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Local permutation stability

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Abstract. We introduce a notion of "local stability in permutations" for finitely generated groups. If a group is sofic and locally stable in our sense, then it is also locally embeddable into finite groups. Our notion is weaker than the "permutation stability" introduced by Glebsky–Rivera and Arzhantseva–Paunescu, which allows one to upgrade soficity to residual finiteness. We prove a necessary and sufficient condition for a finitely generated amenable group to be locally permutation stable, in terms of invariant random subgroups (IRSs), inspired by a similar criterion for permutation stability due to Becker, Lubotzky, and Thom. We apply our criterion to prove that derived subgroups of topological full groups of Cantor minimal subshifts are locally stable, using Zheng's classification of IRSs for these groups. This last result provides continuum-many groups which are locally stable, but not stable.

1. Introduction

The existence of a non-sofic group is one of the major unsolved problems in group theory, and has been a key motivation behind the recent wave of interest in groups which are *stable in permutations* (henceforth *stable*). It is known that a sofic stable group must be residually finite, so to find a non-sofic group it suffices to find a group which is stable but not residually finite [3, 15]. The plausibility of this strategy was demonstrated by De Chiffre, Glebsky, Lubotzky, and Thom [11], who used the analogous notion of *Frobenius stability* to produce examples of groups which are not *Frobenius approximable* (the latter property being analogous to soficity); see also [4] for a construction in the context of "constraint sofic approximations."

One difficulty with using stability in permutations as a path to finding a non-sofic group is that stability is a rather strong property for a group to satisfy, to the extent that the only groups known to be stable have also long been known to be sofic. In this paper we propose a weakening of stability which nonetheless retains the link with soficity: A sofic group satisfying our property, which we call *local stability*, need not be residually finite, but must be locally embeddable into finite groups (LEF). Though it is harder to produce non-LEF groups (LEF being weaker than residual finiteness), this may be a price worth paying if locally stable groups prove to be abundant. To this end, we exhibit many groups which are locally stable but not stable.

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1.1. Terminology and statement of results

Let Γ be a finitely generated group. For $k \in \mathbb{N}$ let d_k be the (normalized) Hamming metric on $\mathrm{Sym}(k)$, given by

$$d_k(\sigma, \tau) = 1 - \frac{1}{k}|\{1 \leq i \leq k : \sigma(i) = \tau(i)\}|$$

for $\sigma, \tau \in \mathrm{Sym}(k)$.

Definition 1.1. Let $(\phi_n : \Gamma \to \mathrm{Sym}(k_n))$ be a sequence of functions.

(i) $(\phi_n)_n$ is an *almost-homomorphism* if, for all $g, h \in \Gamma$,

$$d_{k_n}(\phi_n(g)\phi_n(h), \phi_n(gh)) \to 0 \quad \text{as } n \to \infty.$$

(ii) $(\phi_n)_n$ is a *partial homomorphism* if, for all $g, h \in \Gamma$, there exists $N > 0$ such that for all $n \geq N$, $\phi_n(gh) = \phi_n(g)\phi_n(h)$.

(iii) $(\phi_n)_n$ is *separating* if, for every $e \neq g \in \Gamma$,

$$d_{k_n}(\phi_n(g), \mathrm{id}_{k_n}) \to 1 \quad \text{as } n \to \infty.$$

Definition 1.2. The group Γ is *sofic* if there is a separating almost-homomorphism $\phi_n : \Gamma \to \mathrm{Sym}(k_n)$ for some sequence of positive integers (k_n). Similarly, Γ is *LEF* if it admits a separating partial homomorphism $\phi_n : \Gamma \to \mathrm{Sym}(k_n)$ for some (k_n).

Definition 1.3. The group Γ is *locally stable* (respectively *weakly locally stable*) if, for every almost-homomorphism (respectively every separating almost-homomorphism) $\phi_n : \Gamma \to \mathrm{Sym}(k_n)$, there is a partial homomorphism $\psi_n : \Gamma \to \mathrm{Sym}(n)$ such that for all $g \in \Gamma$,

$$d_{k_n}(\phi_n(g), \psi_n(g)) \to 0 \quad \text{as } n \to \infty.$$

It is immediate from these definitions that a sofic weakly locally stable group is LEF.

Remark 1.4. If we further require in the above definition that $\psi_n : \Gamma \to \mathrm{Sym}(k_n)$ is a homomorphism for all n, then we arrive at the definition of a *stable* (respectively *weakly stable*) group.

Remark 1.5. We have

$$\begin{array}{ccc} \text{Stable} & \Rightarrow & \text{Weakly Stable} \\ \Downarrow & & \Downarrow \\ \text{Locally Stable} & \Rightarrow & \text{Weakly Locally Stable} \end{array}$$

One of the consequences of our results will be that there are no other implications between these four properties. Our main result is the following.

Theorem 1.6 (Theorem 5.19). *There is a continuum of pairwise nonisomorphic finitely generated groups, which are locally stable but not weakly stable.*

To prove Theorem 1.6, we give a criterion for a finitely generated amenable group to be locally stable in terms of invariant random subgroups (IRSs). Here our work is indebted to the influential paper of Becker, Lubotzky, and Thom [6], who gave a necessary and sufficient condition in terms of IRSs for an amenable group to be stable. Recall that an IRS of a discrete group Γ is a probability measure on the Chabauty space $\mathrm{Sub}(\Gamma)$ of subgroups of Γ, which is invariant with respect to the conjugation action of Γ on $\mathrm{Sub}(\Gamma)$. We denote by $\mathrm{IRS}(\Gamma)$ the space of all IRSs of Γ, equipped with the weak* topology. For Γ a group generated by d elements, we fix an epimorphism $\mathbb{F} \twoheadrightarrow \Gamma$ from a rank-d free group $\mathbb{F}$ onto Γ. Then $\mathrm{IRS}(\Gamma)$ is naturally a subspace of $\mathrm{IRS}(\mathbb{F})$.

Theorem 1.7 (Theorem 4.9). *Suppose Γ is a finitely generated amenable group. Then Γ is locally stable if and only if, for every IRS μ of Γ, there exists a sequence of d-marked finite groups Δ_n converging to Γ in the space of marked groups, and an IRS ν_n of Δ_n such that $\nu_n \to \mu$ in* $\mathrm{IRS}(\mathbb{F})$.

It is instructive to compare Theorem 1.7 with [6, Theorem 1.3]. There it was proved that a finitely generated amenable group Γ is stable if and only if every IRS of Γ is the limit of atomic IRSs of Γ supported on finite-index subgroups. Our criterion is weaker in the sense that, although the ν_n are atomic and supported on finite-index subgroups of $\mathbb{F}$ (we may view the ν_n as elements of $\mathrm{IRS}(\mathbb{F})$ since the Δ_n are also quotients of $\mathbb{F}$), they need not lie in $\mathrm{IRS}(\Gamma)$, since the Δ_n need not be quotients of Γ. If Γ is additionally assumed to be finitely presented, however, then the Δ_n are eventually quotients of Γ, and we recover the Becker–Lubotzky–Thom criterion in this special case.

Our main application of Theorem 1.7 is to a family of groups arising from topological dynamics.

Theorem 1.8 (Theorem 5.17). *Let Γ be the topological full group of a Cantor minimal subshift. Then the derived subgroup Γ' of Γ is locally stable.*

Theorem 1.6 is then immediate, since the groups Γ' appearing in Theorem 1.8 are known to fail to be weakly stable, to be finitely generated, and to encompass a continuum of isomorphism types of groups. The IRSs of such groups were classified by Zheng [30]. Theorem 1.8 is proven by showing that the IRSs arising in her classification satisfy the conditions of Theorem 1.7, and using the breakthrough result of Juschenko–Monod that topological full groups are amenable [19]. To apply Theorem 1.7 to the IRSs described in Zheng's result, we need to produce suitable sequences of marked finite groups Δ_n, and this will be achieved using the arguments of [17], where it was proved that topological full groups are LEF, via the construction of partial actions on Kakutani–Rokhlin partitions.

We have one other example of a finitely generated group which is locally stable but not weakly stable.

Theorem 1.9 (Theorem 3.9). *Let $\mathcal{A}(\mathbb{Z})$ be the group of permutations of the set $\mathbb{Z}$ generated by all finitely supported even permutations and the image of the regular representation of the group $\mathbb{Z}$. Then $\mathcal{A}(\mathbb{Z})$ is a finitely generated group which is locally stable but not stable.*

Although this latter result could also be viewed as an application of Theorem 1.7 (see Example 4.13), we shall in fact provide an alternative proof, using the fact that $\mathcal{A}(\mathbb{Z})$ may be obtained as the limit of a directed system of stable groups. The groups appearing in our directed system are drawn from a family constructed by B. H. Neumann [26]; the fact of their stability is a result of Levit–Lubotzky [23].

In view of the relevance of local stability to the search for a non-sofic group, it is disappointing that all the new (that is, non-stable) examples of locally stable groups we produce are amenable (hence sofic), and indeed that amenability plays a key role in our arguments (via Theorem 1.7). Nevertheless, it is intriguing to note that the topological full groups of $\mathbb{Z}$-actions studied in Theorem 1.8 have close relatives which are not known to be sofic. For instance, Thompson's group V is the topological full group of the action of a finitely generated group (namely of V itself). We should note that, since V is finitely presented, it is locally stable iff it is stable (see Lemma 2.14). Nevertheless, given the structural similarities between V and the groups Γ' arising in Theorem 1.8, our results can be seen as evidence that V is stable. If this were the case, then V would be non-sofic.

The paper is structured as follows. After fixing our notation and making some elementary observations about LEF groups, we characterize (weak) local stability, as defined above, in terms of local solutions to "stability challenges"; stability properties of systems of equations; and lifting properties of homomorphisms to metric ultraproducts of finite symmetric groups. This is the subject of Subsection 2.3. In Subsection 2.4, we show that (weak) stability and weak local stability are equivalent in the class of finitely presented groups, and that weak local stability and LEF are equivalent in the class of amenable groups. As a consequence, we exhibit an example of a weakly stable group which is not locally stable. In Section 3, we recall some basic material on the space of marked groups, to be used subsequently, and prove Theorem 1.9. In Section 4, we provide relevant background on IRSs, and prove Theorem 1.7. In Section 5, we discuss topological full groups, prove Theorem 1.8, and deduce Theorem 1.6. We conclude with some open questions and suggested directions for future research.

2. Preliminaries

2.1. Notation

In this paper, all group actions are on the left. For Γ a group, $S \subseteq \Gamma$ a generating set and $n \in \mathbb{N}$, $B_S(n)$ denotes the closed ball of radius n around the identity in the word-metric induced by S on Γ. For n a positive integer, let $[n] = \{1, \dots, n\}$ and let $[\![n]\!] = \{m \in \mathbb{Z} : |m| \leq n\}$.

2.2. LEF groups

The definition of LEF we have given in Definition 1.2 is non-standard. The class of LEF groups is more usually defined in terms of "local embeddings" into finite groups, as follows.

Definition 2.1. Let Γ and Δ be groups and let $A \subseteq \Gamma$. An injective function $\phi : A \to \Delta$ is a *local embedding* if, for all $g, h \in A$, if $gh \in A$ then $\phi(gh) = \phi(g)\phi(h)$.

Proposition 2.2. *A countable group Γ is LEF iff, for all finite $A \subseteq \Gamma$, there exists a finite group Q and a local embedding $\phi : A \to Q$.*

Proof. Let $(\psi_n : \Gamma \to \mathrm{Sym}(k_n))_n$ be a partial homomorphism and let $A \subseteq \Gamma$ be finite. Then for all n sufficiently large, the restriction of ψ_n to A is a local embedding.

Conversely, suppose that (A_n) is an ascending sequence of finite subsets of Γ with union Γ, let Q_n be a finite group affording a local embedding $\phi_n : A_n \to Q_n$, and let $\rho_n : Q_n \to \mathrm{Sym}(|Q_n|)$ be the regular representation of Q_n. Then any function $\psi_n : \Gamma \to \mathrm{Sym}(|Q_n|)$ agreeing with $\rho_n \circ \phi_n$ on A_n is a separating partial homomorphism. ∎

2.3. Characterizations of local stability

Recall that local stability was defined in Definition 1.3. Throughout this subsection, Γ is a group, generated by a finite set $S = \{s_1, \ldots, s_d\}$; $\mathbb{F}$ is the free group on basis S, and $\pi : \mathbb{F} \to \Gamma$ is the standard epimorphism. Where there is no risk of confusion, we shall not distinguish between the free basis S for $\mathbb{F}$ and its image under π in Γ.

2.3.1. Stability challenges.

Definition 2.3. Let X and Y be finite $\mathbb{F}$-sets, with $|X| = |Y|$. For $f : X \to Y$ a bijection, set:

$$\|f\|_{\mathrm{gen}} = \frac{1}{|S|} \sum_{s \in S} \mathrm{Prob}_{x \in X}[f(s(x)) \neq s(f(x))]$$

(where $x \in X$ is uniformly distributed). For X and Y finite $\mathbb{F}$-sets with $|X| = |Y|$, set $d_{\mathrm{gen}}(X, Y)$ to be the minimal value of $\|f\|_{\mathrm{gen}}$, as f ranges over all bijections from X to Y.

Definition 2.4. A *stability challenge* for Γ is a sequence (X_n) of finite $\mathbb{F}$-sets such that, for all $r \in \ker(\pi)$,

$$\mathrm{Prob}_{x \in X_n}[r(x) \neq x] \to 0 \quad \text{as } n \to \infty.$$

The stability challenge (X_n) is *separating* if, for all $w \in \mathbb{F} \setminus \ker(\pi)$,

$$\mathrm{Prob}_{x \in X_n}[w(x) \neq x] \to 1 \quad \text{as } n \to \infty.$$

We call a stability challenge (Y_n) for Γ a *local* Γ*-set* if every $r \in \ker(\pi)$ satisfies the stronger condition:

$$\{y \in Y_n : r(y) \neq y\} = \emptyset \quad \text{for all sufficiently large } n.$$

A *local solution* to the stability challenge (X_n) is a local Γ-set (Y_n) such that $|X_n| = |Y_n|$ for all n, and $d_{\text{gen}}(X_n, Y_n) \to 0$ as $n \to \infty$.

For $k \in \mathbb{N}$, the Hamming metric d_k on $\text{Sym}(k)$ was defined in Subsection 1.1. For an arbitrary finite set X, the Hamming metric d_X on $\text{Sym}(X)$ is defined in a precisely analogous way. In [6, Definitions 3.11 and 7.3], stable groups are characterized in terms of "solutions" to stability challenges. In the same spirit, we have the following.

Proposition 2.5. *The group Γ is locally stable iff every stability challenge for Γ has a local solution. Similarly, Γ is weakly locally stable iff every separating stability challenge for Γ has a local solution.*

Proof. Suppose that Γ is locally stable, and let (X_n) be a stability challenge for Γ, inducing homomorphisms $\widetilde{\phi}_n : \mathbb{F} \to \text{Sym}(X_n)$. Then for every $r \in \ker(\pi)$, we have that $d_{X_n}(\widetilde{\phi}_n(r), \text{id}_{X_n}) \to 0$ as $n \to \infty$. For each $g \in \Gamma$, pick some $w(g) \in \pi^{-1}(g)$, with $w(e) = e$ and $w(s_i) = s_i$. Define $\phi_n : \Gamma \to \text{Sym}(X_n)$ by $\phi_n(g) = \widetilde{\phi}_n(w(g))$. Then for $g, h \in \Gamma$, $w(g)w(h)w(gh)^{-1} \in \ker(\pi)$, so

$$\begin{aligned} d_{X_n}(\phi_n(g)\phi_n(h), \phi_n(gh)) &= d_{X_n}(\phi_n(g)\phi_n(h)\phi_n(gh)^{-1}, \text{id}_{X_n}) \\ &= d_{X_n}(\widetilde{\phi}_n(w(g)w(h)w(gh)^{-1}), \text{id}_{X_n}) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus (ϕ_n) is an almost-homomorphism; by local stability, there is a partial homomorphism (ψ_n) satisfying the conclusion of Definition 1.3. In particular, $\psi_n(e) = \text{id}_{X_n}$ for all n sufficiently large. Define an action of $\mathbb{F}$ on $Y_n = X_n$ by

$$w \cdot y = (\psi_n \circ \pi)(w)[y] \quad \text{for all } w \in \mathbb{F} \text{ and } y \in Y_n.$$

Then each $r \in \ker(\pi)$ acts trivially on Y_n for all sufficiently large n. Thus let $f_n = \text{id} : X_n \to Y_n$, so that if $x \in X_n$ and $s \in S$ are such that $f_n(s \cdot x) \neq s \cdot f_n(x)$, then $\phi(s)[x] \neq \psi(s)[x]$. Thus $d_{\text{gen}}(X_n, Y_n) \leq \| f_n \|_{\text{gen}} \to 0$ as $n \to 0$.

Conversely, suppose that every stability challenge for Γ has a local solution, and let $\phi_n : \Gamma \to \text{Sym}(X_n)$ be an almost-homomorphism, so that X_n is an $\mathbb{F}$-set via $w \cdot x = (\phi_n \circ \pi)(w)[x]$. Then (X_n) is easily seen to be a stability challenge for Γ. Let (Y_n) be a local Γ-set such that there exists a sequence of bijections $(f_n : X_n \to Y_n)$ with $\| f_n \|_{\text{gen}} \to 0$ as $n \to \infty$. Define a function $\psi_n : \Gamma \to \text{Sym}(X_n)$ by

$$\psi_n(g)(x) = f_n^{-1}(w(g) \cdot f_n(x))$$

(it is easily seen that each $\psi_n(g)$ is bijective). Then for $g, h \in \Gamma$, $w(g)w(h)w(gh)^{-1} \in \ker(\pi)$, and since (Y_n) is a local solution, for all n sufficiently large, $w(g)w(h)w(gh)^{-1} \cdot y = y$ for all $y \in Y_n$, from which it easily follows that $\psi_n(g)\psi_n(h) = \psi_n(gh)$. Hence, $(\psi_n)_n$ is a partial homomorphism. Moreover, the fact that $\|f_n\|_{\text{gen}} \to 0$ implies that for every fixed $g \in \Gamma$, $d_{k_n}(\phi_n(g), \psi_n(g)) \to 0$.

The proof of the criterion for weak local stability is essentially identical: It suffices to note (in the forward direction) that if we start with a separating stability challenge (X_n), then the almost-homomorphism (ϕ_n) we constructed is separating also, and (in the reverse direction) that if we start with a separating almost-homomorphism (ϕ_n), then the stability challenge (X_n) we constructed is separating. ∎

2.3.2. Locally stable systems of equations. For $\overline{\sigma} \in \mathrm{Sym}(k)^S$, let $\pi_{\overline{\sigma}} : \mathbb{F} \to \mathrm{Sym}(k)$ be the (unique) homomorphism extending $\overline{\sigma}$, and for $w \in \mathbb{F}$, write $w(\overline{\sigma}) = \pi_{\overline{\sigma}}(w)$ (the evaluation of w at $\overline{\sigma}$). For $R \subseteq \mathbb{F}$, we shall say that $\overline{\sigma}$ is a *solution* to the system of equations $\{r = e\}_{r\in R}$ if $r(\overline{\sigma}) = \mathrm{id}_k$ for all $r \in R$. For $\varepsilon > 0$, $\overline{\sigma}$ is an *ε-almost-solution* to $\{r = e\}_{r\in R}$ if, for all $r \in R$, $d_k(r(\overline{\sigma}), \mathrm{id}_k) < \varepsilon$ (for short, we may also refer to an "ε-almost-solution for R" or a "solution for R," with the same meaning). We shall say that $\overline{\sigma}, \overline{\tau} \in \mathrm{Sym}(k)^S$ are ε-close if

$$\sum_{s\in S} d_k(\overline{\sigma}(s), \overline{\tau}(s)) < \varepsilon.$$

Definition 2.6. For $R \subseteq \mathbb{F}$, we say that the system of equations $\{r = e\}_{r\in R}$ is *locally stable in permutations* if, for all $\varepsilon > 0$ and $R_0 \subseteq R$ finite, there exist $\delta > 0$ and $R_1 \subseteq R$ finite such that, for every $k \in \mathbb{N}$ and every $\overline{\sigma} \in \mathrm{Sym}(k)^S$, if $\overline{\sigma}$ is a δ-almost-solution to $\{r = e\}_{r\in R_1}$, then there is a solution $\overline{\tau} \in \mathrm{Sym}(k)^S$ to $\{r = e\}_{r\in R_0}$ such that $\overline{\sigma}$ and $\overline{\tau}$ are ε-close in $\mathrm{Sym}(k)$.

This definition is a generalization of the concept of a "stable system of equations" first introduced in [15]. In [6, Lemma 3.12], it is shown that the group Γ is stable iff the system of equations $\{r = e : r \in \ker(\pi)\}$ is stable. The analogous statement for local stability is as follows.

Proposition 2.7. *The group Γ is locally stable iff the system of equations $\{r = e : r \in \ker(\pi)\}$ is locally stable in permutations.*

Proof. We use the characterization of local stability for groups in terms of local solutions to stability challenges from Subsection 2.3.1. Suppose first that the system $\{r = e\}_{r\in\ker(\pi)}$ is not locally stable in permutations. Then there exist $\varepsilon > 0$ and $R \subseteq \ker(\pi)$ finite such that for all $n \in \mathbb{N}$, there exist $k_n \in \mathbb{N}$ and a $(1/n)$-almost-solution $\overline{\sigma}^{(n)} \in \mathrm{Sym}(k_n)^S$ to $\ker(\pi) \cap B_S(n)$, which is not ε-close to any solution to $\{r = e\}_{r\in R}$. Let $X_n = [k_n]$ be an $\mathbb{F}$-set, via $w \cdot j = w(\overline{\sigma}^{(n)})(j)$. Then (X_n) is a stability challenge for Γ. If (Y_n) is a local solution, and $f_n : X_n \to Y_n$ are bijections satisfying $\|f_n\|_n \to 0$ as $n \to \infty$, then, defining $\overline{\tau}^{(n)} \in \mathrm{Sym}(k_n)^S$ by $\overline{\tau}^{(n)}(s_i)(j) = f_n^{-1}(s_i \cdot f_n(j))$, we have that for all n sufficiently large, $\overline{\tau}^{(n)}$ is a solution for $\{r = e\}_{r\in R}$ which is ε-close to $\overline{\sigma}^{(n)}$, contradiction.

Conversely, suppose that $\{r = e\}_{r \in \ker(\pi)}$ is locally stable in permutations. For each $m \in \mathbb{N}$, let $\delta_m > 0$ and $a_m \in \mathbb{N}$ be such that $\delta_m \to 0$; (a_m) is an increasing sequence and every δ_m-almost-solution to $\ker(\pi) \cap B_S(a_m)$ is $(1/m)$-close to a solution for $\ker(\pi) \cap B_S(m)$. Let (X_n) be a stability challenge for Γ. Let $k_n = |X_n|$ and fix a bijection $g_n : [k_n] \to X_n$. Define $\overline{\sigma}^{(n)} \in \mathrm{Sym}(k_n)^S$ by $\overline{\sigma}^{(n)}(s_i)(j) = g_n^{-1}(s_i \cdot g_n(j))$. Then there exists a strictly increasing sequence (N_m) such that for all $n \geq N_m$, $\overline{\sigma}^{(n)}$ is a δ_m-solution to $\ker(\pi) \cap B_S(a_m)$. For $N_m \leq n < N_{m+1}$, let $\overline{\tau}^{(n)}$ be a solution to $\ker(\pi) \cap B_S(m)$ which is $(1/m)$-close to $\overline{\sigma}^{(n)}$. Define $Y_n = [k_n]$, which is an $\mathbb{F}$-set via $w \cdot y = w(\overline{\tau}^{(n)})(y)$. Then (Y_n) is a local Γ-set, and the bijections $f_n = g_n^{-1} : X_n \to Y_n$ witness that (Y_n) is a local solution to the stability challenge (X_n). ∎

Remark 2.8. It follows from Proposition 2.7 that for $R \subseteq \mathbb{F}$, whether or not the system $\{r = e : r \in R\}$ is locally stable in permutations depends only on the isomorphism type of the group presented by $\langle S \mid R \rangle$. The corresponding statement for stability follows similarly from [6, Lemma 3.12]. The special case of finitely presented groups appeared already in [3]: As is noted in that paper, stability of a system of relations is unaffected by applying Tietze moves to the system.

We can likewise capture weak local stability in terms of solutions to equations. For the sake of simplicity, we restrict our attention to stability of families of equations constituting a normal subgroup of $\mathbb{F}$.

Definition 2.9. For $R \subseteq \mathbb{F} \setminus \{e\}$, and $\delta > 0$, $\overline{\sigma} \in \mathrm{Sym}(k)^S$ is $(1-\delta)$-*separating* for R if, for all $r \in R$,

$$d_k(r(\overline{\sigma}), \mathrm{id}_k) > 1 - \delta.$$

For $N \lhd \mathbb{F}$, the system $\{r = e : r \in N\}$ is *weakly locally stable in permutations* if, for all $\varepsilon > 0$ and $R_0 \subseteq N$ finite, there exist $\delta > 0$ and finite subsets $R_1 \subseteq N$ and $R_1' \subseteq \mathbb{F} \setminus N$ such that every δ-almost-solution to $\{r = e : r \in R_1\}$, which is $(1-\delta)$-separating for R_1', is ε-close to a solution for $\{r = e : r \in R_0\}$.

Proposition 2.10. *The group* Γ *is weakly locally stable iff the system of equations* $\{r = e : r \in \ker(\pi)\}$ *is locally stable in permutations.*

Proof. This is much the same as the proof of Proposition 2.7. For the "only if" direction, we may assume that the tuple $\overline{\sigma}^{(n)} \in \mathrm{Sym}(k_n)^S$ described in the proof of Proposition 2.7 is $(1 - 1/n)$-separating for $B_S(n) \setminus \ker(\pi)$. For the "if" direction, we choose δ_m and a_m such that every δ_m-almost-solution to $\ker(\pi) \cap B_S(a_m)$, which is $(1-\delta_m)$-separating for $B_S(a_m) \setminus \ker(\pi)$, is $(1/m)$-close to a solution for $\ker(\pi) \cap B_S(m)$, and argue as in the proof of Proposition 2.7. ∎

2.3.3. Lifting properties of homomorphisms to ultraproducts. Let $\mathbf{k} = (k_n)_n$ be an increasing sequence of positive integers. Let

$$G_{\mathbf{k}} = \prod_n \mathrm{Sym}(k_n)$$

be the Cartesian product of the finite groups $\mathrm{Sym}(k_n)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Define the normal subgroups

$$N_{\mathcal{A},\mathbf{k}} = \{(\sigma_n) \in G_{\mathbf{k}} : \{n \in \mathbb{N} : \sigma_n = \mathrm{id}_{k_n}\} \in \mathcal{U}\}$$
$$N_{\mathcal{M},\mathbf{k}} = \{(\sigma_n) \in G_{\mathbf{k}} : \forall \varepsilon > 0, \{n \in \mathbb{N} : d_{k_n}(\sigma_n, \mathrm{id}_{k_n}) < \varepsilon\} \in \mathcal{U}\}$$

of $G_{\mathbf{k}}$ (so that $N_{\mathcal{A},\mathbf{k}} \leq N_{\mathcal{M},\mathbf{k}}$) and define $G_{\mathcal{A},\mathbf{k}} = G_{\mathbf{k}}/N_{\mathcal{A},\mathbf{k}}$, $G_{\mathcal{M},\mathbf{k}} = G/N_{\mathcal{M},\mathbf{k}}$ (called, respectively, the *algebraic ultraproduct* and *metric ultraproduct* of the $\mathrm{Sym}(k_n)$). Let $\pi_{\mathcal{A}} : G_{\mathbf{k}} \twoheadrightarrow G_{\mathcal{A},\mathbf{k}}$, $\pi_{\mathcal{M}} : G_{\mathbf{k}} \twoheadrightarrow G_{\mathcal{M},\mathbf{k}}$, and $\pi_{\mathcal{M},\mathcal{A}} : G_{\mathcal{A},\mathbf{k}} \twoheadrightarrow G_{\mathcal{M},\mathbf{k}}$ be the natural epimorphisms. For any $(\sigma_n), (\tau_n) \in G_{\mathbf{k}}$, there is a well-defined limit $\widetilde{d}_{\mathcal{M},\mathbf{k}}((\sigma_n), (\tau_n)) = \lim_{n\to\mathcal{U}} d_{k_n}(\sigma_n, \tau_n) \in [0, 1]$ of $d_{k_n}(\sigma_n, \tau_n)$ along $\mathcal{U}$. Then $\widetilde{d}_{\mathcal{M},\mathbf{k}}$ is a pseudometric on $G_{\mathbf{k}}$, which descends to a well-defined metric $d_{\mathcal{M},\mathbf{k}}$ on $G_{\mathcal{M},\mathbf{k}}$.

Definition 2.11. For Γ a group, a *sofic representation* of Γ is a homomorphism $\Phi : \Gamma \to G_{\mathcal{M},\mathbf{k}}$ such that, for all $e \neq g \in \Gamma$, $d_{\mathcal{M},\mathbf{k}}(\Phi(g), e) = 1$.

A countable group Γ is sofic iff it admits a sofic representation (see [12, Section 2] for a proof). There is also a characterization of LEF in terms of embeddings into algebraic ultraproducts (see [10, Section 7.2] and the references therein). We include a proof of the exact formulation we need for the reader's convenience.

Lemma 2.12. *A countable group Γ is LEF iff there exists an increasing sequence $\mathbf{k} = (k_n)_n$ and a homomorphism $\Phi : \Gamma \to G_{\mathcal{A},\mathbf{k}}$ such that $\pi_{\mathcal{M},\mathcal{A}} \circ \Phi$ is a sofic representation.*

Proof. Given a separating partial homomorphism $(\phi_n : \Gamma \to \mathrm{Sym}(k_n))$, let $\mathbf{k} = (k_n)_n$ and define the function $\widetilde{\phi} : \Gamma \to G_{\mathbf{k}}$ by $\widetilde{\phi}(g) = (\phi_n(g))_n$. Then $\Phi = \pi_{\mathcal{A}} \circ \widetilde{\phi}$ has the desired properties.

Conversely, suppose such a homomorphism Φ exists, let $A \subseteq \Gamma$ be a finite set, and for each $g \in A$, let $\widetilde{\phi}(g) \in G_{\mathbf{k}}$ be a lift of $\Phi(g)$, so that $\widetilde{\phi}(g)_n \in \mathrm{Sym}(k_n)$. For each $g, h \in A$ satisfying $gh \in A$, we have $\widetilde{\phi}(g)\widetilde{\phi}(h)\widetilde{\phi}(gh)^{-1} \in N_{\mathcal{A},\mathbf{k}}$. Moreover, for each $e \neq g \in A$, since $d_{\mathcal{M},\mathbf{k}}((\pi_{\mathcal{M},\mathcal{A}} \circ \Phi)(g), e) = 1$,

$$\{n \in \mathbb{N} : d_{k_n}(\widetilde{\phi}(g)_n, \mathrm{id}_{k_n}) \geq 1/2\} \in \mathcal{U}.$$

In particular, there exists $n \in \mathbb{N}$ such that $g \mapsto \widetilde{\phi}(g)_n$ defines a local embedding. We apply Proposition 2.2 to conclude. ∎

In [3, Sections 4 and 6], it is proved that Γ is (weakly) stable iff for every $\mathbf{k}$, every homomorphism (respectively every sofic representation) from Γ to $G_{\mathcal{M},\mathbf{k}}$ may be lifted to $G_{\mathbf{k}}$. Similarly, we have the following.

Proposition 2.13. *The group Γ is (weakly) locally stable iff, for every $\mathbf{k}$ and every homomorphism (respectively every sofic representation) $\Phi : \Gamma \to G_{\mathcal{M},\mathbf{k}}$, there is a homomorphism $\widehat{\Phi} : \Gamma \to G_{\mathcal{A},\mathbf{k}}$ such that $\pi_{\mathcal{M},\mathcal{A}} \circ \widehat{\Phi} = \Phi$.*

Proof. We shall use Proposition 2.7. Suppose first that $\{r = e\}_{r \in \ker(\pi)}$ is locally stable in permutations. Let $\Phi : \Gamma \to G_{\mathcal{M},\mathbf{k}}$ be a homomorphism, let $\tilde{\phi} : \mathbb{F} \to G_{\mathbf{k}}$ be a lift of $(\Phi \circ \pi) : \mathbb{F} \to G_{\mathcal{M},\mathbf{k}}$ to $G_{\mathbf{k}}$ (so that $\tilde{\phi}(\ker(\pi)) \leq N_{\mathcal{M},\mathbf{k}}$), and let $\tilde{\phi}_n = p_n \circ \tilde{\phi}$, where $p_n : G_{\mathbf{k}} \to \mathrm{Sym}(k_n)$ is projection to the nth coordinate. For any finite $R \subseteq \ker(\pi)$ and any $\delta > 0$,

$$\{n \in \mathbb{N} : d_{k_n}(\tilde{\phi}_n(r), \mathrm{id}_{k_n}) < \delta \text{ for all } r \in R\} \in \mathcal{U}.$$

For each $m \in \mathbb{N}$, let $\delta_m > 0$ and $a_m \in \mathbb{N}$ be such that $\delta_m \to 0$; (a_m) is an increasing sequence and every δ_m-almost-solution to $\ker(\pi) \cap B_S(a_m)$ is $(1/m)$-close to a solution for $\ker(\pi) \cap B_S(m)$. Let

$$I_m = \{n \in \mathbb{N} : d_{k_n}(\tilde{\phi}_n(r), \mathrm{id}_{k_n}) < \delta_m \text{ for all } r \in \ker(\pi) \cap B_S(a_m)\} \in \mathcal{U}.$$

Then $I_{m+1} \subseteq I_m$ and

$$\bigcap_{m \in \mathbb{N}} I_m = \{n \in \mathbb{N} : \tilde{\phi}_n(r) = e \text{ for all } r \in \ker(\pi)\}.$$

For $n \in I_m \setminus I_{m+1}$, there is a solution $\tilde{\Psi}_n \in \mathrm{Sym}(k_n)^S$ to $\ker(\pi) \cap B_S(m)$ which is $(1/m)$-close to $\tilde{\phi}_n$. Extend $\tilde{\Psi}_n$ to $\mathbb{F}$ and define $\tilde{\Psi} : \mathbb{F} \to G_{\mathbf{k}}$ by $p_n \circ \tilde{\Psi} = \tilde{\Psi}_n$. Then every $r \in \ker(\pi)$ lies in $B_S(m)$ for all m sufficiently large, so $\{n \in \mathbb{N} : \tilde{\Psi}_n(r) = \mathrm{id}_{k_n}\} \in \mathcal{U}$, hence $\tilde{\Psi}(\ker(\pi)) \leq N_{\mathcal{A},\mathbf{k}}$, and $\tilde{\Psi}$ descends to $\Psi : \Gamma \to G_{\mathcal{A},\mathbf{k}}$. Finally, for all $\varepsilon > 0$,

$$\{n \in \mathbb{N} : d_{k_n}(\tilde{\phi}_n(s), \tilde{\Psi}_n(s)) < \varepsilon \text{ for all } s \in S\} \in \mathcal{U},$$

so $\pi_{\mathcal{M},\mathcal{A}} \circ \Psi$ agrees with Φ on S, hence on Γ, and $\hat{\Phi} = \Psi$ is as desired.

Conversely, suppose that $\{r = e\}_{r \in \ker(\pi)}$ is not locally stable in permutations. Then there exist $\varepsilon > 0$ and $R \subseteq \ker(\pi)$ finite such that for all $n \in \mathbb{N}$, there exist $k_n \in \mathbb{N}$ and a $(1/n)$-almost-solution $\overline{\sigma}^{(n)} \in \mathrm{Sym}(k_n)^S$ to $\ker(\pi) \cap B_S(n)$, which is not ε-close to any solution to R. Define $\tilde{\phi} : \mathbb{F} \to G_{\mathbf{k}}$ by $\tilde{\phi}(s) = (\overline{\sigma}^{(n)}(s))_n$. Then for every $r \in \ker(\pi)$, and all n sufficiently large, $d_{k_n}(r(\overline{\sigma}^{(n)}), \mathrm{id}_{k_n}) < 1/n$, so $\tilde{\phi}(r) \in N_{\mathcal{M},\mathbf{k}}$, and $\tilde{\phi}$ descends to $\Phi : \Gamma \to G_{\mathcal{M},\mathbf{k}}$. If Φ lifts to $\hat{\Phi} : \Gamma \to G_{\mathcal{A},\mathbf{k}}$, and $\overline{\tau}^{(n)} \in \mathrm{Sym}(k_n)^S$ is such that $\hat{\Phi}(\pi(s)) = (\overline{\tau}^{(n)}(s))_n N_{\mathcal{A},\mathbf{k}}$ for all $s \in S$, then $(\overline{\sigma}^{(n)}(s)\overline{\tau}^{(n)}(s)^{-1})_n \in N_{\mathcal{M},\mathbf{k}}$ for $s \in S$. Hence, there exists n for which $\overline{\sigma}^{(n)}$ and $\overline{\tau}^{(n)}$ are ε-close and $r(\overline{\tau}^{(n)}) = e$ for all $r \in R$, contradiction.

The argument for weak local stability is much the same, using Proposition 2.10 in the forward direction. ∎

2.4. First properties of the class of locally stable groups

All stable groups are locally stable. This includes all finite groups [15], all polycyclic-by-finite groups and the Baumslag–Solitar groups $\mathrm{BS}(1, n)$ [6], Grigorchuk's group and the Gupta–Sidki p-groups [30], and the (restricted, regular) wreath product of any two finitely generated abelian groups [22]. Note that all groups listed above are amenable; finite-rank nonabelian free groups provide a class of nonamenable groups which are easily seen to be stable: A free group admits a finite presentation with no relations, and the empty set of equations is trivially stable.

We continue to let S be a finite generating set for the group Γ, $\mathbb{F}$ be the free group on basis S, and $\pi : \mathbb{F} \to \Gamma$ be the associated epimorphism.

Lemma 2.14. *Suppose Γ is finitely presented and (weakly) locally stable. Then Γ is (weakly) stable.*

Proof. It suffices to show that, for every partial homomorphism $\phi_n : \Gamma \to \mathrm{Sym}(k_n)$, there is a sequence of homomorphisms $\psi_n : \Gamma \to \mathrm{Sym}(k_n)$ such that, for all $g \in \Gamma$, $d_{k_n}(\phi_n(g), \psi_n(g)) \to 0$ as $n \to \infty$. Indeed, we shall produce ψ_n such that $\phi_n(g) = \psi_n(g)$ for all sufficiently large n.

Let $R \subseteq \mathbb{F}$ be a finite set which normally generates $\ker(\pi)$. Given $m > 0$, let $N = N(m) > 0$ be sufficiently large that for all $g, h \in \Gamma$ of word-length at most m with respect to S, and for all $n \geq N$, $\phi_n(gh) = \phi_n(g)\phi_n(h)$. In particular, for every $w \in \mathbb{F}$ of length at most m, we have

$$\phi_n(\pi(w)) = w(\overline{\sigma}^{(n)}), \tag{1}$$

where $\overline{\sigma}^{(n)} = (\phi_n(s))_{s \in S} \in \mathrm{Sym}(k_n)^S$. In particular, for all $r \in R$, $r(\overline{\sigma}^{(n)}) = \mathrm{id}_{k_n}$, and we have a well-defined homomorphism $\psi_n : \Gamma \to \mathrm{Sym}(k_n)$ sending s to $\phi_n(s)$. By (1), ϕ_n and ψ_n agree on $B_S(m) \subseteq \Gamma$, as desired. ∎

We recall the key observation made following Definition 1.3, which follows immediately from the relevant definitions. The analogous statement for stability appeared in [15, Theorem 2] and [3, Theorem 7.2].

Lemma 2.15. *Suppose Γ is sofic and weakly locally stable. Then Γ is LEF.*

Within the class of amenable groups, there is a simple necessary and sufficient condition for weak stability.

Theorem 2.16 ([3, Theorem 7.2 (iii)]). *Suppose Γ is amenable. Then Γ is weakly stable iff it is residually finite.*

We prove a similar criterion for weak local stability among amenable groups.

Lemma 2.17. *Suppose Γ is amenable. Then Γ is weakly locally stable iff it is LEF.*

Proof. Every amenable group is sofic, as a result the "only if" direction follows from Lemma 2.15. For the "if" direction, we shall use the criterion in terms of ultraproducts from Proposition 2.13. Since Γ is LEF, by Lemma 2.12, there exist an increasing sequence $\mathbf{k} = (k_n)$ and a homomorphism $\psi : \Gamma \to G_{\mathcal{A},\mathbf{k}}$ such that $\pi_{\mathcal{M},\mathcal{A}} \circ \psi$ is a separating homomorphism. Let $\phi : \Gamma \to G_{\mathcal{M},\mathbf{k}}$ be any separating homomorphism. By [13], since Γ is amenable, if $\phi_1, \phi_2 : \Gamma \to G_{\mathcal{M},\mathbf{k}}$ are separating homomorphisms, then there exists $h \in G_{\mathcal{M},\mathbf{k}}$ such that for all $g \in \Gamma$, $\phi_2(g) = h\phi_1(g)h^{-1}$. Apply this result to $\phi_1 = \phi$ and $\phi_2 = \pi_{\mathcal{M},\mathcal{U}} \circ \psi$ to obtain a corresponding $h \in G_{\mathcal{M},\mathbf{k}}$. Letting $\tilde{h} \in \pi_{\mathcal{M},\mathcal{A}}^{-1}(h)$, $\tilde{\phi} : g \mapsto \tilde{h}^{-1}\psi(g)\tilde{h}$ is a homomorphism satisfying $\pi_{\mathcal{M},\mathcal{A}} \circ \tilde{\phi} = \phi$. By Proposition 2.13, Γ is weakly stable. ∎

Corollary 2.18. *There exist groups with the following properties:*

(i) *There exists a finitely generated weakly locally stable group which is not weakly stable.*

(ii) *There exists a finitely generated group which is not weakly locally stable.*

Proof. (i) By Lemma 2.17 and Theorem 2.16, any finitely generated amenable, LEF group which is not residually finite will do. The wreath product $\mathrm{Alt}(5) \wr \mathbb{Z}$ is such a group: It is amenable, being (locally finite)-by-abelian, and is LEF by [29, Theorem 2.4 (ii)].

(ii) By Lemma 2.15, any finitely generated sofic group which is not LEF will do. The Baumslag–Solitar group $\mathrm{BS}(2,3)$ is such a group: It is not LEF by Corollary 4 in [29, Section 2.2] (and the comment following); its soficity is explained, for instance, in [28, Example 4.6]. ∎

Theorem 2.19. *There is a finitely presented soluble group which is weakly stable but not locally stable.*

Proof. Let p be a prime and let $A_p \leq \mathrm{GL}_4(\mathbb{Q})$ be the p-Abels group (see [6, Corollary 8.7]). Then A_p is

(i) finitely presented;

(ii) linear over $\mathbb{Q}$ (and being finitely generated, is therefore residually finite, by Mal'cev's theorem);

(iii) soluble (hence amenable);

(iv) not stable.

By (i) and (iv), we may apply Lemma 2.14 to conclude that A_p is not locally stable. By (ii) and (iii), we may apply Theorem 2.16 and deduce that Γ is weakly stable. ∎

3. Limits in the space of marked groups

The space $\mathcal{G}_d$ of marked d-generated groups was introduced in [16, 18] and may be constructed as follows. Fix $d \in \mathbb{N}$ and an ordered d-element set $\mathbf{X} = \{x_1, \ldots, x_d\}$. We may define $\mathcal{G}_d$ to be the set of all normal subgroups of the free group $\mathbb{F} = F(\mathbf{X})$ on $\mathbf{X}$. Alternatively, the points of $\mathcal{G}_d$ may be described in terms of *d-markings on groups*: If Γ is a d-generated group and $\mathbf{S} = (s_1, \ldots, s_d)$ is an ordered generating d-tuple for Γ, then the pair $(\Gamma, \mathbf{S})$, henceforth to be called a *d-marked group*, determines an epimorphism $\pi_{\mathbf{S}} : F(\mathbf{X}) \to \Gamma$ sending x_i to s_i for $1 \leq i \leq d$, and hence the point $\ker(\pi_{\mathbf{S}}) \in \mathcal{G}_d$. Conversely, every point $N \in \mathcal{G}_d$ determines the d-generated group $\Gamma_N = F(\mathbf{X})/N$ and the generating d-tuple $\mathbf{S}_N = (x_i N)_i \in \Gamma^d$, so that $N = \ker(\pi_{\mathbf{S}_N})$.

By a *quotient of d-marked groups* $(\Gamma, \mathbf{S}) \twoheadrightarrow (\Delta, \mathbf{T})$, we shall mean a homomorphism $\phi : \Gamma \to \Delta$ such that $\phi(s_i) = t_i$ for $1 \leq i \leq d$. By the fact that $\mathbf{S}$ and $\mathbf{T}$ generate, such ϕ is unique (if it exists) and surjective. We note that two d-marked groups $(\Gamma, \mathbf{S})$ and $(\Delta, \mathbf{T})$ determine the same point in $\mathcal{G}_d$ if and only if the quotient of d-marked groups $(\Gamma, \mathbf{S}) \twoheadrightarrow (\Delta, \mathbf{T})$ exists and is an isomorphism. We may give $\mathcal{G}_d$ the structure of a metric space, as follows. For $N, M \lhd F(\mathbf{X})$, we write

$$\nu(N, M) = \max\{n \in \mathbb{N} : N \cap B_{\mathbf{X}}(n) = M \cap B_{\mathbf{X}}(n)\} \in \mathbb{N} \cup \{\infty\}$$

and set

$$d(N, M) = 2^{-\nu(N,M)}.$$

Then d is a well-defined metric on $\mathcal{G}_d$. There is a well-known characterization of the class of d-generated LEF groups in terms of the topology of $\mathcal{G}_d$, as described in the next Lemma, which is proved in [29, Section 1.4].

Lemma 3.1. *Let $(\Gamma, \mathbf{S})$ be a d-marked group and let $(\Delta_n, \mathbf{T}_n)$ be a sequence of marked finite d-generated groups. Then $(\Delta_n, \mathbf{T}_n)$ converges to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$ iff, for all $r \in \mathbb{N}$, for all n sufficiently large there is a local embedding $\phi_n : B_{\mathbf{S}}(r) \to \Delta_n$ satisfying $\phi_n(s_i) = t_{n,i}$ for $1 \leq i \leq d$. In particular, Γ is LEF iff for some (equivalently any) d-marking $\mathbf{S}$ of Γ, $(\Gamma, \mathbf{S})$ lies in the closure in $\mathcal{G}_d$ of the subspace of marked d-generated finite groups.*

Remark 3.2. Let Γ be a d-generated group. If Γ is finitely presented then, for any d-marking $\mathbf{S}$ on Γ, $\ker(\pi_{\mathbf{S}})$ has a finite normal generating set, so there exists $C > 0$ such that, for any $N \lhd \mathbf{F}$, if $\nu(\ker(\pi_{\mathbf{S}}), N) \geq C$, then $\ker(\pi_{\mathbf{S}}) \leq N$. It follows that $(\Gamma, \mathbf{S})$ has an open neighbourhood in $\mathcal{G}_d$ consisting entirely of marked quotients of $(\Gamma, \mathbf{S})$. A slightly more general variant, which follows from the same argument, is: If $(\Gamma, \mathbf{S})$ and $(\widehat{\Gamma}, \widehat{\mathbf{S}})$ are marked d-generated groups, with $\widehat{\Gamma}$ finitely presented, and there is a quotient $(\widehat{\Gamma}, \widehat{\mathbf{S}}) \twoheadrightarrow (\Gamma, \mathbf{S})$ of d-marked groups, then $(\Gamma, \mathbf{S})$ has an open neighbourhood in $\mathcal{G}_d$ consisting entirely of marked quotients of $(\widehat{\Gamma}, \widehat{\mathbf{S}})$.

Definition 3.3. Given a sequence $(\Gamma_m)_m$ of d-generated groups, and given for each m a d-marking $\mathbf{S}_m = (s_{m,1}, \ldots, s_{m,d})$ on Γ_m, let

$$\widetilde{\mathbf{S}} = (\widetilde{s}_1, \ldots, \widetilde{s}_d) \in \Big(\prod_m \Gamma_m\Big)^d$$

be given by $\widetilde{s}_{i,m} = s_{m,i} \in \Gamma_m$. Let $\bigotimes(\Gamma_m, \mathbf{S}_m)$ be the subgroup of $\prod_m \Gamma_m$ generated by the set $\{\widetilde{s}_i : 1 \leq i \leq d\}$ (so that $\widetilde{\mathbf{S}}$ is a d-marking on $\bigotimes(\Gamma_m, \mathbf{S}_m)$). We shall refer to the group $\bigotimes(\Gamma_m, \mathbf{S}_m)$ as the *diagonal product* of the sequence $(\Gamma_m, \mathbf{S}_m)_m$, and $\widetilde{\mathbf{S}}$ as the *diagonal d-marking*.

Note that the projection $p_n : \prod_m \Gamma_m \to \Gamma_n$ restricts to a quotient of d-marked groups $(\bigotimes(\Gamma_m, \mathbf{S}_m), \widetilde{\mathbf{S}}) \twoheadrightarrow (\Gamma_n, \mathbf{S}_n)$.

Proposition 3.4 ([20, Lemma 4.6]). *Suppose the sequence $(\Gamma_m, \mathbf{S}_m)_m$ converges in $\mathcal{G}_d$ to $(\Gamma, \mathbf{S})$. Then there is a quotient of d-marked groups*

$$\tau : \left(\bigotimes(\Gamma_m, \mathbf{S}_m), \widetilde{\mathbf{S}}\right) \twoheadrightarrow (\Gamma, \mathbf{S}),$$

called the tail homomorphism*, with kernel*

$$\ker(\tau) = \left(\bigotimes(\Gamma_m, \mathbf{S}_m)\right) \cap \left(\bigoplus_m \Gamma_m\right).$$

Since the class of amenable groups is closed under subgroups, extensions, and ascending unions, we deduce the following, which will be used in the next section.

Lemma 3.5. *Let $(\Gamma_m, \mathbf{S}_m)_m$ and $(\Gamma, \mathbf{S})$ be as in Proposition 3.4. If Γ and the Γ_m are amenable, then so is $\bigotimes(\Gamma_m, \mathbf{S}_m)$.*

The next proposition gives a sufficient condition for local stability of a group Γ in terms of local stability of a sequence of stable groups converging to Γ in $\mathcal{G}_d$. The hypotheses of the condition are rather strong, but they are general enough to allow us to exhibit our first explicit example of a locally stable finitely generated group which is not stable.

Proposition 3.6. *Let $(\Gamma_m, \mathbf{S}_m)_m$ and $(\Gamma, \mathbf{S})$ be as in Proposition 3.4. Suppose that for all m, there is a quotient of d-marked groups $\pi^{(m)} : (\Gamma_m, \mathbf{S}_m) \to (\Gamma, \mathbf{S})$. If Γ_m is locally stable for all m, then Γ is locally stable.*

Proof. There is a sequence (r_m) in $\mathbb{N}$, tending to ∞, such that the restriction of $\pi^{(m)}$ to $B_{\mathbf{S}_m}(r_m)$ is a bijection onto $B_{\mathbf{S}}(r_m)$. Define $\theta^{(m)} : \Gamma \to \Gamma_m$ such that, for all $h \in B_{\mathbf{S}}(r_m)$, $h = (\pi^{(m)} \circ \theta^{(m)})(h)$ (with $\theta^{(m)}(h)$ defined arbitrarily for $h \in \Gamma \setminus B_{\mathbf{S}}(r_m)$). Then for all $g, h \in B_{\mathbf{S}}(r_m/2)$, $\theta^{(m)}(gh) = \theta^{(m)}(g)\theta^{(m)}(h)$.

Let $(\phi_n : \Gamma \to \mathrm{Sym}(k_n))_n$ be an almost-homomorphism. Then for any fixed m, $(\phi_n \circ \pi^{(m)}) : \Gamma_m \to \mathrm{Sym}(k_n)$ defines an almost-homomorphism, so by local stability, there exists a partial homomorphism $(\psi_n^{(m)} : \Gamma_m \to \mathrm{Sym}(k_n))_n$ such that, for all m, and all $g \in \Gamma_m$,

$$d_{k_n}(\psi^{(m)}(g), (\phi_n \circ \pi^{(m)})(g)) \to 0 \quad \text{as } m \to \infty.$$

Fix a sequence (ε_n) of positive reals, converging to 0. Then there exists a sequence (m_n) in $\mathbb{N}$, tending to ∞, such that:

(i) $(\psi_n^{(m_n)} \circ \theta^{(m_n)} : \Gamma \to \mathrm{Sym}(k_n))_n$ is a partial homomorphism.

(ii) For all $n \in \mathbb{N}$ and $g \in B_{\mathbf{S}_{m_n}}(1/\varepsilon_n)$,

$$d_{k_n}(\psi^{(m_n)}(g), (\phi_n \circ \pi^{(m_n)})(g)) < \varepsilon_n.$$

Now given $h \in \Gamma$, let $l > 0$ be such that $h \in B_{\mathbf{S}}(l)$. Then for all n sufficiently large, $\min(1/\varepsilon_n, r_{m_n}) > l$ and there exists $g \in B_{\mathbf{S}_{m_n}}(l)$ with $h = \pi^{(m_n)}(g)$, so that $(\psi_n^{(m_n)} \circ \theta^{(m_n)})(h) = \psi_n^{(m_n)}(g)$, and

$$d_{k_n}((\psi_n^{(m_n)} \circ \theta^{(m_n)})(h), \phi_n(h)) = d_{k_n}(\psi^{(m_n)}(g), (\phi_n \circ \pi^{(m_n)})(g)) < \varepsilon_n,$$

so $(\psi_n^{(m_n)} \circ \theta^{(m_n)})$ is close to (ϕ_n), as desired. ■

Remark 3.7. The hypothesis in Proposition 3.6 that the convergence of $(\Gamma_m, \mathbf{S}_m)_m$ to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$ is induced by a sequence of epimorphisms $\pi_m : \Gamma_m \to \Gamma$ cannot be entirely removed. For example, consider the group described in the proof of Theorem 2.19: It is not locally stable, but it is residually finite, hence LEF, hence there are *finite* marked groups $(\Gamma_m, \mathbf{S}_m)_m$ converging to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$, and finite groups are stable by [15, Theorem 2].

Let $\mathrm{FSym}(\mathbb{Z}) \leq \mathrm{Sym}(\mathbb{Z})$ be the group of finitely supported permutations of the set $\mathbb{Z}$. $\mathrm{FSym}(\mathbb{Z})$ has a subgroup $\mathrm{FAlt}(\mathbb{Z})$ of index 2, consisting of all finitely supported even permutations. Both $\mathrm{FSym}(\mathbb{Z})$ and $\mathrm{FAlt}(\mathbb{Z})$ are normal in $\mathrm{Sym}(\mathbb{Z})$. $\mathrm{FAlt}(\mathbb{Z})$ is the ascending union of the alternating groups on finite subsets of $\mathbb{Z}$; thus $\mathrm{FAlt}(\mathbb{Z})$ is an infinite simple group. Let $\rho : \mathbb{Z} \to \mathrm{Sym}(\mathbb{Z})$ denote the regular action of the additive group $\mathbb{Z}$; thus $\mathbb{Z}$ acts (through ρ) via conjugation on $\mathrm{Sym}(\mathbb{Z})$. By normality of $\mathrm{FAlt}(\mathbb{Z})$, we can form the semidirect product $\mathcal{A}(\mathbb{Z}) = \mathrm{FAlt}(\mathbb{Z}) \rtimes_\rho \mathbb{Z}$, called the *alternating enrichment* of $\mathbb{Z}$.

For $r \in \mathbb{N}$, let $[\![r]\!] = \{n \in \mathbb{Z} : |n| \leq r\}$ and let $\alpha_r, \beta_r \in \mathrm{Sym}([\![r]\!])$ be given by $\alpha_r = (-r \; 1-r \cdots r-1 \; r), \beta_r = (-1 \; 0 \; 1)$. Then it is easily seen that $\langle \alpha_r, \beta_r \rangle = \mathrm{Alt}([\![r]\!])$, and similarly we have the following.

Lemma 3.8. *The group* $\mathcal{A}(\mathbb{Z})$ *is finitely generated by* $\{(\mathrm{id}_{\mathbb{Z}}, 1), ((-1 \; 0 \; 1), 0)\}$.

We shall prove the following.

Theorem 3.9. $\mathcal{A}(\mathbb{Z})$ *is locally stable but not weakly stable.*

Let $\alpha_r, \beta_r \in \mathrm{Alt}([\![r]\!])$ be as above, and let $\mathbf{T}(r) = (\alpha_r, \beta_r) \in \mathrm{Alt}([\![r]\!])^2$ (a 2-marking on $\mathrm{Alt}([\![r]\!])$). Now let $r : \mathbb{N} \to \mathbb{N}_{\geq 2}$, with r strictly increasing. We consider sequences of the form $(\mathrm{Alt}([\![r(n)]\!]), \mathbf{T}(r(n)))$ in $\mathcal{G}_2$.

Proposition 3.10. *For any function* $r : \mathbb{N} \to \mathbb{N}_{\geq 2}$ *as above,* $(\mathrm{Alt}([\![r(n)]\!]), \mathbf{T}(r(n)))$ *converges in* $\mathcal{G}_2$ *to* $(\mathcal{A}(\mathbb{Z}), \mathbf{S})$*, where* $\mathbf{S} = ((\mathrm{id}_{\mathbb{Z}}, 1), ((-1 \; 0 \; 1), 0))$.

Proof. This is covered, for example, in [25, Remark 5.4]. ■

The groups $G(r) = \bigotimes(\mathrm{Alt}([\![r(n)]\!]), \mathbf{T}(r(n)))$ were originally studied by B. H. Neumann [26], who proved that different functions r as above yield nonisomorphic groups $G(r)$. Our interest here in these groups stems from the following, which is the main result of [23].

Theorem 3.11. *For any increasing function $r : \mathbb{N} \to \mathbb{N}_{\geq 2}$, $G(r)$ is stable.*

Let $\widetilde{\mathbf{S}}(r) \in G(r)^2 = \bigotimes(\mathrm{Alt}(\llbracket r(n) \rrbracket), \mathbf{T}(r(n)))^2$ be the diagonal 2-marking described in Definition 3.3. We now fix an increasing function $r_0 : \mathbb{N} \to \mathbb{N}_{\geq 2}$. For $n \in \mathbb{N}$ we define $r_n(m) = r_0(m+n)$. The key to Theorem 3.9 is the following observation.

Proposition 3.12. *Write $\mathbf{S}_n = \widetilde{\mathbf{S}}(r_n)$. The sequence $(G(r_n), \mathbf{S}_n)_n$ converges in $\mathcal{G}_2$ to $(\mathcal{A}(\mathbb{Z}), \mathbf{S})$, where $\mathbf{S}$ is as in Proposition 3.10.*

Proof of Theorem 3.9. $\mathrm{FAlt}(\mathbb{Z})$ is locally finite, so $\mathcal{A}(\mathbb{Z})$ is amenable (being the extension of a locally finite group by an abelian group). On the other hand, $\mathrm{FAlt}(\mathbb{Z})$ is an infinite simple group, hence $\mathcal{A}(\mathbb{Z})$ is not residually finite. By Theorem 2.16, $\mathcal{A}(\mathbb{Z})$ is not weakly stable.

For fixed n, Proposition 3.10 guarantees that $(\mathrm{Alt}(\llbracket r_n(m) \rrbracket), \mathbf{T}(r_n(m)))_m$ converges in $\mathcal{G}_2$ to $(\mathcal{A}(\mathbb{Z}), \mathbf{S})$. By Proposition 3.4, we have the tail homomorphism $\tau_n : (G(r_n), \mathbf{S}_n) \twoheadrightarrow (\mathcal{A}(\mathbb{Z}), \mathbf{S})$. By Proposition 3.12, we see that the sequence $\pi^{(n)} = \tau_n$ satisfies the conditions of Proposition 3.6, and we conclude that $\mathcal{A}(\mathbb{Z})$ is locally stable. ∎

Proof of Proposition 3.12. Let $K \in \mathbb{N}$, so that by Proposition 3.10, there exists $M \in \mathbb{N}$ such that, for all $m \geq M$,

$$d((\mathrm{Alt}(r_0(m)), \mathbf{T}(r_0(m))), (\mathcal{A}(\mathbb{Z}), \mathbf{S})) \leq 2^{-(K+1)}$$

so that for any $l, m \geq M$,

$$d((\mathrm{Alt}(r_0(l)), \mathbf{T}(r_0(l))), (\mathrm{Alt}(r_0(m)), \mathbf{T}(r_0(m)))) \leq 2^{-K} \tag{2}$$

by the triangle inequality. Let $n \geq N$ and claim

$$d((G(r_n), \mathbf{S}_n), (\mathcal{A}(\mathbb{Z}), \mathbf{S})) \leq 2^{-K} \tag{3}$$

which yields the result. Let

$$\tau_n : (G(r_n), \mathbf{S}_n) \twoheadrightarrow (\mathcal{A}(\mathbb{Z}), \mathbf{S})$$

be the tail homomorphism described in Proposition 3.4. Since $\tau_n \circ \pi_{\mathbf{S}_n} = \pi_{\mathbf{S}}$, if (3) fails, then there exists a nontrivial reduced word $w \in F(\mathbf{X})$ such that $e \neq \pi_{\mathbf{S}_n}(w) \in \ker(\tau_n)$. For $k \in \mathbb{N}$ let

$$p_k : (G(r_n), \mathbf{S}_n) \twoheadrightarrow (\mathrm{Alt}(r_n(k)), \mathbf{T}(r_n(k))) = (\mathrm{Alt}(r_0(k+n)), \mathbf{T}(r_0(k+n)))$$

by projection onto the kth factor. By the conclusion of Proposition 3.4, $\pi_{\mathbf{S}_n}(w) \in \bigoplus_k \mathrm{Alt}(r_0(k+n))$, so there exist $l, m \in \mathbb{N}$ such that

$$\pi_{\mathbf{T}(r(n+l))}(w) = (p_l \circ \pi_{\mathbf{S}_n})(w) \neq e \quad \text{but } \pi_{\mathbf{T}(r(n+m))}(w) = (p_m \circ \pi_{\mathbf{S}_n})(w) = e.$$

It follows that

$$d((\mathrm{Alt}(r_0(n+l)), \mathbf{T}(r_0(n+l))), (\mathrm{Alt}(r_0(n+m)), \mathbf{T}(r_0(n+m)))) > 2^{-K},$$

contradicting (2). ∎

Remark 3.13. One may also define the *symmetric enrichment* $\mathcal{S}(\mathbb{Z}) = \mathrm{FSym}(\mathbb{Z}) \rtimes_\rho \mathbb{Z}$, in which $\mathcal{A}(\mathbb{Z})$ sits as a subgroup of index 2. We do not know whether local stability is preserved under commensurability in general, so one cannot deduce local stability for $\mathcal{S}(\mathbb{Z})$ directly from Theorem 3.9. That said, one can also prove that $\mathcal{S}(\mathbb{Z})$ is locally stable, along the lines of the argument sketched in Example 4.13.

Further, much as in Proposition 3.10, one may construct a sequence of 2-markings of finite symmetric groups converging in $\mathcal{G}_2$ to (a 2-marking of) $\mathcal{S}(\mathbb{Z})$, and thereby produce a family of diagonal product groups admitting epimorphisms onto $\mathcal{S}(\mathbb{Z})$. We expect that these diagonal product groups are also stable groups, and that their stability may be proved using the arguments of [23].

4. Invariant random subgroups

For Γ a countable group, let $\mathrm{Sub}(\Gamma)$ be the space of all subgroups of Γ, a closed subspace of the space $\{0,1\}^\Gamma$ of subsets of Γ (equipped with the Tychonoff topology). The group Γ admits a continuous action on $\mathrm{Sub}(\Gamma)$ by conjugation, which induces an action on the space $\mathrm{Prob}(\Gamma)$ of Borel probability measures on $\mathrm{Sub}(\Gamma)$. An *IRS* of Γ is by definition a fixed point of the action of Γ on $\mathrm{Prob}(\Gamma)$.

Example 4.1. For $H \leq \Gamma$, $\delta_H \in \mathrm{Prob}(\Gamma)$ denotes the point mass on the subgroup H. The measure δ_H is an IRS iff H is normal in Γ. More generally, for $H_1, \dots, H_n \leq \Gamma$ distinct subgroups and $\lambda_1, \dots, \lambda_n \in [0,1]$, with $\sum_i \lambda_i = 1$, we have a probability measure $\sum_i \lambda_i \delta_{H_i} \in \mathrm{Prob}(\Gamma)$. The latter is an IRS iff the H_i form a union of whole conjugacy classes of subgroups, and $\lambda_i = \lambda_j$ whenever H_i and H_j are conjugate in Γ.

We denote by $\mathrm{IRS}(\Gamma)$ the set of all IRSs of Γ; it is a compact metrizable space under the weak* topology. If Γ is finitely generated by the set S, $r \in \mathbb{N}$ and $W \subseteq \Gamma$, we write

$$C_{r,W} = \{H \leq \Gamma : H \cap B_S(r) = W \cap B_S(r)\},$$

a clopen subset of $\mathrm{Sub}(\Gamma)$. For $\mu, \mu_n \in \mathrm{IRS}(\Gamma)$, we have the following useful criterion for convergence in the weak* topology:

$$\mu_n \to \mu \text{ in } \mathrm{IRS}(\Gamma) \text{ iff } \mu_n(C_{r,W}) \to \mu(C_{r,W}) \quad \text{for all } r \in \mathbb{N} \text{ and } W \subseteq B_S(r). \tag{4}$$

It is clear that $\mathrm{IRS}(\Gamma)$ forms a closed and convex subspace of $\mathrm{Prob}(\Gamma)$. An IRS μ of Γ is *ergodic* if, for any $\mu_1, \mu_2 \in \mathrm{IRS}(\Gamma)$ and $t \in (0,1)$, if $\mu = t\mu_1 + (1-t)\mu_2$ then $\mu_1 = \mu_2$. That is, the ergodic IRSs are precisely the extreme points of Γ. Thus, the only closed convex subspace of $\mathrm{IRS}(\Gamma)$ containing all ergodic IRSs of Γ is $\mathrm{IRS}(\Gamma)$ itself.

As in the previous section, fix an ordered basis $\mathbf{X}$ for the rank-d free group $\mathbb{F} = F(\mathbf{X})$. If Γ is d-generated, then a d-marking $\mathbf{S}$ on Γ induces an embedding of $\mathrm{IRS}(\Gamma)$ into $\mathrm{IRS}(\mathbb{F})$: An IRS of $\mathbb{F}$ lies in $\mathrm{IRS}(\Gamma)$ iff it is supported on subgroups of $\mathbb{F}$ containing

$\ker(\pi_{\mathbf{S}} : \mathbb{F} \to \Gamma)$, where as above, $\pi_{\mathbf{S}}$ denotes the epimorphism of d-marked groups $(\mathbb{F}, \mathbf{X}) \twoheadrightarrow (\Gamma, \mathbf{S})$. We write $\mathrm{IRS}(\Gamma, \mathbf{S})$ to specify this embedded copy of $\mathrm{IRS}(\Gamma)$ inside $\mathrm{IRS}(\mathbb{F})$. An IRS of Γ is called *finite-index* if it is an atomic probability measure supported on finite-index subgroups of Γ, and is called *cosofic* in Γ if it is the weak*-limit of finite-index IRSs of Γ. Note that if $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S}) \subseteq \mathrm{IRS}(\mathbb{F}, \mathbf{X})$, then μ may a priori be cosofic in $\mathbb{F}$ but not cosofic in Γ.

For X any Γ-set, there is a map $\mathrm{Stab} : X \to \mathrm{Sub}(\Gamma)$ sending a point $x \in X$ to its stabilizer in Γ. If X is a standard Borel space, and the action of Γ on X is Borel, then Stab is a Borel map. Thus, given a Borel regular probability measure μ on X, there is a pushforward measure $\mathrm{Stab}_*(\mu) \in \mathrm{Prob}(\Gamma)$ on $\mathrm{Sub}(\Gamma)$. If μ is Γ-invariant (i.e., if the action of Γ on (X, μ) is a pmp action), then $\mathrm{Stab}_*(\mu)$ is an IRS of Γ. Although we shall not use the fact in the sequel, it in fact transpires that *every* IRS can be produced in this way (see [1, Proposition 1.4]). In the special case that X is a finite discrete Γ-set, the IRS induced in this way is a finitely supported finite-index IRS of Γ.

Definition 4.2. Let X be a finite Γ-set, and let ν be the uniform probability measure on X (so that ν is Γ-invariant). By the *IRS associated with* X, we shall mean $\mathrm{Stab}_*(\nu) \in \mathrm{IRS}(\Gamma)$. We call the sequence (X_n) of finite $\mathbb{F}$-sets *convergent* if the sequence of IRSs associated with the X_n converges in $\mathrm{IRS}(\mathbb{F})$.

Remark 4.3. If X and Y are finite Γ-sets with associated IRSs μ and ν, respectively, then $X \sqcup Y$ is a finite Γ-set with associated IRS:

$$\left(\frac{|X|}{|X| + |Y|}\right)\mu + \left(\frac{|Y|}{|X| + |Y|}\right)\nu.$$

Repeatedly applying this observation, we have that for any finite Γ-set X and any $N > 0$, there is a finite Γ-set X' with the same associated IRS as X, and $|X'| \geq N$.

Generalizing Remark 4.3, we have the following slight variation of [6, Lemma 7.6], which will be needed in the proof of Theorem 4.9.

Lemma 4.4. *Let $(\Delta_n, \mathbf{T}_n)$ be a sequence of marked d-generated groups and let Y_n be a finite Δ_n-set, with associated IRS ν_n. Suppose that $|Y_n| \to \infty$ as $n \to \infty$ and that (ν_n) converges to $\nu \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$. Then for any sequence (m_k) of positive integers satisfying $m_k \to \infty$ as $k \to \infty$, there exist an unbounded nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ and finite $\Delta_{f(k)}$-sets Y'_k with associated IRSs ν'_k, such that $|Y'_k| = m_k$ for all k, and (ν'_k) converges to ν also.*

Proof. For r a positive integer, let Z_r be a set of size r, on which we let $\mathbb{F}$ act trivially. Then for any quotient Γ of $\mathbb{F}$, Z_r is naturally a Γ-set. Let (i_n) be a strictly increasing sequence of positive integers such that for all $i_n \leq k < i_{n+1}$, $|Y_n|/m_k < 1/n$. Set $f(k) = n$. For each $i_n \leq k < i_{n+1}$, write $m_k = q_k|Y_n| + r_k$, with $q_k \geq n$ and $0 \leq r_k < |Y_n|$, and set

$$Y'_k = Z_{r_k} \sqcup \bigsqcup_{q_k} Y_n$$

a finite Δ_n-set with $|Y'_k| = m_k$. By Remark 4.3, the IRS $\nu'_k \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n)$ associated with Y'_k is given by

$$\nu'_k = \frac{m_k - r_k}{m_k}\nu_n + \frac{r_k}{m_k}\delta_{\Delta_n}.$$

Then (ν'_k) converges to ν, since $r_k/m_k < |Y_n|/m_k < 1/n \to 0$ as $k \to \infty$, because $k < i_{n+1}$. ∎

The IRSs associated with finite Γ-sets are one source of examples of finite-index IRSs of Γ. Not every finite-index IRS is necessarily of this form; for instance, if $\mu \in \mathrm{IRS}(\Gamma)$ is the IRS associated with a finite Γ-set, then for every $r \in \mathbb{N}$ and $W \subseteq \Gamma$, each $\mu(C_{r,W})$ is a rational number. Nevertheless, the next construction (which is a slight modification of [6, Lemma 4.4]) shows that every finite-index IRS may be *approximated* by IRSs associated with actions on finite sets.

Lemma 4.5. *Let $(\Gamma_n, \mathbf{S}_n)$ be a sequence of marked d-generated groups; let $\mu_n \in \mathrm{IRS}(\Gamma_n, \mathbf{S}_n)$ be a finite-index IRS, and suppose the sequence (μ_n) converges to some $\mu \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$. Then for each n, there exists a finite Γ_n-set X_n such that, letting $\nu_n \in \mathrm{IRS}(\Gamma_n, \mathbf{S}_n)$ be the IRS associated with X_n, the sequence (ν_n) converges to μ also.*

Proof. Being finite-index, μ_n is in particular a cosofic IRS, so [6, Lemma 4.4] applies. There is a sequence $(X_{n,m})_m$ of finite Γ_n-sets whose associated sequence $(\mu_{n,m})_m$ of IRSs converges to μ_n. Since (μ_n) converges to μ, there exists increasing $f : \mathbb{N} \to \mathbb{N}$ such that $(\mu_{n,f(n)})$ converges to μ. We may therefore take $X_n = X_{n,f(n)}$. ∎

In [6, Lemma 7.5], the following is proved.

Lemma 4.6. *Let (X_n) be a convergent sequence of finite $\mathbb{F}$-sets, with $\mu \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$ being the limit of the sequence $(\mathrm{Stab}_*(\nu_n))$ of IRS associated with the X_n. Then (X_n) is a stability challenge for Γ iff μ lies in $\mathrm{IRS}(\Gamma, \mathbf{S})$.*

Moreover, it is shown that in proving stability of Γ, it suffices to consider *convergent* stability challenges. In a similar vein, we have the following.

Proposition 4.7. *A finitely generated group Γ is locally stable iff every convergent stability challenge for Γ has a local solution.*

Proof. Let $\mathbf{S}$ be a d-marking on Γ. By Proposition 2.5, it suffices to prove that if every convergent stability challenge has a local solution, then so does every stability challenge. Suppose to the contrary, that there is a stability challenge (X_n) for Γ with no local solution. For m a positive integer and X a finite $\mathbb{F}$-set, we shall describe a finite $\mathbb{F}$-set Y as "m-good" for X if $|X| = |Y|$; $d_{\mathrm{gen}}(X, Y) < 1/m$ and $\ker(\pi_{\mathbf{S}}) \cap B_{\mathbf{X}}(m)$ acts trivially on Y. In this terminology, a sequence (Y_n) of finite $\mathbf{F}$-sets is a local solution for (X_n) iff for all m, Y_n is m-good for X_n for all but finitely many n. Therefore, passing to a subsequence of our (X_n), we may assume that there exists a positive integer m such that for no n does there exist a finite $\mathbb{F}$-set which is m-good for X_n. This (X_n) is then a stability challenge for Γ, no subsequence of which has a local solution. On the other hand, by compactness of $\mathrm{IRS}(\mathbb{F})$, (X_n) has a subsequence which is a convergent stability challenge for Γ, contradiction. ∎

We now come to the main result of this section, which is our necessary and sufficient condition for local stability of a finitely generated amenable group in terms of IRSs.

Definition 4.8. Let Γ be a d-generated group, let $\mathbf{S}$ be a d-marking on Γ, and let $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$. We call μ *partially cosofic in* $(\Gamma, \mathbf{S})$ if there exists a sequence of d-marked finite groups $(\Delta_n, \mathbf{T}_n)$ converging to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$ and $\nu_n \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n)$ such that $\nu_n \to \mu$ in $\mathrm{IRS}(\mathbb{F}, \mathbf{X})$.

Theorem 4.9. *Suppose Γ is a d-generated amenable group and let $\mathbf{S}$ be a d-marking on Γ. Then Γ is locally stable if and only if every $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$ is partially cosofic in $(\Gamma, \mathbf{S})$.*

Remark 4.10. The d-generated group Γ is LEF iff for some (equivalently any) d-marking $\mathbf{S}$ on Γ there exists a sequence of d-marked finite groups $(\Delta_n, \mathbf{T}_n)$ converging to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$. For *any* such sequence, the trivial IRS $\delta_{\{e\}}$ and $\delta_{\Delta_n} \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n)$ converge, respectively, to the trivial IRS $\delta_{\{e\}}$ and $\delta_\Gamma \in \mathrm{IRS}(\Gamma, \mathbf{S})$, so these IRSs are partially cosofic in any LEF group.

Before embarking on the proof of Theorem 4.9, we note one slight refinement, which will be useful in applications, particularly in the next section.

Corollary 4.11. *Let $(\Gamma, \mathbf{S})$ be as in Theorem 4.9. The following are equivalent:*

(i) *Γ is locally stable.*

(ii) *Every $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$ is partially cosofic in $(\Gamma, \mathbf{S})$.*

(iii) *Every ergodic $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$ is partially cosofic in $(\Gamma, \mathbf{S})$.*

Proof. Given Theorem 4.9, the only nontrivial implication is from (iii) to (ii). Let $\mathcal{P} \subseteq \mathrm{IRS}(\Gamma, \mathbf{S})$ be the set of partially cosofic IRSs in $(\Gamma, \mathbf{S})$. As noted when we first defined ergodic IRSs, it suffices to show that $\mathcal{P}$ is closed and convex. For closure, suppose that (μ_n) is a sequence consisting of partially cosofic IRSs in $(\Gamma, \mathbf{S})$ and converging to $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$. For each n, let $(\Delta_m^{(n)}, \mathbf{T}_m^{(n)})$ be finite d-marked groups and $\nu_m^{(n)} \in \mathrm{IRS}(\Delta_m^{(n)}, \mathbf{T}_m^{(n)})$ witness the partial cosoficity of μ_n in $(\Gamma, \mathbf{S})$. Then for some increasing $f : \mathbb{N} \to \mathbb{N}$, the sequences $(\Delta_{f(n)}^{(n)}, \mathbf{T}_{f(n)}^{(n)})$ and $\nu_{f(n)}^{(n)}$ witness that μ is partially cosofic in $(\Gamma, \mathbf{S})$ also.

For convexity, let $\mu_1, \mu_2 \in \mathcal{P}$ and $t \in (0, 1)$. For $i = 1, 2$, let $(\Delta_n^{(i)}, \mathbf{T}_n^{(i)})$ and $\nu_n^{(i)} \in \mathrm{IRS}(\Delta_n^{(i)}, \mathbf{T}_n^{(i)})$ witness the partial cosoficity of μ_i. It is clear that $\nu_n = t\nu_n^{(1)} + (1-t)\nu_n^{(2)}$ converges to $t\mu_1 + (1-t)\mu_2$ in $\mathrm{IRS}(\mathbb{F}, \mathbf{X})$. It therefore suffices to find an associated sequence of finite groups.

For $1 \le j \le d$, let $t_{n,j} = (t_{n,j}^{(1)}, t_{n,j}^{(2)})$, where $\mathbf{T}_n^{(i)} = (t_{n,1}^{(i)}, \ldots, t_{n,d}^{(i)})$, and let:

$$\Delta_n = \langle t_{n,1}, \ldots, t_{n,d} \rangle \le \Delta_n^{(1)} \times \Delta_n^{(2)}$$

(so that the projection of Δ_n to each factor is surjective). It is easy to see that $(\Delta_n, \mathbf{T}_n)$ converges to $(\Gamma, \mathbf{S})$ in $\mathscr{G}_d$, since the $(\Delta_n^{(i)}, \mathbf{T}_n^{(i)})$ do. The $\nu_n^{(i)}$ are finitely supported atomic IRSs, hence ν_n is too. Finally, if $H \le \mathbb{F}$ lies in the support of ν_n, then for one of $i = 1$ or 2, H lies in the support of $\nu_n^{(i)}$, so $\ker(\pi_{\mathbf{T}}) \le \ker(\pi_{\mathbf{T}^{(i)}}) \le H$, and $\nu_n \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n)$, as required. ∎

There is defined in [6, Section 6] a notion of "statistical distance" $d_{\mathrm{stat}}(X, Y)$ between a pair X and Y of Borel probabiliy spaces with Borel $\mathbb{F}$-actions. We will not need the full definition of d_{stat}, only the following consequences.

Proposition 4.12. *Let (X_n) and (Y_n) be sequences of finite $\mathbb{F}$-sets, and let μ_n and $\nu_n \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$ be the IRSs associated with X_n and Y_n, respectively.*

- (i) *If there exists $\lambda \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$ such that $\mu_n \to \lambda$ and $\nu_n \to \lambda$, then we have that $d_{\mathrm{stat}}(X_n, Y_n) \to 0$.*
- (ii) *If $\mu_n \to \lambda \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$ and $d_{\mathrm{stat}}(X_n, Y_n) \to 0$, then $\nu_n \to \lambda$ also.*
- (iii) *If $|X_n| = |Y_n|$ for all n and $d_{\mathrm{gen}}(X_n, Y_n) \to 0$, then $d_{\mathrm{stat}}(X_n, Y_n) \to 0$.*
- (iv) *Suppose there exists a d-marked amenable group $(\Gamma, \mathbf{S})$ such that for all n, the action of $\mathbb{F}$ on Y_n factors through $\pi_{\mathbf{S}}$, and $|X_n| = |Y_n|$ for all n. If we have $d_{\mathrm{stat}}(X_n, Y_n) \to 0$, then $d_{\mathrm{gen}}(X_n, Y_n) \to 0$.*

Proof. Items (i) and (ii) follow from [6, Lemma 6.1]; item (iii) is [6, Proposition 6.3], and item (iv) is the content of [6, Proposition 6.8]. ∎

Proof of Theorem 4.9. Suppose that the hypothesis on the IRSs holds. Let (X_n) be a stability challenge for Γ, which by Proposition 4.7 we can assume to be a convergent stability challenge. Let $\mu_n \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$ be the IRS associated with X_n, so that there exists $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$ with $\mu_n \to \mu$ in $\mathrm{IRS}(\mathbb{F}, \mathbf{X})$. Let $(\Delta_n, \mathbf{T}_n)$ and $\nu_n \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n)$ witness the partial cosoficity of μ in $(\Gamma, \mathbf{S})$. By Lemmas 4.4 and 4.5, we may assume that there exists a finite Δ_n-set Y_n, with $|X_n| = |Y_n|$, whose associated IRS is ν_n. We have $\mu_n, \nu_n \to \mu$, so $d_{\mathrm{stat}}(X_n, Y_n) \to 0$ by Proposition 4.12 (i).

Let $(\widetilde{\Gamma}, \mathbf{T}) = \bigotimes(\Delta_n, \mathbf{T}_n)$. Then we have epimorphisms of marked groups $t : (\widetilde{\Gamma}, \mathbf{T}) \twoheadrightarrow (\Gamma, \mathbf{S})$ and $p_n : (\widetilde{\Gamma}, \mathbf{T}) \twoheadrightarrow (\Delta_n, \mathbf{T}_n)$, so that the Y_n are finite $\widetilde{\Gamma}$-sets. Moreover, $\widetilde{\Gamma}$ is amenable by Lemma 3.5, so by Proposition 4.12 (ii) (applied to $(\widetilde{\Gamma}, \mathbf{T})$ instead of $(\Gamma, \mathbf{S})$), we conclude $d_{\mathrm{gen}}(X_n, Y_n) \to 0$. Viewing X_n and Y_n as finite $\mathbb{F}$-sets, Y_n is thus a solution to the stability challenge (X_n) for $\mathbb{F}$. Finally, since the action of $\mathbb{F}$ on Y_n factors through the marked quotient $(\mathbb{F}, \mathbf{X}) \twoheadrightarrow (\Delta_n, \mathbf{T}_n)$, and $(\Delta_n, \mathbf{T}_n)$ converges to $(\Gamma, \mathbf{S})$ in $\mathscr{G}_d$, every $r \in \ker(\pi_{\mathbf{S}})$ satisfies $\pi_{\mathbf{T}_n}(r) = e$ for all n sufficiently large; hence for such n, r lies in the kernel of the action of $\mathbb{F}$ on Y_n. Thus (Y_n) is a local Γ-set, and hence a local solution for Γ to the stability challenge (X_n). By Proposition 4.7, we conclude that Γ is locally stable.

Conversely, suppose that Γ is locally stable, and let $\mu \in \mathrm{IRS}(\Gamma, \mathbf{S})$. Since Γ is amenable, by [6, Proposition 6.6] there exist finite-index IRSs $\mu_n \in \mathrm{IRS}(\mathbb{F}, \mathbf{X})$ with $\mu_n \to \mu$. By Lemma 4.5, we may assume that there exists a finite $\mathbb{F}$-set X_n with associated IRS μ_n. By Lemma 4.6, (X_n) is a stability challenge for Γ. By local stability, (X_n) has a local solution (Y_n) for Γ, by Proposition 2.5. Now recall that there is a sequence of marked finitely presented groups $(\Gamma_n, \mathbf{S}_n)$ such that:

(i) There exist marked epimorphisms $(\Gamma_n, \mathbf{S}_n) \twoheadrightarrow (\Gamma_{n+1}, \mathbf{S}_{n+1})$ and $(\Gamma_n, \mathbf{S}_n) \twoheadrightarrow (\Gamma, \mathbf{S})$.

(ii) $(\Gamma_n, \mathbf{S}_n)$ converges to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$.

Specifically, let (r_n) be an increasing sequence of positive integers, let $N_n \in \mathcal{G}_d$ be the normal closure in $\mathbb{F}$ of $B_{\mathbf{X}}(r_n) \cap \ker(\pi_{\mathbf{S}})$, and let $(\Gamma_n, \mathbf{S}_n)$ be the d-marked group associated with N_n.

Passing to subsequences, we may assume that Y_n is a Γ_n-set by Remark 3.2 (in its more general version). Let $\mu'_n \in \mathrm{IRS}(\Gamma_n, \mathbf{S}_n)$ be the IRS associated with Y_n (a finitely supported finite-index IRS of Γ_n). Then $\mu'_n \to \mu$ in $\mathrm{IRS}(\mathbb{F}, \mathbf{X})$, by Proposition 4.12 (ii) and (iii). Now, since Γ is amenable locally stable, it is LEF by Lemma 2.17, so there exists a sequence $(\Lambda_n, \mathbf{U}_n)$ of marked finite groups converging to $(\Gamma, \mathbf{S})$ in $\mathcal{G}_d$. Passing to subsequences, we may assume (again, by the more general form of Remark 3.2) that there exists a marked epimorphism $\rho_n : (\Gamma_n, \mathbf{S}_n) \twoheadrightarrow (\Lambda_n, \mathbf{U}_n)$, and that

$$d((\Gamma, \mathbf{S}), (\Gamma_n, \mathbf{S}_n)), d((\Gamma, \mathbf{S}), (\Lambda_n, \mathbf{U}_n)), d((\Gamma_n, \mathbf{S}_n), (\Lambda_n, \mathbf{U}_n)) \leq 2^{-n} \tag{5}$$

in $\mathcal{G}_d$. Write

$$\mu'_n = \sum_{i=1}^{M_n} \lambda_{n,i} \delta_{H_{n,i}}$$

for some finite-index subgroups $H_{n,i}$ of Γ_n. Let $K_{n,i}$ be the normal core of $H_{n,i}$ in Γ_n, and let

$$K_n = \bigcap_{i=1}^{M_n} K_{n,i}.$$

Let $\pi_n : \Gamma_n \twoheadrightarrow \Lambda_n \times (\Gamma_n / K_n)$ be given by $\pi_n(g) = (\rho_n(g), gK_n)$; let $\mathbf{T}_n = \pi_n(\mathbf{S}_n)$ and set $\Delta_n = \mathrm{im}(\pi_n) \leq \Lambda_n \times (\Gamma_n / K_n)$, so that $\pi_n : (\Gamma_n, \mathbf{S}_n) \twoheadrightarrow (\Delta_n, \mathbf{T}_n)$ is a quotient of d-marked groups. Projection to the first factor of $\Lambda_n \times (\Gamma_n / K_n)$ induces a quotient of d-marked groups $(\Delta_n, \mathbf{T}_n) \twoheadrightarrow (\Lambda_n, \mathbf{U}_n)$. Since ρ_n restricts to an isomorphism of balls of radius n, and ρ_n factors through $(\Delta_n, \mathbf{T}_n)$, it follows from (5) that

$$d((\Gamma, \mathbf{S}), (\Delta_n, \mathbf{T}_n)) \leq 2^{-n} \tag{6}$$

and $(\Delta_n, \mathbf{T}_n)$ converges in $\mathcal{G}_d$ to $(\Gamma, \mathbf{S})$. Define

$$\nu_n = \sum_{i=1}^{M_n} \lambda_{n,i} \delta_{\pi_n(H_{n,i})} \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n).$$

Note that ν_n is indeed an IRS: Since μ'_n is an IRS, the subgroups $H_{n,i}$ and the coefficients $\lambda_{n,i}$ satisfy the criterion described in Example 4.1, hence so too do the $\pi_n(H_{n,i})$. We claim that $\nu_n \to \mu$ in IRS($\mathbb{F}, \mathbf{X}$), which will complete the proof. It suffices to show that for all $r \in \mathbb{N}$,

$$B_{\mathbf{X}}(r) \cap \pi_{\mathbf{S}_n}^{-1}(H_{n,i}) = B_{\mathbf{X}}(r) \cap \pi_{\mathbf{T}_n}^{-1}(\pi_n(H_{n,i}))$$

for all n sufficiently large, as in that case we have for all $r \in \mathbb{N}$ and $W \subseteq B_{\mathbf{X}}(r)$,

$$\delta_{H_{n,i}}(C_{r,W}) = \delta_{\pi_n(H_{n,i})}(C_{r,W})$$

for all n sufficiently large, and since $\mu'_n \to \mu$, the criterion (4) applies.

Since $\pi_{\mathbf{T}_n} = \pi_n \circ \pi_{\mathbf{S}_n}$, we have $\pi_{\mathbf{S}_n}^{-1}(H_{n,i}) \subseteq \pi_{\mathbf{T}_n}^{-1}(\pi_n(H_{n,i}))$. For the converse inclusion, let $w \in B_{\mathbf{X}}(r)$ and $h \in H_{n,i}$ be such that $\pi_n(h) = \pi_{\mathbf{T}_n}(w)$. Then there exists $v \in B_{\mathbf{X}}(r)$ such that $\pi_{\mathbf{S}_n}(v) = h$, so $vw^{-1} \in \ker(\pi_{\mathbf{T}_n}) \cap B_{\mathbf{X}}(2r)$. For $n > 2r$, and by (5) and (6), $\pi_{\mathbf{S}_n}(w) = \pi_{\mathbf{S}_n}(v) = h$, as desired. ∎

Example 4.13. Since the group $\mathcal{A}(\mathbb{Z})$ is amenable, given Theorem 3.9 we may apply Theorem 4.9 to deduce that every IRS of $\mathcal{A}(\mathbb{Z})$ is a limit of IRSs of some finite groups converging to $\mathcal{A}(\mathbb{Z})$ in $\mathcal{G}_2$. Alternatively, one could deduce Theorem 3.9 from Theorem 4.9 by constructing such finite groups and their IRSs directly. Let us briefly sketch how this may be done. The ergodic IRSs of $\mathcal{A}(\mathbb{Z})$ are described in [23, Section 4]. They arise either as atomic measures on finite-index normal subgroups of $\mathcal{A}(\mathbb{Z})$ or are supported on subgroups of FAlt($\mathbb{Z}$). By a result of Vershik, the latter class of IRSs arise as the stabilizers of a random colouring of the integers. That is to say, they are pushforwards ν_α under the stabilizer map of the FAlt($\mathbb{Z}$)-invariant ergodic probability measures μ_α on the space of colourings of $\mathbb{Z}$, according to which each integer is independently coloured according to the random variable $\alpha \in [0,1]^C$, for C some countable set of colours.

Now let $\mathbf{T}(n)$ be the 2-markings of the finite groups Alt($[\![n]\!]$) described in Proposition 3.10, so that the sequence (Alt($[\![n]\!]$), $\mathbf{T}(n)$) converges in $\mathcal{G}_2$ to ($\mathcal{A}(\mathbb{Z}), \mathbf{S}$). Given a colour distribution α, colour each point of $[\![n]\!]$ independently according to α to obtain an Alt($[\![n]\!]$)-invariant probability measure $\mu_{\alpha,n}$ on the set of colourings of $[\![n]\!]$. Then $\nu_{\alpha,n} = \mathrm{Stab}_*(\mu_{\alpha,n})$ is an IRS of Alt($[\![n]\!]$), and the sequence $(\nu_{\alpha,n})_n$ converges in IRS($\mathbb{F}$) to ν_α.

5. Topological full groups of minimal subshifts

Let X be the Cantor space, and let $T : X \to X$ be a homeomorphism. We refer to the pair (X, T) as a *Cantor dynamical system*. The system (X, T) is *minimal* if every orbit in X under the action of $\langle T \rangle$ is dense in X. Henceforth, assume that (X, T) is a minimal Cantor dynamical system.

Definition 5.1. The *topological full group* $[\![T]\!]$ of the Cantor dynamical system (X, T) is the set of all homeomorphisms g of X such that there exists a continuous function

$f_g : X \to \mathbb{Z}$ (called the *orbit cocycle* of g) such that for all $x \in X$, $g(x) = T^{f_g(x)}(x)$ (here we assume $\mathbb{Z}$ equipped with the discrete topology).

Equivalently, $g \in \mathrm{Homeo}(X)$ lies in $[\![T]\!]$ if there is a finite clopen partition $C_1, \ldots, C_d$ of X and integers $a_1, \ldots, a_d$ such that for $1 \leq i \leq d$, $g|_{C_i} = T^{a_i}|_{C_i}$ (taking $\{a_1, \ldots, a_d\} = \mathrm{im}(f_g)$, $C_i = f_g^{-1}(a_i)$). It is straightforward to check that $[\![T]\!]$ is a subgroup of $\mathrm{Homeo}(X)$.

Remark 5.2. We note some immediate consequences of the definition:

(i) The orbit cocycle f_g is uniquely determined by $g \in [\![T]\!]$, since, by minimality, T has no finite orbits on X.

(ii) For $g, h \in [\![T]\!]$ and $x \in X$, we have the cocycle relation

$$f_{gh}(x) = f_g(h(x)) + f_h(x).$$

The group $[\![T]\!]$ and its derived subgroup $[\![T]\!]'$ have a remarkable collection of group-theoretic properties.

Theorem 5.3 ([24, Theorems 4.9 and 5.4]). *For any (X, T) as above, $[\![T]\!]'$ is an infinite simple group. If (X, T) is a minimal subshift, then $[\![T]\!]'$ is finitely generated.*

It shall not concern us much exactly what a *minimal subshift* is by definition, beyond the conclusion of Theorems 5.3 and 5.18.

Theorem 5.4 ([17, Theorem 5.1]). *For any minimal Cantor dynamical system (X, T), $[\![T]\!]$ is LEF.*

Theorem 5.5 ([19]). *For any minimal Cantor dynamical system (X, T), $[\![T]\!]$ is amenable.*

Theorems 5.4 and 5.5 already imply that $[\![T]\!]$ is always weakly locally stable, by Lemma 2.17. Further, for (X, T) a minimal subshift, our criterion for local stability from Theorem 4.9 is applicable to $[\![T]\!]'$. The classification of IRSs of $[\![T]\!]'$ is provided by the next result.

Theorem 5.6 ([30, Corollary 1.4]). *Let (X, T) be a minimal Cantor dynamical system. Let μ be an ergodic IRS of $[\![T]\!]'$. Then either*

(i) $\mu = \delta_{\{e\}}$ *or* $\delta_{[\![T]\!]'}$ *or*

(ii) *there exist $k \in \mathbb{N}$ and T-invariant ergodic probability measures ν_i on X such that μ is the pushforward of $\nu_1 \times \cdots \times \nu_k$ under the map* $\mathrm{Stab} : X^k \to \mathrm{Sub}([\![T]\!]')$, *given by*

$$\mathrm{Stab}_k(x_1, \ldots, x_k) = \bigcap_{i=1}^{k} \mathrm{Stab}_{[\![T]\!]'}(x_i).$$

That is, $\mu = \mathrm{Stab}_*(\nu_1 \times \cdots \times \nu_k)$, where $[\![T]\!]'$ acts diagonally on X^k.

Our proof of local stability for the groups $[\![T]\!]'$ (Theorem 5.17) will be based on the proof of Theorem 5.4 given in [17]: We show that the marked finite groups converging to (some marking of) $[\![T]\!]'$ in the space of marked groups, which are constructed in the proof of that theorem, admit IRSs converging to the IRSs of $[\![T]\!]'$ described in Theorem 5.6. The argument goes as follows: A small clopen set B of X determines a clopen partition Ξ of X (the Kakutani–Rokhlin partition), on which $[\![T]\!]'$ admits a partial action by permutations, generating a group $\Delta(\Xi) \leq \mathrm{Sym}(\Xi)$. If ν_i are probability measures on X (as in Theorem 5.6 (ii)), then ν_i imparts a mass to each point in Ξ. Pushing forward under the stabilizer map, we obtain an IRS ν_Ξ of $\Delta(\Xi)$. Taking a nested sequence of clopen sets B_n (intersecting in a point), the sequence of finite groups $\Delta(\Xi_n)$ and their IRSs ν_Ξ will satisfy the conditions of Theorem 4.9.

Definition 5.7. Let (X, T) be a minimal Cantor dynamical system. A T*-tower* is a finite family

$$\xi = \{B, TB, \ldots, T^{h-1}B\},$$

where $B \subseteq X$ is a nonempty clopen set such that the sets $B, TB, \ldots, T^{h-1}B$ are pairwise disjoint. We refer to the positive integer h as the *height* of the T-tower ξ. A *Kakutani–Rokhlin (K–R) partition* of X is a finite clopen partition Ξ of X which is a disjoint union of T-towers, that is, a clopen partition of the form

$$\Xi = \{T^i B_v : 1 \leq v \leq q; 0 \leq i \leq h_v - 1\}. \tag{7}$$

The sets $T^i B_v$ are the *atoms* of the partition Ξ. The sets

$$B(\Xi) = \coprod_{v=1}^{q} B_v \quad \text{and} \quad H(\Xi) = \coprod_{v=1}^{q} T^{h_v - 1} B_v$$

are called, respectively, the *base* and the *roof* of Ξ.

Remark 5.8. We note some easy consequences of the definition.

(i) Any T-tower of height $h \geq 2$ can be written as the disjoint union of two T-towers of smaller height. Thus, the set of T-towers making up a K–R partition Ξ (and their heights) is not intrinsic to the partition Ξ itself; rather, we consider the division of the atoms into T-towers to be part of the data of Ξ.

(ii) Since T is injective, it maps $H(\Xi)$ onto $B(\Xi)$ (any point of X not lying in $B(\Xi)$ is in the image under T of some atom of Ξ disjoint from $H(\Xi)$). Applying the same reasoning to T^{-1}, we have that $TH(\Xi) = B(\Xi)$.

Given a K–R partition Ξ of the form (7), we shall write $h(\Xi) = \min_{1 \leq v \leq q} h_v$ to denote the minimal height among the T-towers appearing in Ξ. The next construction is described in [17, Remark 3.3].

Lemma 5.9. *Let Ξ be a K–R partition of X and let Π be a finite clopen partition of X. Then there exists a K–R partition Ξ' of X which is a common refinement of Ξ and Π, such that $B(\Xi) = B(\Xi')$, $H(\Xi) = H(\Xi')$, and $h(\Xi) = h(\Xi')$.*

Proof. Let $\{B, TB, \ldots, T^{h-1}B\}$ be a T-tower of Ξ. For each $0 \leq i \leq h-1$, $\Pi_i = \{B \cap T^{-i}P : P \in \Pi\} \setminus \{\emptyset\}$ is a finite clopen partition of B. Let $\{C_1, \ldots, C_n\}$ be any finite clopen partition of B which is a common refinement of $\Pi_0, \Pi_1, \ldots, \Pi_{h-1}$. Then for $1 \leq j \leq n$, $\xi_j = \{C_j, TC_j, \ldots, T^{h-1}C_j\}$ is a T-tower of height h; each T^iC_j is contained in T^iB and in a unique element of Π, and

$$\bigcup_{j=1}^{n} \bigcup_{C \in \xi_j} C = \bigcup_{i=0}^{h-1} T^iB.$$

Applying this construction to each T-tower of Ξ yields the desired K–R partition Ξ'. ■

Our next proposition is a summary of the content of [17, Section 3].

Proposition 5.10. *Let (X, T) be a minimal Cantor dynamical system. For any increasing sequence (m_n) of positive integers, there exists a sequence of K–R partitions*

$$\Xi_n = \{T^iB_v^{(n)} : 0 \leq i \leq h_v^{(n)} - 1;\ v = 1, \ldots, v_n\} \tag{8}$$

of X satisfying the following:

(i) *The union of the Ξ_n generates the topology on X.*

(ii) *Ξ_{n+1} refines Ξ_n.*

(iii) *$B(\Xi_{n+1}) \subseteq B(\Xi_n)$ and there exists $x_0 \in X$ such that $\bigcap_n B(\Xi_n) = \{x_0\}$.*

(iv) *For all n, $h(\Xi_n) \geq 2m_n + 2$.*

(v) *For all n and $-m_n - 1 \leq i \leq m_n$,*

$$\operatorname{diam}(T^iB(\Xi_n)) < 1/n.$$

Following [17, Section 4], given a sequence (Ξ_n) of K–R partitions of X as in (8), satisfying (i)–(v) of Proposition 5.10, we say that an element $\pi \in [\![T]\!]$ is an *n-permutation* if the orbit cocycle f_π is constant on each part of Ξ_n and for all $1 \leq v \leq v_n$ and $0 \leq i \leq h_v^{(n)} - 1$, f_π satisfies $-i \leq f_\pi(x) \leq h_v^{(n)} - i - 1$ for all $x \in T^iB_v^{(n)}$. Thus, π preserves each T-tower ξ of Ξ_n, and induces a well-defined permutation on the set of atoms of ξ.

Remark 5.11. The set of all n-permutations in $[\![T]\!]$ forms a subgroup of $[\![T]\!]$. This subgroup is isomorphic to $\operatorname{Sym}(h_1^{(n)}) \times \cdots \times \operatorname{Sym}(h_{v_n}^{(n)})$, since any tuple of permutations of the atoms in each T-tower of Ξ_n may be realized by some n-permutation in $[\![T]\!]$.

There is also defined in [17, Section 4] the notion of an n-*rotation*, and we refer the reader there for the precise definition; the only fact that we require about n-rotations is the following, which is immediate from the definition.

Remark 5.12. If $\rho \in [\![T]\!]$ is an n-rotation, and $x \in X$ is such that $\rho(x) \neq x$, then $|f_\rho(x)| \geq \min(h_1, h_2)$, where h_1 (respectively h_2) is the height of the T-tower of Ξ_n containing x (respectively $\rho(x)$).

Henceforth, $\mathrm{Sym}(\Xi_n)$ denotes the group of all permutations of the finite set Ξ_n. Our next theorem shows how, for large n, an element of $[\![T]\!]$ induces a well-defined n-permutation of Ξ_n, which in turn induces an element of $\mathrm{Sym}(\Xi_n)$. Note, however, that not every element of $\mathrm{Sym}(\Xi_n)$ need arise this way (see Remark 5.11).

Theorem 5.13. *Let (X, T) be a minimal Cantor dynamical system, and let (Ξ_n) be a sequence of K–R partitions satisfying conditions (i)–(v) of Proposition 5.10, for some increasing sequence of integers (m_n).*

(i) *Let $g \in [\![T]\!]$. For all n sufficiently large, there exist unique $\pi_n(g), \rho_n(g) \in [\![T]\!]$ such that $g = \pi_n(g)\rho_n(g)$; $\pi_n(g)$ is an n-permutation and $\rho_n(g)$ is an n-rotation.*

(ii) *For any $A \subseteq [\![T]\!]$ finite, if n is sufficiently large, then there is a local embedding $\phi_n : A \to \mathrm{Sym}(\Xi_n)$, given by $\phi_n(g)(T^i B_v^{(n)}) = \pi_n(g)(T^i B_v^{(n)})$. In particular, $[\![T]\!]$ is LEF.*

Proof. Item (i) is immediate from [17, Theorem 4.7]. Item (ii) is proved as [17, Theorem 5.1] (note that the statement of that theorem does not specify the local embedding, but the local embedding given in the proof is precisely as we have described it). ∎

Henceforth, we assume that (X, T) is a minimal subshift, so that Theorem 5.3 applies. Fix a finite symmetric generating set S for $[\![T]\!]'$, let $n \in \mathbb{N}$, and consider the ball $B_S(n) \subseteq [\![T]\!]'$. Recall that, for $g \in [\![T]\!]$, $f_g : X \to \mathbb{Z}$ is the orbit cocycle of g.

Proposition 5.14. *Let (m_n) be an increasing sequence of positive integers. There is a sequence (Ξ_n) of K–R partitions of X satisfying items (i)–(v) of Proposition 5.10, and additionally satisfying the following, for all $n \in \mathbb{N}$:*

(vi) *For all $g \in B_S(n)$, $h(\Xi_n) \geq 2\max\{|f_g(x)| : x \in X\} + 2$.*

(vii) *For all $g \in B_S(n)$, f_g is constant on each part of Ξ_n.*

(viii) *For all $g \in B_S(n)$, there exist unique $\pi_n(g), \rho_n(g) \in [\![T]\!]$ such that $g = \pi_n(g)\rho_n(g)$; $\pi_n(g)$ is an n-permutation and $\rho_n(g)$ is an n-rotation. Moreover, the map $\phi_n : B_S(n) \to \mathrm{Sym}(\Xi_n)$, given by $\phi_n(g)(T^i B_v^{(n)}) = \pi_n(g)(T^i B_v^{(n)})$, is a local embedding.*

Proof. First, since X is compact and the orbit cocycle is continuous, it is bounded. Therefore (replacing (m_n) with a faster growing sequence if required), we can assume that for all $g \in B_S(n)$,

$$m_n \geq \max\{|f_g(x)| : x \in X\},$$

and then apply Proposition 5.10 (iv).

Second, we inductively refine each Ξ_n such that properties (i)–(vi) still hold, and for all $g \in B_S(n)$, the orbit cocycle f_g is constant on each part of Ξ_n. Supposing that we have already refined Ξ_{n-1}, for each $g \in B_S(n)$ let $\mathcal{C}_g$ be a finite clopen partition of X such that f_g is constant on each part of $\mathcal{C}_g$. Applying Lemma 5.9 repeatedly, we replace Ξ_n with a finer finite clopen partition, which is also a refinement of both Ξ_{n-1} and all $\mathcal{C}_g$. This process clearly preserves properties (i)–(vi) (property (iv) holding by the final part of Lemma 5.9).

Finally, passing to a subsequence of (Ξ_n), and applying Theorem 5.13 (i) to elements $g \in B_S(n)$, we may assume that the decomposition $g = \pi_n(g)\rho_n(g)$ exists and is unique. Moreover, by Theorem 5.13 (ii), applied to the finite subsets $A = B_S(n)$, and again passing to a subsequence of (Ξ_n), we may assume that the given map $\phi_n : B_S(n) \to \mathrm{Sym}(\Xi_n)$ is a well-defined local embedding. Note that passing to a subsequence of (Ξ_n) preserves properties (i)–(vii). ∎

Henceforth, we fix a sequence (Ξ_n) of K–R partitions of X satisfying properties (i)–(viii) of Propositions 5.10 and 5.14, with respect to some increasing sequence (m_n). Let $\phi_n : B_S(n) \to \mathrm{Sym}(\Xi_n)$ be the local embedding as in Proposition 5.14 (viii). For $x \in X$, we write $[x]_n \in \Xi_n$ for the (unique) atom of Ξ_n containing x.

Lemma 5.15. *For all $n \in \mathbb{N}$, for all $g \in B_S(n)$ and all $x \in X$, the following are equivalent:*

(i) $g(x) = x$.

(ii) *For all $y \in [x]_n$, $g(y) = y$.*

(iii) $\phi_n(g)([x]_n) = [x]_n$.

Proof. If $g(x) = x$, then by minimality of T on X, $f_g(x) = 0$. By Proposition 5.14 (vii), for all $y \in B$, $f_g(y) = 0$. Thus (i) and (ii) are equivalent.

Write $g = \pi_n(g)\rho_n(g)$ as in Theorem 5.13, with $\pi_n(g) \in [\![T]\!]$ an n-permutation and $\rho_n(g) \in [\![T]\!]$ an n-rotation, so that $\phi_n(g)([x]_n) = \pi_n(g)([x]_n)$. As in Remark 5.11, $\pi_n(g)^{-1}$ is an n-permutation also. Suppose that (ii) holds, so that $\pi_n(g)^{-1}([x]_n) = \rho_n(g)([x]_n)$. Since an n-permutation preserves each T-tower of Ξ_n, and sends atoms to atoms, $\rho_n(g)([x]_n)$ is an atom of Ξ_n in the same T-tower as $[x]_n$. Let the height of this tower be h. If (iii) fails, so that $\rho_n(g)([x]_n) \neq [x]_n$, then by Remark 5.12, for $y \in [x]_n$, $|f_{\rho_n(g)}(y)| \geq h$. By the cocycle relation (see Remark 5.2 (ii)),

$$0 = f_g(y) = f_{\rho_n(g)}(y) + f_{\pi_n(g)}(\rho_n(g)(y)),$$

so $|f_{\pi_n(g)}(\rho_n(g)(y))| \geq h$ also, contradicting the definition of an n-permutation.

Conversely, suppose (iii) holds and let $y \in [x]_n$. Then $\pi_n(g)$ preserves $[x]_n$, so (by the bound on $f_{\pi_n(g)}$ from the definition of an n-permutation) $\pi_n(g)$ fixes $[x]_n$ pointwise, hence so does $\pi_n(g)^{-1}$. We have $g^{-1} = \rho_n(g)^{-1}\pi_n(g)^{-1}$, so $g^{-1}(y) = \rho_n(g)^{-1}(y)$, hence

$$f_{g^{-1}}(y) = f_{\rho_n(g)^{-1}}(y) = -f_{\rho_n(g)}(\rho_n(g)(y)) \tag{9}$$

(by the cocycle relation). By Proposition 5.14 (vi), the left-hand side of (9) has absolute value less than $h(\Xi_n)$. Applying Remark 5.12 to the right-hand side, we have $f_{g^{-1}}(y) = 0$, so $g(y) = y$. ∎

Recall that S is a finite generating set for $[\![T]\!]'$. Let $S = \{s_1, \ldots, s_d\}$, so that $\mathbf{S} = (s_1, \ldots, s_d)$ is a d-marking on $[\![T]\!]'$.

Proposition 5.16. *Let $\nu_1, \ldots, \nu_k$ be T-invariant ergodic probability measures on X, and let*

$$\mu = (\mathrm{Stab}_k)_*(\nu_1 \times \cdots \times \nu_k) \in \mathrm{IRS}([\![T]\!]', \mathbf{S})$$

be as in Theorem 5.6 (ii). Then μ is partially cosofic.

Proof. Let $\phi_n : B_S(n) \to \mathrm{Sym}(\Xi_n)$ be the local embedding as in Proposition 5.14; let $\Delta_n = \langle \phi_n(S) \rangle \leq \mathrm{Sym}(\Xi_n)$, and let $\mathbf{T}_n = (\phi_n(s_1), \ldots, \phi_n(s_d))$, a d-marking on Δ_n. Then the sequence $(\Delta_n, \mathbf{T}_n)$ converges to $([\![T]\!]', \mathbf{S})$ in $\mathcal{G}_d$, by Lemma 3.1.

For each $n \in \mathbb{N}$ and $1 \leq j \leq k$, there are induced probability measures $\overline{\nu}_j^{(n)}$ on the finite discrete set Ξ_n, given by $\overline{\nu}_j^{(n)}(\{T^i B_v^{(n)}\}) = \nu_j(T^i B_v^{(n)}) = \nu_j(B_v^{(n)})$ (the second equality holding by T-invariance of ν_j). Since each $\phi_n(s_m)$ preserves each T-tower of Ξ_n,

$$\overline{\nu}_j^{(n)}(\phi_n(s_m)\{T^i B_v^{(n)}\}) = \overline{\nu}_j^{(n)}(\{T^i B_v^{(n)}\}),$$

so $\overline{\nu}_j^{(n)}$ is Δ_n-invariant, hence $(\overline{\nu}_1^{(n)} \times \cdots \times \overline{\nu}_k^{(n)})$ is a Δ_n-invariant probability measure on Ξ_n^k (with Δ_n acting diagonally). Thus, $\mu_n = \mathrm{Stab}_*(\overline{\nu}_1^{(n)} \times \cdots \times \overline{\nu}_k^{(n)}) \in \mathrm{IRS}(\Delta_n, \mathbf{T}_n)$. Let $\mathcal{B}_n$ be the family of clopen subsets of X^k which are unions of sets of the form $B_1 \times \cdots \times B_k$, for $B_i \in \Xi_n$. Then $\mathcal{B}_n$ is a (finite) σ-algebra; indeed there is a (unique) isomorphism of σ-algebras $\Psi_n : \mathcal{P}(\Xi_n^k) \to \mathcal{B}_n$ extending $\Psi(\{(B_1, \ldots, B_k)\}) = B_1 \times \cdots \times B_k$. Moreover, since

$$\begin{aligned}(\overline{\nu}_1^{(n)} \times \cdots \times \overline{\nu}_k^{(n)})(\{(B_1, \ldots, B_k)\}) &= \nu_1(B_1) \cdots \nu_k(B_k) \\ &= (\nu_1 \times \cdots \times \nu_k)(B_1 \times \cdots \times B_k)\end{aligned}$$

for any $B_1, \ldots, B_k \in \Xi_n$, we have

$$(\nu_1 \times \cdots \times \nu_k)(\Psi(A)) = (\overline{\nu}_1^{(n)} \times \cdots \times \overline{\nu}_k^{(n)})(A) \tag{10}$$

for all $A \subseteq \Xi_n^k$.

We shall use the criterion (4) from Section 4 to show that the sequence (μ_n) converges to μ in $\mathrm{IRS}(\mathbb{F}, \mathbf{X})$. To this end, let $r \in \mathbb{N}$ and $W \subseteq B_X(r)$. We show that $\mu_n(C_{r,W}) = \mu(C_{r,W})$ for all $n \geq r$. By definition of the pushforward measures, $\mu(C_{r,W})$ is the probability that, for ν_i-random points $x_1, \dots, x_k \in X$, every $w \in W$ satisfies $\pi_{\mathbf{S}}(w)(x_i) = x_i$ for all $1 \leq i \leq k$, but for every $w \in B_X(r) \backslash W$, there exists $1 \leq i \leq k$ such that $\pi_{\mathbf{S}}(w)(x_i) \neq x_i$. Similarly, $\mu_n(C_{r,W})$ is the probability that, for $\overline{\nu}_i^{(n)}$-random points $B_1, \dots, B_k \in \Xi_n$, every $w \in W$ satisfies $\pi_{\mathbf{T}_n}(w)(B_i) = B_i$ for all $1 \leq i \leq k$, but for every $w \in B_X(r) \setminus W$, there exists $1 \leq i \leq k$ such that $\pi_{\mathbf{T}_n}(w)(B_i) \neq B_i$. Our claim is that these probabilities are equal.

By Lemma 5.15, for each $w \in B_X(r)$ the set of points $x \in X$ for which $\pi_{\mathbf{S}}(w)(x) = x$ is a union of whole atoms of Ξ_n. Moreover, since $(\phi_n \circ \pi_{\mathbf{S}})(w) = \pi_{\mathbf{T}_n}(w)$, Lemma 5.15 yields that, for $x \in X$,

$$\{B \in \Xi_n : \pi_{\mathbf{T}_n}(w)(B) = B\} = \{[x]_n : x \in X, \pi_{\mathbf{S}}(w)(x) = x\}.$$

It follows that the set

$$\{(x_1, \dots, x_k) \in X^k : \forall w \in B_X(r), w \in W \Leftrightarrow \forall i, \pi_{\mathbf{S}}(w)(x_i) = x_i\}$$

belongs to the σ-algebra $\mathcal{B}_n$, and is precisely $\Psi_n(A_{r,W})$, where

$$A_{r,W} = \{(B_1, \dots, B_k) \in \Xi_n^k : \forall w \in B_X(r), w \in W \Leftrightarrow \forall i, \pi_{\mathbf{T}_n}(w)(B_i) = B_i\}.$$

By (10) and the discussion immediately following,

$$\mu(C_{r,W}) = (\nu_1 \times \cdots \times \nu_k)(\Psi(A_{r,W})) = (\overline{\nu}_1^{(n)} \times \cdots \times \overline{\nu}_k^{(n)})(A_{r,W}) = \mu_n(C_{r,W}),$$

as desired. ■

Theorem 5.17. *Let (X, T) be a Cantor minimal subshift. Then $[\![T]\!]'$ is locally stable.*

Proof. By Theorems 5.3 and 5.5, the criterion for local stability from Corollary 4.11 (iii) applies. Let $\mu \in \mathrm{IRS}([\![T]\!]', \mathbf{S})$ be an ergodic IRS. If μ is as in Theorem 5.6 (i), then μ is partially cosofic by Theorem 5.4 and Remark 4.10. If μ is as in Theorem 5.6 (ii), the partial cosoficity is precisely the content of Proposition 5.16. ■

Theorem 5.18. *There is a continuum of pairwise nonisomorphic groups of the form $[\![T]\!]'$, for (X, T) a minimal subshift.*

Proof. This is proved, for example, in [24, p. 246]. Alternatively, an explicit continuous family of minimal subshifts $\{(X_r, T_r)\}_{r \in [2,\infty)}$ is constructed in [8, Section 5]. It is shown that for $2 \leq r < r'$, the isomorphism types of $[\![T_r]\!]'$ and $[\![T_{r'}]\!]'$ are distinguished by their *LEF growth functions* (see the statements of [8, Theorems 1.5 and 1.6]). ■

Theorem 5.19. *There is a continuum of pairwise nonisomorphic finitely generated groups, which are locally stable but not weakly stable.*

Proof. Let (X, T) be a Cantor minimal subshift. By Theorem 5.17, $[\![T]\!]'$ is locally stable. By Theorems 5.3 and 5.5, $[\![T]\!]'$ is finitely generated amenable but not residually finite. By Theorem 2.16, we conclude that $[\![T]\!]'$ is not weakly stable. The result then follows from Theorem 5.18. ∎

6. Concluding remarks

There are many sources of examples of amenable LEF groups, and it is interesting to ask which of these may be locally stable. For instance, the (regular restricted) wreath product of any two amenable LEF groups is amenable LEF. It follows immediately from Lemma 2.17 that the class of finitely generated amenable weakly locally stable groups is closed under wreath products. We may therefore ask if the same holds when we strengthen "weakly locally stable" to "locally stable."

Question 6.1. Let Γ and Δ be finitely generated amenable locally stable groups. Must $\Delta \wr \Gamma$ be locally stable?

As a modest first step, one may ask for the following.

Conjecture 6.2. *Let Δ be a finite group. Then $\Delta \wr \mathbb{Z}$ is locally stable.*

Note that by [22], the wreath product of any two finitely generated abelian groups is stable. In particular, the special case of Conjecture 6.2 for which Δ is abelian is known to hold. By contrast, if Δ is nonabelian, then $\Delta \wr \mathbb{Z}$ is amenable but not residually finite, so not even weakly stable.

In a similar vein, one may ask for a generalization of Theorem 3.9. For any group Γ, the group $\mathrm{FAlt}(\Gamma)$ of finitely supported even permutations of the *set* Γ is normalized by the image in $\mathrm{Sym}(\Gamma)$ of the regular representation of Γ. We therefore obtain a semidirect product $\mathcal{A}(\Gamma) = \mathrm{FAlt}(\Gamma) \rtimes \Gamma$, the *alternating enrichment* of Γ. If Γ is respectively finitely generated, amenable or LEF, then so is $\mathcal{A}(\Gamma)$. On the other hand, if Γ is infinite, then $\mathcal{A}(\Gamma)$ is not residually finite.

Question 6.3. Under what conditions on Γ is $\mathcal{A}(\Gamma)$ locally stable?

Next, one may ask for other applications of Proposition 3.6. Many famous examples of "monster" groups are constructed as limits of sequences of marked epimorphisms of d-marked groups. For instance, Tarski monsters and free Burnside groups both arise in this way, as limits of sequences of marked epimorphisms of finitely presented groups satisfying a "small-cancellation" condition (see, for instance, [27]). It remains a well-known open problem whether or not such groups are LEF (see, for instance, [9, Problem 5.14 (ii)]; if it were even the case that such monster groups were not limits in $\mathcal{G}_d$ of nonabelian finite simple groups, then dramatic consequences would follow). It is therefore also interesting to ask whether such groups are locally stable, which would follow from a positive answer to our next question.

Question 6.4. Is every finitely presented small-cancellation group stable?

One may further ask to what extent Proposition 3.6 may be generalized. Although Remark 3.7 shows that local stability is not a closed property in $\mathcal{G}_d$, it is reasonable to see the existence of a sequence of marked (locally) stable groups converging in $\mathcal{G}_d$ to (a d-marking of) Γ as evidence that Γ is locally stable, especially if the groups in the sequence are in some sense "uniformly" stable. For instance, finite rank free groups are surely "at least as stable" as any other finitely generated groups. The closure in $\mathcal{G}_d$ of the set of (d-markings of) free groups of rank $\leq d$ is precisely the set of d-generated *limit groups*. Beyond free groups and free abelian groups, the only limit groups for which any stability results are known are the fundamental groups of closed oriented surfaces (as described in [21]; see below). Moreover, every limit group is finitely presented, so local stability can be upgraded automatically to stability.

Conjecture 6.5. *Every finitely generated limit group is permutation stable.*

There is a generalization of permutation stability, called *flexible stability*, under which we may slightly enlarge the finite domains on which the images of our almost-homomorphisms act before seeking asymptotically equivalent actions on those domains. For example, fundamental groups of closed oriented surfaces are flexibly stable [21], but the question of their stability remains open. In another direction, it is known that the groups $\mathrm{SL}_d(\mathbb{Z})$ ($d \geq 3$) are not stable [5], but unknown whether they are flexibly stable. By [7], if $\mathrm{SL}_d(\mathbb{Z})$ is flexibly stable for some $d \geq 5$, then there exists a non-sofic group. Just as local stability generalizes stability, one may analogously define a notion of flexible local stability, generalizing flexible stability. A flexibly locally stable sofic group is still necessarily LEF.

Problem 6.6. Find examples of groups which are flexibly locally stable but not locally stable.

In a related direction, the following question was posed by A. Lubotzky.

Question 6.7. Does there exist a finitely generated locally stable group with Kazhdan's property (T)?

The corresponding question for stable groups has a negative answer [5]; in particular, no example satisfying Question 6.7 may be found among finitely presented groups. Note that all of the locally stable but non-stable groups we have constructed are amenable, hence far from being property (T) groups.

Finally, there are many other versions of "stability of metric approximations for groups" besides stability in permutations. For any reasonable family of compact groups equipped with bi-invariant metrics d, one can define "almost-homomorphism" in just the same way as for the Hamming metrics on the finite symmetric groups. Corresponding to the definition of a sofic group, there is a notion of "d-approximable" group, and there is a notion of a "d-stable" group corresponding to a group which is stable in permutations. We have alluded to *Frobenius approximable* and *Frobenius stable* groups in Section 1, but one may also consider, for example, the Hilbert–Schmidt metrics on unitary groups (the groups which are HS-approximable are precisely the *hyperlinear* groups), the rank metrics on groups of invertible matrices over fields (leading to the class of *linear sofic* groups), or all finite groups equipped with bi-invariant metrics, leading to the *weakly sofic* groups (see [2] and the references therein; see [3] for a characterization in terms of liftings of homomorphisms to metric ultraproducts). In all these cases, finitely generated groups which are d-approximable and d-stable must be residually finite. One may similarly conceive of a notion of d-local stability, such that d-approximable and d-locally stable implies LEF.

Problem 6.8. For each type of metric approximation discussed above, study the class of "d-locally stable" groups. Produce examples of groups which are d-locally stable but not d-stable.

Since circulation of a preliminary version of the present work, the last three problems have been addressed in [14]. Therein, the following are achieved:

(i) A general framework for metric local stability is described, which incorporates a notion of flexible local stability.

(ii) Lubotzky's Question 6.7 is answered in the negative.

(iii) Problem 6.8 is studied in particular for the Hilbert–Schmidt distance on the unitary groups $U(n)$.

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References

[1] M. Abért, Y. Glasner, and B. Virág, Kesten's theorem for invariant random subgroups. *Duke Math. J.* **163** (2014), no. 3, 465–488 Zbl 1344.20061 MR 3165420

[2] G. Arzhantseva and P.-A. Cherix, Quantifying metric approximations of discrete groups. 2020, arXiv:2008.12954v1

[3] G. Arzhantseva and L. Păunescu, Almost commuting permutations are near commuting permutations. *J. Funct. Anal.* **269** (2015), no. 3, 745–757 Zbl 1368.20025 MR 3350728

[4] G. Arzhantseva and L. Păunescu, Constraint metric approximations and equations in groups. *J. Algebra* **516** (2018), 329–351 Zbl 1427.20054 MR 3863481

[5] O. Becker and A. Lubotzky, Group stability and Property (T). *J. Funct. Anal.* **278** (2020), no. 1, article no. 108298, 20 pp. Zbl 1516.20100 MR 4027744

[6] O. Becker, A. Lubotzky, and A. Thom, Stability and invariant random subgroups. *Duke Math. J.* **168** (2019), no. 12, 2207–2234 Zbl 1516.20007 MR 3999445

[7] L. Bowen and P. Burton, Flexible stability and nonsoficity. *Trans. Amer. Math. Soc.* **373** (2020), no. 6, 4469–4481 Zbl 1476.20046 MR 4105530

[8] H. Bradford and D. Dona, Topological full groups of minimal subshifts and quantifying local embeddings into finite groups. *Ergodic Theory Dynam. Systems* **43** (2023), no. 5, 1492–1510 Zbl 1525.37013 MR 4574146

[9] P.-E. Caprace, Finite and infinite quotients of discrete and indiscrete groups. In *Groups St Andrews 2017 in Birmingham*, pp. 16–69, London Math. Soc. Lecture Note Ser. 455, Cambridge University Press, Cambridge, 2019 Zbl 1514.20115 MR 3931408

[10] T. Ceccherini-Silberstein and M. Coornaert, *Cellular automata and groups*. Springer Monogr. Math., 439 pp. Springer, Berlin, 2010 Zbl 1218.37004 MR 2683112

[11] M. De Chiffre, L. Glebsky, A. Lubotzky, and A. Thom, Stability, cohomology vanishing, and nonapproximable groups. *Forum Math. Sigma* **8** (2020), article no. e18, 37 pp. Zbl 1456.22002 MR 4080477

[12] G. Elek and E. Szabó, Hyperlinearity, essentially free actions and L^2-invariants. The sofic property. *Math. Ann.* **332** (2005), no. 2, 421–441 Zbl 1070.43002 MR 2178069

[13] G. Elek and E. Szabó, Sofic representations of amenable groups. *Proc. Amer. Math. Soc.* **139** (2011), no. 12, 4285–4291 Zbl 1263.43001 MR 2823074

[14] F. Fournier-Facio, M. Gerasimova, and P. Spaas, Local Hilbert–Schmidt stability. *J. Algebra* **663** (2025), 589–629 MR 4811049

[15] L. Glebsky and L. M. Rivera, Almost solutions of equations in permutations. *Taiwanese J. Math.* **13** (2009), no. 2A, 493–500 Zbl 1203.20003 MR 2500002

[16] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 5, 939–985 English translation: *Math. USSR-Izv.* **25** (1985), no. 2, 259–300 Zbl 0583.20023 MR 0764305

[17] R. I. Grigorchuk and K. S. Medynets, On the algebraic properties of topological full groups (in Russian). *Mat. Sb.* **205** (2014), no. 6, 87–108 English translation: *Sbornik: Math.* **205** (2014), no. 6, 843–861 Zbl 1365.37010 MR 3241829

[18] M. Gromov, Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits). Publ. Math. l'IHÉS **53** (1981) 53–78 Zbl 0474.20018 MR 0623534

[19] K. Juschenko and N. Monod, Cantor systems, piecewise translations and simple amenable groups. *Ann. of Math. (2)* **178** (2013), no. 2, 775–787 Zbl 1283.37011 MR 3071509

[20] M. Kassabov and I. Pak, Groups of oscillating intermediate growth. *Ann. of Math. (2)* **177** (2013), no. 3, 1113–1145 Zbl 1283.20027 MR 3034295

[21] N. Lazarovich, A. Levit, and Y. Minsky, Surface groups are flexibly stable. *J. Eur. Math. Soc.* (2024), DOI 10.4171/JEMS/1406

[22] A. Levit and A. Lubotzky, Infinitely presented permutation stable groups and invariant random subgroups of metabelian groups. *Ergodic Theory Dynam. Systems* **42** (2022), no. 6, 2028–2063 Zbl 07543353 MR 4417344

[23] A. Levit and A. Lubotzky, Uncountably many permutation stable groups. *Israel J. Math.* **251** (2022), no. 2, 657–678 Zbl 1512.20102 MR 4527554

[24] H. Matui, Some remarks on topological full groups of Cantor minimal systems. *Internat. J. Math.* **17** (2006), no. 2, 231–251 Zbl 1109.37008 MR 2205435

[25] M. Mimura and H. Sako, Group approximation in Cayley topology and coarse geometry Part I: Coarse embeddings of amenable groups. *J. Topol. Anal.* **13** (2021), no. 1, 1–47 Zbl 1479.20032 MR 4243070

[26] B. H. Neumann, Groups whose elements have bounded orders. *J. London Math. Soc.* **12** (1937), no. 3, 195–198 Zbl 0016.39303 MR 1575072

[27] A. Y. Ol'shanskiĭ, *Geometry of defining relations in groups*. Math. Appl. 70, Kluwer Academic Publishers Group, Dordrecht, 1991 Zbl 0732.20019 MR 1191619

[28] V. G. Pestov, Hyperlinear and sofic groups: a brief guide. *Bull. Symb. Log.* **14** (2008), no. 4, 449–480 Zbl 1206.20048 MR 2460675

[29] A. M. Vershik and E. I. Gordon, Groups that are locally embeddable in the class of finite groups (in Russian). *Algebra i Analiz* **9** (1997), no. 1, 71–97 English translation: *St. Petersburg Math. J.* **9** (1998), no. 1, 49–67 Zbl 0898.20016 MR 1458419

[30] T. Zheng, On rigid stabilizers and invariant random subgroups of groups of homeomorphisms. [v1] 2019, [v3] 2020, arXiv:1901.04428v3

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