Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 31 (2012), 17–30
DOI: 10.4171/ZAA/1446

Limiting Vorticities for Superconducting Thin Films

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Abstract. In the presence of applied magnetic fields H_ε in the order of H_{c_1} the first critical field, we determine the limiting vorticities of the minimal Ginzburg-Landau energy in superconducting thin films having varying thickness.

Keywords. Ginzburg-Landau functional, thin films, vortices
Mathematics Subject Classification (2010). Primary 35J60, secondary 35J20, 35J25, 35B40, 35Q55, 82D55

1. Introduction and main results

Consider a three-dimensional superconducting thin film that occupies the domain $\Omega_\delta = \Omega \times (-\delta a, \delta a)$ where Ω is a bounded smooth planar domain, and a is a smooth function in $\bar{\Omega}$ measuring the variation in the film thickness such that there exist a_0 and a_1 with $0 < a_0 < a_1$ such that $0 < a_0 \leq a(x) \leq a_1$ for all $x \in \bar{\Omega}$. By taking integral averages along the vertical direction and setting δ going to zero, it was shown in [12] that the three-dimensional Ginzburg-Landau model of superconductivity [20, 30] defined on Ω_δ may be reduced to a two-dimensional one given by the minimization in $H^1(\Omega)$ of the functional

$$J_\varepsilon(u) = \int_\Omega a(x) \left(|\nabla u - iA_\varepsilon u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right) \mathrm{d}x, \tag{1}$$

where $A_\varepsilon(x)$, the in-plane component of the magnetic potential, is determined by

$$\begin{cases} -\operatorname{div}(a(x)A_\varepsilon) = 0, \quad \operatorname{curl} A_\varepsilon = H_\varepsilon & \text{in } \Omega \\ A_\varepsilon.\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Here, $H_\varepsilon \geq 0$ is the external magnetic field which is applied vertically to the (x_1, x_2)-plane and independent of (x_1, x_2), ν denotes the outward normal to Ω, u

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is the complex superconducting order parameter with $|u|^2$ representing the density of superconducting electrons ($|u| = 1$ corresponds to the superconducting state, $|u| = 0$ corresponds to the normal state). $\frac{1}{\varepsilon} = \kappa$ is a characteristic of the superconductiong sample. $\nabla_{A_\varepsilon} u = \nabla u - iA_\varepsilon u$, and A_ε is proportional to the coherence length.

Let u be a critical point of the functional J_ε in $H^1(\Omega)$, which satisfies the Euler-Lagrange equations

$$\begin{cases} -(\nabla - iA_\varepsilon)a(x) \cdot (\nabla - iA_\varepsilon)u = \dfrac{a(x)}{\varepsilon^2}(1 - |u|^2)u & \text{in } \Omega \\ (\nabla u - iA_\varepsilon u) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The points where the zeros of u appear, with their topological degrees, are called the vortices of the map u. Understanding the vortex structures in the solutions and describing the vortices as H_ε varies is of great physical relevance and mathematical interests. Discussions on the vortex state in the thin film geometry have been given in [1,17,20,23,25,30], in particular, the variation in the film thickness is thought to provide an effective vortex pinning mechanism [12].

For works related to the mathematical analysis of the various pinning mechanisms, we refer to [2–5, 7, 8, 12–14]. In [10], a rigorous mathematical analysis of vortex solutions has been done for a similar problem with $a(x) = 1, A_\varepsilon = 0$ and Dirichlet boundary condition $u = g : \partial\Omega \longrightarrow S^1$ of degree d. It was proved that, asymptotically, minimizers have d isolated vortices of degree one and their locations are determined by minimizing a renormalized energy. This result was extended to the case $a(x) \neq 1, A_\varepsilon = 0$ with the same Dirichlet boundary conditions in [9] and [19] independently, and the vortices of the minimizers were shown to be located at the minimum of $a(x)$. Some results similar to those in [10] were obtained in [11] for the original Ginzburg-Landau functional $J(u, A)$,

$$J(u, A) = \int_\Omega \left(|\nabla u - iAu|^2 + |\text{curl}A - H|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right) \mathrm{d}x,$$

with $H = 0$ and the gauge invariant Dirichlet conditions (a name given in [27]). This work was later extended in [18] to the case where a weight (thickness) appears in the functional $J(u, A)$, the corresponding renormalized energy was presented in [15].

Similar analysis based on the functional (1) was also presented in [24]. All the available results substantiate the pinning effect of the thickness variation; that is, the vortices turn to stay where the film is thin. In [16], Ding and Du obtained the estimate for the lower critical magnetic field H_{c_1}, in the sense that it is the first critical value of H_ε, for which the minimal energy (1) among vortexless configurations is equal to the minimal energy among single-vortex configurations, moreover, it corresponds to the first phase transition in which vortices appear in the superconductor. They obtained that H_{c_1} has the form

$$H_{c_1} = k_a |\ln \varepsilon| + O(1), \tag{4}$$

where $k_a = \frac{1}{2\max_{x\in\Omega}\left|\frac{\xi_0(x)}{a(x)}\right|}$ with ξ_0 the solution of the following problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla \xi_0}{a}\right) = -1 & \text{in } \Omega \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

For the rest, we let the applied field H_ε be such that

$$\lim_{\varepsilon\longrightarrow 0} \frac{H_\varepsilon}{|\ln \varepsilon|} = \lambda > 0. \tag{6}$$

Our motivation is to study the vortex nucleation for minimizers of J_ε for applied magnetic fields comparable to H_{c_1} the first critical field.

Let $H^{-1}(\Omega)$ be the topological dual of $H_0^1(\Omega)$ and $\mathcal{M}(\Omega)$ be the space of bounded Radon measures on Ω, i.e. the topological dual of $C_0^0(\Omega)$. A measure $\mu \in \mathcal{M}(\Omega)$ can be represented canonically as a difference of two positive measures, $\mu = \mu_+ - \mu_-$. The *total variation* and the *norm* of μ, denoted respectively by $|\mu|$ and $\|\mu\|$, are by definition $|\mu| = \mu_+ + \mu_-$ and $\|\mu\| = |\mu|(\Omega)$.

We introduce an energy E_λ defined on $\mathcal{M}(\Omega) \cap H^{-1}(\Omega)$ as follows. For $\mu \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$, let $h_\mu \in H^1(\Omega)$ be the solution of

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla h_\mu}{a}\right) + 1 = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Now, by definition,

$$E_\lambda(\mu) = \frac{1}{\lambda}\int_\Omega a(x)|\mu|\,\mathrm{d}x + \int_\Omega \frac{|\nabla h_\mu|^2}{a(x)}\,\mathrm{d}x. \tag{8}$$

Let u_ε be a minimizer of J_ε over H^1, which exists under the assumptions (2) and let h_ε be the unique solution of

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla h_\varepsilon}{a}\right) = \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) - \operatorname{curl}(A_\varepsilon |u_\varepsilon|^2) & \text{in } \Omega \\ h_\varepsilon = H_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{9}$$

That h_ε verifies in Ω

$$-\frac{\nabla^\perp h_\varepsilon}{a(x)} = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon). \tag{10}$$

The first main result concerns the Γ-limit of the renormalized minimal energy.

Theorem 1.1. *Given $\lambda > 0$, assume that* $\lim_{\varepsilon\to 0} \frac{H_\varepsilon}{|\ln\varepsilon|} = \lambda$, *then* $\frac{J_\varepsilon}{H_\varepsilon^2} \to E_\lambda$ *in the sense of* Γ*-convergence.*

The convergence in Theorem 1.1 is precisely described in Propositions 2.1 and 3.2 below.

Minimizers of (8) can be characterized by means of minimizers of the following problem,

$$\min_{\substack{h\in H_0^1(\Omega)\\ -\operatorname{div}\left(\frac{\nabla h}{a}\right)+1\in\mathcal{M}(\Omega)}} \int_\Omega \left(\frac{1}{\lambda}\left|-\operatorname{div}\left(\frac{\nabla h}{a(x)}\right)+1\right| + \frac{|\nabla h|^2}{a(x)}\right) \mathrm{d}x. \tag{11}$$

The above functional being strictly convex and lower-semicontinuous, it admits a unique minimizer, and so the functional E_λ. Therefore, as a corollary of Theorem 1.1, we may describe the limiting vorticity measure in terms of the minimizer of the limiting energy E_λ.

Theorem 1.2. *Under the hypothesis of Theorem* 1.1, *if u_ε is a minimizer of* (1) *and h_ε is defined by* (9), *then, denoting by*

$$\mu_\varepsilon = \mu(u_\varepsilon) = H_\varepsilon + \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon - iA_0u_\varepsilon)$$

the "vorticity measure", the following convergences hold

$$\frac{\mu_\varepsilon}{H_\varepsilon} \to \mu_* \quad \text{in } \mathcal{M}(\Omega) \tag{12}$$

$$\frac{h_\varepsilon}{H_\varepsilon} \to h_{\mu_*} \quad \text{weakly in } H^1(\Omega) \text{ and strongly in } W^{1,p}(\Omega),\ \forall\, p<2. \tag{13}$$

Here $\mu_* = -\operatorname{div}\left(\frac{\nabla h_*}{a}\right) + 1$ *is the unique minimizer of E_λ. It corresponds to the limiting measure of vorticity.*

Sketch of the proof. The proof of Theorems 1.1–1.2 is obtained by getting first a lower bound, Proposition 2.1, proved in Section 2, and then an upper bound on the minimal energy of J , Proposition 3.2, proved in Section 3. The upper bound will be done by construction of a test configuration which goes with the same idea of [28].

Remark 1.3.
- The letters $C, \widetilde{C}, M$, etc. will denote positive constants independent of ε.
- For $n\in\mathbb{N}$ and $X\subset\mathbb{R}^n$, $|X|$ denotes the Lebesgue measure of X. $B(x,r)$ denotes the open ball in $\mathbb{R}^n$ of radius r and center x.
- $J_a(u,U)$ means that the energy density of u is integrated only on $U\subset\Omega$.
- For two positive functions $\alpha(\varepsilon)$ and $\beta(\varepsilon)$, we write $\alpha(\varepsilon)\ll\beta(\varepsilon)$ as $\varepsilon\to 0$ to mean that $\lim_{\varepsilon\to 0}\frac{\alpha(\varepsilon)}{\beta(\varepsilon)} = 0$.

2. Lower bound of the energy

First, $\lambda > 0$, so H_ε is of the order of $|\ln\varepsilon|$. The objective of this section is to prove the lower bound stated in Proposition 2.1 below.

Proposition 2.1. *Assume that* $\lim_{\varepsilon\to 0}\frac{H_\varepsilon}{|\ln\varepsilon|} = \lambda > 0$. *Let* u_ε *be a minimizer of* J_ε *and let* h_ε *be defined by* (9). *Then, up to the extraction of a subsequence* ε_n *converging to* 0, *one has,*

$$\frac{\mu(u_{\varepsilon_n})}{H_\varepsilon} \to \mu_0 \quad \text{in } \mathcal{M}(\Omega) \tag{14}$$

$$\frac{h_{\varepsilon_n}}{H_\varepsilon} \rightharpoonup h_0 \quad \text{weakly in } H^1(\Omega). \tag{15}$$

Moreover, $\mu_0 = -\operatorname{div}\left(\frac{\nabla h_0}{a(x)}\right) + 1$, *and* $\liminf_{\varepsilon\longrightarrow 0}\frac{J_\varepsilon(u_\varepsilon)}{H_\varepsilon^2} \geq E_\lambda(\mu_0)$. *Here, the energy* E_λ *is introduced by* (8).

In order to achieve the above lower bound on the minimal energy $J_\varepsilon(u_\varepsilon)$ we adapt results from [22, 26] regarding energy concentration on balls. We recall the hypothesis that there exists a positive constant $C > 0$ such that the applied magnetic field H_ε satisfies

$$H_\varepsilon \leq C|\ln\varepsilon|. \tag{16}$$

Now, we adapt the construction of suitable "vortex-balls", given in [17, Proposition 2.1].

Proposition 2.2. *Assume the hypothesis* (16) *holds. Given a number* $p\in]1,2[$, *there exists a constant* $C > 0$ *and a finite family of disjoint balls* $\{B_i(p_i, r_i)\}_{i\in I}$ *such that,* u *being a configuration satisfying the bound* (19), *the following properties hold:*

1. $\overline{B_i(p_i, r_i)} \subset \Omega$ *for all* i.
2. $w = \{x\in\Omega:\ |u(x)| \leq 1 - |\ln\varepsilon|^{-2}\} \subset \bigcup_{i\in I} B(a_i, r_i)$.
3. $\sum_{i\in I} r_i \leq C\,|\ln\varepsilon|^{-10}$.
4. *Letting* d_i *be the degree of the function* $\frac{u}{|u|}$ *restricted to* $\partial B(p_i, r_i)$ *if* $B_i(p_i, r_i)\subset\Omega$ *and* $d_i = 0$ *otherwise, then we have*

$$\begin{aligned}&\int_{B_i(p_i,r_i)} a(x)|(\nabla - iA_\varepsilon)u|^2\,\mathrm{d}x + \int_{B_i(p_i,r_i)} \frac{a(x)}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\mathrm{d}x \\ &\geq 2\pi a(p_i)|d_i|\left(|\ln\varepsilon| - C\ln|\ln\varepsilon|\right).\end{aligned} \tag{17}$$

5. $\left\|2\pi\sum_{i\in I} d_i\delta_{p_i} - H_\varepsilon - \operatorname{curl}(iu, \nabla_{A_\varepsilon}u)\right\|_{W^{-1,p}(\Omega)} \leq C|\ln\varepsilon|^{-4}$.

Proof of Proposition 2.1. We split the proof in several lemmas. We start with the following

Lemma 2.3. *Let* u_ε *a minimizer of* J_ε *and* h_ε *be defined by* (9)*, then*

$$J_\varepsilon(u_\varepsilon) \geq \int_\Omega \frac{|\nabla h_\varepsilon|^2}{a(x)}\,\mathrm{d}x. \tag{18}$$

Proof. We know that h_ε is solution (9), hence it verifies (10)

$$-\frac{\nabla^\perp h_\varepsilon}{a(x)} = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon) \quad \text{in } \Omega \quad \text{and} \quad h_\varepsilon = H_\varepsilon \quad \text{on } \partial\Omega.$$

A well known inequality is $|u_\varepsilon| \leq 1$, hence $\frac{|\nabla h_\varepsilon|^2}{a(x)} \leq a(x)|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2$. Therefore

$$J_a(u_\varepsilon) = \int_\Omega a(x)\left(|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\mathrm{d}x \geq \int_\Omega \frac{|\nabla h_\varepsilon|^2}{a(x)}\mathrm{d}x. \qquad \square$$

Lemma 2.4. *After extraction of a subsequence there exist* h_0 *and* μ_0 *such that the convergences in* (14)–(15) *hold.*

Proof. By (2) there exists a function $\zeta \in H^2$ such that

$$a(x)A_\varepsilon(x) = \nabla^\perp\zeta = (-\zeta_{x_2}, \zeta_{x_1}) \quad \text{in } \Omega.$$

Thanks to (5) one has $\zeta = H_\varepsilon \xi_0$. By maximum principle, we have $-C < \xi_0 < 0$ where C a positive constant. Notice that, by using $u = 1$ as a test configuration for the energy (1), we deduce an upper bound of the form:

$$J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(1) = \int_\Omega a(x)|A_\varepsilon|^2\mathrm{d}x = \int_\Omega \frac{|\nabla\zeta|^2}{a(x)}\,\mathrm{d}x = H_\varepsilon^2\int_\Omega \frac{|\nabla\xi_0|^2}{a(x)}\,\mathrm{d}x \leq CH_\varepsilon^2. \tag{19}$$

Using (19) and the fact that the function a is bounded above in (18)

$$C\int_\Omega |\nabla(h_\varepsilon - H_\varepsilon)|^2\mathrm{d}x \leq \int_\Omega \frac{|\nabla(h_\varepsilon - H_\varepsilon)|^2}{a(x)}\mathrm{d}x = \int_\Omega \frac{|\nabla h_\varepsilon|^2}{a(x)}\mathrm{d}x \leq J_\varepsilon(u_\varepsilon) \leq CH_\varepsilon^2. \tag{20}$$

We deduce that $\frac{h_\varepsilon - H_\varepsilon}{H_\varepsilon}$ is bounded in H_0^1 independently in ε, hence the existence of h_0 is immediate. Using now the balls concentration and referring to (17)

$$2\pi\sum_i |d_i| a(p_i)(|\ln\varepsilon| - C\ln|\ln\varepsilon|) \leq J(u_\varepsilon, \cup_i B_i) \leq J_\varepsilon(u_\varepsilon, \Omega) \leq CH_\varepsilon^2.$$

Since $a(x) \geq a_0 > 0$ hence, thanks to (16), $2\pi\sum_i |d_i| \leq CH_\varepsilon + o(H_\varepsilon)$. This together with the last assertion in Proposition 2.2 yields easily the existence of the limit measure μ_0. $\square$

Lemma 2.5. *The limit configuration verifies* $\mu_0 = -\operatorname{div}\left(\frac{\nabla h_0}{a(x)}\right) + 1$.

Proof. h_ε verifies (9)

$$-\operatorname{div}\left(\frac{\nabla h_\varepsilon}{a(x)}\right) + H_\varepsilon = \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) + \operatorname{curl}[(1-|u_\varepsilon|^2)A_\varepsilon].$$

Again with the same strategy as in [17, Lemma 2.2] we obtain

$$\left| -\operatorname{div}\left(\frac{\nabla h_\varepsilon}{H_\varepsilon a(x)}\right) + 1 - \mu_0 \right|_{W^{-1,p}_{p<2}} \longrightarrow 0. \tag{21}$$

We deduce then

$$\frac{h_\varepsilon}{H_\varepsilon} - 1 \longrightarrow h_0 - 1 \quad \text{strongly in } W_0^{1,p<2}(\Omega). \tag{22}$$

Passing to the limit in (21) finishes Lemma 2.5. □

We complete the proof of Proposition 2.1 by this lemma.

Lemma 2.6. *We have*

$$\liminf_{\varepsilon \longrightarrow 0} J_\varepsilon(u_\varepsilon) \geq E_\lambda(\mu_0) = \frac{1}{\lambda}\int_\Omega a(x)|\mu_0|\,\mathrm{d}x + \int_\Omega \frac{|\nabla h_0|^2}{a(x)}\,\mathrm{d}x. \tag{23}$$

Proof. (B_i) being the family of balls constructed in Proposition 2.2, then from (17)

$$J_\varepsilon(u_\varepsilon, \Omega) \geq 2\pi \sum_i a(p_i)|d_i|(|\ln\varepsilon| - C\ln|\ln\varepsilon|) + \int_{\Omega\setminus\cup_i B_i} \frac{|\nabla h_\varepsilon|^2}{a(x)}\mathrm{d}x. \tag{24}$$

Thanks to the last assertion in Proposition 2.2, we have approximately $\frac{2\pi\sum_i d_i\delta_{p_i}}{H_\varepsilon} \simeq \mu_\varepsilon = H_\varepsilon + \operatorname{curl}(iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)$. Hence, passing to the lim inf

$$\liminf_{\varepsilon \longrightarrow 0} \frac{J_\varepsilon(u_\varepsilon,\Omega)}{H_\varepsilon^2} \geq \liminf_{\varepsilon \longrightarrow 0} \int_\Omega a(x)|\mu_\varepsilon|\frac{|\ln\varepsilon|}{H_\varepsilon}\,\mathrm{d}x + \int_{\Omega\setminus\cup_i B_i} \frac{|\nabla h_\varepsilon|^2}{H_\varepsilon^2 a(x)}\mathrm{d}x.$$

Thanks to (6) and the convergence of μ_ε to μ_0 in $\mathcal{M}(\Omega)$, one can write

$$\lim_{\varepsilon \longrightarrow 0} \int_\Omega a(x)|\mu_\varepsilon|\mathrm{d}x\frac{|\ln\varepsilon|}{H_\varepsilon} = \frac{1}{\lambda}\int_\Omega a(x)|\mu|\mathrm{d}x,$$

since the function a is continuous on Ω. Now, let $X_\varepsilon = \frac{|\nabla h_\varepsilon|^2}{H_\varepsilon^2 a(x)}$ in $\Omega\setminus(\cup_i B_i)$ and 0 otherwise, so, thanks to (22), $X_\varepsilon \longrightarrow \frac{|\nabla h_0|^2}{a(x)}$ a.e. In particular, using Fatou lemma

$$\liminf_{\varepsilon \longrightarrow 0} \int_{\Omega\setminus(\cup_i B_i)} \frac{|\nabla h_\varepsilon|^2}{H_\varepsilon^2 a(x)}\mathrm{d}x = \liminf_{\varepsilon \longrightarrow 0} \int_\Omega |X_\varepsilon|^2\mathrm{d}x \geq \int_\Omega \frac{|\nabla h_0|^2}{a(x)}\mathrm{d}x.$$

Combining the above relations yields (23). □

3. Upper bound of the energy

Recall that $\lambda > 0$. We write $H_1^1(\Omega)$ for the space of Sobolev functions u such that $u - 1 \in H_0^1(\Omega)$. Recall the expression

$$E_\lambda(f) = \frac{1}{\lambda}\int_\Omega a(x)\left|-\operatorname{div}\left(\frac{\nabla f}{a}\right)+1\right|\mathrm{d}x + \int_\Omega \frac{|\nabla f|^2}{a(x)}\,\mathrm{d}x$$

defined over

$$V = \left\{ f \in H_1^1(\Omega) : \mu = -\operatorname{div}\left(\frac{\nabla f}{a}\right) + 1 \quad \text{is a Radon measure}\right\}.$$

In the next section, the minimum of E_λ will be achieved uniquely over V by the function h_* for which $\mu_* = -\operatorname{div}\left(\frac{\nabla h_*}{a}\right) + 1$ is in fact a positive absolutely continuous measure.

For any $f \in V$, we have $(f-1)(x) = \int_\Omega G(x,y)\mathrm{d}(\mu-1)(y)$, where $G(x,y)$ is the Green solution of

$$-\operatorname{div}\left(\frac{\nabla_x G(x,y)}{a(x)}\right) = \delta_y(x) \quad \text{in } \Omega \quad \text{and} \quad G(x,y)=0 \quad \text{for } x \in \partial\Omega. \tag{25}$$

It is clear that for any $f \in V$

$$E_\lambda(f) = I_\lambda(\mu) = \frac{1}{\lambda}\int_\Omega a(x)\mathrm{d}|\mu| + \int_{\Omega\times\Omega} G(x,y)\mathrm{d}(\mu-1)(x)\mathrm{d}(\mu-1)(y).$$

As in [28, Lemma 2.1], we can state the following

Lemma 3.1. *The function G solution of* (25) *verifies*

i) *$G(x,y)$ is symmetric and positive.*

ii) *$G(x,y) + \frac{a(x)}{2\pi}\ln|x-y|$ is continuous on $\Omega\times\Omega$.*

iii) *There exists $C > 0$ such that for all $x, y \in \Omega\times\Omega\backslash\Delta$*

$$\frac{a(x)}{2\pi}\ln|x-y| - C \le G(x,y) \le C\left(\frac{a(x)}{2\pi}\ln|x-y| + 1\right),$$

where Δ is the diagonal of $\mathbb{R}^2\times\mathbb{R}^2$.

3.1. Main result. The objective of this section is to establish the following upper bound, which corresponds to [28, Proposition 2.1].

Proposition 3.2. *Let H_ε be such that $\lim_{\varepsilon\longrightarrow 0}\frac{H_\varepsilon}{|\ln\varepsilon|} = \lambda > 0$ with the additional condition, if $\lambda = +\infty$, that $H_\varepsilon \ll \frac{1}{\varepsilon^2}$, and μ be a positive Radon measure absolutely continuous with respect to the Lebesgue measure. Then, letting u_ε be a minimizer of J_ε over H^1,*

$$\limsup_{\varepsilon\longrightarrow 0}\frac{J_\varepsilon(u_\varepsilon)}{H_\varepsilon^2} \le I_\lambda(\mu). \tag{26}$$

A consequence of the above results is

Corollary 3.3. *If* $\lambda = +\infty$*, that is,* $|\ln \varepsilon| \ll H_\varepsilon \ll \frac{1}{\varepsilon^2}$*, we have*

$$\lim_{\varepsilon \to 0} \frac{J_\varepsilon(u_\varepsilon)}{H_\varepsilon^2} = 0. \tag{27}$$

Again, $h_{\mu_*} = 1$ *is the strong limit of* $\frac{h_\varepsilon}{H_\varepsilon}$ *in* H^1*, and so* $\mu_* = dx$*. This leads a uniform scattering of vortices.*

Proof. It is clear with the above assumption on the applied field H_ε, that $\lambda = +\infty$, hence it is evident that the minimum of E_λ on V is $h_* = 1$. Thanks to (26), one finds (27). From Lemma 2.3, we get

$$C \int_\Omega |\nabla(h_\varepsilon - H_\varepsilon)|^2 \mathrm{d}x \leq \int_\Omega \frac{|\nabla(h_\varepsilon - H_\varepsilon)|^2}{a(x)} \mathrm{d}x \leq J_\varepsilon(u_\varepsilon) = o(H_\varepsilon^2).$$

It is clear that $\frac{h_\varepsilon - H_\varepsilon}{H_\varepsilon}$ tends strongly to $h_* - 1 = 0$ in H_0^1, so that $\mu_* = dx$. □

Now we can adjust the [28, Proposition 2.2].

Proposition 3.4. *Let* μ*,* H_ε *and* λ *be as in the above proposition. Then, for* $\varepsilon > 0$ *small enough there exist points* a_i^ε*,* $1 \leq i \leq n(\varepsilon)$*, such that*

$$n(\varepsilon) \simeq \frac{H_\varepsilon}{\int_0^{2\pi} a(a_i^\varepsilon + \varepsilon e^{i\theta}) \mathrm{d}\theta} \int_\Omega a(x)\mu \mathrm{d}x, \quad |a_i^\varepsilon - a_i^\varepsilon| > 4_e,$$

and letting μ_ε^i *be the uniform measure on* $\partial B(a_i^\varepsilon, \varepsilon)$ *of mass* 2π*,*

$$\mu_\varepsilon = \frac{1}{H_\varepsilon} \sum_i \mu_\varepsilon^i \longrightarrow \mu$$

in the sense of measures as $\varepsilon \longrightarrow 0$*. Finally,*

$$\limsup_{\varepsilon \longrightarrow 0} \int_{\Omega \times \Omega} G(x,y) \mathrm{d}\mu_\varepsilon(x) \mathrm{d}\mu_\varepsilon(y) \leq \frac{1}{\lambda} \int_\Omega a(x)\mu \mathrm{d}x + \int_{\Omega \times \Omega} G(x,y) \mathrm{d}\mu(x) \mathrm{d}\mu(y). \tag{28}$$

The proof of the above proposition needs a construction of a test configuration for J_a. For more details one can refer to the adjusted of [28, Proposition 2.2] and [7, Lemma 3.9]. In particular, the term $\int_0^{2\pi} a(a_i^\varepsilon + \varepsilon e^{i\theta}) \mathrm{d}\theta$ comes from integration of the irregular term $a(x) \ln |x - y|$ on appropriate sets.

3.2. Proof of Proposition 3.2. One may also follow step by step the proof given in [28]. The only difference in the construction of the test configuration u_ε is in the definition of h_ε. Indeed, let h_ε be the solution to

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla h_\varepsilon}{a(x)}\right) = H_\varepsilon(\mu_\varepsilon - 1) & \text{in } \Omega \\ \qquad\qquad h_\varepsilon = H_\varepsilon & \text{on } \partial\Omega. \end{cases}$$

Here $h_\varepsilon = H_\varepsilon \int_\Omega G(x,y)d(\mu_\varepsilon - 1)(y)$. Therefore,

$$\int_\Omega \frac{|\nabla h_\varepsilon|^2}{a(x)}\mathrm{d}x = H_\varepsilon^2 \int_{\Omega\times\Omega} G(x,y)\mathrm{d}(\mu_\varepsilon - 1)(y)\mathrm{d}(\mu_\varepsilon - 1)(x). \tag{29}$$

Again choosing $x_0 \in \Omega_\varepsilon = \Omega\setminus(\cup_i B(a_i^\varepsilon, \varepsilon))$, we let for any $x \in \Omega_\varepsilon$

$$\phi_\varepsilon(x) = \oint_{(x_0,x)} A_\varepsilon \cdot \tau - \frac{\nabla h_\varepsilon}{a}\cdot \nu,$$

where (x_0, x) is any curve joining x_0 to x in Ω_ε and (τ, ν) is the Frénet frame on the curve. By construction, one can obtain $\nabla\phi_\varepsilon - A_\varepsilon = -\frac{\nabla^\perp h_\varepsilon}{a(x)}$. In other words, we let $\rho_\varepsilon \le 1$ in order to

$$\int_\Omega a(x)|\nabla\rho_\varepsilon|^2\mathrm{d}x + \frac{a(x)}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\mathrm{d}x \le CH_\varepsilon. \tag{30}$$

We take $u_\varepsilon = \rho_\varepsilon e^{i\phi_\varepsilon}$. Consequently thanks to $\rho_\varepsilon \le 1$ with (29)–(30)

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= \int_\Omega a(x)|\nabla\rho_\varepsilon|^2\mathrm{d}x + \frac{a(x)}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\mathrm{d}x + \rho_\varepsilon^2 a(x)|\nabla\phi_\varepsilon - A_\varepsilon|^2\mathrm{d}x \\ &= \int_\Omega a(x)|\nabla\rho_\varepsilon|^2\mathrm{d}x + \frac{a(x)}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\mathrm{d}x + \rho_\varepsilon^2\frac{|\nabla h_\varepsilon|^2}{a(x)}\mathrm{d}x \\ &\le CH_\varepsilon + \int_\Omega \frac{|\nabla h_\varepsilon|^2}{a(x)}\mathrm{d}x \\ &= CH_\varepsilon + H_\varepsilon^2\int_{\Omega\times\Omega} G(x,y)\mathrm{d}(\mu_\varepsilon - 1)(x)\mathrm{d}(\mu_\varepsilon - 1)(y). \end{aligned}$$

It follows that

$$\limsup_{\varepsilon\longrightarrow 0} \frac{J_\varepsilon(u_\varepsilon)}{H_\varepsilon^2} \le \limsup_{\varepsilon\longrightarrow 0}\int_{\Omega\times\Omega} G(x,y)\mathrm{d}(\mu_\varepsilon - 1)(x)\mathrm{d}(\mu_\varepsilon - 1)(y). \tag{31}$$

On the other hand, from the weak convergence of μ_ε to μ, we have

$$\limsup_{\varepsilon\longrightarrow 0}\int_\Omega\left(\int_\Omega G(x,y)\mathrm{d}x\right)\mathrm{d}\mu_\varepsilon(y) = \int_\Omega\left(\int_\Omega G(x,y)\mathrm{d}x\right)\mathrm{d}\mu(y). \tag{32}$$

Combining (28) and (32) yields

$$\begin{aligned}&\limsup_{\varepsilon\longrightarrow 0}\int_{\Omega\times\Omega} G(x,y)\mathrm{d}(\mu_\varepsilon-1)(x)\mathrm{d}(\mu_\varepsilon-1)(y)\\&\le\frac{1}{\lambda}\int_\Omega a(x)\mu\mathrm{d}x+\int_{\Omega\times\Omega}G(x,y)\mathrm{d}(\mu-1)(x)\mathrm{d}(\mu-1)(y)\\&=I_\lambda(\mu),\end{aligned}$$

so from (31), $\limsup_{\varepsilon\longrightarrow 0}\frac{J_\varepsilon(u_\varepsilon)}{H_\varepsilon^2}\le I_\lambda(\mu)$. This completes the proof of Proposition 3.2. □

Remark 3.5. Combining the upper and lower bound of Propositions 3.2 and Proposition 2.1, then by uniqueness of the minimizer μ_* of E_λ (see Section 4 below), it is evident that $\mu_*=\mu_0$ and $h_{\mu_*}=h_0$. Here μ_0 and h_0 are given in Proposition 2.1 above.

4. Minimization of the limiting energy

As we explained in the introduction, by convexity and lower semi-continuity, the limiting energy (8) admits a unique minimizer μ_* which is expressed by means of the unique minimizer h_* of (11) as follows,

$$\mu_*=-\operatorname{div}\left(\frac{\nabla h_*}{a}\right)+1. \tag{33}$$

Proceeding as in [28, 29], we may get an equivalent characterization of h_*.

Proposition 4.1. *The minimizer u_* of*

$$\min_{\substack{u\in H_0^1(\Omega)\\ \mu=-\operatorname{div}(\frac{\nabla u}{a})+1\in\mathcal{M}(\Omega)}}\int_\Omega\left(\frac{a(x)}{\lambda}\left|-\operatorname{div}\left(\frac{\nabla u}{a}\right)+1\right|+\frac{|\nabla u|^2}{a(x)}\right)\mathrm{d}x,$$

is also the unique minimizer of the dual problem

$$\min_{\substack{v\in H_0^1(\Omega)\\ |v|\le\frac{a}{2\lambda}}}\int_\Omega\left(\frac{|\nabla v|^2}{a}+2v\right)\mathrm{d}x.$$

For instance, $h_*=u_*+1$ minimizes the energy,

$$\min_{\substack{f\in H_1^1(\Omega)\\ f-1\ge-\frac{a}{2\lambda}}}\int_\Omega\left(\frac{|\nabla f|^2}{a}+2(f-1)\right)\mathrm{d}x,$$

and satisfies $-\operatorname{div}\left(\frac{\nabla h_*}{a}\right)+1\ge 0$.

Proof. The proof of Proposition 4.1 could be done as in [6, 7]. For more convenience of the reader, we state it as follows: Let us define the lower semi-continuous and convex functional

$$\Phi(u) = \int_\Omega \frac{1}{2\lambda} \left| - \operatorname{div}\left(\frac{\nabla u}{a}\right) + 1 \right| \,\mathrm{d}x$$

in the Hilbert space $H = H_0^1(\Omega)$ endowed with the scalar product $\langle f, g\rangle_H = \int_\Omega \frac{\nabla f}{a} \nabla g$. Let us compute its conjugate Φ^*, i.e.,

$$\Phi^*(f) = \sup_{\{g : \Phi(g) < \infty\}} \langle f, g\rangle - \Phi(g).$$

Indeed, we have, $\Phi^*(f) \geq \sup_{\eta \in L^2} \int_\Omega f\eta \,\mathrm{d}x - \frac{1}{2\lambda}\int_\Omega a(x)|\eta| \,\mathrm{d}x - \int_\Omega f \,\mathrm{d}x$, from which we deduce that

$$\Phi^*(f) = \begin{cases} -\displaystyle\int_\Omega f \,\mathrm{d}x & \text{if} |f| \leq \dfrac{a}{2\lambda}, \\ +\infty & \text{otherwise.} \end{cases}$$

By convex duality (see [29, Lemma 7.2]),

$$\min_{u \in H} \left(\|u\|_H^2 + 2\Phi(u) \right) = -\min_{f \in H} \left(\|f\|_H^2 + 2\Phi^*(-f) \right),$$

and minimizers coincide. Note that the measure $\mu_* = -\operatorname{div}\left(\frac{\nabla h_*}{a}\right) + 1$ is positive and absolutely continuous measure, which is actually a consequence of the weak maximum principle, see [21, p. 131]. One may also follow step by step the proof given in [28]. □

Following [28], the limiting vorticity measure μ_* can be expressed by means of the coincidence set $w_\lambda = \left\{ x \in \Omega : \ 1 - h_*(x) = \frac{a(x)}{2\lambda} \right\}$ as follows,

$$\mu_* = \left(1 - \frac{a(x)}{2\lambda} \right) \mathbf{1}_{w_\lambda} \mathrm{d}x,$$

where $\mathbf{1}_{w_\lambda}$ denotes the Lebesgue measure restricted to w_λ. Furthermore, h_* (the minimizer of (11)) solves,

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla h_*}{a}\right) + 1 = 0 & \text{in } \Omega \setminus \bar{w_\lambda} \\ h_* = 1 - \dfrac{a(x)}{2\lambda} & \text{in } w_\lambda \\ h_* = 1 & \text{on } \partial\Omega. \end{cases}$$

In the limit $\varepsilon \longrightarrow 0$, the vortices are scattered in an inner region w_λ with density μ_*, where $h_* = 1 - \frac{a(x)}{2\lambda}$. In the outer region $\Omega \setminus \bar{w_\lambda}$, there are no vortices. We adjust now [28, Proposition 1.2] to assert that

i) $\Omega \backslash w_\lambda$ is connected,
ii) $w_\lambda = \emptyset \Longleftrightarrow \lambda < k_a = \frac{1}{2 \max_{x\in\Omega} \frac{|\xi_0(x)|}{a(x)|}}$, where ξ_0 is given by (5),
iii) $\mu_* \neq 0 \Longleftrightarrow \lambda > k_a$.

As a conclusion, for $\lambda < k_a$, vortices essentially do not appear, while for $\lambda > k_a$, one has a (non-constant) vortex-density over w_λ, 0 elswhere, that is, the vortices exist and are pinned in w_λ. This completes the vortex nucleation of the minimal energy in superconducting thin films with respect to the applied field H. Note that the case where $\lambda \neq k_a$ is not treated.

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Received April 29, 2010; revised April 14, 2011