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Irreducible representations of the crystallization of the quantized function algebras $C(\mathrm{SU}_q(n+1))$

Manabendra Giri and Arup Kumar Pal

Abstract. The crystallization of the C^*-algebras $C(\mathrm{SU}_q(n+1))$ was introduced by Giri and Pal in [Proc. Indian Acad. Sci. Math. Sci. 134 (2024), article no. 30] as a C^*-algebra $C(\mathrm{SU}_0(n+1))$ given by a finite set of generators and relations. Here we study the representations of the C^*-algebra $C(\mathrm{SU}_0(n+1))$ and prove a factorization theorem for its irreducible representations. This leads to a complete classification of all irreducible representations of this C^*-algebra. As an important consequence, we prove that all the irreducible representations of $C(\mathrm{SU}_0(n+1))$ arise exactly as $q \to 0+$ limits of the irreducible representations of $C(\mathrm{SU}_q(n+1))$. We also present a few other important corollaries of the classification theorem.

Dedicated to the memory of K. R. Parthasarathy (1936–2023).

1. Introduction

The theory of crystal bases in quantum group theory was developed in the early 1990s independently by Kashiwara [11, 12] and Lusztig [14, 15]. In particular, Kashiwara [12] had produced crystal bases for the coordinate ring $\mathcal{O}(G_q)$ of regular functions for the q-deformation of a complex simple Lie group G viewed as a module over the corresponding quantized universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the group G. Loosely speaking, this can be thought of as a $q \to 0$ limit of a basis of the module $\mathcal{O}(G_q)$ in the algebraic setup. In the present paper, we will be concerned with the notion of the crystallization of the algebra (or $*$-algebra) structure of $\mathcal{O}(G_q)$. The $q \to 0$ limits of coordinate function algebras on different quantum spaces have appeared in the literature; see for example [9, 10, 24]. But these were defined in a rather ad hoc manner. A more systematic approach was taken up in [6] where the notion of crystallization for the C^*-algebras $C(K_q)$, $K = \mathrm{SU}(n+1)$, $n \geq 2$, was introduced by the present authors. This was followed by the work of Giselsson [8] who looked at the SU(3) case and proved that $C(\mathrm{SU}_0(3))$ is the C^*-algebra of a graph of rank 2, and Matassa and Yuncken [16] who introduced the notion of crystallization in a wider setup, including in particular all connected simply connected compact Lie groups K and proved that the crystallized algebras $C(K_0)$

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are higher-rank graph C^*-algebras, in the process setting up a link between the theory of crystal bases and the crystallization of coordinate function algebras.

The two definitions of crystallization given in [6, 16] are slightly different. In simple terms, the difference between the two definitions is as follows. Matassa and Yuncken [16] take a specific faithful representation π_q which acts on the same Hilbert space $\mathcal{H}$ for different values of q and define the C^*-subalgebra of the bounded operators on $\mathcal{H}$ formed by the $q \to 0+$ limits of the operators $\pi_q(a)$ for certain elements $a \in \mathcal{O}(K_q)$ to be the crystallization of $C(K_q)$. In [6] on the other hand, we observe that for every irreducible representation of $C(K_q)$, the canonical generating elements, scaled appropriately, obey the same set of relations and use those relations to define the crystallized algebra. In this article, we will study irreducible representations of the crystallization of the C^*-algebras $C(\mathrm{SU}_q(n+1))$ as defined in [6]. In particular, we will classify all irreducible representations of the crystallized algebra. As one of the consequences, we realize our crystallized algebra as a C^*-subalgebra of operators on the Hilbert space on which the Soibelman representation (see Section 3 for the definition) of the C^*-algebras $C(\mathrm{SU}_q(n+1))$ resides.

One immediate outcome of the way the crystal limit of $C(\mathrm{SU}_q(n+1))$ is defined in [6] is that for any irreducible representation π_q of $C(\mathrm{SU}_q(n+1))$, taking the limits of the scaled generators, one gets a representation of the crystallized algebra $C(\mathrm{SU}_0(n+1))$. In the present paper, we prove that the representations that arise in this way are irreducible and inequivalent, and they give all the irreducible representations of $C(\mathrm{SU}_0(n+1))$. Recall [13, Chapter 3] that in the $q \neq 0$ case, Soibelman develops the representation theory of $C(K_q)$ in terms of highest weight modules in classifying all the irreducible representations of $C(K_q)$. It is not clear to us at this point if his techniques can be modified and used for classifying the irreducible representations of $C(\mathrm{SU}_0(n+1))$. The proofs presented here use elementary tools involving Hilbert space operators.

The classification result that we prove here was proved for the $n = 2$ case in [6]. However, the technique of proof used here is different. For the $n = 2$ case, an irreducible representation was classified by extracting the invariant quantities needed to describe it (these correspond to the different cases in the proof of the main theorem in [6]) and then directly setting up a unitary equivalence for the two representations. However, for higher values of n, it becomes difficult to make a similar proof work. The proof we give here is a recursive one. We prove a factorization theorem for irreducible representations, and by the repeated application of it, we are then able to use the classification statement for $n-1$ case, which then gives us the result.

The main motivation behind looking at the crystallizations of the C^*-algebras $C(K_q)$ and their representations comes from the local index computations of Connes for $\mathrm{SU}_q(2)$ [4] and from a subsequent result in Chakraborty and Pal [3] where they prove an approximate equivalence involving the GNS representation of the Haar state of $\mathrm{SU}_q(2)$ and the Soibelman representation of $C(\mathrm{SU}_q(2))$ on the Hilbert space $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N})$. The main observation in [3] was that this approximate equivalence is what is going on structurally behind the computations of Connes [4] on the equivariant spectral triple for $\mathrm{SU}_q(2)$. To obtain the unitary that gives the approximate equivalence, it was observed that 'at $q = 0$', the above two representations give two representations of a $q = 0$ version of the C^*-algebra $C(\mathrm{SU}_q(2))$. One gets this C^*-algebra if one replaces q by 0 in the defining relations of $C(\mathrm{SU}_q(2))$ (see [24]). Up to unitary equivalence, the GNS representation at $q = 0$ is then an amplification of the other representation at $q = 0$, and this same unitary then gives an approximate equivalence for the nonzero values of q. By Soibelmen's classification result, for all connected simply connected compact Lie groups K, the C^*-algebra $C(K_q)$ admits a faithful representation on a nice Hilbert space, just like that of $C(\mathrm{SU}_q(2))$ mentioned above (this representation has been referred to as the Soibelmen representation in [16] – we will adopt the same terminology here). We expect that the study of the crystallizations of $C(K_q)$ for other compact Lie groups K and their representations will shed more light on the relation between the GNS representations of the Haar states for these quantum groups and their Soibelman representations, thereby leading to a better understanding of the equivariant spectral triples for these quantum groups given by Neshveyev and Tuset in [18]. From an application standpoint, the origin of crystallization of quantized function algebras can thus be traced back to the computations by Connes on equivariant spectral triples for $\mathrm{SU}_q(2)$ (and similar computations on equivariant spectral triples for other homogeneous spaces in [20, 23], etc., following Connes' ideas) and the more recent paper [3] where the role of the algebra at $q = 0$ was further clarified.

It is important to note that for the crystallizations of $C(K_q)$ to be useful in the way indicated above, the crystal limits must have the property that the representations of $C(K_q)$ give rise to the representations of the crystallized algebra $C(K_0)$ by sending the generators of $C(K_0)$ to the limits of the (scaled) generators for $C(K_q)$. Our main result in this paper says that in the type A_n case that we have considered, this is the case for irreducible representations of $C(K_q)$.

Let us now give a brief outline of the content of this paper. In Section 2, we recall the crystallzation of the C^*-algebras $C(\mathrm{SU}_q(n+1))$. One consequence of the way the crystallized algebra is defined is that every irreducible representation of $C(\mathrm{SU}_q(n+1))$ gives rise to a representation of the crystallized algebra $C(\mathrm{SU}_0(n+1))$ in the limit $q \to 0+$. In Section 3, we prove that each of these representations is irreducible, and the representations arising from the limits of two inequivalent irreducible representations are inequivalent. Sections 4–7 are devoted to proving that any irreducible representation of $C(\mathrm{SU}_0(n+1))$ must occur in this way. In Section 4, we prove that the generating elements $z_{i,j}$ are all partial isometries. We also prove a few relations that play a very useful role in the subsequent sections. In Section 5, we study the image $\pi(C(\mathrm{SU}_0(n+1)))$ and its closure for a representation π of $C(\mathrm{SU}_0(n+1))$ on a complex separable Hilbert space. In Section 6, we further specialize to an irreducible representation π, which helps us draw stronger conclusions on the operators we define in Section 5. In particular, we are able to decompose the operators $\pi(z_{i,j})$ in a certain way (Theorem 6.8) that helps us prove a factorization theorem for irreducible representations (Theorem 7.1) in Section 7. This factorization theorem, in turn, helps us give a recursive proof of the classification theorem for all irreducible representations of $C(\mathrm{SU}_0(n+1))$. In particular, the classification result tells us that any irreducible representation must occur as a $q \to 0+$ limit of an irreducible representation for $C(\mathrm{SU}_q(n+1))$. In the final section, we present this and a few other important corollaries of the classification theorem.

It will be good to mention a couple of very important questions related to our study that we do not touch upon in this paper. The first is about the two notions of crystallization given in [6, 16] mentioned earlier: whether they result in the same C^*-algebra. We make a few remarks on this in the last section (see Remark 8.6). The other is the possible q-invariance of the C^*-algebras arising as quantized function algebras of a Lie group or its homogeneous space. This has been investigated in various cases in the literature; see for example [9, 10, 17, 21, 24]. For the case of $\mathrm{SU}(n+1)$, Giselsson proved in [7] that the C^*-algebras $C(\mathrm{SU}_q(n+1))$ are isomorphic for $q \in (0,1)$. So the question that remains to be settled in this case is whether the crystallized algebra $C(\mathrm{SU}_0(n+1))$ is isomorphic to the ones for $q \in (0,1)$. The results in the last section, where the crystallized algebra is realized as a C^*-algebra of operators on the same space as the Soibelmen representation of all the $C(\mathrm{SU}_q(n+1))$'s, might be useful in settling this question.

A word about the notations used for the quantized function algebras. For a Lie group K, we will denote the algebra of regular functions on the q-deformation K_q by $\mathcal{O}(K_q)$ and the C^*-algebra of continuous functions on it by $C(K_q)$. For the group $\mathrm{SU}(n+1)$, however, we deviate from this and use the more widely used notations $\mathcal{O}(\mathrm{SU}_q(n+1))$ and $C(\mathrm{SU}_q(n+1))$, respectively.

2. The crystallized algebras $C(\mathrm{SU}_0(n+1))$ and $\mathcal{O}(\mathrm{SU}_0(n+1))$

Let us start by recalling the following theorem from [6].

Theorem 2.1 ([6]). *There is a universal C^*-algebra generated by elements $z_{i,j}$, $1 \le i,j \le n+1$, satisfying the following relations:*

$$z_{i,j} z_{i,l} = 0 \qquad \text{if } j < l, \tag{2.1}$$

$$z_{i,j} z_{k,j} = 0 \qquad \text{if } i < k, \tag{2.2}$$

$$z_{i,l} z_{k,j} - z_{k,j} z_{i,l} = 0 \qquad \text{if } i < k \text{ and } j < l, \tag{2.3}$$

$$z_{i,l} z_{k,j} = 0 \qquad \begin{cases} \text{if } i < k, \quad j < l \\ \text{and } \max\{i,j\} \ge \min\{k,l\}, \end{cases} \tag{2.4}$$

$$z_{i,j} z_{k,l} - z_{k,l} z_{i,j} = z_{i,l} z_{k,j} \qquad \begin{cases} \text{if } i < k, \quad j < l \\ \text{and } \max\{i,j\} + 1 = \min\{k,l\}, \end{cases} \tag{2.5}$$

$$z_{i,j} z_{k,l} - z_{k,l} z_{i,j} = 0 \qquad \begin{cases} \text{if } i < k, \quad j < l \\ \text{and } \max\{i,j\} + 1 < \min\{k,l\}, \end{cases} \tag{2.6}$$

$$z_{1,1} z_{2,2} \cdots z_{n+1,n+1} = 1, \tag{2.7}$$

$$z_{i,j} z_{r,s}^* - z_{r,s}^* z_{i,j} = 0, \qquad \text{if } i \ne r \text{ and } j \ne s, \tag{2.8}$$

$$z_{r,s}^* = \begin{cases} (z_{1,1}\cdots z_{s-1,s-1})(z_{s,s+1}z_{s+1,s+2}\cdots z_{r-1,r})(z_{r+1,r+1}\cdots z_{n+1,n+1}) & \text{if } r > s, \\ (z_{1,1}\cdots z_{r-1,r-1})(z_{r+1,r}z_{r+2,r+1}\cdots z_{s,s-1})(z_{s+1,s+1}\cdots z_{n+1,n+1}) & \text{if } r < s, \\ (z_{1,1}\cdots z_{s-1,s-1})(z_{s+1,s+1}z_{s+2,s+2}\cdots z_{n+1,n+1}) & \text{if } r = s. \end{cases} \tag{2.9}$$

The C^*-algebra in the above theorem is defined to be the crystallization of the family of C^*-algebras $C(\mathrm{SU}_q(n+1))$ in [6]. This will be denoted by $C(\mathrm{SU}_0(n+1))$. The $*$-algebra given by the above set of relations is similarly defined as the crystallization of the quantized algebra of regular functions $\mathcal{O}(\mathrm{SU}_q(n+1))$. We will denote this $*$-algebra by $\mathcal{O}(\mathrm{SU}_0(n+1))$.

The following is an easy consequence of the relations satisfied by the generators of the algebra $\mathcal{O}(\mathrm{SU}_0(n+1))$.

Proposition 2.2. *Let* $1 \le i \le j \le n+1$. *Then,*

$$\begin{aligned}&(z_{i,i}z_{i+1,i+1}\cdots z_{j,j})^* \\ &\quad = (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}).\end{aligned} \tag{2.10}$$

Proof. Note that for $i = j$, it follows from (2.9). We will assume that the equality holds for $i \le j < n+1$ and prove that it holds for $i \le j+1$ also. Using (2.6) and (2.9), we get

$$\begin{aligned}&(z_{i,i}z_{i+1,i+1}\cdots z_{j+1,j+1})^* \\ &\quad = z_{j+1,j+1}^*(z_{i,i}z_{i+1,i+1}\cdots z_{j,j})^* \\ &\quad = (z_{1,1}z_{2,2}\cdots z_{j,j})(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1}) \\ &\qquad \times (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}) \\ &\quad = (z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})(z_{1,1}z_{2,2}\cdots z_{j,j}) \\ &\qquad \times (z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})(z_{1,1}z_{2,2}\cdots z_{i-1,i-1}) \\ &\quad = (z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})(z_{1,1}z_{2,2}\cdots z_{i-1,i-1}) \\ &\quad = (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1}).\end{aligned}$$

This completes the proof. ■

3. $q \to 0+$ limits of representations

In this section, we will be concerned with the representations of $C(\mathrm{SU}_0(n+1))$ that arise as $q \to 0+$ limits of the irreducible representations of the C^*-algebra $C(\mathrm{SU}_q(n+1))$. We will start by recalling briefly the irreducible representations $C(\mathrm{SU}_q(n+1))$ from [13] where the irreducible representations of $C(K_q)$ for all connected simply connected compact Lie groups K were classified. It should be mentioned in this context that following Soibelman's ideas, the irreducible representations of the quantized function algebras for more general classes of spaces were later studied and classified by Dijkhuizen and Stokman [22] and Neshveyev and Tuset [19]. The results in these papers will have an important role to play in studying the crystallizations of quantized function algebras in those cases, exactly like the role Soibelman's results play in the present case.

The Weyl group for $\mathrm{SU}_q(n+1)$ is isomorphic to the permutation group $\mathfrak{S}_{n+1}$ on $n+1$ symbols. Denote by s_i the transposition $(i, i+1)$. Then, $\{s_1, s_2, \ldots, s_n\}$ form a set of generators for $\mathfrak{S}_{n+1}$. For $a, b \in \mathbb{N}$ with $a \leq b$, let us denote by $s_{[a,b]}$ the product $s_b s_{b-1} \cdots s_a$ in $\mathfrak{S}_{n+1}$. Let $1 \leq k \leq n$, $1 \leq b_k < b_{k-1} < \cdots < b_1 \leq n$, and $1 \leq a_i \leq b_i$ for $1 \leq i \leq k$. It follows from the strong exchange condition and the deletion condition in the characterization of Coxeter system (see [1, 5]) that $\omega = s_{[a_k,b_k]} s_{[a_{k-1},b_{k-1}]} \cdots s_{[a_1,b_1]}$ is a reduced word in $\mathfrak{S}_{n+1}$. It is well known that if $k = n$, $a_i = 1$ and $b_i = i$ for $1 \leq i \leq n$, then one gets the longest element ω_0.

Let S be the left shift operator and N the number operator on $\ell^2(\mathbb{N})$:

$$Se_n = \begin{cases} e_{n-1} & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases} \qquad Ne_k = ke_k, \ k \in \mathbb{N}.$$

Let us denote the generators of the C^*-algebra $C(\mathrm{SU}_q(n+1))$ by $u_{i,j}(q)$. Denote by $\psi_{s_r}^{(q)}$ the following representation of $C(\mathrm{SU}_q(n+1))$ on $\ell^2(\mathbb{N})$:

$$\psi_{s_r}^{(q)}\big(u_{i,j}(q)\big) = \begin{cases} S\sqrt{I - q^{2N}} & \text{if } i = j = r, \\ \sqrt{I - q^{2N}}S^* & \text{if } i = j = r+1, \\ -q^{N+1} & \text{if } i = r, j = r+1, \\ q^N & \text{if } i = r+1, j = r, \\ \delta_{i,j} I & \text{otherwise.} \end{cases} \tag{3.1}$$

For a reduced word $\omega = s_{i_1} s_{i_2} \cdots s_{i_m} \in \mathfrak{S}_{n+1}$, define $\psi_\omega^{(q)}$ to be $\psi_{s_{i_1}}^{(q)} * \psi_{s_{i_2}}^{(q)} * \cdots * \psi_{s_{i_m}}^{(q)}$. Here, for two representations ϕ and ψ, $\phi * \psi$ denotes the representation $(\phi \otimes \psi)\Delta_q$, where Δ_q is the comultiplication map on $C(\mathrm{SU}_q(n+1))$.

Next, let $\lambda \equiv (\lambda_1, \ldots, \lambda_n) \in (S^1)^n$. Define

$$\chi_\lambda\big(u_{i,j}(q)\big) = \begin{cases} \lambda_i \delta_{i,j} & \text{if } i = 1, \\ \bar{\lambda}_n \delta_{i,j} & \text{if } i = n+1, \\ \bar{\lambda}_{i-1} \lambda_i \delta_{i,j} & \text{otherwise.} \end{cases} \tag{3.2}$$

A well-known result of Soibelman [13, Theorem 6.2.7, p. 121] says that for any reduced word $\omega \in \mathfrak{S}_{n+1}$ and a tuple $\lambda \equiv (\lambda_1, \ldots, \lambda_n) \in (S^1)^n$, the representation

$$\psi_{\lambda,\omega}^{(q)} := \chi_\lambda * \psi_\omega^{(q)} \tag{3.3}$$

is an irreducible representation of $C(\mathrm{SU}_q(n+1))$, and these give all the irreducible representations of $C(\mathrm{SU}_q(n+1))$. If one takes the longest word ω_0 in $\mathfrak{S}_{n+1}$ and takes the direct integral of the representations $\psi^{(q)}_{\lambda,\omega_0}$ with respect to λ, one gets a faithful representation $\psi^{(q)}$ of $C(\mathrm{SU}_q(n+1))$ acting on $\ell^2(\mathbb{Z})^{\otimes n} \otimes \ell^2(\mathbb{N})^{\otimes \frac{n(n+1)}{2}}$ (or equivalently on $L^2(S^1)^{\otimes n} \otimes \ell^2(\mathbb{N})^{\otimes \frac{n(n+1)}{2}}$). Following [16], this will be called the *Soibelman representation*.

Let us define

$$Z_{i,j} = \lim_{q\to 0+} (-q)^{\min\{i-j,0\}} \psi^{(q)}_{\lambda,\omega}\big(u_{i,j}(q)\big), \tag{3.4}$$

where $\psi^{(q)}_{\lambda,\omega}$ is the representation of $C(\mathrm{SU}_q(n+1))$ defined above. It has been shown in [6, Propositions 2.1, 2.3, 2.4, 2.6, and 2.7] that the above limits exist and the operators $Z_{i,j}$ obey the relations (2.1)–(2.9). Therefore,

$$\psi_{\lambda,\omega}(z_{i,j}) = \lim_{q\to 0+} (-q)^{\min\{i-j,0\}} \psi^{(q)}_{\lambda,\omega}\big(u_{i,j}(q)\big), \quad i,j \in \{1,2,\ldots,n+1\}, \tag{3.5}$$

defines a representation of $C(\mathrm{SU}_0(n+1))$ on the Hilbert space $\ell^2(\mathbb{N}^{\ell(\omega)})$, where $\ell(\omega)$ denotes the length of the element ω. In particular, if ω is the identity element, its reduced word is the empty word and the representation $\psi_{\lambda,\omega}$ in this case is a one-dimensional representation given by

$$\psi_{\lambda,id}(z_{i,j}) = \begin{cases} \lambda_i \delta_{i,j} & \text{if } i = 1, \\ \overline{\lambda}_n \delta_{i,j} & \text{if } i = n+1, \\ \overline{\lambda}_{i-1}\lambda_i \delta_{i,j} & \text{otherwise.} \end{cases} \tag{3.6}$$

If $\lambda = (1,1,\ldots,1)$, we will denote $\psi_{\lambda,\omega}$ by just ψ_ω. The remaining part of this section is devoted to the proof of irreducibility of $\psi_{\lambda,\omega}$ and their inequivalence for different pairs (λ,ω).

Let us start with the observation that the representation ψ_{s_r} is given by

$$\psi_{s_r}(z_{i,j}) = \begin{cases} S & \text{if } i = j = r, \\ S^* & \text{if } i = j = r+1, \\ P_0 & \text{if } i \neq j \text{ and } i,j \in \{r,r+1\}, \\ \delta_{i,j} I & \text{otherwise,} \end{cases} \tag{3.7}$$

where P_0 denotes the rank one projection

$$|e_0\rangle\langle e_0| = I - S^*S.$$

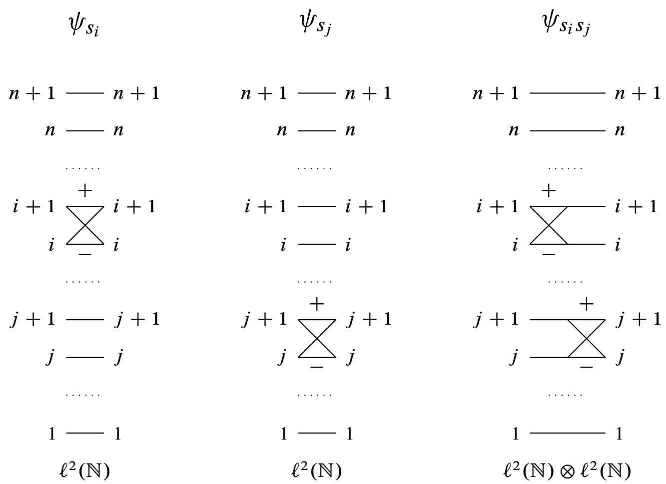
The following lemma is an important observation that will be used in the proof of the main theorem in the last section. The proof is straightforward and hence omitted.

Lemma 3.1. *Let* $\omega = s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\cdots s_{[a_1,b_1]}$ *and let* $\omega' = \omega s_{a_1-1}$. *Then,*

$$\psi_{\lambda,\omega'}(z_{i,j}) = \sum_{m=\min\{i,j\}}^{\max\{i,j\}} \psi_{\lambda,\omega}(z_{i,m}) \otimes \psi_{s_{a_1-1}}(z_{m,j}). \tag{3.8}$$

Diagrams for the representations $\psi_{\lambda,\omega}$. We will next introduce certain diagrams for the representations $\psi_{\lambda,\omega}$ similar to those introduced in [2] for the case of $\mathrm{SU}_q(n+1)$ for nonzero q. These diagrams will be very helpful in understanding the images of the generating elements $z_{i,j}$ of the C^*-algebra $C(\mathrm{SU}_q(n+1))$. In particular, they will play a very important role in proving the irreducibility of the representations $\psi_{\lambda,\omega}$ in the present section and later on while characterizing all irreducible representations, these diagrams will once again be helpful by providing us with the necessary ideas and intuition.

Let us start with the diagrams for the representations ψ_{s_i} and $\psi_{s_i s_j}$.



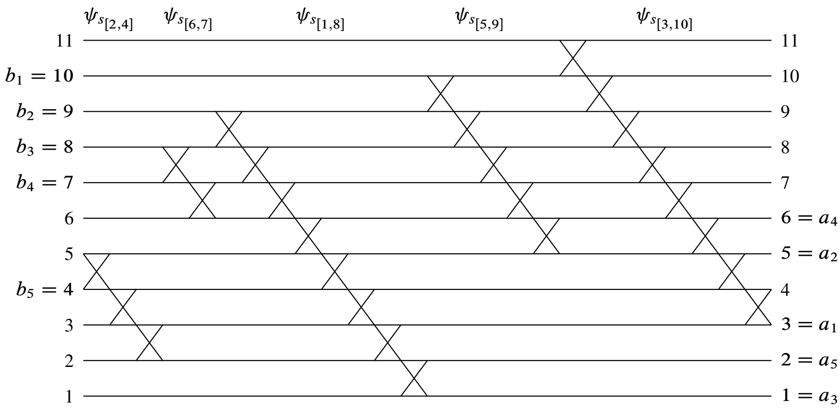
In the leftmost diagram above, each path from a node k on the left to a node l on the right stands for an operator on $\ell^2(\mathbb{N})$. A horizontal unlabeled line stands for the identity operator, a horizontal line labeled with a '$+$' sign stands for the right shift S^*, and one labeled with a '$-$' sign stands for the left shift S. Diagonal lines going upward or downward represent the projection P_0. The image $\psi_{s_i}(z_{m,l})$ is the operator represented by the path from m on the left to l on the right and is zero if there is no such path. Similarly, the diagram in the middle stands for the representation ψ_{s_j}.

The diagram on the right above is for the representation $\psi_{s_i s_j}$, where one simply puts the two diagrams representing ψ_{s_i} and ψ_{s_j} adjacent to each other, with the one for ψ_{s_j} to the right of that for ψ_{s_i}. The image $\psi_{s_i s_j}(z_{m,l})$ of $z_{m,l}$ is an operator on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ given by *the sum of the paths from the node m on the extreme left to the node l on the extreme right which do not contain both upward and downward diagonals*. It would be zero if there is no such path.

Next, we come to the description of ψ_ω. As we have already remarked, we will take a reduced word for ω of the form

$$\omega = s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\cdots s_{[a_1,b_1]}$$

where $1 \le k \le n$, $1 \le b_k < b_{k-1} < \cdots < b_1 \le n$, and $1 \le a_i \le b_i$ for $1 \le i \le k$. To get the diagram for ψ_ω, we simply put the diagrams for the ψ_{s_r}'s appearing in this word adjacent to each other in the order they appear in the word from left to right. Thus, for example, if $\omega = s_{[2,4]}s_{[6,7]}s_{[1,8]}s_{[5,9]}s_{[3,10]}$, then the following is the diagram for ψ_ω:



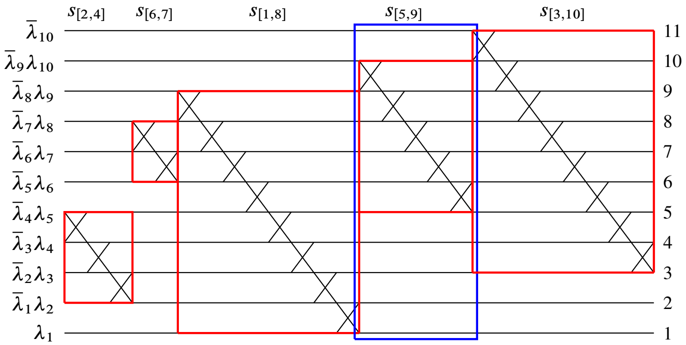
Finally, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in S^1$, the diagram for $\psi_{\lambda,\omega}$ is the same as that for ψ_ω, except that *the paths starting at m on the left get multiplied by a scalar* which is $\overline{\lambda}_{m-1}\lambda_m$ if $1 \le m \le n+1$ (with the convention that $\lambda_0 = \lambda_{n+1} = 1$).

Remark 3.2. In view of the above, we will not need to keep track of the λ_i's except for the last theorem in this section (Theorem 3.9) where we prove that $\psi_{\lambda,\omega}$'s are inequivalent for different pairs (λ, ω). Accordingly, all the operator equalities we write before Theorem 3.9 in this section will be valid only modulo multiplication by a complex number of modulus one.

In the diagram for $\psi_{\lambda,\omega}$, we will refer to the part corresponding to $\psi_{s_{[a_j,b_j]}}$ as the $s_{[a_j,b_j]}$-*section* and the smallest rectangle containing the diagonal lines in the $s_{[a_j,b_j]}$-section as the $s_{[a_j,b_j]}$-*rectangle*. For example, the different $s_{[a_j,b_j]}$-rectangles have been shown in red color and the $s_{[5,9]}$-section has been shown in blue color in the next diagram. Observe that if $i > b_j + 1$, then the horizontal line starting at node i on the left passes above the $s_{[a_m,b_m]}$-rectangles for $m \ge j$. For an operator R given by a path, we will denote by R_j the operator given by the $s_{[a_j,b_j]}$-section of that path. In such a case one has

$$R = R_k \otimes R_{k-1} \otimes \cdots \otimes R_1,$$

where R_j acts on $\ell^2(\mathbb{N})^{\otimes(b_j-a_j+1)}$ for all j.



The diagram for $\psi_{\lambda,\omega}$ and the terminology introduced above will be helpful in understanding the operators in the image $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))$ as well as certain quantities associated with them that we will define in the remaining part of this section. They are also going to be helpful in getting insight into some of the proofs in Sections 5 and 6, though we will not explicitly use or refer to the diagrams there.

Given a reduced word $\omega = s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\cdots s_{[a_1,b_1]}$, we will next define certain elements in the C^*-algebra $C(\mathrm{SU}_0(n+1))$ that will play a crucial role in the proof of irreducibility of the representation $\psi_{\lambda,\omega}$. For $2 \le j \le k$ and $a_j \le i \le b_j + 1$, define two integers $n(j,i)$ and $n'(j,i)$ as follows:

$$n(j,i) = \max\{m : 1 \le m \le j-1 \text{ and } a_m \le i\}, \tag{3.9}$$

$$n'(j,i) = \max\{m : 1 \le m \le j-1 \text{ and } a_m + 1 \le i\}, \tag{3.10}$$

with the usual convention that the maximum of an empty set will be taken as $-\infty$. Thus, $n(j,i)$ is the maximum of all those $m \in \{1,2,\dots,j-1\}$ for which the ith horizontal line (from the bottom) intersects the $s_{[a_m,b_m]}$-rectangle, and $n(j,i) = -\infty$ when the ith horizontal line is below the $s_{[a_m,b_m]}$-rectangles for $1 \le m \le j-1$.

Next, for $1 \le j \le k$ and for $a_j \le i \le b_j + 1$, define elements $v_{j,i}(\omega) \equiv v_{j,i}$ as follows:

$$v_{1,i} = z_{b_1+1,i},$$
$$v_{j,i} = \begin{cases} z_{b_j+1,i}\, v_{n(j,i),i+1}, & \text{if } n(j,i) \text{ is finite,} \\ z_{b_j+1,i} & \text{if } n(j,i) = -\infty, \end{cases} \qquad 2 \le j \le k.$$

Let us denote by $Z_{j,i}$ the operator $\psi_{\lambda,\omega}(z_{j,i})$ for $1 \le j,i \le n+1$, and by $V_{j,i}$ the operator $\psi_{\lambda,\omega}(v_{j,i})$ for $1 \le j \le k$ and $a_j \le i \le b_j + 1$. We will next study the operators $V_{j,i}$. In general, $Z_{b_j+1,i}$ is given by a sum of paths from $b_j + 1$ to i going downwards. What the next proposition says is that the operator $V_{j,i}$ will be given by the lowest among these paths (appearing in the sum for $Z_{b_j+1,i}$) with some of its $s_{[a_r,b_r]}$-sections slightly modified for $1 \le r \le j-1$ so that at most one shift operator occurs in each such section.

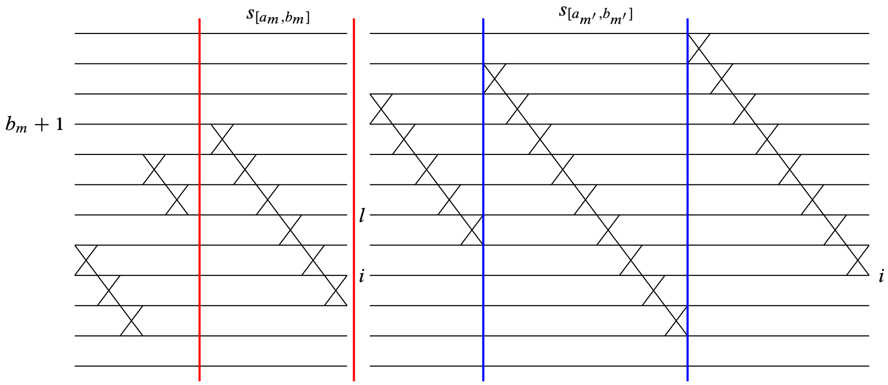
Proposition 3.3. *Let $\ell_r = b_r - a_r + 1$ for $1 \le r \le k$. Then, for $a_j \le i \le b_j + 1$, one has*

$$V_{j,i} = T^{(j)}_{k,i} \otimes T^{(j)}_{k-1,i} \otimes \cdots \otimes T^{(j)}_{1,i}, \tag{3.11}$$

where $T_{r,i}^{(j)}$'s are of the following form:

$I^{\otimes \ell_r}$	*if* $r > j$,	(3.12)
$P_0^{\otimes (b_r+1-i)} \otimes S^* \otimes I^{\otimes (i-a_r-1)}$	*if* $r = j$ *and* $i \geq a_r + 1$,	(3.13)
$P_0^{\otimes (b_r-a_r+1)}$	*if* $r = j$ *and* $i = a_r$,	(3.14)
$P_0^{\otimes (b_r-a_r-1)} \otimes I \otimes S^*$	*if* $r < j$ *and* $i = a_r + 1$,	(3.15)
$P_0^{\otimes (b_r-i)} \otimes I \otimes S^* \otimes I^{\otimes (i-a_r-1)}$	*if* $r < j$ *and* $a_r + 1 < i < b_r$,	(3.16)
tensor product of P_0*'s and* I*'s*	*if* $r < j$ *and* $i \leq a_r$.	(3.17)

Proof. For $j = 1$, it is easy to check from the diagram for $\psi_{\lambda,\omega}$ that (3.11) holds for $a_1 \leq i \leq b_1 + 1$. To help follow the argument, let us keep the following diagram before us in which the $s_{[a_m,b_m]}$-section and the the $s_{[a_{m'},b_{m'}]}$-section have been enclosed by pairs of colored vertical lines.



Let us now assume that (3.11) holds for $j \leq m-1$ and $a_j \leq i \leq b_j + 1$. We will prove the statement for $j = m$ and $a_m \leq i \leq b_m + 1$. Let us look at the case $n(m,i) = -\infty$ first. In this case, $V_{j,i} = Z_{b_m+1,i}$ and $i < a_j$ for all $1 \leq j \leq m-1$; i.e., the ith horizontal line is below the $s_{[a_j,b_j]}$-rectangles for $1 \leq j \leq m-1$. Therefore, $T_{j,i} = I^{\otimes(\ell_j)}$ for $1 \leq j \leq m-1$. Since the horizontal line starting at $b_m + 1$ on the left passes above the $s_{[a_j,b_j]}$-rectangles for $j > m$, there is exactly one path going from $b_m + 1$ on the left to i on the right and it does not intersect the $s_{[a_j,b_j]}$-rectangles for $1 \leq j \leq m-1$. Thus,

$$\begin{aligned} Z_{b_m+1,i} &= I^{\otimes(\sum_{r=m+1}^{k}\ell_r)} \otimes \left(P_0^{\otimes(b_m+1-i)} \otimes S^* \otimes I^{\otimes(i-a_m-1)}\right) \otimes I^{\otimes(\sum_{r=1}^{m-1}\ell_r)} \\ &= T_{k,i}^{(m)} \otimes T_{k-1,i}^{(m)} \otimes \cdots T_{1,i}^{(m)}. \end{aligned}$$

Next, assume $m' := n(m,i)$ is finite. We have $V_{m,i} = Z_{b_m+1,i} V_{m',i+1}$. Since $m' < m$, by our hypothesis,

$$V_{m',i+1} = T_{k,i+1}^{(m')} \otimes T_{k-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}.$$

Note that since $m' < m$, one has $T_{r,i+1}^{(m')} = I^{\otimes(\ell_r)}$ for $m \le r \le k$. It is also immediate that $Z_{b_m+1,i}$ is of the form $I^{\otimes(\sum_{r=m+1}^{k} \ell_r)} \otimes R$ for some operator R that acts on the last $\sum_{r=1}^{m} \ell_r$ copies of $\ell^2(\mathbb{N})$. The operator R is given by $\sum_{l=i}^{b_m+1} R(l)$ where $R(l)$ is the sum of operators given by the paths that go from $b_m + 1$ to l in the $s_{[a_m,b_m]}$-section and then from l to i in the $s_{[a_{m-1},b_{m-1}]} \cdots s_{[a_1,b_1]}$-section.

Since the $s_{[a_r,b_r]}$-rectangle for $m' < r < m$ lies above the ith horizontal line, we conclude that for $l > i$, the $s_{[a_{m'},b_{m'}]}$-section of $R(l)$ is of the form

$$I^{\otimes(b_{m'}-j)} \otimes S \otimes (\text{tensor product of } P_0,\ S^* \text{ and } I\text{'s})$$

for some j with $i + 1 \le j \le l$, while $T_{m',i+1}^{(m')} = P_0^{\otimes(b_{m'}-i)} \otimes S^* \otimes I^{\otimes(i-a_{m'})}$. Therefore,

$$R(l)\big(T_{m,i+1}^{(m')} \otimes T_{m-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}\big) = 0 \quad \text{for } l > i.$$

Hence,

$$R\big(T_{m,i+1}^{(m')} \otimes T_{m-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}\big) = R(i)\big(T_{m,i+1}^{(m')} \otimes T_{m-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}\big).$$

Let us write $T_{r,i} = R(i)_r T_{r,i+1}^{(m')}$ for $1 \le r \le m$. We will show that $T_{r,i}$ is given by (3.12)–(3.17) for $j = m$. Note that the path $R(i)$ consists of the diagonal line coming down from $b_m + 1$ to i in the $s_{[a_m,b_m]}$-section and its horizonal continuation in the $s_{[a_{m-1},b_{m-1}]} \cdots s_{[a_1,b_1]}$-section. Let us now look at the following three cases separately.

Case I. $1 \le r \le m - 1$ and $a_r < i$. Since $a_r < i$, we have $r \le m'$. Therefore, in this case, $R(i)_r = I^{\otimes(b_r-i)} \otimes S \otimes S^* \otimes I^{\otimes(i-a_r-1)}$, and $T_{r,i+1}^{(m')} = P_0^{\otimes(b_r-i)} \otimes S^* \otimes I^{\otimes(i-a_r)}$. Therefore,

$$T_{r,i} = R(i)_r T_{r,i+1}^{(m')} = P_0^{\otimes(b_r-i)} \otimes I \otimes S^* \otimes I^{\otimes(i-a_r-1)}.$$

Since $r < m$ and $i > a_r$, from (3.16), it follows that this is of the form $T_{r,i}^{(m)}$.

Case II. $1 \le r \le m - 1$ and $a_r = i$. As in the earlier case, we have $r \le m'$. Hence, $R(i)_r = I^{\otimes(\ell_r-1)} \otimes S$. Now if $r = m'$, then by (3.13), we have $T_{r,i+1}^{(m')} = P_0^{\otimes(\ell_r-1)} \otimes S^*$ and if $r < m'$, then by (3.15), we have

$$T_{r,i+1}^{(m')} = P_0^{\otimes(\ell_r-1)} \otimes I \otimes S^*.$$

Thus, $T_{r,i} = R(i)_r T_{r,i+1}^{(m')}$ is either $P_0^{\otimes(\ell_r-1)} \otimes I$ or $P_0^{\otimes(\ell_r-2)} \otimes I \otimes I$. Since $r < m$ and $i = a_r$, by (3.17), $T_{r,i}$ is of the form $T_{r,i}^{(m)}$, as required.

Case III. $1 \le r \le m - 1$ and $i < a_r$. In this case, $R(i)_r = I^{\otimes(\ell_r)}$, so that $T_{r,i} = T_{r,i+1}^{(m')}$. If $r > m'$, then $T_{r,i+1}^{(m')} = I^{\otimes \ell_r}$. Since $i + 1 \le a_r$, if $r = m'$, we must have $i + 1 = a_r$. Hence, by (3.16), $T_{r,i+1}^{(m')}$ is a tensor product of P_0's and I's. If $r < m'$ and $i + 1 \le a_r$, then also by (3.17), $T_{r,i+1}^{(m')}$ is a tensor product of P_0's and I's. Since $T_{r,i} = T_{r,i+1}^{(m')}$, in all these subcases, it is of the required form. ∎

We now list a few important corollaries of the above proposition.

Corollary 3.4. *For a bounded operator T, let $|T|$ denote $(T^*T)^{\frac{1}{2}}$. Then, one has*

$$|V_{j,a_j}|\cdot|V_{j-1,a_{j-1}}|\cdots|V_{1,a_1}| = T_k\otimes T_{k-1}\otimes\cdots\otimes T_1,$$

where

$$T_r = \begin{cases} I^{\otimes \ell_r} & \text{if } r > j,\\ P_0^{\otimes(b_r+1-a_r)} & \text{if } r\le j.\end{cases} \tag{3.18}$$

Corollary 3.5. *Let $a_j \le i \le b_j+1$. Assume $n'(j,i) > -\infty$. Then,*

$$V^*_{n'(j,i),i}V_{j,i} = T_k\otimes T_{k-1}\otimes\cdots\otimes T_1, \tag{3.19}$$

where

$$T_r = \begin{cases} I^{\otimes(b_r+1-a_r)} & \text{if } r>j,\\ P_0^{\otimes(b_r+1-a_r)} & \text{if } r=j \text{ and } i=a_j,\\ P_0^{\otimes(b_r+1-i)}\otimes S^*\otimes I^{\otimes(i-a_r-1)} & \text{if } r=j \text{ and } i\ge a_j+1,\\ \text{tensor product of } P_0\text{'s and } I\text{'s} & \text{if } r<j.\end{cases} \tag{3.20}$$

Proof. Observe that for $r<j$, the section $T^{(j)}_{r,i}$ is of the form (3.15) or (3.16) (i.e., contains the right shift) exactly when $r<j$ and $a_r<i$. Let $n'(j,i)$ be as defined in (3.10). Then, for the operator $V_{n'(j,i),i}$ also, the corresponding sections $T^{(s)}_{r,i}$ exactly match $T^{(j)}_{r,i}$ and the remaining ones are tensor products of I and P_0's. Therefore, the result follows. ■

Corollary 3.6. *Let $a_j\le i\le b_j+1$. Define*

$$E_{j,i}\equiv E_{j,i}(\omega) := \begin{cases} V^*_{n'(j,i),i}V_{j,i}|V_{j-1,a_{j-1}}|\cdot|V_{j-2,a_{j-2}}|\cdots|V_{1,a_1}| & \text{if } n'(j,i)>-\infty,\\ V_{j,i}|V_{j-1,a_{j-1}}|\cdot|V_{j-2,a_{j-2}}|\cdots|V_{1,a_1}| & \text{if } n'(j,i)=-\infty.\end{cases}$$

Then,

$$E_{j,i}(\omega) = T_k\otimes T_{k-1}\otimes\cdots\otimes T_1, \tag{3.21}$$

where

$$T_r = \begin{cases} I^{\otimes(b_r+1-a_r)} & \text{if } r>j,\\ P_0^{\otimes(b_r+1-a_r)} & \text{if } r=j \text{ and } i=a_j,\\ P_0^{\otimes(b_r+1-i)}\otimes S^*\otimes I^{\otimes(i-a_r-1)} & \text{if } r=j \text{ and } i\ge a_j+1,\\ P_0^{\otimes(b_r+1-a_r)} & \text{if } r<j.\end{cases} \tag{3.22}$$

Proposition 3.7. *For $1\le j\le k$ and $a_j+1\le i\le b_j+1$, let*

$$R_{j,i} = I^{\otimes(\sum_{r=j+1}^k \ell_r)}\otimes(P_0^{\otimes(b_j+1-i)}\otimes S^*\otimes I^{\otimes(i-a_j-1)})\otimes I^{\otimes(\sum_{r=1}^{j-1}\ell_r)},$$

$$S_{j,i} = I^{\otimes(\sum_{r=j+1}^k \ell_r)}\otimes(I^{\otimes(b_j+1-i)}\otimes S^*\otimes I^{\otimes(i-a_j-1)})\otimes I^{\otimes(\sum_{r=1}^{j-1}\ell_r)}.$$

Then, $R_{j,i}$ and $S_{j,i}$ belong to $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))''$ for $1\le j\le k$ and $a_j+1\le i\le b_j+1$.

Proof. This is easy to see for $j = 1$, as one has $R_{1,i} = V_{1,i}$ for $a_1 + 1 \le i \le b_1 + 1$, $S_{1,1+b_1} = V_{1,1+b_1}$, and

$$S_{1,i} = \sum_{k_i,\dots,k_{b_j}\in\mathbb{N}} S_{1,1+b_1}^{k_{b_1}} \cdots S_{1,i+1}^{k_i} V_{1,i} \big(S_{1,1+b_1}^{k_{b_1}}\big)^* \cdots \big(S_{1,i+1}^{k_i}\big)^* \quad \text{if } a_1 \le i \le b_1.$$

Fix a $j < k$. Assume that $R_{r,i}, S_{r,i} \in \psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))''$ for $1 \le r \le j$. Take i such that $a_{j+1} + 1 \le i \le b_{j+1} + 1$. Let $E_{j,i}$ be as in Corollary 3.6. Then,

$$R_{j+1,i} = \sum_{r=1}^{j-1} \sum_{k_{r,a_r},\dots,k_{r,b_r}\in\mathbb{N}} S_{r,1+b_r}^{k_{r,b_r}} \cdots S_{r,1+a_r}^{k_{r,a_r}} E_{j+1,i} \big(S_{r,1+a_r}^{k_{r,a_r}}\big)^* \cdots \big(S_{r,1+b_r}^{k_{r,b_r}}\big)^*.$$

Since $E_{j+1,i}$ belongs to the image $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))$, the operator $R_{j+1,i}$ belongs to $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))''$. Now note that

$$S_{j+1,i} = \sum_{k_i,\dots,k_{b_j}\in\mathbb{N}} R_{j+1,1+b_j}^{k_{b_j}} \cdots R_{j+1,i+1}^{k_i} R_{j+1,i} \big(R_{j+1,i+1}^{k_i}\big)^* \cdots \big(R_{j+1,1+b_j}^{k_{b_j}}\big)^*.$$

Therefore, it follows that $S_{j+1,i}$ also belongs to the strong operator closure of the image $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))$. ∎

Theorem 3.8. *Let $\lambda \in (S^1)^n$ and $\omega = s_{[a_k,b_k]} s_{[a_{k-1},b_{k-1}]} \cdots s_{[a_1,b_1]}$ be a reduced word in $\mathfrak{S}_{n+1}$. Then, the representation $\psi_{\lambda,\omega}$ is an irreducible representation.*

Proof. This is an immediate consequence of the previous proposition. ∎

Theorem 3.9. *Let $\lambda, \lambda' \in (S^1)^n$ and let*

$$\omega = s_{[a_k,b_k]} s_{[a_{k-1},b_{k-1}]} \cdots s_{[a_1,b_1]}, \quad \omega' = s_{[a'_{k'},b'_{k'}]} s_{[a'_{k'-1},b'_{k'-1}]} \cdots s_{[a'_1,b'_1]}$$

be two reduced words in $\mathfrak{S}_{n+1}$. If $(\lambda,\omega) \neq (\lambda',\omega')$, then the representations $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent.

Proof. Let $1 \le j \le n+1$. Define $m_{k+1}(j,\omega) = j$ and for $1 \le i \le k$, define $m_i(j,\omega)$ recursively as follows:

$$m_i(j,\omega) = \begin{cases} m_{i+1}(j,\omega) + 1 & \text{if } a_i \le m_{i+1}(j,\omega) \le b_i, \\ m_{i+1}(j,\omega) & \text{if } m_{i+1}(j,\omega) \ge b_i + 1 \text{ or } m_{i+1}(j,\omega) < a_i. \end{cases}$$

In the diagram for $\psi_{\lambda,\omega}$, one starts at j and proceeds horizontally and goes one step up whenever one encounters a diagonal going upward. The numbers $m_r(j,\omega)$ simply keep track of the horizontal line one is on when one completes the $s_{[a_r,b_r]}$-section. Thus, $m_1(j,\omega)$ is the maximum k such that $\psi_{\lambda,\omega}(z_{j,k}) \neq 0$.

Let us first assume that $\omega = \omega'$ and $\lambda \neq \lambda'$. Let

$$j := \max\{1 \le i \le n : \lambda_i \neq \lambda'_i\}.$$

If $b_{i+1} < j < b_i$ for some i, then

$$\mathrm{spec}\big(\psi_{\lambda,\omega}(z_{j+1,m_1(j+1,\omega)})\big) = \{0, \overline{\lambda_j}\lambda_{j+1}\},$$
$$\mathrm{spec}\big(\psi_{\lambda',\omega}(z_{j+1,m_1(j+1,\omega)})\big) = \{0, \overline{\lambda'_j}\lambda'_{j+1}\};$$

i.e., the spectrums of the same element are different for the two representations. Therefore, $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent. If $j = b_i$ for some i, then write $m = n'(i, a_i)$. Now using Corollary 3.6 and keeping track of the effect of λ and λ', we get that if $m > -\infty$, then

$$\begin{aligned}\mathrm{spec}\big(\psi_{\lambda',\omega}\big(E_{i,a_i}(\omega)\big)\big) &= \{0, \overline{\lambda}'_{b_m}\overline{\lambda}'_j\lambda'_{b_m+1}\lambda'_{j+1}\}\\ &= \{0, \overline{\lambda}_{b_m}\overline{\lambda}'_j\lambda_{b_m+1}\lambda_{j+1}\},\\ \mathrm{spec}\big(\psi_{\lambda,\omega}\big(E_{i,a_i}(\omega)\big)\big) &= \{0, \overline{\lambda}_{b_m}\overline{\lambda}_j\lambda_{b_m+1}\lambda_{j+1}\},\end{aligned}$$

and if $m = -\infty$, then

$$\mathrm{spec}\big(\psi_{\lambda',\omega}\big(E_{i,a_i}(\omega)\big)\big) = \{0, \overline{\lambda}'_j\lambda'_{j+1}\} = \{0, \overline{\lambda}'_j\lambda_{j+1}\},$$
$$\mathrm{spec}\big(\psi_{\lambda,\omega}\big(E_{i,a_i}(\omega)\big)\big) = \{0, \overline{\lambda}_j\lambda_{j+1}\}.$$

Thus, $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent in this case also.

Let us next assume that $\omega \neq \omega'$. We will split this into different cases and in each case, produce an element of $C(\mathrm{SU}_0(n+1))$ whose image under one representation will be zero while the image under the other is nonzero, thus proving that the two representations are not equivalent.

Case I. There exists $j \le \min\{k, k'\}$ such that $b_i = b'_i$, $a_i = a'_i$ for $1 \le i \le j-1$ and $b_j \neq b'_j$. Assume without loss of generality that $b_j > b'_j$. Then, $m_1(b_j, \omega) > m_1(b_j, \omega')$. Therefore,

$$\psi_{\lambda',\omega'}(z_{b_j,m_1(b_j,\omega)}) = 0, \quad \psi_{\lambda,\omega}(z_{b_j,m_1(b_j,\omega)}) \neq 0.$$

Case II. There exists $j \le \min\{k, k'\}$ such that $b_i = b'_i$, $a_i = a'_i$ for $1 \le i \le j-1$, $b_j = b'_j$, and $a_j \neq a'_j$. Assume without loss of generality that $a_j > a'_j$. Once again using Corollary 3.6, we observe that modulo multiplication by complex numbers of modulus one, $\psi_{\lambda,\omega}(E_{j,a_j}(\omega))$ is a projection, while $\psi_{\lambda',\omega'}(E_{j,a_j}(\omega))$ is a tensor product of shifts and projections. Therefore, $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent.

Case III. $b_i = b'_i$, $a_i = a'_i$ for $1 \le i \le k'$ and $k' < k$. In this case, $m_1(b_k, \omega) > m_1(b_k, \omega')$. Therefore,

$$\psi_{\lambda',\omega'}(z_{b_k,m_1(b_k,\omega)}) = 0, \quad \psi_{\lambda,\omega}(z_{b_k,m_1(b_k,\omega)}) \neq 0.$$

Case IV. $b_i = b'_i$, $a_i = a'_i$ for $1 \le i \le k$ and $k < k'$. This case is identical to Case III with k and k' interchanged. ∎

4. Projections in $C(\mathrm{SU}_0(n+1))$

In this section, we are going to restrict ourselves to the $*$-subalgebra $\mathcal{O}(\mathrm{SU}_0(n+1))$ of $C(\mathrm{SU}_0(n+1))$ generated by the $z_{i,j}$'s and derive certain relations that will in particular show that the generators $z_{i,j}$ are all partial isometries.

Lemma 4.1. *Let us denote by $p_{i,j}$ the element $z_{i,j}^* z_{i,j}$ and by $q_{i,j}$ the element $z_{i,j} z_{i,j}^*$. For $1 \le i \le j \le n+1$, one has the following:*

$$\sum_{k=1}^{i} q_{k,j} = \sum_{k=j}^{n+1} p_{i,k} \tag{4.1}$$

$$\sum_{k=1}^{j} q_{k,j} = \sum_{k=j}^{n+1} p_{j,k} = 1. \tag{4.2}$$

Proof. We will prove that

$$\sum_{k=1}^{i} q_{k,j} = \begin{cases} (z_{1,1} z_{2,2} \cdots z_{i-1,i-1}) z_{i,j} (z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1}) \\ \times (z_{j+1,j+1} \cdots z_{n+1,n+1}) & \text{if } i < j, \\ 1 & \text{if } i = j \end{cases} \tag{4.3}$$

$$\sum_{k=j}^{n+1} p_{i,k} = \begin{cases} (z_{1,1} z_{2,2} \cdots z_{i-1,i-1}) z_{i,j} (z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1}) \\ \times (z_{j+1,j+1} \cdots z_{n+1,n+1}) & \text{if } i < j, \\ 1 & \text{if } i = j. \end{cases} \tag{4.4}$$

Note that for $i = 1$, one has

$$q_{1,j} = z_{1,j} z_{1,j}^* = z_{1,j} (z_{2,1} z_{3,2} \cdots z_{j,j-1})(z_{j+1,j+1} z_{j+2,j+2} \cdots z_{n+1,n+1}).$$

Thus, the first equality in (4.3) holds for $i = 1$. Next, assume that it holds for $i-1$ and $1 \le i < j \le n+1$. Then, using (2.9), we get

$$\begin{aligned} q_{i,j} &+ \sum_{k=1}^{i-1} q_{k,j} \\ &= z_{i,j} (z_{1,1} z_{2,2} \cdots z_{i-2,i-2} z_{i-1,i-1})(z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1})(z_{j+1,j+1} \cdots z_{n+1,n+1}) \\ &\quad + (z_{1,1} z_{2,2} \cdots z_{i-2,i-2}) z_{i-1,j} z_{i,i-1} (z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1})(z_{j+1,j+1} \cdots z_{n+1,n+1}) \\ &= (z_{1,1} z_{2,2} \cdots z_{i-2,i-2}) z_{i,j} z_{i-1,i-1} (z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1})(z_{j+1,j+1} \cdots z_{n+1,n+1}) \\ &\quad + (z_{1,1} z_{2,2} \cdots z_{i-2,i-2}) z_{i-1,j} z_{i,i-1} (z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1}) \\ &\quad \times (z_{j+1,j+1} z_{j+2,j+2} \cdots z_{n+1,n+1}) \\ &= (z_{1,1} \cdots z_{i-2,i-2} z_{i-1,i-1}) z_{i,j} (z_{i+1,i} z_{i+2,i+1} \cdots z_{j,j-1})(z_{j+1,j+1} \cdots z_{n+1,n+1}). \end{aligned}$$

Hence, the first equality in (4.3) follows. Using this, one now gets

$$\begin{aligned}\sum_{k=1}^{j} q_{k,j} &= \sum_{k=1}^{j-1} q_{k,j} + q_{j,j}\\ &= (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{j-1,j}z_{j,j-1}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &\quad + z_{j,j}(z_{1,1}z_{2,2}\cdots z_{j-2,j-2}z_{j-1,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &= (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{j-1,j}z_{j,j-1}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{j,j}z_{j-1,j-1}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &= z_{1,1}z_{2,2}\cdots z_{j-2,j-2}z_{j-1,j-1}z_{j,j}z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1} = 1.\end{aligned}$$

Thus, we have (4.3). The proof of (4.4) is similar. First, observe that for $j = n+1$, one has

$$\begin{aligned}p_{i,n+1} = z_{i,n+1}^{*}z_{i,n+1} &= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{n+1,n})z_{i,n+1}\\ &= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,n+1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{n+1,n}).\end{aligned}$$

Thus, the first equality in (4.4) holds for $j = n+1$. Now, assuming that it holds for $j+1$, we have

$$\begin{aligned}&p_{i,j} + \sum_{k=j+1}^{n+1} p_{i,k}\\ &= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})\\ &\quad \times (z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})z_{i,j}\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,j+1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{j+1,j}(z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})\\ &\quad \times z_{j+1,j+1}z_{i,j}(z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{i,j+1}z_{j+1,j}(z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{i,j}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\ &= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,j}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}).\end{aligned}$$

This proves the first equality in (4.4). Using this we now get

$$\begin{aligned}\sum_{k=i}^{n+1} p_{i,k} &= \sum_{k=i+1}^{n+1} p_{i,k} + p_{i,i}\\ &= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,i+1}z_{i+1,i}(z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i+1}z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})z_{i,i}\\ &= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,i+1}z_{i+1,i}(z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i+1,i+1}z_{i,i}(z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})\\ &= z_{1,1}z_{2,2}\cdots z_{i-1,i-1}z_{i,i}z_{i+1,i+1}z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1} = 1,\end{aligned}$$

which completes the proof. ∎

Proposition 4.2. *The elements $p_{i,j}$ and $q_{i,j}$ are projections for $1 \leq i \leq j \leq n$.*

Proof. We will show that for any $1 \leq j \leq n$, if $\{\sum_{k=1}^{i} q_{k,j} : 1 \leq i \leq j\}$ is a family of projections, then

$$\left\{\sum_{k=1}^{i} q_{k,j+1} : 1 \leq i \leq j+1\right\}$$

is also a family of projections. Then, by an application of equation (4.2) in Lemma 4.1, we will successively conclude for $i = 1, 2, \ldots, n+1$ that the sums $\sum_{k=1}^{i} q_{k,j}$ are all projections for $i \leq j \leq n+1$. This will imply that all $q_{i,j}$'s (and hence all $p_{i,j}$'s) for $1 \leq i \leq j \leq n+1$ are projections.

Now fix $1 \leq j \leq n$ and assume that $\{\sum_{k=1}^{i} q_{k,j} : 1 \leq i \leq j\}$ is a family of projections. Note that by this assumption, it follows that

$$q_{i,j} = \sum_{k=1}^{i} q_{k,j} - \sum_{k=1}^{i-1} q_{k,j},$$

being positive and a difference of two projections, is also a projection. Hence, $p_{i,j}$ is also a projection. Now from Lemma 4.1, for $i \leq j$, we get

$$\sum_{k=1}^{i} q_{k,j+1} = \sum_{k=j+1}^{n+1} p_{i,k} = \sum_{k=j}^{n+1} p_{i,k} - p_{i,j} = \sum_{k=1}^{i} q_{k,j} - p_{i,j}.$$

Thus, the element $\sum_{k=1}^{i} q_{k,j+1}$ is positive and is a difference of two projections. Therefore, it is a projection. For $i = j+1$, $\sum_{k=1}^{j+1} q_{k,j+1} = 1$ and hence is a projection. ∎

Lemma 4.3. *For $1 \leq j \leq i \leq n+1$, one has*

$$\sum_{k=1}^{j} q_{i,k} = \sum_{k=i}^{n+1} p_{k,j}, \tag{4.5}$$

$$\sum_{k=1}^{i} q_{i,k} = \sum_{k=i}^{n+1} p_{k,i} = 1. \tag{4.6}$$

Proof. We will show that

$$\sum_{k=1}^{j} q_{i,k} = \begin{cases} (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i}) & \\ \times (z_{i+1,i+1}\cdots z_{n+1,n+1}), & \text{if } j < i, \\ 1 & \text{if } j = i, \end{cases} \tag{4.7}$$

$$\sum_{k=i}^{n+1} p_{k,j} = \begin{cases} (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i}) & \\ \times (z_{i+1,i+1}\cdots z_{n+1,n+1}), & \text{if } i > j, \\ 1 & \text{if } i = j. \end{cases} \tag{4.8}$$

For $j = 1$, the first equality in (4.7) holds as one has

$$q_{i,1} = z_{i,1}z_{i,1}^{*} = z_{i,1}(z_{1,2}z_{2,3}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}).$$

Next, assuming that it holds for $j-1$, we have

$$\begin{aligned}
& q_{i,j} + \sum_{k=1}^{j-1} q_{i,k} \\
&= z_{i,j}(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&\quad + (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{i,j-1}z_{j-1,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i}) \\
&\quad \times (z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&= (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{i,j}z_{j-1,j-1}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i}) \\
&\quad \times (z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&\quad + (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{i,j-1}z_{j-1,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i}) \\
&\quad \times (z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}).
\end{aligned}$$

Thus, the equality holds. For $i=j$, one has

$$\begin{aligned}
\sum_{k=1}^{i} q_{i,k} = \sum_{k=1}^{i-1} q_{i,k} + q_{i,i} \\
&= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,i-1}z_{i-1,i}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&\quad + z_{i,i}(z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i-1,i-1}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,i-1}z_{i-1,i}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,i}z_{i-1,i-1}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&= z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1}z_{i,i}z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1} = 1.
\end{aligned}$$

Thus, we have (4.7).

Next, note that

$$\begin{aligned}
p_{n+1,j} &= z_{n+1,j}^{*}z_{n+1,j} = (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{n,n+1})z_{n+1,j} \\
&= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{n+1,j}(z_{j,j+1}z_{j+1,j+2}c\cdots z_{n,n+1}).
\end{aligned}$$

Thus, the first equality in (4.8) holds for $i=n+1$. Assume it holds for $i+1$. Then,

$$\begin{aligned}
& p_{i,j} + \sum_{t=i+1}^{n+1} p_{t,j} \\
&= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})z_{i,j} \\
&\quad + (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i+1,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i,i+1}(z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&= (z_{1,1}z_{2,2}\cdots z_{j-2,j-2}z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i}) \\
&\quad \times z_{i+1,i+1}z_{i,j}(z_{i+2,i+2}\cdots z_{n+1,n+1}) \\
&\quad + (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i,i+1}z_{i+1,j}(z_{i+2,i+2}\cdots z_{n+1,n+1})
\end{aligned}$$

$$= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i,j}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})$$
$$= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}).$$

Thus, we have the first equality in (4.8). For $i = j$,

$$\begin{aligned}\sum_{k=j}^{n+1} p_{k,j} &= \sum_{k=j+1}^{n+1} p_{k,j} + p_{j,j}\\ &= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{j+1,j}z_{j,j+1}(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})z_{j,j}\\ &= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{j+1,j}z_{j,j+1}(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})\\ &\quad + (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{j+1,j+1}z_{j,j}(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})\\ &= z_{1,1}z_{2,2}\cdots z_{j-1,j-1}z_{j,j}z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}\\ &= 1.\end{aligned}$$

Thus, the proof is complete. ■

Proposition 4.4. *The elements $p_{i,j}$ and $q_{i,j}$ are projections for $1 \le j \le i \le n$.*

Proof. By using Lemma 4.3, one can prove exactly as in the proof of Proposition 4.2 that if $\{\sum_{k=1}^{j} q_{i,k} : 1 \le j \le i\}$ is a family of projections, then $\{\sum_{k=1}^{j} q_{i+1,k} : 1 \le j \le i+1\}$ is also a family of projections. This leads to the required conclusion. ■

Corollary 4.5. *The elements $z_{i,j}$ are partial isometries for all $1 \le i, j \le n+1$.*

Lemma 4.6. *Let $i < k$ and $j < \ell$. Assume that either $i \ge \ell$ or $k \le j$. Then,*

$$z_{i,\ell}z_{k,j}^* = 0 = z_{i,\ell}^* z_{k,j}. \tag{4.9}$$

Proof. Assume that $i \ge \ell$. Then, using Lemma 4.3, we have

$$\begin{aligned}\sum_{k=i+1}^{n+1}(z_{i,\ell}z_{k,j}^*)(z_{i,\ell}z_{k,j}^*)^* &= z_{i,\ell}\Big(\sum_{k=i+1}^{n+1} p_{k,j}\Big)z_{i,\ell}^* = z_{i,\ell}\Big(\sum_{s=1}^{j} q_{i+1,s}\Big)z_{i,\ell}^*\\ &= \sum_{s=1}^{j} z_{i,\ell}z_{i+1,s}z_{i+1,s}^* z_{i,\ell}^* = 0\end{aligned}$$

since $z_{i,\ell}z_{i+1,s} = 0$ for all $1 \le s \le \ell - 1$. Therefore, $z_{i,\ell}z_{k,j}^* = 0$ for all $j < \ell$ and $i < k$. Similarly, if $k \le j$, then using Lemma 4.1, we get

$$\begin{aligned}\sum_{\ell=j+1}^{n+1}(z_{i,\ell}z_{k,j}^*)^*(z_{i,\ell}z_{k,j}^*) &= z_{k,j}\sum_{\ell=j+1}^{n+1}(p_{i,\ell})z_{k,j}^*\\ &= z_{k,j}\Big(\sum_{s=1}^{i} q_{s,j+1}\Big)z_{k,j}^* = 0.\end{aligned}$$

Therefore, as before, $z_{i,\ell} z^*_{k,j} = 0$. The second equality can be proved in a similar way using Lemmas 4.1 and 4.3. ∎

Lemma 4.7. *Let* $i, j \leq s$ *and* $k \geq s+1$. *Then,*

$$q_{i,j} z_{n+1,k} = z_{n+1,k} q_{i,j}, \quad z_{i,j} p_{n+1,k} = p_{n+1,k} z_{i,j}.$$

Proof. If $\max\{i,j\}+1 < k$, then one has $z_{i,j} z_{n+1,k} = z_{n+1,k} z_{i,j}$ and hence the equality follows. Assume $\max\{i,j\}+1 = k$ (i.e., $k = s+1$ and $\max\{i,j\} = s$). Then,

$$z_{i,j} z_{n+1,k} = z_{n+1,k} z_{i,j} + z_{i,k} z_{n+1,j}.$$

It follows from Lemmas 4.1 and 4.3 that if $i \geq j$, then $p_{n+1,j} p_{i,j} = 0$ and if $i < j$, then $p_{i,k} p_{i,j} = 0$. Therefore, in either case, we get from the above equation that

$$\begin{aligned} q_{i,j} z_{n+1,k} &= z_{i,j} z_{n+1,k} z^*_{i,j} \\ &= z_{n+1,k} z_{i,j} z^*_{i,j} + z_{i,k} z_{n+1,j} z^*_{i,j} \\ &= z_{n+1,k} q_{i,j}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} z_{i,j} p_{n+1,k} &= z_{i,j} z^*_{n+1,k} z_{n+1,k} \\ &= z^*_{n+1,k} z_{i,j} z_{n+1,k} \\ &= p_{n+1,k} z_{i,j} + z^*_{n+1,k} z_{i,k} z_{n+1,j} \\ &= p_{n+1,k} z_{i,j} + z^*_{n+1,k} z_{n+1,j} z_{i,k} \\ &= p_{n+1,k} z_{i,j}. \end{aligned}$$ ∎

Lemma 4.8. *Let* $1 \leq i \leq j \leq n+1$. *Then,*

$$z_{i,j+1} z^*_{n+1,j+1} = z^*_{n+1,j} z_{i,j}.$$

Proof. We have

$$\begin{aligned} &(z_{i,j+1} z^*_{n+1,j+1} - z^*_{n+1,j} z_{i,j})(z_{i,j+1} z^*_{n+1,j+1} - z^*_{n+1,j} z_{i,j})^* \\ &\quad = z_{i,j+1} p_{n+1,j+1} z^*_{i,j+1} - z_{i,j+1} z^*_{n+1,j+1} z^*_{i,j} z_{n+1,j} \\ &\qquad + z^*_{n+1,j} q_{i,j} z_{n+1,j} - z^*_{n+1,j} z_{i,j} z_{n+1,j+1} z^*_{i,j+1} \\ &\quad = z_{i,j+1}(p_{n+1,j} + q_{n+1,j+1}) z^*_{i,j+1} - z_{i,j+1}(z_{n+1,j+1} z_{i,j} + z_{n+1,j} z_{i,j+1})^* z_{n+1,j} \\ &\qquad + z^*_{n+1,j} q_{i,j} z_{n+1,j} - z^*_{n+1,j}(z_{n+1,j+1} z_{i,j} + z_{n+1,j} z_{i,j+1}) z^*_{i,j+1} \\ &\quad = z_{i,j+1} p_{n+1,j} z^*_{i,j+1} - z_{i,j+1} z^*_{i,j+1} z^*_{n+1,j} z_{n+1,j} \\ &\qquad + z^*_{n+1,j} q_{i,j} z_{n+1,j} - z^*_{n+1,j} z_{n+1,j} z_{i,j+1} z^*_{i,j+1} \\ &\quad = p_{n+1,j} q_{i,j+1} + z^*_{n+1,j} q_{i,j} z_{n+1,j} - p_{n+1,j} q_{i,j+1} - z^*_{n+1,j} q_{i,j+1} z_{n+1,j} \\ &\quad = z^*_{n+1,j}(q_{i,j} - q_{i,j+1}) z_{n+1,j}. \end{aligned}$$

Summing over i, we get

$$\begin{aligned}
&\sum_{i=1}^{j}(z_{i,j+1}z_{n+1,j+1}^{*}-z_{n+1,j}^{*}z_{i,j})(z_{i,j+1}z_{n+1,j+1}^{*}-z_{n+1,j}^{*}z_{i,j})^{*}\\
&\quad=z_{n+1,j}^{*}\left(\sum_{i=1}^{j}q_{i,j}-\sum_{i=1}^{j}q_{i,j+1}\right)z_{n+1,j}\\
&\quad=z_{n+1,j}^{*}\left(I-\sum_{k=j+1}^{n+1}p_{j,k}\right)z_{n+1,j}\\
&\quad=z_{n+1,j}^{*}p_{j,j}z_{n+1,j}\\
&\quad=0.
\end{aligned}$$

Therefore, we have the result. ■

5. Operators in $\pi(C(\mathrm{SU}_0(n+1)))''$

In this section, we will study the actions of the generating elements $z_{i,j}$ on a Hilbert space and construct certain operators using the strong operator topology available there. These will turn out to be a very useful tool in analyzing the irreducible representations. Let us start with a representation π of $C(\mathrm{SU}_0(n+1))$ on a Hilbert space $\mathcal{H}$. For all i and j, we will denote the operators $\pi(z_{i,j})$, $\pi(p_{i,j})$, and $\pi(q_{i,j})$ by $Z_{i,j}$, $P_{i,j}$, and $Q_{i,j}$, respectively.

Proposition 5.1. *Let $1 \le s \le m \le n$. Assume $Z_{m+1,s} = 0$ where $s < m+1$. Then, $Z_{i,j} = 0$ for $i \ge m+1$ and $j \le s$.*

Proof. Let us assume $s > 1$ and take $i = m+1$, $j = s-1$. From relation (2.4), we have $Z_{m+1,s-1}Z_{s,s} = 0$, so that

$$Z_{m+1,s-1}Q_{s,s} = 0.$$

For $1 \le k \le s-1$, the commutation relation (2.5) gives us

$$Z_{k,s-1}Z_{m+1,s} - Z_{m+1,s}Z_{k,s-1} = Z_{m+1,s-1}Z_{k,s}.$$

Since $Z_{m+1,s} = 0$, we get $Z_{m+1,s-1}Z_{k,s} = 0$ which implies

$$Z_{m+1,s-1}Q_{k,s} = 0, \quad 1 \le k \le s-1.$$

Therefore,

$$Z_{m+1,s-1} = Z_{m+1,s-1}\left(\sum_{k=1}^{s} Q_{k,s}\right) = 0. \tag{5.1}$$

Next, let us assume $m < n$ and take $i = m+2$, $j = s$. Using relation (2.4), we obtain $Z_{m+1,m+1}Z_{m+2,s} = 0$, so that

$$P_{m+1,m+1}Z_{m+2,s} = 0.$$

For $m+2 \leq k \leq n+1$, we get from (2.5)

$$Z_{m+1,s}Z_{m+2,k} - Z_{m+2,k}Z_{m+1,s} = Z_{m+1,k}Z_{m+2,s}.$$

Since $Z_{m+1,s} = 0$, we get $Z_{m+1,k}Z_{m+2,s} = 0$ which implies

$$P_{m+1,k}Z_{m+2,s} = 0, \quad m+2 \leq k \leq n+1.$$

Therefore,

$$Z_{m+2,s} = \Big(\sum_{k=m+1}^{n+1} P_{m+1,k}\Big)Z_{m+2,s} = 0. \tag{5.2}$$

By the repeated application of (5.1) and (5.2), we get $Z_{i,j} = 0$ for $i \geq m+1$ and $j \leq s$. ■

Lemma 5.2. *Assume $i \neq m$ and $j \neq k$. If $Z_{i,j}Q_{m,k} = Q_{m,k}Z_{i,j}$ and $Z_{m,k}Q_{i,j} = Q_{i,j}Z_{m,k}$, then one has $Z_{i,j}Z_{m,k} = Z_{m,k}Z_{i,j}$.*

Proof. Computing $(Z_{i,j}Z_{m,k} - Z_{m,k}Z_{i,j})(Z_{i,j}Z_{m,k} - Z_{m,k}Z_{i,j})^*$ using the given relations, we get the required equality. ■

Lemma 5.3. *Let F be a subset of $\{1, 2, \ldots, n+1\}$. Let $j \notin F$ and $i \neq m$. Assume $Z_{m,k}Q_{i,j} = Q_{i,j}Z_{m,k}$ for all $k \in F$ and $Z_{i,j}(\sum_{k\in F} Q_{m,k}) = (\sum_{k\in F} Q_{m,k})Z_{i,j}$. Then,*

$$Z_{i,j}Z_{m,k} = Z_{m,k}Z_{i,j} \quad \textit{for all } k \in F.$$

Proof. Note that

$$\begin{aligned}
&\sum_{k\in F}(Z_{i,j}Z_{m,k} - Z_{m,k}Z_{i,j})(Z_{i,j}Z_{m,k} - Z_{m,k}Z_{i,j})^* \\
&= Z_{i,j}\Big(\sum_{k\in F} Q_{m,k}\Big)Z_{i,j}^* + \sum_{k\in F}(Z_{m,k}Q_{i,j}Z_{m,k}^*) \\
&\quad - Z_{i,j}\sum_{k\in F}(Z_{m,k}Z_{i,j}^*Z_{m,k}^*) - \sum_{k\in F}(Z_{m,k}Z_{i,j}Z_{m,k}^*)Z_{i,j}^* \\
&= Q_{i,j}\Big(\sum_{k\in F} Q_{m,k}\Big) + Q_{i,j}\sum_{k\in F}(Z_{m,k}Z_{m,k}^*) \\
&\quad - Z_{i,j}Z_{i,j}^*\sum_{k\in F}(Z_{m,k}Z_{m,k}^*) - Z_{i,j}Z_{i,j}^*\sum_{k\in F}(Z_{m,k}Z_{m,k}^*) \\
&= 0.
\end{aligned}$$

Therefore, we have the result. ■

Given a representation π of $C(\mathrm{SU}_0(n+1))$, we will denote by $r(\pi) \equiv r$ the following

$$r(\pi) \equiv r := \min\{i : 1 \leq i \leq n+1, Z_{n+1,i} \neq 0\}. \tag{5.3}$$

Proposition 5.4. *Assume $2 \leq r \leq n$. Then, one has*

$$Z_{i,j}Z_{n+1,k} = Z_{n+1,k}Z_{i,j} \quad \textit{for } 1 \leq i \leq n,\ 1 \leq j \leq r-1, \textit{ and } r \leq k \leq n+1. \tag{5.4}$$

Proof. Observe that for $1 \le i \le r-1$, $1 \le j \le r-1$, and $r \le k \le n+1$, one has

$$\max\{i, j\} + 1 \le r \le \min\{n+1, k\}.$$

If $\max\{i, j\} + 1 < \min\{n+1, k\}$, then $Z_{i,j} Z_{n+1,k} = Z_{n+1,k} Z_{i,j}$. Since by Proposition 5.1, we have $Z_{n+1,j} = 0$ for $1 \le j \le r-1$, if $\max\{i, j\} + 1 = \min\{n+1, k\}$, then one has

$$Z_{i,j} Z_{n+1,k} = Z_{n+1,k} Z_{i,j} + Z_{n+1,j} Z_{i,k} = Z_{n+1,k} Z_{i,j}.$$

Let us next assume that $Z_{i,j} Z_{n+1,k} = Z_{n+1,k} Z_{i,j}$ for all $1 \le j \le r-1$, $r \le k \le n+1$, and $1 \le i \le t < n$. Then, $P_{i,j} Z_{n+1,k} = Z_{n+1,k} P_{i,j}$ for all such j, k, and i. Therefore,

$$Z_{n+1,k} = Z_{n+1,k} \left(\sum_{i=j}^{n} P_{i,j} \right) = \left(\sum_{i=j}^{n} P_{i,j} \right) Z_{n+1,k}.$$

Since $P_{i,j} Z_{n+1,k} = Z_{n+1,k} P_{i,j}$ for $1 \le i \le t$ and $\sum_{i=t+1}^{n} P_{i,j} = \sum_{m=1}^{j} Q_{t+1,m}$, it follows that

$$Z_{n+1,k} \left(\sum_{m=1}^{j} Q_{t+1,m} \right) = \left(\sum_{m=1}^{j} Q_{t+1,m} \right) Z_{n+1,k}.$$

Therefore,

$$Z_{n+1,k} Q_{t+1,j} = Q_{t+1,j} Z_{n+1,k}$$

for $1 \le j \le r-1$. On the other hand, we also have

$$Z_{t+1,j} = Z_{t+1,j} \left(\sum_{k=r}^{n+1} Q_{n+1,k} \right) = \left(\sum_{k=r}^{n+1} Q_{n+1,k} \right) Z_{t+1,j}.$$

Therefore, by Lemma 5.3, we have

$$Z_{t+1,j} Z_{n+1,k} = Z_{n+1,k} Z_{t+1,j}$$

for $r \le k \le n+1$ and $1 \le j \le r-1$.

By repeated application, we now get the result. ∎

We next look at certain infinite sums of elements from $\pi(C(\mathrm{SU}_0(n+1)))$ that converge in the strong operator topology so that they belong to the strong operator closure $\pi(C(\mathrm{SU}_0(n+1)))''$ of $\pi(C(\mathrm{SU}_0(n+1)))$.

Proposition 5.5. *Let r be as in (5.3) and let $r \le k \le n+1$. If T_k is a partial isometry with $T_k^* T_k \le P_{n+1,k}$ and $T_k T_k^* \le P_{n+1,k}$, then the operator*

$$T_{k+1} := \sum_{m \in \mathbb{N}} Z_{n+1,k+1}^{m} T_k (Z_{n+1,k+1}^{m})^*$$

exists as a strong operator convergent sum and is a partial isometry with $T_{k+1}^* T_{k+1} \le P_{n+1,k+1}$ and $T_{k+1} T_{k+1}^* \le P_{n+1,k+1}$.

Proof. Note that the operators

$$R_{k,m} := Z^m_{n+1,k+1} P_{n+1,k} (Z^m_{n+1,k+1})^*, \quad m \in \mathbb{N},$$

form a family of orthogonal projections with $\sum_{m\in\mathbb{N}} R_{k,m} \leq P_{n+1,k+1}$. Observe also that

$$P_{n+1,k+1} - \sum_{j=0}^{m} R_{k,j} = Z^{m+1}_{n+1,k+1} (Z^{m+1}_{n+1,k+1})^*,$$

and the sum $\sum_{j\in\mathbb{N}} R_{k,j}$ converges in the strong operator topology (SOT). Hence,

$$P_{n+1,k+1} - \sum_{j\in\mathbb{N}} R_{k,j} = \text{SOT-}\lim_{m\to\infty} Z^m_{n+1,k+1} (Z^m_{n+1,k+1})^*. \tag{5.5}$$

Next, write

$$T_{k,m} = Z^m_{n+1,k+1} T_k (Z^m_{n+1,k+1})^*, \quad m \in \mathbb{N}.$$

Then, one has $T_{k,m} = R_{k,m} T_{k,m} R_{k,m}$ for all $m \in \mathbb{N}$. It is easy to check that $T^*_{k,m} T_{k,m'} = 0 = T_{k,m} T^*_{k,m'}$ for $m \neq m'$ and one has

$$T^*_{k,m} T_{k,m} \leq P_{n+1,k+1}, \quad T_{k,m} T^*_{k,m} \leq P_{n+1,k+1}.$$

Therefore, the sum $T := \sum_{m\in\mathbb{N}} T_{k,m}$ converges in the strong operator topology and it is a partial isometry with both initial and range spaces contained in $P_{n+1,k+1}\mathcal{H}$. ∎

Using the above proposition, one can now define recursively the operators $W_{i,j}$ for $1 \leq j \leq n+1$ and $j \leq i \leq n+1$ and the operators U_i for $r+1 \leq i \leq n+1$ as follows:

$$W_{i,j} := \begin{cases} Z_{n+1,j}, & \text{if } i = j, \\ \sum_{k\in\mathbb{N}} Z^k_{n+1,i} W_{i-1,j} (Z^k_{n+1,i})^*, & \text{if } j+1 \leq i \leq n+1, \end{cases} \tag{5.6}$$

$$U_i := \begin{cases} Z_{n+1,r}, & \text{if } i = r+1, \\ \sum_{k\in\mathbb{N}} Z^k_{n+1,i} U_{i-1} (Z^k_{n+1,i})^*, & \text{if } r+2 \leq i \leq n+1. \end{cases} \tag{5.7}$$

Thus, we have for $i \geq r+2$,

$$\begin{aligned} W_{i,r+1} = \sum_{\substack{k_j\in\mathbb{N} \\ (r+2\leq j\leq i)}} & Z^{k_i}_{n+1,i} Z^{k_{i-1}}_{n+1,i-1} \cdots Z^{k_{r+2}}_{n+1,r+2} \\ & \times Z_{n+1,r+1} \big(Z^{k_{r+2}}_{n+1,r+2}\big)^* \cdots \big(Z^{k_{i-1}}_{n+1,i-1}\big)^* \big(Z^{k_i}_{n+1,i}\big)^*, \end{aligned} \tag{5.8}$$

$$\begin{aligned} U_i = \sum_{\substack{k_j\in\mathbb{N} \\ (r+2\leq j\leq i)}} & Z^{k_i}_{n+1,i} Z^{k_{i-1}}_{n+1,i-1} \cdots Z^{k_{r+2}}_{n+1,r+2} \\ & \times Z_{n+1,r} \big(Z^{k_{r+2}}_{n+1,r+2}\big)^* \cdots \big(Z^{k_{i-1}}_{n+1,i-1}\big)^* \big(Z^{k_i}_{n+1,i}\big)^*, \end{aligned} \tag{5.9}$$

and for $i \geq r+1$,

$$W_{i,r} = \sum_{\substack{k_j \in \mathbb{N} \\ (r+1 \leq j \leq i)}} Z_{n+1,i}^{k_i} Z_{n+1,i-1}^{k_{i-1}} \cdots Z_{n+1,r+1}^{k_{r+1}} \times Z_{n+1,r} (Z_{n+1,r+1}^{k_{r+1}})^* \cdots (Z_{n+1,i-1}^{k_{i-1}})^* (Z_{n+1,i}^{k_i})^*. \tag{5.10}$$

We will use these three operators repeatedly in the rest of the paper. We start with a few observations on these operators.

Proposition 5.6. *Let r be as in (5.3). Then,*

1. *U_i is a normal partial isometry for $r+2 \leq i \leq n+1$,*
2. *$W_{i,r}$ is a normal partial isometry for all $r \leq i \leq n+1$,*
3. *$W_{i,r+1}$ is a partial isometry with source projection and range projection contained in $P_{n+1,i}$ for $r+2 \leq i \leq n+1$.*

Proof. By Proposition 5.5, the operators U_i, $W_{i,r}$, and $W_{i,r+1}$ are all partial isometries. The normality of U_i and $W_{i,r}$ follows from the fact that $P_{n+1,r} = Q_{n+1,r}$. It is easy to see from (5.8) and (5.9) that

$$W_{i,r+1}^* W_{i,r+1} \leq P_{n+1,i}, \tag{5.11}$$

$$W_{i,r+1}^* W_{i,r+1} - W_{i,r+1} W_{i,r+1}^* = U_i^* U_i. \tag{5.12}$$

Therefore, part (3) follows. ■

Proposition 5.7. *Let r be as in (5.3). Then, we have the following relations:*

$$Z_{i,r} W_{n+1,r+1} - W_{n+1,r+1} Z_{i,r} = U_{n+1} Z_{i,r+1}, \quad i \leq r, \tag{5.13}$$

$$Z_{i,r} W_{n+1,r+1}^* - W_{n+1,r+1}^* Z_{i,r} = 0, \quad i \leq r. \tag{5.14}$$

Proof. Since $i \leq r$, the operator $Z_{i,r}$ commutes with $Z_{n+1,j}$ for $r+2 \leq j \leq n+1$. Using this and relation (2.5), we get (5.13). Equation (5.14) follows by using (2.5) and (2.8). ■

Theorem 5.8. *Let r be as in (5.3). Then, the operator $W_{n+1,r}$ commutes with all the $Z_{i,j}$'s.*

Proof. For $1 \leq j \leq r-1$, the commutations follow from Proposition 5.4. Assume next that $1 \leq i \leq j = r$. Then,

$$\begin{aligned} Z_{i,r} W_{r+1,r} &= Z_{i,r} \sum_{k \in \mathbb{N}} Z_{n+1,r+1}^k Z_{n+1,r} (Z_{n+1,r+1}^k)^* \\ &= \sum_{k \in \mathbb{N}} Z_{n+1,r+1}^k Z_{i,r} Z_{n+1,r} (Z_{n+1,r+1}^k)^* \\ &\quad + \sum_{k \in \mathbb{N}} Z_{n+1,r+1}^k Z_{n+1,r} Z_{i,r+1} Z_{n+1,r} (Z_{n+1,r+1}^{k+1})^* \\ &= \sum_{k \in \mathbb{N}} Z_{n+1,r+1}^k Z_{n+1,r}^2 Z_{i,r+1} (Z_{n+1,r+1}^{k+1})^*, \end{aligned}$$

and

$$\begin{aligned} W_{r+1,r}Z_{i,r} &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}(Z^*_{n+1,r+1})^k Z_{i,r} \\ &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}Z_{i,r}(Z^k_{n+1,r+1})^*. \end{aligned}$$

Thus, we have

$$\begin{aligned} &Z_{i,r}W_{r+1,r} - W_{r+1,r}Z_{i,r} \\ &\quad = \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}(Z_{n+1,r}Z_{i,r+1}Z^*_{n+1,r+1} - Z_{i,r})(Z^k_{n+1,r+1})^*. \end{aligned}$$

Now,

$$\begin{aligned} &(Z^2_{n+1,r}Z_{i,r+1}Z^*_{n+1,r+1} - Z_{n+1,r}Z_{i,r})(Z^2_{n+1,r}Z_{i,r+1}Z^*_{n+1,r+1} - Z_{n+1,r}Z_{i,r})^* \\ &\quad = Z_{n+1,r}Q_{i,r}Z^*_{n+1,r} - Q_{n+1,r}Q_{i,r+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{r}(Z^2_{n+1,r}Z_{i,r+1}Z^*_{n+1,r+1} - Z_{n+1,r}Z_{i,r})(Z^2_{n+1,r}Z_{i,r+1}Z^*_{n+1,r+1} - Z_{n+1,r}Z_{i,r})^* \\ &\quad = Z_{n+1,r}\left(\sum_{i=1}^{r} Q_{i,r}\right)Z^*_{n+1,r} - Q_{n+1,r}\left(\sum_{i=1}^{r} Q_{i,r+1}\right) \\ &\quad = Q_{n+1,r} - Q_{n+1,r}\left(\sum_{k=r+1}^{n+1} P_{r,k}\right) \\ &\quad = Q_{n+1,r}P_{r,r} = P_{n+1,r}P_{r,r} = 0. \end{aligned}$$

Thus, we have $Z_{i,r}W_{r+1,r} = W_{r+1,r}Z_{i,r}$ for $1 \le i \le r$. Since $Z_{i,r}$ commutes with $Z_{n+1,j}$ for all $j \ge r+2$, we get

$$Z_{i,r}W_{n+1,r} = W_{n+1,r}Z_{i,r}.$$

Next, let $1 \le i \le r+1$. Then, one has

$$\begin{aligned} &Z_{i,r+1}W_{r+2,r} \\ &\quad = Z_{i,r+1}\sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z^k_{n+1,r+2}Z^s_{n+1,r+1}Z_{n+1,r}(Z^s_{n+1,r+1})^*(Z^k_{n+1,r+2})^* \\ &\quad = \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z^k_{n+1,r+2}Z_{i,r+1}Z^s_{n+1,r+1}Z_{n+1,r}(Z^s_{n+1,r+1})^*(Z^k_{n+1,r+2})^* \\ &\qquad + \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z^k_{n+1,r+2}Z_{n+1,r+1}Z_{i,r+2}Z^s_{n+1,r+1}Z_{n+1,r}(Z^s_{n+1,r+1})^*(Z^{k+1}_{n+1,r+2})^* \\ &\quad = \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2}Z_{i,r+1}Z_{n+1,r}(Z^k_{n+1,r+2})^* \\ &\qquad + \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z^k_{n+1,r+2}Z^{s+1}_{n+1,r+1}Z_{i,r+2}Z_{n+1,r}(Z^s_{n+1,r+1})^*(Z^{k+1}_{n+1,r+2})^* \end{aligned}$$

$$= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r} (Z^k_{n+1,r+2})^* Z_{i,r+1}$$
$$+ \sum_{k\in\mathbb{N}} \sum_{s\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r+1} W_{r+1,r} Z_{i,r+2} (Z^{k+1}_{n+1,r+2})^*,$$

and

$$W_{r+2,r} Z_{i,r+1} = \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r} (Z^k_{n+1,r+2})^* Z_{i,r+1}$$
$$+ \sum_{k\in\mathbb{N}} \sum_{s\in\mathbb{N}} Z^k_{n+1,r+2} Z^{s+1}_{n+1,r+1} Z_{n+1,r} (Z^{s+1}_{n+1,r+1})^* (Z^k_{n+1,r+2})^* Z_{i,r+1}$$
$$= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r} (Z^k_{n+1,r+2})^* Z_{i,r+1}$$
$$+ \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r+1} W_{r+1,r} Z^*_{n+1,r+1} Z_{i,r+1} (Z^k_{n+1,r+2})^*.$$

Using Lemma 4.8, we now get

$$Z_{i,r+1} W_{r+2,r} - W_{r+2,r} Z_{i,r+1}$$
$$= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r+1} W_{r+1,r} (Z_{i,r+2} Z^*_{n+1,r+2} - Z^*_{n+1,r+1} Z_{i,r+1})$$
$$\times Z^*_{n+1,r+1} Z_{i,r+1} (Z^k_{n+1,r+2})^*$$
$$= 0.$$

Thus, we have $Z_{i,r+1} W_{r+2,r} = W_{r+2,r} Z_{i,r+1}$. Since $Z_{i,r+1}$ commutes with $Z_{n+1,j}$ for every $j \geq r+3$, we get

$$Z_{i,r+1} W_{n+1,r} = W_{n+1,r} Z_{i,r+1}.$$

Let $j \geq r+2$ and $1 \leq i \leq j$. Then, one has

$$Z_{i,j} W_{j+1,r} = Z_{i,j} \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1} W_{j,r} (Z^*_{n+1,j+1})^k$$
$$= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1} Z_{i,j} W_{j,r} (Z^*_{n+1,j+1})^k$$
$$+ \sum_{k\geq 1} Z^{k-1}_{n+1,j+1} Z_{n+1,j} Z_{i,j+1} W_{j,r} (Z^*_{n+1,j+1})^k$$
$$= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1} W_{j-1,r} (Z^*_{n+1,j+1})^k Z_{i,j}$$
$$+ \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1} Z_{n+1,j} W_{j,r} Z_{i,j+1} (Z^*_{n+1,j+1})^{k+1},$$

and

$$\begin{aligned}
W_{j+1,r}Z_{i,j} &= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j,r}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
&= \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j}^s W_{j-1,r}(Z_{n+1,j}^*)^s (Z_{n+1,j+1}^*)^k Z_{i,j} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j-1,r}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}}\sum_{s\geq 1} Z_{n+1,j+1}^k Z_{n+1,j}^s W_{j-1,r}(Z_{n+1,j}^*)^s (Z_{n+1,j+1}^*)^k Z_{i,j} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j-1,r}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r} Z_{n+1,j}^* (Z_{n+1,j+1}^*)^k Z_{i,j}.
\end{aligned}$$

Use Lemma 4.8 to get

$$\begin{aligned}
&Z_{i,j}W_{j+1,r} - W_{j+1,r}Z_{i,j} \\
&\quad = \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r} Z_{i,j+1}(Z_{n+1,j+1}^*)^{k+1} \\
&\qquad - \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r} Z_{n+1,j}^* (Z_{n+1,j+1}^*)^k Z_{i,j} \\
&\quad = \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r}(Z_{i,j+1}Z_{n+1,j+1}^* - Z_{n+1,j}^* Z_{i,j})(Z_{n+1,j+1}^*)^k \\
&\quad = 0.
\end{aligned}$$

Thus, we have

$$Z_{i,j}W_{j+1,r} = W_{j+1,r}Z_{i,j}.$$

Since $Z_{i,j}$ commutes with $Z_{n+1,k}$ for all $k \geq j+2$, we get

$$Z_{i,j}W_{n+1,r} = W_{n+1,r}Z_{i,j}.$$

Thus, $W_{n+1,r}$ commutes with $Z_{i,j}$ for all $i \leq j$. Therefore, using (2.9), we conclude that $W_{n+1,r}$ commutes with $Z_{i,j}^*$ for all $i > j$. Since $W_{n+1,r}$ is normal, it follows that it commutes with all $Z_{i,j}$'s. ∎

Proposition 5.9. *Let r be as in (5.3). Then,*

$$W_{n+1,r}^* U_{n+1} = U_{n+1}^* U_{n+1}.$$

Proof. We have

$$
\begin{aligned}
&W^*_{n+1,r}U_{n+1}\\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+1\le j\le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+1}}_{n+1,r+1}Z^*_{n+1,r}(Z^*_{n+1,r+1})^{k_{r+1}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\big)\\
&\quad\times \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\big)\\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \Big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}\Big(\sum_{k_{r+1}\in\mathbb{N}} Z^{k_{r+1}}_{n+1,r+1}Z^*_{n+1,r}(Z^*_{n+1,r+1})^{k_{r+1}}\Big)\\
&\qquad\qquad \times Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\Big)\\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}P_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&U^*_{n+1}U_{n+1}\\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}Z^*_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\big)\\
&\quad\times \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\big)\\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}P_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\big).
\end{aligned}
$$

Thus, we have the required equality. ■

6. Implications of irreducibility

We now further restrict our attention to the images of the $z_{i,j}$'s under an irreducible representation π. Irreducibility will help us extract more information on the operators $W_{i,j}$'s, which will be needed in analyzing irreducible representations.

Proposition 6.1. *Assume π is irreducible and let r be as in (5.3); i.e.,*

$$r = \min\{1 \le i \le n+1 : \pi(z_{n+1,i}) \neq 0\}.$$

Then, there is a $\lambda \in S^1$ such that $W_{n+1,r} = \lambda I$.

Proof. Since $W_{n+1,r}$ is normal, by Theorem 5.8 and by the irreducibility of π, it is a scalar operator. But it is also a partial isometry and

$$W_{n+1,r}^* W_{n+1,r} \geq P_{n+1,r} \neq 0.$$

Therefore, we have the result. ■

Our next goal is to show that for an irreducible representation π, the operator $W_{n+1,r+1}$ is an isometry. For that, we start with a few commutation relations.

Proposition 6.2. *Let* $1 \leq j \leq r-1$ *and* $1 \leq i \leq n$. *Then,*

$$W_{n+1,r+1} Z_{i,j} = Z_{i,j} W_{n+1,r+1}, \quad W_{n+1,r+1}^* Z_{i,j} = Z_{i,j} W_{n+1,r+1}^*.$$

Proof. This is an immediate consequence of Proposition 5.4. ■

Proposition 6.3. *Let* r *be as above. Then one has:*

$$W_{i,r+1} Z_{n+1,j} = 0, \qquad r+2 \leq i < j, \tag{6.1}$$

$$Z_{n+1,j} W_{i,r+1} = W_{i,r+1} Z_{n+1,j}, \qquad j \leq i \leq n+1, \tag{6.2}$$

$$Z_{i,j} W_{j,r+1} = Z_{i,j} W_{j-1,r+1} = W_{j-1,r+1} Z_{i,j}, \quad i \leq j,\ r+2 \leq j. \tag{6.3}$$

Proof. The first equality follows from the fact that $Q_{n+1,i} Q_{n+1,j} = 0$ for $i \neq j$.

For the second equality, observe that if $j < i$, then

$$Z_{n+1,j} W_{i,r+1} = Z_{n+1,j} \sum_{k \in \mathbb{N}} Z_{n+1,i}^k W_{i-1,r+1} (Z_{n+1,i}^k)^* = Z_{n+1,j} W_{i-1,r+1},$$

$$W_{i,r+1} Z_{n+1,j} = \sum_{k \in \mathbb{N}} Z_{n+1,i}^k W_{i-1,r+1} (Z_{n+1,i}^k)^* Z_{n+1,j} = W_{i-1,r+1} Z_{n+1,j}.$$

By repeating the above, we get $Z_{n+1,j} W_{i,r+1} = Z_{n+1,j} W_{j,r+1}$ and $W_{i,r+1} Z_{n+1,j} = W_{j,r+1} Z_{n+1,j}$. Finally, for $j = i$, one has

$$\begin{aligned}
Z_{n+1,j} W_{j,r+1} &= Z_{n+1,j} \sum_{k \in \mathbb{N}} Z_{n+1,j}^k W_{j-1,r+1} (Z_{n+1,j}^k)^* \\
&= \sum_{k \in \mathbb{N}} Z_{n+1,j}^{k+1} W_{j-1,r+1} (Z_{n+1,j}^k)^* \\
&= Z_{n+1,j} W_{j-1,r+1} + \sum_{k \geq 1} Z_{n+1,j}^{k+1} W_{j-1,r+1} (Z_{n+1,j}^k)^* \\
&= Z_{n+1,j} W_{j-1,r+1} Z_{n+1,j}^* Z_{n+1,j} + \sum_{k \geq 1} Z_{n+1,j}^{k+1} W_{j-1,r+1} (Z_{n+1,j}^{k+1})^* Z_{n+1,j} \\
&= \sum_{k \geq 1} Z_{n+1,j}^k W_{j-1,r+1} (Z_{n+1,j}^k)^* Z_{n+1,j} \\
&= W_{j,r+1} Z_{n+1,j} - W_{j-1,r+1} Z_{n+1,j} = W_{j,r+1} Z_{n+1,j}.
\end{aligned}$$

The last equation follows from the observation that $Z_{i,i} Z_{n+1,i} = 0$. ■

Proposition 6.4. *Let $j \geq r+2$ and $1 \leq i \leq j$. Then,*

$$Z_{i,j}W_{j+1,r+1} = W_{j+1,r+1}Z_{i,j}, \quad Z_{i,j}W^*_{j+1,r+1} = W_{j+1,r+1}Z^*_{i,j}.$$

Proof. Let $j \geq r+2$ and $1 \leq i \leq j$. Then, one has

$$\begin{aligned}
Z_{i,j}W_{j+1,r+1} &= Z_{i,j}\sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}W_{j,r+1}(Z^*_{n+1,j+1})^k \\
&= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}Z_{i,j}W_{j,r+1}(Z^*_{n+1,j+1})^k \\
&\quad + \sum_{k\geq 1} Z^{k-1}_{n+1,j+1}Z_{n+1,j}Z_{i,j+1}W_{j,r+1}(Z^*_{n+1,j+1})^k \\
&= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}W_{j-1,r+1}(Z^*_{n+1,j+1})^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}Z_{n+1,j}W_{j,r+1}Z_{i,j+1}(Z^*_{n+1,j+1})^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
W_{j+1,r+1}Z_{i,j} &= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}W_{j,r+1}(Z^*_{n+1,j+1})^k Z_{i,j} \\
&= \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z^k_{n+1,j+1}Z^s_{n+1,j}W_{j-1,r+1}(Z^*_{n+1,j})^s(Z^*_{n+1,j+1})^k Z_{i,j} \\
&= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}W_{j-1,r+1}(Z^*_{n+1,j+1})^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}}\sum_{s\geq 1} Z^k_{n+1,j+1}Z^s_{n+1,j}W_{j-1,r+1}(Z^*_{n+1,j})^s(Z^*_{n+1,j+1})^k Z_{i,j} \\
&= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}W_{j-1,r+1}(Z^*_{n+1,j+1})^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}Z_{n+1,j}W_{j,r+1}(Z^*_{n+1,j})(Z^*_{n+1,j+1})^k Z_{i,j}.
\end{aligned}$$

Use Lemma 4.8 to get

$$\begin{aligned}
&Z_{i,j}W_{j+1,r+1} - W_{j+1,r+1}Z_{i,j} \\
&\quad= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}Z_{n+1,j}W_{j,r+1}Z_{i,j+1}(Z^*_{n+1,j+1})^{k+1} \\
&\qquad - \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}Z_{n+1,j}W_{j,r+1}(Z^*_{n+1,j})(Z^*_{n+1,j+1})^k Z_{i,j} \\
&\quad= \sum_{k\in\mathbb{N}} Z^k_{n+1,j+1}Z_{n+1,j}W_{j,r+1}(Z_{i,j+1}Z^*_{n+1,j+1} - Z^*_{n+1,j}Z_{i,j})(Z^*_{n+1,j+1})^k \\
&\quad= 0.
\end{aligned}$$

Thus, we have the required relation. ■

Proposition 6.5. *Let* $1 \le i \le r$. *Then,* $Z_{i,r}$ *commutes with* $W^*_{n+1,r+1} W_{n+1,r+1}$.

Proof. From (5.8), we have for $r+2 \le j \le n+1$,

$$
\begin{aligned}
&W^*_{j,r+1} W_{j,r+1} \\
&= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} Z^{k_j}_{n+1,j} \cdots Z^{k_{r+2}}_{n+1,r+2} (Z^*_{n+1,r+1} Z_{n+1,r+1}) (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_j}_{n+1,j})^*.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&Z_{i,r} W^*_{n+1,r+1} W_{n+1,r+1} - W^*_{n+1,r+1} W_{n+1,r+1} Z_{i,r} \\
&= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} (Z_{i,r} Z^*_{n+1,r+1} Z_{n+1,r+1} - Z^*_{n+1,r+1} Z_{n+1,r+1} Z_{i,r}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^* \big).
\end{aligned}
$$

Since $Q_{n+1,r+1} Q_{n+1,r} = 0$, we have

$$
\begin{aligned}
Z_{i,r} Z^*_{n+1,r+1} Z_{n+1,r+1} &= Z^*_{n+1,r+1} Z_{i,r} Z_{n+1,r+1} \\
&= Z^*_{n+1,r+1} (Z_{n+1,r+1} Z_{i,r} + Z_{n+1,r} Z_{i,r+1}) \\
&= Z^*_{n+1,r+1} Z_{n+1,r+1} Z_{i,r}.
\end{aligned}
$$

Therefore, we have the equality $Z_{i,r} W^*_{n+1,r+1} W_{n+1,r+1} = W^*_{n+1,r+1} W_{n+1,r+1} Z_{i,r}$. ∎

Proposition 6.6. *Let* $1 \le i \le r+1$. *Then,* $Z_{i,r+1}$ *commutes with* $W^*_{n+1,r+1} W_{n+1,r+1}$.

Proof. We have

$$
\begin{aligned}
&Z_{i,r+1} W^*_{r+2,r+1} W_{r+2,r+1} \\
&\quad = Z_{i,r+1} \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} (Z^*_{n+1,r+1} Z_{n+1,r+1}) (Z^k_{n+1,r+2})^* \\
&\quad = \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} (Z_{i,r+1} Z^*_{n+1,r+1} Z_{n+1,r+1}) (Z^k_{n+1,r+2})^* \\
&\qquad + \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} (Z_{n+1,r+1} Z_{i,r+2} Z^*_{n+1,r+1} Z_{n+1,r+1}) (Z^{k+1}_{n+1,r+2})^* \\
&\quad = \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} Z_{i,r+1} (Q_{n+1,r+1} + Q_{n+1,r}) (Z^k_{n+1,r+2})^* \\
&\qquad + \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} (Z_{n+1,r+1} Z_{i,r+2} Z^*_{n+1,r+2}) (Z^k_{n+1,r+2})^* \\
&\quad = \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} Z_{i,r+1} Q_{n+1,r} (Z^k_{n+1,r+2})^* \\
&\qquad + \sum_{k \in \mathbb{N}} Z^k_{n+1,r+2} (Z_{n+1,r+1} Z_{i,r+2} Z^*_{n+1,r+2}) (Z^k_{n+1,r+2})^*,
\end{aligned}
$$

and

$$\begin{aligned}
&W^*_{r+2,r+1}W_{r+2,r+1}Z_{i,r+1}\\
&\quad=\Big(\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}(Z^*_{n+1,r+1}Z_{n+1,r+1})(Z^k_{n+1,r+2})^*\Big)Z_{i,r+1}\\
&\quad=\Big(\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}(Z_{n+1,r}Z^*_{n+1,r}+Z_{n+1,r+1}Z^*_{n+1,r+1})Z_{i,r+1}(Z^k_{n+1,r+2})^*\Big)\\
&\quad=\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}(Q_{n+1,r}Z_{i,r+1})(Z^k_{n+1,r+2})^*\\
&\qquad+\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}(Z_{n+1,r+1}Z^*_{n+1,r+1}Z_{i,r+1})(Z^{k+1}_{n+1,r+2})^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Z_{i,r+1}W^*_{r+2,r+1}W_{r+2,r+1}-W^*_{r+2,r+1}W_{r+2,r+1}Z_{i,r+1}\\
&\quad=\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}(Z_{i,r+1}Q_{n+1,r}-Q_{n+1,r}Z_{i,r+1})(Z^k_{n+1,r+2})^*\\
&\qquad+\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}Z_{n+1,r+1}(Z_{i,r+2}Z^*_{n+1,r+2}-Z^*_{n+1,r+1}Z_{i,r+1})(Z^k_{n+1,r+2})^*\\
&\quad=\sum_{k\in\mathbb{N}}Z^k_{n+1,r+2}Z_{n+1,r+1}(Z_{i,r+2}Z^*_{n+1,r+2}-Z^*_{n+1,r+1}Z_{i,r+1})(Z^k_{n+1,r+2})^*.
\end{aligned}$$

The right-hand side above vanishes by Lemma 4.8. Since $Z_{i,r+1}$ commutes with $Z_{n+1,j}$ and $Z^*_{n+1,j}$ for $j\geq r+3$, it follows that $Z_{i,r+1}$ commutes with $W^*_{n+1,r+1}W_{n+1,r+1}$. ∎

Proposition 6.7. *Let π be irreducible. Then, $W^*_{n+1,r+1}W_{n+1,r+1}=I$.*

Proof. From Propositions 6.2, 6.4, 6.5, and 6.6, it follows that $W^*_{n+1,r+1}W_{n+1,r+1}$ commutes with $Z_{i,j}$ for all $1\leq i\leq j\leq n+1$. Therefore, using (2.9), we conclude that $W^*_{n+1,r+1}W_{n+1,r+1}$ commutes with $Z_{i,j}$ for all i and j. Since $W^*_{n+1,r+1}W_{n+1,r+1}$ is a projection, and

$$W^*_{n+1,r+1}W_{n+1,r+1}\geq P_{n+1,r+1}\neq 0,$$

we get the result. ∎

The above result and Proposition 6.1 tell us that the operator $W^*_{n+1,r}W_{n+1,r+1}$ is also an isometry. We next decompose the operators $Z_{i,j}$ using this isometry. This will be the main tool in proving the theorems in the next section.

Theorem 6.8. *Assume π is irreducible. Define*

$$W:=W^*_{n+1,r}W_{n+1,r+1}=\bar{\lambda}W_{n+1,r+1},$$

$$Y^{(1)}_{i,j}=\begin{cases}Z_{i,j} & \text{if } j\notin\{r,r+1\},\\ Z_{i,r}W & \text{if } j=r,\\ W^*Z_{i,r+1} & \text{if } j=r+1,\end{cases}$$

$$Y^{(2)}_{i,j} = \begin{cases} W & \text{if } i=j=r+1, \\ W^* & \text{if } i=j=r, \\ I - WW^* & \text{if } i,j \in \{r, r+1\} \text{ and } i \neq j, \\ I & \text{if } i=j \notin \{r,r+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $Y^{(1)}_{i,j} \in \{W, W^*\}'$ *for all* i, j *and*

$$Z_{i,j} = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} Y^{(1)}_{i,k} Y^{(2)}_{k,j}. \tag{6.4}$$

Before getting into the proof of the above theorem, we first prove two propositions where we show that the elements $Z_{i,r}W$ and $W^*Z_{i,r+1}$ are in the commutant $\{W, W^*\}'$.

Proposition 6.9. *Let* W *be as defined in Theorem 6.8. Then,* $Z_{i,r}W \in \{W, W^*\}'$ *for* $1 \leq i \leq r$.

Proof. It is enough to show that $Z_{i,r}W_{n+1,r+1} \in \{W_{n+1,r+1}, W^*_{n+1,r+1}\}'$. From Proposition 5.7, we have

$$(Z_{i,r}W_{n+1,r+1} - W_{n+1,r+1}Z_{i,r})W_{n+1,r+1} = U_{n+1}Z_{i,r+1}W_{n+1,r+1}.$$

Now note that

$$\begin{aligned} & U_{n+1}Z_{i,r+1}W_{n+1,r+1} \\ &= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \leq j \leq n+1)}} \left(Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} Z_{n+1,r} (Z^*_{n+1,r+2})^{k_{r+2}} \cdots (Z^*_{n+1,n+1})^{k_{n+1}} \right) \\ &\quad \times Z_{i,r+1} \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \leq j \leq n+1)}} \Big(Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} Z_{n+1,r+1} \\ &\qquad\qquad \times (Z^*_{n+1,r+2})^{k_{r+2}} \cdots (Z^*_{n+1,n+1})^{k_{n+1}} \Big) \\ &= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \leq j \leq n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} (Z_{n+1,r} Z_{i,r+1} Z_{n+1,r+1}) \\ &\qquad \times (Z^*_{n+1,r+2})^{k_{r+2}} \cdots (Z^*_{n+1,n+1})^{k_{n+1}} \\ &= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \leq j \leq n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} (Z_{i,r+1} Z_{n+1,r} Z_{n+1,r+1}) \\ &\qquad \times (Z^*_{n+1,r+2})^{k_{r+2}} \cdots (Z^*_{n+1,n+1})^{k_{n+1}} \\ &= 0. \end{aligned} \tag{6.5}$$

Thus, $Z_{i,r}W_{n+1,r+1}$ commutes with $W_{n+1,r+1}$. One also has

$$(Z_{i,r}W_{n+1,r+1})W^*_{n+1,r+1} = Z_{i,r}(I - U_{n+1}U^*_{n+1}) = Z_{i,r},$$
$$W^*_{n+1,r+1}(Z_{i,r}W_{n+1,r+1}) = Z_{i,r}W^*_{n+1,r+1}W_{n+1,r+1} = Z_{i,r}.$$

Thus, $Z_{i,r}W_{n+1,r+1} \in \{W_{n+1,r+1}, W^*_{n+1,r+1}\}'$ for $1 \leq i \leq r$. ∎

Proposition 6.10. *Let $i \leq r+1$. Then, $W^*Z_{i,r+1} \in \{W, W^*\}'$.*

Proof. Let us first prove that $W^*_{n+1,r+1}Z_{i,r+1}W_{n+1,r+1} = W_{n+1,r+1}W^*_{n+1,r+1}Z_{i,r+1}$. Observe that

$$\begin{aligned}
Z_{i,r+1}W_{r+2,r+1} &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2}Z_{i,r+1}Z_{n+1,r+1}(Z^*_{n+1,r+2})^k \\
&\quad + \sum_{k\geq 1} Z^{k-1}_{n+1,r+2}Z_{i,r+2}Z^2_{n+1,r+1}(Z^*_{n+1,r+2})^k \\
&= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2}Z_{i,r+2}Z^2_{n+1,r+1}(Z^*_{n+1,r+2})^{k+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&W^*_{n+1,r+1}Z_{i,r+1}W_{n+1,r+1} \\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\leq j\leq n+1)}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}(Z^*_{n+1,r+1}Z_{i,r+2}Z^2_{n+1,r+1}Z^*_{n+1,r+2}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^*\cdots(Z^{k_{n+1}}_{n+1,n+1})^* \\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\leq j\leq n+1)}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}(Z^*_{n+1,r+1}Z^2_{n+1,r+1}Z_{i,r+2}Z^*_{n+1,r+2}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^*\cdots(Z^{k_{n+1}}_{n+1,n+1})^* \\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\leq j\leq n+1)}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}(P_{n+1,r+1}Z_{n+1,r+1}Z_{i,r+2}Z^*_{n+1,r+2}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^*\cdots(Z^{k_{n+1}}_{n+1,n+1})^* \\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\leq j\leq n+1)}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}(Z_{n+1,r+1}Z_{i,r+2}Z^*_{n+1,r+2}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^*\cdots(Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

We also have

$$\begin{aligned}
&W_{n+1,r+1}W^*_{n+1,r+1}Z_{i,r+1} \\
&= \sum_{\substack{k_j\in\mathbb{N}\\(r+2\leq j\leq n+1)}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}(Z_{n+1,r+1}Z^*_{n+1,r+1}Z_{i,r+1}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^*\cdots(Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

Hence, using Lemma 4.8, we get

$$\begin{aligned}
&W^*_{n+1,r+1} Z_{i,r+1} W_{n+1,r+1} - W_{n+1,r+1} W^*_{n+1,r+1} Z_{i,r+1} \\
&= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} \big(Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} Z_{n+1,r+1} (Z_{i,r+2} Z^*_{n+1,r+2} - Z^*_{n+1,r+1} Z_{i,r+1}) \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^* \big) \\
&= 0.
\end{aligned}$$

Since W commutes with all the $Z_{i,j}$'s, we get $W^* Z_{i,r+1} W = W W^* Z_{i,r+1}$. It remains to prove that $W^* Z_{i,r+1} W^* = (W^*)^2 Z_{i,r+1}$. Since $W_{n+1,r+1}$ is an isometry, we have

$$\begin{aligned}
W^*_{n+1,r+1} Z_{i,r+1} W^*_{n+1,r+1} W_{n+1,r+1} &= W^*_{n+1,r+1} Z_{i,r+1} \\
&= W^*_{n+1,r+1} W_{n+1,r+1} W^*_{n+1,r+1} Z_{i,r+1} \\
&= (W^*_{n+1,r+1})^2 Z_{i,r+1} W.
\end{aligned}$$

Therefore, it is enough to prove that

$$\begin{aligned}
&W^*_{n+1,r+1} Z_{i,r+1} W^*_{n+1,r+1} (I - W_{n+1,r+1} W^*_{n+1,r+1}) \\
&\qquad = (W^*_{n+1,r+1})^2 Z_{i,r+1} (I - W_{n+1,r+1} W^*_{n+1,r+1}).
\end{aligned}$$

Note that the left-hand side above is 0. Since by (5.12) we have

$$I - W_{n+1,r+1} W^*_{n+1,r+1} = U_{n+1} U^*_{n+1},$$

it suffices to show that $(W^*)^2 Z_{i,r+1} U_{n+1} = 0$. Now,

$$\begin{aligned}
&Z_{i,r+1} U_{n+1} \\
&= Z_{i,r+1} \Big(\sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} Z_{n+1,r} (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^* \Big) \\
&= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+3}}_{n+1,r+3} Z_{i,r+1} Z^{k_{r+2}}_{n+1,r+2} Z_{n+1,r} \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^* \\
&= \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+3}}_{n+1,r+3} Z^{k_{r+2}}_{n+1,r+2} Z_{i,r+1} Z_{n+1,r} \\
&\qquad \times (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^* \\
&\quad + \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \le j \le n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+2}}_{n+1,r+2} Z_{n+1,r+1} Z_{i,r+2} \\
&\qquad \times (Z^{1+k_{r+2}}_{n+1,r+2})^* (Z^{k_{r+3}}_{n+1,r+3})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&(W^*)^2 Z_{i,r+1}U_{n+1}\\
&=\Big(\sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} Z_{n+1,n+1}^{k_{n+1}}\cdots Z_{n+1,r+2}^{k_{r+2}}(Z_{n+1,r+1}^*)^2(Z_{n+1,r+2}^{k_{r+2}})^*\cdots(Z_{n+1,n+1}^{k_{n+1}})^*\Big)\\
&\quad\times\Big(\sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} Z_{n+1,n+1}^{k_{n+1}}\cdots Z_{n+1,r+3}^{k_{r+3}}Z_{n+1,r+2}^{k_{r+2}}Z_{i,r+1}Z_{n+1,r}\\
&\qquad\times(Z_{n+1,r+2}^{k_{r+2}})^*\cdots(Z_{n+1,n+1}^{k_{n+1}})^* + \sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} Z_{n+1,n+1}^{k_{n+1}}\cdots Z_{n+1,r+2}^{k_{r+2}}Z_{n+1,r+1}\\
&\qquad\times Z_{i,r+2}(Z_{n+1,r+2}^{1+k_{r+2}})^*(Z_{n+1,r+3}^{k_{r+3}})^*\cdots(Z_{n+1,n+1}^{k_{n+1}})^*\Big)\\
&=\sum_{\substack{k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} Z_{n+1,n+1}^{k_{n+1}}\cdots Z_{n+1,r+3}^{k_{r+3}}Z_{n+1,r+2}^{k_{r+2}}\\
&\qquad\times(Z_{n+1,r+1}^*)^2 Z_{i,r+1}Z_{n+1,r}(Z_{n+1,r+2}^{k_{r+2}})^*\cdots(Z_{n+1,n+1}^{k_{n+1}})^*\\
&\quad+\sum_{\substack{k,k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z_{n+1,n+1}^{k_{n+1}}\cdots Z_{n+1,r+2}^{k_{r+2}}(Z_{n+1,r+1}^*)^2(Z_{n+1,r+2}^{k_{r+2}})^*(Z_{n+1,r+2})^{k+k_{r+2}}\\
&\qquad\times Z_{n+1,r+1}Z_{i,r+2}(Z_{n+1,r+2}^{k+1+k_{r+2}})^*(Z_{n+1,r+3}^{k_{r+3}})^*\cdots(Z_{n+1,n+1}^{k_{n+1}})^*\big)\\
&=\sum_{\substack{k,k_j\in\mathbb{N}\\(r+2\le j\le n+1)}} \big(Z_{n+1,n+1}^{k_{n+1}}\cdots Z_{n+1,r+2}^{k_{r+2}}(Z_{n+1,r+1}^*)^2(Z_{n+1,r+2})^{k}Z_{n+1,r+1}Z_{i,r+2}\\
&\qquad\times(Z_{n+1,r+2}^{k+1+k_{r+2}})^*(Z_{n+1,r+3}^{k_{r+3}})^*\cdots(Z_{n+1,n+1}^{k_{n+1}})^*\big)\\
&=0.
\end{aligned}
$$

Thus, the proof is complete. ■

For the remaining operators, i.e., $W^*Z_{i,r+1}$ for $i\ge r+2$ and $Z_{i,r}W$ for $i\ge r+1$, we now invoke the relation (2.9). Observe that for $i\ge r+2$, one has

$$
\begin{aligned}
&(W_{n+1,r+1}^*Z_{i,r+1})^*\\
&=(Z_{1,1}Z_{2,2}\cdots Z_{r,r})(Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1})W_{n+1,r+1}\\
&=(Z_{1,1}Z_{2,2}\cdots Z_{r-1,r-1})(Z_{r,r}W_{n+1,r+1})(Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1}),
\end{aligned}
$$

and for $i\ge r+1$, one similarly has

$$
\begin{aligned}
&(Z_{i,r}W_{n+1,r+1})^*\\
&=W_{n+1,r+1}^*(Z_{1,1}Z_{2,2}\cdots Z_{r-1,r-1})(Z_{r,r+1}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1})\\
&=(Z_{1,1}Z_{2,2}\cdots Z_{r-1,r-1})W_{n+1,r+1}^*Z_{r,r+1}(Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1}).
\end{aligned}
$$

Therefore, using the earlier results (Propositions 6.2, 6.9, and 6.4), we now obtain the following proposition.

Proposition 6.11. *Let W be as in Theorem 6.8. Then,*

$$W^* Z_{i,r+1} \in \{W, W^*\}' \quad \textit{for } i \geq r+2,$$
$$Z_{i,r} W \in \{W, W^*\}' \quad \textit{for } i \geq r+1.$$

Proof of Theorem 6.8. From Propositions 6.2, 6.4, and 6.9–6.11, it follows that $Y_{i,j}^{(1)} \in \{W, W^*\}'$ for all i, j. So it remains to prove (6.4). For that, we need to prove the following equalities:

$$Z_{i,r} W W^* = Z_{i,r} \qquad i \leq r, \tag{6.6}$$
$$Z_{i,r} W(I - WW^*) + W^* Z_{i,r+1} W = Z_{i,r+1} \qquad i \leq r, \tag{6.7}$$
$$W^* Z_{i,r+1} W = Z_{i,r+1} \qquad i \geq r+1, \tag{6.8}$$
$$W^* Z_{i,r+1}(I - WW^*) + Z_{i,r} W W^* = Z_{i,r} \qquad i \geq r+1. \tag{6.9}$$

For (6.6), note that

$$Z_{i,r}(I - WW^*) = Z_{i,r}(I - W_{n+1,r+1} W^*_{n+1,r+1}) = Z_{i,r} U_{n+1} U^*_{n+1}.$$

Since $i \leq r$, we have

$$\begin{aligned} Z_{i,r} U_{n+1} \\ = \sum_{\substack{k_j \in \mathbb{N} \\ (r+2 \leq j \leq n+1)}} Z^{k_{n+1}}_{n+1,n+1} \cdots Z^{k_{r+3}}_{n+1,r+3} Z^{k_{r+2}}_{n+1,r+2} (Z_{i,r} Z_{n+1,r}) (Z^{k_{r+2}}_{n+1,r+2})^* \cdots (Z^{k_{n+1}}_{n+1,n+1})^* \\ = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} Z_{i,r} W W^* &= Z_{i,r} - Z_{i,r}(I - WW^*) \\ &= Z_{i,r} - Z_{i,r} U_{n+1} U^*_{n+1} \\ &= 0. \end{aligned}$$

Next, let us prove (6.7). Use Proposition 5.7 to get

$$\begin{aligned} Z_{i,r} W(I - WW^*) &= W^*_{n+1,r} Z_{i,r} W_{n+1,r+1}(I - WW^*) \\ &= W^*_{n+1,r}(W_{n+1,r+1} Z_{i,r} + U_{n+1} Z_{i,r+1})(I - WW^*) \\ &= W Z_{i,r}(I - WW^*) + W^*_{n+1,r} U_{n+1} Z_{i,r+1}(I - WW^*) \\ &= W Z_{i,r} U_{n+1} U^*_{n+1} + U_{n+1} U^*_{n+1} Z_{i,r+1}(I - WW^*) \\ &= (I - WW^*) Z_{i,r+1}(I - WW^*). \end{aligned}$$

Since $W^*Z_{i,r+1}$ commutes with W, we get

$$\begin{aligned} Z_{i,r}W(I-WW^*)+W^*Z_{i,r+1}W &= (I-WW^*)Z_{i,r+1}(I-WW^*)+WW^*Z_{i,r+1} \\ &= Z_{i,r+1}-(I-WW^*)Z_{i,r+1}WW^* \\ &= Z_{i,r+1}-U_{n+1}^*U_{n+1}Z_{i,r+1}WW^*. \end{aligned}$$

By (6.5), the second term in the right-hand side above vanishes. Therefore, we have (6.7).

For proving (6.8) and (6.9), we will need the following equality:

$$W^*Z_{i,r+1}-Z_{i,r+1}W^* = Z_{i,r}(I-WW^*) \quad \text{if } i \geq r+1. \tag{6.10}$$

Here,

$$\begin{aligned} &Z_{i,r+1}^*W_{n+1,r+1}-W_{n+1,r+1}Z_{i,r+1}^* \\ &= (Z_{1,1}Z_{2,2}\cdots Z_{r,r})(Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1})W_{n+1,r+1} \\ &\quad - W_{n+1,r+1}(Z_{1,1}Z_{2,2}\cdots Z_{r,r})(Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1}) \\ &= (Z_{1,1}Z_{2,2}\cdots Z_{r-1,r-1})(Z_{r,r}W_{n+1,r+1}-W_{n+1,r+1}Z_{r,r}) \\ &\quad \times (Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1}) \\ &= (Z_{1,1}Z_{2,2}\cdots Z_{r-1,r-1})(U_{n+1}Z_{r,r+1})(Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1}) \\ &= U_{n+1}(Z_{1,1}Z_{2,2}\cdots Z_{r-1,r-1})(Z_{r,r+1}Z_{r+1,r+2}\cdots Z_{i-1,i})(Z_{i+1,i+1}\cdots Z_{n+1,n+1}) \\ &= U_{n+1}Z_{i,r}^*. \end{aligned}$$

Therefore, we have

$$W^*Z_{i,r+1}-Z_{i,r+1}W^* = Z_{i,r}U_{n+1}^*W_{n+1,r} = Z_{i,r}(I-WW^*).$$

Using the above equality, we get

$$\begin{aligned} W^*Z_{i,r+1}W-Z_{i,r+1} &= (W^*Z_{i,r+1}-Z_{i,r+1}W^*)W \\ &= Z_{i,r}(I-WW^*)W \\ &= 0. \end{aligned}$$

Thus, we have (6.8). Next,

$$\begin{aligned} &W^*Z_{i,r+1}(I-WW^*)+Z_{i,r}WW^* \\ &= \big(Z_{i,r+1}W^*+Z_{i,r}(I-WW^*)\big)(I-WW^*)+Z_{i,r}WW^* \\ &= Z_{i,r}. \end{aligned}$$

Thus, we have proved (6.9). This completes the proof of Theorem 6.8. ∎

7. Classification theorem

We are now in a position to prove a factorization theorem for the irreducible representations of the C^*-algebra $C(\mathrm{SU}_0(n+1))$ that will subsequently lead to a recursive proof of the classification theorem for all irreducible representations of $C(\mathrm{SU}_0(n+1))$.

Theorem 7.1 (Factorization theorem). *Suppose π is an irreducible representation of $C(\mathrm{SU}_0(n+1))$ on a Hilbert space $\mathcal{H}$. Assume $r \equiv r(\pi) \leq n$ where r is as in (5.3). Then, there is an irreducible representation π_1 of $C(\mathrm{SU}_0(n+1))$ acting on a Hilbert space $\mathcal{H}_1$ such that $r(\pi_1) = r+1$ and π is unitarily equivalent to the representation $\pi_1 * \psi_{s_r}$ acting on $\mathcal{H}_1 \otimes \ell^2(\mathbb{N})$ given by*

$$\pi_1 * \psi_{s_r}(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} \pi_1(z_{i,k}) \otimes \psi_{s_r}(z_{k,j}), \quad 1 \leq i, j \leq n+1.$$

Proof. From Proposition 6.7, we know that W is an isometry. Since $U_{n+1}^* U_{n+1} \geq P_{n+1,r}$, it follows from (5.12) that W is not a unitary. By Wold decomposition, there are Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and a unitary U on $\mathcal{H}_0$ such that $(\mathcal{H}, W)$ is unitarily equivalent to $((\mathcal{H}_1 \otimes \ell^2(\mathbb{N})) \oplus \mathcal{H}_0, (I \otimes S^*) \oplus U)$. By Theorem 6.8, the subspace $\mathcal{H}_1 \otimes \ell^2(\mathbb{N})$ is kept invariant by $Y_{i,j}^{(1)}$, $Y_{i,j}^{(2)}$, $(Y_{i,j}^{(1)})^*$, and $(Y_{i,j}^{(2)})^*$ for all i, j. Therefore, the subspace $\mathcal{H}_1 \otimes \ell^2(\mathbb{N})$ is an invariant subspace for π. By the irreducibility of π, it follows that $\mathcal{H}_0 = \{0\}$.

Thus, there are operators $Z_{i,j}^{(1)} \in \mathcal{L}(\mathcal{H}_1)$ and $Z_{i,j}^{(2)} \in \mathcal{L}(\ell^2(\mathbb{N}))$ such that

$$Y_{i,j}^{(1)} = Z_{i,j}^{(1)} \otimes I, \quad Y_{i,j}^{(2)} = I \otimes Z_{i,j}^{(2)}, \quad 1 \leq i, j \leq n+1,$$

and we have

$$Z_{i,j} = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} Z_{i,k}^{(1)} \otimes Z_{k,j}^{(2)}, \quad 1 \leq i, j \leq n+1.$$

In fact, one has

$$Z_{i,j}^{(2)} = \begin{cases} S^* & \text{if } i = j = r+1, \\ S & \text{if } i = j = r, \\ P_0 & \text{if } i, j \in \{r, r+1\} \text{ and } i \neq j, \\ I & \text{if } i = j \notin \{r, r+1\}, \\ 0 & \text{otherwise}. \end{cases} \tag{7.1}$$

Thus, the map $z_{i,j} \mapsto Z_{i,j}^{(2)}$ defines a representation of $C(\mathrm{SU}_0(n+1))$ on $\ell^2(\mathbb{N})$ which is in fact the representation ψ_{s_r} defined in Section 3. Observe that the $Z_{i,j}^{(2)}$'s belong to the Toeplitz algebra $\mathcal{T} \subseteq \mathcal{L}(\ell^2(\mathbb{N}))$. Therefore,

$$\pi(z_{i,j}) \in \mathcal{L}(\mathcal{H}_1) \otimes \mathcal{T} \quad \text{for all } i, j. \tag{7.2}$$

Let $\sigma : \mathscr{T} \to \mathbb{C}$ be the $*$-homomorphism given by $S \mapsto 1$. Then, one has

$$Z^{(1)}_{i,j} = (id \otimes \sigma)Z_{i,j} = (id \otimes \sigma)\pi(z_{i,j}).$$

Thus, the map $\pi_1 : z_{i,j} \mapsto Z^{(1)}_{i,j}$ gives a representation of $C(\mathrm{SU}_0(n+1))$ on $\mathscr{H}_0$ and we have

$$\pi(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} \pi_1(z_{i,k}) \otimes \psi_{s_r}(z_{k,j}), \quad 1 \le i, j \le n+1.$$

If T belongs to the commutant of $\pi_1(C(\mathrm{SU}_0(n+1)))$, then $T \otimes I \in \pi(C(\mathrm{SU}_0(n+1)))'$. By the irreducibility of π, it follows that $T = I$. Thus, π_1 is irreducible. Using (2.1), we get

$$\pi_1(z_{n+1,r}) = Z_{n+1,r}W = 0.$$

Also, note that

$$\begin{aligned}(W^*Z_{n+1,r+1})^*(W^*Z_{n+1,r+1}) &= Z^*_{n+1,r+1}WW^*Z_{n+1,r+1}\\ &= P_{n+1,r+1} - Z^*_{n+1,r+1}(I - WW^*)Z_{n+1,r+1}\\ &= P_{n+1,r+1} - Z^*_{n+1,r+1}U^*_{n+1}U_{n+1}Z_{n+1,r+1}\\ &= P_{n+1,r+1} \ne 0.\end{aligned}$$

Thus, we have

$$\min\left\{i : 1 \le i \le n+1, \pi_1(z_{n+1,i}) \ne 0\right\} = r+1.$$

This completes the proof. ■

We next come to the main result that gives a parametrization of all the irreducible representations.

Theorem 7.2. *Let π be an irreducible representation of $C(\mathrm{SU}_0(n+1))$ on a Hilbert space $\mathscr{H}$. Then, there is a $\lambda = (\lambda_1, \dots, \lambda_n) \in (S^1)^n$ and a reduced word ω in $\mathfrak{S}_{n+1}$ of the form $s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\cdots s_{[a_1,b_1]}$ where $1 \le b_k < b_{k-1} < \cdot < b_1 \le n$, and $1 \le a_i \le b_i$ for $1 \le i \le k$, such that $\pi \cong \psi_{\lambda,\omega}$.*

Proof. Note that when $\pi \cong \psi_{\lambda,\omega}$, then $1 \le r \le n$ implies $b_1 = n$ and $a_1 = r$.

It was proved in [6] that the statement holds for $n = 2$. Let us assume that it holds for $n-1$. Let us first deal with the case $r = n+1$. In this case, from Proposition 5.1, it follows that $Z_{n+1,j} = 0$ for $1 \le j \le n$. Therefore, we have

(1) $P_{n+1,n+1} = Q_{n+1,n+1} = I$,
(2) $Z_{i,n+1} = 0$ for $1 \le i \le n$,
(3) $Z_{i,j}Z_{n+1,n+1} = Z_{n+1,n+1}Z_{i,j}$ for all i, j.

Therefore, $Z_{n+1,n+1}$ is a unitary and any spectral subspace of $Z_{n+1,n+1}$ is an invariant subspace for π. By the irreducibility of π, it follows that $Z_{n+1,n+1} = \mu_0 I$ for some

$\mu_0 \in S^1$. Now define

$$\pi_0(z_{i,j}^{(n-1)}) = \begin{cases} Z_{i,j} & \text{if } 1 \le j \le n \text{ and } 1 \le i \le n-1, \\ \mu_0 Z_{i,j} & \text{if } 1 \le j \le n \text{ and } i = n, \end{cases}$$

where $z_{i,j}^{(n-1)}$'s denote the generators for $C(\mathrm{SU}_0(n))$. Then, π_0 is an irreducible representation of $C(\mathrm{SU}_0(n))$ on $\mathcal{H}$. Therefore, there is a $\mu = (\mu_1, \ldots, \mu_{n-1}) \in (S^1)^{n-1}$ and a reduced word

$$\omega = s_{[a_k,b_k]} \cdots s_{[a_1,b_1]} \in \mathfrak{S}_n \subseteq \mathfrak{S}_{n+1}$$

where $1 \le b_k < b_{k-1} < \cdots < b_1 \le n-1$ such that $\pi_0 \cong \psi_{\mu,\omega}$. We thus have

$$\pi \cong \psi_{\lambda,\omega},$$

where $\lambda = (\mu_1, \mu_2, \ldots, \mu_{n-1}, \overline{\mu_0})$.

Next, assume $1 \le r \le n$. By an application of the second factorization result proved earlier (Theorem 7.1), it follows that there is an irreducible representation π_1 acting on a Hilbert space $\mathcal{H}_1$ such that $r(\pi_1) = r+1$ and $\pi = \pi_1 * \psi_{s_r}$; i.e.,

$$\pi(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} \pi_1(z_{i,k}) \otimes \psi_{s_r}(z_{k,j}).$$

Using Lemma 3.1 and by the repeated application of Theorem 7.1, we conclude that there is an irreducible representation π_0 acting on a Hilbert space $\mathcal{H}_0$ such that $r(\pi_0) = n+1$ and $\pi = \pi_0 * \psi_{s_{[r,n]}}$. By the previous case, the result now follows. ∎

Remark 7.3. Note that one gets the one-dimensional representations $\psi_{\lambda,id}$ given by (3.6) when $r = m+1$ in each of the steps $m = n, n-1, \ldots, 2$ in the above recursive procedure.

8. Corollaries of the classification theorem

In this section, we present a few important consequences of the classification theorem.

Proposition 8.1. *The set*

$$\left\{\psi_{\lambda,\omega} : \lambda \in (S^1)^n,\ \omega \text{ a reduced word in } \mathfrak{S}_{n+1}\right\}$$

gives all inequivalent irreducible representations of $C(\mathrm{SU}_0(n+1))$.

This follows from Theorems 3.8 and 3.9 in Section 3 and Theorem 7.2.

Recall that $\mathcal{O}(\mathrm{SU}_0(n+1))$ denotes the $*$-algebra given by the relations (2.1)–(2.8) in Theorem 2.1. Thus, the space $\mathcal{O}(\mathrm{SU}_0(n+1))$ can be viewed as a dense $*$-subalgebra of $C(\mathrm{SU}_0(n+1))$, and the family $\psi_{\lambda,\omega}$ gives all the irreducible $*$-representations of this algebra.

Proposition 8.2. *$C(\mathrm{SU}_0(n+1))$ is a C^*-algebra of type I.*

Proof. For each $1 \le r \le k$, let i_s and j_s be nonnegative integers for $a_r \le s \le b_r$. Let $|e_i\rangle\langle e_j|$ be the rank one operator $x \mapsto \langle e_j, x\rangle e_i$. Denote by T_r the operator

$$|e_{i_{b_r}}\rangle\langle e_{j_{b_r}}| \otimes |e_{i_{b_r-1}}\rangle\langle e_{j_{b_r-1}}| \otimes \cdots \otimes |e_{i_{a_r}}\rangle\langle e_{j_{a_r}}|.$$

It is enough to prove that the tensor product $T_k \otimes T_{k-1} \otimes \cdots \otimes T_1$ belongs to the image $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))$.

We will use the operators

$$E_{j,i} := V^*_{n'(j,i),i} V_{j,i} |V_{j-1,a_{j-1}}| \cdot |V_{j-2,a_{j-2}}| \cdots |V_{1,a_1}|$$

described in Corollary 3.6 for this. Recall that

$$E_{j,i} = I^{\otimes \sum_{s=j+1}^{k}(b_s-a_s+1)} \otimes (P_0^{\otimes(b_j+1-i)} \otimes S^* \otimes I^{\otimes(i-a_j-1)}) \otimes P_0^{\otimes \sum_{s=1}^{j-1}(b_s-a_s+1)}.$$

Now observe that $|e_i\rangle\langle e_j| = (S^i)^* P_0 S^j$ and hence

$$\begin{aligned} &E_{k,1+b_k}^{i_{b_k}} \cdots E_{k,1+a_k}^{i_{a_k}} E_{k,a_k} (E_{k,1+a_k}^{j_{a_k}})^* \cdots (E_{k,1+b_k}^{j_{b_k}})^* \\ &\quad = T_k \otimes P_0^{\otimes(b_{k-1}-a_{k-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}. \end{aligned}$$

Thus, $T_k \otimes P_0^{\otimes(b_{k-1}-a_{k-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}$ belongs to $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))$. Having proved that

$$T := T_k \otimes T_{k-1} \otimes \cdots \otimes T_{s+1} \otimes P_0^{\otimes(b_s-a_s+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}$$

belongs to $\psi_{\lambda,\omega}(C(\mathrm{SU}_0(n+1)))$, note that

$$\begin{aligned} &E_{s,1+b_s}^{i_{b_s}} \cdots E_{s,1+a_s}^{i_{a_s} T (E_{s,1+a_s}^{j_{a_s}})^* \cdots (E_{s,1+b_s}^{j_{b_s}})^*} \\ &\quad = T_k \otimes T_{k-1} \otimes \cdots \otimes T_s \otimes P_0^{\otimes(b_{s-1}-a_{s-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}. \end{aligned}$$

Thus, by repeating the argument, the result follows. ∎

Proposition 8.3. *For the C^*-algebra $C(\mathrm{SU}_0(n+1))$, we have the following:*

(1) *The map*

$$\Delta(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} z_{i,k} \otimes z_{k,j}$$

extends to a unital C^*-homomorphism from $C(\mathrm{SU}_0(n+1))$ to the tensor product $C(\mathrm{SU}_0(n+1)) \otimes C(\mathrm{SU}_0(n+1))$.

(2) *The C^*-algebra $C(\mathrm{SU}_0(n+1))$ together with Δ and the one-dimensional representation $\varepsilon = \psi_{id}$ is a C^*-bialgebra, i.e., a compact quantum semigroup.*

The proof is identical to the arguments used for the $n = 2$ case in [6]. Observe also that the restriction of Δ and ε to $\mathcal{O}(\mathrm{SU}_0(n+1))$ turns it into a $*$-bialgebra. Using this map Δ, one can write the representations $\psi_{\lambda,\omega}$ as convolution products of χ_λ and ψ_{s_i}'s, just as in the $q \neq 0$ case.

Let $1 \le m < n$. Denote the generators of $C(\mathrm{SU}_0(m+1))$ and $C(\mathrm{SU}_0(n+1))$ by $z_{i,j}^{(m)}$ and $z_{i,j}^{(n)}$, respectively, and their comultiplication maps by Δ_m and Δ_n, respectively. Then, it follows from the defining relations (2.1)–(2.9) that the map ϕ from $C(\mathrm{SU}_0(n+1))$ to $C(\mathrm{SU}_0(m+1))$ given by

$$\phi(z_{i,j}^{(n)}) = \begin{cases} z_{i,j}^{(m)} & \text{if } 1 \le i, j \le m+1, \\ \delta_{i,j} & \text{otherwise} \end{cases}$$

is a surjective C^*-homomorphism and one has

$$(\phi \otimes \phi)\Delta_n = \Delta_m \phi.$$

Thus, $(C(\mathrm{SU}_0(m+1)), \Delta_m)$ is a compact quantum subsemigroup of $(C(\mathrm{SU}_0(n+1)), \Delta_n)$. By the same argument used for $C(\mathrm{SU}_0(3))$ in [6], it follows that $(C(\mathrm{SU}_0(n+1)), \Delta_n)$ is not a compact quantum group.

Theorem 8.4. *The crystallized algebra $C(\mathrm{SU}_0(n+1))$ is isomorphic to the C^*-subalgebra of the space of bounded operators on $\ell^2(\mathbb{Z})^{\otimes n} \otimes \ell^2(\mathbb{N})^{\otimes \frac{n(n+1)}{2}}$ generated by the limits*

$$\lim_{q\to 0+} \psi^{(q)}\big((-q)^{\min\{i-j,0\}} u_{i,j}(q)\big),$$

where $\psi^{(q)}$ is the Soibelman representation of $C(\mathrm{SU}_q(n+1))$.

Proof. Similar to the case of nonzero q, one can prove that the direct integral $\psi^{(0)}$ of the representations ψ_{λ,ω_0} over $\lambda \in (S^1)^n$, where ω_0 is the longest word in $\mathfrak{S}_{n+1}$, is a faithful representation of $C(\mathrm{SU}_0(n+1))$ acting on $\ell^2(\mathbb{Z})^{\otimes n} \otimes \ell^2(\mathbb{N})^{\otimes \frac{n(n+1)}{2}}$. It is simple to check that the limits $\lim_{q\to 0+} \psi^{(q)}((-q)^{\min\{i-j,0\}} u_{i,j}(q))$ exist and one has

$$\psi^{(0)}(z_{i,j}) = \lim_{q\to 0+} \psi^{(q)}\big((-q)^{\min\{i-j,0\}} u_{i,j}(q)\big).$$

Therefore, the result follows. ∎

Finally, we have the following important property of our crystallized algebra.

Theorem 8.5. *Suppose $\pi^{(0)}$ is an irreducible representation of the crystallization $C(\mathrm{SU}_0(n+1))$ (respectively $\mathcal{O}(\mathrm{SU}_0(n+1))$) on a Hilbert space $\mathcal{H}$. Then, there exist irreducible representations $\pi^{(q)}$ of $C(\mathrm{SU}_q(n+1))$ (respectively $\mathcal{O}(\mathrm{SU}_q(n+1))$) on the same Hilbert space $\mathcal{H}$ such that*

$$\pi^{(0)}(z_{i,j}) = \lim_{q\to 0+} \pi^{(q)}\big((-q)^{\min\{i-j,0\}} u_{i,j}(q)\big), \quad i, j \in \{1, 2, \ldots, n+1\}.$$

Proof. This follows from Theorem 3.9, Proposition 8.1, and Soibelman's result [13, Theorem 6.2.7, p. 121]. ■

Remark 8.6. Recall (see the discussion after [6, Remark 2.2.2]) that there is a specialization map

$$\theta_q : \mathcal{O}_t^{\mathbb{Q}[t,t^{-1}]}\big(\mathrm{SU}(n+1)\big) \to \mathcal{O}_q\big(\mathrm{SU}(n+1)\big)$$

such that the limits in Theorem 8.4 are $\lim_{q\to 0+} \psi^{(q)} \circ \theta_q((-t)^{\min\{i-j,0\}} u_{i,j}(t))$. Thus, Theorem 8.4 realizes the crystallization $C(\mathrm{SU}_0(n+1))$ as a C^*-algebra of bounded operators on the Hilbert space that carries the Soibelman representation of the family $C(\mathrm{SU}_q(n+1))$ and is generated by a collection of operators that are $q \to 0+$ limits of images of certain matrix entries from $\mathcal{O}_t^{\mathbb{Q}[t,t^{-1}]}(\mathrm{SU}(n+1))$. This is very similar to the description of the crystallized C^*-algebra given by Mattassa and Yuncken [16]. However, there are crucial differences. In particular, the generating collection of matrix entries used are different, and more importantly, the coordinate function algebras from where they come are over different subrings of $\mathbb{Q}(t)$. We believe that the two crystallized C^*-algebras are isomorphic; however, we do not have a proof of this yet.

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Manabendra Giri
Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi, India;
manabendra991@gmail.com

Arup Kumar Pal
Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi, India;
arup@isid.ac.in, arupkpal@gmail.com