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On the homotopy groups of the automorphism groups of Cuntz–Krieger algebras

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Abstract. In this paper, we first present the homotopy groups of the automorphism groups of Cuntz–Krieger algebras in terms of the underlying matrices defining the Cuntz–Krieger algebras. We also show that the homotopy groups are complete invariants of the isomorphism classes of Cuntz–Krieger algebras. As a result, the isomorphism class of a Cuntz–Krieger algebra is completely determined by the group structure of its weak extension group and strong extension group.

1. Introduction

The study of homotopy groups of endomorphisms of Cuntz–Krieger algebras was initiated by J. Cuntz in [8]. He showed that the homotopy group $\pi_n(\mathrm{End}(\mathcal{O}_A \otimes K(H)))$ of the endomorphisms of the stabilized Cuntz–Krieger algebra $\mathcal{O}_A \otimes K(H)$ is isomorphic to the nth bi-variant weak extension group $\mathrm{Ext}^n_w(\mathcal{O}_A \otimes K(H), \mathcal{O}_A \otimes K(H))$. By using the KK-theoretic machinery, M. Dadarlat in [10] investigated the homotopy groups of endomorphisms and automorphisms of Kirchberg algebras, and showed that the nth homotopy group $\pi_n(\mathrm{Aut}(\mathcal{A}))$ of the automorphism group $\mathrm{Aut}(\mathcal{A})$ of a unital Kirchberg algebra $\mathcal{A}$ is isomorphic to the KK-group $\mathrm{KK}^{n+1}(C_{\mathcal{A}}, \mathcal{A})$, where $C_{\mathcal{A}}$ is the mapping cone of the unital embedding $u_{\mathcal{A}} : \mathbb{C} \to \mathcal{A}$. Related to the classification of bundles of C^*-algebras, Izumi–Sogabe [16] and Sogabe [29] studied the homotopy groups of automorphisms of Cuntz algebras and Cuntz–Toeplitz algebras (see also [11]).

In this paper, we will first present the groups $\pi_i(\mathrm{Aut}(\mathcal{O}_A))$, $i = 1, 2$ for the Cuntz–Krieger algebra $\mathcal{O}_A$ in terms of the four abelian groups $\mathrm{K}_i(\mathcal{O}_A)$, $\mathrm{Ext}^i_s(\mathcal{O}_A)$, $i = 0, 1$ by using the above mentioned Dadarlat's general formulas in [10, Corollary 5.10] for the homotopy groups of the automorphisms of Kirchberg algebras. The former $\mathrm{K}_i(\mathcal{O}_A)$ are the K-groups of $\mathcal{O}_A$ and the latter $\mathrm{Ext}^i_s(\mathcal{O}_A)$ are the strong extension groups of $\mathcal{O}_A$, where the first strong extension group Ext^1_s is the usual strong extension group Ext_s as in [21,23,25]. Since the groups $\mathrm{K}_i(\mathcal{O}_A)$, $\mathrm{Ext}^i_s(\mathcal{O}_A)$, $i = 0, 1$ are written in terms of the matrix A ([6,21]), one may describe the homotopy groups $\pi_i(\mathrm{Aut}(\mathcal{O}_A))$, $i = 1, 2$ by using the matrix A (Corollary 3.4).

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In [30], the second named author introduced the notion of reciprocality for two unital Kirchberg algebras with finitely generated K-groups. Two unital Kirchberg algebras $\mathcal{A}, \mathcal{B}$ are said to be reciprocal if both of the conditions $\mathcal{A} \underset{\mathrm{KK}}{\sim} D(C_{\mathcal{B}})$ and $\mathcal{B} \underset{\mathrm{KK}}{\sim} D(C_{\mathcal{A}})$ hold, where $\underset{\mathrm{KK}}{\sim}$ means KK-equivalence, and the C^*-algebra $D(C_{\mathcal{A}})$ is the Spanier–Whitehead K-dual of $C_{\mathcal{A}}$ defined by Kaminker–Schochet [18] (cf. [17]). The Spanier–Whitehead K-duality is a noncommutative analogue of the classical Spanier–Whitehead duality for finite CW complexes. Related to the homotopy groups of the automorphism groups of unital Kirchberg algebras, the second named author proved in [30] that $\pi_i(\mathrm{Aut}(\mathcal{A}))$ is isomorphic to $\pi_i(\mathrm{Aut}(\mathcal{B}))$ for $i = 1, 2$ if and only if $\mathcal{A}$ is isomorphic to $\mathcal{B}$, or $\mathcal{A}$ and $\mathcal{B}$ are reciprocal. In the present paper, we will show that any pair $\mathcal{O}_A$ and $\mathcal{O}_B$ of two Cuntz–Krieger algebras can never be reciprocal by computing the K-groups of $D(C_{\mathcal{O}_A})$ and $D(C_{\mathcal{O}_B})$ (Lemma 4.5). Hence we obtain that the groups $\pi_i(\mathrm{Aut}(\mathcal{O}_A))$ and $\pi_i(\mathrm{Aut}(\mathcal{O}_B))$ for $i = 1, 2$ are isomorphic if and only if $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$. Since we know that the group structure of the two groups $\mathrm{K}_0(\mathcal{O}_A), \mathrm{Ext}^1_s(\mathcal{O}_A)$ determine the other two groups $\mathrm{K}_1(\mathcal{O}_A), \mathrm{Ext}^0_s(\mathcal{O}_A)$ (Lemma 4.7), the homotopy groups $\pi_i(\mathrm{Aut}(\mathcal{A}))$, $i = 1, 2$ are determined by the only two groups $\mathrm{K}_0(\mathcal{O}_A), \mathrm{Ext}^1_s(\mathcal{O}_A)$. For an $N \times N$ matrix A with entries in $\{0, 1\}$, let us denote by $\widehat{A}$ the $N \times N$ matrix $\widehat{A} = A + R_1 - AR_1$, where R_1 is the $N \times N$ matrix such that its first row is the vector $[1, \dots, 1]$ whose entries are all 1s and the other rows are zero vectors. As a result, we will prove the following theorem in the present paper.

Theorem 1.1 (Theorem 4.8). *Let $A = [A(i,j)]_{i,j=1}^N$, $B = [B(i,j)]_{i,j=1}^M$ be irreducible non-permutation matrices with entries in $\{0, 1\}$. Then the following four conditions are mutually equivalent:*

(i) $\pi_i(\mathrm{Aut}(\mathcal{O}_A)) \cong \pi_i(\mathrm{Aut}(\mathcal{O}_B))$, $i = 1, 2$.

(ii) $\mathcal{O}_A \cong \mathcal{O}_B$.

(iii) $\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_B)$ *and* $\mathrm{Ext}_s(\mathcal{O}_A) \cong \mathrm{Ext}_s(\mathcal{O}_B)$.

(iv) $\mathbb{Z}^N/(I - A)\mathbb{Z}^N \cong \mathbb{Z}^M/(I - B)\mathbb{Z}^M$ *and* $\mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N \cong \mathbb{Z}^M/(I - \widehat{B})\mathbb{Z}^M$.

Since $\mathrm{K}_0(\mathcal{O}_A)$ is isomorphic to the weak extension group $\mathrm{Ext}_w(\mathcal{O}_A)$, condition (iii) is replaced by

$$\mathrm{Ext}_w(\mathcal{O}_A) \cong \mathrm{Ext}_w(\mathcal{O}_B) \quad \text{and} \quad \mathrm{Ext}_s(\mathcal{O}_A) \cong \mathrm{Ext}_s(\mathcal{O}_B).$$

By the equivalence between (i) and (ii), the homotopy groups of $\mathrm{Aut}(\mathcal{O}_A)$ completely determine the isomorphism class of $\mathcal{O}_A$. The equivalence between (ii) and (iii) together with $\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{Ext}_w(\mathcal{O}_A)$ tells us that the isomorphism class of $\mathcal{O}_A$ is determined by the group structure of the two extension groups $\mathrm{Ext}_w(\mathcal{O}_A)$ and $\mathrm{Ext}_s(\mathcal{O}_A)$, which are computed as the abelian groups $\mathbb{Z}^N/(I - A)\mathbb{Z}^N$ and $\mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N$ that are the cokernels in $\mathbb{Z}^N$ of the matrices $I - A$ and $I - \widehat{A}$, respectively ([9,21]). Thus, our theorem implies that the pair $(\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A))$ of the two extension groups has exactly the same information as the pair $(\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0)$ (cf. [26]). The possible range of the pair $(\mathrm{Ext}_w(\mathcal{O}_A),$

$\mathrm{Ext}_s(\mathcal{O}_A))$ is clarified in Corollary 5.2. The relationship between $(\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A))$ and $(\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0)$ is discussed in Proposition 5.4.

Concerning the homotopy groups of the automorphism group of the stabilized Cuntz–Krieger algebra $\mathcal{O}_A \otimes K(H)$, we will show that

$$\pi_i(\mathrm{Aut}(\mathcal{O}_A \otimes K(H))) \cong \pi_i(\mathrm{Aut}(\mathcal{O}_B \otimes K(H))), \quad i = 1, 2$$

if and only if $\mathcal{O}_A \otimes K(H) \cong \mathcal{O}_B \otimes K(H)$ (Proposition 4.13), by using Dadarlat's formulation in [10, Corollary 5.11].

2. Preliminary

Throughout the paper, we mean by a Kirchberg algebra a separable unital nuclear simple purely infinite C^*-algebra. We always assume that a Kirchberg algebra satisfies the Universal Coefficient Theorem (UCT).

2.1. Extension groups as KK-groups

Let $\mathcal{A}$ be a unital Kirchberg algebra. Let $u_{\mathcal{A}} : \mathbb{C} \to \mathcal{A}$ be the unital embedding defined by $u_{\mathcal{A}}(c) = c1_{\mathcal{A}}$ for $c \in \mathbb{C}$, where $1_{\mathcal{A}}$ denotes the unit of $\mathcal{A}$. The mapping cone $C_{\mathcal{A}}$ for the map $u_{\mathcal{A}} : \mathbb{C} \to \mathcal{A}$ is defined by the C^*-algebra

$$C_{\mathcal{A}} := \{ f \in C_0(0,1] \otimes \mathcal{A} \mid f(1) \in \mathbb{C}1_{\mathcal{A}} \}.$$

The suspension $S\mathcal{A}$ is the C^*-algebra $C_0(0,1) \otimes \mathcal{A}$ which is naturally embedded into $C_{\mathcal{A}}$. We then have a short exact sequence

$$0 \to S\mathcal{A} \to C_{\mathcal{A}} \to \mathbb{C} \to 0 \tag{2.1}$$

in a natural way. For separable unital nuclear C^*-algebras $\mathcal{A}, \mathcal{B}$, we write

$$\mathrm{Ext}_s^n(\mathcal{A}, \mathcal{B}) = \mathrm{KK}^{1-n}(C_{\mathcal{A}}, \mathcal{B}), \quad \mathrm{Ext}_w^n(\mathcal{A}, \mathcal{B}) = \mathrm{KK}^n(\mathcal{A}, \mathcal{B}), \quad n = 0, 1,$$

where $\mathrm{KK}^n(\ ,\ )$ means the Kasparov KK-group of degree n ([19], cf. [2]). For separable UCT C^*-algebras $\mathcal{A}, \mathcal{B}$, one can describe the KK-groups in terms of the K-groups by the following short exact sequence, called UCT:

$$0 \to \bigoplus_{i=0,1} \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_i(\mathcal{A}), \mathrm{K}_{i+1}(\mathcal{B})) \to \mathrm{KK}(\mathcal{A}, \mathcal{B}) \to \bigoplus_{i=0,1} \mathrm{Hom}(\mathrm{K}_i(\mathcal{A}), \mathrm{K}_i(\mathcal{B})) \to 0$$

which splits unnaturally (see [2, 3, 27]). We in particular write

$$\mathrm{Ext}_s^n(\mathcal{A}) = \mathrm{Ext}_s^n(\mathcal{A}, \mathbb{C}), \quad \mathrm{Ext}_w^n(\mathcal{A}) = \mathrm{Ext}_w^n(\mathcal{A}, \mathbb{C}), \quad n = 0, 1.$$

The K-homology groups $\mathrm{K}^n(\mathcal{A})$, $n = 0, 1$ are defined by $\mathrm{KK}^n(\mathcal{A}, \mathbb{C})$, so that

$$\mathrm{Ext}_s^n(\mathcal{A}) = \mathrm{K}^{1-n}(C_{\mathcal{A}}), \quad \mathrm{Ext}_w^n(\mathcal{A}) = \mathrm{K}^n(\mathcal{A}), \quad n = 0, 1$$

(see [2, 15, 28] for detail accounts on the relation of Ext-groups to KK-theory). The extension groups $\mathrm{Ext}_s^n(\mathcal{A})$, $\mathrm{Ext}_w^n(\mathcal{A})$ were primary defined and studied to investigate extensions of C^*-algebras motivated by the classification of essentially normal operators on Hilbert spaces (cf. [4, 12], etc.). Let $B(H)$ denote the C^*-algebra of bounded linear operators on a separable infinite dimensional Hilbert space H. The quotient C^*-algebra $Q(H)$ of $B(H)$ by the C^*-subalgebra $K(H)$ of compact operators on H is called the Calkin algebra. For a separable unital nuclear C^*-algebra $\mathcal{A}$, a $*$-homomorphism $\tau : \mathcal{A} \to Q(H)$ is called an extension of $\mathcal{A}$. The extension is said to be unital (resp. essential) if τ is unital (resp. injective). Two extensions $\tau_1, \tau_2 : \mathcal{A} \to Q(H)$ are said to be strongly (resp. weakly) equivalent if there exists a unitary $U \in B(H)$ (resp. $u \in Q(H)$) such that $\tau_2(a) = \pi(U)\tau_1(a)\pi(U)^*$, $a \in \mathcal{A}$ (resp. $\tau_2(a) = u\tau_1(a)u^*$, $a \in \mathcal{A}$), where $\pi : B(H) \to Q(H)$ denotes the natural quotient map. Let us denote by $\mathrm{Ext}_s(\mathcal{A})$ (resp. $\mathrm{Ext}_w(\mathcal{A})$) the set of strong (resp. weak) equivalence classes of unital essential extensions of $\mathcal{A}$. It is well known that both $\mathrm{Ext}_s(\mathcal{A})$ and $\mathrm{Ext}_w(\mathcal{A})$ become abelian groups whose addition is defined by a direct sum of extensions (cf. [4, 12, 15], etc.). It is also well known that $\mathrm{Ext}_s(\mathcal{A})$ and $\mathrm{Ext}_w(\mathcal{A})$ are isomorphic to $\mathrm{Ext}^1_s(\mathcal{A})$ and to $\mathrm{Ext}^1_w(\mathcal{A})$ as abelian groups, respectively (cf. [28, Corollary 2.4], [24, Theorem 4.5]). By a general theory of KK-theory, the short exact sequence (2.1) of C^*-algebras yields the following cyclic six term exact sequence (see [21, 28]):

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Ext}^0_s(\mathcal{A}) & \longrightarrow & \mathrm{Ext}^0_w(\mathcal{A}) \\ \uparrow & & & & \downarrow \\ \mathrm{Ext}^1_w(\mathcal{A}) & \longleftarrow & \mathrm{Ext}^1_s(\mathcal{A}) & \longleftarrow & \mathbb{Z}. \end{array} \tag{2.2}$$

2.2. The cyclic exact sequences for Cuntz–Krieger algebras

In what follows, let A be an $N \times N$ irreducible non-permutation matrix $[A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$. The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix A is defined to be the universal C^*-algebra generated by N-partial isometries $S_1, \dots, S_N$ subject to the operator relations

$$S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1, \dots, N, \qquad \sum_{j=1}^N S_j S_j^* = 1.$$

Let $\hat{\iota}_A : \mathbb{Z} \to \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N$ be the homomorphism of abelian groups defined by $\hat{\iota}_A(m) = [(I - A)[k_i]_{i=1}^N]$ where $\sum_{i=1}^N k_i = m$ with $k_i \in \mathbb{Z}$ (see [21]). Since $I - \hat{A} = (I - A)(I - R_1)$, the map $\hat{\iota}_A$ is well defined. Let us denote by $\mathrm{Ker}(I - A)$, $\mathrm{Ker}(I - \hat{A})$ the subgroups of $\mathbb{Z}^N$ defined by the kernels of the homomorphisms $I - A : \mathbb{Z}^N \to \mathbb{Z}^N$, $I - \hat{A} : \mathbb{Z}^N \to \mathbb{Z}^N$, respectively. In [20], the cyclic six term exact sequence (2.2) for $\mathcal{A} = \mathcal{O}_A$ is computed as

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Ker}(I - \hat{A})/i_1(\mathbb{Z}) & \xrightarrow{j_A} & \mathrm{Ker}(I - A) \\ \uparrow & & & & \downarrow s_A \\ \mathbb{Z}^N/(I - A)\mathbb{Z}^N & \xleftarrow{\hat{q}_A} & \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N & \xleftarrow{\hat{\iota}_A} & \mathbb{Z}, \end{array} \tag{2.3}$$

where the homomorphisms $i_1 : \mathbb{Z} \to \mathrm{Ker}(I - \widehat{A})$, $s_A : \mathrm{Ker}(I - A) \to \mathbb{Z}$, $j_A : \mathrm{Ker}(I - \widehat{A})/i_1(\mathbb{Z}) \to \mathrm{Ker}(I - A)$ are defined by

$$i_1(m) := \begin{bmatrix} m \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad s_A([l_i]_{i=1}^N) := \sum_{i=1}^N l_i,$$

$$j_A([l_i]_{i=1}^N + i_1(\mathbb{Z})) := (I - R_1)([l_i]_{i=1}^N) = \begin{bmatrix} -\sum_{i=2}^N l_i \\ l_2 \\ \vdots \\ l_N \end{bmatrix},$$

respectively, and $\widehat{q}_A : \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N \to \mathbb{Z}^N/(I - A)\mathbb{Z}^N$ denotes the natural quotient map.

Let us consider the Toeplitz algebra $\mathcal{T}_A$ for the matrix A, which is defined by the universal C^*-algebra generated by N-partial isometries $T_1, \ldots, T_N$ and one non-zero projection P_0 subject to the operator relations (see [13, 14])

$$T_i^* T_i = \sum_{j=1}^N A(i,j) T_j T_j^* + P_0, \quad i = 1, \ldots, N, \qquad \sum_{j=1}^N T_j T_j^* + P_0 = 1.$$

The correspondence $T_i \to S_i$, $i = 1, \ldots, N$ yields a short exact sequence

$$0 \to K(H) \to \mathcal{T}_A \to \mathcal{O}_A \to 0 \tag{2.4}$$

called the Toeplitz extension of $\mathcal{O}_A$. The short exact sequence (2.4) for the transposed matrix A^t of A yields the cyclic six term exact sequence of K-groups

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{K}_1(\mathcal{T}_{A^t}) & \longrightarrow & \mathrm{K}_1(\mathcal{O}_{A^t}) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_0(\mathcal{O}_{A^t}) & \longleftarrow & \mathrm{K}_0(\mathcal{T}_{A^t}) & \longleftarrow & \mathbb{Z} \end{array} \tag{2.5}$$

which is nothing but the cyclic six term exact sequence (2.3) ([20, Proposition 4.6]). We summarize the extension groups and the K-groups of Cuntz–Krieger algebras in the following way.

Lemma 2.1 ([6, 7, 9, 20, 21]). *Let A be an $N \times N$ irreducible non-permutation matrix with entries in* $\{0, 1\}$. *Then we have*

$$\begin{aligned}
\mathrm{Ext}_w^1(\mathcal{O}_A) &= \mathrm{Ext}_w(\mathcal{O}_A) = \mathbb{Z}^N/(I - A)\mathbb{Z}^N = \mathrm{K}_0(\mathcal{O}_{A^t}) \cong \mathrm{K}_0(\mathcal{O}_A), \\
\mathrm{Ext}_s^1(\mathcal{O}_A) &= \mathrm{Ext}_s(\mathcal{O}_A) = \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N = \mathrm{K}_0(\mathcal{T}_{A^t}), \\
\mathrm{Ext}_w^0(\mathcal{O}_A) &= \mathrm{Ker}(I - A) = \mathrm{K}_1(\mathcal{O}_{A^t}) \cong \mathrm{K}_1(\mathcal{O}_A), \\
\mathrm{Ext}_s^0(\mathcal{O}_A) &= \mathrm{Ker}(I - \widehat{A})/i_1(\mathbb{Z}) = \mathrm{Ker}(s_A : \mathrm{Ker}(I - A) \to \mathbb{Z}) = \mathrm{K}_1(\mathcal{T}_{A^t}).
\end{aligned}$$

We note that the Smith normal forms for the $\mathbb{Z}$-module maps $I - A$, $I - A^t$ show that $\mathbb{Z}^N/(I - A)\mathbb{Z}^N \cong \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N = K_0(\mathcal{O}_A)$ and $\mathrm{Ker}(I - A) \cong \mathrm{Ker}(I - A^t) = K_1(\mathcal{O}_A)$.

3. Formulas of the homotopy groups

Let us denote by $\mathrm{Aut}(\mathcal{A})$ the group of automorphisms of a C^*-algebra $\mathcal{A}$. It has a topology defined by pointwise norm convergence. The nth homotopy group of $\mathrm{Aut}(\mathcal{A})$ is denoted by $\pi_n(\mathrm{Aut}(\mathcal{A}))$, $n = 1, 2, \ldots$ The following theorem is due to M. Dadarlat [10].

Theorem 3.1 (Dadarlat [10, Corollary 5.10 and Corollary 5.11]). *For a unital Kirchberg algebra $\mathcal{A}$ and $n = 1, 2, \ldots,$ we have the following formulas:*

$$\pi_n(\mathrm{Aut}(\mathcal{A})) \cong \mathrm{KK}^{n+1}(C_{\mathcal{A}}, \mathcal{A}) = \mathrm{Ext}_s^n(\mathcal{A}, \mathcal{A}),$$
$$\pi_n(\mathrm{Aut}(\mathcal{A} \otimes K(H))) \cong \mathrm{KK}^n(\mathcal{A}, \mathcal{A}) = \mathrm{Ext}_w^n(\mathcal{A}, \mathcal{A}).$$

3.1. Formulas of $\pi_i(\mathrm{Aut}(\mathcal{O}_A))$

Lemma 3.2. *Let $\mathcal{A}$ be a separable unital nuclear C^*-algebra with finitely generated* K-*groups. We have short exact sequences of abelian groups,*

$$\begin{aligned} 0 &\to (\mathrm{Ext}_s^1(\mathcal{A}) \otimes \mathrm{K}_0(\mathcal{A})) \oplus (\mathrm{Ext}_s^0(\mathcal{A}) \otimes \mathrm{K}_1(\mathcal{A})) \\ &\to \pi_1(\mathrm{Aut}(\mathcal{A})) \\ &\to \mathrm{Tor}(\mathrm{Ext}_s^1(\mathcal{A}), \mathrm{K}_1(\mathcal{A})) \oplus \mathrm{Tor}(\mathrm{Ext}_s^0(\mathcal{A}), \mathrm{K}_0(\mathcal{A})) \to 0, \end{aligned}$$

and

$$\begin{aligned} 0 &\to (\mathrm{Ext}_s^1(\mathcal{A}) \otimes \mathrm{K}_1(\mathcal{A})) \oplus (\mathrm{Ext}_s^0(\mathcal{A}) \otimes \mathrm{K}_0(\mathcal{A})) \\ &\to \pi_2(\mathrm{Aut}(\mathcal{A})) \\ &\to \mathrm{Tor}(\mathrm{Ext}_s^1(\mathcal{A}), \mathrm{K}_0(\mathcal{A})) \oplus \mathrm{Tor}(\mathrm{Ext}_s^0(\mathcal{A}), \mathrm{K}_1(\mathcal{A})) \to 0, \end{aligned}$$

both of which split unnaturally.

Proof. Since $\pi_1(\mathrm{Aut}(\mathcal{A})) \cong \mathrm{KK}(C_{\mathcal{A}}, \mathcal{A})$, the Künneth theorem (see [2, Theorem 23.1.2], [27]) tells us that there exists a short exact sequence

$$\begin{aligned} 0 &\to (\mathrm{K}^0(C_{\mathcal{A}}) \otimes \mathrm{K}_0(\mathcal{A})) \oplus (\mathrm{K}^1(C_{\mathcal{A}}) \otimes \mathrm{K}_1(\mathcal{A})) \\ &\to \pi_1(\mathrm{Aut}(\mathcal{A})) \\ &\to \mathrm{Tor}(\mathrm{K}^0(C_{\mathcal{A}}), \mathrm{K}_1(\mathcal{A})) \oplus \mathrm{Tor}(\mathrm{K}^1(C_{\mathcal{A}}), \mathrm{K}_0(\mathcal{A})) \to 0. \end{aligned}$$

As $\mathrm{Ext}_s^i(\mathcal{A}) = \mathrm{K}^{1-i}(C_{\mathcal{A}})$, $i = 0, 1$, we have the desired short exact sequence.

The exact sequence for $\pi_2(\mathrm{Aut}(\mathcal{A}))$ is proved similarly. ∎

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible non-permutation $N \times N$ matrix with entries in $\{0,1\}$. By applying Lemma 3.2 to the Cuntz–Krieger algebra $\mathcal{O}_A$, we have the following formulas.

Proposition 3.3. *For the simple Cuntz–Krieger algebra $\mathcal{O}_A$, the following formulas hold:*

$$\begin{aligned}\pi_1(\mathrm{Aut}(\mathcal{O}_A)) &\cong (\mathrm{Ext}_s^1(\mathcal{O}_A) \otimes \mathrm{K}_0(\mathcal{O}_A)) \oplus (\mathrm{Ext}_s^0(\mathcal{O}_A) \otimes \mathrm{K}_1(\mathcal{O}_A)),\\ \pi_2(\mathrm{Aut}(\mathcal{O}_A)) &\cong (\mathrm{Ext}_s^1(\mathcal{O}_A) \otimes \mathrm{K}_1(\mathcal{O}_A)) \oplus (\mathrm{Ext}_s^0(\mathcal{O}_A) \otimes \mathrm{K}_0(\mathcal{O}_A))\\ &\qquad \oplus \mathrm{Tor}(\mathrm{Ext}_s^1(\mathcal{O}_A), \mathrm{K}_0(\mathcal{O}_A)).\end{aligned}$$

Proof. Since $\mathrm{K}_1(\mathcal{O}_A) = \mathrm{Ker}(I - A^t)$ and $\mathrm{Ext}_s^0(\mathcal{O}_A) = \mathrm{Ker}(s_A : \mathrm{Ker}(I - A) \to \mathbb{Z})$ are both torsion free, one has

$$\mathrm{Tor}(\mathrm{Ext}_s^1(\mathcal{O}_A), \mathrm{K}_1(\mathcal{O}_A)) = \mathrm{Tor}(\mathrm{Ext}_s^0(\mathcal{O}_A), \mathrm{K}_0(\mathcal{O}_A)) = 0,$$

and this shows the desired formula for $\pi_1(\mathrm{Aut}(\mathcal{O}_A))$ by Lemma 3.2. The desired formula for $\pi_2(\mathrm{Aut}(\mathcal{O}_A))$ is proved similarly. ∎

For an $N \times N$ matrix A, let us define an $(N+1) \times N$ matrix A_T by

$$A_T = \begin{bmatrix} 1 & \cdots & 1 \\ & I - A & \end{bmatrix}.$$

It is direct to see that $\mathrm{Ker}(A_T : \mathbb{Z}^N \to \mathbb{Z}^{N+1}) = \mathrm{Ker}(s_A : \mathrm{Ker}(I - A) \to \mathbb{Z})$ (cf. [20, Lemma 4.2]).

Corollary 3.4. *Let A be an $N \times N$ irreducible non-permutation matrix with entries in $\{0,1\}$. Then we have*

$$\begin{aligned}\pi_1(\mathrm{Aut}(\mathcal{O}_A)) &\cong (\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \otimes \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N) \oplus (\mathrm{Ker}(A_T : \mathbb{Z}^N \to \mathbb{Z}^{N+1})\\ &\qquad \otimes \mathrm{Ker}(I - A^t)),\\ \pi_2(\mathrm{Aut}(\mathcal{O}_A)) &\cong (\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \otimes \mathrm{Ker}(I - A^t)) \oplus (\mathrm{Ker}(A_T : \mathbb{Z}^N \to \mathbb{Z}^{N+1})\\ &\qquad \otimes \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N) \oplus \mathrm{Tor}(\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N, \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N).\end{aligned}$$

Remark 3.5. We note that direct computation yields

$$\pi_1(\mathrm{Aut}(\mathcal{O}_A)) \cong \pi_2(\mathrm{Aut}(\mathcal{O}_A)) \oplus \mathrm{Tor}(\mathbb{Z}^N/(I - A)\mathbb{Z}^N),$$

where $\mathrm{Tor}(-)$ denotes the torsion part (see examples in Section 6).

3.2. Formulas of $\pi_i(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$

Combining Dadarlat's result

$$\pi_n(\mathrm{Aut}(\mathcal{A} \otimes K(H))) \cong \mathrm{KK}^n(\mathcal{A}, \mathcal{A}), \quad n = 1, 2, \ldots$$

with UCT and Lemma 2.1, we know the following formulas for the homotopy groups of the automorphism groups of stabilized Cuntz–Krieger algebras.

Proposition 3.6. *For a simple Cuntz–Krieger algebra $\mathcal{O}_A$, we have*

$$\begin{aligned}&\pi_1(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))\\ &\qquad\cong (\mathrm{Ext}^1_w(\mathcal{O}_A) \otimes \mathrm{K}_0(\mathcal{O}_A)) \oplus (\mathrm{Ext}^0_w(\mathcal{O}_A) \otimes \mathrm{K}_1(\mathcal{O}_A))\\ &\qquad= (\mathbb{Z}^N/(I-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(I-A^t)\mathbb{Z}^N) \oplus (\mathrm{Ker}(I-A) \otimes \mathrm{Ker}(I-A^t)),\end{aligned}$$

and

$$\begin{aligned}&\pi_2(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))\\ &\qquad\cong (\mathrm{Ext}^1_w(\mathcal{O}_A) \otimes \mathrm{K}_1(\mathcal{O}_A)) \oplus (\mathrm{Ext}^0_w(\mathcal{O}_A) \otimes \mathrm{K}_0(\mathcal{O}_A))\\ &\qquad\quad \oplus \mathrm{Tor}(\mathrm{Ext}^1_w(\mathcal{O}_A), \mathrm{K}_0(\mathcal{O}_A))\\ &\qquad= (\mathbb{Z}^N/(I-A)\mathbb{Z}^N \otimes \mathrm{Ker}(I-A^t)) \oplus (\mathrm{Ker}(I-A) \otimes \mathbb{Z}^N/(I-A^t)\mathbb{Z}^N)\\ &\qquad\quad \oplus \mathrm{Tor}(\mathbb{Z}^N/(I-A)\mathbb{Z}^N, \mathbb{Z}^N/(I-A^t)\mathbb{Z}^N).\end{aligned}$$

4. The homotopy groups $\pi_i(\mathrm{Aut}(\mathcal{O}_A))$ and the reciprocality

4.1. The reciprocality in Spanier–Whitehead K-duals

In [30], the second named author introduced the notion of reciprocality for a pair of unital Kirchberg algebras $\mathcal{A}, \mathcal{B}$ with finitely generated K-groups to investigate the homotopy groups of the automorphism groups of Kirchberg algebras and continuous fields of C^*-algebras, so called bundles of C^*-algebras. Suppose that $\mathcal{C}$ is a separable nuclear UCT C^*-algebra with finitely generated K-groups. Kaminker–Schochet in [18] showed that there exists another separable nuclear UCT C^*-algebra written $D(\mathcal{C})$ unique up to KK-equivalence satisfying the duality

$$\mathrm{KK}^i(\mathcal{C}, \mathbb{C}) \cong \mathrm{KK}^i(\mathbb{C}, D(\mathcal{C})), \quad i = 0, 1.$$

The C^*-algebra $D(\mathcal{C})$ is called the Spanier–Whitehead K-dual of $\mathcal{C}$ (see [17, 18], etc.). By Theorem 3.1 due to M. Dadarlat, we see that for a Kirchberg algebra $\mathcal{A}$ with finitely generated K-groups, the mapping cone $C_{\mathcal{A}}$ has its dual $D(C_{\mathcal{A}})$ such that

$$\pi_i(\mathrm{Aut}(\mathcal{A})) \cong \mathrm{K}_{i+1}(D(C_{\mathcal{A}}) \otimes \mathcal{A}), \quad i = 1, 2, \dots \tag{4.1}$$

Suppose $\pi_i(\mathrm{Aut}(\mathcal{A})) \cong \pi_i(\mathrm{Aut}(\mathcal{B}))$, $i = 1, 2, \dots$ By (4.1), we have $D(C_{\mathcal{A}}) \otimes \mathcal{A} \underset{\mathrm{KK}}{\sim} D(C_{\mathcal{B}}) \otimes \mathcal{B}$. The reciprocality for a pair $\mathcal{A}$ and $\mathcal{B}$ of unital Kirchberg algebras was introduced in [30] in the following way.

Definition 4.1 (Sogabe [30]). Let $\mathcal{A}, \mathcal{B}$ be unital UCT Kirchberg algebras such that both of them have finitely generated K-groups. Then $\mathcal{A}$ and $\mathcal{B}$ are said to be reciprocal if both of the conditions $\mathcal{A} \underset{\mathrm{KK}}{\sim} D(C_{\mathcal{B}})$ and $\mathcal{B} \underset{\mathrm{KK}}{\sim} D(C_{\mathcal{A}})$ hold.

The second named author proved the following theorem in [30].

Theorem 4.2 ([30, Theorem 1.2]). *The homotopy groups* $\pi_i(\mathrm{Aut}(\mathcal{A}))$ *and* $\pi_i(\mathrm{Aut}(\mathcal{B}))$ *are isomorphic for all* $i = 1, 2, \dots$ *if and only if either* $A \cong \mathcal{B}$ *or* $\mathcal{A}$ *and* $\mathcal{B}$ *are reciprocal.*

Let us provide a couple of lemmas to study the reciprocality in Cuntz–Krieger algebras.

Lemma 4.3. *For an irreducible non-permutation matrix* A *with entries in* $\{0, 1\}$*, we have* $D(C_{\mathcal{O}_A}) \underset{\mathrm{KK}}{\sim} \mathcal{T}_{A^t}$*, that is, the Spanier–Whitehead K-dual of the mapping cone* $C_{\mathcal{O}_A}$ *of* $\mathcal{O}_A$ *is* KK*-equivalent to the Toeplitz algebra* $\mathcal{T}_{A^t}$ *defined by the transposed matrix* A^t *of* A.

Proof. Since $\mathcal{O}_A, \mathcal{O}_{A^t}$ satisfy UCT, $C_{\mathcal{O}_A}$ and $\mathcal{T}_{A^t}$ also satisfy UCT. We then have

$$\mathrm{KK}(C_{\mathcal{O}_A}, \mathbb{C}) = \mathrm{Ext}^1_s(\mathcal{O}_A) = \mathrm{Ext}_s(\mathcal{O}_A) = \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N = \mathrm{K}_0(\mathcal{T}_{A^t}) = \mathrm{KK}(\mathbb{C}, \mathcal{T}_{A^t}),$$
$$\mathrm{KK}^1(C_{\mathcal{O}_A}, \mathbb{C}) = \mathrm{Ext}^0_s(\mathcal{O}_A) = \mathrm{Ker}(s_A : \mathrm{Ker}(I - A) \to \mathbb{Z}) = \mathrm{K}_1(\mathcal{T}_{A^t}) = \mathrm{KK}^1(\mathbb{C}, \mathcal{T}_{A^t}),$$

and hence $\mathcal{T}_{A^t}$ is the Spanier–Whitehead K-dual of $C_{\mathcal{O}_A}$. ∎

For a finitely generated abelian group G, we write the rank of the torsion free part of G as $\mathrm{rank}(G)$, which is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} G$.

Lemma 4.4. *For an irreducible matrix* $A = [A(i, j)]_{i,j=1}^N$ *with entries in* $\{0, 1\}$*, we have*

$$\mathrm{rank}(\mathrm{K}_0(\mathcal{T}_{A^t})) = \mathrm{rank}(\mathrm{K}_1(\mathcal{T}_{A^t})) + 1.$$

Proof. Applying the exact functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$ to the sequence (2.5), one has the following exact sequence of $\mathbb{Q}$-vector spaces

$$0 \to \mathbb{Q}^{\mathrm{rank}(\mathrm{K}_1(\mathcal{T}_{A^t}))} \to \mathbb{Q}^{\mathrm{rank}(\mathrm{K}_1(\mathcal{O}_{A^t}))} \to \mathbb{Q} \to \mathbb{Q}^{\mathrm{rank}(\mathrm{K}_0(\mathcal{T}_{A^t}))} \to \mathbb{Q}^{\mathrm{rank}(\mathrm{K}_0(\mathcal{O}_{A^t}))} \to 0,$$

which implies the equality

$$\mathrm{rank}(\mathrm{K}_1(\mathcal{T}_{A^t})) - \mathrm{rank}(\mathrm{K}_1(\mathcal{O}_{A^t})) + 1 - \mathrm{rank}(\mathrm{K}_0(\mathcal{T}_{A^t})) + \mathrm{rank}(\mathrm{K}_0(\mathcal{O}_{A^t})) = 0.$$

The desired equality follows from the equalities

$$\mathrm{rank}(\mathrm{K}_1(\mathcal{O}_{A^t})) = \mathrm{rank}(\mathrm{Ker}(I - A)) = \mathrm{rank}(\mathbb{Z}^N/(I - A)\mathbb{Z}^N) = \mathrm{rank}(\mathrm{K}_0(\mathcal{O}_{A^t})). \quad ∎$$

Lemma 4.5. *For any pair of irreducible non-permutation matrices* A, B *with entries in* $\{0, 1\}$*,* $\mathcal{O}_A$ *is not reciprocal to* $\mathcal{O}_B$.

Proof. Suppose that $\mathcal{O}_A$ and $\mathcal{O}_B$ are reciprocal. Since Lemma 4.3 shows $\mathcal{O}_A \underset{\mathrm{KK}}{\sim} \mathcal{T}_{B^t}$, one has $\mathrm{K}_*(\mathcal{O}_A) \cong \mathrm{K}_*(\mathcal{T}_{B^t})$ and hence

$$\mathrm{rank}(\mathrm{K}_0(\mathcal{T}_{B^t})) = \mathrm{rank}(\mathrm{K}_0(\mathcal{O}_A)) = \mathrm{rank}(\mathrm{K}_1(\mathcal{O}_A)) = \mathrm{rank}(\mathrm{K}_1(\mathcal{T}_{B^t})).$$

This contradicts Lemma 4.4. ∎

Theorem 4.6. *The homotopy groups* $\pi_i(\mathrm{Aut}(\mathcal{O}_A))$ *and* $\pi_i(\mathrm{Aut}(\mathcal{O}_B))$ *are isomorphic for* $i = 1, 2$ *if and only if* $\mathcal{O}_A$ *is isomorphic to* $\mathcal{O}_B$.

Proof. The assertion follows from Theorem 4.2 together with Lemma 4.5. ∎

4.2. Main theorem

Lemma 4.7. *Let* $A = [A(i,j)]_{i,j=1}^N$, $B = [B(i,j)]_{i,j=1}^M$ *be irreducible non-permutation matrices with entries in* $\{0,1\}$. *Suppose that* $\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_B)$ *and* $\mathrm{Ext}_s^1(\mathcal{O}_A) \cong \mathrm{Ext}_s^1(\mathcal{O}_B)$. *Then we have* $\mathrm{K}_1(\mathcal{O}_A) \cong \mathrm{K}_1(\mathcal{O}_B)$ *and* $\mathrm{Ext}_s^0(\mathcal{O}_A) \cong \mathrm{Ext}_s^0(\mathcal{O}_B)$.

Proof. Let us denote by T_A (resp. $\widetilde{T}_A$) the torsion part of $\mathrm{K}_0(\mathcal{O}_{A^t})$ (resp. $\mathrm{K}_0(\mathcal{T}_{A^t})$), and let r_A (resp. R_A) be rank$(\mathrm{K}_1(\mathcal{O}_{A^t}))$ (resp. $\mathrm{K}_1(\mathcal{T}_{A^t})$). Then, one has

$$\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_{A^t}) = \mathbb{Z}^{r_A} \oplus T_A, \quad \mathrm{K}_1(\mathcal{O}_A) \cong \mathrm{K}_1(\mathcal{O}_{A^t}) = \mathbb{Z}^{r_A}.$$

Lemma 4.3 and Lemma 4.4 show that

$$\mathrm{Ext}_s^1(\mathcal{O}_A) = \mathrm{K}_0(\mathcal{T}_{A^t}) = \mathbb{Z}^{R_A+1} \oplus \widetilde{T}_A, \quad \mathrm{Ext}_s^0(\mathcal{O}_A) = \mathrm{K}_1(\mathcal{T}_{A^t}) = \mathbb{Z}^{R_A}.$$

Applying the same formula for the matrix B, the desired assertion follows immediately. ∎

We present the following theorem which is the main theorem in this paper.

Theorem 4.8. *Let* $A = [A(i,j)]_{i,j=1}^N$, $B = [B(i,j)]_{i,j=1}^M$ *be irreducible non-permutation matrices with entries in* $\{0,1\}$. *Then the following five conditions are mutually equivalent.*

(i) $\pi_i(\mathrm{Aut}(\mathcal{O}_A)) \cong \pi_i(\mathrm{Aut}(\mathcal{O}_B))$, $i = 1, 2$.

(ii) $\mathcal{O}_A \cong \mathcal{O}_B$.

(iii) $\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_B)$ *and* $\mathrm{Ext}_s^1(\mathcal{O}_A) \cong \mathrm{Ext}_s^1(\mathcal{O}_B)$.

(iv) $\mathrm{Ext}_w^1(\mathcal{O}_A) \cong \mathrm{Ext}_w^1(\mathcal{O}_B)$ *and* $\mathrm{Ext}_s^1(\mathcal{O}_A) \cong \mathrm{Ext}_s^1(\mathcal{O}_B)$.

(v) $\mathbb{Z}^N/(I-A)\mathbb{Z}^N \cong \mathbb{Z}^M/(I-B)\mathbb{Z}^M$ *and* $\mathbb{Z}^N/(I-\hat{A})\mathbb{Z}^N \cong \mathbb{Z}^M/(I-\hat{B})\mathbb{Z}^M$.

Proof. The equivalence (i) ⇔ (ii) follows from Theorem 4.6. The implication (ii) ⇒ (iii) is clear. Since $\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{Ext}_w^1(\mathcal{O}_A) \cong \mathbb{Z}^N/(I-A)\mathbb{Z}^N$, the equivalences among (iii), (iv) and (v) are obvious. By Lemma 4.7, we know that condition (iii) implies that $\mathrm{K}_i(\mathcal{O}_A) \cong \mathrm{K}_i(\mathcal{O}_B)$ and $\mathrm{Ext}_s^i(\mathcal{O}_A) \cong \mathrm{Ext}_s^i(\mathcal{O}_B)$ for $i = 0, 1$. Hence Proposition 3.3 shows $\pi_i(\mathrm{Aut}(\mathcal{O}_A)) \cong \pi_i(\mathrm{Aut}(\mathcal{O}_B))$, $i = 1, 2$ which is condition (i). ∎

Recall that $\mathrm{Aut}(\mathcal{O}_A)$ is endowed with the topology defined by the pointwise norm convergence, so that it becomes a topological group. The equivalence between (i) and (ii) in Theorem 4.8 tells us the following corollary.

Corollary 4.9. *Let* A, B *be irreducible non-permutation matrices with entries in* $\{0,1\}$. *Then* $\mathrm{Aut}(\mathcal{O}_A)$ *is isomorphic to* $\mathrm{Aut}(\mathcal{O}_B)$ *as topological groups if and only if* $\mathcal{O}_A$ *is isomorphic to* $\mathcal{O}_B$ *as* C^**-algebras.*

For $m \in \mathbb{Z}$, take a unitary $u_m \in Q(H)$ whose Fredholm index is m. Take a trivial extension $\tau : \mathcal{O}_A \to Q(H)$ which means that there exists a $*$-homomorphism $\tau_0 : \mathcal{O}_A \to B(H)$ such that $\tau = \pi \circ \tau_0$, where $\pi : B(H) \to Q(H)$ is the natural quotient map. Let us denote by σ_m the extension $\mathrm{Ad}(u_m) \circ \tau : \mathcal{O}_A \to Q(H)$. Let $q_A : \mathrm{Ext}_w(\mathcal{O}_A) \to \mathrm{Ext}_s(\mathcal{O}_A)$ denote the natural quotient map. Then the map $\iota_A : m \in \mathbb{Z} \to [\sigma_m]_s \in \mathrm{Ext}_s(\mathcal{O}_A)$ yields a homomorphism of groups such that the sequence

$$\mathbb{Z} \xrightarrow{\iota_A} \mathrm{Ext}_s(\mathcal{O}_A) \xrightarrow{q_A} \mathrm{Ext}_w(\mathcal{O}_A) \tag{4.2}$$

is exact at the middle so that we have $\mathrm{Ext}_s(\mathcal{O}_A)/\iota_A(\mathbb{Z})$ is isomorphic to $\mathrm{Ext}_w(\mathcal{O}_A)$. The sequence (4.2) is rephrased as

$$\mathbb{Z} \xrightarrow{\hat{\iota}_A} \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \xrightarrow{\hat{q}_A} \mathbb{Z}^N/(I - A)\mathbb{Z}^N$$

([20, Lemma 3.1]).

Corollary 4.10. *The pair* $(\mathrm{Ext}_s(\mathcal{O}_A), \iota_A(1))$ *of the group* $\mathrm{Ext}_s(\mathcal{O}_A)$ *and the position* $\iota_A(1)$ *in* $\mathrm{Ext}_s(\mathcal{O}_A)$ *is a complete invariant of the isomorphism class of* $\mathcal{O}_A$. *It means that the class* $[(I - A)[e_1]]$ *of the vector* $(I - A)[e_1] \in \mathbb{Z}^N$, *where* $e_1 = [1, 0, \dots, 0]^t$, *in the quotient group* $\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N$ *completely determines the isomorphism class of* $\mathcal{O}_A$.

4.3. More about $\pi_i(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$

In this subsection, we will refer to the homotopy groups $\pi_i(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$, $i = 1, 2$ of the automorphism group of the stabilized Cuntz–Krieger algebra $\mathcal{O}_A \otimes K(H)$.

Lemma 4.11. $\pi_1(\mathrm{Aut}(\mathcal{O}_A \otimes K(H))) \cong \pi_2(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$.

Proof. Let r_A be the rank of $\mathrm{K}_0(\mathcal{O}_A)$. We may write $\mathrm{K}_0(\mathcal{O}_A) = \mathbb{Z}^{r_A} \oplus T_A$ with a finite abelian group T_A. By Proposition 3.6, it is direct to see that both $\pi_1(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$ and $\pi_2(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$ are isomorphic to the abelian group

$$(T_A \otimes T_A) \oplus (T_A \otimes \mathbb{Z}^{r_A}) \oplus (\mathbb{Z}^{r_A} \otimes T_A) \oplus (\mathbb{Z}^{r_A} \otimes \mathbb{Z}^{r_A}) \oplus (\mathbb{Z}^{r_A} \otimes \mathbb{Z}^{r_A}). \tag{4.3}$$

∎

Remark 4.12. One can also check that, for a Kirchberg algebra $\mathcal{A}$ with finitely generated K-groups, the isomorphism $\pi_1(\mathrm{Aut}(\mathcal{A} \otimes K(H))) \cong \pi_2(\mathrm{Aut}(\mathcal{A} \otimes K(H)))$ holds if and only if $\mathrm{rank}(\mathrm{K}_0(\mathcal{A})) = \mathrm{rank}(\mathrm{K}_1(\mathcal{A}))$ holds.

We see that the group structure of the abelian group of (4.3) determines the torsion group T_A and the free abelian group $\mathbb{Z}^{r_A}$ as in the following proposition.

Proposition 4.13. $\pi_1(\mathrm{Aut}(\mathcal{O}_A \otimes K(H))) \cong \pi_1(\mathrm{Aut}(\mathcal{O}_B \otimes K(H)))$ *if and only if* $\mathcal{O}_A \otimes K(H) \cong \mathcal{O}_B \otimes K(H)$.

Proof. We show the only if part. Assume that $\pi_1(\mathrm{Aut}(\mathcal{O}_A \otimes K(H)))$ is isomorphic to $\pi_1(\mathrm{Aut}(\mathcal{O}_B \otimes K(H)))$. It suffices to prove that $\mathrm{K}_0(\mathcal{O}_A)$ is isomorphic to $\mathrm{K}_0(\mathcal{O}_B)$, because of [26, Theorem 6.5]. We use the same notation as in the proof of Lemma 4.11. There are integers $\{n_i\}_{i=1}^k, \{m_j\}_{j=1}^l$ such that

$$\begin{aligned}
1 < n_1 \le n_2 \le \cdots \le n_k, \quad n_1|n_2|\cdots|n_k,\\
1 < m_1 \le m_2 \le \cdots \le m_l, \quad m_1|m_2|\cdots|m_l,\\
T_A = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z},\\
T_B = \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_l\mathbb{Z}.
\end{aligned}$$

By the fundamental theorem of finite abelian groups, these integers are uniquely determined for the groups T_A, T_B, respectively. The assumption $\pi_1(\mathrm{Aut}(\mathcal{O}_A \otimes K(H))) \cong \pi_1(\mathrm{Aut}(\mathcal{O}_B \otimes K(H)))$ and (4.3) yield

$$\begin{aligned}
r_A = r_B =: r,\\
(T_A \otimes T_A) \oplus T_A^{2r} \cong (T_B \otimes T_B) \oplus T_B^{2r}.
\end{aligned} \tag{4.4}$$

Comparing the following expressions:

$$\begin{aligned}
\bigoplus_{i=1}^{k} (\mathbb{Z}/n_i\mathbb{Z})^{2(k+r-i)+1} &= (T_A \otimes T_A) \oplus T_A^{2r}\\
&= (T_B \otimes T_B) \oplus T_B^{2r}\\
&= \bigoplus_{j=1}^{l} (\mathbb{Z}/m_j\mathbb{Z})^{2(l+r-j)+1},
\end{aligned}$$

we have $n_k = m_l =: n$. For the groups

$$T'_A := \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_{k-1}\mathbb{Z}, \quad T'_B := \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_{l-1}\mathbb{Z},$$

one has $T'_A \otimes \mathbb{Z}/n\mathbb{Z} = T'_A$, $T'_B \otimes \mathbb{Z}/n\mathbb{Z} = T'_B$, and (4.4) implies

$$(T'_A \otimes T'_A) \oplus {T'_A}^{2(r+1)} \cong (T'_B \otimes T'_B) \oplus {T'_B}^{2(r+1)}.$$

Now we can inductively obtain

$$n_{k-1} = m_{l-1}, \quad n_{k-2} = m_{l-2}, \ldots, \quad k = l,$$

and this proves $\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_B)$. ■

Remark 4.14. The same statement as the above proposition for general stable Kirchberg algebras $\mathcal{A}, \mathcal{B}$ with finitely generated K-groups does not hold. Let P_∞ be the stable Kirchberg algebra KK-equivalent to $S(= C_0(0,1))$. Then, two algebras $\mathcal{A} = \mathcal{O}_A \otimes K(H)$ and $\mathcal{B} = P_\infty \otimes \mathcal{O}_A \otimes K(H)$ have the same homotopy groups of their automorphism groups, but they are not isomorphic in general (see the case that $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_N$, $N \ge 3$).

5. The invariants $(\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A))$

In this section, we will determine the possible range of the pair $(\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A))$.

Proposition 5.1. *For a finitely generated abelian group M and an element $e \in \mathbb{Z} \oplus M$, there exists a Cuntz–Krieger algebra $\mathcal{O}_A$ for an irreducible non-permutation matrix A with entries in $\{0, 1\}$ such that*

$$\mathrm{Ext}_w(\mathcal{O}_A) \cong (\mathbb{Z} \oplus M)/\mathbb{Z}e, \quad \mathrm{Ext}_s(\mathcal{O}_A) \cong \mathbb{Z} \oplus M.$$

Before going to the proof of Proposition 5.1, let us recall the range of K-groups of the purely infinite simple Cuntz–Krieger algebras ([26]). For a finite abelian group

$$T := \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}, \quad n_i \geq 2,$$

and an integer $r \geq 0$, one can find a matrix A of size $N := \sum_{i=1}^{k}(1 + n_i) + r + 3$ satisfying

$$\mathbb{Z}^N/(I - A)\mathbb{Z}^N = \mathbb{Z}^r \oplus T, \quad \mathrm{Ker}(I - A) = \mathbb{Z}^r$$

as follows. Let N_i, $i = 1, 2, \ldots, k$ denote the $(1 + n_i) \times (1 + n_i)$-matrix whose entries are all 1 and let I_r denote the identity matrix of size r. Define the matrix A by setting

$$A := \begin{bmatrix} & & & 0 & 0 & 1 \\ & \mathbf{D} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 1 \\ 1 & \ldots & 1 & 0 & 0 & 1 \\ 0 & \ldots & 0 & 0 & 1 & 1 \\ 0 & \ldots & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{where} \quad D := \begin{bmatrix} I_r & & & \\ & N_1 & & \\ & & \ddots & \\ & & & N_k \end{bmatrix}.$$

It is straightforward to check that

$$\mathbb{Z}^N/(I_N - A)\mathbb{Z}^N \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}^{1+n_i}/(I_{1+n_i} - N_i)\mathbb{Z}^{1+n_i} = \mathbb{Z}^r \oplus T,$$
$$\mathrm{Ker}(I_N - A) = \mathbb{Z}^r.$$

In the directed graph with N-vertices determined by A, every two vertices are connected by a directed path passing through the Nth vertex, which implies that A is an irreducible, non-permutation matrix. Combining this with [26] (see also [1]), for an arbitrary element $e \in \mathbb{Z}^r \oplus T$, one can find a purely infinite simple Cuntz–Krieger algebra $\mathcal{O}_B$ (i.e., $\mathcal{O}_B$ with an irreducible non-permutation matrix B) satisfying

$$(\mathrm{K}_0(\mathcal{O}_B), [1_{\mathcal{O}_B}]_0, \mathrm{K}_1(\mathcal{O}_B)) \cong (\mathbb{Z}^r \oplus T, e, \mathbb{Z}^r).$$

Proof of Proposition 5.1. We first show the assertion when $e \in \mathbb{Z} \oplus M$ is a non-torsion element. In this case one has a short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus M \to (\mathbb{Z} \oplus M)/\mathbb{Z}e \to 0$$

of abelian groups that gives an element of $\mathrm{Ext}^1_{\mathbb{Z}}((\mathbb{Z} \oplus M)/\mathbb{Z}e, \mathbb{Z})$. Let $\mathcal{O}_B$ be a purely infinite simple Cuntz–Krieger algebra with $\mathrm{K}_0(\mathcal{O}_B) \cong (\mathbb{Z} \oplus M)/\mathbb{Z}e$. Then, the following UCT

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}((\mathbb{Z} \oplus M)/\mathbb{Z}e, \mathbb{Z}) \to \mathrm{Ext}_w(\mathcal{O}_B) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(\mathcal{O}_B), \mathbb{Z}) \to 0$$

implies that there is an essential unital extension $0 \to K(H) \to E \to \mathcal{O}_B \to 0$ whose cyclic six term exact sequence for K-groups splits into the following short exact sequences

$$\begin{aligned} &0 \to \mathbb{Z} \to \mathrm{K}_0(E) \to (\mathbb{Z} \oplus M)/\mathbb{Z}e \to 0, \\ &0 \to \mathrm{K}_1(E) \to \mathrm{K}_1(\mathcal{O}_B) \to 0 \end{aligned}$$

such that the first short exact sequence is equivalent to the extension $0 \to \mathbb{Z} \to \mathbb{Z} \oplus M \to (\mathbb{Z} \oplus M)/\mathbb{Z}e \to 0$. Hence one has $\mathbb{Z} \oplus M \cong K_0(E)$, and the strong K-theoretic duality (see [20, 24]) implies that there is a unital Kirchberg algebra $\mathcal{B}$ satisfying

$$\mathrm{Ext}_s(\mathcal{B}) = K_0(E), \quad \mathrm{Ext}_w(\mathcal{B}) = \mathrm{K}_0(\mathcal{O}_B), \quad \mathcal{B} \underset{\mathrm{KK}}{\sim} D(S\mathcal{O}_B) \underset{\mathrm{KK}}{\sim} \mathcal{O}_B.$$

Applying [26] (cf. [1]) to $\mathcal{B} \otimes K(H) \cong \mathcal{O}_B \otimes K(H)$, the algebra $\mathcal{B}$ is isomorphic to a purely infinite simple Cuntz–Krieger algebra $\mathcal{O}_A$ (i.e., $\mathcal{O}_A$ with an irreducible non-permutation matrix A).

We next show the assertion when e is a torsion element. We write $M := \mathbb{Z}^m \oplus \widetilde{T}$, $e := (0, \tilde{t}) \in M$, $(\mathbb{Z} \oplus M)/\mathbb{Z}e = \mathbb{Z}^{1+m} \oplus T$ where $m \geq 0$ is an integer and $\widetilde{T}, T = \widetilde{T}/\mathbb{Z}\tilde{t}$ are finite abelian groups. By [30, Corollary 3.10], there exists a non-torsion element $d \in \mathbb{Z} \oplus T$ satisfying

$$\widetilde{T} = (\mathbb{Z} \oplus T)/\mathbb{Z}d.$$

For a purely infinite simple Cuntz–Krieger algebra $\mathcal{O}_A$ with

$$(\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0, K_1(\mathcal{O}_A)) \cong (\mathbb{Z}^m \oplus (\mathbb{Z} \oplus T), (0, d), \mathbb{Z}^{m+1}),$$

one has

$$\mathrm{Ext}_w(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_A) \cong (\mathbb{Z} \oplus M)/\mathbb{Z}e, \quad \mathrm{K}_1(C_{\mathcal{O}_A}) \cong \mathbb{Z}^m \oplus (\mathbb{Z} \oplus T)/\mathbb{Z}d = \mathbb{Z}^m \oplus \widetilde{T}.$$

We have the cyclic six term exact sequence of the K-groups

$$\begin{array}{ccccc} \mathrm{K}_1(\mathcal{O}_A) & \longrightarrow & \mathrm{K}_0(C_{\mathcal{O}_A}) & \longrightarrow & \mathrm{K}_0(\mathbb{C}) \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & \mathrm{K}_1(C_{\mathcal{O}_A}) & \longleftarrow & \mathrm{K}_0(\mathcal{O}_A) \end{array}$$

for the short exact sequence $0 \to S\mathcal{O}_A \to C_{\mathcal{O}_A} \to \mathbb{C} \to 0$, where the vertical map $\mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_0(\mathcal{O}_A)$ is identified with the induced homomorphism

$$\mathrm{K}_0(u_{\mathcal{O}_A}) : \mathrm{K}_0(\mathbb{C}) \ni [1]_0 \mapsto [1_{\mathcal{O}_A}]_0 = (0, d) \in \mathrm{K}_0(\mathcal{O}_A)$$

from the unital embedding $u_{\mathcal{O}_A} : \mathbb{C} \to \mathcal{O}_A$. Since d is a non-torsion element, the map $\mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_0(\mathcal{O}_A)$ is injective, and the above cyclic six term exact sequence shows $\mathbb{Z}^{m+1} = \mathrm{K}_1(\mathcal{O}_A) \cong \mathrm{K}_0(C_{\mathcal{O}_A})$. Thus, the UCT

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(C_{\mathcal{O}_A}), \mathbb{Z}) \to \mathrm{Ext}_s(\mathcal{O}_A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(C_{\mathcal{O}_A}), \mathbb{Z}) \to 0$$

implies $\mathrm{Ext}_s(\mathcal{O}_A) \cong \widetilde{T} \oplus \mathbb{Z}^{m+1} = \mathbb{Z} \oplus M$. ∎

Since $\mathrm{Ext}_w(\mathcal{O}_A) = \mathrm{K}_0(\mathcal{O}_{A^t})$ and $\mathrm{Ext}_s(\mathcal{O}_A) = \mathrm{K}_0(\mathcal{T}_{A^t})$, Lemma 4.4 together with (2.5) yields the following corollary.

Corollary 5.2. *The possible range of* $(\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(OA))$ *is determined in the following way:*

$$\begin{aligned}\{(\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A)) \mid A &: \textit{irreducible non-permutation matrix with entries in } \{0,1\}\} \\ = \{(G, \mathbb{Z} \oplus M) \mid M &: \textit{finitely generated abelian group,} \\ & G \cong (\mathbb{Z} \oplus M)/\mathbb{Z}e \textit{ for } e \in \mathbb{Z} \oplus M\}.\end{aligned}$$

One may notice that complete invariants of $\mathcal{O}_A$ appear in two different ways. One is a pair (G, d) of G and $d \in G$ (i.e., $(\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0)$ due to [26]), and the other is the pair $(G, \mathbb{Z} \oplus M)$ of abelian groups as in the above corollary. We will explain the relationship between these two aspects via the reciprocality.

Lemma 5.3 (Cf. [30, Corollary 3.10]). *Let M be a finitely generated abelian group. For any $e \in \mathbb{Z} \oplus M$, there exists an element $\widetilde{d} \in \mathbb{Z} \oplus M$ satisfying*

$$M \cong (\mathbb{Z} \oplus M)/\langle e, \widetilde{d} \rangle.$$

Proof. We write $M = \mathbb{Z}^m \oplus T$ for a finite abelian group T and an integer $m \geq 0$ so that $\mathbb{Z} \oplus M = \mathbb{Z} \oplus (\mathbb{Z}^m \oplus T)$. We may assume that e is of the form $(n, (0, t))$, $n \in \mathbb{Z}$. By [30, Corollary 3.10], one may find an element $\widetilde{d} \in \mathbb{Z} \oplus 0 \oplus T$ satisfying

$$(\mathbb{Z} \oplus 0 \oplus T)/\langle e, \widetilde{d} \rangle \cong T.$$

We see that the elements $e, \widetilde{d}$ satisfy $(\mathbb{Z} \oplus M)/\langle e, \widetilde{d} \rangle \cong M$. ∎

For $d \in G$ and $d' \in G'$, we say that (G, d) and (G', d') are equivalent if there is an isomorphism $\theta : G \to G'$ with $\theta(d) = d'$, and denote by $[G, d]$ the equivalence class. We write

$$\begin{aligned}\mathcal{K} &:= \{[G, d] \mid G : \text{finitely generated abelian group}, d \in G\} \\ &(= \{[\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0] \mid A : \text{ irreducible non-permutation matrix} \\ &\qquad \text{with entries in } \{0,1\}\}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{E} := \{[G, \mathbb{Z} \oplus M] \mid M &: \text{finitely generated abelian group,}\\ G &\cong (\mathbb{Z} \oplus M)/\mathbb{Z}e \text{ for } e \in \mathbb{Z} \oplus M\}\\ (= \{[\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A)] \mid A &: \text{ irreducible non-permutation matrix}\\ &\text{with entries in}\{0,1\}\}),\end{aligned}$$

where, in the definition of $\mathcal{E}$, two pairs $(G, \mathbb{Z} \oplus M)$ and $(G', \mathbb{Z} \oplus M')$ are said to be isomorphic if $G \cong G'$ and $\mathbb{Z} \oplus M \cong \mathbb{Z} \oplus M'$ hold. The isomorphism class is written as $[G, \mathbb{Z} \oplus M]$.

Proposition 5.4. *Let G, M be finitely generated abelian groups.*

(i) *For any $[G, d] \in \mathcal{K}$, one has $(G, \mathbb{Z} \oplus (G/\mathbb{Z}d)) \in \mathcal{E}$.*

(ii) *For any $[G, \mathbb{Z} \oplus M] \in \mathcal{E}$, there exist two elements $e, \tilde{d} \in \mathbb{Z} \oplus M$ satisfying*

$$G \cong (\mathbb{Z} \oplus M)/\mathbb{Z}e, \quad (\mathbb{Z} \oplus M)/\langle e, \tilde{d}\rangle \cong M, \tag{5.1}$$

and the class $[(\mathbb{Z} \oplus M)/\mathbb{Z}e, \tilde{d} + \mathbb{Z}e] \in \mathcal{K}$ does not depend on the choice of $e, \tilde{d}$ as long as they satisfy (5.1).

(iii) *The two constructions above yield a bijective correspondence between $\mathcal{K}$ and $\mathcal{E}$ by which $[\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0]$ corresponds to $[\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A)]$.*

Proof. We first prove (i). Applying Lemma 5.3 to $d \in \mathbb{Z} \oplus G$, there exists an element $\tilde{e} \in \mathbb{Z} \oplus G$ satisfying

$$G = (\mathbb{Z} \oplus G)/\langle d, \tilde{e}\rangle = (\mathbb{Z} \oplus (G/\mathbb{Z}d))/\langle(\tilde{e} + \mathbb{Z}d)\rangle,$$

and one has $[G, \mathbb{Z} \oplus (G/\mathbb{Z}d)] \in \mathcal{E}$.

We next show (ii). Existence of $e, \tilde{d}$ follows from the definition of $\mathcal{E}$ and Lemma 5.3. Since $(\mathbb{Z} \oplus M)/\mathbb{Z}e \cong G$ and $((\mathbb{Z} \oplus M)/\mathbb{Z}e)/\langle\tilde{d} + \mathbb{Z}e\rangle \cong M$ hold, the class $[(\mathbb{Z} \oplus M)/\mathbb{Z}e, \tilde{d} + \mathbb{Z}e] \in \mathcal{K}$ does not depend on the choice of $e, \tilde{d}$ as long as they satisfy (5.1), because of [30, Proposition 2.19].

We finally show (iii). The assertions (i) and (ii) provide two correspondences $\mathcal{K} \xrightarrow{\text{(i)}} \mathcal{E}$ and $\mathcal{E} \xrightarrow{\text{(ii)}} \mathcal{K}$, and it is straightforward to check that the composition $\mathcal{E} \xrightarrow{\text{(ii)}} \mathcal{K} \xrightarrow{\text{(i)}} \mathcal{E}$ is identical.

We show that the composition $\mathcal{K} \xrightarrow{\text{(i)}} \mathcal{E} \xrightarrow{\text{(ii)}} \mathcal{K}$ is identical. For a given $[G, d] \in \mathcal{K}$, the correspondence $\mathcal{K} \xrightarrow{\text{(i)}} \mathcal{E}$ gives the pair $(G, \mathbb{Z} \oplus (G/\mathbb{Z}d))$, and the correspondence $\mathcal{E} \xrightarrow{\text{(ii)}} \mathcal{K}$ sends the pair to $[G, d']$ satisfying $G/\mathbb{Z}d' \cong G/\mathbb{Z}d$. By [30, Proposition 2.19], we obtain $[G, d'] = [G, d]$.

Since $\mathrm{Ext}_w(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_A)$, the following computation implies that $\mathcal{K} \xrightarrow{\text{(i)}} \mathcal{E}$ sends $[\mathrm{K}_0(\mathcal{O}_A), [1_{\mathcal{O}_A}]_0]$ to $[\mathrm{Ext}_w(\mathcal{O}_A), \mathrm{Ext}_s(\mathcal{O}_A)]$:

$$\begin{aligned}\mathrm{Ext}_s(\mathcal{O}_A) &= \mathrm{KK}(C_{\mathcal{O}_A}, \mathbb{C})\\ &\cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(C_{\mathcal{O}_A}), \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(C_{\mathcal{O}_A}), \mathbb{Z})\\ &\cong \mathbb{Z}^{\mathrm{rank}(\mathrm{K}_0(C_{\mathcal{O}_A}))} \oplus \mathrm{Tor}(\mathrm{K}_1(C_{\mathcal{O}_A}))\end{aligned}$$

$$\begin{aligned} &= \mathbb{Z} \oplus \mathbb{Z}^{\mathrm{rank}(\mathrm{K}_1(C_{\mathcal{O}_A}))} \oplus \mathrm{Tor}(\mathrm{K}_1(C_{\mathcal{O}_A})) \\ &= \mathbb{Z} \oplus \mathrm{K}_1(C_{\mathcal{O}_A}) \\ &\cong \mathbb{Z} \oplus (\mathrm{K}_0(\mathcal{O}_A)/\mathbb{Z}[1_{\mathcal{O}_A}]_0). \end{aligned}$$ ∎

6. Examples

1. $\mathcal{O}_N$: Let $\mathcal{O}_N$ be the Cuntz algebra of order $1 < N \in \mathbb{N}$ (see [5]). It was shown that $\mathrm{K}_0(\mathcal{O}_N) = \mathbb{Z}/(1-N)\mathbb{Z}$, $\mathrm{K}_1(\mathcal{O}_N) = 0$ in [7] and also $\mathrm{Ext}_s^1(\mathcal{O}_N) = \mathbb{Z}$, $\mathrm{Ext}_s^0(\mathcal{O}_N) = 0$ in [23, 25]. By Proposition 3.3 and Corollary 3.4, we have

$$\begin{aligned} \pi_1(\mathrm{Aut}(\mathcal{O}_N)) &\cong (\mathrm{Ext}_s^1(\mathcal{O}_N) \otimes \mathrm{K}_0(\mathcal{O}_A)) \oplus (\mathrm{Ext}_s^0(\mathcal{O}_N) \otimes \mathrm{K}_1(\mathcal{O}_A)) \\ &\cong \mathbb{Z} \otimes \mathbb{Z}/(1-N)\mathbb{Z} \\ &\cong \mathbb{Z}/(1-N)\mathbb{Z}, \\ \pi_2(\mathrm{Aut}(\mathcal{O}_N)) &\cong (\mathrm{Ext}_s^1(\mathcal{O}_N) \otimes \mathrm{K}_1(\mathcal{O}_A)) \oplus (\mathrm{Ext}_s^0(\mathcal{O}_N) \otimes \mathrm{K}_0(\mathcal{O}_A)) \\ &\qquad \oplus \mathrm{Tor}(\mathrm{Ext}_s^1(\mathcal{O}_N), \mathrm{K}_0(\mathcal{O}_A)) \cong 0, \\ \pi_1(\mathrm{Aut}(\mathcal{O}_N \otimes K(H))) &\cong \pi_2(\mathrm{Aut}(\mathcal{O}_N \otimes K(H))) \cong \mathbb{Z}/(1-N)\mathbb{Z}. \end{aligned}$$

2. $\mathcal{O}_N \otimes M_k(\mathbb{C})$: We note that $\mathcal{O}_N \otimes M_k(\mathbb{C})$ is realized as the Cuntz–Krieger algebra $\mathcal{O}_{A^{\langle k \rangle}}$ for the $Nk \times Nk$ matrix $A^{\langle k \rangle}$ defined by

$$A^{\langle k \rangle} := \left[\begin{array}{ccc|c} 0_N & \cdots & 0_N & [N] \\ \hline I_N & & & 0_N \\ & \ddots & & \vdots \\ & & I_N & 0_N \end{array}\right] \quad \text{where} \quad [N] := \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix},$$

and $[N], 0_N, I_N$ are $N \times N$ matrices. Hence the formulas of Proposition 3.3 and Corollary 3.4 are applicable for $\mathcal{O}_N \otimes M_k(\mathbb{C})$. Since $\mathrm{K}_1(\mathcal{O}_N \otimes M_k(\mathbb{C})) = \mathrm{K}_1(\mathcal{O}_N) = 0$ and hence $\mathrm{Ext}_s^0(\mathcal{O}_N) = 0$, we have

$$\pi_1(\mathrm{Aut}(\mathcal{O}_N \otimes M_k(\mathbb{C}))) \cong \mathrm{Ext}_s^1(\mathcal{O}_N \otimes M_k(\mathbb{C})) \otimes \mathbb{Z}/(1-N)\mathbb{Z}.$$

By Paschke and Salinas [23], we know that $\mathrm{Ext}_s^1(\mathcal{O}_N \otimes M_k(\mathbb{C})) \cong \mathbb{Z} \oplus \mathbb{Z}/(N-1,k)\mathbb{Z}$, where $(N-1,k)$ is the greatest common divisor of $N-1$ and k. Hence we have

$$\pi_1(\mathrm{Aut}(\mathcal{O}_N \otimes M_k(\mathbb{C}))) \cong \mathbb{Z}/(1-N)\mathbb{Z} \oplus \mathbb{Z}/(N-1,k)\mathbb{Z}.$$

Similarly we have

$$\pi_2(\mathrm{Aut}(\mathcal{O}_N \otimes M_k(\mathbb{C}))) \cong \mathbb{Z}/(N-1,k)\mathbb{Z}.$$

As $(\mathcal{O}_N \otimes M_k(\mathbb{C})) \otimes K(H) \cong \mathcal{O}_N \otimes K(H)$, we have

$$\pi_i(\mathrm{Aut}((\mathcal{O}_N \otimes M_k(\mathbb{C})) \otimes K(H))) \cong \mathbb{Z}/(1-N)\mathbb{Z} \quad \text{for } i = 1, 2.$$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = A^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

As in [21], we know that

$$\mathrm{K}_0(\mathcal{O}_A) \cong \mathrm{K}_0(\mathcal{O}_B) \cong \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{K}_1(\mathcal{O}_A) = \mathrm{K}_1(\mathcal{O}_B) = \mathrm{Ext}_s^0(\mathcal{O}_A) = \mathrm{Ext}_s^0(\mathcal{O}_B) = 0,$$

and

$$\mathrm{Ext}_s^1(\mathcal{O}_A) \cong \mathbb{Z}, \quad \mathrm{Ext}_s^1(\mathcal{O}_B) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

so that

$$\pi_1(\mathrm{Aut}(\mathcal{O}_A)) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_1(\mathrm{Aut}(\mathcal{O}_B)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Hence we know that $\mathcal{O}_A$ is not isomorphic to $\mathcal{O}_B$. Similarly we have

$$\pi_2(\mathrm{Aut}(\mathcal{O}_A)) \cong 0, \quad \pi_2(\mathrm{Aut}(\mathcal{O}_B)) \cong \mathbb{Z}/2\mathbb{Z}$$

and

$$\pi_i(\mathrm{Aut}(\mathcal{O}_A \otimes K(H))) \cong \pi_i(\mathrm{Aut}(\mathcal{O}_B \otimes K(H))) \cong \mathbb{Z}/2\mathbb{Z}, \quad i = 1, 2.$$

The main result of the present paper is generalized to a wider class of Kirchberg algebras in our recent preprint [22].

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