J. Noncommut. Geom. 20 (2026), 489–505
DOI 10.4171/JNCG/607

Quantum automorphism groups of trees

Josse van Dobben de Bruyn, Prem Nigam Kar, David E. Roberson,
Simon Schmidt, and Peter Zeman

Abstract. We give a characterization of quantum automorphism groups of trees. In particular, for every tree, we show how to iteratively construct its quantum automorphism group using free products and free wreath products. This can be considered a quantum version of Jordan's theorem for the automorphism groups of trees. We use ideas from algebraic combinatorics, quantum groups, and quantum information theory. This is one of the first characterizations of quantum automorphism groups of a natural class of graphs with quantum symmetry.

1. Introduction

Attempts to characterize automorphism groups of trees can be traced back to Jordan [12] and Pólya [17]. It is well known that the class of automorphism groups of trees can be constructed from the trivial group using two types of group products: the direct product and the wreath product with a symmetric group $\mathbb{S}_n$. We prove an analogue of this result for *quantum* automorphism groups of trees.

Banica [3] and Bichon [7] defined the quantum automorphism group of a finite graph to be a quotient of the quantum symmetric group $\mathbb{S}_n^+$, introduced by Wang [21]. The theory of quantum automorphism groups of graphs is still a relatively young field and we refer to [18] for a survey of the state of the art. Most of the known results answer the question of whether a given graph has quantum symmetry, i.e., whether the graph's quantum automorphism group differs from its automorphism group. Explicit computations are known only for some very special families of graphs, including hypercubes [5], folded hypercubes [19], and more generally, Cayley graphs of abelian groups [10].

Although trees are the most basic graph class, there have not been many attempts to calculate quantum automorphism groups of trees explicitly. Fulton [9] proved that trees with automorphism groups isomorphic to $\mathbb{S}_2^k$, for $k \geq 2$, have quantum symmetry. Further, whereas almost all graphs do not have quantum symmetry [14], almost all trees do have quantum symmetry [13]. This makes the study of quantum automorphism groups of trees particularly interesting.

In this paper, we provide a way to compute the quantum automorphism group of every tree explicitly, as an expression using the trivial quantum group $\mathbb{1}$, the free product, and

Mathematics Subject Classification 2020: 46L67 (primary); 05C05 (secondary).
Keywords: quantum automorphism groups of graphs, trees, free wreath product, free product.

the free wreath product with a quantum symmetric group $\mathbb{S}_n^+$. This gives the following classification of the quantum automorphism groups of trees.

Theorem 1.1. *The class $\mathcal{T}$ of all quantum automorphism groups of trees can be constructed inductively as follows:*

(i) $\mathbb{1} \in \mathcal{T}$.

(ii) *If* $\mathbb{G}, \mathbb{H} \in \mathcal{T}$, *then* $\mathbb{G} * \mathbb{H} \in \mathcal{T}$.

(iii) *If* $\mathbb{G} \in \mathcal{T}$, *then* $\mathbb{G} \wr_* \mathbb{S}_n^+ \in \mathcal{T}$.

The proof of Theorem 1.1 gives a polynomial-time algorithm to explicitly calculate the quantum automorphism group of any tree. The proof goes as follows: First, we prove that the center of a tree is preserved by quantum automorphisms, so we can restrict our attention to rooted trees. Then, we recursively compute the quantum autormorphism group of the rooted tree from the quantum automorphism groups of its subtrees. Intuitively, the free products in Theorem 1.1 (ii) stem from non-isomorphic subtrees, whereas the free wreath products with $\mathbb{S}_n^+$ in (iii) stem from isomorphic subtrees.

Remark 1.2. Following the initial appearance of this work on arXiv, another work containing similar results (including a different proof of Theorem 1.1) was announced on arXiv by Meunier [15]. These two papers were written independently and use different techniques. Contrary to our paper, Meunier's paper does not give an algorithm to compute quantum automorphism groups of trees, but it proves many other things and considers other classes of graphs in addition to trees.

2. Preliminaries

A *graph* is a tuple $X = (V(X), E(X))$, where $V(X)$ is a finite set of *vertices* of X and $E(X) \subseteq \binom{V(X)}{2}$ is the set of *edges* of X. (Here, $\binom{S}{2}$ denotes the set of all 2-element subsets of an arbitrary set S.) A *complete graph* is a graph X with $E(X) = \binom{V(X)}{2}$ and an *empty graph* is a graph X with $E(X) = \varnothing$. For $S \subseteq V(X)$, the graph $X[S] = (S, E(X) \cap \binom{S}{2})$ is the subgraph of X *induced* by S. A set $S \subseteq V(X)$ is called an *independent set* if $X[S]$ is an empty graph, S is called a *clique* if $X[S]$ is a complete graph.

For two subsets $A, B \subseteq V(X)$, let $E(A, B) = \{\{i, j\} \in E(X) : i \in A, j \in B\}$ and let $e(A, B) := |E(A, B)|$. For singleton sets, we write $E(i, B)$ instead of $E(\{i\}, B)$ and so on. For a subset $A \subseteq V(X)$ and a vertex $i \in A$, we write $d_A(i) := e(i, A)$. If $A = V(X)$, then we write $d(i) := d_{V(X)}(i)$ and say that $d(i)$ is the *degree of i in X*, or just the *degree* of i if X is clear from the context. We will refer to a vertex of degree 1 in any graph (not just trees) as a *leaf*.

A *path of length k* in a graph X is a finite sequence

$$(x_0, e_1, x_1, \dots, e_k, x_k)$$

such that $x_0, \dots, x_k \in V(X)$ are pairwise distinct and for every $i = 1, 2, \dots, k$, we have $e_i = \{x_{i-1}, x_i\} \in E(X)$. We say that a graph X is *connected* if $V(X) \neq \varnothing$ and for every

$x, y \in V(X)$, there is a path from x to y. For a graph X, we define *distance* to be the function $\text{dist}_X \colon V(X) \times V(X) \to \mathbb{R}$ defined by

$$\text{dist}_X(x, y) = \min\{k \in \mathbb{N}_0 : \text{ there is a path of length } k \text{ from } x \text{ to } y\}.$$

If X is clear from the context, we write $\text{dist} := \text{dist}_X$. Note that the function dist is a metric. A *cycle of length* k in a graph X is a sequence

$$(x_0, e_1, x_1, \dots, e_{k-1}, x_{k-1}, e_k, x_0)$$

such that $x_0, \dots, x_{k-1} \in V(X)$ are pairwise distinct and for every $i \in \{1, 2, \dots, k-1\}$, we have $e_i = \{x_{i-1}, x_i\} \in E(X)$ and also $e_k = x_{k-1}, x_0 \in E(X)$.

A *forest* is a graph with no cycles. A *tree* is a connected forest.

Definition 2.1. A *rooted tree* is a tuple (T, r) where T is a tree and $r \in V(T)$ is a designated vertex, called the *root*. A *forest of rooted trees* is a forest in which every connected component is a rooted tree.

Definition 2.2. A *(vertex-)colored graph* is a tuple (X, c) where X is a graph and $c : V(X) \to C$ is a function that assigns to each vertex $x \in V(X)$ a color $c(x)$ from some set of colors C. The set of vertices assigned to a fixed color c is called *color class*. The color classes partition the vertex set, but we do not require that adjacent vertices receive distinct colors.

Every uncolored graph will be thought of as a colored graph where all vertices have the same color, and every forest of rooted trees will be thought of as a colored graph where all roots have color c_1 and all non-roots have color $c_0 \neq c_1$.

Definition 2.3 ([22]). A *compact quantum group* $\mathbb{G}$ is a tuple $(\mathcal{A}, \Delta)$, where $\mathcal{A}$ is a unital C^*-algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a unital $*$-homomorphism satisfying the following conditions:

(1) $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$;

(2) $\Delta(\mathcal{A})(1 \otimes \mathcal{A})$ and $\Delta(\mathcal{A})(\mathcal{A} \otimes 1)$ are dense in $\mathcal{A} \otimes \mathcal{A}$.

The map Δ is called the *comultiplication* of $\mathbb{G}$.

Let G be a compact group and let $C(G)$ be the commutative C^*-algebra of continuous complex-valued functions on G. Since $C(G) \otimes C(G) \cong C(G \times G)$, $\Delta : C(G) \to C(G) \otimes C(G)$ defined by $\Delta(f)(g \times h) = f(gh)$ turns $(C(G), \Delta)$ into a compact quantum group. All compact quantum groups $\mathbb{G}$ where the C^*-algebra is commutative are of this type. In view of this example, it is customary to denote the C^*-algebra $\mathcal{A}$ associated with a compact quantum group $\mathbb{G} = (\mathcal{A}, \Delta)$ as $C(\mathbb{G})$. The *trivial quantum group* $\mathbb{1}$ is the trivial group 1, viewed as a quantum group.

Definition 2.4. Let $C(\mathbb{G})$ be a unital C^*-algebra generated by $\{u_{ij} : i, j \in [n]\}$ such that the matrices $[u_{ij}]_{i,j=1}^n$ and $[u_{ij}^*]_{i,j=1}^n$ are invertible and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$-homomorphism satisfying

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad i, j \in [n]. \tag{2.1}$$

Then, $\mathbb{G} = (C(\mathbb{G}), \Delta)$ forms a compact quantum group (see [16, Proposition 1.1.4]). Such compact quantum groups are knows as *compact matrix quantum groups*. They are usually specified by the pair $\mathbb{G} = (C(\mathbb{G}), u)$, as the comultiplication only depends on u. The matrix u will be called the *fundamental representation* of the compact matrix quantum group $\mathbb{G}$.

We now define the compact matrix quantum groups that are of interest to us.

Definition 2.5 ([21]). The *quantum symmetric group* $\mathbb{S}_n^+ = (C(\mathbb{S}_n^+), u)$ is the compact matrix quantum group, where $C(\mathbb{S}_n^+)$ is the universal C^*-algebra with generators u_{ij} and relations

$$\begin{aligned} u_{ij}^2 = u_{ij} = u_{ij}^* \qquad & \text{for all } i, j = 1, \dots, n \\ \sum_{j=1}^{n} u_{ij} = \sum_{i=1}^{n} u_{ij} = 1 \quad & \text{for all } i, j = 1, \dots, n. \end{aligned} \tag{2.2}$$

A matrix satisfying the conditions in (2.2) is called a *magic unitary*.

Definition 2.6. Let $\mathbb{G} = (C(\mathbb{G}), \Delta_{\mathbb{G}})$ and $\mathbb{H} = (C(\mathbb{H}), \Delta_{\mathbb{H}})$ be compact quantum groups. We say that $\mathbb{G}$ is a *quantum subgroup* of $\mathbb{H}$ if there is a surjective $*$-homomorphism $\phi : C(\mathbb{H}) \to C(\mathbb{G})$ such that $\Delta_{\mathbb{G}} \circ \phi = (\phi \otimes \phi) \circ \Delta_{\mathbb{H}}$. Quantum subgroups of $\mathbb{S}_n^+$ are known as *quantum permutation groups*.

Any compact matrix quantum group $\mathbb{G} = (C(\mathbb{G}), u)$, where u is a magic unitary, is a quantum permutation group, due to the universal property of $\mathbb{S}_n^+$.

Definition 2.7. We define the quantum automorphism group of colored graphs:

(I) The *quantum automorphism group* $\mathrm{Qut}(X)$ *of a graph* X is the compact matrix quantum group $(C(\mathrm{Qut}(X)), u)$, where $C(\mathrm{Qut}(X))$ is the universal C^*-algebra with generators u_{ij}, for $i, j \in V(X)$, and relations

$$\begin{aligned} u_{ij} = u_{ij}^* = u_{ij}^2, \qquad & i, j \in V(X), \\ \sum_k u_{ik} = 1 = \sum_k u_{kj}, \quad & i, j \in V(X), \\ u_{ij} u_{kl} = 0 \quad & \text{if } ik \in E, jl \notin E \text{ or vice versa.} \end{aligned} \tag{2.3}$$

Assuming the first two conditions of (2.3), the third condition is equivalent to $A_X u = u A_X$, where A_X is the adjacency matrix of X. This can be rewritten as

$$\sum_{kj \in E} u_{ik} = \sum_{ik \in E} u_{kj}, \tag{2.4}$$

for each i, j.

(II) The *quantum automorphism group* $\mathrm{Qut}_c(X)$ *of a graph* X *with coloring* c is the compact matrix quantum group that is obtained by taking the quotient of $C(\mathrm{Qut}(X))$ by the relations $u_{ij} = 0$ if $c(i) \neq c(j)$. This is equivalent to adding the relations $D^a U = uD^a$ for every vertex color a, where D^a is a diagonal matrix with $D^a_{ii} = 1$ if i has color a and $D^a_{ii} = 0$ otherwise.

(III) The *quantum automorphism group of a rooted tree* R is $\mathrm{Qut}_c(R)$. We shall denote $\mathrm{Qut}_c(R)$ as $\mathrm{Qut}_r(R)$ for convenience.

Definition 2.8 ([20]). Let $\mathbb{G} = (C(\mathbb{G}), u)$ and $H = (C(\mathbb{H}), v)$ be compact matrix quantum groups. Then, their *free product* $\mathbb{G} * \mathbb{H}$ is defined as the compact matrix quantum group $(C(\mathbb{G}) * C(\mathbb{H}), u \oplus v)$, where $C(\mathbb{G}) * C(\mathbb{H})$ is the universal C^*-algebra with generators u_{ij} and v_{kl} such that u_{ij} and v_{kl} satisfy the relations of $\mathbb{G}$ and $\mathbb{H}$, respectively, in addition to the relation $1_{\mathbb{G}} = 1_{\mathbb{H}}$.

Definition 2.9 ([8]). Let $\mathbb{G} = (C(\mathbb{G}), u)$ and $\mathbb{H} = (C(\mathbb{H}), v)$ be two quantum permutation groups such that their fundamental representations are magic unitaries. The *free wreath product of* $\mathbb{G}$ *and* $\mathbb{H}$ is the quantum permutation group $\mathbb{G} \wr_* \mathbb{H}$ with the C^*-algebra

$$C(\mathbb{G} \wr_* \mathbb{H}) = \big(C(\mathbb{G})^{*n} * C(\mathbb{H})\big)/\langle [u^{(a)}_{ij}, v_{ab}] = 0\rangle,$$

where $n \times n$ is the size of matrix v, and the fundamental representation of $\mathbb{G} \wr_* \mathbb{H}$ given by the magic unitary

$$w_{ia,jb} = u^{(a)}_{ij} v_{ab}.$$

A closely related concept to quantum automorphism groups of graphs is quantum isomorphism of graphs. This notion was originally defined in terms of quantum strategies for the so-called *isomorphism game* [2]. However, for this work, the following equivalent definition from [14] is better suited.[1]

Definition 2.10. Let X and Y be (possibly colored) graphs. We say that X and Y are *quantum isomorphic*, and write $X \cong_q Y$ if there is a nonzero unital C^*-algebra $\mathcal{A}$ and a magic unitary $u = [u_{xy}]_{x \in V(X), y \in V(Y)}$ with entries from $\mathcal{A}$ such that $A_X u = uA_Y$ and $u_{xy} = 0$ if $c(x) \neq c(y)$.

It is clear that if two graphs are isomorphic, then they are quantum isomorphic (the magic unitary in the definition above can be taken to be the permutation matrix encoding the isomorphism). But it is a nontrivial result that there are non-isomorphic graphs that are quantum isomorphic [2]. However, isomorphism and quantum isomorphism coincide for forests.

Two graphs X and Y are said to be *fractionally isomorphic* if there exists a doubly stochastic matrix D satisfying $A_X D = DA_Y$. It follows from [2, Theorem 4.5] that if two graphs are quantum isomorphic, then they are fractionally isomorphic. It is also

[1] In both [2] and [14], only uncolored graphs were considered, but adapting their results to include vertex colors is trivial.

known [11] that any two forests are fractionally isomorphic if and only if they are isomorphic. Therefore, we have the following.

Lemma 2.11. *If F and F' are forests, then they are quantum isomorphic if and only if they are isomorphic.*

Definition 2.12. Let $\mathbb{G} = (C(\mathbb{G}), u)$ be a quantum permutation group. We define a relation $\sim$ on $[n]$ by $i \sim j$ if and only if $u_{ij} \neq 0$. In [6] and [14], it was shown that $\sim$ is an equivalence relation. The partitions of $[n]$ induced by the equivalence relation $\sim$ are known as the *orbits* or *quantum orbits* of $\mathbb{G}$.

Remark 2.13. Note that by the definition of the quantum automorphism group of a colored graph, vertices of different colors must be in different quantum orbits. In other words, the color classes of a colored graph are unions of quantum orbits.

In addition to the above, we will need to use the following property of orbits of $\mathrm{Qut}_c(X)$.

Lemma 2.14. *Let X be a (possibly colored) graph and suppose that $S \subseteq V(X)$ is a union of orbits of* $\mathrm{Qut}_c(X)$. *If Y is the (possibly colored) subgraph of X induced by S, then the orbits of* $\mathrm{Qut}_c(Y)$ *are unions of orbits of X.*

Proof. We will first prove the uncolored case and then remark on the additions needed when colors are present. Let u and v be the fundamental representations of $\mathrm{Qut}(X)$ and $\mathrm{Qut}(Y)$, respectively. Since S is a union of orbits of $\mathrm{Qut}(X)$, the fundamental representation u can be written as

$$u = \begin{pmatrix} \hat{v} & 0 \\ 0 & \hat{u} \end{pmatrix}$$

where $\hat{v}$ is a magic unitary indexed by the elements of S. We will show that $A_Y \hat{v} = \hat{v} A_Y$. Let D be the diagonal matrix such that $D_{ii} = 1$ if $i \in S$ and $D_{ii} = 0$ otherwise. Then, $Du = uD$, and therefore $DA_X Du = uDA_X D$. Of course,

$$DA_X D = \begin{pmatrix} A_Y & 0 \\ 0 & 0 \end{pmatrix},$$

and therefore $DA_X Du = uDA_X D$ is equivalent to $A_Y \hat{v} = \hat{v} A_Y$. Thus, we have proven our claim.

Now, by the universality of $C(\mathrm{Qut}(Y))$, we have that there is a $*$-homomorphism φ from $C(\mathrm{Qut}(Y))$ to $C(\mathrm{Qut}(X))$ such that $\varphi(v_{ij}) = \hat{v}_{ij}$. Thus, if $v_{ij} = 0$, then $\hat{v}_{ij} = 0$; i.e., if $i, j \in V(Y)$ are in different orbits of $\mathrm{Qut}(Y)$, then they are in different orbits of $\mathrm{Qut}(X)$. This is the contrapositive of the lemma statement.

If the graph X is vertex colored, then we must additionally show that for any color a appearing in Y, the $V(Y)$-indexed diagonal matrix D^a that indicates whether a vertex of Y is colored a commutes with $\hat{v}$. However, if $\hat{D}^a$ is the similarly defined $V(X)$-indexed diagonal matrix, then $D\hat{D}^a D = D^a$ and the argument works the same as for A_Y. ■

3. Modifications that do not change the quantum automorphism group

In this section, we show that the quantum automorphism group of a colored graph is invariant under certain modifications to the graph. This will greatly simplify the proofs in the remainder of this paper since most proofs can be reduced to a combination of the modifications from this section. The main result in this section is the following lemma.

Lemma 3.1. *Let X be a colored graph.*

(I) *Let $S \subseteq V(X)$ be an independent set which is a union of color classes. Then, adding $\binom{|S|}{2}$ edges to X, one between every pair of distinct vertices in S, does not change the quantum automorphism group.*

(II) *Let $S, T \subseteq V(X)$ be disjoint vertex sets such that $S \cup T$ is an independent set and each of S and T is a union of color classes. Then, adding $|S| \times |T|$ edges to X, one from every $s \in S$ to every $t \in T$, does not change the quantum automorphism group.*

(III) *Let $S \subseteq V(X)$ be a monochromatic vertex set that is a union of quantum orbits of X. Then, changing the color of S to a new color (that does not occur elsewhere) does not change the quantum automorphism group.*

(IV) *Adding an isolated vertex in a new color (that does not occur elsewhere) does not change the quantum automorphism group.*

Proof. (I) Let $Y = X \cup \binom{S}{2}$ be the colored graph obtained from X by adding all possible edges between vertices in S. Let $u = (u_{ij})_{i,j \in V(X)}$ be a magic unitary in an arbitrary C^*-algebra that respects the color classes of X (i.e., $u_{ij} = 0$ whenever $c(i) \neq c(j)$). We prove that u satisfies the relations of $\mathrm{Qut}_c(X)$ if and only if it satisfies the relations of $\mathrm{Qut}_c(Y)$. With respect to the ordered partition $V(X) = V(Y) = S \sqcup R$ with $R = V(X) \setminus S$, we see that u and $A_Y - A_X$ are in block diagonal form

$$u = \begin{pmatrix} u_S & 0 \\ 0 & u_R \end{pmatrix}, \quad A_Y - A_X = \begin{pmatrix} J_S - I_S & 0 \\ 0 & 0 \end{pmatrix},$$

where u_S and u_R denote the restrictions of u to $S \times S$ and $R \times R$, and where J_S and I_S denote the all-ones matrix and the identity matrix on $S \times S$, respectively. Since rows and columns in u_S sum to 1, we have $u_S J_S = J_S = J_S u_S$, and therefore

$$\begin{pmatrix} u_S & 0 \\ 0 & u_R \end{pmatrix} \begin{pmatrix} J_S - I_S & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} J_S - u_S & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} J_S - I_S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_S & 0 \\ 0 & u_R \end{pmatrix}.$$

This shows that u commutes with $A_Y - A_X$. Therefore, u commutes with A_X if and only if it commutes with A_Y. The color classes remain the same, so we conclude that u satisfies the relations of $\mathrm{Qut}_c(X)$ if and only if it satisfies the relations of $\mathrm{Qut}_c(Y)$. By the universality of the respective C^*-algebras, we get natural $*$-homomorphisms

$$\phi : C\big(\mathrm{Qut}_c(X)\big) \to C\big(\mathrm{Qut}_c(Y)\big) \quad \text{and} \quad \psi : C\big(\mathrm{Qut}_c(Y)\big) \to C\big(\mathrm{Qut}_c(X)\big)$$

that map the fundamental representation of one to the fundamental representation of the other. Clearly, ϕ and ψ are each other's inverses and preserve the comultiplication, so we have $\mathrm{Qut}_c(X) \cong \mathrm{Qut}_c(Y)$.

(II) Let Y' be the colored graph obtained from X by adding all possible edges between S and T. Since S and T are independent sets in Y', it follows from (I) that

$$\mathrm{Qut}_c(Y') \cong \mathrm{Qut}_c\left(Y' \cup \binom{S}{2} \cup \binom{T}{2}\right).$$

But we have $Y' \cup \binom{S}{2} \cup \binom{T}{2} = X \cup \binom{S\cup T}{2}$, so another application of (I) shows that

$$\mathrm{Qut}_c(Y') \cong \mathrm{Qut}_c\left(Y' \cup \binom{S}{2} \cup \binom{T}{2}\right) = \mathrm{Qut}_c\left(X \cup \binom{S \cup T}{2}\right) \cong \mathrm{Qut}_c(X).$$

(III) Let Y'' be the colored graph obtained from X by recoloring the vertices of S to a new color (that does not occur elsewhere). Let $R \subseteq V(X) \setminus S$ be the remainder of the color class containing S, and let $u = (u_{ij})_{i,j\in V(X)}$ and $v'' = (v''_{ij})_{i,j\in V(Y'')}$ denote the fundamental representations of $\mathrm{Qut}_c(X)$ and $\mathrm{Qut}_c(Y'')$, respectively. The only difference between X and Y'' is that the latter splits the color class $S \cup R$ of X into separate color classes S and R, so the set of relations defining $C(\mathrm{Qut}_c(Y''))$ is the union of the set of relations defining $C(\mathrm{Qut}_c(X))$ and the relations $\{v''_{sr} = v''_{rs} = 0 : s \in S,\ r \in R\}$. Since S is a union of quantum orbits of X, the latter (added) relations are also satisfied by u, so u satisfies all the relations of $C(\mathrm{Qut}_c(Y''))$. Conversely, since the relations defining $C(\mathrm{Qut}_c(Y''))$ contain the relations defining $C(\mathrm{Qut}_c(X))$, clearly v'' satisfies all the relations of $C(\mathrm{Qut}_c(X))$. By the universality of the respective C^*-algebras, we get natural $*$-homomorphisms

$$\phi : C\big(\mathrm{Qut}_c(X)\big) \to C\big(\mathrm{Qut}_c(Y'')\big) \quad \text{and} \quad \psi : C\big(\mathrm{Qut}_c(Y'')\big) \to C\big(\mathrm{Qut}_c(X)\big)$$

that map the fundamental representation of one to the fundamental representation of the other. Clearly, ϕ and ψ are each other's inverses and preserve the comultiplication, so we have $\mathrm{Qut}_c(X) \cong \mathrm{Qut}_c(Y'')$.

(IV) Let v_0 be the newly added vertex and Y''' the colored graph thus obtained. Since v_0 has a different color from all other vertices, we have $u_{v_0 i} = u_{i v_0} = 0$ for all $i \neq v_0$, and $u_{v_0 v_0} = 1$. From this it readily follows that $\mathrm{Qut}_c(X) \cong \mathrm{Qut}_c(Y''')$. ■

Remark 3.2. In Lemma 3.1 (I) and (II), the assumption that S or $S \cup T$ is an independent set can be omitted by using multigraphs. To do so, one has to define the quantum automorphism group of a multigraph in the same way as Definition 2.7, namely, as the magic unitaries that commute with the adjacency matrix (which can now contain entries other than 0 and 1). Note that this might be considered to be the "wrong" definition of the quantum automorphism group of a multigraph since it does not see (quantum) automorphisms that permute parallel edges.

For simple graphs, Lemma 3.1 (II) remains true if the condition "$S \cup T$ is an independent set" is weakened to "there are no edges between S and T", by the same proof as above, but this proof now uses multigraphs in an intermediate step.

4. Reduction to rooted trees

In this section, we show how to transform a tree into a rooted tree with the same quantum automorphism group. Recall the following definitions.

Definition 4.1. Let X be a graph. The *eccentricity* of a vertex $v \in V(X)$, denoted by $\varepsilon(v)$, is the maximum distance from v to any other vertex in X. In other words,

$$\varepsilon(v) = \max \left\{ \mathrm{dist}(v, x) \mid x \in V(X) \right\}.$$

The (*Jordan*) *center* of X, which we denote by $Z(X)$, is the set of vertices of minimum eccentricity.

It is well known that the Jordan center of a tree coincides with the vertex set obtained by simultaneously removing all leaves from the graph and repeating this procedure until only a single vertex or edge remains.[2] It follows that the Jordan center of a tree consists of either a single vertex or two adjacent vertices. In the latter case, we refer to the edge between these two vertices as the *central edge*.

In the 19th century, Jordan [12] already recognized that the center of a tree must be preserved by every automorphism. We extend this to quantum automorphisms, and use it as the basis for our reduction from trees to rooted trees. For this we use the following construction. Note that we say that we *subdivide* an edge $e = (u, v)$ in a graph X if we delete the edge e from X and add a vertex w as well as edges (u, w) and (w, v) to X.

Definition 4.2. Let T be a tree. The *rootification* of T is the rooted tree (T_r, r) obtained from T in the following way:

- if $|Z(T)| = 1$, set $T_r := T$ and let r be the Jordan center of T;
- if $|Z(T)| = 2$, let T_r be the tree obtained from T by subdividing the central edge of T, and let $r \in V(T_r)$ be the new vertex thus created.

Note that one always has $Z(T_r) = r$. Below, we will show that $\mathrm{Qut}(T) \cong \mathrm{Qut}_r(T_r)$ for any tree T. The key to this result is the following lemma.

Lemma 4.3. *Let X be a graph, and let $S \subseteq V(X)$ be a vertex set obtained by simultaneously deleting all leaves and repeating this process any number of times. Then, S is a union of quantum orbits.*

Proof. Let X_1 be the graph obtained from X by deleting all of its leaves. In other words, X_1 is the subgraph of X induced by the set $U = \{v \in V(X) : d(v) \neq 1\}$. By [18, Proposition 2.4.14], it follows that if $d(x) \neq d(y)$, then $u_{xy} = 0$; i.e., x and y are in different quantum orbits. Hence, U is a union of quantum orbits of X since its complement is the set of vertices with degree 1 in X which is a union of quantum orbits. Therefore, by Lemma 2.14, the quantum orbits of X_1 are unions of the quantum orbits of X.

[2] In fact, Jordan's original definition of the center (see [12, Section 4]) was based on this iterative procedure and not on the modern definition in terms of distances. To prove that these notions are equivalent, note that the Jordan center of a tree T with $|V(T)| \geq 3$ is equal to the Jordan center of the tree T' obtained from T by removing all leaves.

If X_2 is the graph obtained from X_1 by removing all of its leaves, then by the same reasoning as above, $V(X_2)$ is a union of quantum orbits of X_1, and therefore the union of quantum orbits of X. Iterating this argument proves the lemma. ∎

Corollary 4.4. *Let T be a tree. Then, $Z(T)$ is a union of quantum orbits.*

Corollary 4.5. *Let T be a tree, and let (T_r, r) be its rootification. Then, $\{r\}$ is a quantum orbit of T_r.*

Proof. Since $Z(T_r) = \{r\}$, this follows from Corollary 4.4. ∎

Corollary 4.6. *Let T be a tree, and let (T_r, r) be its rootification. Then, $\mathrm{Qut}(T_r) \cong \mathrm{Qut}_r(T_r)$; that is, the quantum automorphism group of T_r as a graph is isomorphic to the quantum automorphism group of T_r as a rooted tree.*

Proof. This follows from Corollary 4.5 and Lemma 3.1 (III). ∎

We now come to the main result of this section.

Proposition 4.7. *Let T be a tree, and let (T_r, r) be the rootification of T. Then, $\mathrm{Qut}(T) \cong \mathrm{Qut}_r(T_r)$.*

Proof. If $|Z(T)| = 1$, then we have $T_r = T$, so the result follows from Corollary 4.6. Otherwise, $|Z(T)| = 2$, and we let $Z(T) = \{z_1, z_2\}$. Let $X_0 := T$ and $X_5 := T_r$. We transform X_0 into X_5 through the following sequence of modifications from Lemma 3.1, which is also depicted in Figure 1.

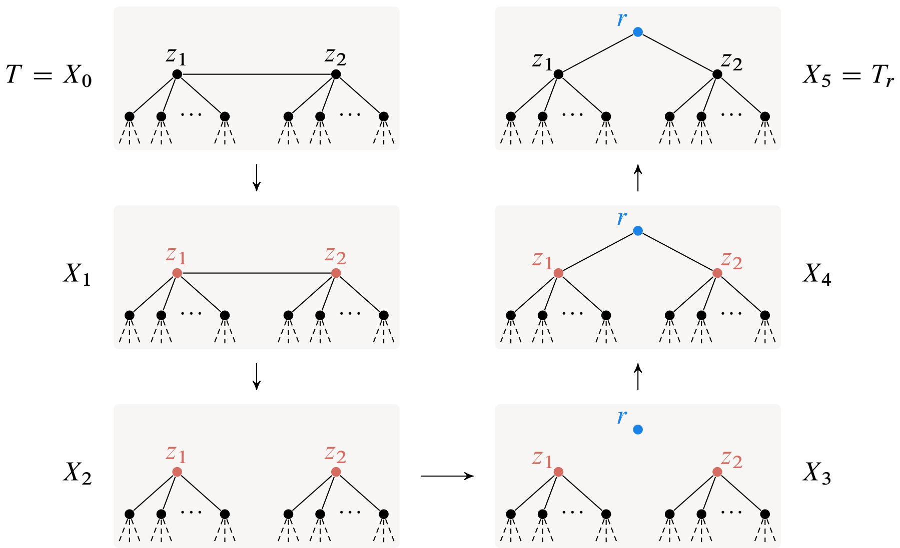


Figure 1. Transforming any tree T with Jordan center of size 2 to its rootification T_r.

Let X_1 be the colored graph obtained from the (uncolored) graph $X_0 := T$ by coloring $V(T) \setminus Z(T)$ with the color c_0 and coloring $Z(T)$ in a different color $c_1 \neq c_0$. Then, by Corollary 4.4 and Lemma 3.1 (III), we have $\mathrm{Qut}(X_0) \cong \mathrm{Qut}_c(X_1)$.

Let X_2 be the colored graph obtained from X_1 by removing the edge between z_1 and z_2. Then, by Lemma 3.1 (I) (applied to X_2), we have $\mathrm{Qut}_c(X_1) \cong \mathrm{Qut}_c(X_2)$.

Let X_3 be the colored graph obtained from X_2 by adding an isolated vertex r in a new color $c_2 \neq c_0, c_1$. Then, by Lemma 3.1 (IV), we have $\mathrm{Qut}_c(X_2) \cong \mathrm{Qut}_c(X_3)$.

Let X_4 be the colored graph obtained from X_3 by connecting all vertices of color c_1 (namely, z_1 and z_2) to all vertices of color c_2 (namely, r). Then, by Lemma 3.1 (II), we have $\mathrm{Qut}_c(X_3) \cong \mathrm{Qut}_c(X_4)$.

Finally, let $X_5 := T_r$ be the rootification of T. Since $Z(T)$ can be obtained from T by repeatedly removing all leaves, doing the same in T_r shows that $\{z_1, z_2, r\}$ can be obtained from T_r by repeatedly removing all leaves. Hence, it follows from Lemma 4.3 that $\{z_1, z_2, r\}$ is a union of quantum orbits of T_r. Since $\{r\}$ is a quantum orbit by itself, it follows that $\{z_1, z_2\}$ is a union of quantum orbits of T_r. Since X_4 is (isomorphic to) the graph obtained from $X_5 = T_r$ by recoloring $\{z_1, z_2\}$ in a new color, it follows from Lemma 3.1 (III) (applied to X_5) that $\mathrm{Qut}_c(X_4) \cong \mathrm{Qut}_c(X_5)$. Hence, $\mathrm{Qut}(T) = \mathrm{Qut}(X_0) \cong \mathrm{Qut}_c(X_1) \cong \cdots \cong \mathrm{Qut}_c(X_5) = \mathrm{Qut}_r(T_r)$. ∎

5. Quantum automorphisms of trees

In this section, we will prove the main theorem of this paper (Theorem 1.1). To do that, we first need some lemmas.

Lemma 5.1 ([18, Lemma 3.2.2]). *Let X be a finite, undirected vertex-colored graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(Qut(X))$. If the distance between i and k is different to the distance between j and l, then $u_{ij}u_{kl} = 0$.*

Lemma 5.2. *Let F be a forest of rooted trees, and let $\widetilde{F}$ be the rooted tree obtained by connecting the roots of the individual trees of F to a single new root. Then,* $\mathrm{Qut}_c(F) \cong \mathrm{Qut}_r(\widetilde{F})$.

Proof. Let $R \subseteq V(F)$ denote the set of roots of F, and write $X_0 := F$ and $X_3 := \widetilde{F}$. Without loss of generality, assume that the non-root vertices in F and $\widetilde{F}$ are colored with color c_0, the roots in F are colored c_1, and the root in $\widetilde{F}$ is colored c_2 (with c_0, c_1, and c_2 distinct). We transform X_0 into X_3 using the following sequence of modifications from Lemma 3.1, which is also depicted in Figure 2.

Let X_1 be the graph obtained from $X_0 := F$ by adding an isolated vertex r in a new color c_2 (that does not occur elsewhere in X_0). Then, by Lemma 3.1 (IV), we have $\mathrm{Qut}_c(X_0) \cong \mathrm{Qut}_c(X_1)$.

Let X_2 be the graph obtained from X_1 by connecting all vertices of color c_1 (namely, the roots R from F) to all vertices of color c_2 (namely, the newly added isolated vertex r). Then, by Lemma 3.1 (II), we have $\mathrm{Qut}_c(X_1) \cong \mathrm{Qut}_c(X_2)$.

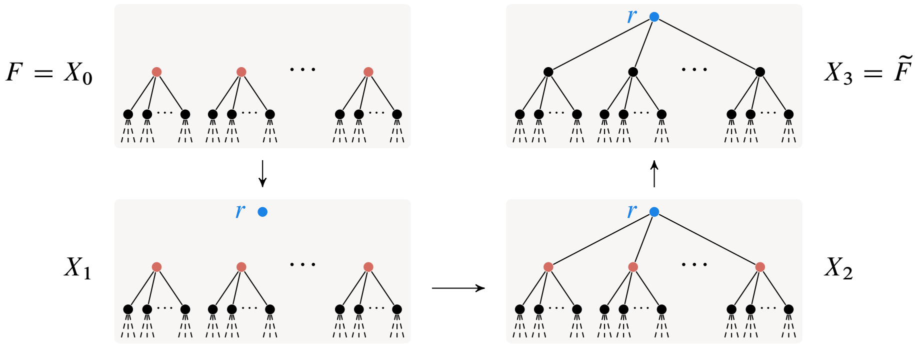


Figure 2. Transforming a forest of rooted trees to a single rooted tree.

Finally, let $X_3 := \widetilde{F}$. Since R is the set of all neighbors of r, it follows from Lemma 5.1 that $u_{rr}u_{xy} = 0$ if $x \in R$ and $y \notin R$. However, since r is the only vertex with color c_2, we also have that $u_{rr} = 1$. Hence, $u_{xy} = 0$ if $x \in R$ and $y \notin R$. Therefore, R is a union of quantum orbits of $\widetilde{F}$. Since X_2 is (isomorphic to) the graph obtained from $X_3 = \widetilde{F}$, by recoloring R in a new color c_1, it follows from Lemma 3.1 (III) (applied to X_3) that $\mathrm{Qut}_c(X_2) \cong \mathrm{Qut}_c(X_3)$. ∎

Lemma 5.3. *Let X be a colored graph with connected components $X_1, \ldots, X_n$. If $u_{st} \neq 0$ for some $s \in V(X_i)$, $t \in V(X_j)$, $i \neq j$, in the fundamental representation of* $\mathrm{Qut}(X)$, *then X_i and X_j are quantum isomorphic.*

Proof. We adjust the proof of [14, Theorem 4.4] to our setting. Assume that we have i, j, $i \neq j$ such that there exist $s \in V(X_i)$, $t \in V(X_j)$ with $u_{st} \neq 0$. We have $u_{xy}u_{x'y'} = 0$ for $x, x' \in V(X_a)$, $y \in V(X_b)$, $y' \in V(X_c)$, $b \neq c$ by Lemma 5.1 since we know that there is a path between x and x' as they are in the same connected component but no path between y and y'. Define $p_{x,j} = \sum_{y \in V(X_j)} u_{xy}$, $x \in V(X)$. It holds that $p_{x,j} = p_{x',j}$ for $x, x' \in V(X_i)$ because of the following. We have

$$p_{x,j}(1 - p_{x',j}) = \Big(\sum_{y \in V(X_j)} u_{xy}\Big)\Big(\sum_{y' \in V(X_k), k \neq j} u_{x'y'}\Big) = \sum_{\substack{y \in V(X_j),\\ y' \in V(X_k), k \neq j}} u_{xy}u_{x'y'} = 0$$

since $u_{xy}u_{x'y'} = 0$ for $x, x' \in V(X_i)$, $y \in V(X_j)$, $y' \in V(X_k)$. We get $(1 - p_{x,j})p_{x',j} = 0$ in the same way and therefore $p_{x,j} = p_{x,j}\,p_{x',j} = p_{x',j}$. We define $p_j := p_{x,j}$. Similarly, we can define $p'_{y,i} = \sum_{x \in V(X_i)} u_{xy}$ and get $p'_{y,i} = p'_{y',i}$ for $y, y' \in V(X_j)$. We then define $p'_i := p'_{y,i}$.

Note that $p_j \neq 0$ since we have $p_j \geq u_{st} \neq 0$ by assumption. It holds that

$$\big|V(X_i)\big| p_j = \sum_{x \in V(X_i)} p_{x,j} = \sum_{\substack{x \in V(X_i)\\ y \in V(X_j)}} u_{xy} = \sum_{y \in V(X_j)} p'_{y,i} = \big|V(X_j)\big| p'_i. \qquad (5.1)$$

Then, we have

$$p_j = p_j^2 = \left(\frac{|V(X_j)|}{|V(X_i)|}\right)^2 p_i' = \frac{|V(X_j)|}{|V(X_i)|} p_j,$$

and thus we must have $|V(X_i)| = |V(X_j)|$ since $p_j \neq 0$. Therefore, $p_j = p_i'$ by equation (5.1). We define $p := p_j$.

Let $\mathcal{A}$ be the C^*-subalgebra of $C(\mathrm{Qut}(X))$ generated by the elements u_{xy}, $x \in V(X_i)$, $y \in V(X_j)$. Note that $\mathcal{A}$ is nonzero since $p \neq 0$ as noted above. Further, p is the identity in $\mathcal{A}$ since

$$u_{xy} p = u_{xy} p_{x,j} = u_{xy} \sum_{y' \in V(X_j)} u_{xy'} = u_{xy}$$

and similarly $u_{xy} = p u_{xy}$. Moreover, we have $u_{xy} u_{x'y'} = 0$ for $xx' \in E$, $yy' \notin E$ or vice versa, and $u_{xy} = 0$ for $c(x) \neq c(y)$, since this holds in $C(\mathrm{Qut}(X))$. Thus, $\hat{u} = (u_{xy})_{x \in V(X_i), y \in V(X_j)}$ is a magic unitary with $A_{X_i} \hat{u} = \hat{u} A_{X_j}$. This means that X_i and X_j are quantum isomorphic. ■

The following lemma is a generalization of [18, Corollary 7.1.4].

Lemma 5.4. *Let $X_1, \dots, X_n$ be vertex colored graphs such that for any $i \neq j$, no connected component of X_i is quantum isomorphic to a connected component of X_j. Then,*

$$\mathrm{Qut}_c\left(\bigsqcup_{i=1}^n X_i\right) = \mathop{\ast}_{i=1}^n \mathrm{Qut}_c(X_i) \tag{5.2}$$

where $\bigsqcup_{i=1}^n X_i$ denotes the disjoint union of $X_1, \dots, X_n$.

Proof. Define $X := \bigsqcup_{i=1}^n X_i$ and let A_X and A_{X_i} be the adjacency matrices of the corresponding graphs. Label A_X in the following way:

$$A_X = \begin{pmatrix} A_{X_1} & 0 & \cdots & 0 \\ 0 & A_{X_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_{X_n} \end{pmatrix}.$$

Since all connected components of X_i and X_j, $i \neq j$, are not quantum isomorphic, we know by Lemma 5.3 that $u_{xy} = 0$ for all $x \in V(X_i)$, $y \in V(X_j)$, where u is the fundamental representation of $\mathrm{Qut}(X)$. This yields

$$u = \begin{pmatrix} u^{(1)} & 0 & \cdots & 0 \\ 0 & u^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & u^{(n)} \end{pmatrix},$$

where $u^{(i)} \in \mathrm{M}_{|V(X_i)|}(C(\mathrm{Qut}(X)))$. Furthermore, we see that $uA_X = A_X u$ is equivalent to $u^{(i)} A_{X_i} = A_{X_i} u^{(i)}$ for all i. Therefore, we get the desired surjective $*$-homomorphisms in both directions by using the respective universal properties. ■

Lemma 5.5. *For every rooted tree, there exists a tree with an isomorphic quantum automorphism group.*

Proof. Let T be a rooted tree. We construct a tree T' with $\mathrm{Qut}_r(T) \cong \mathrm{Qut}(T')$.

First, suppose that T is a path rooted at one of its endpoints. Then, by induction, it follows from Lemma 5.2 that $\mathrm{Qut}_r(T) \cong \mathrm{Qut}_r(K_1) = \mathbb{1}$. Since $\mathrm{Qut}(K_1) = \mathbb{1}$, the choice of $T' := K_1$ suffices.

Now suppose that T is not a path rooted at one of its endpoints. Let X_1 be the forest of rooted trees given by $X_1 := T \sqcup P$, where P is a path of length $2|V(T)|$ rooted at one of its leaves. Let X_2 be the rooted tree formed from X_1 by adding a new vertex r adjacent to only the roots of T and P. Let T' be the unrooted tree obtained from X_2 by ignoring the root.

From the choice of P and the definition of the centre, it follows that the centre of T' lies on the path P. Hence, by Proposition 4.7, we may conclude that $\mathrm{Qut}(T') = \mathrm{Qut}_r(T'_r)$, where T'_r is the rootification of T'. Let the $\tilde{r}$ be the root of T'_r. We have that $\deg(\tilde{r}) = 2$. Let X_3 and X_4 be the rooted trees obtained by deleting $\tilde{r}$ and designating its neighbors as roots. Relabelling if necessary, we know that X_3 is a rooted tree that is obtained by adjoining a path to the root of R and designating the other end as the root, and X_4 is a path. It follows from applying Lemma 5.2 multiple times that $\mathrm{Qut}_r(X_3) = \mathrm{Qut}_r(T)$. Additionally, by the argument for the case where T was a path rooted at one of its endpoints, we know that $\mathrm{Qut}_r(X_4) = \mathbb{1}$. Finally, from Lemma 5.2 and Lemma 5.4, we may conclude that $\mathrm{Qut}(T') = \mathrm{Qut}_r(T'_r) = \mathrm{Qut}_r(X_3) * \mathrm{Qut}_r(X_4) = \mathrm{Qut}_r(T)$. This finishes the proof. ∎

Theorem 5.6. *Let X be a connected vertex colored graph and $n \in \mathbb{N}$. Let $\bigsqcup_{i=1}^n X$ denote the disjoint union of n copies of X, all with the same coloring. Then,* $\mathrm{Qut}_c(\bigsqcup_{i=1}^n X) = \mathrm{Qut}_c(X) \wr_* \mathbb{S}_n^+$, *where $\wr_*$ denotes the free wreath product.*

The proof is similar to the proof of [4, Theorem 6.1] and will be omitted. For connected uncolored graphs, a proof of the above can be found in [8, Theorem 4.2] for a different definition of the quantum automorphism group. For the version of the definition we are working with, the uncolored case Theorem 5.6 follows from [4, Theorem 7.1].

We now come to the proof of our main theorem.

Proof of Theorem 1.1. Let $\mathcal{S}$ be the family of compact quantum groups that is generated by (i)–(iii), and let $\mathcal{R}$ denote the set of all quantum automorphism groups of rooted trees. It follows from Proposition 4.7 and Lemma 5.5 that $\mathcal{T} = \mathcal{R}$. Hence, it is sufficient to show that $\mathcal{S} = \mathcal{R}$.

First, we show that $\mathcal{S} \subseteq \mathcal{R}$. We do this by showing that $\mathbb{1} \in \mathcal{R}$ (which is trivial) and that $\mathcal{R}$ is closed under (ii) and (iii) from the theorem statement. For (ii), we split it into the two cases of $\mathbb{G} \ncong \mathbb{H}$ and $\mathbb{G} \cong \mathbb{H}$. For the former, consider any two non-isomorphic elements $\mathbb{G}, \mathbb{H} \in \mathcal{R}$. Let $T_{\mathbb{G}}$ and $T_{\mathbb{H}}$ be two rooted trees such that $\mathrm{Qut}(T_{\mathbb{G}}) \cong \mathbb{G}$ and $\mathrm{Qut}_r(T_{\mathbb{H}}) \cong \mathbb{H}$. Let us denote by $T_{\mathbb{G}\sqcup\mathbb{H}}$ the rooted tree formed from $T_{\mathbb{G}} \sqcup T_{\mathbb{H}}$ by applying the construction in Lemma 5.2. It follows from Lemma 5.2 that $\mathrm{Qut}_r(T_{\mathbb{G}\sqcup\mathbb{H}}) \cong \mathrm{Qut}_c(T_{\mathbb{G}} \sqcup T_{\mathbb{H}})$.

Since $\mathbb{G}$ and $\mathbb{H}$ are distinct, $T_{\mathbb{G}} \ncong T_{\mathbb{H}}$, and hence they are also not quantum isomorphic by Lemma 2.11. It now follows from Lemma 5.4 that $\mathrm{Qut}_r(T_{\mathbb{G}\sqcup\mathbb{H}}) \cong \mathbb{G} * \mathbb{H}$, so that $\mathbb{G} * \mathbb{H} \in \mathcal{R}$.

For the latter, let $\mathbb{G} \in \mathcal{R}$ be a compact quantum group, and let $T_{\mathbb{G}}$ be a rooted tree such that $\mathrm{Qut}_r(T_{\mathbb{G}}) \cong \mathbb{G}$. Let $\widetilde{T_{\mathbb{G}}}$ be the rooted tree constructed by joining the root of $T_{\mathbb{G}}$ to a new vertex, which is designated as the root of $\widetilde{T_{\mathbb{G}}}$. It follows from Lemma 5.2 that $\mathrm{Qut}_r(\widetilde{T_{\mathbb{G}}}) \cong \mathbb{G}$. Since $T_{\mathbb{G}}$ and $\widetilde{T_{\mathbb{G}}}$ have a different number of vertices, they are not quantum isomorphic. Let T be the rooted tree that is formed by applying the construction in Lemma 5.2 to $T_{\mathbb{G}} \sqcup \widetilde{T_{\mathbb{G}}}$. Then, from Lemmas 5.2 and 5.4, $\mathrm{Qut}_r(T) \cong \mathrm{Qut}_c(T_{\mathbb{G}} \sqcup \widetilde{T_{\mathbb{G}}}) \cong \mathbb{G} * \mathbb{G}$, so that $\mathbb{G} * \mathbb{G} \in \mathcal{R}$.

Similarly, it follows from Theorem 5.6 that $\mathrm{Qut}_c(\bigsqcup_{i=1}^{n} T_{\mathbb{G}}) \cong \mathbb{G} \wr_* \mathbb{S}_n^+$ for any $n \in \mathbb{N}$. Let T' be the rooted tree that is formed from $\bigsqcup_{i=1}^{n} T_{\mathbb{G}}$ by the operation in Lemma 5.2. Then, it follows from Lemma 5.2 that $\mathrm{Qut}_r(T') \cong \mathrm{Qut}_c(\bigsqcup_{i=1}^{n} T_{\mathbb{G}}) \cong \mathbb{G} \wr_* \mathbb{S}_n^+$, so that $\mathbb{G} \wr_* \mathbb{S}_n^+ \in \mathcal{R}$. Hence, we may conclude that any compact quantum group that is inductively constructed using (i)–(iii) is the quantum automorphism group of a rooted tree, which implies that $\mathcal{S} \subseteq \mathcal{R}$.

Now we show that $\mathcal{R} \subseteq \mathcal{S}$, i.e., that for any rooted tree T, $\mathrm{Qut}(T)$ can be constructed iteratively through (i)–(iii). Let $\tilde{T}$ be the forest of rooted trees that is constructed by deleting the root of T, and designating the neighbors of T as the roots of the trees that are formed. If we have n equivalence classes of rooted trees (where two trees are equivalent if they are isomorphic), and the ith equivalence class has m_i isomorphic copies of the rooted tree T_i, it follows from Theorem 5.6, Lemma 5.4, and Lemma 5.2 that

$$\mathrm{Qut}_r(T) \cong \mathop{\ast}_{i=1}^{n} \big(\mathrm{Qut}_r(T_i) \wr_* \mathbb{S}_{m_i}^+\big).$$

Now, we can iteratively apply the same deconstruction to the rooted trees $\{T_i\}_{i=1}^{n}$ until we end up with trees with only one vertex and no edges, whose quantum automorphism group is the trivial quantum group $\mathbb{1}$. Hence, $\mathrm{Qut}_r(T)$ can be constructed iteratively using (i)–(iii). We have now proved that $\mathcal{R} \subseteq \mathcal{S}$. ■

The proof of Theorem 1.1 contains an algorithm for determining the quantum automorphism group of a rooted tree T. We explicitly outline the algorithm here:

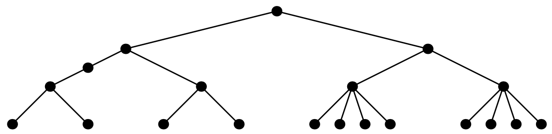
(1) Delete the root from T and designate its neighbors as roots to form a forest $\tilde{T}$ of rooted trees.

(2) Determine the isomorphism classes of these rooted trees, which can be done by standard techniques in linear time (see [1] for example).

(3) Now express the quantum automorphism group of T in terms of the quantum automorphism groups of the rooted trees of the rooted forest $\tilde{T}$ as described in the proof of Theorem 1.1.

(4) Determine the quantum automorphism groups of the rooted trees in $\tilde{T}$ recursively.

Figure 3 shows an example of a tree and its quantum automorphism group, computed using the algorithm described above. The details of this computation are left to the reader.



$$((\mathbb{1} \wr_* \mathbb{S}_2^+) * (\mathbb{1} \wr_* \mathbb{S}_2^+)) * ((\mathbb{1} \wr_* \mathbb{S}_4^+) \wr_* \mathbb{S}_2^+) \cong (\mathbb{S}_2^+ * \mathbb{S}_2^+) * (\mathbb{S}_4^+ \wr_* \mathbb{S}_2^+)$$

Figure 3. An example of a tree with its quantum automorphism group.

Remark 5.7. It follows from Lemma 5.4 and Theorem 5.6 that the quantum automorphism group of a forest can be obtained by free products and free wreath products of quantum automorphism groups of trees. However, it follows from Theorem 1.1 that the class of quantum automorphism groups of trees is closed under free products and free wreath products. Hence, both Theorem 1.1 and the poynomial-time algorithm to compute the quantum automorphism group can be extended to forests.

Funding. J. van Dobben de Bruyn, P. N. Kar, D. E. Roberson, and P. Zeman are supported by the Carlsberg Foundation Young Researcher Fellowship CF21-0682 – "Quantum Graph Theory". S. Schmidt was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC 2092 CASA - 390781972.

References

[1] A. V. Aho, J. E. Hopcroft, and J. D. Ullman, *The design and analysis of computer algorithms*. Addison-Wesley Ser. Comput. Sci. Inform. Process, Addison-Wesley Publishing, Reading, MA, 1974 Zbl 0326.68005 MR 0413592

[2] A. Atserias, L. Mančinska, D. E. Roberson, R. Šámal, S. Severini, and A. Varvitsiotis, Quantum and non-signalling graph isomorphisms. *J. Combin. Theory Ser. B* **136** (2019), 289–328 Zbl 1414.05197 MR 3926289

[3] T. Banica, Quantum automorphism groups of homogeneous graphs. *J. Funct. Anal.* **224** (2005), no. 2, 243–280 Zbl 1088.46040 MR 2146039

[4] T. Banica and J. Bichon, Free product formulae for quantum permutation groups. *J. Inst. Math. Jussieu* **6** (2007), no. 3, 381–414 Zbl 1188.16029 MR 2329759

[5] T. Banica, J. Bichon, and B. Collins, The hyperoctahedral quantum group. *J. Ramanujan Math. Soc.* **22** (2007), no. 4, 345–384 Zbl 1185.46046 MR 2376808

[6] T. Banica and A. Freslon, Modeling questions for quantum permutations. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **21** (2018), no. 2, article no. 1850009 Zbl 1404.46065 MR 3819809

[7] J. Bichon, Quantum automorphism groups of finite graphs. *Proc. Amer. Math. Soc.* **131** (2003), no. 3, 665–673 Zbl 1013.16032 MR 1937403

[8] J. Bichon, Free wreath product by the quantum permutation group. *Algebr. Represent. Theory* **7** (2004), no. 4, 343–362 Zbl 1112.46313 MR 2096666

[9] M. B. Fulton, *The quantum automorphism group and undirected trees*. Ph.D. thesis, Virginia Polytechnic Institute and State University, 2006

[10] D. Gromada, Quantum symmetries of Cayley graphs of abelian groups. *Glasg. Math. J.* **65** (2023), no. 3, 655–686 Zbl 1526.20075 MR 4653714

[11] N. Immerman and E. Lander, Describing graphs: A first-order approach to graph canonization. In *Complexity theory retrospective*, pp. 59–81, Springer, New York, 1990 MR 1060782

[12] C. Jordan, Sur les assemblages de lignes. *J. Reine Angew. Math.* **70** (1869), 185–190 Zbl 02.0344.01 MR 1579443

[13] L. Junk, S. Schmidt, and M. Weber, Almost all trees have quantum symmetry. *Arch. Math. (Basel)* **115** (2020), no. 4, 367–378 Zbl 1451.05041 MR 4145763

[14] M. Lupini, L. Mančinska, and D. E. Roberson, Nonlocal games and quantum permutation groups. *J. Funct. Anal.* **279** (2020), no. 5, article no. 108592 Zbl 1508.81313 MR 4097284

[15] P. Meunier, Quantum properties of $\mathcal{F}$-cographs. [v1] 2023, [v2] 2024, arXiv:2312.01516v2

[16] S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*. Cours Spec. 20, Société Mathématique de France, Paris, 2013 Zbl 1316.46003 MR 3204665

[17] G. Pólya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Math.* **68** (1937), no. 1, 145–254 Zbl 0017.23202 MR 1577579

[18] S. Schmidt, *Quantum automorphism groups of finite graphs*. Ph.D. thesis, Universität des Saarlandes, 2020

[19] S. Schmidt, Quantum automorphisms of folded cube graphs. *Ann. Inst. Fourier (Grenoble)* **70** (2020), no. 3, 949–970 Zbl 1465.46077 MR 4117054

[20] S. Wang, Free products of compact quantum groups. *Comm. Math. Phys.* **167** (1995), no. 3, 671–692 Zbl 0838.46057 MR 1316765

[21] S. Wang, Quantum symmetry groups of finite spaces. *Comm. Math. Phys.* **195** (1998), no. 1, 195–211 Zbl 1013.17008 MR 1637425

[22] S. L. Woronowicz, Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pp. 845–884, North-Holland, Amsterdam, 1998 Zbl 0997.46045 MR 1616348

Received 15 March 2024; revised 24 September 2024.

Josse van Dobben de Bruyn
DTU Compute, Technical University of Denmark, Richard Petersens Plads Building 322, 2800 Kongens Lyngby, Denmark; jdob@dtu.dk

Prem Nigam Kar
DTU Compute, Technical University of Denmark, Richard Petersens Plads Building 322, 2800 Kongens Lyngby, Denmark; pkar@dtu.dk

David E. Roberson
DTU Compute, Technical University of Denmark, Richard Petersens Plads Building 322, 2800 Kongens Lyngby; QMATH, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark; dero@dtu.dk

Simon Schmidt
Faculty of Computer Science, Ruhr-Universität Bochum, Universitätsstraße 150, 44801 Bochum, Germany; s.schmidt@rub.de

Peter Zeman
DTU Compute, Technical University of Denmark, Richard Petersens Plads Building 322, 2800 Kongens Lyngby, Denmark; pezem@dtu.dk