O = \sum_{1 \le i < j \le d} [a_i, a_j]^{M_{i,j} + O_{i,j}} = \sum_{1 \le i < j \le d} [a_i, a_j]^{\mathbb{Z}[Q^{(j)}]} = N$$

and the surjectivity is proved.

Again by Proposition 4.7 (uniqueness), we have

$$\begin{aligned} M \cap O &= \Bigg(\sum_{1 \le i < j \le d} [a_i, a_j]^{M_{i,j}} \Bigg) \cap \Bigg(\sum_{1 \le i < j \le d} [a_i, a_j]^{O_{i,j}} \Bigg) \\ &= \sum_{1 \le i < j \le d} [a_i, a_j]^{M_{i,j} \cap O_{i,j}} = 0. \end{aligned}$$

The injectivity is proved. ■

5. Proof of Theorem 1.3

Let $G = F_d / F_d''$ for $d \geq 2$. Set $N = G' = F_d' / F_d''$ and $Q = G/N$. We follow Notation 4.1.

Let $R \leq Q$ be a subgroup. By Corollary 2.7, it suffices to construct a Følner sequence (F_n) of G satisfying property $(\#_R)$ in Corollary 2.7.

5.1. Construction of the Følner sequence

Take subgroups $U, V \leq Q$ such that $Q = U \times V$ and R is a finite index subgroup of U. Set $c = \operatorname{rank}(U)$. Let $\{s_1, \dots, s_c\}$ and $\{s_{c+1}, \dots, s_d\}$ be generators of U and V, respectively. We may assume that $\overline{a_i} = s_i$ for every $1 \leq i \leq d$ by taking a suitable free generator

$\{a_1, \dots, a_d\}$ of G since the natural map $\mathrm{Aut}(F_d) \to \mathrm{Aut}(F_d/F_d')$ is surjective [12, Chapter I, Proposition 4.4]. Set $\Sigma_Q = \{s_1, \dots, s_d\}$.

Define the section $q \in Q \mapsto \widehat{q} \in G$ of the quotient map by $\widehat{q} = a_1^{k_1} \cdots a_d^{k_d}$ if $q = s_1^{k_1} \cdots s_d^{k_d}$ with $k_1, \dots, k_d \in \mathbb{Z}$.

Lemma 5.1. *Let $g \in G$ and $q = s_1^{\varepsilon_1 k_1} \cdots s_d^{\varepsilon_d k_d} \in Q$ with $\varepsilon_i \in \{\pm 1\}$ and $k_i \geq 0$ for $1 \leq i \leq d$. Then we have*

$$[g, \widehat{q}] = \sum_{\substack{1 \leq i \leq d \\ \text{s.t. } k_i \geq 1}} \sum_{l=0}^{k_i - 1} [g, a_i^{\varepsilon_i}]^{s_i^{\varepsilon_i l} s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}}.$$

Proof. By equation (4.1), we have

$$\begin{aligned}
[g, \widehat{q}] &= \left[g, a_1^{\varepsilon_1 k_1} \cdots a_d^{\varepsilon_d k_d}\right] \\
&= \left[g, a_1^{\varepsilon_1 k_1}\right]^{s_2^{\varepsilon_2 k_2} \cdots s_d^{\varepsilon_d k_d}} + \left[g, a_2^{\varepsilon_2 k_2} \cdots a_d^{\varepsilon_d k_d}\right] \\
&= \cdots \\
&= \sum_{i=1}^{d} \left[g, a_i^{\varepsilon_i k_i}\right]^{s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}}.
\end{aligned}$$

Then for every i with $k_i \geq 1$, we have

$$\left[g, a_i^{\varepsilon_i k_i}\right] = \sum_{l=0}^{k_i - 1} \left[g, a_i^{\varepsilon_i}\right]^{s_i^{\varepsilon_i l}}$$

by equation (4.2). Hence, the lemma holds. ■

In this subsection, we construct subgroups $Q_n \leq Q$ and $M_n \leq N$, and finite subsets $I_n \subset Q$ and $T_n, P_n \subset M_n$ such that $F_n = \widehat{I_n} P_n$ is a Følner sequence of G. We will apply Theorem 2.16 to them in the next subsection.

For $n \in \mathbb{N}$, we set $m_n = n \cdot [U : R] \in \mathbb{N}$, $V_n = \{v^{2m_n} \mid v \in V\} \leq V$, and $Q_n = R \times V_n \leq Q$. Recall Notation 4.12. The next lemma is clear.

Lemma 5.2. *The following conditions hold:*

- (i) *$R \leq Q_n$ and $Q_n \to R$ in* $\mathrm{Sub}(Q)$.
- (ii) *$I_n := \mathrm{B}_Q(2m_n, \Sigma_Q)$ is a finite-to-one transversal for Q_n in Q.*

Let O_n be the $\mathbb{Z}[Q]$-submodule of N generated by

$$\left\{[g, a_i^{2m_n}] \mid g \in G,\ c + 1 \leq j \leq d\right\}.$$

We regard N/O_n as a $\mathbb{Z}[Q]$-module. Let $\pi_n : N \to N/O_n$ be the quotient map. Let $M_n \leq N$ be the U-residue of O_n (Definition 4.10). The following is a consequence of our observation in Section 4.

Lemma 5.3. *The following conditions hold:*

(i) *The action of V_n on N/O_n is trivial.*

(ii) *The subgroup M_n is R-invariant.*

(iii) *The map $\pi_n|_{M_n} : M_n \to N/O_n$ is a bijection.*

Proof. (i) Since V_n is generated by $\{s_i^{2m_n}\}_{i=c+1}^{d}$, it follows from Lemma 4.8.

(ii) Since $R \leq U$ and M_n is U-invariant by Lemma 4.13, the claim holds.

(iii) This follows from Proposition 4.14. ■

Lemma 5.4. *Set $\pi'_n = (\pi_n|_{M_n})^{-1} \circ \pi_n : N \to M_n$. If P is a finite-to-one transversal for a subgroup $K \leq M_n$ in M_n, then P is also a finite-to-one transversal for $\pi_n'^{-1}(K)$ in N.*

Proof. Let P be a transversal for a subgroup $K \leq M_n$ in M_n. Then $M_n = \bigsqcup_{p \in P} pK$ holds. Since $\pi'_n|_{M_n} = \mathrm{id}_{M_n}$, we have

$$N = \pi_n'^{-1}(M_n) = \pi_n'^{-1}\left(\bigsqcup_{p \in P} pK\right) = \bigsqcup_{p \in P} p \cdot \pi_n'^{-1}(K).$$

Hence, P is a transversal for $\pi_n'^{-1}(K)$ in N. ■

We set

$$\begin{aligned} X &= \{[a_i, a_j] \in N \mid 1 \leq i < j \leq d\}, \\ Z_n &= \{x^q \in N \mid x \in X,\, q \in \mathrm{B}_Q(2m_n - 1, \Sigma_Q)\}, \end{aligned}$$

and

$$T_n = \left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \leq n^2 \text{ for every } z \in Z_n \right\}.$$

Note that $T_n \subset T_{n+1}$ for every n and $\bigcup_n T_n = N$. By Lemma 4.13, $Z_n \subset M_n$ and thus $T_n \subset M_n$ holds.

Lemma 5.5. *The sequence T_n is adapted to the sequence $\widehat{I_n}$, that is, for all $g \in G$ and all $\Phi \subset N$ finite,*

$$\frac{|\{h \in F_n \mid [g, h] + \Phi \subset T_n\}|}{|F_n|} \to 1 \quad (n \to \infty). \tag{5.1}$$

Proof. Let $g \in G$ and let $\Phi \subset N$ be a finite subset. Take $n_0 \in \mathbb{N}$ so that

$$\{[g, a_i^{\pm 1}] \mid 1 \leq i \leq d\} \cup \Phi \subset T_{n_0}.$$

Set $J_n = \mathrm{B}_Q(2(m_n - m_{n_0}) + 1, \Sigma_Q) \subset I_n$ for $n > n_0$. Note that $Z_{n_0}^q \subset Z_n$ for every $q \in J_n$ since $J_n \mathrm{B}_Q(2m_{n_0} - 1, \Sigma_Q) \subset \mathrm{B}_Q(2m_n - 1, \Sigma_Q)$. Thus for every $q \in J_n$, we have

$$T_{n_0}^q \subset \left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \leq n_0^2 \text{ for every } z \in Z_n \right\}.$$

Since $n^2/m_n \to \infty$, we have $n_0^2(m_n d + 1) \leq n^2$ for every n sufficiently large. Fix such $n > n_0$. We show that $[g, \widehat{q}] + \Phi \subset T_n$ for every $q \in J_n$. If $q = s_1^{\varepsilon_1 k_1} \cdots s_d^{\varepsilon_d k_d} \in J_n$ with $\varepsilon_i \in \{\pm 1\}$ and $k_i \geq 0$, then by Lemma 5.1 we have

$$[g, \widehat{q}] = \sum_{1 \leq i \leq d} \sum_{l=0}^{k_i - 1} [g, a_i^{\varepsilon_i}]^{s_i^{\varepsilon_i l} s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}}. \tag{5.2}$$

Then each term $[g, a_i^{\varepsilon_i}]^{s_i^{\varepsilon_i l} s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}}$ of the right-hand side is in the set

$$\left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \leq n_0^2 \text{ for every } z \in Z_n \right\}$$

since $[g, a_i^{\varepsilon_i}] \in T_{n_0}$ and $s_i^{\varepsilon_i l} s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d} \in J_n$. Also since the right-hand side of equation (5.2) has at most $m_n d$ terms,

$$[g, \widehat{q}] \in \left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \leq n_0^2 m_n d \text{ for every } z \in Z_n \right\}.$$

It follows from $\Phi \subset T_{n_0}$ and $n_0^2(m_n d + 1) \leq n^2$ that

$$[g, \widehat{q}] + \Phi \subset \left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \leq n_0^2(m_n d + 1) \text{ for every } z \in Z_n \right\} \subset T_n.$$

Hence, we have

$$\frac{|\{q \in I_n \mid [g, \widehat{q}] + \Phi \subset T_n\}|}{|I_n|} \geq \frac{|J_n|}{|I_n|} \to 1 \quad (n \to \infty). \qquad \blacksquare$$

Let L_n be the subgroup of N generated by Z_n. Let $Y_n \subset L_n$ be a basis of L_n as a free abelian group. Note that the sequence

$$\left\{ \sum_{y \in Y_n} k_y y \in L_n \;\middle|\; k_y \in [0, l] \text{ for every } y \in Y_n \right\} \quad (l = 1, 2, \ldots)$$

is a Følner sequence of L_n. Thus, we can take $l_n \in \mathbb{N}$ so that the set

$$P_n := \left\{ \sum_{y \in Y_n} k_y y \in L_n \;\middle|\; k_y \in [0, l_n n! - 1] \text{ for every } y \in Y_n \right\}$$

is $(T_n, 1/n)$-invariant, that is, $|P_n \cap (f + P_n)| \geq (1 - 1/n)|P_n|$ for every $f \in T_n$. Since $Z_n \subset M_n$, we have $P_n \subset L_n \subset M_n$.

Lemma 5.6. *The sequence $F_n := \widehat{I_n} P_n$ is a Følner sequence of G.*

Proof. Since G is generated by $\{a_i\}_{i=1}^d$, it suffices to show that

$$\frac{|F_n \cap a_i F_n|}{|F_n|} \to 1 \quad (n \to \infty)$$

for every $1 \le i \le d$. Fix $1 \le i \le d$. Take $n_0 \in \mathbb{N}$ so that $\{[a_i, a_j^{\pm 1}]\}_{j=1}^{i} \subset T_{n_0}$. Set $J_n = \mathrm{B}_Q(2(m_n - m_{n_0}) + 1, \Sigma_Q) \subset I_n$ for $n > n_0$. Then for every $q \in J_n$, we have

$$T_{n_0}^q \subset \left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \le n_0^2 \text{ for every } z \in Z_n \right\}$$

as proved in the proof of Lemma 5.5. Since $n^2/m_n \to \infty$, we have $n_0^2 m_n i \le n^2$ for every n sufficiently large. Fix such $n > n_0$. Let $q = s_1^{\varepsilon_1 k_1} \cdots s_d^{\varepsilon_d k_d} \in Q$ with $\varepsilon_j \in \{\pm 1\}$ and $k_j \ge 0$ for $1 \le j \le d$. Then we have

$$\begin{aligned}
a_i \widehat{q} &= a_i a_1^{\varepsilon_1 k_1} \cdots a_d^{\varepsilon_d k_d} \\
&= a_i a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i} \cdot a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d} \\
&= a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i} a_i \big[a_i, a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i}\big] \cdot a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d} \\
&= a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i} a_i a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d} \cdot \big[a_i, a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i}\big]^{s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}},
\end{aligned}$$

where the last equation follows from

$$\begin{aligned}
&\big(a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d}\big)^{-1} \big[a_i, a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i}\big] \big(a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d}\big) \\
&\qquad = \big[a_i, a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i}\big]^{s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}}.
\end{aligned}$$

Set $f_q = [a_i, a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i}]^{s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}}$. Then

$$a_i \widehat{q} = a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i} a_i a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d} \cdot f_q \tag{5.3}$$

holds. We show that $f_q \in T_n$ if $q \in J_n$. By Lemma 5.1, we have

$$f_q = \big[a_i, a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i}\big]^{s_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots s_d^{\varepsilon_d k_d}} = \sum_{1 \le j \le i} \sum_{l=0}^{k_j - 1} \big[a_i, a_j^{\varepsilon_j}\big]^{s_j^{\varepsilon_j l} s_{j+1}^{\varepsilon_{j+1} k_{j+1}} \cdots s_d^{\varepsilon_d k_d}}. \tag{5.4}$$

If $q \in J_n$, then each term $[a_i, a_j^{\varepsilon_j}]^{s_j^{\varepsilon_j l} s_{j+1}^{\varepsilon_{j+1} k_{j+1}} \cdots s_d^{\varepsilon_d k_d}}$ of the right-hand side is in

$$\left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \le n_0^2 \text{ for every } z \in Z_n \right\}$$

since $[a_i, a_j^{\varepsilon_j}] \in T_{n_0}$ and $s_j^{\varepsilon_j l} s_{j+1}^{\varepsilon_{j+1} k_{j+1}} \cdots s_d^{\varepsilon_d k_d} \in J_n$. Also since the right-hand side of equation (5.4) has at most $m_n i$ terms, we have

$$f_q \in \left\{ \sum_{z \in Z_n} k_z z \in N \;\middle|\; |k_z| \le n_0^2 m_n i \text{ for every } z \in Z_n \right\} \subset T_n$$

as desired. The last inclusion follows from $n_0^2 m_n i \leq n^2$.

Now since P_n is $(T_n, 1/n)$-invariant, we have

$$|P_n \cap (-f_q + P_n)| \geq (1 - 1/n)|P_n| \tag{5.5}$$

for every $q \in J_n$. On the other hand, if $q \in s_i^{-1} I_n$, then we have

$$a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i} a_i a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d} = \widehat{s_i q} \in \widehat{I_n}$$

and thus by equation (5.3)

$$a_i \widehat{q}(-f_q + P_n) = a_1^{\varepsilon_1 k_1} \cdots a_i^{\varepsilon_i k_i} a_i a_{i+1}^{\varepsilon_{i+1} k_{i+1}} \cdots a_d^{\varepsilon_d k_d} \cdot (f_q - f_q + P_n) \subset \widehat{I_n} P_n,$$

where we are using multiplicative notation in G and additive notation in N. It follows that

$$\bigsqcup_{q \in I_n \cap s_i^{-1} I_n} a_i \widehat{q}(P_n \cap (-f_q + P_n)) \subset F_n \cap a_i F_n.$$

Then by inequality (5.5), we have

$$|F_n \cap a_i F_n| \geq \sum_{q \in J_n \cap s_i^{-1} I_n} |P_n \cap (-f_q + P_n)| \geq |J_n \cap s_i^{-1} I_n| \cdot (1 - 1/n)|P_n|.$$

Hence, we have

$$\frac{|F_n \cap a_i F_n|}{|F_n|} \geq \frac{|J_n \cap s_i^{-1} I_n|}{|I_n|} \cdot (1 - 1/n) \to 1 \quad (n \to \infty). \qquad \blacksquare$$

The following lemma is used in the next subsection to show that F_n is a finite-to-one transversal for some subgroups in G. The proof is inspired by the proof of Lemma 10.9 of [9].

Lemma 5.7. *If $K \leq N$ is a finite index subgroup, then P_n is a finite-to-one transversal for $K \cap M_n$ in M_n for every n sufficiently large.*

Proof. Let $K \leq N$ be a finite index subgroup and set $k = [N : K]$. Since $L_n \to N$ in $\mathrm{Sub}(N)$, there exists $n_0 \in \mathbb{N}$ such that $[L_n : L_n \cap K] = k$ for every $n \geq n_0$. Let $n \geq \max\{k, n_0\}$. Then

$$kL_n = \left\{ \sum_{y \in Y_n} k_y y \in L_n \;\middle|\; k_y \in k\mathbb{Z} \text{ for every } y \in Y_n \right\} \leq L_n \cap K$$

holds since all k-th power is trivial in $L_n/(L_n \cap K)$. Recall that Y_n is a basis of the free abelian group L_n and

$$P_n = \left\{ \sum_{y \in Y_n} k_y y \in L_n \;\middle|\; k_y \in [0, l_n n! - 1] \text{ for every } y \in Y_n \right\}.$$

Since $n \geq k$, the integer k divides $l_n n!$ and thus P_n is a finite-to-one transversal for kL_n in L_n. Hence, it is a finite-to-one transversal for $L_n \cap K$ in L_n. Now we have $L_n \leq M_n$ and $k = [L_n : L_n \cap K] \leq [M_n : M_n \cap K] \leq [N : K] = k$. Thus $[L_n : L_n \cap K] = [M_n : M_n \cap K]$ holds. It follows that every transversal for $L_n \cap K$ in L_n is also a transversal for $M_n \cap K$ in M_n. Hence, P_n is a finite-to-one transversal for $M_n \cap K$ in M_n. ■

5.2. Construction of the finite index subgroups

Recall Notation 2.8. In this subsection, we prove that the Følner sequence (F_n) constructed in Lemma 5.6 satisfies the following:

($\#_R$) If $H \leq G$ is a subgroup with $Q_H = R$ and $[N : \mathrm{N}_N(H)] < \infty$, then there exists a sequence of finite index subgroups $K_n \leq G$ such that the sequence of pairs (K_n, F_n) is a Weiss approximation of H.

This part is essentially the same as [9, Section 10], and the lemmas below are proved quite analogously. However, the setting is changed at some points, so we give their proofs for the reader's convenience.

Let $H \leq G$ be a subgroup with $Q_H = R$ and $[N : \mathrm{N}_N(H)] < \infty$. We will construct a sequence of finite index subgroups $K_n \leq G$ with $[K_n] = [Q_n, N_n, \alpha_n]$. The subgroups $Q_n = R \times V_n$ are already defined. Now we define N_n.

Lemma 5.8 ([9, Lemma 10.4]). *For every $n \in \mathbb{N}$, there exists a finite index subgroup $N_n \leq N$ such that $O_n \leq N_n$, $N_H \cap M_n \leq N_n$, $N_H \cap T_n = N_n \cap T_n$, and N_n is Q_n-invariant.*

Proof. Let $\Gamma_n := Q \ltimes N/O_n$ be the semidirect product induced from the $\mathbb{Z}[Q]$-structure on N/O_n. It is finitely generated since N/O_n is a finitely generated $\mathbb{Z}[Q]$-module. The subgroup $\pi_n(N_H \cap M_n) \leq N/O_n$ is Q_n-invariant. Indeed, N_H is R-invariant since $Q_H = R$, M_n is R-invariant by Lemma 5.3 (ii), and V_n acts on N/O_n trivially by Lemma 5.3 (i). Hence,

$$\widetilde{H_n} := Q_n \cdot \pi_n(N_H \cap M_n) \subset \Gamma_n$$

is a subgroup. Since Q_n is finite index in Q, the subgroup $Q_n \cdot N/O_n \leq \Gamma_n$ is finitely generated. Then by Proposition 2.11 (take G, N, H, T of Proposition 2.11 to be $\Gamma_n, N/O_n, \widetilde{H_n}, \pi_n(T_n)$), there exists a finite index subgroup $\widetilde{N_n} \leq N/O_n$ such that $\pi_n(N_H \cap M_n) \cap \pi_n(T_n) = \widetilde{N_n} \cap \pi_n(T_n)$, $\pi_n(N_H \cap M_n) \leq \widetilde{N_n}$, and $\widetilde{N_n}$ is Q_n-invariant. Now we set $N_n = \pi_n^{-1}(\widetilde{N_n})$. Clearly $N_H \cap M_n \leq N_n$ and N_n is Q_n-invariant. Also, $\ker \pi_n = O_n \leq N_n$ holds. Finally, since $T_n \subset M_n$ and $\pi_n|_{M_n}$ is injective by Lemma 5.3 (iii), $N_H \cap T_n = N_n \cap T_n$ holds. ■

Next we define α_n and K_n.

Lemma 5.9 ([9, Lemma 10.6]). *There exist $n_0 \in \mathbb{N}$ and finite index subgroups $K_n \leq G$ with $[K_n] = [Q_n, N_n, \alpha_n]$ defined for $n \geq n_0$ such that the sequence $\alpha_n|_R$ is consistent with α_H.*

Proof. Since $N_H \cap T_n = N_n \cap T_n$ and the sequence T_n is increasing to N, we have $N_n \to N_H$ in $\mathrm{Sub}(N)$. Then by Proposition 2.13, there exist homomorphisms $\alpha'_n : R \to \mathrm{N}_G(N_n)/N_n$ for all n sufficiently large such that the sequence α'_n is consistent with α_H. Then for each n, we extend α'_n to $\alpha_n : Q_n \to \mathrm{N}_G(N_n)/N_n$ by $\alpha_n(s_i^{2m_n}) = a_i^{2m_n} N_n$ for $c+1 \le i \le d$. This is well defined. Indeed, since $Q_n = R \times V_n$ and V_n is a free abelian group generated by $\{s_i^{2m_n}\}_{i=c+1}^{d}$, it suffices to show $[g, a_i^{2m_n}] \in N_n$ for every $g \in G$ and $c+1 \le i \le d$. By the definition of O_n, $[g, a_i^{2m_n}] \in O_n \le N_n$ holds, and it is done. Then α_n satisfies $\overline{\alpha_n(q)} = q$ for every $q \in Q_n$. There exists a unique $K_n \le G$ such that $[K_n] = [Q_n, N_n, \alpha_n]$ by Proposition 2.9. ■

Now we show that the sequence (K_n, F_n) is a Weiss approximation of H.

Lemma 5.10. *We have* $F_n * H - F_n * K_n \to 0$ *in* $C(\mathrm{Sub}(G))^*$.

Proof. It suffices to verify conditions (i)–(v) of Theorem 2.16. Recall that $Q_H = R$. Condition (i) follows from Lemma 5.2 (i). Condition (ii) follows from Lemma 5.3 (ii). Condition (iii) follows from Lemma 5.9. Condition (iv) follows from Lemma 5.8. Finally, condition (v) follows from Lemma 5.5. ■

Lemma 5.11 ([9, Lemma 10.8]). *For every* $n \ge n_0$, $\mathrm{N}_N(H) \cap M_n \le \mathrm{N}_N(K_n)$ *holds.*

Proof. Let $f \in \mathrm{N}_N(H) \cap M_n$ and $g \in K_n$. Set $q = \overline{g} \in Q_n$. Then $g^f = g \cdot [g, f]$ and $[g, f] = f^{1-q} \in N$. It suffices to show $f^{1-q} \in N_n$. Write $q = rv$ with $r \in R$ and $v \in V_n$. We have

$$f^{1-q} = f^{1-r} + f^{r(1-v)}.$$

Since $Q_H = R$, we can take $h \in H$ such that $\overline{h} = r$. Then $f^{1-r} = [h, f] = h^{-1} \cdot f^{-1} h f \in H$ since $f \in \mathrm{N}_N(H)$. Also, we have $f^{1-r} \in M_n$ since $f \in M_n$ and M_n is R-invariant. It follows that $f^{1-r} \in N_H \cap M_n \le N_n$. On the other hand, $f^{r(1-v)} = f^r - f^{rv} \in O_n \le N_n$ since f^r and f^{rv} are in the same O_n-cosets by Lemma 5.3 (i). Hence, $f^{1-q} \in N_n$ holds. ■

Lemma 5.12 ([9, Lemma 10.9]). *For every* n *sufficiently large,* F_n *is a finite-to-one transversal for* $\mathrm{N}_G(K_n)$ *in* G.

Proof. Since $[N : \mathrm{N}_N(H)] < \infty$, the subset P_n is a finite-to-one transversal for $\mathrm{N}_N(H) \cap M_n$ in M_n for every n sufficiently large by Lemma 5.7. Fix such $n \ge n_0$. Then P_n is also a finite-to-one transversal for $\mathrm{N}_N(K_n) \cap M_n$ in M_n by the previous lemma. By Lemma 5.4, P_n is a finite-to-one transversal for $\pi_n'^{-1}(\mathrm{N}_N(K_n) \cap M_n)$ in N. Since $\ker \pi'_n = O_n \le N_n \le \mathrm{N}_N(K_n)$ and $\pi'_n|_{M_n} = \mathrm{id}_{M_n}$, we have $\pi_n'^{-1}(\mathrm{N}_N(K_n) \cap M_n) \le \mathrm{N}_N(K_n)$. Hence, P_n is a finite-to-one transversal for $\mathrm{N}_N(K_n)$ in N. On the other hand, I_n is a finite-to-one transversal for $Q_{\mathrm{N}_G(K_n)}$ in Q by Lemma 5.2 (ii) and $Q_n \le Q_{\mathrm{N}_G(K_n)}$. Hence, $F_n = \widehat{I_n} P_n$ is a finite-to-one transversal for $\mathrm{N}_G(K_n)$ in G by Proposition 2.10. ■

We now have all the ingredients for the proof of our main theorem.

Proof of Theorem 1.3. Let H be a subgroup of G with $Q_H = R$ and $[N : \mathrm{N}_N(H)] < \infty$, and let K_n be a sequence of finite index subgroups of G constructed as above. By Lemmas 5.10 and 5.12, the sequence (K_n, F_n) is a Weiss approximation of H by ignoring small n. Hence, property $(\#_R)$ holds for every $R \leq Q$. By Corollary 2.7, G is P-stable. ■

6. Limits of P-stable groups

Lemma 6.1. *Let G be a countable group and $G_k \leq G$ a sequence of subgroups such that $G_k \to G$ in* $\mathrm{Sub}(G)$. *Assume that for every k, there exists a homomorphism $p_k : G \to G_k$ such that $p_k|_{G_k} = \mathrm{id}_{G_k}$. If G_k is P-stable for every k, then G is P-stable.*

Proof. Let $\phi_n : G \to \mathrm{Sym}(n)$ be an almost homomorphism. Let $F \subset G$ be a finite subset. It suffices to show that there exists a sequence of homomorphisms $\psi_n : G \to \mathrm{Sym}(n)$ such that $d_{\mathrm{H}}(\phi_n(g), \psi_n(g)) \to 0$ for every $g \in F$. Fix k such that $F \subset G_k$. Since $\phi_n|_{G_k} : G_k \to \mathrm{Sym}(n)$ is an almost homomorphism, there exists a sequence of homomorphisms $\psi'_n : G_k \to \mathrm{Sym}(n)$ such that $d_{\mathrm{H}}(\phi_n(g), \psi'_n(g)) \to 0$ for every $g \in F$. Then the sequence of homomorphisms $\psi_n := \psi'_n \circ p_k$ satisfies the required condition. ■

Proof of Corollary 1.4. Let $F_\infty = \langle a_1, a_2, \ldots \rangle$ and $F_k = \langle a_1, \ldots, a_k \rangle \leq F_\infty$ for every $k \in \mathbb{N}$. Let $i_k : F_k/F_k'' \to F_\infty/F_\infty''$ be the homomorphism induced from the inclusion $F_k \to F_\infty$. Also let $p_k : F_\infty/F_\infty'' \to F_k/F_k''$ be the homomorphism induced from the natural projection $F_\infty \to F_k$. Then we have $p_k \circ i_k = \mathrm{id}_{F_k/F_k''}$. Since F_k/F_k'' is P-stable for every k by Theorem 1.3, so is F_∞/F_∞'' by Lemma 6.1. ■

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