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The centrally extended Heisenberg double of quantum SL_2

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Abstract. The centrally extended Heisenberg double of quantum SL_2 is a smash product of the quantized coordinate ring $\mathcal{O}_q(M_2)$ of 2×2 matrices and the Hopf algebra $U_q(\mathfrak{sl}_2)$. For this algebra, we determine its centre and show that it satisfies the quantum Gelfand–Kirillov conjecture. We give an explicit description of its prime, primitive and maximal ideals. We study a class of non-weight modules known as quasi-Whittaker modules. We classify the simple quasi-Whittaker modules and determine the annihilator of each simple quasi-Whittaker module.

1. Introduction

The concept of quantum groups was first introduced by Drinfeld and independently by Jimbo in the 1980s as a generalization of Lie groups and Lie algebras. Quantum groups, as a mathematical structure, have their origins in many problems studied in theoretical physics, such as solutions to the Yang–Baxter equations, the description of monodromy of vertex operators in conformal field theory, and integrable systems. They naturally arise as Hopf algebras depending on a deformation parameter q, which specialize to the universal enveloping algebras of certain Lie algebras as q approaches 1. The relationship between quantum groups and the corresponding classical groups is important in understanding the transition from classical to quantum physics.

Let G be a reductive algebraic group, and $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. There is a non-degenerate Hopf pairing between the universal enveloping algebra $U(\mathfrak{g})$ and the coordinate algebra $\mathcal{O}(G)$. The Lie algebra $\mathfrak{g}$ acts by derivations on $\mathcal{O}(G)$, which makes $\mathcal{O}(G)$ a left $U(\mathfrak{g})$-module algebra. With this action, one can construct the smash product algebra $D(G) := \mathcal{O}(G) \# U(\mathfrak{g})$, called the Heisenberg double of G. This setting can be extended to quantum groups, and then one obtains the Heisenberg double of $\mathcal{O}_q(G)$. More precisely, there is a non-degenerate dual pairing between the quantized coordinate algebra $\mathcal{O}_q(G)$ and the quantized universal enveloping algebra $U_q(\mathfrak{g})$ which turns $\mathcal{O}_q(G)$ into a $U_q(\mathfrak{g})$-module algebra. Then one can form the smash product algebra $D_q(G) := \mathcal{O}_q(G) \# U_q(\mathfrak{g})$, termed the Heisenberg double of $\mathcal{O}_q(G)$. The algebra $D_q(G)$ is interpreted as the ring of quantum differential operators on $\mathcal{O}_q(G)$, see, for instance, [2]. The Heisenberg double is

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a generalization of the Heisenberg–Weyl algebra, it captures interactions between a Hopf algebra and its dual, leading to interesting algebraic and geometric phenomena. For the general construction of the Heisenberg double of a Hopf algebra, see [12]. A twisted version of the Heisenberg double, constructed from a twisted Hopf algebra and a twisted pairing was introduced by Rosso and Savage [15].

This paper is concerned with a central extension of the Heisenberg double $D_q(\mathrm{SL}_2)$ of the quantized coordinate algebra of SL_2. More precisely, we study a smash product $A = \mathcal{O}_q(M_2) \# U_q(\mathfrak{sl}_2)$ of the quantized coordinate ring $\mathcal{O}_q(M_2)$ of 2×2 matrices and the Hopf algebra $U_q(\mathfrak{sl}_2)$. The algebra A can be seen as a central extension of the Heisenberg double $D_q(\mathrm{SL}_2)$ in the sense that $D_q(\mathrm{SL}_2)$ is a central factor of A. The primary goal of this paper is to describe the prime, primitive and maximal spectra of the algebra A. Additionally, we study a class of non-weight modules known as quasi-Whittaker modules. We classify the simple quasi-Whittaker modules and determine the annihilator of each simple quasi-Whittaker module. For an infinite-dimensional noncommutative algebra, it is a difficult and often intractable problem to classify its irreducible representations. Dixmier proposed that a basic first step towards tackling this problem would be to find the annihilators of the irreducible representations (that is, the primitive ideals), and then for each primitive ideal P, find at least one irreducible representation with annihilator P. Understanding these annihilators provides deep insight into the representation theory of the algebra.

As an abstract algebra, the generators and defining relations of A are given in the following.

Definition 1.1. The *centrally extended Heisenberg double of quantum* SL_2 is the $\mathbb{k}$-algebra A generated by K, K^{-1}, E, F, x_1, x_2, y_1, y_2 subject to the following defining relations:

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2E, \\ &KFK^{-1} = q^{-2}F, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \end{aligned} \tag{1}$$

$$\begin{aligned} &y_1x_1 = qx_1y_1, \qquad y_2x_2 = qx_2y_2, \\ &y_1y_2 = qy_2y_1, \qquad x_1x_2 = qx_2x_1, \\ &x_1y_2 = y_2x_1, \qquad y_1x_2 - x_2y_1 = (q - q^{-1})x_1y_2, \end{aligned} \tag{2}$$

$$\begin{aligned} &Kx_i = qx_iK, \qquad Ex_i = x_iE, \qquad Fx_i = q^{-1}x_iF + y_i, \\ &Ky_i = q^{-1}y_iK, \qquad Ey_i = y_iE + x_iK, \qquad Fy_i = qy_iF, \qquad (i = 1, 2). \end{aligned} \tag{3}$$

The algebra A is a Noetherian domain of Gelfand–Kirillov dimension 7, and it contains some remarkable subalgebras. First of all, the subalgebra generated by K, K^{-1}, E and F is isomorphic to the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$; and the subalgebra generated by y_1, x_1, y_2 and x_2 is isomorphic to the quantum matrix algebra $\mathcal{O}_q(M_2)$. Next, for $i = 1, 2$, let $A[i]$ be the subalgebra of A generated by y_i, x_i and $U_q(\mathfrak{sl}_2)$. Then $A[i]$ can be presented as a smash product of the quantum plane with the Hopf algebra $U_q(\mathfrak{sl}_2)$. We shall see that the subalgebras $A[i]$ are isomorphic to the main object studied in [3]. Some ring-theoretic properties and representations of the algebra A can be reduced to those of its subalgebras $A[i]$.

The algebra A is a quantum analogue of the universal enveloping algebra of the non-semisimple Lie algebra $\mathfrak{a} = \mathfrak{sl}_2 \ltimes (V_2 \oplus V_2)$, where $V_2 \oplus V_2$ is a direct sum of two copies of the 2-dimensional simple $\mathfrak{sl}_2$-module V_2. The universal enveloping algebra $U(\mathfrak{a})$ is known as the centrally extended Heisenberg double of SL_2 (see [16]). The name comes from the fact that the Heisenberg double of SL_2, denoted $D(\mathrm{SL}_2)$, is a central factor of $U(\mathfrak{a})$. The algebra $U(\mathfrak{a})$ can be presented as a smash product of the polynomial algebra in four variables with the universal enveloping algebra $U(\mathfrak{sl}_2)$. Basic properties and important families of simple modules over $U(\mathfrak{a})$ have been studied in [16], and the prime and primitive spectra of $U(\mathfrak{a})$ was obtained in [18].

The Heisenberg double $D_q(\mathrm{SL}_2)$ can be seen as a quantization of the algebra $D(\mathrm{SL}_2)$. The two algebras share many similar properties: their centres are trivial, they have no finite-dimensional representations, and they cannot have a Hopf algebra structure. However, the two algebras also exhibit some distinct features in both structure and representation theory. For instance, it is proven in [16] that $D(\mathrm{SL}_2)$ is a simple algebra. In contrast, we shall see that $D_q(\mathrm{SL}_2)$ is not a simple algebra, and its prime spectrum is homeomorphic to that of the quantum Weyl algebra. In [17], the Heisenberg double $D_q(E_2)$ of the quantum Euclidean group and its representations have been investigated. As is well known, the quantum Euclidean group is obtained by the contraction of the quantum group $\mathrm{SU}_q(2)$, see [7] and [19]. It can be expected that, in some sense, $D_q(E_2)$ is a potential contraction of the algebra $D_q(\mathrm{SL}_2)$.

Let us now briefly describe the contents of this paper. In Section 2, we present the construction of the smash product algebra A and the Heisenberg double $D_q(\mathrm{SL}_2)$. We also equip both A and $D_q(\mathrm{SL}_2)$ with an involution. In Section 3, we show that the centre of the algebra A is a polynomial algebra in one variable and that A satisfies the quantum Gelfand–Kirillov conjecture (Theorem 3.5). To achieve this, we consider some subalgebras and their localizations, and introduce four distinguished elements C_{ij} $(i, j = 1, 2)$. These elements are of special importance for the structure and representation theory of A. In particular, the element C_{21} is normal in A. In Section 4, we give a classification of the prime ideals of the algebra A (Theorem 4.3). The set $\mathrm{Spec}(A)$ of all prime ideals of A is decomposed into a disjoint union of four subsets, with each subset described explicitly. It is proved that the factor algebra A/AC_{21} is a domain (Proposition 4.4), and its centre is determined (Proposition 4.6). In Section 5, we give a classification of the primitive and maximal ideals of the algebra A (Theorem 5.11 and Corollary 5.12). For each primitive ideal, a set of explicit generators is given. Section 6 is devoted to the study of a class of non-weight A-modules, namely, the quasi-Whittaker modules. We obtain a classification of all simple quasi-Whittaker A-modules and determine the annihilator for each simple quasi-Whittaker module.

2. Preliminaries

In this paper, a module means a left module, $\Bbbk$ is an algebraically closed field of characteristic zero, $\Bbbk^* := \Bbbk \setminus \{0\}$, and $q \in \Bbbk^*$ is not a root of unity.

In this section, we present the construction of the smash product algebra A and the Heisenberg double $D_q(\mathrm{SL}_2)$. We introduce some elements (ϕ_i and ψ_i) that are useful in investigating the structures of A. Additionally, we equip the algebras A and $D_q(\mathrm{SL}_2)$ with an involution.

The algebra A. We start by recalling the notion of smash product algebras. Let H be a Hopf algebra and let R be a unital algebra. We say that R is a left H-module algebra if R is a left H-module with the H-action satisfying $h \cdot 1_R = \varepsilon(h)1_R$ and $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ for all $h \in H$ and $a, b \in R$. Here $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ is the coproduct in H. Given a left H-module algebra R, one can form the smash product algebra $R \# H$; that is, $R \# H = R \otimes H$ as a vector space, with multiplication given by

$$(a \# g)(b \# h) = \sum a(g_{(1)} \cdot b) \# g_{(2)}h \quad \text{for all } a, b \in R,\ g, h \in H.$$

As usual, we identify R with $R \# 1$, and H with $1 \# H$ for simplicity. For basic facts on smash product algebras, see, for instance, [14, Chap. 4].

The quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is the $\Bbbk$-algebra generated by the elements K, K^{-1}, E, F subject to the relations in (1). The centre of $U_q(\mathfrak{sl}_2)$ is a polynomial algebra generated by the Casimir element

$$\Delta_q := FE + (qK + q^{-1}K^{-1})/(q^{-1} - q)^2.$$

There exists a unique Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ with the coproduct Δ, counit ε, and antipode S such that

$$\begin{aligned}
&\Delta(K) = K \otimes K, && S(K) = K^{-1}, && \varepsilon(K) = 1,\\
&\Delta(E) = E \otimes K + 1 \otimes E, && S(E) = -EK^{-1}, && \varepsilon(E) = 0,\\
&\Delta(F) = F \otimes 1 + K^{-1} \otimes F, && S(F) = -KF, && \varepsilon(F) = 0.
\end{aligned}$$

The quantized coordinate ring of the 2×2 matrices, denoted $\mathcal{O}_q(M_2)$, is the $\Bbbk$-algebra generated by the indeterminates x_1, x_2, y_1, y_2 subject to the relations in (2). It is well known that the centre of $\mathcal{O}_q(M_2)$ is a polynomial algebra $\Bbbk[\mathtt{D}]$, where $\mathtt{D} := y_1x_2 - qx_1y_2$ is called the quantum determinant of $\mathcal{O}_q(M_2)$. Basic properties and representations of $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(M_2)$ can be found, for instance, in [10]. There is a natural action of $U_q(\mathfrak{sl}_2)$ on the quantum matrix algebra $\mathcal{O}_q(M_2)$ which turns $\mathcal{O}_q(M_2)$ into a $U_q(\mathfrak{sl}_2)$-module algebra. On generators, the action is given by the following rule:

$$\begin{aligned}
&K \cdot x_i = qx_i, && E \cdot x_i = 0, && F \cdot x_i = y_i,\\
&K \cdot y_i = q^{-1}y_i, && E \cdot y_i = x_i, && F \cdot y_i = 0.
\end{aligned} \tag{4}$$

The action (4) makes $\mathcal{O}_q(M_2)$ a left $U_q(\mathfrak{sl}_2)$-module algebra, so one can form the smash product algebra $A := \mathcal{O}_q(M_2) \# U_q(\mathfrak{sl}_2)$. The explicit description of A via generators and defining relations was given in Definition 1.1. The algebra A can be presented as an iterated Ore extension over $\mathcal{O}_q(M_2)$ in the form $A = \mathcal{O}_q(M_2)[K^{\pm 1}; \sigma_1][F; \sigma_2, \delta_2][E; \sigma_3, \delta_3]$ where the σ_i are automorphisms and the δ_i are left σ_i-derivations of the appropriate subalgebras. In particular, A is a Noetherian domain of Gelfand–Kirillov dimension 7. Note that the algebra A admits a PBW type basis.

The Heisenberg double $D_q(\mathrm{SL}_2)$. For a ring R and a subset $\mathtt{S} \subset R$, we indicate by $\langle \mathtt{S} \rangle$ the two-sided ideal of R generated by the elements of $\mathtt{S}$. As usual, we denote by the same letters their images in the factor algebra $R/\langle \mathtt{S} \rangle$. The quantized coordinate algebra $\mathcal{O}_q(\mathrm{SL}_2)$ is the $\Bbbk$-algebra given by the quotient $\mathcal{O}_q(\mathrm{SL}_2) := \mathcal{O}_q(M_2)/\langle \mathtt{D} - 1 \rangle$. There exists a Hopf algebra structure on the algebra $\mathcal{O}_q(\mathrm{SL}_2)$ with coproduct Δ, counit ε and antipode S given by

$$\Delta \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} \otimes \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix}, \qquad \varepsilon \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$S \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_2 & -q^{-1}x_1 \\ -qy_2 & y_1 \end{bmatrix}.$$

There exists a unique dual pairing $\langle \cdot, \cdot \rangle$ of the Hopf algebras $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(\mathrm{SL}_2)$ such that the nonvanishing pairings between the generators are

$$\langle K, y_1 \rangle = q^{-1}, \quad \langle K^{-1}, y_1 \rangle = q, \quad \langle K, x_2 \rangle = q,$$
$$\langle K^{-1}, x_2 \rangle = q^{-1}, \qquad \langle E, y_2 \rangle = \langle F, x_1 \rangle = 1.$$

If q is not a root of unity, the pairing is non-degenerate. In particular, $U_q(\mathfrak{sl}_2)$ can be embedded into the dual algebra $\mathcal{O}_q(\mathrm{SL}_2)^*$. The map $\rho : U_q(\mathfrak{sl}_2) \to \operatorname{End} \mathcal{O}_q(\mathrm{SL}_2)$ defined by

$$\rho(u)(f) := \sum \langle u, f_{(2)} \rangle f_{(1)} \quad \text{where } u \in U_q(\mathfrak{sl}_2),\ f \in \mathcal{O}_q(\mathrm{SL}_2)$$

endows $\mathcal{O}_q(\mathrm{SL}_2)$ with the structure of a left $U_q(\mathfrak{sl}_2)$-module algebra. It is easily checked that, on generators, this action coincides with the one given in (4). With this action, the smash product algebra $D_q(\mathrm{SL}_2) := \mathcal{O}_q(\mathrm{SL}_2) \# U_q(\mathfrak{sl}_2)$ is called the Heisenberg double of $\mathcal{O}_q(\mathrm{SL}_2)$.

The algebra $D_q(\mathrm{SL}_2)$ has a close connection with the algebra A. In fact, it can be verified that *the quantum determinant* $\mathtt{D}$ *lies in the centre of* A. As a result, we have

$$D_q(\mathrm{SL}_2) \simeq \frac{\mathcal{O}_q(M_2) \# U_q(\mathfrak{sl}_2)}{\langle \mathtt{D} - 1 \rangle}.$$

Thus, A is a central extension of $D_q(\mathrm{SL}_2)$, which explains the name of the algebra A.

The elements ϕ_i and ψ_i. We first introduce the elements ϕ_i and ψ_i $(i = 1, 2)$, and compute their commutation relations with the generators of A. For $i = 1, 2$, define

$$\begin{aligned}\phi_i &:= (1-q^2)Fx_i + q^2 y_i,\\ \psi_i &:= (1-q^{-2})Ey_i + q^{-2}x_i K.\end{aligned} \tag{5}$$

Note that we can also write $\phi_i = Fx_i - qx_iF$, and $\psi_i = Ey_i - q^{-2}y_iE$.

The commutation relations of ϕ_i with the generators of A are given in the following:

$$\begin{aligned}&\phi_i K = qK\phi_i, \qquad \phi_i E = E\phi_i - qK^{-1}x_i, \qquad \phi_i F = qF\phi_i, \quad (i = 1, 2);\\ &\phi_1 y_1 = y_1\phi_1, && \phi_1 x_1 = q^{-1}x_1\phi_1,\\ &\phi_1 y_2 = qy_2\phi_1, && \phi_1 x_2 = x_2\phi_1;\\ &\phi_2 y_1 = q^{-1}y_1\phi_2 - (q^{-1}-q)^2 F\mathtt{D}, && \phi_2 x_1 = q^{-2}x_1\phi_2 + (q^{-1}-q)\mathtt{D},\\ &\phi_2 y_2 = y_2\phi_2, && \phi_2 x_2 = q^{-1}x_2\phi_2.\end{aligned} \tag{6}$$

From (6) we see that the element ϕ_1 has nice commutation relations with the elements of $\mathcal{O}_q(M_2)$, as it q-commutes with the generators of $\mathcal{O}_q(M_2)$. The commutation relations of ψ_i with the generators of A are given in the following:

$$\begin{aligned}&\psi_i K = q^{-1}K\psi_i, \quad \psi_i E = E\psi_i, \quad \psi_i F = q^{-1}F\psi_i - q^{-2}K^{-1}y_i, \quad (i = 1, 2);\\ &\psi_1 y_1 = y_1\psi_1, && \psi_1 x_1 = qx_1\psi_1,\\ &\psi_1 y_2 = qy_2\psi_1 + (1-q^{-2})K\mathtt{D}, && \psi_1 x_2 = q^2x_2\psi_1 - (q^{-1}-q)^2E\mathtt{D};\\ &\psi_2 y_1 = q^{-1}y_1\psi_2, && \psi_2 x_1 = x_1\psi_2,\\ &\psi_2 y_2 = y_2\psi_2, && \psi_2 x_2 = qx_2\psi_2.\end{aligned} \tag{7}$$

From (7) we see that the element ψ_2 q-commutes with the generators of $\mathcal{O}_q(M_2)$.

An involution on A. Now we equip the algebra A with an involution. Recall that an involution $*$ on an algebra R is a $\Bbbk$-algebra anti-automorphism such that $(r^*)^* = r$ for all $r \in R$. It is easy to check that the following map defines an involution τ on A:

$$\begin{aligned}&\tau(K) = K, \qquad \tau(K^{-1}) = K^{-1}, \qquad \tau(E) = -KF, \qquad \tau(F) = -EK^{-1},\\ &\tau(x_1) = y_2, \qquad \tau(x_2) = y_1, \qquad \tau(y_1) = x_2, \qquad \tau(y_2) = x_1.\end{aligned} \tag{8}$$

For the construction of a canonical involution on a smash product algebra $R \# H$ that extends the involutions on R and H, see [1, Lemma 3.2.3]. Notice that $\tau(\mathtt{D}) = \mathtt{D}$, thus τ also defines an involution on the Heisenberg double $D_q(\mathrm{SL}_2)$. Applying the involution τ to the elements ϕ_i and ψ_i $(i = 1, 2)$, we obtain

$$\tau(\phi_1) = q^2\psi_2K^{-1}, \quad \tau(\phi_2) = q^2\psi_1K^{-1}, \quad \tau(\psi_1) = q^{-2}K\phi_2, \quad \tau(\psi_2) = q^{-2}K\phi_1. \tag{9}$$

The following lemma provides some connections between the elements ϕ_i's and ψ_i's.

Lemma 2.1. *In the algebra A, the following identities hold:*

$$\phi_1 x_2 - q\phi_2 x_1 = q^2 \mathtt{D}, \qquad \phi_1 y_2 - q\phi_2 y_1 = q(q^2-1)\mathtt{D}F,$$
$$\psi_1 y_2 - q\psi_2 y_1 = -q^{-2}\mathtt{D}K, \qquad \psi_1 x_2 - q\psi_2 x_1 = (1-q^{-2})\mathtt{D}E.$$

Proof. A straightforward verification. ∎

3. The centre of A

The aim of this section is to show that the centre of the algebra A is a polynomial algebra $Z(A) = \Bbbk[\mathtt{D}]$, and that A satisfies the quantum Gelfand–Kirillov conjecture. Four distinguished elements C_{ij} $(i, j = 1, 2)$ are introduced, which are of special importance for the structure and representation theory of A. Additionally, some subalgebras and their localizations are considered, which are instrumental for our purpose.

The elements C_{ij}. For $i, j \in \{1, 2\}$, define

$$C_{ij} := \psi_i \phi_j - y_i x_j K^{-1}. \tag{10}$$

Using the expressions of ψ_i and ϕ_j (see (5)), we can write C_{ij} explicitly in the following form:

$$\begin{aligned} C_{ij} = {} & -q^{-1}(q^{-1}-q)^2 EF y_i x_j + (q^2-1)E y_i y_j + (q^{-2}-1)KF x_i x_j \\ & + y_i x_j((1-q^{-2})K - K^{-1}) + q^{-1} x_i y_j K. \end{aligned} \tag{11}$$

The elements C_{ij} behave nicely under the involution τ of the algebra A. More precisely, applying the involution τ to C_{ij} and using (9), we obtain

$$\tau(C_{11}) = C_{22}, \quad \tau(C_{22}) = C_{11}, \quad \tau(C_{12}) = C_{12}, \quad \tau(C_{21}) = C_{21}. \tag{12}$$

Recall that, an element a of a ring R is normal if $aR = Ra$. The following lemma shows that the elements C_{ij} commute with the elements of $U_q(\mathfrak{sl}_2)$. Moreover, the element C_{21} is normal in A.

Lemma 3.1. (1) *The elements C_{ij} $(i, j \in \{1, 2\})$ commute with $K^{\pm 1}$, E and F. Moreover, for $i \in \{1, 2\}$, the element C_{ii} commutes with y_i and x_i.*

(2) *The element C_{21} is normal in the algebra A.*

Proof. (1) Clearly, C_{ij} commutes with K. Let us verify that $C_{ij}E = EC_{ij}$ by using (6) and (7):

$$\begin{aligned} C_{ij}E &= \psi_i \phi_j E - y_i x_j K^{-1} E \\ &= \psi_i (E\phi_j - qK^{-1} x_j) - y_i x_j K^{-1} E \\ &= E\psi_i \phi_j - q\psi_i K^{-1} x_j - y_i x_j K^{-1} E \\ &= E\psi_i \phi_j - \psi_i \cdot x_j K^{-1} - q^{-2} y_i E \cdot x_j K^{-1} \end{aligned}$$

$$= E\psi_i\phi_j - (\psi_i + q^{-2}y_iE)x_jK^{-1}$$
$$= E\psi_i\phi_j - Ey_ix_jK^{-1}$$
$$= EC_{ij}.$$

Applying the involution τ (see (8)) and invoking (12), we obtain $C_{ij}F = FC_{ij}$. It is easily checked that C_{ii} commutes with y_i and x_i.

(2) Note that the following identities hold in the algebra A:

$$C_{21}x_1 = q^{-1}x_1C_{21}, \qquad C_{21}x_2 = qx_2C_{21},$$
$$C_{21}y_1 = q^{-1}y_1C_{21}, \qquad C_{21}y_2 = qy_2C_{21}.$$

The result then follows from statement (1) and the above identities. ■

It is noteworthy that the elements C_{11}, C_{22} and C_{12} are not normal in A. The following identity provides a relation between the elements C_{12} and C_{21}:

$$C_{12} = qC_{21} - q^{-1}(q^{-1} - q)^2\mathtt{D}\Delta_q \tag{13}$$

where Δ_q is the Casimir element of $U_q(\mathfrak{sl}_2)$. Moreover, based on the definition of C_{ij} given in (10) and utilizing (6) and (7), one verifies that the elements C_{11} and C_{22} satisfy the following relations:

$$\begin{aligned} C_{11}x_2 &= q^2x_2C_{11} + (q - q^{-1})\mathtt{D}\omega_1, \\ C_{11}y_2 &= q^2y_2C_{11} + (q - q^{-1})\mathtt{D}K\phi_1, \end{aligned} \tag{14}$$

where $\omega_1 = (q^{-1} - q)E\phi_1 + qx_1K^{-1}$, and

$$\begin{aligned} C_{22}x_1 &= q^{-2}x_1C_{22} + (q^{-1} - q)\mathtt{D}\psi_2, \\ C_{22}y_1 &= q^{-2}y_1C_{22} + (q^{-1} - q)\mathtt{D}\omega_2, \end{aligned} \tag{15}$$

where $\omega_2 = (q - q^{-1})\psi_2F + y_2K^{-1}$. Applying the involution τ, we get $\tau(\omega_1) = q^2\omega_2$. The next lemma gives some connections between the elements C_{11}, C_{22} and C_{21}.

Lemma 3.2. *In the algebra A, the following identities hold*

$$C_{11}y_2 = q^2C_{21}y_1 - q^{-1}\mathtt{D}K\phi_1, \qquad C_{11}x_2 = q^2C_{21}x_1 - q^{-1}\mathtt{D}\omega_1,$$
$$C_{22}y_1 = q^{-1}C_{21}y_2 - q\mathtt{D}\omega_2, \qquad C_{22}x_1 = q^{-1}C_{21}x_2 - q\mathtt{D}\psi_2.$$

Proof. The identities are obtained from the definition of C_{ij} (see (10)) and Lemma 2.1. In fact,

$$\begin{aligned} C_{11}y_2 &= (\psi_1\phi_1 - y_1x_1K^{-1})y_2 \\ &= q\psi_1y_2 \cdot \phi_1 - q^2y_2x_1K^{-1}y_1 \\ &= q(q\psi_2y_1 - q^{-2}\mathtt{D}K)\phi_1 - q^2y_2x_1K^{-1}y_1 \\ &= q^2(\psi_2\phi_1 - y_2x_1K^{-1})y_1 - q^{-1}\mathtt{D}K\phi_1 \\ &= q^2C_{21}y_1 - q^{-1}\mathtt{D}K\phi_1. \end{aligned}$$

Then applying the involution τ to the above identity, one gets $C_{22}x_1 = q^{-1}C_{21}x_2 - q\mathtt{D}\psi_2$. Similarly,

$$\begin{aligned}C_{11}x_2 &= (\psi_1\phi_1 - y_1x_1K^{-1})x_2\\ &= \psi_1\cdot\phi_1x_2 - qy_1x_2K^{-1}x_1\\ &= \psi_1(q\phi_2x_1 + q^2\mathtt{D}) - qy_1x_2K^{-1}x_1\\ &= q(\psi_1\phi_2 - y_1x_2K^{-1})x_1 + q^2\mathtt{D}\psi_1\\ &= qC_{12}x_1 + q^2\mathtt{D}\psi_1.\end{aligned}$$

Substituting (13) into the above identity yields that $C_{11}x_2 = q^2C_{21}x_1 - q^{-1}\mathtt{D}\omega_1$. Then applying the involution τ to this equality, we get $C_{22}y_1 = q^{-1}C_{21}y_2 - q\mathtt{D}\omega_2$, as required. ∎

Some subalgebras of A. We will consider the following chain of subalgebras $\mathcal{O}_q(M_2) \subset \Gamma \subset \mathcal{A} \subset A$, as well as some of their localizations. The aim is to embed A into a quantum polynomial algebra that is isomorphic to a certain localization of A. First, let us recall the definition of generalized Weyl algebras which was originally introduced by Bavula [5]. Let D be a ring, σ an automorphism of D, and a an element of the centre of D. The generalized Weyl algebra $\mathcal{A} := D[x, y; \sigma, a]$ is a ring generated by D, x and y subject to the following relations

$$\begin{aligned}xd &= \sigma(d)x, & yd &= \sigma^{-1}(d)y, \quad \text{for all } d \in D,\\ yx &= a, & xy &= \sigma(a).\end{aligned}$$

The generalized Weyl algebra $\mathcal{A}$ is a domain if and only if D is a domain and $a \neq 0$.

Next, we recall the definition of quantum polynomial algebras. Let $\mathbf{Q} = (q_{ij}) \in M_n(\mathbb{k}^*)$ be a multiplicatively skew-symmetric matrix (that is $q_{ii} = q_{ij}q_{ji} = 1$ for all $1 \le i, j \le n$). If there is a skew-symmetric matrix $M = (\mu_{ij})$ such that $q_{ij} = q^{\mu_{ij}}$ for all $1 \le i, j \le n$, then we denote $\mathbf{Q} = q^M$. Let r be an integer $\le n$, the quantum polynomial algebra $\Lambda := \mathbb{k}_{\mathbf{Q}}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}, X_{r+1}, \ldots, X_n]$ is the associated $\mathbb{k}$-algebra generated by the indeterminates $X_1, \ldots, X_n, X_1^{-1}, \ldots, X_r^{-1}$ subject to the following defining relations:

$$X_iX_j = q_{ij}X_jX_i, \quad (1 \le i, j \le n), \quad \text{and} \quad X_iX_i^{-1} = X_i^{-1}X_i = 1, \quad (i = 1, \ldots, r).$$

In particular, Λ is a quantum affine space if $r = 0$, and a quantum torus if $r = n$.

The quantum matrix algebra $\mathcal{O}_q(M_2)$ can be presented as a generalized Weyl algebra in the form $\mathcal{O}_q(M_2) = \mathbb{k}[x_1, y_2, \mathtt{D}][x_2, y_1; \sigma, a = qx_1y_2 + \mathtt{D}]$ where σ is the automorphism of the polynomial algebra $\mathbb{k}[x_1, y_2, \mathtt{D}]$ such that $\sigma(x_1) = q^{-1}x_1$, $\sigma(y_2) = q^{-1}y_2$ and $\sigma(\mathtt{D}) = \mathtt{D}$. Let $\mathcal{O}_q(M_2)_{x_2}$ be the localization of $\mathcal{O}_q(M_2)$ at the powers of the element x_2. It is easily seen that $\mathcal{O}_q(M_2)_{x_2}$ can be expressed as the tensor product of the polynomial algebra $\mathbb{k}[\mathtt{D}]$ and a skew Laurent polynomial algebra, that is,

$$\mathcal{O}_q(M_2)_{x_2} = \mathbb{k}[\mathtt{D}] \otimes \mathbb{k}[x_1, y_2][x_2^{\pm 1}; \sigma] \tag{16}$$

where $\sigma(x_1) = q^{-1}x_1$ and $\sigma(y_2) = q^{-1}y_2$. A simple calculation verifies that the centre of the skew Laurent polynomial algebra is trivial. Then we conclude from (16) that the centre of $\mathcal{O}_q(M_2)_{x_2}$ is equal to $\Bbbk[\mathtt{D}]$, and it follows that $Z(\mathcal{O}_q(M_2)) = \Bbbk[\mathtt{D}]$.

Let Γ be the subalgebra of A generated by the elements x_1, y_1, x_2, y_2, and F. Clearly, Γ is an Ore extension over $\mathcal{O}_q(M_2)$ that can be presented as $\Gamma = \mathcal{O}_q(M_2)[F;\sigma,\delta]$. Let Γ_{x_1,x_2} be the localization of Γ at the Ore set generated by the elements x_1 and x_2. From the expression of the element ϕ_1 (see (5)), we see that in the algebra Γ_{x_1,x_2}, one can replace the generator F by the element ϕ_1. Then from (16), it follows that Γ_{x_1,x_2} can be presented as the tensor product of $\Bbbk[\mathtt{D}]$ with a quantum polynomial algebra, that is,

$$\Gamma_{x_1,x_2} = \Bbbk[\mathtt{D}] \otimes \Bbbk_{q^N}[x_1^{\pm 1}, x_2^{\pm 1}, y_2, \phi_1], \quad \text{where } N = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}. \tag{17}$$

It is easy to check that the centre of the quantum polynomial algebra in (17) is trivial. Thus $Z(\Gamma_{x_1,x_2}) = \Bbbk[\mathtt{D}]$, and hence $Z(\Gamma) = \Bbbk[\mathtt{D}]$. Note that the elements y_2 and ϕ_1 are normal in Γ.

Recall that an ideal $\mathfrak{p}$ of a ring R is said to be completely prime if $R/\mathfrak{p}$ is a domain.

Lemma 3.3. *The ideals $\langle y_2\rangle$ and $\langle \phi_1\rangle$ of Γ are completely prime.*

Proof. Since y_2 is normal in Γ, the factor algebra $\Gamma/\langle y_2\rangle$ can be presented as an Ore extension in the form $\Gamma/\langle y_2\rangle \simeq (\mathcal{O}_q(M_2)/\langle y_2\rangle)[F;\sigma,\delta]$. Clearly, the coefficient ring $\mathcal{O}_q(M_2)/\langle y_2\rangle$ is a domain. As a result, $\Gamma/\langle y_2\rangle$ is a domain, i.e., the ideal $\langle y_2\rangle$ of Γ is completely prime. Let $\mathbb{F}$ be the subalgebra of Γ generated by the elements F, x_1 and y_1. Then $\mathbb{F}$ can be presented as a generalized Weyl algebra of the following form:

$$\mathbb{F} = \Bbbk[y_1,\phi_1][x_1, F;\sigma, a = (\phi_1 - q^2 y_1)/(1-q^2)]$$

where σ is the automorphism of $\Bbbk[y_1,\phi_1]$ such that $\sigma(y_1) = q^{-1}y_1$ and $\sigma(\phi_1) = q\phi_1$. It follows that $\mathbb{F}/\langle\phi_1\rangle$ is a generalized Weyl algebra that can be expressed as

$$\mathbb{F}/\langle\phi_1\rangle \simeq \Bbbk[y_1][x_1, F;\sigma, a = -q^2 y_1/(1-q^2)].$$

In particular, $\mathbb{F}/\langle\phi_1\rangle$ is a domain. Notice that the algebra Γ can be presented as an iterated Ore extension over $\mathbb{F}$, that is $\Gamma = \mathbb{F}[y_2;\sigma_1][x_2;\sigma_2,\delta_2]$. Since the element ϕ_1 is normal in Γ, the factor algebra $\Gamma/\langle\phi_1\rangle$ is isomorphic to the iterated Ore extension $(\mathbb{F}/\langle\phi_1\rangle)[y_2;\sigma_1][x_2;\sigma_2,\delta_2]$. Since the coefficient ring $\mathbb{F}/\langle\phi_1\rangle$ is a domain, we conclude that $\Gamma/\langle\phi_1\rangle$ is a domain. In other words, $\langle\phi_1\rangle$ is a completely prime ideal of Γ. ∎

Let $\mathcal{A}$ be the subalgebra of A generated by Γ and $K^{\pm 1}$. Then $\mathcal{A}$ is a skew Laurent polynomial algebra over Γ that can be presented as $\mathcal{A} = \Gamma[K^{\pm 1};\sigma]$. If we denote by U_q^- the (Hopf) subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $K^{\pm 1}$ and F, then $\mathcal{A} = \mathcal{O}_q(M_2) \# U_q^-$. Let

$\mathcal{A}_{x_1,x_2}$ be the localization of $\mathcal{A}$ at the Ore set generated by x_1 and x_2. Then $\mathcal{A}_{x_1,x_2} = \Gamma_{x_1,x_2}[K^{\pm 1};\sigma]$. Invoking (17), we obtain

$$\mathcal{A}_{x_1,x_2} = \Bbbk[\mathtt{D}] \otimes \Bbbk_{q^{\mathbf{N}}}[x_1^{\pm 1}, x_2^{\pm 1}, y_2, \phi_1, K^{\pm 1}], \quad \text{where } \mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}. \tag{18}$$

It is easy to verify that the centre of the second tensor component is trivial. We conclude that $Z(\mathcal{A}_{x_1,x_2}) = \Bbbk[\mathtt{D}]$, and hence $Z(\mathcal{A}) = \Bbbk[\mathtt{D}]$. Clearly, the elements y_2 and ϕ_1 are normal in $\mathcal{A}$. The next lemma shows that the ideals $\langle y_2 \rangle$ and $\langle \phi_1 \rangle$ of the algebra $\mathcal{A}$ are completely prime.

Lemma 3.4. *The ideals $\langle y_2 \rangle$ and $\langle \phi_1 \rangle$ of $\mathcal{A}$ are completely prime. Moreover, $\langle y_2\phi_1 \rangle = \langle y_2 \rangle \cap \langle \phi_1 \rangle$.*

Proof. Notice that $\mathcal{A}/\langle y_2 \rangle \simeq \Gamma/\langle y_2 \rangle[K^{\pm 1};\sigma]$ is an Ore extension over $\Gamma/\langle y_2 \rangle$. It is also obvious that $\mathcal{A}/\langle \phi_1 \rangle \simeq \Gamma/\langle \phi_1 \rangle[K^{\pm 1};\sigma]$ is an Ore extension over $\Gamma/\langle \phi_1 \rangle$. From Lemma 3.3 it follows that the ideals $\langle y_2 \rangle$ and $\langle \phi_1 \rangle$ of $\mathcal{A}$ are completely prime. It remains to show that $\langle y_2\phi_1 \rangle = \langle y_2 \rangle \cap \langle \phi_1 \rangle$. The inclusion $\langle y_2\phi_1 \rangle \subseteq \langle y_2 \rangle \cap \langle \phi_1 \rangle$ is obvious. Conversely, if $u \in \langle y_2 \rangle \cap \langle \phi_1 \rangle$ then $u = y_2 v = \phi_1 w$ for some $v, w \in \mathcal{A}$, since y_2 and ϕ_1 are normal in $\mathcal{A}$. Therefore, $y_2 v \in \langle \phi_1 \rangle$, and this implies that $v \in \langle \phi_1 \rangle$, since $\langle \phi_1 \rangle$ is completely prime and $y_2 \notin \langle \phi_1 \rangle$. Then we have $v = \phi_1 r$ for some $r \in \mathcal{A}$, and thus $u = y_2\phi_1 r \in \langle y_2\phi_1 \rangle$. ∎

The centre of A. The classical Gelfand–Kirillov conjecture states that for a finite-dimensional algebraic Lie algebra $\mathfrak{g}$ over an algebraically closed field $\Bbbk$ of characteristic zero, the skew field $\mathrm{Frac}(U(\mathfrak{g}))$ should be isomorphic to the skew field of some Weyl algebra over a purely transcendental field extension of $\Bbbk$. Drawing inspiration from the concept of birational equivalence in algebraic geometry, this conjecture extends that idea into the realm of noncommutative algebra. It not only highlights a deep connection between enveloping algebras and Weyl algebras but also provides a general framework for comparing different noncommutative algebras through their skew fields of fractions. Furthermore, the conjecture has been extended to the setting of quantum algebras. Recall that a quantum Weyl field is a skew field of fractions of a quantum affine space. We say that a $\Bbbk$-algebra R admitting a skew field of fractions $\mathrm{Frac}(R)$ satisfies the quantum Gelfand–Kirillov conjecture if $\mathrm{Frac}(R)$ is isomorphic to a quantum Weyl field over a purely transcendental field extension of $\Bbbk$. In the next theorem, we show that the centre of A is a polynomial algebra, and A satisfies the quantum Gelfand–Kirillov conjecture.

Theorem 3.5. $Z(A) = \Bbbk[\mathtt{D}]$, *and A satisfies the quantum Gelfand–Kirillov conjecture.*

Proof. Note that the algebra A is an Ore extension over $\mathcal{A}$ that can be written as $A = \mathcal{A}[E;\sigma,\delta]$ for some proper automorphism σ and a σ-derivation δ. Let S be the Ore set of A generated by the elements x_1, x_2, y_2 and ϕ_1, and let AS^{-1} be the localization of A at the Ore set S. Then $AS^{-1} = \mathcal{A}_{x_1,x_2,y_2,\phi_1}[E;\sigma,\delta]$. By (10) and (5), we can write the element C_{21} as follows:

$$\begin{aligned} C_{21} &= \psi_2\phi_1 - y_2x_1K^{-1} \\ &= ((1-q^{-2})Ey_2 + q^{-2}x_2K)\phi_1 - y_2x_1K^{-1} \\ &= (1-q^{-2})E \cdot y_2\phi_1 + q^{-3}x_2\phi_1K - y_2x_1K^{-1}. \end{aligned}$$

Thus in the algebra AS^{-1} one can replace the generator E by the element C_{21}. From (18) it follows that AS^{-1} is a tensor product of the polynomial algebra $\mathbb{k}[\mathtt{D}]$ with a quantum polynomial algebra. More precisely, let $Q = q^M$ where the skew-symmetric matrix M is as follows:

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}.$$

Then, we have

$$AS^{-1} = \mathbb{k}[\mathtt{D}] \otimes \mathbb{k}_Q[x_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \phi_1^{\pm 1}, K^{\pm 1}, C_{21}]. \tag{19}$$

It is easy to check that the centre of the quantum polynomial algebra in (19) is equal to $\mathbb{k}$. Thus $Z(AS^{-1}) = \mathbb{k}[\mathtt{D}]$, and we conclude that $Z(A) = \mathbb{k}[\mathtt{D}]$. Since $\mathrm{Frac}(A) = \mathrm{Frac}(AS^{-1})$, it follows immediately from (19) that A satisfies the quantum Gelfand–Kirillov conjecture. ■

For any $\chi \in \mathbb{k}$, we define $A(\chi) := A/\langle \mathtt{D} - \chi\rangle$. In particular, $A(\chi)$ specializes to the Heisenberg double $D_q(\mathrm{SL}_2)$ if we take $\chi = 1$. The following corollary shows that the centre of $A(\chi)$ is trivial for any $\chi \in \mathbb{k}$. In particular, the Heisenberg double $D_q(\mathrm{SL}_2)$ has a trivial centre.

Corollary 3.6. *For any $\chi \in \mathbb{k}$, the algebra $A(\chi)$ is a Noetherian domain of Gelfand–Kirillov dimension 6, and $Z(A(\chi)) = \mathbb{k}$.*

Proof. Note that the factor algebra $\mathcal{O}_q(\chi) := \mathcal{O}_q(M_2)/\langle \mathtt{D} - \chi\rangle$ is a domain of Gelfand–Kirillov dimension 3 for any $\chi \in \mathbb{k}$. It is clear that the algebra $A(\chi) = \mathcal{O}_q(\chi) \,\#\, U_q(\mathfrak{sl}_2)$ can be presented as an iterated Ore extension over the algebra $\mathcal{O}_q(\chi)$ with the generators added in the order F, $K^{\pm 1}$, and E. As a consequence, $A(\chi)$ is a Noetherian domain of Gelfand–Kirillov dimension 6. Let $A(\chi)S^{-1}$ be the localization of $A(\chi)$ at the Ore set S generated by the elements x_1, x_2, y_2 and ϕ_1. It is clear that $A(\chi)S^{-1} \simeq AS^{-1}/\langle \mathtt{D} - \chi\rangle$. Then it follows from (19) that, for any $\chi \in \mathbb{k}$, the algebra $A(\chi)S^{-1}$ is isomorphic to the quantum polynomial algebra $\mathbb{k}_Q[x_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \phi_1^{\pm 1}, K^{\pm 1}, C_{21}]$ whose centre is equal to $\mathbb{k}$ (see (19) for the matrix Q). As a result, $Z(A(\chi)) = \mathbb{k}$. ■

4. The prime spectrum of A

In this section, we provide a description of the prime spectrum of the algebra A (Theorem 4.3). We prove that the factor algebra A/AC_{21} is a domain (Proposition 4.4) and determine its centre (Proposition 4.6).

Prime ideals of the algebra A. Let R be a ring. A proper ideal $\mathfrak{p}$ of R is called a prime ideal whenever, for all ideals $\mathfrak{a}, \mathfrak{b}$ of R, if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ then either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. As is well known, a completely prime ideal is prime, but in general, the converse does not hold. We denote by $\mathrm{Spec}(R)$ the set of its prime ideals. The set $\mathrm{Spec}(R)$ is a partially ordered set with respect to inclusion of prime ideals. Let M be an R-module, the annihilator of M is an ideal of R defined by

$$\mathrm{ann}_R\, M = \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

Recall that an ideal $\mathfrak{p}$ of a ring R is called a primitive ideal if $\mathfrak{p}$ is the annihilator of some simple R-module, and the set $\mathrm{Prim}(R)$ of all of them is called the primitive spectrum of R. It is well known that all primitive ideals of R are prime, and all maximal ideals of R are primitive. Our next goal is to describe the prime spectrum of the algebra A. We will see that the elements C_{ij} are of special importance in the description of $\mathrm{Spec}(A)$ where the elements C_{ij} are defined in (10).

Let us first recall some results from [3]. The main object investigated in [3] is a smash product of the quantum plane with the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, denoted $\mathbf{A} := \mathbb{k}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$. Some subalgebras or factor algebras of A are isomorphic to $\mathbf{A}$. In fact, for $i = 1, 2$, let $A[i]$ be the subalgebra of A generated by y_i, x_i and $U_q(\mathfrak{sl}_2)$. Then $A[i] := \mathbb{k}_q[y_i, x_i] \# U_q(\mathfrak{sl}_2)$ is a smash product algebra where $\mathbb{k}_q[y_i, x_i] = \mathbb{k}\langle y_i, x_i \mid y_i x_i = q x_i y_i\rangle$ is the quantum plane. The following map gives an isomorphism of algebras from $\mathbf{A}$ to $A[i]$ (see [3] for the defining relations of $\mathbf{A}$):

$$\mathbf{A} \to A[i], \quad K \mapsto K^{-1}, \quad K^{-1} \mapsto K, \quad E \mapsto F, \quad F \mapsto E, \quad X \mapsto y_i, \quad Y \mapsto x_i. \tag{20}$$

Notice that in the algebra A we have $\langle x_i \rangle = \langle y_i, x_i \rangle$ $(i = 1, 2)$. Then it is clear that

$$A/\langle x_1 \rangle \simeq \mathbb{k}_q[y_2, x_2] \# U_q(\mathfrak{sl}_2) \quad \text{and} \quad A/\langle x_2 \rangle \simeq \mathbb{k}_q[y_1, x_1] \# U_q(\mathfrak{sl}_2). \tag{21}$$

The algebra $\mathbf{A}$ is a Noetherian domain of Gelfand–Kirillov dimension 5. In [3], the prime, primitive, completely prime and maximal ideals of $\mathbf{A}$ have been determined, and a classification of its simple weight modules was obtained. In view of the isomorphism given in (20), the results obtained for $\mathbf{A}$ can be transferred to that of the algebras $A[i]$ $(i = 1, 2)$. For convenience, we restate some algebraic results from [3] regarding the algebras $A[i]$ that are relevant to our investigation in the following proposition.

Proposition 4.1. *Let $i = 1, 2$.*

(1) ([3, Theorem 2.10]) *The centre of $A[i]$ is a polynomial algebra $Z(A[i]) = \mathbb{k}[C_{ii}]$, where the element C_{ii} is given in* (11).

(2) ([3, Theorem 3.7]) *The prime spectrum of* $A[i]$ *is as follows:*

$$\mathrm{Spec}(A[i]) = \{\langle 0\rangle\} \sqcup \{\langle C_{ii} - \alpha\rangle \mid \alpha \in \Bbbk\} \sqcup \{\langle x_i, \mathfrak{p}\rangle \mid \mathfrak{p} \in \mathrm{Spec}(U_q(\mathfrak{sl}_2))\}.$$

(3) ([3, Corollary 3.8]) *If I is a non-zero ideal of the algebra $A[i]$ then $I \cap \Bbbk[C_{ii}] \neq 0$.*

(4) ([3, Corollary 3.9]) *For $\alpha \in \Bbbk$, the ideal $\langle C_{ii} - \alpha\rangle$ of $A[i]$ is maximal if and only if $\alpha \in \Bbbk^*$.*

(5) ([3, Theorem 3.11]) *The primitive spectrum of* $A[i]$ *is as follows:*

$$\mathrm{Prim}(A[i]) = \{\langle C_{ii} - \alpha\rangle \mid \alpha \in \Bbbk\} \sqcup \{\langle x_i, \mathfrak{p}\rangle \mid \mathfrak{p} \in \mathrm{Prim}(U_q(\mathfrak{sl}_2))\}.$$

(6) ([3, Corollary 3.12]) *For any $\alpha \in \Bbbk$, the ideal $\langle C_{ii} - \alpha\rangle$ of $A[i]$ is completely prime.*

Lemma 4.2. *Let $i = 1, 2$.*

(1) *In the algebra A, it holds that $\langle x_i\rangle = Ax_i + Ay_i = \langle y_i\rangle$.*

(2) *For any positive integer n, $\langle x_i\rangle^n = \langle x_i^n\rangle = \langle y_i^n\rangle = \langle y_i\rangle^n$.*

(3) *For any positive integer n, $\langle \phi_i\rangle^n = \langle \phi_i^n\rangle = \langle x_i^n\rangle = \langle x_i\rangle^n$.*

Proof. (1) From (3), it is obvious that $\langle x_i\rangle = \langle x_i, y_i\rangle = \langle y_i\rangle$. Using the defining relations and the PBW basis of A, one verifies that $x_i A \subseteq Ax_i + Ay_i$. Thus $\langle x_i\rangle = Ax_iA \subseteq Ax_i + Ay_i \subseteq \langle x_i\rangle$, and the result follows.

(2) Note that for any positive integer n, the following identities hold in the algebra A:

$$Fx_i^n = q^{-n}x_i^nF + (1-q^{-2n})/(1-q^{-2})y_ix_i^{n-1}, \tag{22}$$
$$Ey_i^n = y_i^nE + (q^{-n} - q^n)/(1-q^2)Ky_i^{n-1}x_i. \tag{23}$$

By (22) and (23), one obtains $y_ix_i^{n-1} \in \langle x_i^n\rangle$ and $x_iy_i^{n-1} \in \langle y_i^n\rangle$, respectively. Applying (22) and (23) repeatedly yields that $y_i^n \in \langle x_i^n\rangle$ and $x_i^n \in \langle y_i^n\rangle$, and therefore $\langle x_i^n\rangle = \langle y_i^n\rangle$. Let us prove the identity $\langle x_i\rangle^n = \langle x_i^n\rangle$ by induction on n. The case $n = 1$ is obvious. Suppose it holds for all positive integers $\le n-1$. Then

$$\langle x_i\rangle^n = \langle x_i\rangle\langle x_i^{n-1}\rangle = (Ax_i + Ay_i)x_i^{n-1}A = \langle x_i^n\rangle + \langle y_ix_i^{n-1}\rangle = \langle x_i^n\rangle.$$

The proof is completed by statement (1).

(3) It can be verified by induction that, for any positive integer n,

$$E\phi_i^n = \phi_i^nE + q(1-q^{-2n})/(1-q^{-2})K^{-1}x_i\phi_i^{n-1}. \tag{24}$$

It is clear that $\langle \phi_i\rangle \subseteq \langle x_i\rangle$. By (24) with $n = 1$, one gets $\langle \phi_i\rangle = \langle x_i\rangle$. Then by statement (2), we have $\langle \phi_i^n\rangle \subseteq \langle \phi_i\rangle^n = \langle x_i\rangle^n = \langle x_i^n\rangle$. To complete the proof, it suffices to show that $x_i^n \in \langle \phi_i^n\rangle$. This follows from a repeated application of equality (24). ∎

Recall that we denote by AS^{-1} the localization of A at the Ore set S generated by the elements x_1, x_2, y_2, and ϕ_1. For a subset $\mathscr{G}$ of AS^{-1}, we indicate by $\langle \mathscr{G} \rangle_S$ the two-sided ideal of AS^{-1} generated by $\mathscr{G}$. In the following theorem, we determine the prime ideals of the algebra A by considering various quotients and localizations.

Theorem 4.3. *The prime spectrum of A is a disjoint union of the following four sets:*

- $T_1 = \{\langle x_1 \rangle\} \sqcup \{\langle x_1, C_{22} - \beta \rangle \mid \beta \in \Bbbk\} \sqcup \{\langle x_1, x_2, \mathfrak{a} \rangle \mid \mathfrak{a} \in \mathrm{Spec}(U_q(\mathfrak{sl}_2))\}$,
- $T_2 = \{\langle x_2 \rangle\} \sqcup \{\langle x_2, C_{11} - \alpha \rangle \mid \alpha \in \Bbbk\}$,
- $T_3 = \{A \cap \langle C_{21}, \mathfrak{a} \rangle_S \mid \mathfrak{a} \in \mathrm{Spec}(\Bbbk[\mathtt{D}, Z^{\pm 1}])\}$ *where* $Z := K\phi_1 y_2^{-1}$,
- $T_4 = \{\langle 0 \rangle\} \sqcup \{\langle \mathtt{D} - \chi \rangle \mid \chi \in \Bbbk\}$.

Proof. Note that for any prime ideal $\mathfrak{p}$ of A, exactly one of the following four cases holds:

- $x_1 \in \mathfrak{p}$,
- $x_1 \notin \mathfrak{p}$, $x_2 \in \mathfrak{p}$,
- $x_1 \notin \mathfrak{p}$, $x_2 \notin \mathfrak{p}$, $C_{21} \in \mathfrak{p}$,
- $x_1 \notin \mathfrak{p}$, $x_2 \notin \mathfrak{p}$, $C_{21} \notin \mathfrak{p}$.

We will describe the sets T_i that correspond to the above four cases, respectively.

Case 1: Description of the set $T_1 = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid x_1 \in \mathfrak{p}\}$. It is clear that the prime ideals containing x_1 are in natural bijection with the prime ideals of the factor algebra $A/\langle x_1 \rangle \simeq A[2]$. From Proposition 4.1 (2), it follows that $T_1 = \{\langle x_1 \rangle\} \sqcup \{\langle x_1, C_{22} - \beta \rangle \mid \beta \in \Bbbk\} \sqcup \{\langle x_1, x_2, \mathfrak{a} \rangle \mid \mathfrak{a} \in \mathrm{Spec}(U_q(\mathfrak{sl}_2))\}$.

Case 2: Description of the set $T_2 = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid x_1 \notin \mathfrak{p},\ x_2 \in \mathfrak{p}\}$. Let A_{x_1} be the localization of A at the powers of the element x_1. By Lemma 4.2 (2), for any positive integer n, one has $\langle x_1 \rangle^n = \langle x_1^n \rangle$. Thus the prime ideals of A that contain x_2 but do not contain x_1 are in natural bijection with the prime ideals of the algebra $A_{x_1}/\langle x_2 \rangle$. From (21), it follows that the algebra $A_{x_1}/\langle x_2 \rangle$ is a Noetherian domain that is isomorphic to the algebra $\Bbbk_q[y_1, x_1^{\pm 1}] \# U_q(\mathfrak{sl}_2)$. Thus the prime ideals of $A_{x_1}/\langle x_2 \rangle$ are in bijection with the prime ideals of $A[1]$ that do not contain the element x_1, and by Proposition 4.1 (2), these prime ideals of $A[1]$ in turn are in bijection with the prime ideals of the polynomial algebra $\Bbbk[C_{11}]$. Consequently, $T_2 = \{\langle x_2 \rangle\} \sqcup \{\langle x_2, C_{11} - \alpha \rangle \mid \alpha \in \Bbbk\}$.

Case 3: Description of the set $T_3 = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid x_1 \notin \mathfrak{p},\ x_2 \notin \mathfrak{p},\ C_{21} \in \mathfrak{p}\}$. Let $\mathfrak{p}$ be a prime ideal of A that does not contain the elements x_1 and x_2. Then by Lemma 4.2, $y_2^n \notin \mathfrak{p}$ and $\phi_1^n \notin \mathfrak{p}$ for any positive integer n. It follows that the prime ideals of A that contain C_{21} but do not contain x_1, x_2 are in natural bijection with the prime ideals of the algebra $AS^{-1}/\langle C_{21} \rangle$ where AS^{-1} is the localization of A at the Ore set S generated by the elements x_1, x_2, y_2, ϕ_1. It can be easily deduced from (19) that the factor algebra $AS^{-1}/\langle C_{21} \rangle$ is isomorphic to the tensor product of the polynomial algebra $\Bbbk[\mathtt{D}, Z^{\pm 1}]$ with a quantum torus. More precisely,

$$AS^{-1}/\langle C_{21} \rangle \simeq \Bbbk[\mathtt{D}, Z^{\pm 1}] \otimes \Bbbk_{q^N}[x_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \phi_1^{\pm 1}] \quad \text{where } Z = K\phi_1 y_2^{-1} \tag{25}$$

and the matrix N is the same as the one in (17). By [8, Corollary 1.5], contraction and extension provide mutually inverse isomorphisms between the ideals of a quantum torus and its centre. A simple computation shows that the centre of the quantum torus $\Bbbk_{q^N}[x_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \phi_1^{\pm 1}]$ is trivial, and therefore, it is a simple algebra. Now from the isomorphism given in (25) we see that the prime ideals of $AS^{-1}/\langle C_{21}\rangle$ are in natural bijection with the prime ideals of the algebra $\Bbbk[\mathtt{D}, Z^{\pm 1}]$. It follows that $T_3 = \{A \cap \langle C_{21}, \mathfrak{a}\rangle_S \mid \mathfrak{a} \in \mathrm{Spec}(\Bbbk[\mathtt{D}, Z^{\pm 1}])\}$.

Case 4: Description of the set $T_4 = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid x_1 \notin \mathfrak{p},\ x_2 \notin \mathfrak{p},\ C_{21} \notin \mathfrak{p}\}$. By Lemma 4.2, if $\mathfrak{p}$ is a prime ideal of A that does not contain the elements x_1 and x_2, then $y_2^n \notin \mathfrak{p}$ and $\phi_1^n \notin \mathfrak{p}$ for any positive integer n. Thus there is a natural bijection between the prime ideals of A that do not contain x_1, x_2, C_{21} and the prime ideals of the algebra $A\mathcal{S}^{-1}$ (the localization of A at the Ore set $\mathcal{S}$ generated by the elements x_1, x_2, y_2, ϕ_1 and C_{21}). It follows from (19) that $A\mathcal{S}^{-1}$ is a tensor product of the polynomial algebra $\Bbbk[\mathtt{D}]$ with a quantum torus,

$$A\mathcal{S}^{-1} = \Bbbk[\mathtt{D}] \otimes \Lambda \quad \text{where } \Lambda := \Bbbk_Q[x_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \phi_1^{\pm 1}, K^{\pm 1}, C_{21}^{\pm 1}],$$

where the matrix Q is the same as the one in (19). It is easily shown that the centre of the quantum torus Λ is trivial, and then by [8, Corollary 1.5], Λ is a simple algebra. As a consequence, the prime ideals of $A\mathcal{S}^{-1}$ are in bijection with the prime ideals of the polynomial algebra $\Bbbk[\mathtt{D}]$. More precisely, $T_4 = \{A \cap \langle \mathfrak{a}\rangle_{\mathcal{S}} \mid \mathfrak{a} \in \mathrm{Spec}(\Bbbk[\mathtt{D}])\}$. Note that for any prime ideal $\mathfrak{a}$ of $\Bbbk[\mathtt{D}]$, the factor algebra $A/\langle \mathfrak{a}\rangle$ is a domain. From this fact, it is easily seen that $A \cap \langle \mathfrak{a}\rangle_{\mathcal{S}} = \langle \mathfrak{a}\rangle$. Since $\Bbbk$ is algebraically closed, we have $T_4 = \{\langle 0\rangle\} \sqcup \{\langle \mathtt{D} - \chi\rangle \mid \chi \in \Bbbk\}$, as required. ∎

The next proposition shows that the ideals $\langle C_{21}\rangle$ and $\langle \mathtt{D} - \chi, C_{21}\rangle$ (where $\chi \in \Bbbk^*$) of the algebra A are completely prime, whereas the ideal $\langle \mathtt{D}, C_{21}\rangle$ is not completely prime.

Proposition 4.4. (1) *The algebra* A/AC_{21} *is a domain.*

(2) *The algebra* $A/\langle \mathtt{D} - \chi, C_{21}\rangle$ *is a domain if and only if* $\chi \in \Bbbk^*$.

Proof. (1) Recall that A is an Ore extension over the subalgebra $\mathcal{A}$, that is, $A = \mathcal{A}[E; \sigma, \delta]$, and the element C_{21} is normal in A. Note that the element C_{21} can be written as $C_{21} = (1 - q^{-2})y_2\phi_1 \cdot E + e$ where $e := q^{-1}K\phi_1 x_2 - q^{-2}K^{-1}x_1 y_2 \in \mathcal{A}$. By [9, Proposition 1], to prove that A/AC_{21} is a domain, it suffices to show that the element e is regular in the factor algebra $\mathcal{A}/\langle y_2\phi_1\rangle$. Recall that a regular element of a ring is an element that is neither a left nor a right zero divisor. Assume that $ea \in \langle y_2\phi_1\rangle$ for some $a \in \mathcal{A}$, then by Lemma 3.4, we have $ea \in \langle y_2\rangle \cap \langle \phi_1\rangle$. It is easy to see that $e \notin \langle y_2\rangle$ and $e \notin \langle \phi_1\rangle$. Therefore, $a \in \langle y_2\rangle \cap \langle \phi_1\rangle = \langle y_2\phi_1\rangle$, since the ideals $\langle y_2\rangle$ and $\langle \phi_1\rangle$ of the algebra $\mathcal{A}$ are completely prime (see Lemma 3.4). This proves that e is right regular in $\mathcal{A}/\langle y_2\phi_1\rangle$. Similarly one can prove that e is left regular in $\mathcal{A}/\langle y_2\phi_1\rangle$.

(2) Note that $A/\langle \mathtt{D} - \chi, C_{21}\rangle \simeq A(\chi)/A(\chi)C_{21}$, and the algebra $A(\chi)$ is an Ore extension over the subalgebra $\mathcal{A}(\chi) = \mathcal{A}/\langle \mathtt{D} - \chi\rangle$. When $\chi \in \Bbbk^*$, the proof that $A(\chi)/A(\chi)C_{21}$ is a domain follows the same lines as the proof of statement (1). In the case where $\chi = 0$, the assertion that $A/\langle \mathtt{D}, C_{21}\rangle$ is not a domain can be seen from Lemma 3.2. Indeed, according to Lemma 3.2, in the algebra A, we have $C_{11}y_2 \equiv 0 \mod \langle \mathtt{D}, C_{21}\rangle$, and $C_{11}x_2 \equiv 0$

mod $\langle \mathtt{D}, C_{21}\rangle$. In particular, C_{11} is not regular in $A/\langle \mathtt{D}, C_{21}\rangle$, thus the factor algebra $A/\langle \mathtt{D}, C_{21}\rangle$ is not a domain. ∎

Corollary 4.5. *The height one prime ideals of A are as follows:*

$$\{\langle \mathtt{D}-\chi\rangle \mid \chi \in \Bbbk\} \cup \{\langle C_{21}\rangle\}.$$

Proof. By the principal ideal theorem, see [13, Theorem 4.1.11], the prime ideals $\langle \mathtt{D}-\chi\rangle$ and $\langle C_{21}\rangle$ have height one. From Theorem 4.3, we see that all prime ideals in $T_1 \cup T_2$ contain $\langle \mathtt{D}\rangle$, and all prime ideals in T_3 contain $\langle C_{21}\rangle$. We conclude that the height one primes of A must be the ones given in the statement. ∎

The centre of the algebra A/AC_{21}. The next proposition determines the centre of the factor algebra A/AC_{21}. It turns out that $Z(A/AC_{21})$ is isomorphic to the coordinate ring of the Klein singularity of type A_1.

Proposition 4.6. *The centre of A/AC_{21} is given by*

$$Z(A/AC_{21}) = \Bbbk[C_{11}, C_{22}, \mathtt{D}]/\langle C_{11}C_{22} - q^{-1}\mathtt{D}^2\rangle.$$

Proof. Recall from Proposition 4.4 (1) that $\bar{A} := A/AC_{21}$ is a domain. Let $\bar{A}S^{-1}$ be the localization of $\bar{A}$ at the Ore set S generated by the elements x_1, x_2, y_2 and ϕ_1. Then by (25), $\bar{A}S^{-1}$ can be presented as a tensor product of algebras in the following form:

$$\bar{A}S^{-1} \simeq AS^{-1}/\langle C_{21}\rangle \simeq \Bbbk[\mathtt{D}, Z^{\pm 1}] \otimes \mathbb{T}, \quad \text{where } \mathbb{T} := \Bbbk_{q^N}[x_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \phi_1^{\pm 1}] \tag{26}$$

and the element $Z = K\phi_1 y_2^{-1}$. Moreover, the quantum torus $\mathbb{T}$ is a central simple algebra. It follows that $Z(\bar{A}S^{-1}) = \Bbbk[\mathtt{D}, Z^{\pm 1}]$. As a result, $Z(\bar{A}) = \bar{A} \cap \Bbbk[\mathtt{D}, Z^{\pm 1}]$. We proceed to show that the elements C_{11} and C_{22} belong to $Z(\bar{A})$. First, we note that the element C_{21} can be written as $C_{21} = q^{-1}\phi_1\psi_2 - q^{-2}K^{-1}x_1y_2$. Therefore, in the algebra $\bar{A}S^{-1}$, one has $K\phi_1 y_2^{-1} \cdot q\psi_2 x_1^{-1} = 1$. It means that $Z^{-1} = q\psi_2 x_1^{-1}$. Recall from Lemma 3.2 that, in the algebra A, we have

$$C_{11}y_2 = q^2 C_{21}y_1 - q^{-1}\mathtt{D}K\phi_1 \quad \text{and} \quad C_{22}x_1 = q^{-1}C_{21}x_2 - q\mathtt{D}\psi_2.$$

These two equalities imply, respectively, that in the algebra $\bar{A}S^{-1}$, one has

$$C_{11} = -q^{-1}\mathtt{D}K\phi_1 y_2^{-1} = -q^{-1}\mathtt{D}Z \quad \text{and} \quad C_{22} = -q\mathtt{D}\psi_2 x_1^{-1} = -\mathtt{D}Z^{-1}. \tag{27}$$

It follows from (27) that C_{11} and C_{22} are central in $\bar{A}$, and the relation $C_{11}C_{22} = q^{-1}\mathtt{D}^2$ holds.

Let us now show that the centre $Z(\bar{A})$ is generated by the elements C_{11}, C_{22} and $\mathtt{D}$. First of all, we claim that the monomials $\mathtt{D}^i Z^j$ ($i \in \mathbb{N}$, and $j \in \mathbb{Z}$) belong to $\bar{A}$ if and only if $i \geq |j|$. Indeed, if $i \geq j \geq 0$, then by (27), $\mathtt{D}^i Z^j = \mathtt{D}^{i-j}(\mathtt{D}Z)^j = \mathtt{D}^{i-j}(-qC_{11})^j \in \bar{A}$; if $j < 0$ and $i \geq |j|$, then by (27), $\mathtt{D}^i Z^j = \mathtt{D}^{i+j}(\mathtt{D}Z^{-1})^{-j} = \mathtt{D}^{i+j}(-C_{22})^{-j} \in \bar{A}$. For the converse, suppose $i < |j|$ and the element $u = \mathtt{D}^i Z^j \in \bar{A}$, we seek a contradiction. Then we have either $i < j$ or $i + j < 0$. If $i < j$ then the element u can be written as $u = (\mathtt{D}Z)^i Z^{j-i} = (-qC_{11})^i (K\phi_1 y_2^{-1})^{j-i} \in \overline{A}$. In particular, the element $uy_2^{j-i} = (-q)^i C_{11}^i (K\phi_1)^{j-i}$ belongs to the ideal $\langle y_2 \rangle$ of $\overline{A}$. However, the ideal $\langle y_2 \rangle$ of $\overline{A}$ is completely prime (since $\overline{A}/\langle y_2 \rangle \simeq A[1]$ is a domain), and the images of the elements C_{11} and $K\phi_1$ in $\overline{A}/\langle y_2 \rangle$ are non-zero, a contradiction. Similarly, if $i + j < 0$ then the element u can be written as $u = (\mathtt{D}Z^{-1})^i Z^{i+j} = (-C_{22})^i (q\psi_2 x_1^{-1})^{-(i+j)}$. Therefore, one has $ux_1^{-(i+j)} = q^{-(i+j)}(-C_{22})^i \psi_2^{-(i+j)}$ belonging to the ideal $\langle x_1 \rangle$ of $\overline{A}$. However, the ideal $\langle x_1 \rangle$ of the algebra $\overline{A}$ is completely prime (since $\overline{A}/\langle x_1 \rangle \simeq A[2]$ is a domain), and the images of the elements C_{22} and ψ_2 in $\overline{A}/\langle x_1 \rangle$ are non-zero, a contradiction. This confirms our claim. Furthermore, it is not difficult to show that any non-zero linear combination of the monomials $\mathtt{D}^i Z^j$ (where $i < |j|$) does not belong to the algebra $\overline{A}$. As a consequence, we obtain

$$
\begin{aligned}
Z(\overline{A}) = \overline{A} \cap \Bbbk[\mathtt{D}, Z^{\pm 1}] &= \bigoplus_{i \geq j \geq 0} \Bbbk \mathtt{D}^i Z^j \oplus \bigoplus_{i \geq j > 0} \Bbbk \mathtt{D}^i (Z^{-1})^j \\
&= \bigoplus_{i \geq j \geq 0} \Bbbk C_{11}^j \mathtt{D}^{i-j} \oplus \bigoplus_{i \geq j > 0} \Bbbk C_{22}^j \mathtt{D}^{i-j} \\
&= \bigoplus_{j \geq 0} \Bbbk[\mathtt{D}] C_{11}^j \oplus \bigoplus_{j > 0} \Bbbk[\mathtt{D}] C_{22}^j.
\end{aligned}
$$

In particular, the above equality provides a basis for $Z(\overline{A})$, and from which we see that the algebra $Z(\overline{A})$ is generated by the elements C_{11}, C_{22} and $\mathtt{D}$. To sum up, we have shown that there is a natural epimorphism of algebras $\pi : \Bbbk[C_{11}, C_{22}, \mathtt{D}]/\langle C_{11}C_{22} - q^{-1}\mathtt{D}^2 \rangle \twoheadrightarrow Z(\overline{A})$. Comparing the bases of the two algebras yields that π must be an isomorphism. This completes the proof. ∎

Corollary 4.7. *If* $\chi \in \Bbbk^*$, *then* $Z(A(\chi)/\langle C_{21} \rangle) = \Bbbk[C_{11}, C_{22}]/\langle C_{11}C_{22} - q^{-1}\chi^2 \rangle$.

Proof. From Proposition 4.4 (2), it follows that $R := A(\chi)/\langle C_{21} \rangle$ is a domain. Let RS^{-1} be the localization of R at the Ore set S generated by the elements x_1, x_2, y_2 and ϕ_1. We conclude from (26) that $RS^{-1} = \Bbbk[Z, Z^{-1}] \otimes \mathbb{T}$ where $Z = K\phi_1 y_2^{-1}$ and the quantum torus $\mathbb{T}$ is a central simple algebra. Thus $Z(RS^{-1}) = \Bbbk[Z, Z^{-1}]$. Therefore, $Z(R) = R \cap \Bbbk[Z, Z^{-1}]$. By (27), the elements Z and Z^{-1} lie in the algebra R, since one has $C_{11} = -q^{-1}\chi Z$ and $C_{22} = -\chi Z^{-1}$. The result follows. ∎

5. The primitive spectrum of A

In this section, we give a classification of the primitive and maximal ideals of the algebra A. For each primitive ideal, a set of explicit generators is given.

Since $\Bbbk$ is algebraically closed, by Quillen's lemma, any central element of A acts as a scalar on a simple A-module. Thus any primitive ideal of A contains a central element $\mathtt{D} - \chi$ for some $\chi \in \Bbbk$. We shall distinguish between the cases where $\chi = 0$ and $\chi \in \Bbbk^*$. In the following, we identify the primitive spectrum of $A(\chi)$ with the set of primitive ideals of A that contain $\mathtt{D}-\chi$. Then $\mathrm{Prim}(A)$ decomposes into a disjoint union of the primitive spectra of the central factors $A(\chi)$, that is,

$$\mathrm{Prim}(A) = \mathrm{Prim}(A(0)) \sqcup \bigsqcup_{\chi\in\Bbbk^*} \mathrm{Prim}(A(\chi)). \tag{28}$$

Recall that all primitive ideals are prime, and the prime spectrum of the algebra A has been obtained in Theorem 4.3. The next step is to identify the primitive ideals among the prime ideals. In representation theory, primitive ideals are of particular importance compared to general prime ideals.

The prime ideals $\mathfrak{p}_\mu$. For any $\mu \in \Bbbk$, let $\mathfrak{p}_\mu := \langle \phi_1 - \mu K^{-1} y_2\rangle$ be the ideal of A generated by the element $\phi_1 - \mu K^{-1} y_2$. We aim to show that the ideal $\mathfrak{p}_\mu$ is completely prime. To simplify notation, for $\mu \in \Bbbk$, we denote

$$\begin{aligned} X_\mu &:= \mu\psi_2 - q^{-1}x_1 = \mu[(1-q^{-2})Ey_2 + q^{-2}x_2K] - q^{-1}x_1,\\ Y_\mu &:= \phi_1 - \mu K^{-1}y_2 = (1-q^2)Fx_1 + q^2y_1 - \mu K^{-1}y_2. \end{aligned} \tag{29}$$

In particular, $\mathfrak{p}_\mu = \langle Y_\mu\rangle$. The following lemma gives some equivalent descriptions of the ideal $\mathfrak{p}_\mu$. Additionally, it demonstrates that some important elements belong to $\mathfrak{p}_\mu$.

Lemma 5.1. *Let $\mu \in \Bbbk$.*

(1) *It holds that $\mathfrak{p}_\mu = \langle Y_\mu\rangle = \langle X_\mu, Y_\mu\rangle = \langle X_\mu\rangle$.*

(2) *The elements C_{21}, $C_{11} + q^{-1}\mu\mathtt{D}$, and $\mu C_{22} + \mathtt{D}$ belong to $\mathfrak{p}_\mu$.*

Proof. (1) The statement follows from the following identities:

$$EY_\mu = Y_\mu E - q^2K^{-1}X_\mu \quad \text{and} \quad FX_\mu = qX_\mu F - q^{-1}Y_\mu.$$

(2) If $\mu = 0$ then, by Lemma 4.2, one has $\mathfrak{p}_0 = \langle\phi_1\rangle = \langle x_1, y_1\rangle$. In this case, it is clear that the elements $\mathtt{D}$, C_{21} and C_{11} belong to $\mathfrak{p}_0$. Note that in the algebra A, the following identity holds:

$$X_\mu Y_\mu - qY_\mu X_\mu = (1-q^2)\mu C_{21}.$$

Thus for $\mu \in \Bbbk^*$, we also have $C_{21} \in \mathfrak{p}_\mu$. For any $\mu \in \Bbbk$, one verifies that

$$C_{11} + q^{-1}\mu\mathtt{D} = \psi_1 Y_\mu + q^2K^{-1}X_\mu y_1 \quad \text{and} \quad \mu C_{22} + \mathtt{D} = X_\mu\phi_2 + Y_\mu x_2. \tag{30}$$

From the above identities, we see that the elements $C_{11} + q^{-1}\mu\mathtt{D}$ and $\mu C_{22} + \mathtt{D}$ belong to $\mathfrak{p}_\mu$. ■

The next proposition shows that $\mathfrak{p}_\mu$ is a completely prime ideal of A, and the factor algebra $A/\mathfrak{p}_\mu$ is isomorphic to a smash product of the quantum plane with the Hopf algebra $U_q(\mathfrak{sl}_2)$.

Proposition 5.2. *For any $\mu \in \Bbbk$, the ideal $\mathfrak{p}_\mu$ of A is completely prime. Furthermore, we have*

$$A/\mathfrak{p}_\mu \simeq \Bbbk_q[y_2, x_2] \# U_q(\mathfrak{sl}_2).$$

Proof. If $\mu = 0$ then $\mathfrak{p}_0 = \langle x_1, y_1 \rangle$. It is obvious that $A/\mathfrak{p}_0$ is isomorphic to the algebra $A[2]$, see (21). Now assume $\mu \in \Bbbk^*$. From (29), one deduces that

$$x_1 \equiv q\mu\psi_2 \mod \mathfrak{p}_\mu \quad \text{and} \quad y_1 \equiv (q - q^{-1})\mu F\psi_2 + q^{-2}\mu K^{-1} y_2 \mod \mathfrak{p}_\mu.$$

Hence $A/\mathfrak{p}_\mu$ is generated by the images of the elements $y_2, x_2, K^{\pm 1}, E$ and F in $A/\mathfrak{p}_\mu$. As usual, for simplicity we denote by the same letter its image in the factor algebra $A/\mathfrak{p}_\mu$. Then there is a natural epimorphism of algebras $\pi : A[2] \twoheadrightarrow A/\mathfrak{p}_\mu$. We have to show that π is an injection. By Theorem 4.3, $P_\mu := A \cap \langle C_{21}, Z - \mu \rangle_S \in T_3$ is a prime ideal of A. Note that $P_\mu = A \cap \langle C_{21}, Y_\mu \rangle_S$, and hence $\mathfrak{p}_\mu \subseteq P_\mu$. So the factor algebra $A/\mathfrak{p}_\mu$ cannot collapse completely. Suppose π is not an injection, we seek a contradiction. Then $\ker \pi$ is a proper ideal of $A[2]$. By Proposition 4.1 (3), $\ker \pi$ has a nontrivial intersection with the centre of $A[2]$, that is, $\ker \pi \cap \Bbbk[C_{22}] \neq 0$. Thus there is a non-zero polynomial $f(t) \in \Bbbk[t]$ such that $f(C_{22}) \equiv 0 \mod \mathfrak{p}_\mu$. By the second identity of (30), one has $C_{22} \equiv -\mu^{-1}\mathtt{D} \mod \mathfrak{p}_\mu$. It follows that $f(-\mu^{-1}\mathtt{D}) \in \mathfrak{p}_\mu \subseteq P_\mu$. Since P_μ is a prime ideal of A and the field $\Bbbk$ is algebraically closed, there exists $\chi \in \Bbbk$ such that $\mathtt{D} - \chi \in P_\mu$. This leads to a contradiction, since the prime ideals in T_3 are in bijection with the prime spectrum of $\Bbbk[\mathtt{D}, Z^{\pm 1}]$. As a result, $A/\mathfrak{p}_\mu$ is isomorphic to the algebra $A[2]$, and so $\mathfrak{p}_\mu$ is completely prime. ∎

For $\chi \in \Bbbk$ and $\mu \in \Bbbk^*$, define

$$\mathfrak{m}_\mu^\chi := A(\mathtt{D} - \chi) + \mathfrak{p}_\mu = \langle \mathtt{D} - \chi, \phi_1 - \mu K^{-1} y_2 \rangle.$$

Corollary 5.3. *For any $\chi \in \Bbbk$ and $\mu \in \Bbbk^*$, the ideal $\mathfrak{m}_\mu^\chi$ of A is completely prime. Moreover, the ideal $\mathfrak{m}_\mu^\chi$ is maximal if and only if $\chi \in \Bbbk^*$.*

Proof. By the second identity of (30), $\mathtt{D} \equiv -\mu C_{22} \mod \mathfrak{p}_\mu$. From Proposition 5.2, it follows that

$$A/\mathfrak{m}_\mu^\chi \simeq A/\mathfrak{p}_\mu/\mathfrak{m}_\mu^\chi/\mathfrak{p}_\mu \simeq \frac{\Bbbk_q[y_2, x_2] \# U_q(\mathfrak{sl}_2)}{\langle \mu C_{22} + \chi \rangle}. \tag{31}$$

We conclude from Proposition 4.1 (6) that the factor algebra $A/\mathfrak{m}_\mu^\chi$ is a domain. The fact that $\mathfrak{m}_\mu^\chi$ is maximal if and only if $\chi \in \Bbbk^*$ follows from Proposition 4.1 (4). ∎

Remark 5.4. For $\chi \in \Bbbk^*$ and $\mu \in \Bbbk^*$, the ideal $\mathfrak{m}_\mu^\chi$ has the following equivalent descriptions:

$$\mathfrak{m}_\mu^\chi = \langle \mathtt{D} - \chi, C_{11} + q^{-1}\mu\mathtt{D} \rangle \quad \text{and} \quad \mathfrak{m}_\mu^\chi = \langle \mathtt{D} - \chi, \mu C_{22} + \mathtt{D} \rangle. \tag{32}$$

In fact, utilizing (14) and (15), we can verify that the following identities hold in the algebra A:

$$(C_{11} + q^{-1}\mu\mathtt{D})y_2 - q^2 y_2(C_{11} + q^{-1}\mu\mathtt{D}) = (q - q^{-1})\mathtt{D}KY_\mu,$$
$$(\mu C_{22} + \mathtt{D})x_1 - q^{-2}x_1(\mu C_{22} + \mathtt{D}) = (q^{-1} - q)\mathtt{D}X_\mu.$$

Then (32) follows from the above identities and Lemma 5.1.

Prime and primitive spectra of $A(\chi)$ where $\chi \in \mathbb{k}^*$. Recall that a prime ideal P of an algebra R is rational provided the field $Z(\mathrm{Frac}(R/P))$ is algebraic over $\mathbb{k}$, and P is locally closed in $\mathrm{Spec}(R)$ if and only if the intersection of all prime ideals properly containing P is an ideal properly containing P. If R is a Noetherian algebra satisfying the Nullstellensatz over $\mathbb{k}$, then, by [6, Lemma II.7.15], the following implications for prime ideals hold:

$$\text{locally closed} \;\Rightarrow\; \text{primitive} \;\Rightarrow\; \text{rational}.$$

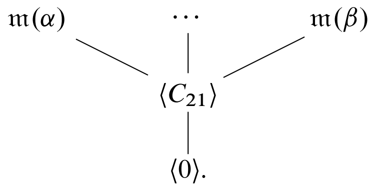
In the following theorem, we give explicit descriptions of the prime, primitive and maximal spectra of the algebra $A(\chi)$ where $\chi \in \mathbb{k}^*$. In particular, by setting $\chi = 1$, we obtain the prime and primitive spectra of the Heisenberg double $D_q(\mathrm{SL}_2)$.

Theorem 5.5. *Fix $\chi \in \mathbb{k}^*$.*

(1) *The prime spectrum of $A(\chi)$ is as follows:*

$$\mathrm{Spec}(A(\chi)) = \{\langle 0\rangle,\ \langle C_{21}\rangle,\ \mathfrak{m}(\mu) \mid \mu \in \mathbb{k}^*\} \quad \textit{where } \mathfrak{m}(\mu) := \langle \phi_1 - \mu K^{-1} y_2\rangle.$$

The inclusions of prime ideals of $A(\chi)$ are depicted in the following (where $\alpha, \dots, \beta \in \mathbb{k}^*$):



(2) $\mathrm{Max}(A(\chi)) = \{\mathfrak{m}(\mu) \mid \mu \in \mathbb{k}^*\}$.

(3) $\mathrm{Prim}(A(\chi)) = \{\langle 0\rangle\} \cup \{\mathfrak{m}(\mu) \mid \mu \in \mathbb{k}^*\}$.

Proof. (1) Since $A(\chi) = A/A(\mathtt{D} - \chi)$, the prime ideals of $A(\chi)$ are in bijection with the prime ideals of A containing the element $\mathtt{D} - \chi$. Recall that we denote by $A(\chi)S^{-1}$ the localization of $A(\chi)$ at the Ore set S generated by the elements x_1, x_2, y_2 and ϕ_1. From Theorem 4.3, we see that the prime ideals of A containing $\mathtt{D} - \chi$ (where $\chi \in \mathbb{k}^*$) lie in the sets T_3 and T_4. We conclude that $\mathrm{Spec}(A(\chi)) = \{\langle 0\rangle\} \cup \{A(\chi) \cap \langle C_{21}, \mathfrak{b}\rangle_S \mid \mathfrak{b} \in \mathrm{Spec}(\mathbb{k}[Z, Z^{-1}])\}$ where $Z = K\phi_1 y_2^{-1}$ and $\langle C_{21}, \mathfrak{b}\rangle_S$ is the ideal of $A(\chi)S^{-1}$ generated by C_{21} and $\mathfrak{b}$. Since the field $\mathbb{k}$ is algebraically closed, $\mathrm{Spec}(\mathbb{k}[Z, Z^{-1}]) = \{\langle 0\rangle\} \cup \{\langle Z - \mu\rangle \mid \mu \in \mathbb{k}^*\}$. As a result,

$$\mathrm{Spec}(A(\chi)) = \{\langle 0\rangle\} \cup \{A(\chi) \cap \langle C_{21}\rangle_S\} \cup \{A(\chi) \cap \langle C_{21}, Z - \mu\rangle_S \mid \mu \in \mathbb{k}^*\}.$$

By Proposition 4.4 (2), the ideal $\langle C_{21}\rangle$ of the algebra $A(\chi)$ is completely prime. It is easy to see that the generators x_1, x_2, y_2, ϕ_1 of the Ore set S do not belong to the ideal $\langle C_{21}\rangle$, it follows that $A(\chi) \cap \langle C_{21}\rangle_S = \langle C_{21}\rangle$. It remains to show that $A(\chi) \cap \langle C_{21}, Z - \mu\rangle_S = \mathfrak{m}(\mu)$ for any $\mu \in \mathbb{k}^*$. Notice that $A(\chi) \cap \langle C_{21}, Z - \mu\rangle_S = A(\chi) \cap \langle C_{21}, \phi_1 - \mu K^{-1} y_2\rangle_S$. Thus the inclusion $\mathfrak{m}(\mu) \subseteq A(\chi) \cap \langle C_{21}, Z - \mu\rangle_S$ is obvious. To show that the reverse inclusion holds, it suffices to show that $\mathfrak{m}(\mu)$ is a maximal ideal of $A(\chi)$. This follows from Corollary 5.3, since $A(\chi)/\mathfrak{m}(\mu) \simeq A/\mathfrak{m}_\mu^\chi$. All prime ideals of $A(\chi)$ are shown in the diagram, the inclusions are obvious.

(2) Statement (2) follows from the inclusions of prime ideals given in statement (1).

(3) First, the ideals $\mathfrak{m}(\mu)$ ($\mu \in \Bbbk^*$) of $A(\chi)$ are maximal, and hence primitive. Next, the ideal $\langle C_{21}\rangle$ of $A(\chi)$ is not primitive, since the factor algebra $A(\chi)/\langle C_{21}\rangle$ is not a primitive ring as it has a nontrivial centre (see Corollary 4.7). It remains to show that the ideal $\langle 0\rangle$ is primitive. By [6, Proposition II.7.16], the algebra $A(\chi)$ satisfies the noncommutative Nullstellensatz over $\Bbbk$. It follows from [6, Lemma II.7.15] that any locally closed prime ideal of $A(\chi)$ is primitive. The ideal $\langle 0\rangle$ is locally closed, since the intersection of all non-zero primes is the ideal $\langle C_{21}\rangle$ which properly contains $\langle 0\rangle$, and therefore, $\langle 0\rangle$ is primitive. ∎

Corollary 5.6. *For $\chi \in \Bbbk^*$, the algebra $A(\chi)$ has no finite-dimension modules. Moreover, $A(\chi)$ cannot have a Hopf algebra structure.*

Proof. For each maximal ideal $\mathfrak{m}(\mu)$ of $A(\chi)$, the corresponding factor algebra $A(\chi)/\mathfrak{m}(\mu)$ is an infinite-dimensional simple algebra. Thus $A(\chi)$ has no finite-dimensional modules. Suppose $A(\chi)$ has a Hopf algebra structure. Then the kernel of the counit $\varepsilon : A(\chi) \to \Bbbk$ is a maximal ideal of $A(\chi)$ with codimension one. However, from Theorem 5.5 (2), we see that this is impossible. ∎

Primitive spectrum of A. Recall that we identify the primitive spectrum of $A(\chi)$ with the set of primitive ideals of A containing $\mathtt{D} - \chi$. For $\chi \in \Bbbk^*$, the primitive spectrum of $A(\chi)$ has been obtained in Theorem 5.5 (3). It follows that the primitive ideals of A containing $\mathtt{D} - \chi$ for some $\chi \in \Bbbk^*$ are as follows:

$$\bigsqcup_{\chi\in\Bbbk^*} \operatorname{Prim}(A(\chi)) = \{\langle \mathtt{D} - \chi\rangle \mid \chi \in \Bbbk^*\} \sqcup \{\mathfrak{m}_\mu^\chi \mid \chi \in \Bbbk^*,\ \mu \in \Bbbk^*\} \tag{33}$$

where $\mathfrak{m}_\mu^\chi = \langle \mathtt{D} - \chi, \phi_1 - \mu K^{-1} y_2\rangle$, see also (32) for some equivalent descriptions of the ideal $\mathfrak{m}_\mu^\chi$. According to (28), in order to give a classification of the primitive ideals of A, it remains to describe the primitive spectrum of $A(0)$, i.e., the set of primitive ideals of A containing the central element $\mathtt{D}$. To this end, we apply Theorem 4.3 where the prime spectrum of A is obtained. We have to select the primitive ideals containing $\mathtt{D}$ from the four sets T_i ($i = 1, 2, 3, 4$). Let $\operatorname{Prim}_{T_i}(A(0))$ be the set of primitive ideals of A in T_i containing $\mathtt{D}$. Then we have

$$\operatorname{Prim}(A(0)) = \bigsqcup_{i=1}^{4} \operatorname{Prim}_{T_i}(A(0)). \tag{34}$$

We describe the sets $\operatorname{Prim}_{T_i}(A(0))$ ($i = 1, 2, 3, 4$) in the following four lemmas.

Lemma 5.7. *The set of primitive ideals in T_1 containing $\mathtt{D}$ is as follows:*

$$\operatorname{Prim}_{T_1}(A(0)) = \{\langle x_1, C_{22} - \beta\rangle \mid \beta \in \Bbbk\} \cup \{\langle x_1, x_2, \mathfrak{a}\rangle \mid \mathfrak{a} \in \operatorname{Prim}(U_q(\mathfrak{sl}_2))\}.$$

Proof. The prime ideals in the set T_1 are exactly the primes of A that contain x_1. In particular, all prime ideal in T_1 contain the element $\mathtt{D}$. Moreover, the primitive ideals in T_1 are in bijection with the primitive ideals of the factor algebra $A/\langle x_1\rangle \simeq A[2]$. The result then follows from Proposition 4.1 (5). We note that all non-zero prime ideals of $U_q(\mathfrak{sl}_2)$ are primitive. For a complete list of prime ideals of $U_q(\mathfrak{sl}_2)$, see, for instance, [11, Theorem 4.6]. ∎

Lemma 5.8. *The set of primitive ideals in T_2 containing $\mathtt{D}$ is as follows:*

$$\mathrm{Prim}_{T_2}(A(0)) = \{\langle x_2, C_{11} - \alpha\rangle \mid \alpha \in \Bbbk\}.$$

Proof. Notice that all prime ideals in T_2 contain $\langle x_2\rangle$ and $\mathtt{D} \in \langle x_2\rangle$. Thus all prime ideals in T_2 contain the central element $\mathtt{D}$. The ideal $\langle x_2\rangle$ of A is not primitive, since the factor algebra $A/\langle x_2\rangle \simeq A[1]$ has a nontrivial centre, see Proposition 4.1 (1). Notice that

$$A/\langle x_2, C_{11} - \alpha\rangle \simeq \frac{\Bbbk_q[y_1, x_1] \,\#\, U_q(\mathfrak{sl}_2)}{\langle C_{11} - \alpha\rangle}.$$

By Proposition 4.1 (5), for any $\alpha \in \Bbbk$, the ideal $\langle C_{11} - \alpha\rangle$ of the algebra $A[1]$ is primitive. We conclude that $\langle x_2, C_{11} - \alpha\rangle$ is a primitive ideal of A. ∎

Lemma 5.9. *The set of primitive ideals in T_3 containing $\mathtt{D}$ is as follows:*

$$\mathrm{Prim}_{T_3}(A(0)) = \{\langle \mathtt{D}, \phi_1 - \mu K^{-1} y_2\rangle \mid \mu \in \Bbbk^*\}.$$

Proof. First, we note that the prime ideals in T_3 containing $\mathtt{D}$ are as follows:

$$\mathcal{T}_3 = \{A \cap \langle C_{21}, \mathtt{D}\rangle_S\} \cup \{A \cap \langle C_{21}, \mathtt{D}, Z - \mu\rangle_S \mid \mu \in \Bbbk^*\}. \tag{35}$$

Let $Q_\mu := A \cap \langle C_{21}, \mathtt{D}, Z - \mu\rangle_S$, and $\mathfrak{m}_\mu^0 = \langle \mathtt{D}, \phi_1 - \mu K^{-1} y_2\rangle = A\mathtt{D} + \mathfrak{p}_\mu$ where $\mathfrak{p}_\mu = \langle \phi_1 - \mu K^{-1} y_2\rangle$. We claim that $Q_\mu = \mathfrak{m}_\mu^0$. The inclusion $Q_\mu \supseteq \mathfrak{m}_\mu^0$ is obvious. By Lemma 5.1 (2), the elements C_{21}, C_{11} and C_{22} lie in the ideal $\mathfrak{m}_\mu^0$. Moreover, by Corollary 5.3, the ideal $\mathfrak{m}_\mu^0$ is completely prime. If $u \in Q_\mu$, then $u x_1^i x_2^j y_2^k \phi_1^\ell \in \mathfrak{m}_\mu^0$ for some $i, j, k, \ell \in \mathbb{N}$. Since the elements x_1, x_2, y_2, ϕ_1 do not belong to $\mathfrak{m}_\mu^0$, we must have $u \in \mathfrak{m}_\mu^0$. This confirms our claim. The fact that $\mathfrak{m}_\mu^0$ is a primitive ideal follows from (31) and Proposition 4.1 (5). It remains to show that $\mathfrak{p} = A \cap \langle C_{21}, \mathtt{D}\rangle_S$ is not a primitive ideal of A. Suppose $\mathfrak{p}$ is primitive, then by [6, Lemma II.7.15], it must be rational, that is, the field $Z(\mathrm{Frac}(A/\mathfrak{p}))$ is algebraic over $\Bbbk$. However, we conclude from (25) that $Z(\mathrm{Frac}(A/\mathfrak{p})) = Z(\mathrm{Frac}(AS^{-1}/\langle \mathtt{D}, C_{21}\rangle_S)) = \Bbbk(Z)$ is a field of rational functions, which is transcendental over $\Bbbk$. This leads to a contradiction. ∎

Lemma 5.10. *The set of primitive ideals in T_4 containing $\mathtt{D}$ is as follows:*

$$\mathrm{Prim}_{T_4}(A(0)) = \{\langle \mathtt{D}\rangle\}.$$

Proof. Clearly, $\langle \mathtt{D} \rangle$ is the only prime ideal in T_4 containing the element $\mathtt{D}$. We have to show that $\langle \mathtt{D} \rangle$ is a primitive ideal of A. Note that the set of prime ideals properly containing $\mathtt{D}$ is $T_1 \cup T_2 \cup \mathfrak{T}_3$ where the subset $\mathfrak{T}_3$ is given in (35). Notice that for any prime ideal $\mathfrak{p} \in T_1 \cup T_2 \cup \mathfrak{T}_3$ one has $\langle \mathtt{D}, C_{21} \rangle \subseteq \mathfrak{p}$. We conclude that $\langle \mathtt{D} \rangle$ is locally closed in $\mathrm{Spec}(A)$, since the intersection of all prime ideals properly containing $\langle \mathtt{D} \rangle$ is an ideal that properly contains $\langle \mathtt{D} \rangle$. Hence $\langle \mathtt{D} \rangle$ is primitive. ■

The following theorem gives an explicit description of the primitive spectrum of A.

Theorem 5.11. *The primitive spectrum of A is as follows:*

$$\begin{aligned}\mathrm{Prim}(A) = \{\langle \mathtt{D} - \chi \rangle \mid \chi \in \Bbbk\} \cup \{\mathfrak{m}_\mu^\chi \mid \chi \in \Bbbk,\ \mu \in \Bbbk^*\} \cup \{\langle x_2, C_{11} - \alpha \rangle \mid \alpha \in \Bbbk\} \\ \cup \{\langle x_1, C_{22} - \beta \rangle \mid \beta \in \Bbbk\} \cup \{\langle x_1, x_2, \mathfrak{a} \rangle \mid \mathfrak{a} \in \mathrm{Prim}(U_q(\mathfrak{sl}_2))\}\end{aligned}$$

where $\mathfrak{m}_\mu^\chi = \langle \mathtt{D} - \chi, \phi_1 - \mu K^{-1} y_2 \rangle$.

Proof. The result follows from (28), (33), (34) and Lemmas 5.7–5.10. ■

The following corollary describes the maximal ideals of A.

Corollary 5.12. *The maximal spectrum of A is as follows:*

$$\begin{aligned}\mathrm{Max}(A) = \{\mathfrak{m}_\mu^\chi \mid \chi \in \Bbbk^*,\ \mu \in \Bbbk^*\} \cup \{\langle x_2, C_{11} - \alpha \rangle \mid \alpha \in \Bbbk^*\} \\ \cup \{\langle x_1, C_{22} - \beta \rangle \mid \beta \in \Bbbk^*\} \cup \{\langle x_1, x_2, \mathfrak{a} \rangle \mid \mathfrak{a} \in \mathrm{Max}(U_q(\mathfrak{sl}_2))\}.\end{aligned}$$

Proof. Since all maximal ideals are primitive, the result follows from Theorem 5.11, Corollary 5.3, and Proposition 4.1 (4). ■

Remark 5.13. For primitive ideals of A that are not of the form $\langle \mathtt{D} - \chi \rangle$ ($\chi \in \Bbbk$), let us summarize the description of the corresponding primitive factors in the following.

(1) For $\chi \in \Bbbk$ and $\mu \in \Bbbk^*$, we have $A/\mathfrak{m}_\mu^\chi \simeq A[2]/\langle \mu C_{22} + \chi \rangle$, see (31).

(2) For $\alpha \in \Bbbk$, we have $A/\langle x_2, C_{11} - \alpha \rangle \simeq A[1]/\langle C_{11} - \alpha \rangle$.

(3) For $\beta \in \Bbbk$, we have $A/\langle x_1, C_{22} - \beta \rangle \simeq A[2]/\langle C_{22} - \beta \rangle$.

(4) For $\mathfrak{a} \in \mathrm{Prim}(U_q(\mathfrak{sl}_2))$, we have $A/\langle x_1, x_2, \mathfrak{a} \rangle \simeq U_q(\mathfrak{sl}_2)/\mathfrak{a}$.

The quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ can be presented as a generalized Weyl algebra, for a classification of its simple modules, see, e.g., [4,5]. A classification of simple weight modules over the algebra $A[i] = \Bbbk_q[y_i, x_i] \# U_q(\mathfrak{sl}_2)$ ($i = 1, 2$) was obtained in [3]. This gives rise to plenty of simple modules over the algebra A.

6. Classification of simple quasi-Whittaker A-modules

The aim of this section is to study a class of non-weight A-modules, termed quasi-Whittaker modules. A classification of all simple quasi-Whittaker A-modules is obtained. For each simple quasi-Whittaker module, its annihilator is determined.

Definition 6.1. Let $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ be an algebra homomorphism, and let M be an A-module. A vector $v \in M$ is called a quasi-Whittaker vector of type ς if $xv = \varsigma(x)v$ for all $x \in \mathcal{O}_q(M_2)$. The module M is called a quasi-Whittaker module of type ς if it is generated by a quasi-Whittaker vector of type ς.

The universal module $\mathbf{M}(\varsigma)$. Note that the algebra homomorphisms $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ are in bijection with the maximal ideals of $\mathcal{O}_q(M_2)$ with codimension one. From the defining relations of $\mathcal{O}_q(M_2)$ given by (2), it is easy to see that ς must be one of the following forms:

$$\varsigma \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}, \quad \alpha, \delta \in \Bbbk; \tag{36a}$$

$$\varsigma \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}, \quad \beta \in \Bbbk; \tag{36b}$$

$$\varsigma \begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}, \quad \gamma \in \Bbbk. \tag{36c}$$

Let $\Bbbk_\varsigma := \mathcal{O}_q(M_2)/\ker(\varsigma)$ be the one-dimensional $\mathcal{O}_q(M_2)$-module associated to ς. We define the universal quasi-Whittaker module of type ς as follows:

$$\mathbf{M}(\varsigma) := \mathrm{Ind}^{A}_{\mathcal{O}_q(M_2)} \Bbbk_\varsigma = A \otimes_{\mathcal{O}_q(M_2)} \Bbbk_\varsigma.$$

We use the term universal to refer to the following property: if M is a quasi-Whittaker module of type ς, then M is an epimorphic image of $\mathbf{M}(\varsigma)$. Clearly, we have

$$\mathbf{M}(\varsigma) \simeq A/\mathbf{I}_\varsigma, \quad \text{where } \mathbf{I}_\varsigma = A\big(y_1 - \varsigma(y_1), x_1 - \varsigma(x_1), y_2 - \varsigma(y_2), x_2 - \varsigma(x_2)\big).$$

Let $\bar{1} = 1 + \mathbf{I}_\varsigma$ (the coset of 1) be the canonical generator of $\mathbf{M}(\varsigma)$. From the PBW basis of A, we see that $\mathbf{M}(\varsigma)$ is a free $U_q(\mathfrak{sl}_2)$-module of rank one, that is, $\mathbf{M}(\varsigma) = U_q(\mathfrak{sl}_2)\bar{1}$, and the elements $K^i F^j E^k \bar{1}$ ($i \in \mathbb{Z}$, $j, k \in \mathbb{N}$) form a basis of $\mathbf{M}(\varsigma)$.

Lemma 6.2. *If ς is the zero homomorphism, then V is a simple quasi-Whittaker A-module of type ς if and only if V is a simple $U_q(\mathfrak{sl}_2)$-module.*

Proof. In this case, $\mathbf{M}(\varsigma) \simeq A/A(y_1, x_1, y_2, x_2)$. By Lemma 4.2 (1), the left ideal $A(y_1, x_1, y_2, x_2)$ is identical to the two-side ideal $I := \langle y_1, x_1, y_2, x_2 \rangle$. Thus $\mathbf{M}(\varsigma)$ is annihilated by the ideal I. Since V is an epimorphic image of $\mathbf{M}(\varsigma)$, V is also annihilated by I. Notice that A/I is isomorphic to $U_q(\mathfrak{sl}_2)$, thus V is a simple module over $U_q(\mathfrak{sl}_2)$. The converse is obvious. ∎

The structures of $\mathbf{M}(\varsigma)$. In the following, we always assume that ς is a non-zero algebra homomorphism. By the universal property of the module $\mathbf{M}(\varsigma)$, classifying simple quasi-Whittaker modules requires finding simple quotients of $\mathbf{M}(\varsigma)$. The next proposition studies the structure of the module $\mathbf{M}(\varsigma)$ where ς is given by (36a). In this case, $\mathbf{M}(\varsigma) \simeq A/A(y_1 - \alpha, x_1, y_2, x_2 - \delta)$ where $\alpha, \delta \in \Bbbk$.

Proposition 6.3. *Let $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ be a non-zero algebra homomorphism of the form* (36a).

(1) *If $\alpha \in \Bbbk^*$ and $\delta \in \Bbbk^*$, then $\mathbf{M}(\varsigma)$ is a simple A-module with* $\operatorname{ann}_A \mathbf{M}(\varsigma) = \langle \mathtt{D} - \alpha\delta \rangle$.

(2) *If $\alpha \in \Bbbk^*$ and $\delta = 0$, then for any $\lambda \in \Bbbk$, $\mathbf{M}(\varsigma)$ has the following chain of submodules:*

$$\mathbf{M}(\varsigma) = M^0 \supset M^1 \supset \cdots \supset M^n \supset \cdots$$

where $M^i := (C_{11} - \lambda)^i \mathbf{M}(\varsigma)$ is a quasi-Whittaker module of type ς. Moreover, the quotient

$$L(\alpha, \lambda) := \mathbf{M}(\varsigma)/M^1 \simeq A/A(y_1 - \alpha, x_1, y_2, x_2, C_{11} - \lambda)$$

is a simple module with $\operatorname{ann}_A L(\alpha, \lambda) = \langle x_2, C_{11} - \lambda \rangle$, *and $M^i/M^{i+1} \simeq L(\alpha, \lambda)$ for all $i \in \mathbb{N}$.*

(3) *If $\alpha = 0$ and $\delta \in \Bbbk^*$, then for any $\mu \in \Bbbk$, $\mathbf{M}(\varsigma)$ has the following chain of submodules:*

$$\mathbf{M}(\varsigma) = \mathbb{M}^0 \supset \mathbb{M}^1 \supset \cdots \supset \mathbb{M}^n \supset \cdots$$

where $\mathbb{M}^i := (C_{22} - \mu)^i \mathbf{M}(\varsigma)$ is a quasi-Whittaker module of type ς. Moreover, the quotient

$$\mathbb{L}(\delta, \mu) := \mathbf{M}(\varsigma)/\mathbb{M}^1 \simeq A/A(y_1, x_1, y_2, x_2 - \delta, C_{22} - \mu)$$

is a simple module with $\operatorname{ann}_A \mathbb{L}(\delta, \mu) = \langle x_1, C_{22} - \mu \rangle$, *and $\mathbb{M}^i/\mathbb{M}^{i+1} \simeq \mathbb{L}(\delta, \mu)$ for all $i \in \mathbb{N}$.*

Proof. (1) Recall that we denote by $\bar{1} = 1 + \mathbf{I}_\varsigma$ the canonical generator of $\mathbf{M}(\varsigma)$. Using the explicit expressions of C_{11} and C_{21} given in (11), we obtain

$$C_{11}\bar{1} = (q^2 - 1)\alpha^2 E\bar{1} \quad \text{and} \quad C_{21}\bar{1} = q^{-1}\alpha\delta K\bar{1}. \tag{37}$$

Set $\widetilde{F} = KF$. Since $\alpha \in \Bbbk^*$, it follows that

$$\mathbf{M}(\varsigma) = \bigoplus_{i\in\mathbb{Z}, j,k\in\mathbb{N}} \Bbbk K^i \widetilde{F}^j E^k \bar{1} = \bigoplus_{i\in\mathbb{Z}, j,k\in\mathbb{N}} \Bbbk K^i \widetilde{F}^j C_{11}^k \bar{1}.$$

Let V be a non-zero submodule of $\mathbf{M}(\varsigma)$, and $v = \sum K^i h_i(\widetilde{F}, C_{11})\bar{1}$ a non-zero element of V where $h_i(\widetilde{F}, C_{11})$ are polynomials in $\Bbbk[\widetilde{F}, C_{11}]$. Since y_1 commutes with the elements $\widetilde{F}$ and C_{11}, the non-zero terms of v are eigenvectors of y_1 with distinct eigenvalues. More precisely, $y_1 \cdot K^i h_i(\widetilde{F}, C_{11})\bar{1} = q^i\alpha \cdot K^i h_i(\widetilde{F}, C_{11})\bar{1}$. Thus there exists a non-zero element $w \in \Bbbk[\widetilde{F}, C_{11}]\bar{1}$ that belongs to V. Write $w = \sum g_i(C_{11})\widetilde{F}^i\bar{1}$ where $g_i(C_{11})$ are polynomials in $\Bbbk[C_{11}]$. It can be verified by induction that for all positive integers i,

$$x_1\widetilde{F}^i = \widetilde{F}^i x_1 - (1 - q^{2i})/(1 - q^2)K\widetilde{F}^{i-1}y_1. \tag{38}$$

From the above identity and the fact that x_1 commutes with C_{11}, we see that there is a non-zero element $v' \in \Bbbk[C_{11}]\bar{1}$ that belongs to V. Write $v' = \sum k_i C_{11}^i \bar{1}$ where $k_i \in \Bbbk$. In the algebra A, we have $C_{21}C_{11} = q^{-2}C_{11}C_{21}$. Thus, by (37), the non-zero terms of v' are $K^{-1}C_{21}$-eigenvectors with distinct eigenvalues. More precisely, $K^{-1}C_{21} \cdot C_{11}^i \bar{1} = q^{-2i} C_{11}^i K^{-1} C_{21} \bar{1} = q^{-2i-1}\alpha\delta \cdot C_{11}^i \bar{1}$. As a result, we have $C_{11}^n \bar{1} \in V$ for some $n \in \mathbb{N}$. From (37) it follows that $E^n \bar{1} \in V$ for some $n \in \mathbb{N}$. Note that for all positive integers n, we have

$$y_2 E^n = E^n y_2 + (1 - q^{-2n})/(q^{-1} - q) K E^{n-1} x_2.$$

From the above identity and the inclusion $y_2 E^n \bar{1} \in V$, we obtain $E^{n-1}\bar{1} \in V$. Applying the above identity repeatedly yields that $\bar{1} \in V$, and therefore $V = \mathbf{M}(\varsigma)$. Thus $\mathbf{M}(\varsigma)$ is simple. It is obvious that $\langle \mathtt{D} - \alpha\delta \rangle \subseteq \operatorname{ann}_A \mathbf{M}(\varsigma)$. Notice that $(\phi_1 - \mu K^{-1} y_2)\bar{1} = \phi_1 \bar{1} = q^2 \alpha \bar{1}$. Thus, for any $\mu \in \Bbbk$, the element $\phi_1 - \mu K^{-1} y_2$ does not belong to $\operatorname{ann}_A \mathbf{M}(\varsigma)$. Since $\operatorname{ann}_A \mathbf{M}(\varsigma)$ is a primitive ideal, we conclude from Theorem 5.11 that $\operatorname{ann}_A \mathbf{M}(\varsigma) = \langle \mathtt{D} - \alpha\delta \rangle$.

(2) By Lemma 4.2, the left ideal $A(y_2, x_2)$ is equal to the two-sided ideal $\langle y_2, x_2 \rangle$ of A. Thus the module $\mathbf{M}(\varsigma)$ is annihilated by $\langle y_2, x_2 \rangle$. It follows that $\mathbf{M}(\varsigma) \simeq A[1]/A[1](y_1 - \alpha, x_1)$. Since C_{11} is central in $A[1]$, it is clear that M^i are quasi-Whittaker modules of type ς, and moreover, $M^i = U_q(\mathfrak{sl}_2)(C_{11} - \lambda)^i \bar{1}$. Clearly,

$$L(\alpha, \lambda) \simeq A/A(y_1 - \alpha, x_1, y_2, x_2, C_{11} - \lambda).$$

Let $\bar{\mathbf{1}}$ (the coset of 1) be the canonical generator of $L(\alpha, \lambda)$. Then $\lambda \bar{\mathbf{1}} = C_{11}\bar{\mathbf{1}} = (q^2 - 1)\alpha^2 E \bar{\mathbf{1}}$. From the PBW basis of A, and recalling that $\widetilde{F} = KF$, we obtain

$$L(\alpha, \lambda) = \bigoplus_{i \in \mathbb{Z}, j \in \mathbb{N}} \Bbbk K^i \widetilde{F}^j \bar{\mathbf{1}}.$$

Let W be a non-zero submodule of $L(\alpha, \lambda)$, and $w = \sum K^i g_i(\widetilde{F})\bar{\mathbf{1}}$ be a non-zero element of W where $g_i(\widetilde{F}) \in \Bbbk[\widetilde{F}]$. Notice that the non-zero terms of w are eigenvectors of y_1 with distinct eigenvalues, since $y_1 \cdot K^i g_i(\widetilde{F})\bar{\mathbf{1}} = q^i \alpha \cdot K^i g_i(\widetilde{F})\bar{\mathbf{1}}$. Thus there is a non-zero element $w' \in \Bbbk[\widetilde{F}]\bar{\mathbf{1}}$ that belongs to W. The simplicity of $L(\alpha, \lambda)$ then follows from identity (38). In particular, $\operatorname{ann}_A L(\alpha, \lambda)$ is a primitive ideal of A. The inclusion $\langle x_2, C_{11} - \lambda \rangle \subseteq \operatorname{ann}_A L(\alpha, \lambda)$ is obvious. We conclude from Theorem 5.11 that it must be an equality.

Since $M^i = U_q(\mathfrak{sl}_2)(C_{11} - \lambda)^i \bar{1}$, the module M^i / M^{i+1} is a cyclic module generated by the image of the element $v = (C_{11} - \lambda)^i \bar{1}$. Since v is a quasi-Whittaker vector of type ς and $(C_{11} - \lambda)v \equiv 0 \mod M^{i+1}$, there is a natural epimorphism of modules $\varrho : L(\alpha, \lambda) \twoheadrightarrow M^i / M^{i+1}$. The simplicity of $L(\alpha, \lambda)$ yields that ϱ must be an isomorphism.

(3) The result can be proved in a way similar to that of statement (2). ∎

Proposition 6.4. (1) *Let $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ be a non-zero algebra homomorphism of the form* (36b). *In this case, $\mathbf{M}(\varsigma) \simeq A/A(y_1, x_1 - \beta, y_2, x_2)$ where $\beta \in \Bbbk^*$. For any $\lambda \in \Bbbk$, $\mathbf{M}(\varsigma)$ has the following chain of submodules:*

$$\mathbf{M}(\varsigma) = W^0 \supset W^1 \supset \cdots \supset W^n \supset \cdots$$

where $W^i = (C_{11} - \lambda)^i \mathbf{M}(\varsigma)$ *is a quasi-Whittaker module of type* ς. *Moreover, the quotient*

$$N(\beta, \lambda) := \mathbf{M}(\varsigma)/W^1 \simeq A/A(y_1, x_1 - \beta, y_2, x_2, C_{11} - \lambda)$$

is a simple module with $\operatorname{ann}_A N(\beta, \lambda) = \langle x_2, C_{11} - \lambda \rangle$, *and* $W^i/W^{i+1} \simeq N(\beta, \lambda)$ *for all* $i \in \mathbb{N}$.

(2) *Let* $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ *be a non-zero algebra homomorphism of the form* (36c). *In this case,* $\mathbf{M}(\varsigma) \simeq A/A(y_1, x_1, y_2 - \gamma, x_2)$ *where* $\gamma \in \Bbbk^*$. *For any* $\mu \in \Bbbk$, $\mathbf{M}(\varsigma)$ *has the following chain of submodules:*

$$\mathbf{M}(\varsigma) = \mathcal{W}^0 \supset \mathcal{W}^1 \supset \cdots \supset \mathcal{W}^n \supset \cdots$$

where $\mathcal{W}^i = (C_{22} - \mu)^i \mathbf{M}(\varsigma)$ *is a quasi-Whittaker module of type* ς. *Moreover, the quotient*

$$S(\gamma, \mu) := \mathbf{M}(\varsigma)/\mathcal{W}^1 \simeq A/A(y_1, x_1, y_2 - \gamma, x_2, C_{22} - \mu)$$

is a simple module with $\operatorname{ann}_A S(\gamma, \mu) = \langle x_1, C_{22} - \mu \rangle$, *and* $\mathcal{W}^i/\mathcal{W}^{i+1} \simeq S(\gamma, \mu)$ *for all* $i \in \mathbb{N}$.

Proof. The proof is similar to that of Proposition 6.3. ■

Classification of simple quasi-Whittaker modules. The following theorem gives a classification of simple quasi-Whittaker A-modules of non-zero types.

Theorem 6.5. *Let* $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ *be a non-zero algebra homomorphism, and* V *be a simple quasi-Whittaker* A*-module of type* ς.

(1) *Suppose* ς *is of the form* (36a). *Then* V *is isomorphic to* $\mathbf{M}(\varsigma)$ *if* $(\alpha, \delta) \in (\Bbbk^*)^2$; *and* V *is isomorphic to* $L(\alpha, \lambda)$ *for some* $\lambda \in \Bbbk$ *if* $(\alpha, \delta) \in \Bbbk^* \times \{0\}$; *and* V *is isomorphic to* $\mathbb{L}(\delta, \mu)$ *for some* $\mu \in \Bbbk$ *if* $(\alpha, \delta) \in \{0\} \times \Bbbk^*$.

(2) *If* ς *is of the form* (36b), *then* V *is isomorphic to* $N(\beta, \lambda)$ *for some* $\lambda \in \Bbbk$.

(3) *If* ς *is of the form* (36c), *then* V *is isomorphic to* $S(\gamma, \mu)$ *for some* $\mu \in \Bbbk$.

Proof. (1) First, note that V is an epimorphic image of the universal module $\mathbf{M}(\varsigma)$. If $(\alpha, \delta) \in (\Bbbk^*)^2$, then by Proposition 6.3, $\mathbf{M}(\varsigma)$ is simple, so in this case we must have $V \simeq \mathbf{M}(\varsigma)$. If $(\alpha, \delta) \in \Bbbk^* \times \{0\}$, then $\mathbf{M}(\varsigma)$ is annihilated by the ideal $\langle x_2, y_2 \rangle$ of A. Consequently, V is annihilated by $\langle x_2, y_2 \rangle$. Thus V is a simple module over the algebra $A[1] \simeq A/\langle x_2, y_2 \rangle$. By Proposition 4.1 (1), the element C_{11} is central in $A[1]$. Since V is simple, Quillen's lemma tells us that C_{11} acts as a scalar, say λ, on V. It follows that V is an epimorphic image of the module $L(\alpha, \lambda)$, see Proposition 6.3 (2). Then the simplicity of $L(\alpha, \lambda)$ yields that $V \simeq L(\alpha, \lambda)$. If $(\alpha, \delta) \in \{0\} \times \Bbbk^*$, then $\mathbf{M}(\varsigma)$ is annihilated by the ideal $\langle x_1, y_1 \rangle$, and so V is annihilated by $\langle x_1, y_1 \rangle$. This means that V is a simple module over $A[2] \simeq A/\langle x_1, y_1 \rangle$. The element C_{22} is central in $A[2]$, so by Quillen's lemma, C_{22}

acts on V as a scalar. Therefore, V is an epimorphic image of $\mathbb{L}(\delta, \mu)$ for some $\mu \in \Bbbk$, see Proposition 6.3 (3). The simplicity of $\mathbb{L}(\delta, \mu)$ implies that $V \simeq \mathbb{L}(\delta, \mu)$.

(2) The result follows from Proposition 6.4 (1).

(3) The result follows from Proposition 6.4 (2). ■

Remark 6.6. Since any algebra homomorphism $\varsigma : \mathcal{O}_q(M_2) \to \Bbbk$ must be one of the forms (36a)–(36c), a classification of all simple quasi-Whittaker A-modules is obtained by Lemma 6.2 and Theorem 6.5. For each simple quasi-Whittaker module of non-zero type, its annihilator is determined, see Propositions 6.3 and 6.4. By Lemma 6.2, simple quasi-Whittaker modules of type zero are actually simple $U_q(\mathfrak{sl}_2)$-modules, whose annihilators are obtained from the primitive ideals of $U_q(\mathfrak{sl}_2)$.

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