

Tracial states on groupoid C^* -algebras and essential freeness

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Abstract. Let \mathcal{G} be a locally compact Hausdorff étale groupoid. We call a tracial state τ on a general groupoid C^* -algebra $C_v^*(\mathcal{G})$ *canonical* if $\tau = \tau|_{C_0(\mathcal{G}^{(0)})} \circ E$, where $E : C_v^*(\mathcal{G}) \rightarrow C_0(\mathcal{G}^{(0)})$ is the canonical conditional expectation. In this paper, we consider so-called fixed point traces on $C_c(\mathcal{G})$, and prove that \mathcal{G} is essentially free if and only if any tracial state on $C_v^*(\mathcal{G})$ is canonical and any fixed point trace is extendable to $C_v^*(\mathcal{G})$.

As applications, we obtain the following: (1) a group action is essentially free if every tracial state on the reduced crossed product is canonical and every isotropy group is amenable; (2) if the groupoid \mathcal{G} is second-countable, amenable and essentially free then every (not necessarily faithful) tracial state on the reduced groupoid C^* -algebra is quasidiagonal.

1. Introduction

In recent years, there has been increasing interest in the classification of crossed products of C^* -algebras $C_0(X) \rtimes_v \Gamma$ arising from discrete amenable group actions on locally compact Hausdorff spaces $\Gamma \curvearrowright X$ (see, e.g., [2, 10, 12, 15–17, 19, 22, 25, 26, 29, 30]). One of the key ingredients in those proofs is that every tracial state τ on $C_0(X) \rtimes_v \Gamma$ is *canonical* in the sense that $\tau = \tau|_{C_0(X)} \circ E$, where $E : C_0(X) \rtimes_v \Gamma \rightarrow C_0(X)$ is the canonical conditional expectation. Actually, it was shown in [14, Theorem 2.7] that every tracial state on the *maximal* crossed product $C(X) \rtimes \Gamma$ of an action on a compact Hausdorff space X is canonical if and only if the action is essentially free with respect to all invariant probability Radon measures on X . On the other hand, every tracial state on the *reduced* crossed product $C(X) \rtimes_r \Gamma$ is canonical if and only if the action of the *amenable radical* $R_a(\Gamma)$ of Γ (i.e., the largest amenable normal subgroup in Γ) on X is essentially free with respect to all invariant probability Radon measures on X (see [34, Corollary 1.12]).

On the other hand, X. Li was able to show that all classifiable C^* -algebras necessarily arise from twisted étale groupoids (see [20]). Hence, it is natural to consider canonical tracial states on general groupoid C^* -algebras $C_v^*(\mathcal{G})$ of locally compact Hausdorff étale groupoids \mathcal{G} . Similarly, a tracial state τ on $C_v^*(\mathcal{G})$ is canonical if $\tau = \tau|_{C_0(\mathcal{G}^{(0)})} \circ E$, where $E : C_v^*(\mathcal{G}) \rightarrow C_0(\mathcal{G}^{(0)})$ is the canonical conditional expectation. If we let μ be

the uniquely associated invariant probability Radon measure on $\mathcal{G}^{(0)}$ to $\tau|_{C_0(\mathcal{G}^{(0)})}$, then it follows from [27, Corollary 1.2] and [28, Corollary 2.4] that if \mathcal{G} is second-countable, then a tracial state τ on the maximal groupoid C^* -algebra $C^*(\mathcal{G})$ is canonical if and only if \mathcal{G} is essentially free with respect to the associated measure μ (see Definition 3.3).

In this article, we would like to consider the relationship between the essential freeness of \mathcal{G} and tracial states on a general groupoid C^* -algebra $C_v^*(\mathcal{G})$ with respect to any C^* -norm $\|\cdot\|_v$ dominating the reduced C^* -norm. More precisely, we ask the following question.

Question 1.1. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid and $\|\cdot\|_v$ be a C^* -norm on $C_c(\mathcal{G})$ dominating the reduced C^* -norm. If μ is an invariant probability Radon measure on $\mathcal{G}^{(0)}$, can we characterise the essential freeness of \mathcal{G} with respect to μ in terms of tracial states on $C_v^*(\mathcal{G})$?*

One of the crucial ingredients to answer Question 1.1 is the extendibility of the so-called fixed point trace τ_μ^{Fix} from $C_c(\mathcal{G})$ to $C_v^*(\mathcal{G})$ (see Definition 3.9). If $\mathcal{G} = X \rtimes \Gamma$ is a transformation groupoid, then τ_μ^{Fix} has a simplified form (see equation (5.1)),

$$\tau_\mu^{\text{Fix}}\left(\sum_{i=1}^n f_i \gamma_i\right) = \sum_{i=1}^n \int_{\text{Fix}(\gamma_i)} f_i d\mu,$$

where $f_i \in C_0(X)$ and $\gamma_i \in \Gamma$ for $i = 1, \dots, n$, and $\text{Fix}(\gamma_i) = \{x \in X : \gamma_i x = x\}$ is the set of fixed points of γ_i . The key observation is that τ_μ^{Fix} defined on $C_c(\mathcal{G})$ is canonical if and only if \mathcal{G} is essentially free with respect to μ (see Lemma 3.14). Using the fixed point trace, we prove the following main result of this paper which answers Question 1.1.

Theorem A (Theorem 3.16). *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

- (1) \mathcal{G} is essentially free with respect to μ ;
- (2) for any C^* -norm $\|\cdot\|_v$ on $C_c(\mathcal{G})$ dominating the reduced C^* -norm, any tracial state τ on $C_v^*(\mathcal{G})$ with the associated measure being μ is canonical and τ_μ^{Fix} can be extended to a tracial state on $C_v^*(\mathcal{G})$;
- (3) there exists a C^* -norm $\|\cdot\|_v$ on $C_c(\mathcal{G})$ dominating the reduced C^* -norm such that any tracial state τ on $C_v^*(\mathcal{G})$ with the associated measure being μ is canonical and τ_μ^{Fix} can be extended to a tracial state on $C_v^*(\mathcal{G})$;
- (4) any tracial state τ on $C^*(\mathcal{G})$ with the associated measure being μ is canonical.

On reduced crossed products of C^* -algebras, the extendibility of τ_μ^{Fix} can be reformulated using a recent result in [11] (see Proposition 4.6) as follows.

Corollary B (Corollary 5.5). *Let Γ be a discrete group acting on a locally compact Hausdorff space X with an invariant probability Radon measure μ . We consider the following conditions:*

- (1) the action is essentially free with respect to μ ;
- (2) any tracial state τ on $C_0(X) \rtimes_r \Gamma$ with the associated measure being μ is canonical;
- (3) the isotropy group Γ_x is amenable for μ -almost every $x \in X$.

Then (1) \Rightarrow (2)¹ and (2) + (3) \Rightarrow (1).

If additionally Γ is countable and X is second-countable, then (1) \Leftrightarrow (2) + (3).

We end this paper with the following result about quasidiagonal traces, which play a crucial role in the classification of simple nuclear C^* -algebras (see [36]).

Corollary C (Corollary 5.16²). *Let \mathcal{G} be a locally compact, Hausdorff, second-countable, amenable and étale groupoid, which is also essentially free. Then every (not necessarily faithful) tracial state on $C_r^*(\mathcal{G})$ is quasidiagonal.*

2. Preliminaries

Given a locally compact Hausdorff space X , we denote by $C(X)$ the set of complex-valued continuous functions on X . Recall that the *support* of a function $f \in C(X)$ is the closure of $\{x \in X : f(x) \neq 0\}$, written as $\text{supp } f$. Denote by $C_c(X)$ the set of complex-valued continuous functions with compact support, and by $C_0(X)$ the set of complex-valued continuous functions vanishing at infinity, which is the closure of $C_c(X)$ with respect to the supremum norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$.

2.1. Basic notions for groupoids

Let us start with some basic notions and terminologies about groupoids. For details we refer the reader to [31, 32].

Recall that a *groupoid* is a small category, in which every morphism is invertible. Roughly speaking, a groupoid consists of a set \mathcal{G} , a subset $\mathcal{G}^{(0)}$ called the *unit space*, two maps $s, r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ called the *source* and *range* maps respectively, a *composition law*

$$\mathcal{G}^{(2)} := \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\} \ni (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \in \mathcal{G},$$

and an *inverse* map on \mathcal{G} given by $\gamma \mapsto \gamma^{-1}$. These operations satisfy a couple of axioms, including the associativity law and the fact that elements in $\mathcal{G}^{(0)}$ act as units. For $x \in \mathcal{G}^{(0)}$, we define $\mathcal{G}^x := r^{-1}(x)$ and $\mathcal{G}_x := s^{-1}(x)$. Moreover, $\mathcal{G}_x^x = \mathcal{G}^x \cap \mathcal{G}_x$ is called the *isotropy group* at $x \in \mathcal{G}^{(0)}$. A subset $Y \subseteq \mathcal{G}^{(0)}$ is called *invariant* if $r^{-1}(Y) = s^{-1}(Y)$, and we define $\mathcal{G}_Y := s^{-1}(Y)$. For $A, B \subseteq \mathcal{G}$, we define

$$A^{-1} := \{\gamma^{-1} \in \mathcal{G} : \gamma \in A\};$$

$$AB := \{\gamma \in \mathcal{G} : \gamma = \gamma_1 \gamma_2 \text{ where } \gamma_1 \in A, \gamma_2 \in B \text{ and } s(\gamma_1) = r(\gamma_2)\}.$$

¹Note that the implication “(2) \Rightarrow (1)” in Corollary B does not hold in general (see Remark 5.6).

²We also refer the reader to Theorem 5.15 for a more general result.

A *locally compact Hausdorff groupoid* is a groupoid \mathcal{G} endowed with a locally compact and Hausdorff topology for which the composition, inversion, source and range maps are continuous with respect to the induced topologies.

We say that a locally compact Hausdorff groupoid \mathcal{G} is *étale* if the range (and hence the source) map is a local homeomorphism, i.e., for any $\gamma \in \mathcal{G}$ there exists an open neighbourhood U of γ such that $r(U)$ is open and $r|_U$ is a homeomorphism. In this case, the fibers \mathcal{G}_x and \mathcal{G}^x with the induced topologies are discrete for each $x \in \mathcal{G}^{(0)}$, and $\mathcal{G}^{(0)}$ is clopen in \mathcal{G} .

A subset A in an étale groupoid \mathcal{G} is called a *bisection* if the restrictions of s, r to A are homeomorphisms onto their respective images. It follows from definitions that all open bisections form a basis for the topology of \mathcal{G} . As a direct consequence, any function $f \in C_c(\mathcal{G})$ can be written as a linear combination of continuous functions whose supports are contained in pre-compact open bisections.

We record the following known result (see, e.g., [32, Lemma 8.4.11]). For convenience of the reader, we provide here a short proof.

Lemma 2.1. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid. Then the multiplication map $\mathcal{G}^{(2)} \rightarrow \mathcal{G}$ is open.*

Proof. Given two open subsets U and V of \mathcal{G} , we need to show that UV is also open. Without loss of generality, we can assume that U and V are open bisections with $s(U) = r(V)$. Fix $\alpha \in U$ and $\beta \in V$ with $s(\alpha) = r(\beta)$. Since the multiplication map is continuous at $(\alpha^{-1}, \alpha\beta)$, there exists an open neighbourhood $U_1 \subseteq U$ of α and an open bisection $W \subseteq r^{-1}(r(U_1))$ containing $\alpha\beta$ such that $U_1^{-1}W \subseteq V$. As $U_1 \subseteq U$ is also a bisection, we obtain that $W \subseteq UV$. ■

Convention: Throughout the paper, we always assume that \mathcal{G} is a locally compact Hausdorff and étale groupoid.

2.2. Groupoid C^* -algebras

Let us now recall different constructions of groupoid C^* -algebras and their basic properties. Given a locally compact Hausdorff and étale groupoid \mathcal{G} , the space $C_c(\mathcal{G})$ can be turned into a $*$ -algebra with the following operations: given $f, g \in C_c(\mathcal{G})$, we define their *convolution* and *involution* by

$$(f * g)(\gamma) := \sum_{\alpha \in \mathcal{G}_s(\gamma)} f(\gamma\alpha^{-1})g(\alpha),$$

$$f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

To tell the difference, we denote the point-wise product by $f \cdot g$.

Recall that for each $x \in \mathcal{G}^{(0)}$, the *left regular representation at x* , denoted by $\lambda_x : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\ell^2(\mathcal{G}_x))$, is defined as follows:

$$(\lambda_x(f)\xi)(\gamma) = \sum_{\alpha \in \mathcal{G}_x} f(\gamma\alpha^{-1})\xi(\alpha), \quad \text{where } \gamma \in \mathcal{G}_x, f \in C_c(\mathcal{G}) \text{ and } \xi \in \ell^2(\mathcal{G}_x).$$

It is routine to check that λ_x is a well-defined $*$ -homomorphism. The *reduced C^* -norm* on $C_c(\mathcal{G})$ is defined by

$$\|f\|_r := \sup_{x \in \mathcal{G}^{(0)}} \|\lambda_x(f)\|,$$

and the *reduced groupoid C^* -algebra* $C_r^*(\mathcal{G})$ is defined to be the completion of the $*$ -algebra $C_c(\mathcal{G})$ with respect to the reduced C^* -norm $\|\cdot\|_r$. It is clear that each left regular representation λ_x can be extended automatically to a $*$ -homomorphism $\lambda_x : C_r^*(\mathcal{G}) \rightarrow \mathcal{B}(\ell^2(\mathcal{G}_x))$.

We also consider the following norm on $C_c(\mathcal{G})$ defined by:

$$\|f\|_I := \max \left\{ \sup_{x \in \mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sup_{x \in \mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} |f^*(\gamma)| \right\}.$$

The completion of $C_c(\mathcal{G})$ with respect to the norm $\|\cdot\|_I$ is denoted by $L^1(\mathcal{G})$. Recall that the *maximal groupoid C^* -algebra* $C^*(\mathcal{G})$ is defined to be the completion of $C_c(\mathcal{G})$ with respect to the C^* -norm

$$\|f\|_{\max} := \sup \|\pi(f)\|,$$

where the supremum is taken over all bounded $*$ -representations π of $L^1(\mathcal{G})$. It is clear that there is a surjective $*$ -homomorphism

$$q_{\max} : C^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}),$$

which is the identity on $C_c(\mathcal{G})$. We say that \mathcal{G} has the *weak containment property* if q_{\max} is an isomorphism.

We will also consider other C^* -norms between the reduced and the maximal ones. More precisely, we say that a C^* -norm $\|\cdot\|_v$ *dominates the reduced C^* -norm* if $\|f\|_v \geq \|f\|_r$ for all $f \in C_c(\mathcal{G})$. It is worth noticing that $\|\cdot\|_v \leq \|\cdot\|_{\max}$ always holds. We denote the C^* -completion of $C_c(\mathcal{G})$ with respect to $\|\cdot\|_v$ by $C_v^*(\mathcal{G})$, called a *groupoid C^* -algebra* of \mathcal{G} . Similarly, we have a surjective $*$ -homomorphism

$$q_v : C_v^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}), \tag{2.1}$$

which is the identity on $C_c(\mathcal{G})$.

We remark that there is an inclusion map $\iota_0 : C_c(\mathcal{G}^{(0)}) \rightarrow C_c(\mathcal{G})$ given by extending functions by zero on $\mathcal{G} \setminus \mathcal{G}^{(0)}$, and it was recorded in [3, Section 2.2] that ι_0 can be extended to an isometric $*$ -homomorphism $\iota : C_0(\mathcal{G}^{(0)}) \hookrightarrow C_r^*(\mathcal{G})$, where the norm on $C_0(\mathcal{G}^{(0)})$ is the supremum norm. The same fact holds for any C^* -norm $\|\cdot\|_v$ dominating the reduced C^* -norm. Hence, we will in what follows regard $C_0(\mathcal{G}^{(0)})$ as a C^* -subalgebra in $C_v^*(\mathcal{G})$ without further explanation.

From [31, Proposition II.4.2] (see also [5, Section 2.2]) we have that any element of $C_r^*(\mathcal{G})$ can be regarded as a C_0 -function on the groupoid \mathcal{G} . Indeed, there exists a linear and contractive map $j : C_r^*(\mathcal{G}) \rightarrow C_0(\mathcal{G})$ given by

$$j(a)(\gamma) := \langle \lambda_{s(\gamma)}(a) \delta_{s(\gamma)}, \delta_\gamma \rangle_{\ell^2(\mathcal{G}_{s(\gamma)})}$$

for $a \in C_r^*(\mathcal{G})$ and $\gamma \in \mathcal{G}$. On $C_c(\mathcal{G}) \cup C_0(\mathcal{G}^{(0)})$ the map j is nothing but the identity map. The reduced groupoid C^* -algebra also admits a faithful conditional expectation $E : C_r^*(\mathcal{G}) \rightarrow C_0(\mathcal{G}^{(0)})$ defined by

$$E(a)(u) := \langle \lambda_u(a)\delta_u, \delta_u \rangle_{\ell^2(\mathcal{G}_u)}$$

for $a \in C_r^*(\mathcal{G})$ and $u \in \mathcal{G}^{(0)}$ (see, e.g., [3, Section 2.2]). Intuitively, E is given by restriction of functions in the sense that $j(E(a)) = j(a)|_{\mathcal{G}^{(0)}}$ for all $a \in C_r^*(\mathcal{G})$. Hence, it follows that $E \circ \iota = \text{Id}_{C_0(\mathcal{G}^{(0)})}$. For any C^* -norm $\|\cdot\|_v$ dominating the reduced C^* -norm, we can compose E with q_v and obtain a conditional expectation $E \circ q_v : C_v^*(\mathcal{G}) \rightarrow C_0(\mathcal{G}^{(0)})$ on $C_v^*(\mathcal{G})$.

We end this subsection with an elementary fact, and leave its proof (which is a relatively straightforward computation) to the reader.

Lemma 2.2. *For $a \in C_r^*(\mathcal{G})$, $f, g \in C_0(\mathcal{G}^{(0)})$ and $\gamma \in \mathcal{G}$, we have*

$$j(f * a * g)(\gamma) = f(r(\gamma)) \cdot j(a)(\gamma) \cdot g(s(\gamma)).$$

2.3. Tracial states

Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and $\|\cdot\|_v$ be a C^* -norm on $C_c(\mathcal{G})$ dominating the reduced C^* -norm.

Definition 2.3. A *tracial state* on the groupoid C^* -algebra $C_v^*(\mathcal{G})$ is a state $\tau : C_v^*(\mathcal{G}) \rightarrow \mathbb{C}$ satisfying $\tau(ab) = \tau(ba)$ for any $a, b \in C_v^*(\mathcal{G})$.

If τ is a tracial state on the groupoid C^* -algebra $C_v^*(\mathcal{G})$, then $\tau|_{C_0(\mathcal{G}^{(0)})}$ is a state on $C_0(\mathcal{G}^{(0)})$, which corresponds to a (positive) probability Radon measure μ on $\mathcal{G}^{(0)}$ according to the Riesz representation theorem. In other words, we have

$$\tau(f) = \int_{\mathcal{G}^{(0)}} f \, d\mu \quad \text{for any } f \in C_0(\mathcal{G}^{(0)}). \tag{2.2}$$

We call μ the *measure associated* to τ and we also denote this measure by μ_τ . It is actually invariant in the following sense (see, e.g., [21, Lemma 4.1]):

Given a bisection $B \subseteq \mathcal{G}$, we consider the homeomorphism

$$\alpha_B : s(B) \rightarrow r(B) \quad \text{given by } x \mapsto r((s|_B)^{-1}(x)) \quad \text{for } x \in s(B).$$

A Borel measure μ on $\mathcal{G}^{(0)}$ is called *invariant* (cf. [31, Definition I.3.12]) if for any open bisection B in \mathcal{G} , we have $\mu|_{r(B)} = (\alpha_B)_*(\mu|_{s(B)})$. For an invariant measure μ on $\mathcal{G}^{(0)}$, its support $\text{supp } \mu$ is an invariant subset of $\mathcal{G}^{(0)}$. We also define

$$\text{Fix}(\alpha_B) := \{x \in s(B) : \alpha_B(x) = x\},$$

which is an intersection of an open set and a closed set (hence measurable) if B is an open bisection and \mathcal{G} is étale.

Conversely, let μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then it follows from [21, Lemma 4.2] (see also [31, Proposition II.5.4]) that

$$\tau_\mu : a \mapsto \int_{\mathcal{G}^{(0)}} E(a) d\mu \quad \text{for } a \in C_r^*(\mathcal{G}) \tag{2.3}$$

is a tracial state on $C_r^*(\mathcal{G})$, called the *tracial state associated to μ* . Similarly, we can also consider the tracial state associated to μ on $C_v^*(\mathcal{G})$ (with the same notation)

$$\tau_\mu : a \mapsto \int_{\mathcal{G}^{(0)}} E(q_v(a)) d\mu \quad \text{for } a \in C_v^*(\mathcal{G}),$$

where $q_v : C_v^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$ is the canonical quotient map mentioned in (2.1).

3. Canonical tracial states and essential freeness

In this section, we study the essential freeness of étale groupoids via their canonical tracial states. Let us start with the following definition.

Definition 3.1. Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and $\| \cdot \|_v$ be a C^* -norm on $C_c(\mathcal{G})$ dominating the reduced C^* -norm. A tracial state τ on $C_v^*(\mathcal{G})$ is called *canonical* if $\tau = \tau_\mu$, where μ is the measure associated to τ on $\mathcal{G}^{(0)}$.

The following is elementary but useful in what follows.

Lemma 3.2. A tracial state τ on $C_v^*(\mathcal{G})$ is canonical if and only if $\tau(f) = 0$ for all $f \in C_c(\mathcal{G} \setminus \mathcal{G}^{(0)})$.

Proof. The forward implication is clear. Now we assume that $\tau(f) = 0$ for all $f \in C_c(\mathcal{G} \setminus \mathcal{G}^{(0)})$. As $C_c(\mathcal{G})$ is dense in $C_v^*(\mathcal{G})$, it suffices to show that $\tau(f) = \tau_\mu(f)$ for any $f \in C_c(\mathcal{G})$, where μ is the measure associated to τ . Since \mathcal{G} is étale, we have the decomposition $f = f|_{\mathcal{G}^{(0)}} + f|_{\mathcal{G} \setminus \mathcal{G}^{(0)}}$ in $C_c(\mathcal{G})$. Hence by assumption, we have $\tau(f) = \tau(f|_{\mathcal{G}^{(0)}})$, which finishes the proof by the definition of the canonical trace (2.2). ■

It is interesting to know when every tracial state on $C_v^*(\mathcal{G})$ is canonical. Recall from [21, Lemma 4.3] (see also [31, Proposition II.5.4]) that if the groupoid \mathcal{G} is principal, then every tracial state on $C_r^*(\mathcal{G})$ is canonical. We would like to weaken the condition of being principal to the following notion of essential freeness.

Definition 3.3. For a locally compact Hausdorff and étale groupoid \mathcal{G} and an invariant probability Radon measure μ on $\mathcal{G}^{(0)}$, we say that \mathcal{G} is *essentially free with respect to μ* if for any pre-compact open bisection $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$, we have $\mu(\text{Fix}(\alpha_B)) = 0$. We say that \mathcal{G} is *essentially free* if \mathcal{G} is essentially free with respect to any invariant probability Radon measure on $\mathcal{G}^{(0)}$.

The following proposition is perhaps known to experts at least for the maximal C^* -norm (see [27, Corollary 1.2]). We provide here a self-contained proof, because we cannot find the explicit statement we need in the literature.

Proposition 3.4. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and $\|\cdot\|_v$ be a C^* -norm on $C_c(\mathcal{G})$ dominating the reduced C^* -norm. If τ is a tracial state on $C_v^*(\mathcal{G})$ with the associated measure μ on $\mathcal{G}^{(0)}$ such that \mathcal{G} is essentially free with respect to μ , then τ is canonical.*

Proof. Our proof here is mainly inspired by the one for [18, Proposition 1.1]. By Lemma 3.2, it suffices to show that $\tau(f) = 0$ for any $f \in C_c(\mathcal{G} \setminus \mathcal{G}^{(0)})$. By decomposing f into its positive part and negative part, it suffices to show that $\tau(f) = 0$ for any $f \in C_c(\mathcal{G} \setminus \mathcal{G}^{(0)})$ which is point-wise non-negative. Using an argument of partitions of unity, we can additionally assume that $\text{supp } f \subseteq B$ for some pre-compact open bisection B such that $\overline{B} \subseteq B_0$ for another pre-compact open bisection $B_0 \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$.

First we suppose that $\text{supp } f \cap \{\gamma \in B : s(\gamma) = r(\gamma)\} = \emptyset$. Then for any $\gamma \in \text{supp } f$, there exists an open neighbourhood W_γ of γ such that $s(W_\gamma) \cap r(W_\gamma) = \emptyset$. Since $\text{supp } f$ is compact, we can choose a finite cover $\{W_{\gamma_1}, \dots, W_{\gamma_N}\}$ for $\text{supp } f$, and take a partition of unity $\{\rho_{\gamma_1}, \dots, \rho_{\gamma_N}\}$. Then we can write $f = \sum_{n=1}^N (\rho_n \cdot f)$, where $\rho_n \cdot f$ denotes the point-wise product of ρ_n and f as in Section 2.2. For each $n = 1, \dots, N$, take $h_n \in C_c(s(W_{\gamma_n}))$ such that $h_n|_{s(\text{supp}(\rho_n \cdot f))} = 1$ and $0 \leq h_n \leq 1$. A direct calculation as in the proof of [21, Lemma 4.3] shows that $(\rho_n \cdot f) * h_n = \rho_n \cdot f$ while $h_n * (\rho_n \cdot f) = 0$. Hence, we obtain $\tau(\rho_n \cdot f) = \tau((\rho_n \cdot f) * h_n) = \tau(h_n * (\rho_n \cdot f)) = 0$, which implies that $\tau(f) = \sum_{n=1}^N \tau(\rho_n \cdot f) = 0$, as required.

Now we suppose that $\text{supp } f \cap \{\gamma \in B : s(\gamma) = r(\gamma)\} \neq \emptyset$. Note that

$$\text{Fix}(\alpha_B) = \{x \in s(B) : \alpha_B(x) = x\} \subseteq \{x \in s(\overline{B}) : \alpha_{\overline{B}}(x) = x\} = \text{Fix}(\alpha_{\overline{B}}),$$

and $\text{Fix}(\alpha_{\overline{B}})$ is closed in $s(\overline{B})$. It follows that $\overline{\text{Fix}(\alpha_B)}$ is compact and is contained in $\text{Fix}(\alpha_{B_0})$. By the essential freeness of μ , we know that $\mu(\text{Fix}(\alpha_{B_0})) = 0$ and hence $\mu(\overline{\text{Fix}(\alpha_B)}) = 0$ as well.

Since μ is a Radon measure, it is outer regular. Hence given $\varepsilon > 0$, we can take an open set $U \subseteq \mathcal{G}^{(0)}$ containing the closure of $s(\text{supp } f) \cap \text{Fix}(\alpha_B)$ and an open set $V \supseteq \overline{U}$ such that $\mu(V) < \varepsilon$. Since $s(\text{supp } f)$ is compact and V is open, $s(\text{supp } f) \setminus V$ is also compact. Hence we can take $\rho_\varepsilon \in C_c(\mathcal{G}^{(0)})$ such that $0 \leq \rho_\varepsilon \leq 1$, $\rho_\varepsilon|_{s(\text{supp } f) \setminus V} \equiv 1$ and $\rho_\varepsilon|_{\overline{U}} \equiv 0$.

We aim to apply the argument of the second paragraph above to the function $f * \rho_\varepsilon$, which has support in B by Lemma 2.2. Hence we must first show that

$$\text{supp}(f * \rho_\varepsilon) \cap \{\gamma \in B : s(\gamma) = r(\gamma)\} = \emptyset.$$

Note that

$$\begin{aligned} s(\text{supp}(f * \rho_\varepsilon) \cap \{\gamma \in B : s(\gamma) = r(\gamma)\}) &\subseteq s(\text{supp } f \cap \{\gamma \in B : s(\gamma) = r(\gamma)\}) \\ &= s(\text{supp } f) \cap \text{Fix}(\alpha_B) \subseteq U. \end{aligned}$$

Assume that there exists $\gamma \in \text{supp}(f * \rho_\varepsilon) \cap \{\gamma \in B : s(\gamma) = r(\gamma)\}$. Since $\gamma \in (s|_B)^{-1}(U) \cap \text{supp}(f * \rho_\varepsilon)$, we can choose a net $\{\gamma_\lambda\}_\lambda$ in $(s|_B)^{-1}(U)$ converging to γ such that $(f * \rho_\varepsilon)(\gamma_\lambda) \neq 0$, then Lemma 2.2 implies that $\rho_\varepsilon(s(\gamma_\lambda)) \neq 0$ for each λ . We reach a contradiction since $s(\gamma_\lambda) \in U$ and $\rho_\varepsilon|_U \equiv 0$. Therefore, the same analysis in the second paragraph of this proof shows that $\tau(f * \rho_\varepsilon) = 0$.

Take $\eta \in C_c(B_0)$ such that $\eta|_{\overline{B}} \equiv 1$ and $\|\eta\|_\infty = 1$, and define $f_0 \in C_c(\mathcal{G}^{(0)})$ by $f_0(x) = f \circ (s|_{B_0})^{-1}(x)$ if $x \in s(B_0)$ and zero otherwise. As $\text{supp } f \subseteq B_0$, f_0 is a continuous function on $\mathcal{G}^{(0)}$ such that $\text{supp } f_0 = s(\text{supp } f)$. Then Lemma 2.2 implies that

$$(\eta * f_0)(\gamma) = \eta(\gamma) \cdot f_0(s(\gamma)) = \eta(\gamma) \cdot f(\gamma) = f(\gamma), \quad \forall \gamma \in \mathcal{G},$$

which means that $\eta * f_0 = f$. Afterwards we obtain

$$|\tau(f)| = |\tau(f) - \tau(f * \rho_\varepsilon)| = |\tau(\eta * (f_0 - f_0 \cdot \rho_\varepsilon))|. \tag{3.1}$$

Since $f_0 - f_0 \cdot \rho_\varepsilon$ is point-wise non-negative, we write

$$\eta * (f_0 - f_0 \cdot \rho_\varepsilon) = (\eta * (f_0 - f_0 \cdot \rho_\varepsilon)^{1/2}) * (f_0 - f_0 \cdot \rho_\varepsilon)^{1/2}.$$

Using the Cauchy–Schwarz inequality $|\tau(y^*x)|^2 \leq \tau(x^*x) \cdot \tau(y^*y)$ for the elements $x = (f_0 - f_0 \cdot \rho_\varepsilon)^{1/2}$ and $y^* = \eta * (f_0 - f_0 \cdot \rho_\varepsilon)^{1/2}$, we obtain

$$|\tau(\eta * (f_0 - f_0 \cdot \rho_\varepsilon))|^2 \leq \tau(f_0 - f_0 \cdot \rho_\varepsilon) \cdot \tau(\eta * (f_0 - f_0 \cdot \rho_\varepsilon) * \eta^*). \tag{3.2}$$

Using properties of tracial states, we have

$$\begin{aligned} \tau(\eta * (f_0 - f_0 \cdot \rho_\varepsilon) * \eta^*) &= \tau((f_0 - f_0 \cdot \rho_\varepsilon)^{1/2} * \eta^* * \eta * (f_0 - f_0 \cdot \rho_\varepsilon)^{1/2}) \\ &\leq \tau(f_0 - f_0 \cdot \rho_\varepsilon), \end{aligned} \tag{3.3}$$

where we use $\|\eta\|_\infty = 1$ for the second inequality. Combining (3.1), (3.2) and (3.3), we obtain

$$|\tau(f)| \leq \tau(f_0 - f_0 \cdot \rho_\varepsilon).$$

Finally, from $\mu(V) < \varepsilon$ and $\rho_\varepsilon|_{s(\text{supp } f) \setminus V} \equiv 1$ we have that

$$\tau(f_0 - f_0 \cdot \rho_\varepsilon) = \int_{\mathcal{G}^{(0)}} f_0 \cdot (1 - \rho_\varepsilon) d\mu = \int_{s(\text{supp } f) \cap V} f_0 \cdot (1 - \rho_\varepsilon) d\mu \leq \varepsilon \cdot \|f_0\|_\infty,$$

which goes to 0 as $\varepsilon \rightarrow 0$. Therefore, we conclude that $\tau(f) = 0$, as desired. ■

Remark 3.5. If $\text{Iso}(\mathcal{G}) := \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma)\}$ denotes the isotropy groupoid of an étale groupoid \mathcal{G} , then we have

$$s(\text{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}) = \bigcup \{\text{Fix}(\alpha_B) : B \text{ is a pre-compact open bisection in } \mathcal{G} \setminus \mathcal{G}^{(0)}\}.$$

If \mathcal{G} is σ -compact, then $s(\text{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)})$ is a countable union of measurable sets. In particular, it is measurable. Therefore, \mathcal{G} is in this case essentially free with respect to μ if and only if $\mu(s(\text{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)})) = 0$.

If we assume that $\mathcal{G}^{(0)}$ is compact then the convex set $M(\mathcal{G})$ of invariant probability Radon measures on $\mathcal{G}^{(0)}$ and the tracial state space $T(C_v^*(\mathcal{G}))$ of $C_v^*(\mathcal{G})$ are both compact in the weak*-topology. Then the following corollary of Proposition 3.4 generalises [1, Proposition 3.1], because almost finite ample groupoids with compact unit space are always essentially free by [23, Remark 6.6].

Corollary 3.6. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid with compact unit space which is also essentially free. Then the canonical map $\tau \mapsto \mu_\tau$ from $T(C_v^*(\mathcal{G}))$ to $M(\mathcal{G})$ is an affine homeomorphism, and hence we can identify their extreme boundaries $\partial_e T(C_v^*(\mathcal{G})) = \partial_e M(\mathcal{G})$. In particular, this holds for both maximal and reduced C^* -norms.*

In the following, we would like to study Question 1.1 for a locally compact Hausdorff and étale groupoid such that $\mathcal{G}^{(0)}$ is not necessarily compact. For that we consider an auxiliary trace $\tau_\mu^{\text{Fix}} : C_c(\mathcal{G}) \rightarrow \mathbb{C}$ associated to a given invariant probability Radon measure μ on $\mathcal{G}^{(0)}$. The key point is that τ_μ^{Fix} reveals the complete information of essential freeness with respect to μ (see Lemma 3.14 for details). Since its construction is a bit complicated, we divide it into several steps.

Firstly, for any pre-compact open bisection B and $g \in C_c(B)$ we define $g_B \in C_c(s(B))$ by $g_B(x) := g((s|_B)^{-1}(x))$ for $x \in s(B)$. Similarly, we define $g^B \in C_c(r(B))$ by $g^B(x) := g((r|_B)^{-1}(x))$ for $x \in r(B)$. For such g and B , we define

$$\tau_\mu^{\text{Fix}}(g) := \int_{\text{Fix}(\alpha_B)} g_B d\mu. \tag{3.4}$$

The following observation shows that $\tau_\mu^{\text{Fix}}(g)$ is well defined.

Lemma 3.7. *Assume that B_1, B_2 are pre-compact open bisections and $g \in C_c(B_1) \cap C_c(B_2)$. Then we have*

$$\int_{\text{Fix}(\alpha_{B_1})} g_{B_1} d\mu = \int_{\text{Fix}(\alpha_{B_2})} g_{B_2} d\mu.$$

Proof. Taking $B = B_1 \cap B_2$, it suffices to show that $\int_{\text{Fix}(\alpha_{B_i})} g_{B_i} d\mu = \int_{\text{Fix}(\alpha_B)} g_B d\mu$ for $i = 1, 2$. Note that $\text{supp}(g) \subseteq B_1 \cap B_2 = B$, and hence $\text{supp}(g_B) \subseteq s(B)$ and $g_B = g_{B_i}$ for $i = 1, 2$. Therefore, for $i = 1, 2$ we have

$$\int_{\text{Fix}(\alpha_{B_i})} g_{B_i} d\mu = \int_{\text{Fix}(\alpha_{B_i}) \cap s(B)} g_B d\mu.$$

We note that $x \in \text{Fix}(\alpha_{B_i}) \cap s(B)$ if and only if there exists $\gamma_i \in B_i$ such that $s(\gamma_i) = r(\gamma_i) = x \in s(B)$, which implies that $\gamma_i \in B$ and hence $x \in \text{Fix}(\alpha_B)$. It follows that $\text{Fix}(\alpha_{B_i}) \cap s(B) = \text{Fix}(\alpha_B)$ for $i = 1, 2$, as desired. ■

Moreover, we have the following lemma.

Lemma 3.8. *Let $g \in C_c(\mathcal{G})$ with $g = \sum_{i=1}^n g_i = \sum_{j=1}^m h_j$, where $g_i \in C_c(B_i)$ for some pre-compact open bisection B_i and $h_j \in C_c(D_j)$ for some pre-compact open bisection D_j . Then we have*

$$\sum_{i=1}^n \tau_\mu^{\text{Fix}}(g_i) = \sum_{j=1}^m \tau_\mu^{\text{Fix}}(h_j).$$

Proof. Since $K := \bigcup_{i=1}^n \text{supp}(g_i) \cup \bigcup_{j=1}^m \text{supp}(h_j)$ is compact, we can take a finite open cover $\mathcal{U} = \{U_k : k = 1, \dots, N\}$ of K such that each U_k is a pre-compact open bisection. Then we take a partition of unity $\{\rho_k : k = 1, \dots, N\}$ subordinate to \mathcal{U} such that $\sum_{k=1}^N \rho_k \equiv 1$ on K . In particular, we have $g_i = \sum_{k=1}^N (\rho_k \cdot g_i)$ for any $i = 1, \dots, n$, where $\rho_k \cdot g_i$ means the point-wise product as in Section 2.2. As both $\text{supp}(g_i)$ and $\text{supp}(\rho_k \cdot g_i)$ are contained in B_i , it follows from Lemma 3.7 that

$$\tau_\mu^{\text{Fix}}(g_i) = \int_{\text{Fix}(\alpha_{B_i})} (g_i)_{B_i} d\mu \quad \text{and} \quad \tau_\mu^{\text{Fix}}(\rho_k \cdot g_i) = \int_{\text{Fix}(\alpha_{B_i})} (\rho_k \cdot g_i)_{B_i} d\mu. \quad (3.5)$$

For each $x \in s(B_i)$, we have $(\rho_k \cdot g_i)_{B_i}(x) = \rho_k((s|_{B_i})^{-1}(x)) \cdot g_i((s|_{B_i})^{-1}(x))$. Hence, we obtain $\sum_{k=1}^N (\rho_k \cdot g_i)_{B_i} = (g_i)_{B_i}$, which together with (3.5) implies that

$$\tau_\mu^{\text{Fix}}(g_i) = \int_{\text{Fix}(\alpha_{B_i})} (g_i)_{B_i} d\mu = \sum_{k=1}^N \int_{\text{Fix}(\alpha_{B_i})} (\rho_k \cdot g_i)_{B_i} d\mu = \sum_{k=1}^N \tau_\mu^{\text{Fix}}(\rho_k \cdot g_i).$$

So we obtain

$$\sum_{i=1}^n \tau_\mu^{\text{Fix}}(g_i) = \sum_{i=1}^n \sum_{k=1}^N \tau_\mu^{\text{Fix}}(\rho_k \cdot g_i) = \sum_{k=1}^N \sum_{i=1}^n \tau_\mu^{\text{Fix}}(\rho_k \cdot g_i). \quad (3.6)$$

Similarly, we also have

$$\sum_{j=1}^m \tau_\mu^{\text{Fix}}(h_j) = \sum_{k=1}^N \sum_{j=1}^m \tau_\mu^{\text{Fix}}(\rho_k \cdot h_j). \quad (3.7)$$

For a fixed $k \in \{1, \dots, N\}$, using a similar argument as above we have

$$\begin{aligned} \sum_{i=1}^n \tau_\mu^{\text{Fix}}(\rho_k \cdot g_i) &= \tau_\mu^{\text{Fix}}\left(\sum_{i=1}^n \rho_k \cdot g_i\right) = \tau_\mu^{\text{Fix}}(\rho_k \cdot g) \\ &= \tau_\mu^{\text{Fix}}\left(\sum_{j=1}^m \rho_k \cdot h_j\right) = \sum_{j=1}^m \tau_\mu^{\text{Fix}}(\rho_k \cdot h_j). \end{aligned} \quad (3.8)$$

Finally, we conclude the proof by (3.6), (3.7) and (3.8). ■

Lemma 3.8 allows us to give the following definition of τ_μ^{Fix} on $C_c(\mathcal{G})$.

Definition 3.9. Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. The associated *fixed point trace* is the linear map $\tau_\mu^{\text{Fix}} : C_c(\mathcal{G}) \rightarrow \mathbb{C}$ defined as follows: Suppose that $g \in C_c(\mathcal{G})$ with $g = \sum_{i=1}^n g_i$ such that each $g_i \in C_c(B_i)$ for some pre-compact open bisection B_i . Then we define

$$\tau_\mu^{\text{Fix}}(g) := \sum_{i=1}^n \tau_\mu^{\text{Fix}}(g_i),$$

where $\tau_\mu^{\text{Fix}}(g_i)$ is given in (3.4).

We record the following fact, which is straightforward from the construction.

Lemma 3.10. For $f \in C_c(\mathcal{G}^{(0)})$, we have $\tau_\mu^{\text{Fix}}(f) = \int_{\mathcal{G}^{(0)}} f d\mu$.

The next lemma shows that τ_μ^{Fix} is positive and has the tracial property.

Lemma 3.11. For any $a \in C_c(\mathcal{G})$, we have $\tau_\mu^{\text{Fix}}(a^*a) = \tau_\mu^{\text{Fix}}(aa^*) \geq 0$.

Proof. Assume that $a = \sum_{i=1}^n g_i$ such that each $g_i \in C_c(B_i)$ for some pre-compact open bisection B_i . Then we have $a^*a = \sum_{i,j=1}^n g_i^* * g_j$ and $aa^* = \sum_{i,j=1}^n g_j * g_i^*$. Note that for any $\gamma \in \mathcal{G}$ and $i, j = 1, \dots, n$, we have

$$(g_j * g_i^*)(\gamma) = \sum_{\alpha \in \mathcal{G}_{s(\gamma)}} g_j(\gamma\alpha^{-1}) \cdot g_i^*(\alpha) = \sum_{\alpha \in \mathcal{G}_{s(\gamma)}} g_j(\gamma\alpha^{-1}) \cdot \overline{g_i(\alpha^{-1})}.$$

If $(g_j * g_i^*)(\gamma) \neq 0$, then there exists a unique $\alpha^{-1} \in B_i$ such that $r(\alpha^{-1}) = s(\gamma)$ and $(g_j * g_i^*)(\gamma) = g_j(\gamma\alpha^{-1}) \cdot \overline{g_i(\alpha^{-1})} \neq 0$. This shows that

$$\text{supp}(g_j * g_i^*) \subseteq \text{supp}(g_j) \cdot (\text{supp}(g_i))^{-1} \subseteq B_j \cdot B_i^{-1}.$$

Since multiplication and inversion are continuous and \mathcal{G} is Hausdorff, the set $B_j \cdot B_i^{-1}$ is pre-compact, and it follows from Lemma 2.1 that $B_j \cdot B_i^{-1}$ is also an open bisection. Moreover, for any $\gamma \in B_j \cdot B_i^{-1}$ there exist unique $\beta \in B_j$ and $\alpha \in B_i^{-1}$ such that $\gamma = \beta\alpha$, and hence $(g_j * g_i^*)(\gamma) = g_j(\beta) \cdot \overline{g_i(\alpha^{-1})}$.

It is also worth noticing that

$$s(B_j \cdot B_i^{-1}) = r((s|_{B_i})^{-1}(s(B_j) \cap s(B_i)))$$

and

$$r(B_j \cdot B_i^{-1}) = r((s|_{B_j})^{-1}(s(B_j) \cap s(B_i))).$$

Thus by definition, a direct calculation shows that $\alpha_{B_j \cdot B_i^{-1}} = \alpha_{B_j} \circ \alpha_{B_i}^{-1}$. Moreover, if $x \in \text{Fix}(\alpha_{B_j} \circ \alpha_{B_i}^{-1})$ then we take $\gamma \in B_j \cdot B_i^{-1}$ such that $s(\gamma) = x$. Writing $\gamma = \beta\alpha^{-1}$ for $\beta \in B_j$ and $\alpha \in B_i$, then $s(\alpha) = s(\beta)$ and $r(\alpha) = x = \alpha_{B_j} \circ \alpha_{B_i}^{-1}(x) = \alpha_{B_j}(s(\alpha)) = \alpha_{B_j}(s(\beta)) = r(\beta) = r(\gamma)$. So for $x \in \text{Fix}(\alpha_{B_j} \circ \alpha_{B_i}^{-1})$ we obtain

$$(g_j * g_i^*)_{B_j \cdot B_i^{-1}}(x) = (g_j)^{B_j}(x) \cdot \overline{(g_i)^{B_i}(x)}.$$

Therefore, we obtain

$$\begin{aligned} \tau_\mu^{\text{Fix}}(aa^*) &= \sum_{i,j=1}^n \tau_\mu^{\text{Fix}}(g_j * g_i^*) = \sum_{i,j=1}^n \int_{\text{Fix}(\alpha_{B_j \cdot B_i^{-1}})} (g_j * g_i^*)_{B_j \cdot B_i^{-1}}(x) d\mu(x) \\ &= \sum_{i,j=1}^n \int_{\text{Fix}(\alpha_{B_j} \circ \alpha_{B_i}^{-1})} (g_j)^{B_j}(x) \cdot \overline{(g_i)^{B_i}(x)} d\mu(x). \end{aligned} \tag{3.9}$$

On the other hand, we have that

$$\begin{aligned} \tau_\mu^{\text{Fix}}(a^*a) &= \sum_{i,j=1}^n \tau_\mu^{\text{Fix}}(g_i^* * g_j) = \sum_{i,j=1}^n \int_{\text{Fix}(\alpha_{B_i^{-1} \cdot B_j})} (g_i^* * g_j)_{B_i^{-1} \cdot B_j}(x) d\mu(x) \\ &= \sum_{i,j=1}^n \int_{\alpha_{B_i} \text{Fix}(\alpha_{B_i}^{-1} \circ \alpha_{B_j})} (g_i^* * g_j)_{B_i^{-1} \cdot B_j} \circ \alpha_{B_i}^{-1}(x) d\mu(x), \end{aligned} \tag{3.10}$$

where the last equality holds since μ is invariant. For $x \in \alpha_{B_i} \text{Fix}(\alpha_{B_i}^{-1} \circ \alpha_{B_j})$, we have $\alpha_{B_i}^{-1} \circ \alpha_{B_j} \circ \alpha_{B_i}^{-1}(x) = \alpha_{B_i}^{-1}(x)$, which implies that $x = \alpha_{B_j} \circ \alpha_{B_i}^{-1}(x)$. Hence, we conclude that $\alpha_{B_i} \text{Fix}(\alpha_{B_i}^{-1} \circ \alpha_{B_j}) = \text{Fix}(\alpha_{B_j} \circ \alpha_{B_i}^{-1})$. Let $y := \alpha_{B_i}^{-1}(x)$, then we have $y \in s(B_j) \cap s(B_i)$. Take the unique $\beta' \in B_j$ with $s(\beta') = y$, and the unique $\alpha' \in B_i$ with $s(\alpha') = y$. Hence, we have $r(\alpha') = x$. As $x = \alpha_{B_j}(y) = \alpha_{B_j}(s(\beta')) = r(\beta')$, we obtain

$$\begin{aligned} (g_i^* * g_j)_{B_i^{-1} \cdot B_j} \circ \alpha_{B_i}^{-1}(x) &= (g_i^* * g_j)_{B_i^{-1} \cdot B_j}(y) = \overline{g_i(\alpha')} \cdot g_j(\beta') \\ &= \overline{(g_i)^{B_i}(x)} \cdot (g_j)^{B_j}(x). \end{aligned}$$

From (3.10) we obtain

$$\tau_\mu^{\text{Fix}}(a^*a) = \sum_{i,j=1}^n \int_{\text{Fix}(\alpha_{B_j} \circ \alpha_{B_i}^{-1})} \overline{(g_i)^{B_i}(x)} \cdot (g_j)^{B_j}(x) d\mu(x). \tag{3.11}$$

By (3.9) and (3.11) we conclude $\tau_\mu^{\text{Fix}}(aa^*) = \tau_\mu^{\text{Fix}}(a^*a)$, as desired.

Now we aim to show that $\tau_\mu^{\text{Fix}}(aa^*) \geq 0$. By (3.9) we have

$$\begin{aligned} \tau_\mu^{\text{Fix}}(aa^*) &= \sum_{i,j=1}^n \int_{\text{Fix}(\alpha_{B_j} \circ \alpha_{B_i}^{-1})} (g_j)^{B_j}(x) \cdot \overline{(g_i)^{B_i}(x)} d\mu(x) \\ &= \int_{\mathcal{G}^{(0)}} \sum_{i,j=1}^n \chi_{B_{ij}}(x) \cdot (g_j)^{B_j}(x) \cdot \overline{(g_i)^{B_i}(x)} d\mu(x), \end{aligned}$$

where $B_{ij} := \{x \in r(B_i) \cap r(B_j) : \alpha_{B_i}^{-1}(x) = \alpha_{B_j}^{-1}(x)\}$. Let us consider the function

$$F(x) := \sum_{i,j=1}^n \chi_{B_{ij}}(x) \cdot (g_j)^{B_j}(x) \cdot \overline{(g_i)^{B_i}(x)} \quad \text{for } x \in \mathcal{G}^{(0)}.$$

It suffices to show that $F(x) \geq 0$ for each $x \in \mathcal{G}^{(0)}$. Fix an $x \in \mathcal{G}^{(0)}$, and let

$$I := \{i \in \{1, \dots, n\} : x \in r(B_i)\}.$$

Note that there is an equivalence relation on elements of I given by $i \sim j$ if and only if $x \in B_{ij}$. Hence we can decompose $I = I_1 \sqcup I_2 \sqcup \dots \sqcup I_N$ such that for any $i, j \in I_k$ (where $k = 1, \dots, N$) we have $\alpha_{B_i}^{-1}(x) = \alpha_{B_j}^{-1}(x)$, and for any $i \in I_k$ and $j \in I_l$ where $k \neq l$ and $k, l = 1, \dots, N$ we have $\alpha_{B_i}^{-1}(x) \neq \alpha_{B_j}^{-1}(x)$. Hence, we have

$$F(x) = \sum_{k=1}^N \sum_{i,j \in I_k} (g_j)^{B_j}(x) \cdot \overline{(g_i)^{B_i}(x)} = \sum_{k=1}^N \left| \sum_{i \in I_k} (g_i)^{B_i}(x) \right|^2 \geq 0,$$

where we use the elementary calculation $|\sum_{i=1}^n \lambda_i|^2 = \sum_{i,j=1}^n \lambda_i \overline{\lambda_j}$ for $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ for the second equality. So we have completed the proof. ■

We also need the following property of τ_μ^{Fix} .

Lemma 3.12. *Let $\{u_i\}_{i \in I} \subseteq C_c(\mathcal{G}^{(0)})$ be any approximate unit for $C_0(\mathcal{G}^{(0)})$. Then we have $\lim_{i \in I} \tau_\mu^{\text{Fix}}(u_i) = 1$.*

Proof. By Lemma 3.10, $\tau_\mu^{\text{Fix}}|_{C_c(\mathcal{G}^{(0)})}$ can be extended to a tracial state on $C_0(\mathcal{G}^{(0)})$ (with the same notation). Hence, we have $\lim_{i \in I} \tau_\mu^{\text{Fix}}(u_i) = \|\tau_\mu^{\text{Fix}}|_{C_0(\mathcal{G}^{(0)})}\| = 1$ (see, e.g., [24, Theorem 3.3.3]). ■

Using the GNS construction together with Lemma 3.12, we obtain the following.

Corollary 3.13. *The map $\tau_\mu^{\text{Fix}} : C_c(\mathcal{G}) \rightarrow \mathbb{C}$ defined in Definition 3.9 can be extended to a tracial state on $C^*(\mathcal{G})$, still denoted by τ_μ^{Fix} .*

Proof. Define (\cdot, \cdot) on $C_c(\mathcal{G})$ by $(a, b) := \tau_\mu^{\text{Fix}}(b^*a)$, and set $N := \{a \in C_c(\mathcal{G}) : (a, a) = 0\}$. Set $\langle \cdot, \cdot \rangle$ on $C_c(\mathcal{G})/N$ by $\langle [a], [b] \rangle := (a, b)$ for $a, b \in C_c(\mathcal{G})$, which is well defined. Then let \mathcal{H} be the completion of $C_c(\mathcal{G})/N$ with respect to $\langle \cdot, \cdot \rangle$, which is a Hilbert space. Also, define the $*$ -representation $\pi : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$ by $\pi(a)([b]) := [ab]$, where $a, b \in C_c(\mathcal{G})$. By definition, π can be extended to a $*$ -representation (with the same notation) $\pi : C^*(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$. For each compact $K \subseteq \mathcal{G}^{(0)}$, choose $\rho_K \in C_c(\mathcal{G}^{(0)})$ with range in $[0, 1]$ and $\rho_K|_K \equiv 1$. Then $\{\rho_K : K \text{ is a compact subset of } \mathcal{G}^{(0)}\}$ forms an approximate unit for $C_0(\mathcal{G}^{(0)})$.

We claim that $\{[\rho_K^{1/2}]\}_K$ is a Cauchy net in \mathcal{H} . In fact, we have

$$\begin{aligned} \|[\rho_K^{1/2}] - [\rho_L^{1/2}]\|_{\mathcal{H}}^2 &= \tau_\mu^{\text{Fix}}((\rho_K^{1/2} - \rho_L^{1/2})(\rho_K^{1/2} - \rho_L^{1/2})) \\ &= \tau_\mu^{\text{Fix}}(\rho_K) + \tau_\mu^{\text{Fix}}(\rho_L) - 2\tau_\mu^{\text{Fix}}(\rho_K^{1/2} \rho_L^{1/2}). \end{aligned}$$

Taking limits for K and L , Lemma 3.12 implies that $\tau_\mu^{\text{Fix}}(\rho_K) \rightarrow 1$ and $\tau_\mu^{\text{Fix}}(\rho_L) \rightarrow 1$. Moreover, Lemma 3.10 shows that

$$\tau_\mu^{\text{Fix}}(\rho_K^{1/2} \rho_L^{1/2}) = \int_{\mathcal{G}^{(0)}} \rho_K^{1/2} \rho_L^{1/2} d\mu \rightarrow 1$$

since μ is a probability Radon measure. This concludes the proof that $\{[\rho_K^{1/2}]\}_K$ is a Cauchy net in \mathcal{H} , and hence converges to a unit vector $\xi \in \mathcal{H}$. For $a \in C^*(\mathcal{G})$, we define $\tau_\mu^{\text{Fix}}(a) := \langle \pi(a)\xi, \xi \rangle$. It is routine to check that τ_μ^{Fix} is a tracial state on $C^*(\mathcal{G})$ which extends $\tau_\mu^{\text{Fix}} : C_c(\mathcal{G}) \rightarrow \mathbb{C}$. ■

The following lemma is a crucial step towards an answer to Question 1.1.

Lemma 3.14. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

- (1) \mathcal{G} is essentially free with respect to μ ;
- (2) $\tau_\mu^{\text{Fix}}(a) = \tau_\mu(a)$ for any $a \in C_c(\mathcal{G})$.

Proof. “(1) \Rightarrow (2)” follows directly from definitions. For “(2) \Rightarrow (1)”: If not, then there exists a pre-compact open bisection $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ such that $\mu(\text{Fix}(\alpha_B)) \neq 0$. Then for any $g \in C_c(B)$ with $g > 0$, we have $\tau_\mu^{\text{Fix}}(g) = \int_{\text{Fix}(\alpha_B)} g_B d\mu > 0$, while $\tau_\mu(g) = \int_{\mathcal{G}^{(0)}} E(g)d\mu = 0$. This leads to a contradiction. ■

By Proposition 3.4 and Lemma 3.14 we obtain the following.

Corollary 3.15. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid with a C^* -norm $\|\cdot\|_v$ on $C_c(\mathcal{G})$ dominating the reduced C^* -norm, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Assume that $\tau_\mu = \tau_\mu^{\text{Fix}}$ on $C_c(\mathcal{G})$. Then any tracial state on $C_v^*(\mathcal{G})$ with the associated measure being μ is canonical.*

Finally, we obtain the following answer to Question 1.1.

Theorem 3.16. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

- (1) \mathcal{G} is essentially free with respect to μ ;
- (2) for any C^* -norm $\|\cdot\|_v$ on $C_c(\mathcal{G})$ dominating the reduced C^* -norm, any tracial state τ on $C_v^*(\mathcal{G})$ with the associated measure being μ is canonical and τ_μ^{Fix} can be extended to a tracial state on $C_v^*(\mathcal{G})$;
- (3) there exists a C^* -norm $\|\cdot\|_v$ on $C_c(\mathcal{G})$ dominating the reduced C^* -norm such that any tracial state τ on $C_v^*(\mathcal{G})$ with the associated measure being μ is canonical and τ_μ^{Fix} can be extended to a tracial state on $C_v^*(\mathcal{G})$;
- (4) any tracial state τ on $C^*(\mathcal{G})$ with the associated measure being μ is canonical.

Proof. “(1) \Rightarrow (2)”: The first half of the sentence follows from Proposition 3.4. For the second statement, Lemma 3.14 shows that $\tau_\mu^{\text{Fix}}(a) = \tau_\mu(a)$ for $a \in C_c(\mathcal{G})$. Since τ_μ can be extended to a tracial state on $C_v^*(\mathcal{G})$, we conclude (2).

“(2) \Rightarrow (3)” and “(2) \Rightarrow (4)” are trivial.

“(3) \Rightarrow (1)”: By (3), τ_μ^{Fix} can be extended to a tracial state on $C_v^*(\mathcal{G})$. We note that the associated measure of τ_μ^{Fix} is μ by Lemma 3.10. It follows from (3) again that $\tau_\mu^{\text{Fix}} = \tau_\mu$. Finally, we conclude (1) by Lemma 3.14.

“(4) \Rightarrow (1)”: By Corollary 3.13, τ_μ^{Fix} can be extended to a tracial state on $C^*(\mathcal{G})$. We note that the associated measure of τ_μ^{Fix} is μ by Lemma 3.10. From assumption (4), we obtain $\tau_\mu^{\text{Fix}} = \tau_\mu$. Finally, we conclude (1) by Lemma 3.14. ■

Recall that a groupoid \mathcal{G} has the *weak containment property* if $C^*(\mathcal{G}) \cong C_r^*(\mathcal{G})$ canonically³. In this case, we obtain the following directly from Corollary 3.13 and Theorem 3.16.

Corollary 3.17. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid with the weak containment property, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

- (1) \mathcal{G} is essentially free with respect to μ ;
- (2) any tracial state τ on $C_r^*(\mathcal{G})$ with the associated measure being μ is canonical.

4. Extension of τ_μ^{Fix}

This section is devoted to discussing when the fixed point trace τ_μ^{Fix} introduced in Section 3 can be extended to a tracial state on the reduced groupoid C^* -algebra $C_r^*(\mathcal{G})$. For future use, we need a decomposition formula for general tracial states essentially from [11, 27].

Let μ be a probability Radon measure on $\mathcal{G}^{(0)}$. Assume that for μ -almost everywhere $x \in \mathcal{G}^{(0)}$, we are given a state τ_x on $C^*(\mathcal{G}_x^x)$. Denote the canonical unitary generators of $C^*(\mathcal{G}_x^x)$ by u_γ , for $\gamma \in \mathcal{G}_x^x$. Recall that the field of states $\{\tau_x\}_{x \in \mathcal{G}^{(0)}}$ is μ -measurable if for any $f \in C_c(\mathcal{G})$, the function

$$\mathcal{G}^{(0)} \rightarrow \mathbb{C}, \quad x \mapsto \sum_{\gamma \in \mathcal{G}_x^x} f(\gamma)\tau_x(u_\gamma)$$

is μ -measurable.

We recall the following special case of [27, Theorem 1.1 and Theorem 1.3].

Proposition 4.1 ([27]). *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and τ be a tracial state on $C^*(\mathcal{G})$ with the associated measure μ . Then there exists a unique μ -measurable field $\{\tau_x\}_{x \in \mathcal{G}^{(0)}}$, where τ_x is a tracial state on $C^*(\mathcal{G}_x^x)$ such that*

$$\tau(f) = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x^x} f(\gamma)\tau_x(u_\gamma)d\mu(x) \quad \text{for } f \in C_c(\mathcal{G}). \tag{4.1}$$

In the case of τ_μ^{Fix} , we can directly calculate its associated measurable field of states as follows.

³Amenable groupoids have the weak containment property, but there are also non-amenable groupoids with the weak containment property [35].

Lemma 4.2. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. For each $x \in \mathcal{G}^{(0)}$, we denote by $\tau_x^{\text{triv}} : C_c(\mathcal{G}_x^x) \rightarrow \mathbb{C}$ the trivial representation of \mathcal{G}_x^x . Then for every $f \in C_c(\mathcal{G})$, we have*

$$\tau_\mu^{\text{Fix}}(f) = \int_{\mathcal{G}^{(0)}} \tau_x^{\text{triv}}(\eta_x(f)) d\mu(x), \tag{4.2}$$

where $\eta_x : C_c(\mathcal{G}) \rightarrow C_c(\mathcal{G}_x^x)$, $f \mapsto f|_{\mathcal{G}_x^x}$ is the restriction map for each $x \in \mathcal{G}^{(0)}$.

Proof. It suffices to verify (4.2) for $g \in C_c(B)$, where B is a pre-compact open bisection. By definition, $\tau_x^{\text{triv}}(\eta_x(g)) \neq 0$ if and only if there exists $\gamma \in B$ such that $s(\gamma) = r(\gamma) = x$ and $g(\gamma) \neq 0$. This happens if and only if $s(\gamma) \in \text{Fix}(\alpha_B)$ and $g(\gamma) \neq 0$. Hence, we obtain

$$\int_{\mathcal{G}^{(0)}} \tau_x^{\text{triv}}(\eta_x(g)) d\mu(x) = \int_{\text{Fix}(\alpha_B)} g((s|_B)^{-1}(x)) d\mu(x) = \int_{\text{Fix}(\alpha_B)} g_B(x) d\mu(x) = \tau_\mu^{\text{Fix}}(g),$$

which concludes the proof of (4.2). The argument also shows that for such $g \in C_c(B)$, the map $x \mapsto \tau_x^{\text{triv}}(\eta_x(g)) = \chi_{\text{Fix}(\alpha_B)}(x) \cdot g((s|_B)^{-1}(x))$ is μ -measurable. Hence, the family $\{\tau_x^{\text{triv}}\}_{x \in \mathcal{G}^{(0)}}$ is μ -measurable. ■

Remark 4.3. It is worth noticing that we do not need second-countability of \mathcal{G} in the hypothesis of Lemma 4.2. The reason is that we can simply find the associated measurable field of tracial states $\{\tau_x^{\text{triv}}\}_{x \in \mathcal{G}^{(0)}}$ and then directly verify (4.2) without using Renault’s disintegration theorem. However, Renault’s disintegration theorem was used in the proof of Proposition 4.1.

Remark 4.4. S. Neshveyev has pointed out to us that the fixed point trace τ_μ^{Fix} in the current paper coincides with φ''_μ considered in [28, equation (2.3)] by Lemma 4.2.

Proposition 4.1 and Lemma 4.2 deal with tracial states on the maximal groupoid C^* -algebra $C^*(\mathcal{G})$. To study the reduced case, we need to consult the newly developed tool in [11]. More precisely, we need an exotic C^* -norm $\|\cdot\|_e$ on $C_c(\mathcal{G}_x^x)$ for each $x \in \mathcal{G}^{(0)}$ introduced in [11, Definition 2.1] which dominates the reduced C^* -norm. If we fix $x \in \mathcal{G}^{(0)}$, then we have the restriction map

$$\eta_x : C_c(\mathcal{G}) \rightarrow C_c(\mathcal{G}_x^x), \quad f \mapsto f|_{\mathcal{G}_x^x}.$$

By [11, Lemma 1.2], η_x extends to completely positive contractions

$$\vartheta_x : C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}_x^x) \quad \text{and} \quad \vartheta_{x,r} : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}_x^x).$$

According to [11, Theorem 2.4], the exotic C^* -norm $\|\cdot\|_e$ can be (equivalently) defined as follows: for each $h \in C_c(\mathcal{G}_x^x)$, we define

$$\|h\|_e := \inf\{\|f\|_r : f \in C_c(\mathcal{G}) \text{ and } \eta_x(f) = h\}.$$

Denote by $C_e^*(\mathcal{G}_x^x)$ the C^* -completion of $C_c(\mathcal{G}_x^x)$ with respect to $\|\cdot\|_e$.

Proposition 4.5 ([11, Proposition 3.1]). *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and τ be a tracial state on $C^*(\mathcal{G})$ with the associated measure μ on $\mathcal{G}^{(0)}$. Then τ factors through $C_r^*(\mathcal{G})$ if and only if for the associated μ -measurable field of tracial states $\{\tau_x\}_{x \in \mathcal{G}^{(0)}}$ (as in Proposition 4.1), τ_x factors through $C_e^*(\mathcal{G}_x^x)$ (denoted by $\tau_{x,e}$) for μ -almost every $x \in \mathcal{G}^{(0)}$.*

Combining Lemma 4.2 with Proposition 4.5, we reach the following.

Proposition 4.6. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then τ_μ^{Fix} extends to a tracial state on $C_r^*(\mathcal{G})$ if and only if the trivial representation τ_x^{triv} on $C_c(\mathcal{G}_x^x)$ extends to a tracial state on $C_e^*(\mathcal{G}_x^x)$ for μ -almost every $x \in \mathcal{G}^{(0)}$.*

Consequently, we have the following.

Corollary 4.7. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and μ be an invariant probability Rodan measure on $\mathcal{G}^{(0)}$. Then \mathcal{G} is essentially free with respect to μ if and only if the following conditions hold:*

- \mathcal{G}_x^x is amenable for μ -almost all $x \in \mathcal{G}^{(0)}$;
- for any tracial state τ on $C_r^*(\mathcal{G})$ with associated measure μ , $\tau_{x,e}$ is faithful on $C_e^*(\mathcal{G}_x^x)$ for μ -almost all $x \in \mathcal{G}^{(0)}$.

Proof. By definition, \mathcal{G} is essentially free with respect to μ if and only if \mathcal{G}_x^x is trivial for μ -almost all $x \in \mathcal{G}^{(0)}$, which concludes the proof of the necessity. For sufficiency, we consider the fixed point trace τ_μ^{Fix} . By Proposition 4.6 and the assumption of amenability of \mathcal{G}_x^x for μ -almost all $x \in \mathcal{G}^{(0)}$, τ_μ^{Fix} extends to a tracial state on $C_r^*(\mathcal{G})$. Moreover, for $x \in \mathcal{G}^{(0)}$, $(\tau_\mu^{\text{Fix}})_{x,e}$ is the trivial representation by Lemma 4.2. Hence, it is faithful if and only if it is injective, which is equivalent to that \mathcal{G}_x^x is trivial. ■

As another direct corollary of Proposition 4.5, we also obtain the following.

Corollary 4.8. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and τ be a tracial state on $C_r^*(\mathcal{G})$ with the associated measure μ on $\mathcal{G}^{(0)}$. If $\|\cdot\|_e = \|\cdot\|_r$ on $C_c(\mathcal{G}_x^x)$ for μ -almost every $x \in \mathcal{G}^{(0)}$, then for the associated μ -measurable field of tracial states $\{\tau_x\}_{x \in \mathcal{G}^{(0)}}$ (as in Proposition 4.1), τ_x factors through a state $\tau_{x,r}$ on $C_r^*(\mathcal{G}_x^x)$ for μ -almost every $x \in \mathcal{G}^{(0)}$.*

For future use, we record the following decomposition formula extending (4.1) for general elements in the reduced groupoid C^* -algebra $C_r^*(\mathcal{G})$.

Proposition 4.9. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and τ be a tracial state on $C_r^*(\mathcal{G})$ with the associated measure μ on $\mathcal{G}^{(0)}$. Assume that for μ -almost every $x \in \mathcal{G}^{(0)}$, we have $\|\cdot\|_e = \|\cdot\|_r$ on $C_c(\mathcal{G}_x^x)$.*

Then for $a \in C_r^*(\mathcal{G})$, we have

$$\tau(a) = \int_{\mathcal{G}^{(0)}} \tau_{x,r}(\vartheta_{x,r}(a))d\mu(x), \tag{4.3}$$

where $\tau_{x,r}$ is the associated tracial state on $C_r^*(\mathcal{G}_x^x)$ in Corollary 4.8.

Proof. By Proposition 4.1 and Corollary 4.8, we know that (4.3) holds for any $f \in C_c(\mathcal{G})$. Given $a \in C_r^*(\mathcal{G})$, we take a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C_c(\mathcal{G})$ such that $\|a - f_n\|_r \rightarrow 0$ as $n \rightarrow \infty$. For each $x \in \mathcal{G}^{(0)}$, we have that

$$|\tau_{x,r}(\vartheta_{x,r}(f_n)) - \tau_{x,r}(\vartheta_{x,r}(a))| \leq \|\vartheta_{x,r}(f_n - a)\|_r \leq \|f_n - a\|_r \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$|\tau_{x,r}(\vartheta_{x,r}(f_n))| \leq \|f_n\|_r \leq \|f_n - a\|_r + \|a\|_r \leq 2\|a\|_r$$

for sufficiently large n . Hence, Lebesgue’s dominated convergence theorem implies that

$$\int_{\mathcal{G}^{(0)}} \tau_{x,r}(\vartheta_{x,r}(f_n))d\mu(x) \rightarrow \int_{\mathcal{G}^{(0)}} \tau_{x,r}(\vartheta_{x,r}(a))d\mu(x) \quad \text{as } n \rightarrow \infty.$$

Since τ is continuous, we have $\tau(f_n) \rightarrow \tau(a)$ as $n \rightarrow \infty$. Therefore, we have completed the proof. ■

Using an identical argument as in the proof of Proposition 4.9, (4.1) can be always extended from $C_c(\mathcal{G})$ to $C^*(\mathcal{G})$ without requiring $\|\cdot\|_e = \|\cdot\|_r$ on $C_c(\mathcal{G}_x^x)$ for μ -almost every $x \in \mathcal{G}^{(0)}$. More precisely, we have the following.

Corollary 4.10. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is second-countable, and τ be a tracial state on $C^*(\mathcal{G})$ with the associated measure μ . Then there exists a unique μ -measurable field $\{\tau_x\}_{x \in \mathcal{G}^{(0)}}$, where τ_x is a tracial state on $C^*(\mathcal{G}_x^x)$ such that*

$$\tau(a) = \int_{\mathcal{G}^{(0)}} \tau_x(\vartheta_x(a))d\mu(x) \quad \text{for } a \in C^*(\mathcal{G}).$$

5. Applications

5.1. Transformation groupoids

In this subsection, we apply the results in previous sections to transformation groupoids. If Γ is a discrete group acting on a locally compact Hausdorff space X , then we form the *transformation groupoid* $X \rtimes \Gamma$. In this case, the reduced groupoid C^* -algebra $C_r^*(X \rtimes \Gamma)$ is $*$ -isomorphic to the reduced crossed product $C_0(X) \rtimes_r \Gamma$, and the maximal groupoid C^* -algebra $C^*(X \rtimes \Gamma)$ is $*$ -isomorphic to the maximal crossed product $C_0(X) \rtimes \Gamma$.

For each $\gamma \in \Gamma$, we use the same notation to denote the associated homeomorphism $\gamma : X \ni x \mapsto \gamma x \in X$. Therefore, a probability Radon measure μ on X is invariant as defined in Section 2.3 if and only if γ is μ -preserving for each $\gamma \in \Gamma$.

The notion of essential freeness for transformation groupoids can be easily simplified as follows.

Lemma 5.1. *Let Γ be a discrete group acting on a locally compact Hausdorff space X with an invariant probability Radon measure μ on X . Then the transformation groupoid $X \rtimes \Gamma$ is essentially free with respect to μ (in the sense of Definition 3.3) if and only if the action is essentially free with respect to μ in the sense that $\mu(\text{Fix}(\gamma)) = 0$ for every $\gamma \neq 1$ in Γ .*

Proof. The necessity follows from the inner regularity of the Radon measure μ , while the sufficiency follows from the fact that every pre-compact bisection in $X \rtimes \Gamma$ is contained in $X \times F$ for some finite $F \subseteq \Gamma$. ■

Moreover, one can calculate directly that the fixed point trace τ_μ^{Fix} has the following form:

$$\tau_\mu^{\text{Fix}} : C_c(\Gamma, C_0(X)) \rightarrow \mathbb{C}, \quad \sum_{i=1}^n f_i \gamma_i \mapsto \sum_{i=1}^n \int_{\text{Fix}(\gamma_i)} f_i d\mu, \tag{5.1}$$

where $f_i \in C_0(X)$ and $\gamma_i \in \Gamma$ for $i = 1, \dots, n$.

In the maximal crossed products, Theorem 3.16 recovers [14, Theorem 2.7].

Proposition 5.2. *Let Γ be a discrete group acting on a locally compact Hausdorff space X with an invariant probability Radon measure μ on X . Then the following are equivalent:*

- (1) *the action is essentially free with respect to μ ;*
- (2) *any tracial state τ on $C_0(X) \rtimes \Gamma$ with the associated measure being μ is canonical.*

In the reduced crossed products, Theorem 3.16 can be translated as follows:

Proposition 5.3. *Let Γ be a discrete group acting on a locally compact Hausdorff space X with an invariant probability Radon measure μ on X . Then the following are equivalent:*

- (1) *the action is essentially free with respect to μ ;*
- (2) *any tracial state τ on $C_0(X) \rtimes_r \Gamma$ with the associated measure being μ is canonical and τ_μ^{Fix} can be extended to a tracial state on $C_0(X) \rtimes_r \Gamma$.*

Using the discussions in Section 4, we obtain the following.

Lemma 5.4. *Let Γ be a discrete group acting on a locally compact Hausdorff space X with an invariant probability Radon measure μ . Consider the following conditions:*

- (1) τ_μ^{Fix} *extends to a tracial state on $C_0(X) \rtimes_r \Gamma$;*

(2) for μ -almost every $x \in X$, the isotropy group $\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$ is amenable.

Then (2) \Rightarrow (1).

If Γ is countable and X is second-countable, then we actually have (1) \Leftrightarrow (2).

Proof. First, we show that “(2) \Rightarrow (1)”. Recall from Lemma 4.2 (see also Remark 4.3) that

$$\tau_\mu^{\text{Fix}}(f) = \int_{\mathcal{G}(0)} \tau_x^{\text{triv}}(\eta_x(f)) d\mu(x),$$

holds for any $f \in C_c(\mathcal{G})$. By (2), the trivial representation τ_x^{triv} on $C_c(\Gamma_x)$ extends to a state $\tau_{x,r}^{\text{triv}}$ on $C_r^*(\Gamma_x)$ for μ -almost every $x \in X$. By the same argument as in the proof of Proposition 4.9 without requiring the second-countability of $X \rtimes \Gamma$ (see Remark 4.3), the map

$$C_0(X) \rtimes_r \Gamma \rightarrow \mathbb{C}, \quad a \mapsto \int_{\mathcal{G}(0)} \tau_{x,r}^{\text{triv}}(\vartheta_{x,r}(a)) d\mu(x)$$

is well defined and extends τ_μ^{Fix} .

Now we show that “(1) \Rightarrow (2)” if Γ is countable and X is second-countable. By Proposition 4.6, τ_μ^{Fix} extends to a tracial state on $C_0(X) \rtimes_r \Gamma$ if and only if for μ -almost every $x \in X$, the trivial representation on $C_c((X \rtimes \Gamma)_x^x)$ extends to a state on $C_e^*((X \rtimes \Gamma)_x^x)$. Note that $(X \rtimes \Gamma)_x^x$ is isomorphic to Γ_x , and it follows from [11, Proposition 2.10] that $\|\cdot\|_e = \|\cdot\|_r$ on $C_c((X \rtimes \Gamma)_x^x)$. It is also known that the trivial representation on $C_c(\Gamma_x)$ extends to a state on $C_r^*(\Gamma_x)$ if and only if Γ_x is an amenable group. Thus, we have concluded the proof. ■

Combining the previous lemma with Proposition 5.3, we have obtained the main result of this subsection.

Corollary 5.5. *Let Γ be a discrete group acting on a locally compact Hausdorff space X with an invariant probability Radon measure μ . We consider the following conditions:*

- (1) *the action is essentially free with respect to μ ;*
- (2) *any tracial state τ on $C_0(X) \rtimes_r \Gamma$ with the associated measure being μ is canonical;*
- (3) *the isotropy group Γ_x is amenable for μ -almost every $x \in X$.*

Then (1) \Rightarrow (2) and (2) + (3) \Rightarrow (1). In particular, if the action is essentially free then any tracial state τ on $C_0(X) \rtimes_r \Gamma$ is canonical.

If additionally Γ is countable and X is second-countable, then (1) \Leftrightarrow (2) + (3).

Remark 5.6. Note that “(2) \Rightarrow (1)” in Corollary 5.5 does not hold in general. Indeed, it was shown in [9, Corollary 1.4] that if the amenable radical of Γ (i.e., the largest amenable normal subgroup of Γ) is trivial (e.g., Γ is C^* -simple), then any tracial state on the reduced crossed product is canonical. Actually, it was shown in [34, Corollary 1.12] that any tracial state on the reduced crossed product is canonical if and only if the action of the amenable radical of Γ on X is essentially free. On the other hand, trivial actions by non-trivial groups cannot be essentially free.

Finally, we focus on the case of trivial actions. In this case, each measure on X is invariant. Moreover, we have

$$\tau_\mu^{\text{Fix}}\left(\sum_{i=1}^n f_i \gamma_i\right) = \sum_{i=1}^n \int_{\text{Fix}(\gamma_i)} f_i d\mu = \sum_{i=1}^n \int_X f_i d\mu, \quad \text{for } \sum_{i=1}^n f_i \gamma_i \in C_c(\Gamma, C_0(X)).$$

This shows that on $C_c(\Gamma, C_0(X)) \cong C_c(\Gamma) \otimes C_0(X)$ we have

$$\tau_\mu^{\text{Fix}} = \tau^{\text{triv}} \otimes \tau_\mu,$$

where τ^{triv} is the trivial representation of $C_c(\Gamma)$, and $\tau_\mu(f) = \int_X f d\mu$ for $f \in C_0(X)$. It is worth noticing that τ_μ is the same as (2.3) in the special case when $\mathcal{G} = X$.

Lemma 5.7. *Let Γ be a discrete group acting trivially on a compact Hausdorff space X with a probability Radon measure μ . Then the following are equivalent:*

- (1) τ_μ^{Fix} can be extended to a tracial state on $C(X) \rtimes_r \Gamma$;
- (2) Γ is amenable;
- (3) $X \rtimes \Gamma$ has the weak containment property.

In particular, if Γ is not amenable then τ_μ^{Fix} cannot be extended to a tracial state on $C(X) \rtimes_r \Gamma$.

Proof. “(1) \Leftrightarrow (2)”: In this case, we already know that $\tau_\mu^{\text{Fix}} = \tau^{\text{triv}} \otimes \tau_\mu$ and $C(X) \rtimes_r \Gamma \cong C_r^*(\Gamma) \otimes C(X)$. So τ_μ^{Fix} can be extended to a tracial state on $C(X) \rtimes_r \Gamma$ if and only if τ^{triv} can be extended to a tracial state on $C_r^*(\Gamma)$, which is well known to be equivalent to amenability of Γ .

“(2) \Leftrightarrow (3)”: If Γ is amenable, then $X \rtimes \Gamma$ is amenable and hence has the weak containment property. Conversely, if we assume that $X \rtimes \Gamma$ has the weak containment property, then the canonical quotient map $C(X) \otimes_{\max} C^*(\Gamma) = C(X) \rtimes \Gamma \rightarrow C(X) \rtimes_r \Gamma = C(X) \otimes_{\min} C_r^*(\Gamma)$ is injective, which implies that $C^*(\Gamma) \rightarrow C_r^*(\Gamma)$ is injective by a diagram chase. Therefore, Γ is amenable. ■

We note that trivial actions by non-trivial groups cannot be essentially free. Hence, Proposition 5.3 and Lemma 5.7 together imply the following.

Corollary 5.8. *Let Γ be a non-trivial discrete group acting trivially on a compact Hausdorff space X with a probability Radon measure μ . Then at least one of the following conditions fails:*

- (1) Γ is amenable;
- (2) any tracial state τ on $C(X) \rtimes_r \Gamma$ with the associated measure being μ is canonical.

5.2. Tracial ideals and quasidiagonal traces

Given a tracial state on the reduced groupoid C^* -algebra, we can consider its tracial ideal as follows.

Definition 5.9. Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and τ be a tracial state on $C_r^*(\mathcal{G})$. The *tracial ideal associated to τ* is defined to be $I_\tau := \{a \in C_r^*(\mathcal{G}) : \tau(a^*a) = 0\}$, which is a closed two-sided ideal in $C_r^*(\mathcal{G})$. For an invariant probability Radon measure μ on $\mathcal{G}^{(0)}$, we simply write $I_\mu := I_{\tau_\mu}$, where τ_μ is the tracial state associated to μ defined in (2.3).

Remark 5.10. For a tracial state τ , we always have $I_\tau \subseteq \text{Ker}(\tau)$. Indeed, let a be any positive element in I_τ . If $a = b^*b$ for some $b \in I_\tau$, then by definition we have $0 = \tau(b^*b) = \tau(a)$.

The following proposition is the key observation in this subsection.

Proposition 5.11. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid, and μ be an invariant probability Radon measure on $\mathcal{G}^{(0)}$. Then we have the following short exact sequence:*

$$0 \rightarrow I_\mu \rightarrow C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}_{\text{supp } \mu}) \rightarrow 0.$$

Proof. Firstly, we note that μ being invariant implies that $\text{supp } \mu$ is invariant. According to [6, Proposition 5.4], it suffices to show that

$$I_\mu = \{a \in C_r^*(\mathcal{G}) : E(a^*a) \in C_0(\mathcal{G}^{(0)} \setminus \text{supp } \mu)\}.$$

Given $a \in C_r^*(\mathcal{G})$, we note that $E(a^*a)$ is a continuous non-negative function on $\mathcal{G}^{(0)}$. Hence, it is clear that $\int_{\mathcal{G}^{(0)}} E(a^*a) d\mu = 0$ if and only if $\{x : E(a^*a)(x) \neq 0\} \subseteq \mathcal{G}^{(0)} \setminus \text{supp } \mu$. This concludes the proof. ■

Corollary 5.12. *Let \mathcal{G} be a locally compact Hausdorff and étale groupoid which is essentially free. Then for any tracial state τ on $C_r^*(\mathcal{G})$, we have $C_r^*(\mathcal{G})/I_\tau \cong C_r^*(\mathcal{G}_{\text{supp } \mu_\tau})$.*

Proof. It follows directly from Theorem 3.16 and Proposition 5.11. ■

A question of N. Brown asks whether every amenable tracial state is quasidiagonal (see [7, Question 6.7 (2)]) in the following sense.

Definition 5.13. A tracial state τ on a C^* -algebra A is *quasidiagonal* if there is a net of contractive completely positive maps $\phi_i : A \rightarrow M_{k(i)}$ such that

- $\tau(a) = \lim_i \text{tr} \circ \phi_i(a)$ for all $a \in A$;
- $\lim_i \|\phi_i(ab) - \phi_i(a)\phi_i(b)\| = 0$ for all $a, b \in A$.

Substantial progress on this question has recently been made in [13, 33] as follows.

Theorem 5.14 (see [13, 33]). *Any faithful, amenable tracial state on a separable, exact C^* -algebra satisfying the UCT is quasidiagonal.*

We refer the reader to [7, 13, 33] for relevant definitions and the question of N. Brown. We end this paper by removing the condition “faithful” on reduced groupoid C^* -algebras. More precisely, we prove the following result.

Theorem 5.15. *Let \mathcal{G} be a locally compact, Hausdorff, second-countable and étale groupoid such that $C_r^*(\mathcal{G})$ is an exact C^* -algebra and \mathcal{G} satisfies the strong Baum–Connes conjecture with all coefficients in the sense of [4, Definition 3.6]. If \mathcal{G} is also essentially free, then every amenable tracial state on $C_r^*(\mathcal{G})$ is quasidiagonal.*

Proof. Let τ be any amenable tracial state on $C_r^*(\mathcal{G})$. Then τ vanishes on the tracial ideal I_τ by Remark 5.10, and hence there is an induced faithful tracial state $\dot{\tau}$ on $C_r^*(\mathcal{G})/I_\tau$ such that $\tau = \dot{\tau} \circ p$, where $p : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})/I_\tau$ is the quotient map. To see that τ is quasidiagonal, it suffices to show that $\dot{\tau}$ is quasidiagonal.

Since τ is amenable and $C_r^*(\mathcal{G})$ is exact, it follows that $\dot{\tau}$ is also amenable (see [8, Proposition 6.3.5 (4)]). By Corollary 5.12, there is a closed invariant subset D of $\mathcal{G}^{(0)}$ such that $C_r^*(\mathcal{G})/I_\tau \cong C_r^*(\mathcal{G}_D)$, which is exact as quotients of exact C^* -algebras are exact by [8, Theorem 10.2.5]. As \mathcal{G} is second-countable and satisfies the strong Baum–Connes conjecture with all coefficients, $C_r^*(\mathcal{G}_D)$ is separable and satisfies the UCT by [4, Theorem 4.11 and Corollary 4.2]. Thus, we conclude the quasidiagonality of $\dot{\tau}$ from Theorem 5.14. ■

As a consequence, we obtain the following corollary (see a similar result in [3, Remark 3.13]).

Corollary 5.16. *Let \mathcal{G} be a locally compact, Hausdorff, second-countable, amenable and étale groupoid, which is also essentially free. Then every tracial state on $C_r^*(\mathcal{G})$ is quasidiagonal.*

Proof. If G is amenable, then \mathcal{G} satisfies the strong Baum–Connes conjecture with all coefficients by [4, Corollary 3.15] and $C_r^*(G)$ is nuclear (hence also exact). As every tracial state on a nuclear C^* -algebra is amenable [8, Proposition 6.3.4], we complete the proof by Theorem 5.15. ■

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