J. Noncommut. Geom. 20 (2026), 507–560
DOI 10.4171/JNCG/600

On singular foliations tangent to a given hypersurface

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Abstract. We consider a class of singular foliations in the sense of Androulidakis and Skandalis that we call transverse order k foliations. These have a finite number of leaves: one hypersurface (the singular leaf) together with the components of its complement (open leaves). The positive integer parameter k encodes the "order of tangency" of the leafwise vector fields to L. We show that a loop in the singular leaf induces a well-defined holonomy transformation at the level of $(k-1)$-jets. The resulting holonomy invariant can be used to give a complete classification of these foliations and obtain concrete descriptions of their associated groupoids and algebras.

Contents

1. Introduction

Note that, except for minor changes, the contents of this article are duplicated the author's Ph.D. dissertation [12, Chapters 1, 2, and 5].

The interplay between foliation theory and operator theory is a significant aspect of Connes' noncommutative geometry program [6]. A key construction in this area is the

Mathematics Subject Classification 2020: 53C12 (primary); 22A22, 46L87 (secondary).
Keywords: singular foliation, Lie groupoid, jet, holonomy.

C*-algebra $C^*(\mathcal{F})$ of a regular foliation $\mathcal{F}$. This begins with Winkelnkemper's construction [30] of a (possibly non-Hausdorff) Lie groupoid $G(\mathcal{F})$ called the *holonomy groupoid* or *graph* of $\mathcal{F}$.

Since many natural examples of foliations are not regular, but instead present singularities of some type or another, it is desirable to extend these constructions so that they also apply in singular cases. A number of authors have done work on this topic, see [2, 4, 7, 8, 21–23]. The most broadly applicable construction is the one given by Androulidakis and Skandalis in [2] and it is their approach that we are concerned with here.

We also follow [2] in understanding a foliation to be any locally finitely-generated $C^\infty(M)$-module $\mathcal{F}$ of compactly-supported vector fields on M that is closed under Lie bracket. A leaf of $\mathcal{F}$ is then the set of points accessible from a given point by composing flows of vector fields in $\mathcal{F}$. By work of Stefan and Sussmann ([28, 29]), the leaves of $\mathcal{F}$ constitute a partition of M into immersed submanifolds (generalizing the Frobenius theorem). A foliation is *regular* if all its leaves have the same dimension and *singular* otherwise. In the regular setting, the module of vector fields can be recovered from the partition, but this fails in the singular setting. In fact, varying the module $\mathcal{F}$ while keeping the partition the same will be a prominent theme in this work.

In [2], given any singular foliation $\mathcal{F}$, Androulidakis and Skandalis constructed the following objects.

(1) A holonomy groupoid $G(\mathcal{F})$. In general, this is only a topological groupoid, and its topology can be very wild. It is a Lie groupoid if and only if $\mathcal{F}$ is *almost regular*, a hypothesis satisfied by all foliations studied in this article.

(2) A smooth convolution algebra $\mathcal{A}(\mathcal{F})$. Note $\mathcal{A}(\mathcal{F}) \cong C_c^\infty(G(\mathcal{F}))$ in the almost regular case, choosing a smooth Haar system in order to make sense of convolution.

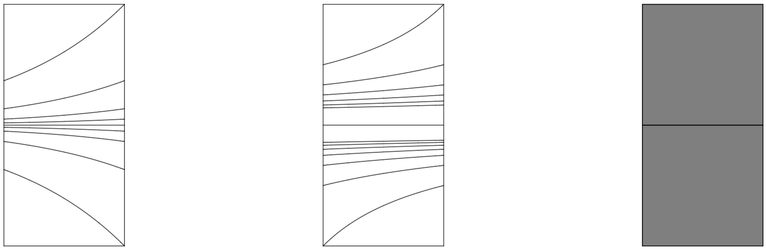
(3) A C*-algebra $C^*(\mathcal{F})$, obtained by completing $\mathcal{A}(\mathcal{F})$. In the almost regular case, this is the usual groupoid C*-algebra in the sense of [24].

We now give an informal discussion of holonomy to indicate how things change in the singular setting. Figure 1 depicts the leaves of the following three foliations of the cylinder $S^1 \times \mathbb{R}$, which we regard as having coordinates (x, y) where x is $\mathbb{Z}$-periodic:

$$\mathcal{F}\left\{\frac{d}{dx} + y\frac{d}{dy}\right\}, \quad \mathcal{F}\left\{\frac{d}{dx} + y^2\frac{d}{dy}\right\}, \quad \mathcal{F}\left\{\frac{d}{dx}, y\frac{d}{dy}\right\}.$$

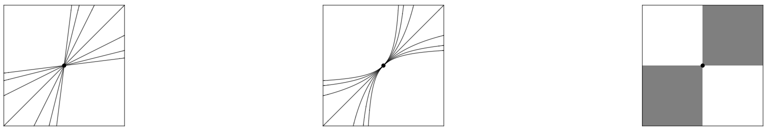
Here, the notation $\mathcal{F}\{X_1, \dots, X_n\}$ refers to the foliation generated by a finite set of vector fields $X_1, \dots, X_n$. The first two of these are regular foliations with 1-dimensional leaves while the third is a singular foliation whose leaves are $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$, and $S^1 \times \mathbb{R}_-$, where $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_- := (-\infty, 0)$.

All three of these foliations determine nonsmooth equivalence relations. The issue becomes apparent when we restrict the leaf equivalence relations to the slice $T = \{0\} \times \mathbb{R}$ passing through $p = (0, 0)$. The resulting subsets of $T \times T$ are depicted in Figure 2.



$\mathcal{F}\{\frac{d}{dx}+y\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx}+y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx},y\frac{d}{dy}\}$

Figure 1. Leaves of some foliations of $S^1\times\mathbb{R}$.



$\mathcal{F}\{\frac{d}{dx}+y\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx}+y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx},y\frac{d}{dy}\}$

Figure 2. Equivalence relations of some foliations of $S^1\times\mathbb{R}$, restricted to $T=\{0\}\times\mathbb{R}$.

Despite the singularities apparent in Figure 2, the holonomy groupoid of each of these foliations is smooth. In terms of these pictures, what occurs is that the problematic point (p,p) at the origin gets blown up and replaced with the *holonomy group* at p. For a regular foliation, given a point p and a transversal submanifold T through the leaf of p, the holonomy group at p may be viewed as the discrete group consisting of all (germs of) diffeomorphisms of T fixing p which can be obtained using flows of vector fields in $\mathcal{F}$. For both of the regular foliations in Figure 1, this holonomy group is infinite cyclic, and it is easy to imagine how such a replacement can resolve the singularity. To put it more plainly, for each of the two regular foliations, the equivalence relation in Figure 2 is the union of the graphs of a countable family of graphs of diffeomorphisms which intersect only at the origin. After performing the blowup, one is left instead with the *disjoint union* of these graphs.

A striking difference between the regular and singular settings is that, whereas for regular foliations holonomy is purely a discrete phenomenon, for singular foliations one can also have *continuous holonomy*. For the singular foliation $\mathcal{F}\{\frac{d}{dx},y\frac{d}{dy}\}$ shown above, the group of diffeomorphism germs of T which can be obtained using compositions of flows is infinite-dimensional. One perspective is that the work of Androulidakis and Skandalis identifies the correct way to take a quotient of this infinite-dimensional group and obtain a finite-dimensional Lie group which serves as the natural generalization of the usual holonomy group.

For the foliation $\mathcal{F}\{\frac{d}{dx},y\frac{d}{dy}\}$ above, the holonomy group at $p=(0,0)$ may be identified with the Lie group of linear, orientation-preserving diffeomorphisms of T and, in particular, it is isomorphic to $\mathbb{R}$. However, this is just one of many foliations of $S^1 \times \mathbb{R}$ whose leaves are $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$, and $S^1 \times \mathbb{R}_-$. With the exception of some pathological examples, the holonomy group at p of any such foliation is naturally realized, for some positive integer k encoding the "transverse order" of the foliation, as a one-dimensional subgroup of the group J^k of k-jets of diffeomorphisms of $\mathbb{R}$ which fix 0. Explicitly,

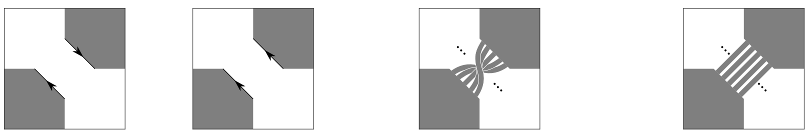
$$J^k = \{a_1 y + a_2 y^2 + \cdots + a_\ell y^k : a_i \in \mathbb{R}, a_1 \neq 0\}$$

under the operation "compose and truncate." Some examples of holonomy groups which can occur are tabulated in Table 1.

Foliation	Holonomy group	Ambient group
$\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$	$\{e^t y : t \in \mathbb{R}\}$	J^1
$\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\}$	$\{y + ty^2 : t \in \mathbb{R}\}$	J^2
$\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\}$	$\{e^n y + ty^2 : n \in \mathbb{Z}, t \in \mathbb{R}\}$	J^2
$\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\}$	$\{y + ny^2 + n^2y^3 + ty^4 : n \in \mathbb{Z}, t \in \mathbb{R}\}$	J^4

Table 1. Holonomy groups at $p = (0, 0)$ of several different foliations of $S^1 \times \mathbb{R}$, all of which have leaves $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$.

The precise details of how (p, p) is blown up into a copy of the holonomy group at p also depends on two natural orderings of the group J^k, associated to the positive and negative half lines. As Figure 3 shows, the topological possibilities for the blowup space are actually quite rich, especially given how simple the leaf space of these foliations is. The last two surfaces, for example, are not homeomorphic, as can be seen by counting the number of topological ends.



$\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\}$

Figure 3. Holonomy groupoids of several foliations of $S^1 \times \mathbb{R}$ whose leaves are $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$, restricted to $T = \{0\} \times \mathbb{R}$.

The following family of one-dimensional foliations appeared in [2, Example 1.3 (3)] and received a detailed analysis in [11]. Discounting examples whose construction involves the use of bump functions, these are all of the singular foliations of $\mathbb{R}$ having leaves $\mathbb{R}_-$, $\mathbb{R}_+$, and $\{0\}$.

Definition 1.1. Let the coordinate function of $\mathbb{R}$ be y. For each positive integer k, we denote by $\mathcal{F}_{\mathbb{R}}^k$ the singular foliation of $\mathbb{R}$ singly-generated by $y^k \frac{\partial}{\partial y}$. That is, $\mathcal{F}_{\mathbb{R}}^k$ is all compactly supported vector fields on $\mathbb{R}$ which vanish to k-th order at the origin.

In this article, we analyze a class of singular foliations which we call *transverse order k foliations* (Definition 5.1) which are the generalizations to higher codimension of the above foliations $\mathcal{F}_{\mathbb{R}}^k$. Suppose $\mathcal{F}$ is a foliation of a connected manifold M whose leaves consist of a single codimension-1 submanifold $L \subseteq M$ together with the components of $M \setminus L$. The total number of leaves is therefore either two or three. Thanks to a splitting principle for singular foliations ([2, Proposition 1.12] and [4, Proposition 1.2]), the local structure of such a foliation is completely determined by a foliation of $\mathbb{R}$ modelling the transverse structure of the foliation near the leaf L. If this transverse foliation is $\mathcal{F}_{\mathbb{R}}^k$, we say that $(M, \mathcal{F})$ is a foliation of *transverse order k* (Definition 5.1). The foliations of $S^1 \times \mathbb{R}$ with singular leaf $S^1 \times \{0\}$ that appeared earlier in Table 1 were examples of transverse order k foliations. The purpose of this article is to classify transverse order k foliations and provide explicit descriptions of their groupoids and algebras.

If M and L are given, there is a unique foliation $\mathcal{F}$ of transverse order $k = 1$ whose singular leaf is L, namely the collection of all compactly-supported vector fields which are tangent along L. However, when $k \geq 2$, the structure of transverse order k foliations becomes much more interesting. For this reason, we will generally assume k is an integer ≥ 2 in this article.

If $X_1, \ldots, X_n$ are smooth vector fields on a manifold M such that $[X_i, X_j]$ is a $C^\infty(M)$-linear combination of $X_1, \ldots, X_n$ for all i, j, we use the notation

$$\mathcal{F}\{X_1, \ldots, X_n\} := \operatorname{span}_{C_c^\infty(M)}\{X_1, \ldots, X_n\}$$

for the foliation they generate.

Example 1.2. Consider $\mathbb{R}^2$ with usual coordinates (x, y). Then

$$\mathcal{F}\left\{y^2 \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right\} \quad \mathcal{F}\left\{y^2 \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right\}$$

are distinct transverse order 2 foliations with singular leaf the x-axis. Although these foliations are distinct, they are still isomorphic to each other; the pushforward of the first foliation by the diffeomorphism $(x, y) \mapsto (x, e^x y)$ is the second foliation.

The above example exposes the basic point that the module of vector fields on M which are "tangent to L to order 2" is not in fact well defined when working up to smooth coordinate changes. To get nonisomorphic examples of transverse order 2 foliations with the same leaves, one needs the singular leaf to not be simply connected.

Example 1.3. In the previous example, consider x as a $\mathbb{Z}$-periodic coordinate so that the space is the cylinder $S^1 \times \mathbb{R}$. Since the generating vector fields are invariant under horizontal translation, they also define transverse order 2 foliations on $S^1 \times \mathbb{R}$ with singular leaf the equator $S^1 \times \{0\}$. These foliations are not isomorphic to each other and this article will provide a framework explaining this nonisomorphism. Briefly, whereas for the first foliation the holonomy around the equator is trivial, for the second, the holonomy, suitably interpreted, is multiplication by e.

Suppose $(M, \mathcal{F})$ is a foliation of transverse order $k \geq 2$ with singular leaf L. We give a brief overview of the notion of holonomy to be introduced. Let c be a path in L from a point x to a point x' and fix small, one-dimensional transversals T and T' at the endpoints. If $\mathcal{F}$ were a regular foliation of codimension-one, the classical notion of holonomy would assign to c a diffeomorphism germ $T \to T'$ sending $x \mapsto x'$. This does not occur in the case at hand, but it turns out one does have a well-defined holonomy mapping at the level of $(k-1)$-jets (this is related to the picture of the holonomy groupoid obtained in [4]). In particular, taking $x = x'$, $T = T'$, and choosing an identification of T with $\mathbb{R}$, one obtains from this assignment a homomorphism

$$\gamma\colon \pi_1(L, x) \to J^{k-1}, \tag{1}$$

where J^r denotes the group of r-jets at 0 of diffeomorphisms of $\mathbb{R}$ fixing 0. Concretely, J^r is the group of polynomials of the form $a_1 y + \cdots + a_r y^r$ with $a_i \in \mathbb{R}$, $a_1 \neq 0$ under the operation "compose and truncate." There is a canonical quotient map $J^r \to J^{r-1}$ for all $r \geq 2$ whose kernel is the group $\mathbb{R}$, embedded in J^r by way of $t \mapsto y + ty^r$. We obtain the following.

Theorem (Definition 7.6). *The homomorphism (1) is well defined up to inner automorphisms of J^{k-1} and gives rise to a holonomy invariant*

$$h(\mathcal{F}) \in [\pi_1(L), J^{k-1}] \tag{2}$$

for the foliation.

Here, if A and B are groups, we use $[A, B]$ to denote the quotient set of $\mathrm{Hom}(A, B)$ by the conjugation action of B, in the spirit of the similar notation frequently employed for homotopy classes of maps. Note it is not necessary to specify a basepoint for the fundamental group in (2) because L is connected, so its different fundamental groups are canonically isomorphic when working up to inner automorphisms.

The holonomy invariant (2) is "L-local," in the sense that it only depends on the restriction of the foliation to a neighbourhood of the singular leaf, and natural with respect to isomorphisms of transverse order k foliations, in an appropriate way. In fact, it is a complete invariant for the structure of the foliation nearby to the singular leaf.

Theorem (Theorem 11.2). *If $(M_i, \mathcal{F}_i)$ is a foliation of transverse order k with singular leaf L_i for $i = 1, 2$ and there is a diffeomorphism $\theta_0\colon L_1 \to L_2$ carrying $h(\mathcal{F}_1)$ to $h(\mathcal{F}_2)$, then θ_0 can be extended to a foliation-preserving diffeomorphism $\theta\colon U_1 \to U_2$, where U_i is neighbourhood of L_i in M_i.*

Furthermore, the possible values of this holonomy invariant are exhausted.

Theorem (Theorem 12.1). *Given any connected manifold L and any group homomorphism $\pi_1(L) \to J^{k-1}$, there exists a transverse order k singular foliation with singular leaf L whose holonomy invariant is (the class of) the given homomorphism.*

We formalize the above ideas about holonomy using principal bundles.[1] Given a transverse order $k \geq 2$ singular foliation $(M, \mathcal{F})$ with singular leaf L, we construct a sequence of principal bundles $P^r(\mathcal{F}) \to L$, $r = 1, 2, \ldots, k$. The elements of $P^r(\mathcal{F})$ are r-jets of certain submersions $M \to \mathbb{R}$ and may be thought of as dual versions of transversals. When $r \leq k-1$, the structure group of $P^r(\mathcal{F})$ is J^r_d, the underlying discrete group of J^r. This captures the idea that the $(k-1)$-jets of transversals can be parallel transported along paths in L. This rigidity breaks down at $r = k$; the structure group of $P^k(\mathcal{F})$ is $J^k_{\mathbb{R}}$, the one-dimensional Lie group structure on J^k obtained by decomposing it into the fibers of the natural projection $J^k \to J^{k-1}$. The main applications of these principal bundles $P^r(\mathcal{F})$ are as follows.

Theorem (Sections 7, 8, 9). *The following facts hold.*

(1) *The monodromy of the principal J^{k-1}_d-bundle $P^{k-1}(\mathcal{F})$ is exactly the holonomy invariant $h(\mathcal{F})$ of (2).*

(2) *The gauge groupoid of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F})$ reconstructs the holonomy groupoid of $\mathcal{F}$, restricted to L.*

(3) *The monodromy of the principal J^1_d-bundle $P^1(\mathcal{F})$ determines a flat connection on the conormal bundle of L in M (note that $J^1 = \mathrm{GL}(1, \mathbb{R})$) which it is appropriate to call the* Bott connection.

In [2], in addition to the (singular analog of the) usual holonomy groupoid $G(\mathcal{F})$, the authors construct a *full holonomy groupoid* $G_{\mathrm{full}}(\mathcal{F})$. This is a “big groupoid” containing the usual (i.e., minimal) holonomy groupoid, as well as various intermediate groupoids. It is helpful to work inside this big groupoid initially and later extract the usual one as its s-connected component.

The full holonomy groupoid of $\mathcal{F}^k_{\mathbb{R}}$ can be thought of as a smooth blowup of the singular equivalence relation $(\mathbb{R} \setminus \{0\})^2 \cup \{(0,0)\} \subseteq \mathbb{R}^2$ wherein the singular point $(0,0)$ is replaced by a copy of $J^k_{\mathbb{R}}$:

$$G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}}) \cong (\mathbb{R} \setminus \{0\})^2 \cup J^k_{\mathbb{R}}.$$

More generally, the full holonomy groupoid of any transversely order k foliation $(M, \mathcal{F})$ with singular leaf L can be thought of as a smooth blowup of the singular equivalence relation $(M \setminus L)^2 \cup L^2 \subseteq M^2$ wherein the singular locus L^2 is replaced by a copy of the gauge groupoid of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F}) \to L$:

$$G_{\mathrm{full}}(\mathcal{F}) \cong (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F})).$$

[1] Our principal bundles will always be smooth, with structure group acting from the left.

The full holonomy groupoid of a transverse order k foliation gives an interesting example of a topological space equipped with a smooth atlas that is nearly, but not quite, a smooth manifold.

Theorem (Theorem 4.14). *The topology of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is Hausdorff, regular and separable, but not normal.*

There are known constructions for "manifolds" of this type (see [16, Chapter 14]), but it is interesting to encounter such a beast "in the wild."

The holonomy groupoid of a transverse order $k \geq 2$ foliation $(M, \mathcal{F})$ is the s-connected component of the full holonomy groupoid. We can give more concrete information: let $\gamma\colon \pi_1(L, x_0) \to J^{k-1}$ be any homomorphism representing the holonomy invariant $h(\mathcal{F})$ in $[\pi_1(L), J^{k-1}]$. Let Γ be the range of γ, a countable subgroup of J^{k-1}, and let $\Gamma_{\mathbb{R}}$ be the preimage of Γ by the natural projection $J^k \to J^{k-1}$. The relationships between these various groups are shown in the following diagram:

$$\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & J^k_{\mathbb{R}} & \longrightarrow & J^{k-1}_d \\
\| & & \uparrow & & \uparrow \\
\mathbb{R} & \longrightarrow & \Gamma_{\mathbb{R}} & \longrightarrow & \Gamma \\
 & & & & \uparrow{\scriptstyle \gamma} \\
 & & & & \pi_1(L, x_0)
\end{array}$$

Theorem (Theorem 9.5). *The isotropy groups of $G(\mathcal{F})$ at points on the singular leaf are isomorphic to $\Gamma_{\mathbb{R}}$.*

Meanwhile, the restriction of $G(\mathcal{F})$ to one of the open leaves (there are at most two) is a pair groupoid so, applying standard results on groupoid C*-algebras, we obtain information about the structure of the foliation C*-algebra.

Theorem (Theorem 9.2, Corollary 9.6). *The foliation C*-algebra $C^*(\mathcal{F})$ fits into an extension*

$$0 \to I \to C^*(G(\mathcal{F})) \to C^*(\Gamma_{\mathbb{R}}) \otimes \mathbb{K} \to 0, \tag{3}$$

where $\mathbb{K}$ is the C-algebra of compact operators on a separable Hilbert and I denotes either $\mathbb{K}$ or $\mathbb{K} \oplus \mathbb{K}$, according to whether $\mathcal{F}$ has one open leaf or two.*

It would be interesting to analyse the extension (3). The one-dimensional Lie group $\Gamma_{\mathbb{R}}$ is solvable, so this problem is likely to be tractable.

Since we are mainly concerned with what is happening near the singular leaf L, it is often sufficient to consider the case where M is the total space of a line bundle $\pi\colon E \to L$, with L embedded in E as the zero section. In this case, we already have a natural principal J^r-bundle $J^r(E, \mathbb{R}) \to L$, even without specifying a transverse order k-foliation.

Theorem (Theorem 10.4). *There is a one-to-one correspondence between*

(1) *flat connections on the principal J^{k-1}-bundle $J^{k-1}(E, \mathbb{R})$ and*

(2) *singular foliations of transverse order k on E whose singular leaf is L (embedded as the zero section).*

Once a flat connection on $J^{k-1}(E, \mathbb{R})$ has been fixed, the resulting J_d^{k-1}-bundle structure on $J^{k-1}(E, \mathbb{R})$ is canonically isomorphic to the principal J_d^{k-1}-bundle $P^{k-1}(\mathcal{F})$.

Relation to other work. What were introduced in [12] as *transverse order k foliations* are quite similar to what Scott previously introduced as *b^k-manifolds* [25]. A *b^k-manifold* is an oriented, smooth manifold M with an oriented hypersurface L plus the data of the $(k-1)$-jet along L of a smooth, positively-oriented defining function for L, i.e., an oriented submersion $p: \Omega \to \mathbb{R}$ with $p = f^{-1}(0)$, where Ω is a neighbourhood of L. Such a choice of $(k-1)$-jet determines an associated transverse order k foliation by taking for $\mathcal{F}$ the collection of all compactly-supported vector fields X such that Xp vanishes to k-th order along L, where p is some defining function representing the given $(k-1)$-jet. However, this assignment is neither injective nor surjective when $k \geq 2$. Replacing p by $2p$ changes its $(k-1)$-jet, but not its associated transverse order k foliation. Furthermore, the existence of an inducing $(k-1)$-jet globally-defined on L implies the triviality of the holonomy invariant studied in [12], thus not all transverse order k foliations arise from Scott's setup. Indeed, p is an $\mathcal{F}$-$\mathcal{F}_{\mathbb{R}}^k$-submersion (Definition 5.1) defined on all of L and yields a global section in the principal bundle $P^{k-1}(\mathcal{F})$ (Section 7) whose monodromy is the holonomy of $\mathcal{F}$.

In the recent preprint [5], Bischoff, del Pino, and Witte introduce independently under the name *k-th order foliations* the same objects that [12] called *transverse order k foliations*, and also consider the case of arbitrary submanifolds in addition to that of hypersurfaces. The authors of [5] independently obtain a local classification theorem essentially the same as the one in [12]. The other aspects of the theory of this class of singular foliations considered by the present author and by Bischoff, del Pino, and Witte are rather complementary. For example, [12] emphasizes the holonomy groupoids and C*-algebras whereas [5] is more focused on the Lie algebroid along the leaf, also considering symplectic structures and going on to develop and apply a cohomology theory related to that in [25].

Even more recently, the preprint [10] has appeared, in which Fischer and Laurent-Gengoux carry the idea of classifying singular foliations with a given transverse model in a neighbourhood of given leaf by holonomy data much further than is done in either [12] or [5]. In [10], a complete classification is given, at least at the formal level, for any codimension and any transverse model.

Structure of article. We now briefly summarize the contents that follow this introduction. Section 2 is a rather long preliminary section that gathers for convenience various definitions and results, especially relating to Androulidakis and Skandalis's construction of the holonomy groupoid of a singular foliation. Furthermore, a description (Theorem 2.43) of the holonomy groupoid closely related to the description in [4] is obtained. In Section 3, we introduce the groups in which our holonomy invariant will take values. In Section 4, we study the full holonomy groupoid of the model foliation $\mathcal{F}_{\mathbb{R}}^k$ and its point-set topological properties (Theorem 4.14). In Section 5, we define precisely transverse order k foliations (Definition 5.1). In Section 6, we determine which local transformations and submersions are compatible with transverse order k foliations (Theorem 6.6). The rigidity in lower order Taylor expansions uncovered in this section is at the root of the holonomy invariants to be defined. In Section 7, we construct certain principal bundles and use their monodromy to precisely define our holonomy invariant (Definition 7.6). In Section 8, we use the principal bundles of the preceding section to give a gauge groupoid description of the full holonomy groupoid of a transverse order k foliation. In Section 9 we describe the holonomy groupoid (Theorem 9.2) which sits inside the full holonomy groupoid as the *s-connected component*. In Section 10, we consider transverse order k foliations in the particular case where the total space is a line bundle and the singular leaf is its zero section. In this case, transverse order-k foliations can be put into a natural correspondence with certain flat connections (Theorems 10.4 and 10.5). In Section 11, we prove that our holonomy invariant is a complete invariant (Theorem 11.2). Finally, in Section 12, we prove that our holonomy invariant takes on all of its possible values (Theorem 12.1).

2. Preliminaries

In this lengthy section, we gather for ease of reference various definitions and results which will be needed. We review the work [2], defining precisely what is meant by a (singular) foliation $(M, \mathcal{F})$ and giving the constructions of the holonomy groupoid $G(\mathcal{F})$ as well as the *full holonomy groupoid* $G_{\text{full}}(\mathcal{F})$, a larger groupoid containing $G(\mathcal{F})$. We also give a picture of the full holonomy groupoid as a groupoid of (equivalence classes of germs of) holonomy transformations that is similar in spirit to the picture obtained in [4], but slightly different because we do not make use of slices. Proofs are omitted when they can be found in [2]. Readers already familiar with the literature on holonomy groupoids of singular foliations will most likely wish to skip this section and refer back to it whenever necessary.

2.1. Manifolds

In this article, a *smooth manifold* refers to a topological space equipped with a smooth atlas of some constant, finite dimension that is furthermore metrizable (or, equivalently, paracompact and Hausdorff). Many authors require their manifolds to be second-countable, but that assumption is not convenient here. In any event, the distinction is only relevant for highly disconnected manifolds; every smooth manifold in our sense is a (possibly uncountable) disjoint union of second-countable smooth manifolds.

We sometimes employ the term *smooth space* to refer to a topological space that is equipped with a smooth atlas. Note that the same disclaimers which apply when one speaks of two manifolds being “equal” apply to smooth spaces also. This is to say, an atlas for a smooth space is not really an innate part of its structure. Rather, one introduces the usual notion of equivalence of two atlases and works either with equivalence classes of atlases, or with the unique maximal atlas in each class. Note the topology of a smooth space is uniquely determined by any atlas, so the former need not be specified in advance.

2.2. Groupoids

We shall tend to use calligraphic characters such as $\mathcal{G}$ to denote abstract groupoids (no topology) and reserve roman characters such as G for topological groupoids. For brevity, we often write $\mathcal{G} \rightrightarrows X$ to indicate that $\mathcal{G}$ is an (abstract) groupoid with unit space X. We typically denote the source and target projections of $\mathcal{G}$ by s and t, respectively. Multiplication is performed from right to left so that, given $a, b \in \mathcal{G}$, the product ab is defined if and only if $s(a) = t(b)$. The inversion map is denoted $\iota\colon \mathcal{G} \to \mathcal{G}$ or, frequently, just $a \mapsto a^{-1}$. We use (standard) notations such as $\mathcal{G}_x := s^{-1}(x)$ and $\mathcal{G}^x := t^{-1}(x)$ for the source and target fibers. We only speak of morphisms between groupoids that have the same unit space and always require that the underlying map on the unit space is the identity. Accordingly, the way in which we understand quotients of groupoids is constrained to the following.

Definition 2.1. A *normal subgroupoid* of a groupoid $\mathcal{G} \rightrightarrows X$ is a union $\mathcal{N} = \bigcup_{x \in M} \mathcal{N}_x$, where $\mathcal{N}_x$ is a subgroup of the isotropy group $\mathcal{G}_x^x$ such that, if $x, y \in X$, $a \in \mathcal{G}_x^y$, $b \in \mathcal{N}_x$, then $aba^{-1} \in \mathcal{N}_y$.

Lemma 2.2. *Let $\mathcal{N}$ be normal subgroupoid of a groupoid $\mathcal{G} \rightrightarrows X$. Given $x, y \in X$ and $a, b \in \mathcal{G}_x^y$, put $a \approx b$ if and only if $a^{-1}b \in \mathcal{N}$. Then, $\approx$ is an equivalence relation and the groupoid operations of $\mathcal{G}$ descend in a well-defined way to give the quotient set $\mathcal{G}/\approx$ the structure of a groupoid on X.* ∎

Definition 2.3. Given a normal subgroupoid $\mathcal{N}$ of a groupoid $\mathcal{G}$, the *quotient groupoid* $\mathcal{G}/\mathcal{N}$ is the groupoid $\mathcal{G}/\approx$ of the above lemma.

2.3. Lie groupoids

A *Lie groupoid* is a groupoid $G \rightrightarrows B$ where G and B are smooth manifolds, the source and target maps $s, t\colon G \to B$ are submersions and all structure maps are smooth. If G is only a smooth space (see Section 2.1), we call $G \rightrightarrows B$ a *smooth groupoid*. One may refer to [18] for a detailed treatment of Lie groupoids.

Example 2.4. For any manifold M, the *pair groupoid* refers to the Lie groupoid structure on cartesian product $M \times M$ with source projection pr_2, target projection pr_1, and multiplication defined by $(x_3, x_2)(x_2, x_1) = (x_3, x_1)$ for all $x_1, x_2, x_3 \in M$.

Example 2.5. Given a smooth action of a Lie group H on a smooth manifold M, the *transformation groupoid* $H \ltimes M$ is the Lie groupoid whose underlying manifold is

$H \times M$ with groupoid operations defined as follows:

$$\begin{aligned} \text{source projection:} \quad & (h, x) \mapsto x; \\ \text{target projection:} \quad & (h, x) \mapsto hx; \\ \text{multiplication:} \quad & (h_2, h_1 x)(h_1, x) = (h_2 h_1, x). \end{aligned}$$

We sometimes find it convenient to reverse the order of the factors; the transformation groupoid $M \rtimes H$ has underlying manifold $M \times H$ and its Lie groupoid structure is such that $(h, x) \mapsto (x, h)$: $H \ltimes M \to M \rtimes H$ is a Lie groupoid isomorphism.

In this article, all principal bundles are assumed to be smooth, with structure group acting on the left. Every principal bundle determines a so-called *gauge groupoid* (also known as the *Atiyah groupoid*) as described below. This construction will play an important role in this article.

Lemma 2.6. *Let $\pi\colon P \to B$ be a (smooth, left) principal H-bundle, where H is a Lie group. There is a unique Lie groupoid structure on the quotient manifold $(P \times P)/H$, where H acts diagonally, whose operations are determined as follows:*

$$\begin{aligned} \text{source projection:} \quad & [q, p] \mapsto \pi(p); \\ \text{target projection:} \quad & [q, p] \mapsto \pi(q); \\ \text{multiplication:} \quad & [r, q][q, p] = [r, p]. \end{aligned}$$

for all $p, q, r \in P$. Here $[q, p]$ denotes the class of (q, p) in $(P \times P)/H$. ∎

Definition 2.7. The *gauge groupoid* $\mathrm{Gauge}(P) \rightrightarrows B$ of a (smooth, left) principal bundle $P \to B$ is the Lie groupoid constructed in the above lemma.

Example 2.8. In the case of a trivial left H-bundle $B \times H$, it is easy to see there is an isomorphism $B \times B \times H \to \mathrm{Gauge}(B \times H)$ defined by $(y, x, h) \mapsto [(y, 1), (x, h)]$.

Gauge groupoids are always transitive and, in fact, every transitive Lie groupoid $G \rightrightarrows B$ is isomorphic to a gauge groupoid; for any choice of $x \in B$, one has that G^x is a principal G^x_x-bundle and $G \cong \mathrm{Gauge}(G^x)$. In [1], the correspondence between transitive Lie groupoids and principal bundles is extended in a way that takes extensions into account.

2.4. Monodromy of flat bundles

Given two groups A and B, we denote by $[A, B]$ the quotient of the set $\mathrm{Hom}(A, B)$ by the conjugation action of B.

Let M be a connected, smooth manifold and let Γ be a (discrete) group. Given basepoints $x, y \in M$, there is a canonical bijection between the sets $[\pi_1(M, x), \Gamma]$ and $[\pi_1(M, y), \Gamma]$. This is so because the isomorphisms $\pi_1(M, x) \to \pi_1(M, y)$ determined by two different choices of paths from x to y only differ by an inner automorphism of

$\pi_1(M, x)$, and it follows that their induced bijections

$$\mathrm{Hom}(\pi_1(M, x), \Gamma) \to \mathrm{Hom}(\pi_1(M, y), \Gamma)$$

differ by an inner automorphism of Γ. It therefore makes sense to speak of the set $[\pi_1(M), G)]$ without specifying a choice of basepoint. In a similar vein, we have the following.

Proposition 2.9. *If $\theta\colon M_1 \to M_2$ is a diffeomorphism of connected manifolds, then pushing forward loops by θ determines a well-defined bijection*

$$\theta_*\colon [\pi_1(M_1), \Gamma] \to [\pi_1(M_2), \Gamma]. \qquad \blacksquare$$

In fact, the procedure of the above proposition makes

$$M \mapsto [\pi_1(M), \Gamma]$$

into a functor from the category of connected, smooth manifolds and diffeomorphisms to the category of sets and bijections.

Let $\pi\colon Q \to M$ be a smooth principal bundle with connected base manifold M and discrete structure group Γ.[2] Fix a point $q_0 \in Q$ and put $x_0 := \pi(q_0)$. One may then define a group homomorphism

$$\gamma\colon \pi_1(M, x_0) \to \Gamma$$

in the following way. Given any loop $c\colon [0,1] \to M$ based at M, let $\tilde{c}\colon [0,1] \to Q$ be the unique lift of c with $c(0) = q_0$ and define $\gamma([c])$ by

$$\tilde{c}(1) = \gamma([c]) \cdot q_0.$$

One may check this gives a well-defined homomorphism γ and, moreover, that the class in $[\pi_1(M), \Gamma]$ of this homomorphism does not depend on the chosen point $q_0 \in M$. For example, if $h \in \Gamma$ and $q_0' = h \cdot q_0$, then the monodromy homomorphism $\gamma'\colon \pi_1(M, x_0) \to \Gamma$ determined by q_0' satisfies $\gamma' = \mathrm{Ad}_h \circ \gamma$. It therefore makes sense to introduce the following definition.

Definition 2.10. Let Q be a smooth principal bundle with connected base manifold M and discrete structure group Γ. The *monodromy invariant* of Q is the element $h(Q)$ in $[\pi_1(M), \Gamma]$ represented by the homomorphism $\pi_1(M, x_0) \to \Gamma$ constructed above.

It is a standard result that principal bundles with discrete structure group (or equivalently flat bundles) are completely classified by their monodromy invariants. To be more precise, the following result holds.

[2] This is the same thing as a normal covering space whose group of deck transformations has been identified with Γ.

Theorem 2.11. *For $i = 1, 2$, let M_i be a connected, smooth manifold and let $Q_i \to M_i$ be a smooth left principal bundle with discrete structure group Γ.*

(1) *If $\theta\colon Q_1 \to Q_2$ is a Γ-bundle isomorphism and $\theta_0\colon M_1 \to M_2$ is the underlying map of the base, then $(\theta_0)_*(h(Q_1)) = h(Q_2)$.*

(2) *If there exists a diffeomorphism $\theta_0\colon M_1 \to M_2$ such that $(\theta_0)_*(h(Q_1)) = h(Q_2)$, then there exists a Γ-bundle isomorphism $\theta\colon Q_1 \to Q_2$ covering θ_0.* ■

2.5. Modules of smooth sections

Throughout the following, E is a smooth vector bundle over a smooth manifold M and $\mathcal{F}$ is a $C^\infty(M)$-submodule of $C_c^\infty(M; E)$, the smooth, compactly-supported sections of E.

Definition 2.12. The *fiber* of a submodule $\mathcal{F} \subseteq C_c^\infty(M; E)$ at $x \in M$ is the vector space $A_x\mathcal{F} := \mathcal{F}/I_x\mathcal{F}$, where $I_x \subseteq C^\infty(M)$ denotes the ideal of functions vanishing at the point x. We denote the quotient map $\mathcal{F} \to A_x\mathcal{F}$ by $X \mapsto [X]_x$.

From this definition, we have the relation

$$[fX]_x = f(x)[X]_x \quad f \in C^\infty(M), X \in \mathcal{F}.$$

If $X \in \mathcal{F}$ has $X \in I_x\mathcal{F}$ for all $x \in M$, then $X = 0$, so one may legitimately regard $\mathcal{F}$ as a module of sections of the bundle

$$A\mathcal{F} := \bigsqcup_{x \in M} A_x\mathcal{F}.$$

However, it should be noted that $A\mathcal{F}$ need not have the structure of a smooth vector bundle and, indeed, the dimensions of its fibers may vary from point to point. Note also that, for each $x \in M$, the evaluation map $\mathcal{F} \to E_x$ contains $I_x\mathcal{F}$ in its kernel and therefore descends to a well-defined map $A_x\mathcal{F} \to E_x$ on the fiber. Thus, the “singular bundle” $A\mathcal{F}$ is equipped with a “singular bundle map” $A\mathcal{F} \to E$. Furthermore, this bundle map sends $\mathcal{F}$, viewed as a module of sections of $A\mathcal{F}$, identically onto $\mathcal{F}$, viewed as a module of sections of E.

Definition 2.13. A submodule $\mathcal{F} \subseteq C_c^\infty(M; E)$ is *finitely-generated* if there exist sections $X_1, \ldots, X_n \in C^\infty(M; E)$ such that every $X \in \mathcal{F}$ has a representation $X = f_1X_1 + \cdots + f_nX_n$ with $f_i \in C_c^\infty(M)$ and *free of rank n* if the latter representations are also unique.

Remark 2.14. Note that, in the above definition, the generators are not required to be compactly-supported. So, for example, we consider $\mathfrak{X}_c(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n; T\mathbb{R}^n)$, the $C^\infty(\mathbb{R}^n)$-module of compactly-supported vector fields on $\mathbb{R}^n$, to be freely-generated by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, even though $\frac{\partial}{\partial x_i} \notin \mathfrak{X}_c(\mathbb{R}^n)$.

Definition 2.15. Given an open set $U \subseteq M$, the *restriction* of a submodule $\mathcal{F} \subseteq C_c^\infty(M; E)$ to U is defined to be the $C^\infty(U)$-module $\mathcal{F}_U \subseteq C_c^\infty(U; E_U)$ given by $\mathcal{F}_U := C_c^\infty(U)\mathcal{F}$.

We say that $\mathcal{F}$ is *locally finitely-generated* (resp. *locally free of rank* n) if each point of M belongs to some open set U such that $\mathcal{F}_U$ is finitely-generated (resp. free of rank n).

If $\mathcal{F}$ is locally finitely-generated, then each fiber $A_x\mathcal{F}$ is a finite-dimensional vector space. Furthermore, the dimension of $A_x\mathcal{F}$ equals the minimum number of generators required for $\mathcal{F}_U$, when U is any sufficiently small neighbourhood of x ([2, Proposition 1.5]).

Let us now say a bit more about the locally free case.

Proposition 2.16. *Let $\mathcal{F}$ be a locally finitely-generated $C^\infty(M)$-submodule of the module $C_c^\infty(M; E)$. Then, the following are equivalent:*

(1) $\dim(A_x\mathcal{F}) = k$ *for all* $x \in M$*;*

(2) $\mathcal{F}$ *is locally free of rank* k*;*

(3) *there exists a k-dimensional smooth vector bundle $A \to M$ and a vector bundle map $A \to E$ which is injective over a dense subset of M such that the image of the induced map $C_c^\infty(M; A) \to C_c^\infty(M; E)$ is $\mathcal{F}$.*

Moreover, when these equivalent conditions hold, we may take $A = A\mathcal{F}$, equipped with the unique smooth structure for which $\mathcal{F}$, realized as a module of sections of $A\mathcal{F}$, coincides with $C_c^\infty(M; A\mathcal{F})$.

Proof. If (2) holds, then (a version of) the Serre–Swan theorem gives that $A\mathcal{F}$ is a vector bundle with respect to the unique smooth structure for which $\mathcal{F}$, realized as a module of sections of $A\mathcal{F}$, coincides with $C_c^\infty(M; A\mathcal{F})$. From this, it is simple to deduce (1), (3), and the "moreover" statement.

Suppose (3) holds. Note the "almost injectivity" assumption on the bundle map $A \to E$ is equivalent to injectivity of the induced map $C_c^\infty(M; A) \to C_c^\infty(M; E)$. Therefore, $\mathcal{F}$ is isomorphic to $C_c^\infty(M; A)$ as a $C^\infty(M)$-module and (2) follows.

Finally, assume (1) holds. Let $U \subseteq M$ be open and $X_1, \dots, X_k \in C_c^\infty(U; E_U)$ be generators for $\mathcal{F}_U$. Suppose $f_1, \dots, f_k \in C_c^\infty(U)$ have $f_1X_1 + \cdots + f_kX_k = 0$. Then, for any $x \in U$, we get $f_1(x)[X_1]_x + \cdots f_k(x)[X_k]_x$. Since the k vectors $[X_i]_x$ span the k-dimensional vector space $A_x\mathcal{F}$, they are a basis, and we obtain $f_1(x) = \cdots = f_k(x) = 0$. This shows that (2) holds. ∎

2.6. Foliations

In this section, we precisely define what will be meant by the word "foliation" and discuss related notions and constructions.

Definition 2.17 ([2, Definition 1.1]). A *foliation* $\mathcal{F}$ of a smooth manifold M is a locally finitely-generated $C^\infty(M)$-module of compactly-supported vector fields on M that is furthermore stable under taking Lie brackets.

The choice to work with compactly-supported vector fields, though not totally essential, is convenient in several ways. Flows of compactly-supported vector fields are automatically complete. Furthermore, a compactly-supported vector field defined on an open subset can always be extended by zero to the whole space. Alternative approaches include dropping the compact-support assumption altogether or working with the sheaf of locally-defined vector fields. The approach via compactly-supported vector fields is something of a compromise between a fully global and fully local approach.

Definition 2.18 ([2, Definition 1.7]). A *leaf* of a foliation $(M, \mathcal{F})$ is an orbit of $\exp(\mathcal{F})$, the group of diffeomorphisms of M generated by $\exp(X)$, $X \in \mathcal{F}$.

By work of Stefan and Sussmann ([28, 29]), the leaves of a foliation $(M, \mathcal{F})$ constitute a partition of M into immersed submanifolds. See [2, Section 1.3] for more detailed information on the leafwise smooth structure.[3]

Example 2.19. If A is any Lie algebroid over a smooth manifold M, then the image of the map $C_c^\infty(M; A) \to \mathfrak{X}_c(M)$ induced by the anchor map is a foliation of M. Presently, it seems not to be known whether in fact *all* foliations can (perhaps only locally) be obtained in this way. It was noted in [3, Proposition 1.3] that one can construct foliations $(M, \mathcal{F})$, with M noncompact, such that $\sup_{x \in M} \dim(A_x\mathcal{F}) = \infty$. Obviously, such a foliation cannot be induced by a single Lie algebroid on M. However, it appears to not be known whether a foliation whose fibers are bounded in dimension (which always happens if M is compact) must be induced by a Lie algebroid, nor does the local version of this question seem to be settled. The article [17] contains some partial work on this problem; see [17, Proposition 4.33].

Example 2.20. Specializing the above example, any Lie groupoid $G \rightrightarrows M$ induces a foliation by way of its Lie algebroid. If G is s-connected, the leaves of the foliation induced by G are exactly the orbits of G.

Definition 2.21. Let $(M, \mathcal{F})$ be a foliation and let $x \in M$.

- The *fiber* of $\mathcal{F}$ at x is the finite-dimensional vector space $A_x\mathcal{F} := \mathcal{F}/I_x\mathcal{F}$ (this is a particular case of Definition 2.12).
- The *tangent space* of $\mathcal{F}$ at x is the subspace $T_x\mathcal{F} := \{X(x) : X \in \mathcal{F}\}$ of T_xM.
- The *isotropy Lie algebra* of $\mathcal{F}$ at x is the kernel $\mathfrak{g}_x\mathcal{F}$ of the surjective linear map $A_x\mathcal{F} \to T_x\mathcal{F}$ that descends from the evaluation map $\mathcal{F} \to T_x\mathcal{F}$.

Consequent to these definitions, for each point $x \in M$, there is an exact sequence

$$0 \to \mathfrak{g}_x\mathcal{F} \to A_x\mathcal{F} \to T_x\mathcal{F} \to 0.$$

As the notation and terminology would suggest, $\mathfrak{g}_x\mathcal{F}$ is a Lie algebra. The bracket on $\mathcal{F}$ descends to a well-defined bracket on $\mathfrak{g}_x\mathcal{F}$.

[3] There is an unimportant error in [2, Remark 1.15 (1)]. Let $N = \mathbb{R}$ and let $M = \mathbb{R}$ with the singular foliation singly-generated by $x\frac{d}{dx}$. Define $f : N \to M$ by $f(x) = x^3$. Then $(df_x)(T_xN) \subseteq F_{f(x)}$ is satisfied, but f is not leafwise in the sense of [2].

2.7. Regular and almost regular foliations

Definition 2.22. A foliation $(M, \mathcal{F})$ is called *regular* if the dimensions of its tangent spaces $T_x\mathcal{F}$, $x \in M$ are constant. A foliation which is not regular is said to be *singular.* A foliation is called *almost regular* if the dimensions of its fibers $A_x\mathcal{F}$, $x \in M$ are constant.

A foliation is regular if and only if its leaves all have the same dimension. For a regular foliation, the tangent spaces $T_x\mathcal{F}$, $x \in M$ form a subbundle of TM and $\mathcal{F}$ is equal to the compactly-supported sections of this subbundle. See [2, Example 1.3 (2)] for further details.

Every regular foliation is almost regular. As explained in Proposition 2.16, if $\mathcal{F}$ is almost regular, $A\mathcal{F} = \bigsqcup_{x\in M} A_x\mathcal{F}$ is a vector bundle. Indeed, transferring the bracket of $\mathcal{F}$ to $C_c^\infty(M; A\mathcal{F})$ makes $A\mathcal{F}$ into Lie algebroid whose anchor map is moreover injective on a dense open subset of M. In fact, one may equivalently define almost regular foliations as precisely the ones arising from a Lie algebroid with an almost injective anchor map. See also the discussion in [2, Section 3.2].

2.8. Pullbacks and automorphisms

Foliations can be pulled back by submersions (or, more generally, by maps satisfying an appropriate transversality assumption; see [2, Definition 1.9]).

Definition 2.23. If $(N, \mathcal{E})$ is a foliation and $p\colon M \to N$ is a submersion, the *pullback foliation* $p^{-1}(\mathcal{E})$ is the foliation of M consisting of $C_c^\infty(M)$-linear combinations of vector fields on M which are p-projectable and project to elements of $\mathcal{E}$. In particular, if ι is the inclusion of an open set U into M, we write $\iota^{-1}(\mathcal{F}) = \mathcal{F}_U$ and call $\mathcal{F}_U$ the *restriction* of $\mathcal{F}$ to U.

It is clear that a foliation can be pushed forward or pulled back by a diffeomorphism, simply by pushing forward or pulling back its constituent vector fields. Indeed, this may be considered a special case of pullback by a submersion. We use the following terminology and notations.

Definition 2.24. Let $(M, \mathcal{F})$ be a foliation.

- An *$\mathcal{F}$-automorphism* is a diffeomorphism $\theta\colon M \to M$ satisfying $\theta_*(\mathcal{F}) = \mathcal{F}$. We denote the group of $\mathcal{F}$-automorphisms by $\mathrm{Aut}(\mathcal{F})$.
- A *local $\mathcal{F}$-automorphism* is a diffeomorphism $\theta\colon U \to V$, where U and V are open subsets of M, satisfying $\theta_*(\mathcal{F}_U) = \mathcal{F}_V$. The collection of all local $\mathcal{F}$-automorphisms is a pseudogroup.
- We write $\mathrm{GermAut}(\mathcal{F})$ for the groupoid over the base M consisting of germs of local $\mathcal{F}$-automorphisms.

One has that $\exp(\mathcal{F})$ (Definition 2.18) is a normal subgroup of $\mathrm{Aut}(\mathcal{F})$ (see [2, Proposition 1.6]).

2.9. Gluing foliations

Even though we never view *individual* foliations as sheaves, in Section 10 it will be useful for us to know that one can compare or construct foliations on the same manifold using a sheaf property. The proof, which we omit, is a routine verification using partitions of unity.

Proposition 2.25. *Let M be a smooth manifold and let $(U_i)_{i\in I}$ be an open cover of M.*

(1) *If $\mathcal{E}$ and $\mathcal{F}$ are foliations of M and $\mathcal{E}_{U_i} = \mathcal{F}_{U_i}$ for all $i \in I$, then $\mathcal{E} = \mathcal{F}$.*

(2) *Suppose $\mathcal{F}_i$ is a foliation of U_i for each $i \in I$. If $(\mathcal{F}_i)_{U_i\cap U_j} = (\mathcal{F}_j)_{U_i\cap U_j}$ is satisfied for all $i, j \in I$, then there exists a (unique by (1)) foliation $\mathcal{F}$ of M such that $\mathcal{F}_{U_i} = \mathcal{F}_i$ for all $i \in I$.* ■

2.10. Bisubmersions

The following definition is the basic ingredient in Androulidakis and Skandalis's construction of the holonomy groupoid.

Definition 2.26 ([2, Definition 2.1]). An $\mathcal{F}$*-bisubmersion* of a foliation $(M, \mathcal{F})$ is a triple (W, t, s) where W is a smooth manifold and $s, t\colon W \to M$ are submersions satisfying $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C_c^\infty(W; \ker(ds)) + C_c^\infty(W; \ker(dt))$.

We sometimes abuse notation and denote an $\mathcal{F}$-bisubmersion (W, t, s) simply by W. It is easy to see that, if $U \subseteq W$ is open, then $(U, t|_U, s|_U)$ is also an $\mathcal{F}$-submersion.

Definition 2.27. A *morphism* of $\mathcal{F}$-bisubmersions (W_1, t_1, s_1), (W_2, t_2, s_2) is a smooth map $f\colon W_1 \to W_2$ such that $s_1 = s_2 \circ f$ and $t_1 = t_2 \circ f$. A *local morphism* from W_1 to W_2 is a morphism from an open subset of W_1 to W_2.

It is clear that morphisms of bisubmersions can be composed and that the identity is always a morphism of bisubmersions. In [2, Corollary 2.11 (c)], it is shown that, if there is a local morphism of bisubmersions $W_1 \to W_2$ sending $w_1 \mapsto w_2$, then there is also a morphism of bisubmersions $W_2 \to W_1$ sending $w_2 \mapsto w_1$. The following definition is therefore justified.

Definition 2.28. Let $(M, \mathcal{F})$ be a foliation, and $(W_i)_{i\in I}$ be a collection of $\mathcal{F}$-bisubmerions. Then, we denote by $\sim$ the equivalence relation on $\bigsqcup_{i\in I} W_i$ given by $W_i \ni w_i \sim w_j \in W_j$ if and only if there exists a local morphism from W_i to W_j sending w_i to w_j. We denote the quotient map $\bigsqcup_{i\in I} W_i \to \big(\bigsqcup_{i\in I} W_i\big)/\sim$ by $Q = (Q_i)_{i\in I}$.

Definition 2.29. Let $(M, \mathcal{F})$ be a foliation and let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be collections of $\mathcal{F}$-bisubmersions. Put $U = \bigsqcup_{i\in I} U_i$ and $V = \bigsqcup_{j\in J} V_j$. We say that $\mathcal{U}$ is *adapted* to $\mathcal{V}$ if every $u \in U$ is $\sim$ to some $v \in V$. We say that $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* if they are adapted to each other.

The following simple proposition is helpful in clarifying certain issues relating to forming the quotient by $\sim$.

Proposition 2.30. *The quotient map Q of the above definition is an open map.*

Proof. Let $W := \bigsqcup_{i \in I} W_i$. We need to prove that the $\sim$-saturation of any open set $U \subseteq W$ is open. It suffices to consider the case where $U \subseteq W_i$ for some $i \in I$. To this end, suppose $w \in U$, $w' \in W_j$ for some $j \in I$ and $w \sim w'$. Therefore, there exists a local morphism f from W_j to W_i with $f(w') = w$. Then, $f^{-1}(U)$ is a neighbourhood of w' with the property that each point in $f^{-1}(U)$ is $\sim$-equivalent (by way of f) to a point in U. ∎

Corollary 2.31. *Let $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ be collections of bisubmersions of $(M, \mathcal{F})$. Put $U = \bigsqcup_{i \in I} U_i$ and $V = \bigsqcup_{j \in J} V_j$. If $\mathcal{U}$ is adapted to $\mathcal{V}$, then the map $U/\sim \to V/\sim$ sending $[u] \mapsto [v]$ whenever $u \sim v$ is an open embedding. If $\mathcal{U}$ and $\mathcal{V}$ are equivalent, this map $U/\sim \to V/\sim$ is a homeomorphism.*

Proof. Note that the restriction of an open mapping to an open set is an open mapping. In particular, this can be applied to U or V sitting in $U \sqcup V$. It follows that both $U/\sim$ and $V/\sim$ sit as open subsets in $(U \sqcup V)/\sim$. If $\mathcal{U}$ is adapted to $\mathcal{V}$, then V meets every equivalence class in $U \sqcup V$, so that $V/\sim$ is homeomorphic to $(U \sqcup V)/\sim$. If $\mathcal{U}$ and $\mathcal{V}$ are equivalent, then $U/\sim$ is homeomorphic to $(U \sqcup V)/\sim$ as well. ∎

Remark 2.32. Note that, in general, the restriction of a quotient map of topological spaces to an open set which meets every equivalence class is not a quotient map. For example, the map $[0, 1] \to S^1 : t \mapsto e^{2\pi i t}$ is a quotient map, but its restriction to $[0, 1)$ is not.

2.11. Construction of the holonomy groupoid

There are natural notions of inverse and composition for bisubmersions. See [2, Proposition 2.4].

Definition 2.33. Let $(M, \mathcal{F})$ be a foliation.

- The *composition* of two $\mathcal{F}$-bisubmersions W_1, W_2 is the $\mathcal{F}$-bisubmersion $W_2 \circ W_1$ whose underlying manifold is the fiber product $W_2 \,{}_{s_2}\!\times_{t_1} W_1$ and whose source and target maps are given by $s(w_2, w_1) = s_1(w_1)$ and $t(w_2, w_1) = t_2(w_2)$, in an obvious notation.
- The *inverse* of an $\mathcal{F}$-bisubmersion W is the $\mathcal{F}$-bisubmersion W^{-1} obtained by keeping the same underlying manifold, but interchanging the source and target maps.

Definition 2.34 ([2, Definition 3.1]). Let $(M, \mathcal{F})$ be a foliation. A collection $\mathcal{W} = (W_i)_{i \in I}$ of $\mathcal{F}$-bisubmersions is called a *holonomy atlas* provided that

(i) $\bigcup_{i \in I} s_i(W_i) = M$;

(ii) if $W \in \mathcal{W}$ and $w \in W$, then there exists $W' \in \mathcal{W}$ and $w' \in W'$ such that $w \in W^{-1}$ is $\sim$ to $w' \in W'$;

(iii) if $W_1, W_2 \in \mathcal{W}$, $w_1 \in W_1$, $w_2 \in W_2$, and $s_1(w_1) = t_2(w_2)$, then there exists $W' \in \mathcal{W}$ and $w' \in W'$ such that $(w_1 \circ w_2) \in W_1 \circ W_2$ is $\sim$ to $w' \in W'$.

Items (ii) and (iii) amount to saying $\mathcal{W}$ is closed under inverse and composition, if we work up to $\sim$.

Working up to equivalence (Definition 2.29), every foliation $(M, \mathcal{F})$ has a smallest atlas $\mathcal{W}_{\text{path}}$ called the *path holonomy atlas* ([2, Example 3.4.3]) and a largest atlas $\mathcal{W}_{\text{full}}$ called the *full holonomy atlas* ([2, Example 3.4.1]). The path holonomy atlas is generated by an crucial family of bisubmersions introduced by Androulidakis and Skandalis in [2, Proposition 2.10]. The full holonomy atlas is, effectively, the atlas of all bisubmersions. If $\mathcal{W}$ is any holonomy atlas for $(M, \mathcal{F})$, then $\mathcal{W}_{\text{path}}$ is adapted to $\mathcal{W}$ and $\mathcal{W}$ is adapted to $\mathcal{W}_{\text{full}}$.

Theorem 2.35 ([2, Proposition 3.2]). *Suppose $\mathcal{W} = (W_i)_{i \in I}$ is a holonomy atlas for a foliation $(M, \mathcal{F})$. Let $G(\mathcal{W}) := \left(\bigsqcup_{i \in I} W_i\right)/\sim$. Then there is a groupoid structure on $G(\mathcal{W})$ such that*

$$Q_{W_2}(w_2) Q_{W_1}(w_1) = Q_{W_2 \circ W_1}(w_2, w_1)$$

whenever W_1 and W_2 are bisubmersions adapted to $\mathcal{W}$ and $w_1 \in W_1$, $w_2 \in W_2$ are such that $s_2(w_2) = s_1(w_1)$. ■

Definition 2.36 ([2, Definition 3.5, Example 3.4 (1)]). Let $(M, \mathcal{F})$ be a foliation with path holonomy atlas $\mathcal{W}_{\text{path}}$ and full holonomy atlas $\mathcal{W}_{\text{full}}$. Then, the *path holonomy groupoid*, or simply the *holonomy groupoid*, of $\mathcal{F}$ is $G(\mathcal{F}) := G(\mathcal{W}_{\text{path}})$. Similarly, the *full holonomy groupoid* of $\mathcal{F}$ is $G_{\text{full}}(\mathcal{F}) := G(\mathcal{W}_{\text{full}})$.

Using Corollary 2.31, given any holonomy atlas $\mathcal{W}$ for $\mathcal{F}$, there are canonical open inclusions $G(\mathcal{F}) \subseteq G(\mathcal{W}) \subseteq G_{\text{full}}(\mathcal{F})$.

2.12. Bisections

Definition 2.37. A *bisection* of a bisubmersion (W, t, s) of a foliation $(M, \mathcal{F})$ is a locally closed submanifold $N \subseteq W$ such that the restrictions of s and t to N are diffeomorphisms onto open subsets of M.

Proposition 2.38. *Suppose N is a bisection of a bisubmersion (W, t, s) of a foliation $(M, \mathcal{F})$. Then, $t|_N \circ (s|_N)^{-1}$ is a local $\mathcal{F}$-automorphism.*

Proof. See [2, Proposition 2.9]. ■

Definition 2.39. Suppose that (W, t, s) is a bisubmersion of a foliation $(M, \mathcal{F})$. We say that a local $\mathcal{F}$-automorphism θ is *carried* by (W, t, s) at a point $w \in W$ if there is a bisection N of (W, t, s) with $w \in N$ such that θ has the same germ as $t|_N \circ (s|_N)^{-1}$ at $s(w)$.

2.13. Holonomy transformation picture of full holonomy groupoid

In [4], it is shown that the holonomy groupoid of [2] can be realized as a groupoid of (equivalence classes of) holonomy transformations of a family of transversal slices. It is pointed out in [4, Remark 2.10 (c)] that the use of slices is essential because of an issue which can arise from nonorientable leaves. In this section, we lay out a rather cheap way to realize the groupoid of [2] without introducing slices which, though not very different from the original description in terms of bisubmersions, still has some of the flavour of a description by holonomy transformations.

Proposition 2.40. *Let W_1, W_2 be bisubmersions of $(M, \mathcal{F})$ and fix $w_i \in W_i$. If $w_1 \sim w_2$, then the set of local automorphisms carried by W_1 at w_1 is exactly the equal to the set of local automorphisms carried by W_2 at w_2. Conversely, if there exists a local automorphism carried at both w_1 and w_2, then $w_1 \sim w_2$.*

Proof. Suppose $w_1 \sim w_2$ and let f be a local morphism with $f(w_1) = w_2$. Let $N_1 \subseteq W_1$ be a bisection with $w_1 \in N_1$. Since N_1 is a section of s_1, we have $T_w W_1 = T_w N_1 \oplus \ker(ds_1)_w$ for all $w \in N_1$. From $s_1 = s_2 \circ f$, one may deduce that $T_w N_1 \cap \ker(df)_w = \{0\}$, and that $df_w(T_w N_1) \cap \ker(ds_2) = \{0\}$. Thus, $f|_{N_1}$ is an immersion which is transverse to s_2. In the same way, $f|_{N_1}$ is transverse to r_2. It follows that the image of a small neighbourhood of w_1 in N_1 is a bisection $N_2 \subseteq W_2$ with $w_2 \in N_2$. Clearly, the local diffeomorphisms induced by N_1 and N_2 have the same germ at $s_1(w_1) = s_2(w_2)$, and we obtain that every local automorphism carried at w_1 is also carried at w_2. Interchanging the roles of w_1 and w_2, we get that same local automorphisms are carried at the two points. The converse statement is exactly [2, Corollary 2.11 (b)]. ∎

By the above proposition, the following defines an equivalence relation.

Definition 2.41. Let $(M, \mathcal{F})$ be a foliation. Let θ_1 and θ_2 be germs at $x \in M$ of local $\mathcal{F}$-automorphisms with $\theta_1(x) = \theta_2(x)$. We write $\theta_1 \approx \theta_2$ if there exists an $\mathcal{F}$-bisubmersion W and a point $w \in W$ such that both θ_1 and θ_2 are carried by W at w.

Definition 2.42. Let $(M, \mathcal{F})$ be a foliation. A local $\mathcal{F}$-automorphism θ_0 is *null* at $x \in M$ if there exists an $\mathcal{F}$-bisubmersion W and a point $w \in W$ with $s(w) = x$ such that both θ_0 and id_M are carried by W at w. We denote the group of germs at x of local $\mathcal{F}$-automorphisms which are null at x by $\mathrm{NullAut}(\mathcal{F})_x$ and put $\mathrm{NullAut}(\mathcal{F}) := \bigsqcup_{x \in M} \mathrm{NullAut}(\mathcal{F})_x$.

Theorem 2.43. *Let $(M, \mathcal{F})$ be a foliation. Then $\mathrm{NullAut}(\mathcal{F})$ is a normal subgroupoid of $\mathrm{GermAut}(\mathcal{F})$ and there is an abstract groupoid isomorphism*

$$G_{\mathrm{full}}(\mathcal{F}) \to \mathrm{GermAut}(\mathcal{F}) / \mathrm{NullAut}(\mathcal{F})$$

such that, if $W = (W, t, s)$ is an $\mathcal{F}$-bisubmersion and $w \in W$, then $Q_W(w) \in G_{\mathrm{full}}(\mathcal{F})$ is mapped to the germ at $s(w)$ of any local $\mathcal{F}$-automorphism that is carried by W at w.

Proof. That this map is a bijection follows from Proposition 2.40. Multiplication and inversion are preserved by [2, Proposition 2.8]. ∎

2.14. Smoothness of the holonomy groupoid of an almost regular foliation

The holonomy groupoid constructed in [2] can be quite poorly-behaved for general foliations. Almost regular foliations are precisely the foliations whose holonomy groupoids are Lie groupoids. This case was previously treated by Debord in [7], with a different approach than that of [2]. The following result follows from [2, Section 3.2] and can also be obtained in a more direct manner by adapting the arguments in [9].

Proposition 2.44. *Let M be an n-dimensional smooth manifold, and $\mathcal{F}$ an almost regular singular foliation of M with constant fiber dimension k. Then there is a unique smooth structure on $G_{\text{full}}(\mathcal{F})$ such that, for any $\mathcal{F}$-bisubmersion W, the map $Q_W\colon W \to G_{\text{full}}(\mathcal{F})$ is smooth. The groupoid operations are smooth with respect to this smooth structure.* ■

When $\mathcal{F}$ is almost regular, $\mathcal{A}(\mathcal{F})$ is isomorphic to $C_c^\infty(G(\mathcal{F}))$, the smooth convolution algebra of the groupoid, and $C^*(\mathcal{F})$ is isomorphic to $C^*(G(\mathcal{F}))$, the C*-algebra of the groupoid. Here we are implicitly fixing a smooth Haar system on $G(\mathcal{F})$ in order to make sense of convolution and bypass any discussion of densities.

3. Groups of jets on the line

In this section we introduce certain groups of jets of diffeomorphisms of the real line which will play an important role. Let us briefly recall the concept of the jet of a smooth mapping. For more information, one may refer the exposition in [15, Section 12].

Definition 3.1. Let M be a smooth manifold and f be a smooth real-valued function on M. Given $x_0 \in M$ and r a positive integer, we say that f *vanishes to order* **r** at x_0 if $f \in (I_{x_0})^r$, where $I_{x_0} \subseteq C^\infty(M, \mathbb{R})$ denotes the ideal of functions which vanish at x_0.

Lemma 3.2. *Let M, N be smooth manifolds, let r be a nonnegative integer and let $x_0 \in M$, $y_0 \in N$. Choose a diffeomorphism $\phi = (\phi_1, \dots, \phi_n)$ from an open neighbourhood $V \subseteq N$ of y_0 onto an open set in $\mathbb{R}^n$ and define an equivalence relation $\sim_{r,x_0}$ on the set of smooth functions $M \to N$ that send $x_0 \mapsto y_0$ by $f \sim_{r,x_0} g$ if and only if $\phi_i \circ f - \phi_i \circ g$ vanishes to order $r+1$ at x_0 for $i = 1, \dots, n$. Then, the equivalence relation $\sim_{r,x_0}$ does not depend on the choice of chart ϕ.* ■

Definition 3.3. Let M, N be smooth manifolds and $f\colon M \to N$ a smooth function. Given $x_0 \in M$ and r a nonnegative integer, the **r**-*jet* of f at x_0 is the equivalence class $j^r_{x_0}(f)$ of f under the relation $\sim_{r,x_0}$ of above lemma.

If $M = \mathbb{R}^n$ and $N = \mathbb{R}$ in the above definition, then $f \sim_{r,x_0} g$ if and only if the r-th order Taylor polynomials of f and g at x_0 are the same. For this reason, it makes sense to identify $j^r_{x_0}(f)$ with the r-th order Taylor polynomial of f. We shall make such identifications without comment.

There is a well-defined composition operation on jets. If $f, f'\colon M_1 \to M_2$ have the same r-jet at x_0 in M_1 and $g, g'\colon M_2 \to M_3$ have the same r-jet at $y_0 := f(x_0) = f'(x_0)$ in M_2, then $g \circ f$ and $g' \circ f'$ have the same r-jet at x_0. It therefore makes sense to define $j^r_{y_0}(g) \circ j^r_{x_0}(f) := j^r_{x_0}(g \circ f)$. When working in coordinates, this operation on jets is the usual "compose and truncate" operation on r-th order Taylor polynomials.

The following groups of jets will play an important role for us.

Definition 3.4. For r a positive integer, J^r denotes the group of r-jets at 0 of diffeomorphisms of $\mathbb{R}$ fixing 0.

The group J^r has a canonical r-dimensional Lie group structure coming from its identification with the group of real polynomials of the form $a_1 y + \cdots + a_r y^r$ where $a_1 \neq 0$ with respect to the "compose and truncate" operation.

For each $k \geq 2$, there is canonical exact sequence of Lie groups

$$0 \to \mathbb{R} \to J^k \to J^{k-1} \to 0$$

where the projection map $J^k \to J^{k-1}$ is given by deleting the order k term and the inclusion map $\mathbb{R} \to J^k$ is defined by $t \mapsto y + ty^k$.

We will frequently want to equip these jet groups with a nonstandard topology.

Definition 3.5. For $r \geq 1$, we write J^r_d for J^r considered as an (uncountable) discrete group. For $k \geq 2$, we write $J^k_{\mathbb{R}}$ for J^k equipped with the one-dimensional Lie group structure arising from its partition into the cosets of $\mathbb{R}$ in J^k (of which there are uncountably many).

We then have an extension of (non-second-countable) Lie groups

$$0 \to \mathbb{R} \to J^k_{\mathbb{R}} \to J^{k-1}_d \to 0,$$

where $\mathbb{R}$ has its standard smooth structure.

The following proposition is intended to show that, from the perspective of abstract group theory, these jet groups are quite tame.

Proposition 3.6. *For every $k \geq 2$, the group J^k is solvable. Indeed, we may express J^k as the semidirect product of a nilpotent group by an abelian group.*

Proof. For every $k \geq 2$, there is an evident exact sequence

$$0 \to J^{2,k} \to J^k \to J^1 \to 0$$

where $J^{2,k} := \{y + a_2 y^2 + \cdots + a_k y^k : a_i \in \mathbb{R}\}$. This sequence is split on the right by the map $ay \mapsto ay$, so J^k is the semidirect product of $J^{2,k}$ by the abelian group J^1. For $k = 2$, we have $J^{2,k} \cong \mathbb{R}$. For $k \geq 3$, we have a central extension

$$0 \to \mathbb{R} \to J^{2,k} \to J^{2,k-1} \to 0.$$

By induction, $J^{2,k}$ is nilpotent for all $k \geq 2$. ∎

The groups J^k for $k \geq 2$ are not themselves nilpotent. In fact, the center of J^k is trivial. It is interesting to notice that J^2 is isomorphic to the "ax+b group" of affine bijections of the real line. An example of an isomorphism is $ay + b \mapsto a^{-1}y + ba^{-2}y^2$. This is conceptually related to the fact that the inversion map $y \mapsto \frac{1}{y}$ conjugates $y^2\frac{d}{dy}$ to $-\frac{d}{dy}$.

Remark 3.7. It will later be relevant to take a countable subgroup $\Gamma \subseteq J_d^{k-1}$ and consider its preimage $\Gamma_{\mathbb{R}} \subseteq J_{\mathbb{R}}^k$ under the projection $J_{\mathbb{R}}^k \to J_d^{k-1}$. Because subgroups of solvable groups are solvable, the above proposition gives that the one-dimensional group $\Gamma_{\mathbb{R}}$ is second-countable and amenable. One then has that extensions of $C^*(\Gamma_{\mathbb{R}})$ can be described in terms of K-theoretic data (by the universal coefficient theorem) and that the K-theory of $C^*(\Gamma_{\mathbb{R}})$ can be geometrically computed (by the Baum–Connes conjecture).

4. The full holonomy groupoid of $\mathcal{F}_{\mathbb{R}}^k$

In this section, we describe the full holonomy groupoid (Definition 2.35) $G_{\text{full}}(\mathcal{F}_{\mathbb{R}}^k)$ of $\mathcal{F}_{\mathbb{R}}^k$ and discuss its point-set topological properties. When $k = 1$, it is simple to see that $G_{\text{full}}(\mathcal{F}_{\mathbb{R}}^1) \cong \mathrm{GL}(1,\mathbb{R}) \ltimes \mathbb{R}$, so we concentrate our discussion on the case $k \geq 2$ where the groupoid is larger. The minimal holonomy groupoid of $\mathcal{F}_{\mathbb{R}}^k$ was discussed in [11]; there is a unique Lie groupoid isomorphism

$$G(\mathcal{F}_{\mathbb{R}}^k) \cong \mathbb{R} \ltimes_\phi \mathbb{R},$$

where ϕ denotes the flow of any complete vector field X generating $\mathcal{F}_{\mathbb{R}}^k$. It will be more convenient, however, to replace $\mathbb{R} \ltimes_\phi \mathbb{R}$ by an isomorphic bisubmersion with polynomial structure maps.

Definition 4.1. Let $\Omega = \{(t,y) \in \mathbb{R}^2 : 1 + ty^{k-1} > 0\}$ and define $\sigma, \tau \colon \Omega \to \mathbb{R}$ by $\sigma(t,y) = y$ and $\tau(t,y) = y + ty^k$.

Note the inequality $1 + ty^{k-1} > 0$ is precisely the condition guaranteeing $\sigma(t,y)$ and $\tau(t,y) = y + ty^k$ have the same sign.

Lemma 4.2. *The triple $\Omega = (\Omega, \tau, \sigma)$ is an $\mathcal{F}_{\mathbb{R}}^k$-bisubmersion adapted to the path holonomy atlas and the natural map $Q_\Omega \colon \Omega \to G(\mathcal{F}_{\mathbb{R}}^k)$ is a diffeomorphism.*

Proof. Identify $G(\mathcal{F}_{\mathbb{R}}^k)$ with $\mathbb{R} \ltimes_\phi \mathbb{R}$, where ϕ is the flow of some complete vector field X on $\mathbb{R}$ generating $\mathcal{F}_{\mathbb{R}}^k$. We have $X = f(y)y^k\frac{d}{dy}$ where f is smooth and nonvanishing. From this, it follows that we can write

$$\phi_t(y) = y + h(t,y)y^k, \quad t, y \in \mathbb{R}, \tag{4}$$

where $h \colon \mathbb{R}^2 \to \mathbb{R}$ is smooth. We claim that

$$(t,y) \mapsto (h(t,y), y) \colon \mathbb{R} \ltimes_\phi \mathbb{R} \to \Omega$$

defines an isomorphism of bisubmersions. When $y \neq 0$, note that $\{t \in \mathbb{R} : 1 + ty^{k-1} > 0\}$ equals the set of all $t \in \mathbb{R}$ such that $y + ty^k$ has the same sign as y. Since ϕ is free and transitive on the positive and negative half lines, it follows that $(t, y) \mapsto (h(t, y), y)$ is a bijection from $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$ to $\Omega \setminus (\mathbb{R} \times \{0\})$.

Differentiating (4) with respect to t and rearranging gives

$$\frac{\partial h}{\partial t}(t, y) = \left(\frac{\phi_t(y)}{y}\right)^k > 0 \quad t, y \in \mathbb{R}, y \neq 0,$$

so $(t, y) \mapsto (h(t, y), y)$ is a diffeomorphism from $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}) \to \Omega \setminus (\mathbb{R} \times \{0\})$.

Finally, for the sake of simplicity, choose X to coincide with $y^k \frac{d}{dy}$ on a neighbourhood of 0. Then,

$$\phi_t(y) = \frac{y}{\sqrt[k-1]{1 - (k-1)ty^{k-1}}}$$

holds on a neighbourhood of $\mathbb{R} \times \{0\}$. In particular, the Taylor series of ϕ_t begins $\phi_t(y) \sim y + ty^k + \frac{1}{2}t^2 y^{2k-1} + \cdots$ and we have $h(t, 0) = t$ for all $t \in \mathbb{R}$. The rest follows. ■

By Theorem 2.43, the full holonomy groupoid of a foliation $(M, \mathcal{F})$ is isomorphic to the groupoid $\mathrm{GermAut}(\mathcal{F})$ of germs of local $\mathcal{F}$-automorphisms modulo the normal subgroupoid $\mathrm{NullAut}(\mathcal{F})$ of germs of null automorphisms (Definition 2.42). Therefore, to determine $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ as an abstract groupoid, we need only determine the automorphisms and null automorphisms of $\mathcal{F}^k_{\mathbb{R}}$.

Lemma 4.3. *Let θ be a diffeomorphism of $\mathbb{R}$ defined on a neighbourhood of* 0.

(1) *θ preserves $\mathcal{F}^k_{\mathbb{R}}$ if and only if $\theta(0) = 0$.*

(2) *θ is null at* 0 *if and only if $j^k_0(\theta) = y$.*

Proof. If $f \in C^\infty_c(\mathbb{R})$, we have $\theta_*\left(f \frac{d}{dy}\right) = (f \circ \theta^{-1})\theta_*\left(\frac{d}{dy}\right)$. Since $f \mapsto \theta^{-1}$ preserves the ideal of functions that vanish to order k at 0 and $\theta_*\left(\frac{d}{dy}\right)$ is a positive, smooth function-multiple of $\frac{d}{dy}$, assertion (1) follows. If $j^k_0(\theta) = y$, we may write $\theta(y) = y + f(y)y^k$, where f is a smooth function with $f(0) = 0$. Then, $\{(f(y), y) : y \in (-\varepsilon, \varepsilon)\}$ is a bisection of Ω containing $(0, 0)$ inducing θ for small enough $\varepsilon > 0$. Conversely, any bisection passing through $(0, 0)$ is locally of the form $\{(f(y), y) : y \in (-\varepsilon, \varepsilon)\}$ for some smooth f, and (2) follows. ■

Proposition 4.4. *There is a unique isomorphism of abstract groupoids*

$$G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}}) \to (\mathbb{R} \setminus \{0\})^2 \cup J^k$$

such that, if (W, t, s) is any $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion and $w \in W$ has $s(w) = 0$, then $Q_W(w) \mapsto j^k_0(\theta)$ where θ is any diffeomorphism of $\mathbb{R}$ carried by W at w.

Proof. Clearly, the restriction of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ to $\mathbb{R} \setminus \{0\}$ is uniquely isomorphic to the pair groupoid $(\mathbb{R} \setminus \{0\})^2$. By Lemma 4.3, the group $\mathrm{GermAut}(\mathcal{F}^k_{\mathbb{R}})_0$ of germs at 0 of local $\mathcal{F}^k_{\mathbb{R}}$

automorphisms is the group of germs of diffeomorphisms θ of $\mathbb{R}$ with $\theta(0) = 0$ and the normal subgroup $\mathrm{NullAut}(\mathcal{F}^k_{\mathbb{R}})_0$ of germs of null automorphisms at 0 is the group of germs of diffeomorphisms θ of $\mathbb{R}$ whose k-jet at 0 is y. Thus, by Theorem 2.43, $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})_0$ is isomorphic to $\mathrm{GermAut}(\mathcal{F}^k_{\mathbb{R}})_0 / \mathrm{NullAut}(\mathcal{F}^k_{\mathbb{R}})_0 = J^k$. ∎

Remark 4.5. The orbits of the minimal holonomy groupoid $G(\mathcal{F})$ of a foliation $(M, \mathcal{F})$ are exactly the leaves of $\mathcal{F}$. Similarly, if $\mathcal{W}$ is any holonomy atlas for $\mathcal{F}$, then the orbits of the holonomy groupoid $G(\mathcal{W})$ are unions of leaves related by $\mathcal{W}$. This explains why $G(\mathcal{F}^k_{\mathbb{R}})$ is a blow up of $(\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2 \cup \{(0,0)\}$ and $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is a blowup of $(\mathbb{R} \setminus \{0\})^2 \cup \{(0,0)\}$.

It is quite easy to see that the above identification of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})_{\mathbb{R}\setminus\{0\}}$ with $(\mathbb{R} \setminus \{0\})^2$ is also a diffeomorphism. However, we shall see that the isotropy group $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})_0$ is in fact diffeomorphic to the one-dimensional Lie group $J^k_{\mathbb{R}}$, rather than the k-dimensional Lie group J^k. In order to illuminate the topological structure of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ near its isotropy group $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})_0$, we need to introduce an explicit full holonomy atlas for $\mathcal{F}^k_{\mathbb{R}}$. We use freely the results and terminology of Section 2.11.

Definition 4.6. For each $\theta \in \mathrm{Diff}_0(\mathbb{R})$, put $\Omega_\theta := (\Omega, \theta \circ \tau, \sigma)$.

By construction, Ω_θ is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion carrying θ at the point $(0,0)$.

Proposition 4.7. *The following facts hold.*

(1) $\{\Omega_\theta : \theta \in \mathrm{Diff}_0(\mathbb{R})\}$ *is a full holonomy atlas for* $\mathcal{F}^k_{\mathbb{R}}$. *That is, any* $\mathcal{F}^k_{\mathbb{R}}$*-bisubmersion is adapted to this holonomy atlas (see Section 2.11).*

(2) *For each* $\theta \in \mathrm{Diff}_0(\mathbb{R})$, *the canonical map* $Q_{\Omega_\theta} \colon \Omega_\theta \to G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ *is a diffeomorphism onto its image.*

Proof. If θ is orientation-preserving, then the restriction of Ω_θ to $\mathbb{R} \setminus \{0\}$ is isomorphic to $(\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2$. If θ is orientation-reversing, then the restriction of θ to $\mathbb{R} \setminus \{0\}$ is isomorphic to $(\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-)$. This implies that, for $y \in \mathbb{R} \setminus \{0\}$, every $\mathcal{F}^k_{\mathbb{R}}$-automorphism germ at y is carried by some Ω_θ. By construction, if $\theta \in \mathrm{Diff}_0(\mathbb{R})$, then θ is carried by Ω_θ at $(0,0)$. We have shown that every local automorphism of $\mathcal{F}^k_{\mathbb{R}}$ is carried by some Ω_θ so, by Proposition 2.40, the Ω_θ form a full holonomy atlas.

For (2), note the dimension of Ω_θ equals the dimension of $\mathbb{R}$ plus the fiber dimension of $\mathcal{F}^k_{\mathbb{R}}$, so the map of Ω_θ to $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is a local diffeomorphism (see [2, Proposition 3.11 (b)]). Since the set of points in Ω_θ with trivial isotropy is dense, every local morphism $\Omega_\theta \to \Omega_\theta$ is the identity and the map $\Omega_\theta \to G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is injective. ∎

One rudimentary property of the topology of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is that it has two connected components. The basic idea of the proof, which we omit, already appeared in the first paragraph of the proof of Proposition 4.7.

Proposition 4.8. *The full holonomy groupoid* $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ *has two connected components*[4] *and they are mapped onto* $J^k_+ \cup (\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2$ *and* $J^k_- \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-)$ *by the isomorphism of Proposition 4.4. Here,* J^k_+ *and* J^k_- *denote the* k*-jets of the orientation-preserving and orientation-reversing diffeomorphisms, respectively.* ∎

Next we describe the smooth structure of the isotropy group $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$.

Proposition 4.9. *Giving* J^k *its one-dimensional Lie group structure* $J^k_{\mathbb{R}}$ *(Definition 3.5) makes the group isomorphism* $G(\mathcal{F}^k_{\mathbb{R}})_0 \to J^k_{\mathbb{R}}$ *provided by Proposition 4.4 into a Lie group isomorphism.*

Proof. If $\theta \in \mathrm{Diff}_0(\mathbb{R})$ and $t \in \mathbb{R}$, then $(\{t\} \times \mathbb{R}) \cap \Omega_\theta$ is a bisection of Ω_θ carrying the local diffeomorphism $y \mapsto \theta(y + ty^k)$ at the point $(t, 0)$. The composition

$$\Omega_\theta \xrightarrow{\Omega_\theta} G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}}) \to (\mathbb{R} \setminus \{0\})^2 \cup J^k$$

therefore sends

$$\Omega_\theta \ni (t, 0) \mapsto j^k_0(\theta) \circ (y + ty^k) \in J^k$$

so that $\Omega_\theta \cap (\mathbb{R} \times \{0\})$ is carried diffeomorphically onto the coset of $\mathbb{R} \subseteq J^k_{\mathbb{R}}$ containing $j^k_0(\theta)$. ∎

Proposition 4.9 shows that $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is necessarily a somewhat strange space; it is a blowup of the singular equivalence relation $(\mathbb{R} \setminus \{0\})^2 \cup \{(0, 0)\} \subseteq \mathbb{R}^2$ for which the singular point at the origin is replaced by continuum-many copies of the real line. This already shows it is not a manifold (even in our relaxed sense of the word, see Section 2.1) because it has only two components (Proposition 4.8), but is not second-countable.

The remainder of this section is devoted to investigating the separation properties of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$. The basic observation is as follows: if $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ have different Taylor series at 0, then the intersection of their graphs with some punctured neighbourhood of the origin are disjoint and, therefore, can be separated open subsets of the punctured plane. The following lemma shows that, by choosing neighbourhoods carefully, we can separate the different cosets of $\mathbb{R}$ in $J^k_{\mathbb{R}}$ from each other by open sets in $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$.

Lemma 4.10. *For any* $\theta \in \mathrm{Diff}_0(\mathbb{R})$*, there is a unique smooth function* f_θ *on* $\mathbb{R}^2$ *such that* $\theta(y + ty^k) = \theta(y) + f_\theta(t, y)y^k$ *for all* $(t, y) \in \mathbb{R}^2$*. Define*

$$U_\theta := \{(t, y) \in \Omega_\theta : |y|^{1/2} |f_\theta(t, y)| < 1\},$$

so that U_θ *is an open subset of* Ω_θ *containing* $\mathbb{R} \times \{0\}$*. Then, given* $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ *with different* $(k-1)$*-jets at* 0*, there exists* $\varepsilon > 0$ *such that* U_{θ_1} *and* $U_{\theta_2} \cap (\mathbb{R} \times (-\varepsilon, \varepsilon))$*, viewed as* $\mathcal{F}^k_{\mathbb{R}}$*-bisubmersions, have disjoint images in* $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$.

[4] Or, equivalently, path components, since $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ is locally Euclidean.

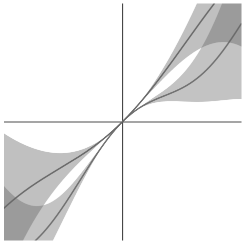


Figure 4. The geometry behind Lemma 4.10.

Proof. We identify $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ with $(\mathbb{R}\setminus\{0\})^2 \cup J^k_{\mathbb{R}}$ without comment. It is straightforward to deduce the existence of f_θ from Taylor's theorem.

Suppose $\theta_1, \theta_2 \in \text{Diff}_0(\mathbb{R})$ and $j_0^{k-1}(\theta_1) \neq j_0^{k-1}(\theta_2)$. The images of $U_{\theta_1} \cap (\mathbb{R}\times\{0\})$ and $U_{\theta_2} \cap (\mathbb{R}\times\{0\})$ are the (disjoint) cosets of $\mathbb{R}$ in $J^k_{\mathbb{R}}$ which contain $j_0^k(\theta_1)$ and $j_0^k(\theta_2)$, respectively. We need therefore only need to check that there is some $\varepsilon > 0$ such that, whenever $(t_1, y) \in U_{\theta_1}$, $(t_2, y) \in U_{\theta_2}$ and $0 < |y| < \varepsilon$, we have $\theta_1(y + t_1 y^k) \neq \theta_2(y + t_2 y^k)$. For any $(t_1, y) \in U_{\theta_1}$ and $(t_2, y) \in U_{\theta_2}$, we have

$$\begin{aligned}|\theta_1(y + t_1 y^k) - \theta_2(y + t_2 y^k)| &\geq |\theta_1(y) - \theta_2(y)| - |y^k f_{\theta_1}(t_1, y)| - |y^k f_{\theta_2}(t_2, y) \\ &\geq |\theta_1(y) - \theta_2(y)| - 2|y|^{k-\frac{1}{2}}.\end{aligned}$$

Because the $(k-1)$-jet of $\theta_1 - \theta_2$ at 0 is nonzero, $2|y|^{k-\frac{1}{2}}$ vanishes more quickly than $|\theta_1(y) - \theta_2(y)|$ at $y = 0$. It follows that there exists an $\varepsilon > 0$ such that, for $0 < |y| < \varepsilon$, the right-hand side of the above inequality is strictly positive. ∎

Corollary 4.11. *Let A be any coset of $\mathbb{R}$ in $J^k_{\mathbb{R}}$ and put $B = J^k_{\mathbb{R}} \setminus A$. Then, there are disjoint open sets $U, V \subseteq G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}}) \cong (\mathbb{R}\setminus\{0\})^2 \cup J^k_{\mathbb{R}}$ such that $A \subseteq U$ and $B \subseteq V$ (see Figure 5).*

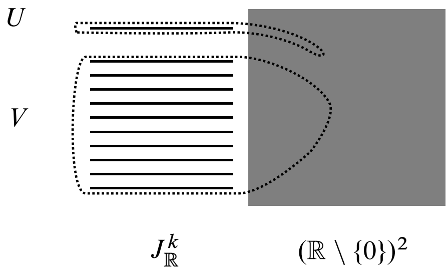


Figure 5. Separating one component of $J^k_{\mathbb{R}}$ from the rest inside $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$.

Proof. Fix $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$. In the notation of Lemma 4.10 above, for each $\theta \in \mathrm{Diff}_0(\mathbb{R})$ with $j_0^{k-1}(\theta) \neq j_0^{k-1}(\theta_0)$, there exists an ε_θ such that U_{θ_0} and $U_\theta \cap (\mathbb{R} \times (-\varepsilon_\theta, \varepsilon_\theta))$ have disjoint images in $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$. Let U be the image of U_θ and let V be the union of the images of the $U_\theta \cap (\mathbb{R} \times (-\varepsilon_\theta, \varepsilon_\theta))$, ranging over $\theta \in \mathrm{Diff}_0(\mathbb{R})$ with $j_0^{k-1}(\theta) \neq j_0^{k-1}(\theta_0)$. ■

Corollary 4.12. *The topology of $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ is Hausdorff and regular.*

For a review of the countablility and separation axioms of point-set topology, the reader may refer to [20] or another standard source. Note that in [20], regular and normal topological spaces are assumed to be Hausdorff by definition. Nonetheless, since this convention is not universal, we will repeat the Hausdorff assumption even when it is redundant to avoid possible confusion.

Proposition 4.13. *For every $k \geq 2$, the topology of $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ is not normal.*

Proof. For notational convenience we take $k = 2$; the argument for $k > 2$ is essentially the same. Let $A \subseteq J_{\mathbb{R}}^2$ consist of all $a_1 y + a_2 y^2 \in J^2$ with a_1 rational. Put $B = J_{\mathbb{R}}^2 \setminus A$. Then A and B are disjoint closed sets partitioning $J_{\mathbb{R}}^2$ (each is a union of cosets). In a similar spirit to Niemytzki's tangent disc (see [26, p. 100]), one can use the Baire category theorem to show that A and B cannot be separated by disjoint open sets in $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^2)$. ■

We summarize various topological properties of $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ discussed above in the following theorem.

Theorem 4.14. *For every $k \geq 2$, the full holonomy groupoid $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$*

(1) *is equipped with a smooth atlas (i.e., is a smooth space, in the sense of Section 2.1);*

(2) *is Hausdorff and regular, but not normal;*

(3) *is separable, but not second-countable;*

(4) *has two connected components.*

Proof. (1) holds for the full holonomy groupoid of any almost regular foliation (see Section 2.14). (2) is a repetition of Corollary 4.12 and Corollary 4.13. $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ is separable because it contains $(\mathbb{R} \setminus \{0\})^2$ as a dense open subset. It is not second-countable because it contains $J_{\mathbb{R}}^k$ and $J_{\mathbb{R}}^k$ is not second-countable. (4) is a repetition of Proposition 4.8. ■

5. Transverse order k foliations

We now define this articles's main objects of study and give some of their basic properties. Recall (Definition 1.1) that $\mathcal{F}_{\mathbb{R}}^k := \mathcal{F}\{y^k \frac{d}{dy}\}$. This is the foliation of $\mathbb{R}$ consisting of all compactly-supported vector fields on $\mathbb{R}$ which vanish to order k or more at 0.

Definition 5.1. Let M be a connected, smooth manifold. A foliation $(M, \mathcal{F})$ is a *transverse order k* foliation if

(1) for each $x \in M$, there exists an open set $U \subseteq M$ with $x \in U$ and a local submersion $p: U \to \mathbb{R}$ such that $p^{-1}(\mathcal{F}_{\mathbb{R}}^k) = \mathcal{F}_U$;

(2) $\mathcal{F}$ has exactly one (connected) singular leaf L.

Given a transverse order k foliation $(M, \mathcal{F})$, we call a submersion such as the one in (1) a *local $\mathcal{F}$-$\mathcal{F}_{\mathbb{R}}^k$-submersions.*

Remark 5.2. The important assumption above is (1). Assumption (2) is included mainly for convenience.

The prototypical example of a transverse order k foliation is the following.

Example 5.3. Let ℓ be a positive integer and $n := \ell + 1$. Equip $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ with coordinates $(x, y) = (x_1, \ldots, x_\ell, y)$. Then

$$\mathcal{F}_{\mathbb{R}^n}^k := \mathcal{F}\left\{y^k \frac{\partial}{\partial y}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\ell}\right\}$$

is a transverse order k foliation of $\mathbb{R}^n$. Indeed, $\mathcal{F}_{\mathbb{R}^n}^k = \mathrm{pr}_2{}^{-1}(\mathcal{F}_{\mathbb{R}}^k)$, where pr_2 is the final coordinate projection $(x, y) \mapsto y$. The singular leaf of $\mathcal{F}_{\mathbb{R}^n}^k$ is the horizontal hyperplane $\mathbb{R}^\ell \times \{0\}$.

Every transverse order k foliation of an n-dimensional manifold is locally isomorphic to the foliation in Example 5.3.

Proposition 5.4. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L and $n := \dim(M) \geq 2$. Then, for any $x_0 \in L$, there exists a diffeomorphism $\theta: U \to V$, where $U \subseteq M$ is an open neighbourhood of x_0 and $V \subseteq \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ is an open neighbourhood of $(0, 0)$, such that $\theta(x_0) = (0, 0)$ and $\theta_*(\mathcal{F}_U) = (\mathcal{F}_{\mathbb{R}^n}^k)_V$. Moreover, given any local $\mathcal{F}$-$\mathcal{F}_{\mathbb{R}}^k$-submersions p and any local retraction π onto L, both defined near x_0, there exists a diffeomorphism θ under which p becomes pr_2 and π becomes pr_1.*

Proof. Let $p: U_0 \to \mathbb{R}$ be a local $\mathcal{F}$-$\mathcal{F}_{\mathbb{R}}^k$-submersions with $x_0 \in U_0$. Any submersion is locally a Euclidean projection so, shrinking U_0 and choosing coordinates appropriately, we may identify U_0 with a ball in $\mathbb{R}^n$ centred at $x_0 = (0, 0)$ in such a way that $p = \mathrm{pr}_2|_{U_0}$. We then have $\mathcal{F}_{U_0} = (\mathrm{pr}_2|_{U_0})^{-1}(\mathcal{F}_{\mathbb{R}}^k) = (\mathcal{F}_{\mathbb{R}^n}^k)_{U_0}$. In particular, $L \cap U_0 = (\mathbb{R}^\ell \times \{0\}) \cap U_0$. Now, possibly shrinking U_0 further, let $\pi: U_0 \to \mathbb{R}^\ell$ be a submersion satisfying $\pi(x, 0) = x$ for all $(x, 0) \in U_0 \cap (\mathbb{R}^\ell \times \{0\})$. By the inverse function theorem, there is a smaller ball $U \subseteq U_0$ centred on the origin such that $(x, y) \mapsto (\pi(x, y), y)$ defines a diffeomorphism $\theta: U \to V$, where $V = \theta(U) \subseteq U_0$ as well. Since, by construction, $\mathrm{pr}_2|_V \circ \theta = \mathrm{pr}_2|_U$, we have $\theta_*(\mathcal{F}_U) = \theta_*((\mathrm{pr}_2|_U)^{-1}(\mathcal{F}_{\mathbb{R}}^k)) = (\mathrm{pr}_2|_V)^{-1}(\mathcal{F}_{\mathbb{R}}^k) = (\mathcal{F}_{\mathbb{R}^m}^k)_V$. ∎

Corollary 5.5. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. Then, L is a closed submanifold of codimension-1.*

Proof. The above proposition shows that L is a codimension-1 embedded submanifold of M. Since the other leaves of $\mathcal{F}$ are open, it follows that L is closed. ∎

6. Local results on transverse order k foliations

In this section, we study the prototypical transverse order k foliation $\mathcal{F}^k_{\mathbb{R}^n}$ of Example 5.3. We assume throughout that $k \geq 2$. The main results for $\mathcal{F}^k_{\mathbb{R}^n}$ are Theorem 6.6, which characterizes $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions (Definition 5.1) in terms of their infinitesimal behaviour along the singular hyperplane, and Theorem 6.15, which computes the restriction of the full holonomy groupoid to the singular hyperplane. Also of importance is Theorem 6.12 which shows that, for any transverse order k foliation $(M, \mathcal{F})$ and x a point in the singular leaf, k-jet equivalence of $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions at x is the same as orbit equivalence under the action of the group of null $\mathcal{F}$-automorphisms at x.

Let $n \geq 2$ be an integer and put $\ell := n - 1$. We equip $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ with coordinates $(x, y) = (x_1, \ldots, x_\ell, y)$. The transverse order k foliation under discussion is

$$\mathcal{F}^k_{\mathbb{R}^n} := \mathcal{F}\left\{y^k \frac{\partial}{\partial y}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\ell}\right\} = \mathrm{pr}_2^{-1}(\mathcal{F}^k_{\mathbb{R}}).$$

The singular leaf of $\mathcal{F}^k_{\mathbb{R}^n}$ is the horizontal hyperplane

$$L := \mathbb{R}^\ell \times \{0\}.$$

The following result shows that both $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms and $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions are closely related.

Proposition 6.1. *Let $\theta\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then, θ preserves $\mathcal{F}^k_{\mathbb{R}^n}$ if and only if $p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = \mathcal{F}^k_{\mathbb{R}^n}$, where $p := \mathrm{pr}_2 \circ \theta\colon \mathbb{R}^n \to \mathbb{R}$.*

Proof. We have $\theta^*(\mathcal{F}^k_{\mathbb{R}^n}) = \theta^{-1}(\mathrm{pr}_2^{-1}(\mathcal{F}^k_{\mathbb{R}})) = p^{-1}(\mathcal{F}^k_{\mathbb{R}})$. ∎

The following terminology will be convenient.

Definition 6.2. Let θ be a diffeomorphism of $\mathbb{R}^n$. We say that θ is *vertical* if it has the form $\theta(x, y) = (x, \theta_x(y))$ where θ_x, $x \in \mathbb{R}^\ell$ is a smoothly-varying diffeomorphism of $\mathbb{R}$. Similarly, we say that θ is *horizontal* if it has the form $\theta(x, y) = (\theta_y(x), y)$ where θ_y, $y \in \mathbb{R}$ is a smoothly varying diffeomorphism of $\mathbb{R}^\ell$.

Since a horizontal diffeomorphism θ satisfies $\mathrm{pr}_2 \circ \theta = \theta$, the following is an immediate consequence of Proposition 6.1.

Lemma 6.3. *Any horizontal diffeomorphism preserves $\mathcal{F}^k_{\mathbb{R}^n}$.* ∎

On the other hand, an inverse function theorem argument gives the following.

Lemma 6.4. *Let θ be any diffeomorphism of $\mathbb{R}^n$ which preserves L. Then, locally near any point of L, we can write $\theta = \theta_h \circ \theta_v$ where θ_v is vertical and θ_h is horizontal.* ■

The crux, therefore, is to understand which vertical diffeomorphisms θ preserve $\mathcal{F}^k_{\mathbb{R}^n}$. It is obviously necessary that θ preserve L, but this does not suffice. For example, the diffeomorphism $(x, y) \mapsto (x, e^x y)$ does not preserve $\mathcal{F}^2_{\mathbb{R}^2}$ (see Example 1.2).

Lemma 6.5. *Let $\theta(x, y) = (x, \theta_x(y))$ be a vertical diffeomorphism of $\mathbb{R}^n$ which preserves L. Then θ preserves $\mathcal{F}^k_{\mathbb{R}^n}$ if and only if $\theta'_x(0), \theta''_x(0), \dots, \theta^{(k-1)}_x(0)$ are independent of x.*

Proof. Suppose that θ is vertical. Firstly, even without the assumptions on the derivatives, one has that θ preserves the foliation singly-generated by $y^k \frac{\partial}{\partial y}$. Indeed, for any $f \in C^\infty_c(\mathbb{R}^n)$, we have $\theta^*\big(f \frac{d}{dy}\big) = (f \circ \theta^{-1})\theta^*\big(\frac{\partial}{\partial y}\big)$. Since $f \mapsto f \circ \theta^{-1}$ preserves the ideal $I^k_L \subseteq C^\infty_c(\mathbb{R}^n)$ of functions vanishing to order k on L, and since $\theta_*\big(\frac{\partial}{\partial y}\big) = \theta'_x(y) \frac{d}{dy}$ where $\theta'_x(y)$ is nowhere vanishing, the claim follows.

Next, $\theta_*\big(\frac{\partial}{\partial x_i}\big) = \frac{\partial}{\partial x_i} + (f_i \circ \theta^{-1}) \frac{\partial}{\partial y}$, where $f_i(x, y) = \frac{\partial}{\partial x_i}\theta_x(y)$. The foliation $\mathcal{F}^k_{\mathbb{R}^n}$ is preserved by θ if and only if $f_i \circ \theta^{-1}$, or equivalently f_i, belongs to I^k_L for $i = 1, \dots, \ell$. In other words, we need $\frac{\partial}{\partial x_i}\theta^{(r)}_x(0) = 0$ for all $x \in \mathbb{R}^\ell$, $i = 1, \dots, \ell$, $r = 1, \dots, k - 1$, proving (2). ■

We now translate Lemma 6.5 into the following result about submersions which will play an important role.

Theorem 6.6. *Let $U \subseteq \mathbb{R}^n$ be a convex open set containing $(0, 0)$. Let $p\colon U \to \mathbb{R}$ be a submersion with $p^{-1}(0) = L \cap U$. Then, the following are equivalent:*

(1) *$p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = (\mathcal{F}^k_{\mathbb{R}^n})_U$, i.e., p is a local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion;*

(2) *$\frac{\partial^r p}{\partial y^r}$ is constant on $L \cap U$ for $r = 1, \dots, k - 1$;*

(3) *there exist constants $a_1, \dots, a_{k-1} \in \mathbb{R}$ with $a_1 \neq 0$ and a smooth function f on $L \cap U$ such that $j^k_{(x_0,0)}(p) = a_1 y + \cdots a_{k-1} y^{k-1} + f(x_0) y^k$ for all $(x_0, 0) \in L \cap U$;*

(4) *there exist constants $a_1, \dots, a_{k-1} \in \mathbb{R}$ with $a_1 \neq 0$ and a smooth function f on U such that $p(x, y) = a_1 y + \cdots a_{k-1} y^{k-1} + f(x, y) y^k$ for all $(x, y) \in U$.*

Proof. Define $\theta\colon U \to \mathbb{R}^n$ by $\theta(x, y) = (x, p(x, y))$. By the inverse function theorem, θ is a diffeomorphism in a neighbourhood of L. Shrinking U, we may assume that θ maps U diffeomorphically onto $\theta(U)$. According to Proposition 6.1, θ preserves $\mathcal{F}^k_{\mathbb{R}^m}$ if and only if $p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = (\mathcal{F}^k_{\mathbb{R}^n})_U$. The equivalence of statements (1) and (2) then follows from Lemma 6.5. Obviously, (3) implies (2). Conversely, (2) and the fact that p itself vanishes on L imply that

$$\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_\ell}\Big)^{\alpha_\ell} \Big(\frac{\partial}{\partial y}\Big)^{\beta} p(x, 0) = 0$$

whenever $\beta \leq k-1$ and at least one of $\alpha_1, \ldots, \alpha_\ell$ is nonzero. Statement (3) follows. Clearly, (4) implies (3) and, by a Taylor series argument, (4) implies (3) as well. ∎

One may interpret the above theorem as saying that the infinitesimal structure of local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions is very rigid along the singular leaf L: their k-th order Taylor expansions involve only the variable y, and none of the variables $x_1, \ldots, x_\ell$. Furthermore, the coefficients of $y, \ldots, y^{k-1}$ remain constant as the basepoint of the Taylor expansion varies in L. As a further demonstration of this rigidity principle, the following corollary says that, working locally and up to order $k-1$, any two local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions are related by a unique polynomial.

Corollary 6.7. *Let $(M, \mathcal{F})$ be any transverse order k foliation with singular leaf L. Let p and q be local $\mathcal{F}$-submersions defined at a point $x_0 \in L$. Then, on some neighbourhood U of x_0, there exist unique constants $a_1, \ldots, a_{k-1} \in \mathbb{R}$ and a unique smooth, real-valued function $f : U \to \mathbb{R}$ such that*

$$q = a_1 p + \cdots a_{k-1} p^{k-1} + f p^k$$

holds on U. Necessarily, $a_1 \neq 0$.

Proof. By Proposition 5.4, we may assume $M = \mathbb{R}^n$, $\mathcal{F} = \mathcal{F}^k_{\mathbb{R}^n}$, $x_0 = (0,0)$ and $p = \mathrm{pr}_2$, whence the claim follows from Theorem 6.6 (4). ∎

Taking Proposition 6.1 and Theorem 6.6 together gives a good understanding of the structure of $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms and $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. Our next task is to mod out by null automorphisms (Definition 2.42). It is therefore necessary to involve some $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersions. We can easily convert an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion into an $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion by taking the product with the pair groupoid L^2.

Lemma 6.8. *If (W, t, s) is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion, then $(L^2 \times W, \mathrm{pr}_1 \times t, \mathrm{pr}_2 \times s)$ is an $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion.*

Proof. This conclusion follows from consideration of the commutative diagrams:

$$\begin{CD} L^2 \times W @>>> W \\ @V{\mathrm{pr}_2 \times \sigma}VV @VV{\sigma}V \\ \mathbb{R}^n @>>> \mathbb{R} \end{CD} \qquad \begin{CD} L^2 \times W @>>> W \\ @V{\mathrm{pr}_1 \times \tau}VV @VV{\tau}V \\ \mathbb{R}^n @>>> \mathbb{R} \end{CD}$$

∎

In particular, recall (Definition 4.1) that

$$\Omega = \{(t,y) \in \mathbb{R}^2 : 1 + t y^{k-1} > 0\}, \quad \sigma(t,y) = y, \quad \tau(t,y) = y + t y^k$$

is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion $\Omega := (\Omega, \tau, \sigma)$. More generally (Definition 4.6), we have that $\Omega_\theta := (\Omega, \theta \circ \tau, \sigma)$ is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion for any $\theta \in \mathrm{Diff}_0(\mathbb{R})$.

Definition 6.9. Let $\widetilde{\Omega}$ denote the $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion $L^2 \times \Omega$. More generally, for any diffeomorphism $\theta \in \mathrm{Diff}_0(\mathbb{R})$, let $\widetilde{\Omega}_\theta$ denote the $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion $L^2 \times \Omega_\theta$.

Note that $N_0 := \{(x, x, 0, y) : (x, y) \in \mathbb{R}^n\}$ is a bisection of $\widetilde{\Omega}_\theta$ inducing the constant vertical diffeomorphism $(x, y) \mapsto (x, \theta(y))$. In particular, $\Omega = \Omega_{\mathrm{id}}$ carries the identity map and we may characterize the local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms which are null at a point $x \in L$ as the ones which are carried by $\widetilde{\Omega}$ at the point $(x, x, 0, 0)$. For example, we have the following result.

Lemma 6.10. *Let θ be a local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism with $\theta(0,0) = (0,0)$. If θ is horizontal, then θ is null (Definition 2.42) at $(0,0)$.*

Proof. Near $(0,0)$, since θ is horizontal, we may write $\theta(x, y) = (\theta_y(x), y)$ where θ_y depends smoothly on y. Let $N = \{(\theta_y(x), x, 0, y) \in \widetilde{\Omega} : (x, y) \in U\}$, where $U \subseteq \mathbb{R}^n$ is an appropriately chosen neighbourhood of $(0,0)$. Then, N is a bisection of $\widetilde{\Omega}$ through the point $(0,0,0,0)$ inducing the germ of θ at $(0,0)$. ■

In a similar spirit, we have the following generalization of Lemma 4.3 (2).

Lemma 6.11. *Let $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$ be a diffeomorphism of $\mathbb{R}$ defined near 0 and define a constant, vertical diffeomorphism θ of $\mathbb{R}^n$ by $(x, y) \mapsto (x, \theta(y))$. Then, θ preserves $\mathcal{F}^k_{\mathbb{R}^n}$ and is null at $(0,0)$ if and only if $j^k_0(\theta) = y$.* ■

We state the next result for general foliations of transverse order k, though we quickly reduce to coordinates in the proof.

Theorem 6.12. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. Let p and q be local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $x_0 \in L$. Then, $j^k_{x_0}(p) = j^k_{x_0}(q)$ if and only if there exists a local $\mathcal{F}$-automorphism θ which is null at x_0 such that $q = p \circ \theta$ on a neighbourhood of x_0.*

Proof. Using Proposition 5.4, suppose without loss of generality that $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$, $x_0 = (0,0)$ and $p = \mathrm{pr}_2$. By the inverse function theorem, $\eta(x, y) = (x, q(x, y))$ defines a diffeomorphism nearby to $(0,0)$. By definition, $q = p \circ \eta$ holds near $(0,0)$. It remains to confirm that η is null at $(0,0)$. Since $j^k_{(0,0)}(q) = y$, Theorem 6.6 (4) implies that we can write $q(x, y) = y + f(x, y)y^k$ near $(0,0)$ for a smooth function f satisfying $f(0,0) = 0$. Then, the bisection

$$N_f = \{(x, x, f(x, y), y) : (x, y) \in U\},$$

where U is an appropriate neighbourhood of $(0,0) \in \mathbb{R}^n$, induces η. Since $(0,0,0,0) \in N_f$ and the identity is also carried at $(0,0,0,0) \in \widetilde{\Omega}$, we have that η is null at $(0,0)$. ■

Along similar lines, we now show that all local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms can be brought, modulo a null automorphism, into a simple form.

Proposition 6.13. *Suppose that $x_1, x_2 \in L$ and let θ be a local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism with $\theta(x_1, 0) = (x_2, 0)$. Then, there is a local $\mathcal{F}_{\mathbb{R}^n}$ automorphism η which is null at $(x_1, 0)$ and a diffeomorphism $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$ such that $(\theta \circ \eta)(x, y) = (x - x_1 + x_2, \theta_0(y))$ holds on a neighbourhood of $(x_1, 0)$.*

Proof. By translating, we may reduce to the case $x_1 = x_2 = 0$. Using Lemmas 6.4 and 6.10, we may furthermore assume θ is a vertical $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism $(x, y) \mapsto (x, \theta_x(y))$. Composing with the constant vertical diffeomorphism $(x, y) \mapsto (x, \theta_0^{-1}(y))$, we may reduce to the case where $\theta_0(y) = y$. The result then follows from Theorem 6.12, taking $p(x, y) = y$ and $q(x, y) = \theta_x(y)$. ∎

Proposition 6.14. *The following facts hold.*

(1) *$\{\widetilde{\Omega}_\theta : \theta \in \mathrm{Diff}_0(\mathbb{R})\}$ is a full holonomy atlas for $\mathcal{F}^k_{\mathbb{R}^n}$. That is to say, any $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion is adapted to this holonomy atlas (see Section 2.11).*

(2) *For each $\theta \in \mathrm{Diff}_0(\mathbb{R})$, the canonical map $Q_{\widetilde{\Omega}_\theta} : \widetilde{\Omega}_\theta \to G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})$ is a diffeomorphism onto its image.*

Proof. Denote by $\mathbb{R}^n_+ := \mathbb{R}^\ell \times \mathbb{R}_+$ and $\mathbb{R}^n_- := \mathbb{R}^\ell \times \mathbb{R}_-$ the open upper and lower half spaces of $\mathbb{R}^n$. It is easy to see that the restriction of $\widetilde{\Omega}_\theta$ to $\mathbb{R}^n \setminus L$ is isomorphic to $(\mathbb{R}^n_-)^2 \cup (\mathbb{R}^n_+)^2$ if θ is orientation-preserving and $(\mathbb{R}^n_- \times \mathbb{R}^n_+) \cup (\mathbb{R}^n_+ \times \mathbb{R}^n_-)$ if θ is orientation-reversing. Thus, every local diffeomorphism of $\mathbb{R}^n \setminus L$ is already carried by $\widetilde{\Omega}_{\mathrm{id}}$ and $\widetilde{\Omega}_{-\mathrm{id}}$. It remains to show that, given any point $x_0 \in L$ and any local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism θ defined at x_0, the germ of θ at x_0 can by induced by a bisection of one of $\{\widetilde{\Omega}_\theta : \theta \in \mathrm{Diff}_0(\mathbb{R})\}$. After an easy reduction, we may take $x_0 = (0, 0)$ and assume $\theta(0, 0) = (0, 0)$. By Proposition 6.13, after composing θ with a local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism that is null at $(0, 0)$, we may assume that $\theta(x, y) = (x, \theta_0(y))$ holds on a neighbourhood U of $(0, 0)$ for some $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$. Then, $N = \{(x, x, 0, y) : (x, y) \in U\}$ is a bisection of $\widetilde{\Omega}_\theta$ which induces $\theta|_U$, proving (1). The proof of (2) is the same as in Proposition 4.7. ∎

We can now show that the restriction of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})$ is isomorphic, as a Lie groupoid, to $L^2 \times J^k_{\mathbb{R}}$, the product of the pair groupoid L^2 and the 1-dimensional Lie group $J^k_{\mathbb{R}}$ (see Definition 3.5).

Theorem 6.15. *There is a unique isomorphism of Lie groupoids*

$$G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})_L \to L^2 \times J^k_{\mathbb{R}}$$

such that

$$Q_{\widetilde{\Omega}_\theta}(x_2, x_1, t, 0) \mapsto (x_2, x_1, j^k_0(\theta) \circ (y + t y^k))$$

for all $\theta \in \mathrm{Diff}_0(\mathbb{R})$, $x_1, x_2 \in L$ and $t \in \mathbb{R}$.

Proof. Theorem 2.43 gives an isomorphism $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})_L \to \mathcal{G}/\mathcal{N}$ where, for the sake of brevity, we write $\mathcal{G} := \mathcal{G}(\mathcal{F}^k_{\mathbb{R}^n})_L$ and $\mathcal{N} := \mathcal{N}(\mathcal{F}^k_{\mathbb{R}^n})_L$. Given $x_1, x_2 \in L$, $\theta \in \mathrm{Diff}_0(\mathbb{R})$, let $T_{x_2, x_1, \theta}$ denote the germ at x_1 of the diffeomorphism $(x, y) \mapsto (x + x_2 - x_1, \theta(x))$.

The set of all $T_{x_2,x_1,\theta}$ constitute a subgroupoid $\mathcal{H} \subseteq \mathcal{G}$. It is easy to see that $T_{x_2,x_1,\theta} \mapsto (x_2, x_1, j_0^k(\theta))$ is a surjective groupoid homomorphism $\mathcal{H} \to L^2 \times J^k$. By Lemma 6.11, the kernel of the latter homomorphism is exactly $\mathcal{H} \cap \mathcal{N}$, so that

$$\frac{\mathcal{H}\mathcal{N}}{\mathcal{N}} \cong \frac{\mathcal{H}}{\mathcal{H} \cap \mathcal{N}} \cong L^2 \times J^k.$$

By Proposition 6.13, $\mathcal{H}\mathcal{N} = \mathcal{G}$, so we have an isomorphism

$$\mathcal{G}/\mathcal{N} \ni [T_{x_2,x_1,\theta}] \mapsto (x_2, x_1, j_0^k(\theta)) \to L^2 \times J^k.$$

Finally, note that the bisection $N_{x_1,x_2,t} := \{(x + x_2 - x_1, x, t, y) \in \widetilde{\Omega}_\theta : (x, y) \in \mathbb{R}^n\}$ induces the diffeomorphism $(x, y) \mapsto (x + x_2 - x_1, \theta(y + ty^k))$ so that the resulting isomorphism $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n}) \to L^2 \times J^k$ indeed sends

$$Q_{\widetilde{\Omega}_\theta}(x_2, x_1, t, 0) \mapsto (x_2, x_1, j_0^k(\theta) \circ (y + ty^k)) \in L^2 \times J^k.$$

By Proposition 6.14, the $\widetilde{\Omega}_\theta$ constitute a full holonomy atlas for $\mathcal{F}^k_{\mathbb{R}^n}$ and each $Q_{\widetilde{\Omega}_\theta}$ is a diffeomorphism onto its image. It follows that this groupoid isomorphism becomes a Lie groupoid isomorphism when J^k is replaced by $J^k_{\mathbb{R}}$. ∎

7. Principal bundles of a transverse order *k* foliation

In what follows, $(M, \mathcal{F})$ denotes a transverse order k foliation with singular leaf L and $k \geq 2$. The purpose of this section is to leverage the rigidity phenomena encountered in the preceding section to construct certain natural principal bundles over L whose elements are jets of local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. The main result is Theorem 7.3.

Because the group $\mathrm{Diff}_0(\mathbb{R})$ of diffeomorphisms of $\mathbb{R}$ fixing 0 preserves $\mathcal{F}^k_{\mathbb{R}}$, the group $\mathrm{Diff}_0(\mathbb{R})$ acts by composition from the left on the set of local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $x \in M$. Since composition of jets is well defined, the following definition makes sense.

Definition 7.1. Given $x \in L$ and $r \in \{1, \ldots, k\}$, we write $P^r_x(\mathcal{F})$ for the set of r-jets at x of local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions endowed with the left action of J^r that descends from the left action of $\mathrm{Diff}_0(\mathbb{R})$. Put also $P^r(\mathcal{F}) := \bigsqcup_{x \in L} P^r_x(\mathcal{F})$, so that $P^r(\mathcal{F})$ is a bundle of sets over L with an action of J^r on each fiber.

Our goal is to equip $P^r(\mathcal{F})$ with the structure of a smooth principal bundle over L. Our constructions will rely on the following elementary lemma which provides a mechanism by which a smooth principal bundle structure may be induced from an appropriate family of sections. In essence, this is the construction of a principal bundle from a 1-cocycle.

Lemma 7.2. *Let H be a Lie group and M a smooth manifold. Suppose that P is a set equipped with a map $\pi\colon P \to M$ and an action of H that is free and transitive on each fiber of π. Let $\{U_i\}_{i \in I}$ be an open cover of M and let $\{s_i\colon U_i \to P\}_{i \in I}$ be a collection* *of local sections of* π *such that the transition maps* $\{h_{ij}: U_i \cap U_j \to H\}_{i,j \in I}$*, uniquely defined by* $s_i(x) = h_{ij}(x)s_j(x)$ *for all* $x \in U_i \cap U_j$*, are smooth. There is a unique smooth structure on* P *making it a smooth principal* H*-bundle with respect to which each* s_i *is a smooth section.* ∎

Recall (Definition 3.5) that J^r_d denotes J^r considered as a discrete group and $J^r_{\mathbb{R}}$ denotes J^r with a certain one-dimensional Lie group structure.

Theorem 7.3. *Let* $(M, \mathcal{F})$ *be a transverse order* k *foliation with singular leaf* L *and* $k \geq 2$.

(1) *There is a unique principal* $J^k_{\mathbb{R}}$*-bundle structure on* $P^k(\mathcal{F})$ *such that, for any local* $\mathcal{F}$*-*$\mathcal{F}^k_{\mathbb{R}}$*-submersions* $p: U \to \mathbb{R}$*, the map* $x \mapsto j^k_x(p): U \cap L \to P^k(\mathcal{F})$ *is a smooth section.*

(2) *If* $1 \leq r \leq k-1$*, there is a unique principal* J^r_d*-bundle structure on* $P^r(\mathcal{F})$ *such that, for any local* $\mathcal{F}$*-*$\mathcal{F}^k_{\mathbb{R}}$*-submersions* $p: U \to \mathbb{R}$*, the map* $x \mapsto j^r_x(p): U \cap L \to P^r(\mathcal{F})$ *is a smooth section.*

Proof. First we claim that, for any $r \in \{1, \dots, k\}$, the action of J^r on $P^r(\mathcal{F})$ is free and transitive on fibers. By Proposition 5.4, it suffices to consider the case $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$ and work near the point $(0,0) \in \mathbb{R}^n$. By Theorem 6.6, given a local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion p defined at $(0,0)$, the jet $j^k_{(0,0)}(p)$ (a priori a polynomial of degree $\leq r$ in $x_1, \dots, x_\ell, y$) has the form $a_1 y + \cdots + a_k y^k$ where $a_i \in \mathbb{R}$, $a_1 \neq 0$. The action of J^k, meanwhile, is the usual compose and truncate operation. It follows that J^r acts freely and transitively on $P^r_{(0,0)}(\mathcal{F})$ for $r = 1, \dots, k$.

Next, suppose that $p, q: U \to \mathbb{R}$ are local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions, where $U \subseteq \mathbb{R}^n$ is a convex, open neighbourhood of $(0,0)$. Without loss of generality (Proposition 5.4) we may take $p = \mathrm{pr}_2$. By Theorem 6.6, we have

$$j^k_{(0,x_0)}(q) = a_1 y + \cdots + a_{k-1} y^{k-1} + f(x_0) y^k$$

for $x_0 \in U \cap \mathbb{R}^\ell$, where $f: U \cap \mathbb{R}^\ell \to \mathbb{R}$ is smooth. The map $h: U \cap \mathbb{R}^\ell \to J^k_{\mathbb{R}}$ defined by $h(x_0) = a_1 y + \cdots + a_{k-1} y^{k-1} + f(x_0) y^k$ is smooth (the smooth structure on $J^{k-1}_{\mathbb{R}}$ only permits us to vary the coefficient of y^k). We have $j^k_{(x_0,0)}(q) = h \circ j^k_{(x_0,0)}(p)$ for all x_0 in $U \cap \mathbb{R}^\ell$ and so, applying Lemma 7.2, we obtain (1). The proof of (2) is similar, keeping fewer terms of the Taylor expansions. ∎

By construction, these bundles are functorial for foliation-preserving diffeomorphisms defined near the singular leaf.

Proposition 7.4. *For* $i = 1, 2$*, let* $(M_i, \mathcal{F}_i)$ *be transverse order* k *singular foliations with singular leaves* L_i*. Suppose* $U_i \subseteq M_i$ *is an open set containing* L_i *and* $\theta: U_1 \to U_2$ *is a diffeomorphism with* $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$*. Then* θ *can be restricted to a diffeomorphism* $\theta_0: L_1 \to L_2$ *and, for* $r = 1, \dots, n$*, there is a unique isomorphism of principal bundles*

$P^r(\theta)\colon P^r(\mathcal{F}_1) \to P^r(\mathcal{F}_2)$ covering θ_0 sending $j^r_x(p) \mapsto j^r_{\theta(x)}(p \circ \theta^{-1})$, whenever p is a local $\mathcal{F}_1$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $x \in L$:

$$\begin{array}{ccc} P^r(\mathcal{F}_1) & \xrightarrow{P^r(\theta)} & P^r(\mathcal{F}_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\theta|_L} & L_2 \end{array}$$

Proof. We may assume $U_1 = M_1$, $U_2 = M_2$. Pushforward by $\theta\colon M_1 \to M_2$ gives a bijection $p \mapsto p \circ \theta^{-1}$ from the set of local $\mathcal{F}_1$-$\mathcal{F}^k_{\mathbb{R}}$-submersions to the set of local $\mathcal{F}_2$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. This bijection commutes with the left action of $\mathrm{Diff}_0(\mathbb{R})$ by composition. Passing to r-jets gives the result. ∎

Since $P^{k-1}(\mathcal{F})$ has discrete structure group J^{k-1}_d, taking monodromy immediately gives the following. It is appropriate to think of an $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion at a point $x \in L$ as a dual version of a transversal. Accordingly, the statement below may be interpreted as saying that a path in L induces a holonomy transformation between transversals *at the level of* $(k-1)$*-jets*, in alignment with the classical notion of holonomy for regular foliations.

Corollary 7.5. *Suppose $(M, \mathcal{F})$ is a transverse order k singular foliation with singular leaf L. Then a path $c\colon [0,1] \to L$ from a point x to a point y induces a J^{k-1}_d-equivariant map $P^{k-1}_x(\mathcal{F}) \to P^{k-1}_y(\mathcal{F})$ defined by sending $p_0 \mapsto p(1)$ where $p\colon [0,1] \to P^{k-1}(\mathcal{F})$ is the unique lift of c with $p(0) = p_0$.*

We furthermore use the monodromy of the J^{k-1}_d-bundle $P^{k-1}(\mathcal{F})$ to define the following invariant of a transverse order k foliation. See Section 2.4 for notation and terminology.

Definition 7.6. Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. The *holonomy invariant* of $\mathcal{F}$ is

$$h(\mathcal{F}) := h(P^{k-1}(\mathcal{F})) \in [\pi_1(L), J^{k-1}].$$

That is, $h(\mathcal{F})$ is the monodromy invariant (Definition 2.10) of $P^{k-1}(\mathcal{F})$.

Here, $[\pi_1(L), J^{k-1}]$ denotes the quotient of the set $\mathrm{Hom}(\pi_1(M, x_0), J^{k-1})$ by the conjugation action of J^{k-1}, where $x_0 \in L$ is any basepoint, as in Section 2.4.

The following proposition shows that $h(\mathcal{F})$ is indeed an *invariant* of $\mathcal{F}$. More precisely, $h(\mathcal{F})$ is an "L-local invariant" in the sense that it only depends on the restriction of $\mathcal{F}$ to any neighbourhood of its singular leaf.

Proposition 7.7. *For $i = 1, 2$, let $(M_i, \mathcal{F}_i)$ be transverse order k singular foliations with singular leaves L_i. Suppose $U_i \subseteq M_i$ is an open set containing L_i and $\theta\colon U_1 \to U_2$ is a diffeomorphism with $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$. Then θ can be restricted to a diffeomorphism $\theta_0\colon L_1 \to L_2$ and $(\theta_0)_*(h(\mathcal{F}_1)) = h(\mathcal{F}_2)$. We refer to Proposition 2.9 for the definition of the induced map $(\theta_0)_*\colon [\pi_1(M_1), J^{k-1}] \to [\pi_1(M_2), J^{k-1}]$.*

Proof. First apply Theorem 7.4 and then Theorem 2.11 (1). ∎

If we take the underlying J^1-bundle of the J^1_d-bundle $P^1(\mathcal{F}) \to L$ and identify J^1 with $\mathrm{GL}(1,\mathbb{R})$ in the obvious way, then we see that, through the usual correspondence between vector bundles and their frame bundles, $P^1(\mathcal{F})$ determines a flat line bundle over L. As one might guess (since a 1-jet $j^1_x(p)$ of a real-valued map is essentially the same thing as its differential dp_x), this line bundle is canonically isomorphic to $\nu^*_M(L)$, the conormal bundle of L in M. For convenience, we identify $\nu^*_M(L)$ with the subbundle of $T^*M|_L$ which annihilates $TL \subseteq TM|_L$. We omit the verification of the following.

Proposition 7.8. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. View $P^1(\mathcal{F})$ as a* $\mathrm{G}(1,\mathbb{R})$*-bundle, as described above. Then*

$$\mathbb{R} \underset{\mathrm{GL}(1,\mathbb{R})}{\times} P^1(\mathcal{F}) \ni [\lambda, j^1_x(p)] \mapsto \lambda dp_x \in \nu^*_M(L)$$

is an isomorphism of line bundles. ∎

Thinking of $P^1(\mathcal{F})$ as a flat $\mathrm{GL}(1,\mathbb{R})$-bundle, we may therefore make the following definition.

Definition 7.9. Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. The *Bott connection* of $\mathcal{F}$ is the flat connection $b(\mathcal{F})$ on the conormal bundle $\nu^*_M(L)$ induced by Proposition 7.8 above.

We conclude this section with the observation that one really only needs to construct the bundle $P^k(\mathcal{F})$; the bundles $P^r(\mathcal{F})$ for $r < k$ can be recovered as quotients of the former. Recall that if P is a principal H-bundle and K is a closed normal subgroup of H, then P/K is naturally a principal H/K bundle over the same base.

Proposition 7.10. *Let r be an integer with $2 \leq r \leq k$ and identify J^{r-1} with $J^r/\mathbb{R}$. Then the map $P^r(\mathcal{F}) \to P^{r-1}(\mathcal{F})$ given by taking the underlying $(r-1)$-jet of an r-jet induces an isomorphism of J^{r-1}_d-bundles $P^r(\mathcal{F})/\mathbb{R} \to P^{r-1}(\mathcal{F})$.* ∎

8. Gauge groupoid description of full holonomy groupoid

Suppose $(M, \mathcal{F})$ is a transversely order k singular foliation with singular leaf L. Since the restriction $G_{\mathrm{full}}(\mathcal{F})_L$ of the full holonomy groupoid to L is transitive, it must be a gauge groupoid (see Definition 2.7). In this section, we show that it is isomorphic to the gauge groupoid of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F})$ constructed in Section 7. The main result is Theorem 8.6.

The following lemma indicates the method we use to produce elements of the gauge groupoid. The proof is a simple algebraic verification, which we omit.

Lemma 8.1. *Let $P \to B$ be a (smooth, left) principal H-bundle. Given $x, y \in B$, denote by Λ_x^y the set of all maps $\lambda \colon P_y \times P_x \to H$ that satisfy*

$$\lambda(kq, hp) = k\lambda(q, p)h^{-1}$$

for all $h, k \in H$, $p \in P_x$, $q \in P_y$.

(1) *Given $\lambda \in \Lambda_x^y$, the element*

$$\bar{\lambda} := [q, \lambda(q, p)p] \in \mathrm{Gauge}(P)_x^y$$

is independent of choice of $p \in P_x$, $q \in P_y$.

(2) *$\lambda \mapsto \bar{\lambda}$ defines a bijection $\Lambda_x^y \to \mathrm{Gauge}(P)_x^y$.*

(3) *Given $x, y, z \in B$; $\lambda_1 \in \Lambda_x^y$; $\lambda_2 \in \Lambda_y^z$; $\lambda \in \Lambda_x^z$, the following are equivalent:*
   (a) *$\bar{\lambda}_2\bar{\lambda}_1 = \bar{\lambda}$;*
   (b) *$\lambda_2(r, q)\lambda_1(q, p) = \lambda(r, p)$ for all $p \in P_x$, $q \in P_y$, $r \in P_z$.*

We need the following simple lemma concerning bisubmersions.

Lemma 8.2. *Let $(M, \mathcal{F})$ and $(N, \mathcal{F}_N)$ be singular foliations. Let $U, V \subseteq M$ be open and let $p \colon U \to N$ and $q \colon V \to N$ be submersions such that $p^{-1}(\mathcal{F}_N) = \mathcal{F}_U$ and $q^{-1}(\mathcal{F}_N) = \mathcal{F}_V$.*

(1) *If (W, t, s) is an $\mathcal{F}$-bisubmersion, then $W_{q,p} := (s^{-1}(U) \cap t^{-1}(V), q \circ t, p \circ s)$ is an $\mathcal{F}_N$-bisubmersion.*

(2) *Suppose W' is another $\mathcal{F}$-bisubmersion and let $w \in W_{q,p}$, $w' \in W'_{q,p}$. If there is a local morphism of $\mathcal{F}$-bisubmersions from W to W' sending $w_1 \mapsto w_2$, then there is also a local morphism of $\mathcal{F}_N$-bisubmersions from $W_{q,p}$ to $W'_{q,p}$ sending $w_1 \mapsto w_2$.*

Proof. For brevity, put $s_N := p \circ s$, $t_N := q \circ t$ and $\mathcal{F}_W := s_N^{-1}(\mathcal{F}_N) = t_N^{-1}(\mathcal{F}_N)$. Since $C_c^\infty(\ker(ds_N)) \subseteq s_N^{-1}(\mathcal{F}_N)$ and $C_c^\infty(\ker(dt_N)) \subseteq t_N^{-1}(\mathcal{F}_N)$, we have

$$C_c^\infty(\ker(ds_N)) + C_c^\infty(\ker(dt_N)) \subseteq \mathcal{F}_W.$$

On the other hand, $\ker(ds) \subseteq \ker(ds_N)$ and $\ker(dt) \subseteq \ker(dt_N)$, so

$$\mathcal{F}_W = C_c^\infty(\ker(ds)) + C_c^\infty(\ker(dt)) \subseteq C_c^\infty(\ker(ds_N)) + C_c^\infty(\ker(dt_N)),$$

establishing (1). For (2), one needs simply to restrict the domain of the given morphism appropriately. ∎

Remark 8.3. The above lemma shows that $p \colon U \to N$ and $q \colon V \to N$ determine a well-defined function (typically not a groupoid morphism) $G_{\mathrm{full}}(\mathcal{F})_U^V \to G_{\mathrm{full}}(\mathcal{F}_N)$ sending $Q_W(w) \mapsto Q_{W_{q,p}}(w)$. This function is smooth when $\mathcal{F}$ and $\mathcal{F}_N$ are almost regular.

The notation in the following definition will facilitate the statement of Theorem 8.6.

Definition 8.4. Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. Let (W, t, s) be an $\mathcal{F}$-bisubmersion, let $w \in W$ and assume that $x_1 := s(w)$ and $x_2 := t(w)$ belong to L. Let p and q be $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at x_1 and x_2 respectively. Then, we put

$$\lambda_W(w, q, p) := j^k_0(\theta) \in J^k,$$

where θ is any diffeomorphism of $\mathbb{R}$ carried by the associated $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion $W_{q,p}$ (see Lemma 8.2, (1)) at w. Equivalently, $\lambda_W(w, q, p)$ is the image of $Q_{W_{q,p}}(w) \in G(\mathcal{F}^k_{\mathbb{R}})_0$ under the isomorphism $G(\mathcal{F}^k_{\mathbb{R}})_0 \to J^k$ of Proposition 4.9.

Lemma 8.5. *The k-jet $\lambda_W(w, q, p)$ in Definition 8.4 above only depends on*

(1) *the groupoid element $Q_W(w) \in G_{\mathrm{full}}(\mathcal{F})$, and*

(2) *the k-jets $j^k_{x_1}(p) \in P^k_{x_1}(\mathcal{F})$ and $j^k_{x_2}(q) \in P^k_{x_2}(\mathcal{F})$.*

Proof. Lemma 8.2 (2) gives immediately that $\lambda_W(w, q, p)$ only depends on $Q_W(w)$ (this is in the same vein as Remark 8.3). It remains to investigate the dependence of $\lambda_W(w, q, p)$ on p and q. To this end, suppose that p' and q' are local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions and that $j^k_{x_1}(p) = j^k_{x_1}(p')$ and $j^k_{x_2}(q) = j^k_{x_2}(q')$. Then, by Theorem 6.12, there exists, for $i = 1, 2$, a local $\mathcal{F}$-automorphism θ_i which is null at x_i, such that $p' = p \circ \theta_1$ near x_1 and $q' = q \circ \theta_2$ near x_2. Because θ_1, θ_2 are null at w, the $\mathcal{F}$-bisubmersion $W' := W_{\theta_2,\theta_2}$ carries the same local $\mathcal{F}$-automorphisms at w as does W. Equivalently, by Lemma 2.40, there exists a local morphism from W to W' sending $w \mapsto w$. Thus, applying Lemma 8.2, (2), there is a local morphism from $W_{q,p}$ to $W'_{q,p} = W_{q\circ\theta_2, p\circ\theta_1} = W_{q',p'}$ sending $w \mapsto w$ and it follows that $\lambda_W(w, q, p) = \lambda_W(w, q', p')$. ∎

We have come to the main result of this section.

Theorem 8.6. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. There is a unique isomorphism of Lie groupoids*

$$G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$$

such that if (W, t, s) is an $\mathcal{F}$-bisubmersion, and if $w \in W$ is such that $x_1 := s(w)$ and $x_2 := t(w)$ belong to L, then

$$Q_W(w) \mapsto [j^k_{x_2}(q), \lambda_W(w, q, p) \cdot j^k_{x_1}(p)],$$

where p and q are any local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at x_1 and x_2, respectively.

Proof. Suppose that W is a $\mathcal{F}$-bisubmersion and $w \in W$ has $s(w) \in L$. Let p be an $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion defined at $s(w)$ and q and $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion defined at $t(w)$. If $W_{q,p}$ carries θ at w and $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ are given, then it is easy to see $W_{q\circ\theta_2, p\circ\theta_1}$ carries $\theta_2 \circ \theta \circ \theta_1^{-1}$ at w. Therefore, combining Lemma 8.5 and Lemma 8.1 (2), the procedure in the theorem statement determines a well-defined map $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$.

Next we claim the map $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$ under discussion preserves groupoid multiplication. Let (W, t, s) and (W', t', s') be $\mathcal{F}$-bisubmersions and suppose $w \in W$ and $w' \in W'$ have $s'(w') = t(w) \in L$. Put $x_1 := s(w)$, $x_2 := s'(w') = t(w)$, $x_3 := t'(w')$. Let p_i be an $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at x_i for $i = 1, 2, 3$. Observe that $W'_{p_3,p_2} \circ W_{p_2,p_1}$ is a closed submanifold of $(W' \circ W)_{p_3,p_1}$. The latter observation implies that, if θ is carried by W_{p_2,p_1} at w and θ' is carried by W'_{p_3,p_2} at w', then $\theta' \circ \theta$ is carried by $(W' \circ W)_{p_3,p_1}$ at (w', w). Thus,

$$\lambda_{W'}(w', p_3, p_2)\lambda_W(w, p_2, p_1) = \lambda_{W' \circ W}((w', w), p_3, p_1)$$

so that, by Lemma 8.1 (3), the map $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$ preserves the groupoid multiplication.

It remains to establish that this groupoid morphism $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$ is a diffeomorphism. This problem is local in nature and it is enough to consider the case $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$, $L = \mathbb{R}^\ell \times \{0\}$ which was studied in detail in Section 6. In a bit more detail, this reduction is arrived at by showing that any pair of points $x_1, x_2 \in L$ admit open neighbourhoods $U_1, U_2 \subseteq M$ such that the restriction

$$G_{\mathrm{full}}(\mathcal{F})^{U_2 \cap L}_{U_1 \cap L} \to \mathrm{Gauge}(P^k(\mathcal{F}))^{U_2 \cap L}_{U_1 \cap L}$$

of the groupoid morphism is a diffeomorphism. Taking U_i, $i = 1, 2$, to be sufficiently small, Proposition 5.4 implies there is a diffeomorphism $\theta_i \colon U_i \to \mathbb{R}^n$ such that we have $(\theta_i)_*(\mathcal{F}_{U_i}) = \mathcal{F}^k_{\mathbb{R}^n}$. The foliation isomorphisms θ_1, θ_2 yield a bijection between the set of bisubmersions from $(U_1, \mathcal{F}_{U_1})$ to $(U_2, \mathcal{F}_{U_2})$ and the set of bisubmersions from $(\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$ to itself. This bijection respects the relation $\sim$ of Section 2.28 and yields a diffeomorphism from $G_{\mathrm{full}}(\mathcal{F})^{U_2}_{U_1}$ to $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})$. Similarly, θ_i induces a bijection from the set of $\mathcal{F}_{U_i}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions to the set of $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions which, in turn, induces a $J^k_{\mathbb{R}}$-bundle isomorphism from $P^k(\mathcal{F})|_{U_i \cap L}$ to $P^k(\mathcal{F}^k_{\mathbb{R}^n})$. These bundle isomorphisms permit $\mathrm{Gauge}(P^k(\mathcal{F}))^{U_2 \cap L}_{U_1 \cap L}$ to be identified with $\mathrm{Gauge}(P^k(\mathcal{F}^k_{\mathbb{R}^n}))$ and lead to the following commutative diagram:

$$\begin{array}{ccc} G_{\mathrm{full}}(\mathcal{F})^{U_2 \cap L}_{U_1 \cap L} & \longrightarrow & \mathrm{Gauge}(P^k(\mathcal{F}))^{U_2 \cap L}_{U_1 \cap L} \\ \downarrow & & \downarrow \\ G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n}) & \longrightarrow & \mathrm{Gauge}(P^k(\mathcal{F}^k_{\mathbb{R}^n})) \end{array}$$

Because the vertical maps in this diagram are diffeomorphisms, the upper map is a diffeomorphism if and only if the lower map is a diffeomorphism.

For the remainder of the proof, we assume therefore $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$, $L = \mathbb{R}^\ell \times \{0\}$, as in Section 6. In this case, the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F}) \to L$ is isomorphic to $L \times J^k_{\mathbb{R}}$, courtesy of the global section induced by the global $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion $\mathrm{pr}_2 \colon \mathbb{R}^n \to \mathbb{R}$. Correspondingly (Example 2.8), there is an isomorphism of Lie groupoids

$$L^2 \times J^k_{\mathbb{R}} \ni (x_2, x_1, \alpha) \mapsto [j^k_{x_2}(\mathrm{pr}_2), \alpha \cdot j^k_{x_2}(\mathrm{pr}_2)] \in \mathrm{Gauge}(P^k(\mathcal{F})).$$

We also have already a Lie groupoid isomorphism

$$G_{\mathrm{full}}(\mathcal{F}) \to (\mathbb{R}^{\ell})^2 \times J^k_{\mathbb{R}}$$

by Theorem 6.15. It remains, therefore, to check that the diagram

$$\begin{array}{ccc} G_{\mathrm{full}}(\mathcal{F}) & \longrightarrow & \mathrm{Gauge}(P^k(\mathcal{F})) \\ & \searrow \qquad \nearrow & \\ & L^2 \times J^k_{\mathbb{R}} & \end{array}$$

is commutative. For this purpose, recall the full holonomy atlas $\{\tilde{\Omega}_\theta : \theta \in \mathbb{R}\}$ (Definition 6.9). Fix $\theta \in \mathrm{Diff}_0(\mathbb{R})$; $x_1, x_2 \in \mathbb{R}^{\ell}$; $t \in \mathbb{R}$ and put $w := (x_2, x_1, t, 0) \in \tilde{\Omega}_\theta$. It is straightforward to chase $Q_{\tilde{\Omega}_\theta}(w) \in G_{\mathrm{full}}(\mathcal{F})$ both ways around the above diagram and confirm that result is the same. ■

9. Extracting the holonomy groupoid from the full holonomy groupoid

The holonomy groupoid of a singular foliation is contained in the full holonomy groupoid as an open subgroupoid. To be precise, $G(\mathcal{F})$ is the s-connected component $G_{\mathrm{full}}(\mathcal{F})$ (cf. Proposition 2.30 and [2, Theorem 0.1]).

Definition 9.1. The **s**-*connected component* of a Lie groupoid G is the subgroupoid G_0 consisting of all $g \in G$ that can be connected to $s(g)$ by a path in $G_{s(g)}$.

In the preceding section, we obtained a description

$$G_{\mathrm{full}}(\mathcal{F}) = (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F}))$$

for the full holonomy groupoid (Theorem 8.6). As a corollary, we have the following result.

Theorem 9.2. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. If $M \setminus L$ is connected, then*

$$G(\mathcal{F}) \cong (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F}))_0.$$

If $M \setminus L$ has two connected components U and V, then

$$G(\mathcal{F}) \cong U^2 \cup V^2 \cup \mathrm{Gauge}(P^k(\mathcal{F}))_0.$$

Here, $\mathrm{Gauge}(P^k(\mathcal{F}))_0$ *denotes the s-connected component of* $\mathrm{Gauge}(P^k(\mathcal{F}))$. *The smooth structure of $G(\mathcal{F})$ is the one inherited from $G_{\mathrm{full}}(\mathcal{F})$.* ■

Although the bundle $P^k(\mathcal{F})$ is in fact a manifold in our sense (i.e., is metrizable), it is still very large; both it and its structure group have continuum-many components. One may therefore desire a more concrete description of $G(\mathcal{F})_L$ which avoids this bundle. The following proposition, whose proof we omit, shows that we in fact only need to deal with one connected component of $P_0 \subseteq P^k(\mathcal{F})$, which is automatically a second-countable manifold.

Proposition 9.3. *Let $P \to B$ be a (smooth, left) principal H-bundle, where H is a (possibly disconnected) Lie group.*

(1) *If $p \in P$, then the set of $h \in H$ for which hp belongs to the same component of P as p is a closed and open subgroup $H_p \subseteq H$.*

(2) *If p and q belong to the same component of P, then $H_p = H_q$.*

(3) *Let P_0 be a component of P. Then P_0 is a principal H_0-bundle, where H_0 is the stabilizer of P_0 or, equivalently, any particular fiber of P_0.*

(4) *The inclusion $P_0 \times P_0 \to P \times P$ descends to a Lie groupoid isomorphism from* $\mathrm{Gauge}(P_0)$ *onto the s-connected component* $\mathrm{Gauge}(P)_0 \subseteq \mathrm{Gauge}(P)$. ■

Corollary 9.4. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. Then $G(\mathcal{F})_L$ is isomorphic to* $\mathrm{Gauge}(P_0)$*, where P_0 is any connected component of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F})$.*

The following result gives additional insight into the holonomy *groups* of a transverse order k foliation at points of its singular leaf.

Theorem 9.5. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. Fix $x_0 \in L$ and $q_0 \in P^{k-1}(\mathcal{F})_{x_0}$. Let $\gamma\colon \pi_1(L, x_0) \to J^{k-1}$ be the monodromy homomorphism determined by q_0 and let $\Gamma \subseteq J^{k-1}$ be the range of γ. Finally, let $\Gamma_{\mathbb{R}} \subseteq J^k$ be the preimage of Γ under the projection $P^k(\mathcal{F}) \to P^{k-1}(\mathcal{F})$. Then, the isotropy group of $G(\mathcal{F})$ at any point of L is isomorphic, as a Lie group, to the one-dimensional group $\Gamma_{\mathbb{R}}$.*

For convenience, the following diagram summarizes the relationships between the various groups appearing in Theorem 9.5. We remark that Γ, being a quotient of a fundamental group of a manifold, is countable. Accordingly, the 1-dimensional Lie group $\Gamma_{\mathbb{R}}$ is second-countable.

$$\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & J^k_{\mathbb{R}} & \longrightarrow & J^{k-1}_d \\
\| & & \uparrow & & \uparrow \\
\mathbb{R} & \longrightarrow & \Gamma_{\mathbb{R}} & \longrightarrow & \Gamma \\
 & & & & \gamma \uparrow \\
 & & & & \pi_1(L, x_0)
\end{array}$$

Proof. Let Q_0 denote the connected component of $P^{k-1}(\mathcal{F})$ containing q_0. By standard covering space theory, the subgroup Γ equals the stabilizer of Q_0 so that, by Proposition 9.3, Q_0 becomes a principal Γ-bundle. Let P_0 be the preimage of Q_0 under the projection $P^k(\mathcal{F}) \to P^{k-1}(\mathcal{F})$. Since the fibers of this projection are copies of $\mathbb{R}$, it follows that P_0 is a connected component of $P^k(\mathcal{F})$. Thus, by Proposition 9.3 (4),

$$\mathrm{Gauge}(P_0) \cong \mathrm{Gauge}(P^k(\mathcal{F}))_0 \cong G(\mathcal{F})_L.$$

The subgroup of $J^k_{\mathbb{R}}$ stabilizing P_0 is $\Gamma_{\mathbb{R}}$. By Proposition 9.3 (3), P_0 is a principal $\Gamma_{\mathbb{R}}$-bundle, whence every isotropy group of $\mathrm{Gauge}(P_0)$ is isomorphic to $\Gamma_{\mathbb{R}}$. ∎

Corollary 9.6. *With notation as in Theorem 9.5, we have*

$$C^*(G(\mathcal{F})_L) \cong C^*(\Gamma_{\mathbb{R}}) \otimes \mathbb{K},$$

where $\mathbb{K}$ denotes the compact operators on the canonical L^2 space of L.

Proof. From [19, Theorem 3.1], it follows that, if $G \rightrightarrows M$ is any transitive Lie groupoid (hence a gauge groupoid), then $C^*(G) \cong C^*(H) \otimes \mathbb{K}(L^2(M))$, where H denotes any isotropy group of G. ∎

Remark 9.7. The homomorphisms γ in Theorem 9.5 are exactly the same ones used to define the holonomy invariant $h(\mathcal{F})$ (Definition 7.6), so Corollary 9.6 shows that knowledge of the invariant $h(\mathcal{F})$ is enough to compute $C^*(G(\mathcal{F})_L)$.

10. Transverse order k foliations on line bundles

Throughout this section, L is a connected, smooth manifold. If $E \to L$ is a smooth line bundle, we always identity L with the zero section of E, so that $L \subseteq E$.

Definition 10.1. Fix a positive integer r.

- If $E \to L$ is a line bundle, we write $J^r(E,\mathbb{R}) \to L$ for the natural smooth, left principal J^r-bundle whose fiber over $x \in L$ consists of all r-jets at x of local diffeomorphisms $E_x \to \mathbb{R}$ sending $x \mapsto 0$.
- If $U \subseteq E$ is open and $p\colon U \to \mathbb{R}$ is a submersion with $p^{-1}(0) = U \cap L$, we write $j^r(p)\colon U \cap L \to J^r(E,\mathbb{R})$ for the smooth section defined by $j^r(p)(x) := j^r_x(p|_{E_x \cap U})$.
- If $\pi_i\colon E_i \to L$ is a line bundle for $i = 1, 2$, we write $J^r(E_1, E_2) \to L$ for the smooth fiber bundle whose fiber over $x \in L$ consists of r-jets at x of (local) diffeomorphisms $(E_1)_x \to (E_2)_x$ sending $x \mapsto x$.
- Let $\theta\colon U_1 \to U_2$ be a diffeomorphism, where $U_i \subseteq E_i$ is open for $i = 1, 2$. We say that θ is *fiberwise* if $\pi_2(\theta(e)) = \pi_1(e)$ for all $e \in U_1$ and $\theta(L \cap U_1) = L \cap U_2$. If θ is fiberwise, we write $j^r(\theta)\colon U \cap L \to J^r(E_1, E_2)$ for the smooth section defined by $j^r(\theta)(x) = j^r_x(\theta|_{E_x \cap U})$.

The following proposition shows that all local sections of the jet bundles of Definition 10.1 can be lifted to honest maps.

Proposition 10.2. *Let r be a positive integer.*

(1) *Let E be a line bundle over L. Then, every local section of $J^r(E, \mathbb{R})$ is equal to $j^r(p)$ for some local submersion $p: U \to L$ with $p^{-1}(0) = L \cap U$.*

(2) *Let E_1 and E_2 be line bundles over L. Then, every local section of $J^r(E_1, E_2)$ is equal to $j^r(\theta)$ for some local fiberwise diffeomorphism θ.*

Proof. Both statements follow from the fact that each r-jet of a local diffeomorphism between a pair of one-dimensional vector spaces preserving 0 is represented by a unique polynomial mapping of degree $\leq r$. ∎

If $P \to M$ and $Q \to M$ are smooth principal H-bundles, we write $\mathrm{Hom}_H(P, Q)$ for the usual smooth fiber bundle over M whose fiber over $x \in M$ is the set of H-equivariant maps $P_x \to Q_x$. We note the following without proof.

Lemma 10.3. *If E_1 and E_2 are line bundles over L, then there is a fiber bundle isomorphism $\alpha \mapsto \alpha_*: J^r(E_1, E_2) \to \mathrm{Hom}_{J^r}(J^r(E_1, \mathbb{R}), J^r(E_2, \mathbb{R}))$ defined by $\alpha_*(\beta) = \beta \circ \alpha^{-1}$.* ∎

In particular, if θ is a fiberwise diffeomorphism from E_1 to E_2 defined on a neighbourhood of L, the corresponding global section $j^r(\theta): L \to J^r(E_1, E_2)$ determines a principal bundle isomorphism $j^r(\theta)_*: J^r(E_1, \mathbb{R}) \to J^r(E_2, \mathbb{R})$. Thus, $E \mapsto J^r(E, \mathbb{R})$ is functorial for fiberwise diffeomorphisms. In fact, by Proposition 10.2, *every* isomorphism $J^r(E_1, \mathbb{R}) \to J^r(E_2, \mathbb{R})$ is induced by a fiberwise diffeomorphism.

The following two results are the main findings of this section.

Theorem 10.4. *Let $\pi: E \to L$ be a smooth line bundle with connected base manifold L. There is a unique bijection $\mathcal{F} \mapsto \nabla_{\mathcal{F}}$ from*

(1) *the set of transverse order k foliations of E with singular leaf L to*

(2) *the set of (smooth) flat connections on $J^{k-1}(E, \mathbb{R})$*

such that a submersion $p: U \to \mathbb{R}$, where $U \subseteq E$ is open, satisfying $p^{-1}(0) = U \cap L$ is a local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion if and only if the associated section $j^{k-1}(p): U \cap L \to J^{k-1}(E, \mathbb{R})$ is parallel for $\nabla_{\mathcal{F}}$.

Proof. Suppose $U \subseteq E$ is open and $p, q: U \to \mathbb{R}$ are a pair of submersions satisfying $p^{-1}(0) = q^{-1}(0) = U \cap L$. Up to shrinking U around $U \cap L$, we may assume $p_x := p|_{U \cap E_x}$ and $q_x := q|_{U \cap E_x}$ are diffeomorphisms onto their images for all $x \in U \cap L$. Define then $\theta_x := q_x \circ (p_x)^{-1}$ for all $x \in U \cap L$ so that, by construction, θ_x is a smooth family of local diffeomorphisms of $\mathbb{R}$ satisfying $\theta_{\pi(e)}(p(e)) = q(e)$ on a neighbourhood of $U \cap L$. Define $h_{qp}: U \cap L \to J^{k-1}$ by $h_{qp}(x) = j^{k-1}_0(\theta_x) \in J^{k-1}$. By construction, h_{qp} is the transition function for the sections $j^{k-1}(p)$ and $j^{k-1}(q)$ (Definition 10.1).

Now, suppose $\mathcal{F}$ is a transverse order k foliation of E with singular leaf L and fix local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions $p, q: U \to \mathbb{R}$. We claim that the transition map $h_{qp}: U \cap L \to J^{k-1}$

is locally constant. By Proposition 5.4, given any point $x_0 \in U \cap L$, we may choose coordinates so that x_0 is the point $(0,0)$ in $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ and the foliation is $\mathcal{F}^k_{\mathbb{R}^n}$. Moreover, we may perform this coordinate change in such a way that $p\colon E \to \mathbb{R}$ and $\pi\colon E \to L$ are given by $p(x,y) = y$ and $\pi(x,y) = x$ (the linear structure of the fibers is not likely to be preserved under this coordinate change, but that is irrelevant). By Theorem 6.6, there are constants $a_1, \ldots, a_{k-1}$ and a smooth function f such that

$$\theta_x(y) = q(x,y) = a_1 y + \cdots + a_{k-1} y^{k-1} + f(x,y) y^k$$

holds on a neighbourhood of $(0,0)$, whence $h_{qp}(x) = j_0^{k-1}(\theta_x)$ is constant on a neighbourhood of any $x_0 \in U \cap L$. Therefore, applying Lemma 7.2, there is a unique J_d^{k-1}-bundle structure on $J^{k-1}(E,\mathbb{R})$ for which $j^{k-1}(p)$ is a smooth section for every local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$ submersion p. This shows the map $\mathcal{F} \mapsto \nabla_{\mathcal{F}}$ is well defined, where $\nabla_{\mathcal{F}}$ is the flat connection characterized in the theorem statement.

Conversely, suppose that ∇ is a flat connection on $J^{k-1}(E,\mathbb{R})$. For each local submersion $p\colon U_p \to \mathbb{R}$ with $p^{-1}(0) = L \cap U_p$ satisfying $j^{k-1}(p) = U_p \cap L$, define $\mathcal{F}_p = p^{-1}(\mathcal{F}^k_{\mathbb{R}})$. Thus, $\mathcal{F}_p$ is a foliation of U_p. We use Proposition 2.25 to glue these foliations together into a foliation $\mathcal{F}$ of M. To this end, it suffices to consider p and q with $U_p = U_q = U$. It is easy to see that $\mathcal{F}_p$ and $\mathcal{F}_q$ both restrict to the trivial one leaf foliation on $U \setminus L$. It remains only to check the foliations agree on a neighbourhood of each point $x_0 \in L$. Using that $j^{k-1}(p)$ and $j^{k-1}(q)$ are ∇-parallel and shrinking U about x_0, there exists $h \in J^{k-1}$ with

$$j^{k-1}(q) = h \cdot j^{k-1}(p).$$

We may pass again to coordinates such that $x_0 = (0,0) \in \mathbb{R}^\ell \times \mathbb{R}$ and $\pi(x,y) = x$, $p(x,y) = y$ near $(0,0)$. Writing $h = a_1 y + \cdots + a_{k-1} y^{k-1}$, the equation above says that $\frac{\partial^r q}{\partial y^r}(x,0) = a_r$ holds near $(0,0)$ for $r = 1, \ldots, k-1$. Thus, applying Theorem 6.6 again, we have that, in these coordinates, $p^{-1}(\mathcal{F}^k_{\mathbb{R}})$ and $q^{-1}(\mathcal{F}^k_{\mathbb{R}})$ agree with $\mathcal{F}^k_{\mathbb{R}^n}$ in a neighbourhood of $(0,0)$.

It is easy to see that these mappings between foliations and flat connections are inverses of each other. ∎

Next we show that the correspondence of the above theorem is natural with respect to fiberwise diffeomorphisms.

Theorem 10.5. *Let L be a connected, smooth manifold. For $i = 1, 2$, let E_i be a smooth line bundle over L equipped with a transverse order k foliation $\mathcal{F}_i$ with singular leaf L. Suppose that $\theta\colon U_1 \to U_2$ is a fiberwise diffeomorphism, where U_i is an open subset of E_i containing L_i. Then, the following are equivalent:*

1. $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$;
2. *the induced J^{k-1}-bundle isomorphism $j^{k-1}(\theta)_* : J^{k-1}(E_1,\mathbb{R}) \to J^{k-1}(E_2,\mathbb{R})$ pushes forward $\nabla_{\mathcal{F}_1}$ to $\nabla_{\mathcal{F}_2}$.*

Proof. Assume for convenience that $U_1 = E_1$, $U_2 = E_2$. Let $U \subseteq L$ be open and let $p: U \to L$ be a submersion satisfying $p^{-1}(0) = U \cap L$. Clearly, (1) is equivalent to the assertion: p is a local $\mathcal{F}_1$-$\mathcal{F}^k_{\mathbb{R}}$-submersion if and only if $p \circ \theta^{-1}$ is a local $\mathcal{F}_2$-$\mathcal{F}^k_{\mathbb{R}}$-submersion. Meanwhile (applying Proposition 10.2 (2)), (2) is equivalent to the assertion: $j^{k-1}(p)$ is $\nabla_{\mathcal{F}_1}$ parallel if and only if $j^{k-1}(\theta)_*(j^{k-1}(p))$ is $\nabla_{\mathcal{F}_2}$ parallel. Since

$$j^{k-1}(\theta)_*(j^{k-1})(p) = j^{k-1}(p) \circ j^{k-1}(\theta)^{-1} = j^{k-1}(p \circ \theta^{-1}),$$

the desired conclusion is an immediate consequence of Theorem 10.4. ■

11. Completeness of the holonomy invariant

Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. In this section, we show that the holonomy invariant (Definition 7.6)

$$h(\mathcal{F}) \in [\pi_1(L), J^{k-1}]$$

together with the diffeomorphism type of L is a complete invariant for the structure of $\mathcal{F}$ nearby to L.

Given a line bundle equipped with a transverse order k foliation whose singular leaf is the zero section, there are ostensibly two principal J^{k-1}_d-bundles at play: the one from Section 7 and the one from Section 10. As one would probably suspect, these two bundles are in fact the same.

Proposition 11.1. *Let $E \to L$ be a smooth line bundle and let $\mathcal{F}$ be a transverse order k foliation of E with singular leaf L. There is an isomorphism of principal J^{k-1}_d-bundles $P^{k-1}(\mathcal{F}) \to J^{k-1}(E, \mathbb{R})$ such that, if $p: U \to \mathbb{R}$ is a local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion and $x \in U \cap L$, then $j^k_x(p) \mapsto j^k_x(p|_{E_x \cap U})$. Here, $J^{k-1}(E, \mathbb{R})$ is given the structure of a J^{k-1}_d-bundle using the flat connection $\nabla_{\mathcal{F}}$ of Section 10.*

Proof. The map is clearly well defined and J^{k-1}-equivariant. Furthermore, for any local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion $p: U \to \mathbb{R}$, the sections

$$x \mapsto j^{k-1}_x(p): U \cap L \to P^{k-1}(\mathcal{F}), \quad x \mapsto j^{k-1}_x(p|_{E_x \cap U}): U \cap L \to J^{k-1}(E, \mathbb{R})$$

are smooth as part of the definition of those bundles. The first of these smooth local sections is clearly mapped to the second, and it follows that the map $P^{k-1}(\mathcal{F}) \to J^{k-1}(E, \mathbb{R})$ is smooth. ■

We now state and prove the main result of the section. One may refer to Section 2.4 and Definition 7.6 for context.

Theorem 11.2. *For $i = 1, 2$, let $(M_i, \mathcal{F}_i)$ be a transverse order k foliations with singular leaf L_i. Suppose there is a diffeomorphism $\theta_0: L_1 \to L_2$ for which the induced* *map* $[\pi_1(L_1), J^{k-1}] \to [\pi_1(L_2), J^{k-1}]$ *given by pushing forward loops sends* $h(\mathcal{F}_1)$ *to* $h(\mathcal{F}_2)$*. Then, we can extend* θ_0 *to a diffeomorphism* $\theta: U_1 \to U_2$*, where* $U_i \subseteq M_i$ *are open neighbourhoods of* L_i*, so as to have* $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$*.*

Proof. Without loss of generality, $L = L_1 = L_2$ and $\theta_0 = \mathrm{id}_L$. Thus, $h(\mathcal{F}_1) = h(\mathcal{F}_2)$. By Definition 7.6, this means exactly that the principal J_d^{k-1}-bundles $P^{k-1}(\mathcal{F}_1)$ and $P^{k-1}(\mathcal{F}_2)$ have the same monodromy invariant so, by Proposition 2.11, there is an isomorphism of principal J_d^{k-1}-bundles $P^{k-1}(\mathcal{F}_1) \to P^{k-1}(\mathcal{F}_2)$ covering the identity map on L.

By passing to tubular neighbourhoods, we may assume that M_i is the total space of a line bundle $\pi_i: E_i \to L$, with L contained in E_i as the zero section. Then, applying Proposition 11.1, there is a J^{k-1}-bundle isomorphism $J^{k-1}(E_1, \mathbb{R}) \to J^{k-1}(E_2, \mathbb{R})$, equivalently a section of $J^{k-1}(E_1, E_2)$, which moreover maps $\nabla_{\mathcal{F}_1}$ to $\nabla_{\mathcal{F}_2}$. By Proposition 10.2, we can take this section of the form $j^{k-1}(\theta)$ for θ some fiberwise diffeomorphism $E_1 \to E_2$ preserving L. Then, by Theorem 10.5, $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$, as desired. ∎

As a corollary, when L is simply connected, there is only one "L-local" isomorphism class of transverse order k foliation with singular leaf L.

Corollary 11.3. *Let* $(M, \mathcal{F})$ *be a transverse order k foliation with singular leaf L. If L is simply connected, then there exists a diffeomorphism* $\theta: U \to L \times \mathbb{R}$*, where* $U \subseteq M$ *is an open neighbourhood of L, such that* $\theta|_L = \mathrm{id}_L$ *and* $\theta_*(\mathcal{F}_U)$ *is the product the trivial one-leaf foliation of L with* $\mathcal{F}_{\mathbb{R}}^k$*.*

More generally, the same conclusion holds when there is no nontrivial homomorphism $\pi_1(L) \to J^{k-1}$. For example, this is the case when $\pi_1(L)$ is a simple, nonsolvable group (see Proposition 3.6).

12. Range of the holonomy invariant

In this section we show that the range of the holonomy invariant (Definition 7.6) is as large as one could hope it to be. Specifically, we prove the following result.

Theorem 12.1. *Fix a connected, smooth manifold L, a basepoint* $x_0 \in L$ *and a group homomorphism* $\gamma: \pi_1(L, x_0) \to J^{k-1}$*. Then, there exists a line bundle* $E \to L$ *and a transverse order k foliation* $\mathcal{F}$ *on E with singular leaf L (identified with the zero section of E) whose holonomy invariant* $h(\mathcal{F})$ *is equal to the class of* γ *in* $[\pi_1(L), J^{k-1}]$*.*

Together, Theorems 11.2 and 12.1 imply that the problem of enumerating the "L-local" isomorphism classes of transverse order k-foliations with a fixed singular leaf L is equivalent to the problem of enumerating homomorphisms $\gamma: \pi_1(L) \to J^{k-1}$ modulo a certain equivalence relation. Specifically,

$$\gamma \sim \alpha \circ \gamma \circ \beta$$

as α ranges over inner automorphisms of J^{k-1} and β ranges over automorphisms of $\pi_1(L)$ that can be induced by a diffeomorphism of L. Under a variety of assumptions on L, one is assured of being able to implement any automorphism of $\pi_1(L)$ with a diffeomorphism of L. For example, if L is a surface, then the homomorphism from the mapping class group to the group of outer automorphisms of $\pi_1(L)$ is an isomorphism (the Dehn–Nielsen–Baer theorem). Another example: for L a hyperbolic 3-manifold of dimension ≥ 3, one has that $\mathrm{Isom}(L) \to \mathrm{Out}(\pi_1(L))$ is an isomorphism by the Mostow rigidity theorem. In cases such as these, the L-local isomorphism classes of transverse order k foliations with singular leaf L can be put into correspondence with the quotient set

$$\mathrm{Inn}(J^{k-1}) \setminus \mathrm{Hom}(\pi_1(L), J^{k-1}) / \mathrm{Aut}(\pi_1(L)).$$

The proof of Theorem 12.1 will be an exercise in manipulating principal bundles and will have little to do with foliations per se. Let us first recall the associated principal bundle construction.

Definition 12.2. If $\varphi: G \to H$ is a homomorphism of Lie groups and P is a principal G-bundle, then $H \times_\varphi P$ denotes the principal H-bundle whose total space is the quotient of $H \times P$ by the equivalence relation defined by $(h, p) \sim (hg, g^{-1}p)$ for all $g \in G$ and equipped with the H-action defined by $h[h', p] = [hh', p]$. Here we denote the class of (h, p) by $[h, p]$.

See for example [13, Chapter 5]. The following instances of this construction will be relevant.

Example 12.3. If $1 \to N \to G \xrightarrow{\pi} H \to 1$ is an exact sequence of Lie groups and if P is a principal G-bundle, then there is a principal H-bundle isomorphism $P/N \to H \times_\pi P$ sending $[p] \mapsto [1, p]$.

Definition 12.4. Given a line bundle E, the *coframe bundle* $F^*(E)$ of E is the principal $\mathrm{GL}(1) := \mathrm{GL}(1, \mathbb{R})$-bundle whose fiber at x is the collection of nonzero linear functionals $E_x \to \mathbb{R}$.

Example 12.5. Let E be a smooth line bundle and r a positive integer. Denote by ι the natural inclusion $a \mapsto ay: \mathrm{GL}(1) \to J^r$. Then there is a unique principal J^r-bundle isomorphism $J^r \times_\iota F^*(E) \to J^r(E, \mathbb{R})$ that sends $[j^r_0(\theta), \varphi] \mapsto j^r_x(\theta \circ \varphi)$ for all $j^r_0(\theta) \in J^r$ and all linear isomorphisms $\varphi: E_x \to \mathbb{R}$.

Example 12.6. Let M be a connected smooth manifold with basepoint x_0. Let $\widetilde{M}$ be the universal cover of M with corresponding basepoint $\widetilde{x}_0$. Let Γ be a discrete group and let $\varphi: \pi_1(M, x_0) \to \Gamma$ be a homomorphism. View $\widetilde{M}$ as a principal $\pi_1(M, x_0)$-bundle in the usual way using deck transformations. Put $P = \Gamma \times_\varphi \widetilde{M}$ and $p_0 = [1, \widetilde{x}_0]$. Then, P is a principal Γ-bundle over M for which the monodromy homomorphism $\pi_1(M, x_0) \to \Gamma$ determined by the point p_0 is exactly φ.

Suppose we have an exact sequence of Lie groups $1 \to N \to G \xrightarrow{\pi} H \to 1$ that is split by a homomorphism $\iota\colon H \to G$. In other words, suppose G is presented as a semidirect product $N \rtimes H$. Then, using the fact that $\pi \circ \iota = \mathrm{id}_H$, there is a natural isomorphism

$$P \cong H \times_\pi (G \times_\iota P) \tag{5}$$

for every principal H-bundle P. On the other hand, $\iota \circ \pi \neq \mathrm{id}_G$ (unless π is an isomorphism), so there is no reason to expect the opposite relation

$$Q \cong G \times_\iota (H \times_\pi Q)$$

to hold for every principal G-bundle Q. With some knowledge of classifying space theory, however, one may prove the following.

Proposition 12.7. *Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an exact sequence of Lie groups and let $\iota\colon H \to G$ satisfy $\pi \circ \iota = \mathrm{id}_H$. If either ι or π is a homotopy equivalence, then, for every principal G-bundle Q, one has $Q \cong G \times_\iota (H \times_\pi Q)$.*

Note that, unlike the isomorphism (5), the isomorphism given by the above proposition is not natural.

Proof sketch. A homomorphism between G and H which is also a homotopy equivalence induces a homotopy equivalence at the level of classifying spaces. It follows that the associated bundle construction for such a homomorphism induces a bijection between isomorphism classes of G-bundles and isomorphism classes of H-bundles. The desired claim then follows from the principle that a one-sided inverse for an invertible map is the two-sided inverse. We refer the reader to [14, Chapter 4] or [27, Part II] for relevant aspects of classifying space theory. ∎

In the following two lemmas, we give an alternative proof of a special case of the above proposition which avoids the technology of classifying space theory.

Lemma 12.8. *Let $1 \to N \to G \to H \to 1$ be short exact sequence of Lie groups. Let $P \to B$ and $Q \to B$ be smooth principal G-bundles. If $P/N \cong Q/N$ as principal H-bundles and N is diffeomorphic to a finite-dimensional Euclidean space, then $P \cong Q$ as principal G-bundles.*

Proof. Recall $\mathrm{Hom}_G(P, Q)$ is defined as the fiber bundle over B whose fiber at $x \in B$ is the set of G-equivariant maps from P_x to Q_x. Reformulated slightly, the statement to be proven is that $\mathrm{Hom}_G(P, Q) \to B$ admits a (smooth) global section. We are given that the fiber bundle $\mathrm{Hom}_H(P/N, Q/N) \to B$ has a global section. We may instead view $\mathrm{Hom}_G(P, Q)$ as a fiber bundle over $\mathrm{Hom}_H(P/N, Q/N)$ with typical fiber N. Since N is Euclidean, it follows that the latter fiber bundle has a global section.

Indeed, the Tietze extension theorem can be used to show that every fiber bundle over a manifold with Euclidean space fibers has a continuous section (see [27, Theorem 12.2]) and routine smoothing arguments can then be applied. Composing the section $B \to \mathrm{Hom}_H(P/N, Q/N)$ with the section $\mathrm{Hom}_H(P/N, Q/N) \to \mathrm{Hom}_G(P, Q)$ gives the result. ∎

Lemma 12.9. *Let $1 \to N \to G \to H \to 1$ be a short exact sequence of Lie groups and let $\iota\colon H \to G$ satisfy $\pi \circ \iota = \mathrm{id}_H$. Let $P \to B$ be a smooth principal G bundle. If is N diffeomorphic to a finite-dimensional Euclidean space, then there is a smooth isomorphism of principal G-bundles from P to $G \times_\iota (H \times_\pi P)$.*

Proof. Let $Q := H \times_\pi P$. Since $\pi \circ \iota = \mathrm{id}_H$, we have $Q \cong H \times_\pi (G \times_\iota Q)$ as principal H-bundles. Noting that $Q \cong P/N$ and $H \times_\pi (G \times_\iota Q) \cong (G \times_\iota Q)/N$ (see Example 12.3), we have $P/N \cong (G \times_\iota Q)/N$. Thus, applying Lemma 12.8, P is isomorphic to $G \times_\iota Q$, as desired. ∎

In particular, the above lemma applies to the extension

$$1 \to N \to J^r \xrightarrow{\pi} \mathrm{GL}(1) \to 1,$$

where the quotient map is $a_1 y + \cdots + a_r y^r \mapsto a_1$, N is its kernel and $\iota(a) = ay$ provides a splitting on the right. From this we obtain the following result.

Proposition 12.10. *Let $P \to B$ be any principal J^r bundle, where r is a positive integer. Then, there exists a line bundle $E \to B$ such that $P \cong J^r(E, \mathbb{R})$ as principal J^r-bundles.*

Proof. Applying Lemma 12.9 to P with the extension $N \to J^r \to \mathrm{GL}(1)$ described above, we obtain $P \cong J^r \times_\iota Q$ where $Q := \mathrm{GL}(1) \times_\pi P$. Any GL(1)-bundle is isomorphic to the coframe bundle of its associated vector bundle, so $Q \cong F^*(E)$ where $E := \mathbb{R} \times_{\mathrm{GL}(1)} Q$. Thus, $P \cong J^r \times_\iota F^*(E) \cong J^r(E, \mathbb{R})$ (see Example 12.5). ∎

We conclude with the proof of the main result of this section.

Proof of Theorem 12.1. Let $\gamma\colon \pi_1(L, x_0) \to J^{k-1}$ be any homomorphism. Let $P \to L$ be any principal J_d^{k-1}-bundle such that the monodromy mapping associated to some point $p_0 \in P_{x_0}$ is γ. For instance, we may use $P = J^{k-1} \times_\gamma \widetilde{L}$, as in Example 12.6. We may equivalently view P as a principal J^{k-1}-bundle equipped with a flat connection. According to the preceding lemma, there exists a line bundle $E \to L$ and an isomorphism $P \to J^{k-1}(E, \mathbb{R})$ of principal J^{k-1}-bundles. Push forward the flat connection on P to $J^{k-1}(E, \mathbb{R})$ through this isomorphism, and then use Theorem 10.4 to produce a corresponding transverse order k foliation $\mathcal{F}$ of E with singular leaf L. By Proposition 11.1, $J^{k-1}(E, \mathbb{R})$ and $P^{k-1}(\mathcal{F})$ are isomorphic as J_d^{k-1}-bundles, so the holonomy invariant of $\mathcal{F}$ is as desired. ∎

Funding. This work was done as part of the author's Ph.D. dissertation which was completed at Pennsylvania State University and partially supported by the Natural Sciences and Engineering Research Council of Canada through a PGS-D award.

References

[1] I. Androulidakis, Classification of extensions of principal bundles and transitive Lie groupoids with prescribed kernel and cokernel. *J. Math. Phys.* **45** (2004), no. 10, 3995–4012 Zbl 1071.58017 MR 2095684

[2] I. Androulidakis and G. Skandalis, The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.* **626** (2009), 1–37 Zbl 1161.53020 MR 2492988

[3] I. Androulidakis and M. Zambon, Smoothness of holonomy covers for singular foliations and essential isotropy. *Math. Z.* **275** (2013), no. 3-4, 921–951 Zbl 1292.57020 MR 3127043

[4] I. Androulidakis and M. Zambon, Holonomy transformations for singular foliations. *Adv. Math.* **256** (2014), 348–397 Zbl 1294.53025 MR 3177296

[5] F. Bischoff, Álvaro del Pino, and A. Witte, Jets of foliations and b^k-algebroids. 2023, arXiv:2311.17045v1

[6] A. Connes, A survey of foliations and operator algebras. In *Operator algebras and applications, Part* 1 *(Kingston, Ont., 1980)*, pp. 521–628, Proc. Sympos. Pure Math. 38, Amer. Math. Soc., Providence, RI, 1982 MR 0679730

[7] C. Debord, Holonomy groupoids of singular foliations. *J. Differential Geom.* **58** (2001), no. 3, 467–500 Zbl 1034.58017 MR 1906783

[8] C. Debord, Local integration of Lie algebroids. In *Lie algebroids and related topics in differential geometry (Warsaw, 2000)*, pp. 21–33, Banach Center Publ. 54, Polish Acad. Sci. Inst. Math., Warsaw, 2001 Zbl 1007.22008 MR 1881646

[9] C. Debord, Longitudinal smoothness of the holonomy groupoid. *C. R. Math. Acad. Sci. Paris* **351** (2013), no. 15-16, 613–616 Zbl 1285.53019 MR 3119886

[10] S.-R. Fischer and C. Laurent-Gengoux, A classification of neighborhoods around leaves of a singular foliation. 2024, arXiv:2401.05966v2

[11] M. Francis, The smooth algebra of a one-dimensional singular foliation. 2020, arXiv:2011.08422v1

[12] M. D. Francis, *Groupoids and Algebras of Certain Singular Foliations with Finitely Many Leaves*. Ph.D. thesis, The Pennsylvania State University, University Park, PA, 2021 MR 4327081

[13] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology. Vol.* II: *Lie groups, principal bundles, and characteristic classes*. Pure and Applied Mathematics, Vol. 47-II, Academic Press Harcourt Brace Jovanovich, Publishers, New York and London, 1973 Zbl 0335.57001 MR 0336651

[14] D. Husemoller, *Fibre bundles*. Third edn., Grad. Texts in Math. 20, Springer, New York, 1994 Zbl 0794.55001 MR 1249482

[15] I. Kolář, P. W. Michor, and J. Slovák, *Natural operations in differential geometry*. Springer, Berlin, 1993 Zbl 0782.53013 MR 1202431

[16] K. Kunen and J. E. Vaughan (eds.), *Handbook of set-theoretic topology*. North-Holland Publishing Co., Amsterdam, 1984 Zbl 0546.00022 MR 0776619

[17] C. Laurent-Gengoux, S. Lavau, and T. Strobl, The universal Lie ∞-algebroid of a singular foliation. *Doc. Math.* **25** (2020), 1571–1652 Zbl 1453.53033 MR 4164730

[18] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*. London Math. Soc. Lecture Note Ser. 124, Cambridge University Press, Cambridge, 1987 Zbl 0683.53029 MR 0896907

[19] P. S. Muhly, J. N. Renault, and D. P. Williams, Equivalence and isomorphism for groupoid C^*-algebras. *J. Operator Theory* **17** (1987), no. 1, 3–22 Zbl 0645.46040 MR 0873460

[20] J. R. Munkres, *Topology*. 2nd edn., Prentice Hall, Upper Saddle River, NJ, 2000 Zbl 0951.54001 MR 3728284

[21] J. Pradines, Feuilletages: holonomie et graphes locaux. *C. R. Acad. Sci. Paris Sér. I Math.* **298** (1984), no. 13, 297–300 Zbl 0568.57018 MR 0765427

[22] J. Pradines, How to define the differentiable graph of a singular foliation. *Cahiers Topologie Géom. Différentielle Catég.* **26** (1985), no. 4, 339–380 Zbl 0576.57023 MR 0816646

[23] J. Pradines and B. Bigonnet, Graphe d'un feuilletage singulier. *C. R. Acad. Sci. Paris Sér.* I *Math.* **300** (1985), no. 13, 439–442 Zbl 0581.57013 MR 0794018

[24] J. Renault, *A groupoid approach to C^*-algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl 0433.46049 MR 0584266

[25] G. Scott, The geometry of b^k manifolds. *J. Symplectic Geom.* **14** (2016), no. 1, 71–95 Zbl 1348.53077 MR 3523250

[26] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology*. Second edn., Springer, Berlin etc., 1978 MR 0507446

[27] N. Steenrod, *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol 14, Princeton University Press, Princeton, NJ, 1951 Zbl 0054.07103 MR 0039258

[28] P. Stefan, Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc. (3)* **29** (1974), 699–713 Zbl 0342.57015 MR 0362395

[29] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.* **180** (1973), 171–188 Zbl 0274.58002 MR 0321133

[30] H. E. Winkelnkemper, The graph of a foliation. *Ann. Global Anal. Geom.* **1** (1983), no. 3, 51–75 Zbl 0526.53039 MR 0739904

Received 1 April 2024; revised 9 September 2024.

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