

# From length-preserving pushouts of graphs to one-surjective pullbacks of graph algebras

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**Abstract.** The unions of directed graphs are the simplest examples of pushouts of directed graphs. The conditions under which they contravariantly induce surjective gauge-equivariant pullbacks of graph  $C^*$ -algebras have been well studied and vastly instantiated in noncommutative topology (e.g., quantum balls and spheres). Herein, we go beyond the unions of graphs to systematically determine optimal conditions for more general length-preserving pushouts of graphs under which they contravariantly induce graded pullbacks of path algebras, Leavitt path algebras, and graph  $C^*$ -algebras. Our pullbacks are surjective only on one side, as dictated by natural examples and  $K$ -theory. The proposed new approach enlarges the scope of applications from admissible subgraphs (also called quotient graphs) to generalizations of unlabeled foldings of Stallings and collapsing the line graphs of graphs to initial graphs. Moreover, we introduce the concept of locally derived graphs, which substantially extends the paradigm of derived graphs (or skew products of graphs), and use the projection foldings from locally derived graphs to their base (or voltage) graphs to obtain one-surjective pullbacks of graph  $C^*$ -algebras.

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## 1. Introduction

Graph theory is one of the most accessible parts of combinatorics, and one often uses graphs to visualize and study abstract mathematical objects. For instance, the structure of a group can be encoded in its Cayley graph. In the same vein, with every (unital, basic,

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connected) finite-dimensional associative algebra over an algebraically closed field, we can associate a directed graph (or a quiver) from which the algebra in question can be recovered via a path-algebra construction (e.g., see [5]). Better still, every finite-dimensional hereditary associative algebra over an algebraically closed field is Morita equivalent to a path algebra. These two results show the importance of path algebras in the classification and representation theory of finite-dimensional associative algebras. Moreover, Leavitt path algebras, which provide an algebraic backbone of graph  $C^*$ -algebras, are defined as quotients of path algebras (e.g., see [1]).

The construction of path algebras, Leavitt path algebras, and graph  $C^*$ -algebras can be considered as a functor from a category of directed graphs to the category of algebras and algebra homomorphisms in two different ways: covariant and contravariant. The former was explored in [4, 13, 18, 31]. This paper is concerned with the latter. The standard category of graphs and graph homomorphism was spectacularly successful in the work of Stallings [32], and the contravariant induction for admissible subgraphs (also called quotient graphs) is ubiquitous, including natural examples in noncommutative topology explored by Hong and Szymański [20, 21]. However, only considering subgraphs restricts the standard contravariant functor to injective graph homomorphisms, which is at odds with an unlabeled Stallings folding (Example 2.8), collapsing the line graph of a Hawaiian earring graph (Equation (7.10)), and shrinking loops (Example 5.6), which all identify edges and vertices.

The first aim of this paper is to unravel optimal conditions for graph homomorphisms to contravariantly induce graded algebra homomorphisms between path algebras, Leavitt path algebras, and graph  $C^*$ -algebras. We achieve it in Lemma 4.2, Theorem 4.5, and Corollary 4.8, respectively, by fine-tuning subcategories of directed graphs. We thus arrive at the category of graphs and admissible graph homomorphisms (see Section 2) as a domain of a contravariant functor to the category of  $C^*$ -algebras and  $*$ -homomorphisms. It turns out that this contravariant functor is a special case of Katsura's contravariant functor from the category of topological graphs and factor maps to the category of  $C^*$ -algebras and  $*$ -homomorphisms.

We introduce and study new types of admissible graph homomorphism. In particular, as a basic non-trivial example of a non-injective graph homomorphism contravariantly inducing a gauge-equivariant  $*$ -homomorphism of graph  $C^*$ -algebras, we have an unlabeled Stallings folding. In this spirit, we define a generalized folding (Definition 5.1) as an example of a non-injective (except in the trivial case) admissible graph homomorphism. Better still, we show that a well-known isomorphism between the graph  $C^*$ -algebra of the line graph of a row-finite graph without sinks and the graph  $C^*$ -algebra of the initial graph is contravariantly induced from a generically non-injective graph homomorphism. Moreover, we significantly extend the concept of derived graphs (which include all Cayley graphs of finite groups) by defining locally derived graphs, and show that, for families of non-trivial finite groups, projection foldings from locally derived graphs to their base graphs are non-injective admissible graph homomorphisms. For starters, we exemplify such a graph homomorphism by shrinking vertex-simple loops of length

$n$  to the loop of length one, which induces inclusions of the circle  $C^*$ -algebra  $C(S^1)$  in  $C(S^1) \otimes M_n(\mathbb{C})$ . Then, we construct  $M_2(\mathcal{O}_3)$  as the graph  $C^*$ -algebra of a locally derived graph of the Hawaiian earring graph with two loops, and obtain a unital  $*$ -homomorphism  $\mathcal{O}_2 \rightarrow M_2(\mathcal{O}_3)$ .

Furthermore, an unexpected and important application of the contravariant induction was found recently in [16]. Therein, the authors construct a  $U(1)$ -equivariant unital  $*$ -homomorphism  $\mathcal{O}_N \rightarrow M_k(\mathcal{O}_M)$  whenever  $M - 1 = k(N - 1)$ , which is a necessary condition given by K-theory. The construction is given by the contravariant functor applied to a non-injective admissible graph homomorphism. It clearly exemplifies in action the target-bijectivity condition, which is the pivotal condition of admissibility. This application of contravariant functoriality complements the application of covariant functoriality [18] unraveling the unital  $*$ -homomorphisms  $\mathcal{O}_M \rightarrow \mathcal{O}_N$  of Kawamura [24, Lemma 2.1 and Section 6.1] (cf. [9, Section 3.3]) constructed whenever the same congruence  $M - 1 = k(N - 1)$ , necessitated by K-theory, is satisfied.

The second and principal outcome of this article are pushout-to-pullback theorems: for path algebras (Theorem 6.1), Leavitt path algebras (Theorem 6.4), and graph  $C^*$ -algebras (Theorem 6.5). These are key applications of the aforementioned contravariant-induction results. To the best of our knowledge, these are the first pushout-to-pullback theorems where pullbacks are surjective only on one side, as is the case in the mixed-pullback theorems of [8, 18]. However, even in these mixed-pullback theorems, gluing of vertices is not allowed, ruling out the above-mentioned examples. It is worth mentioning here that one-surjective pullbacks of  $C^*$ -algebras form an ideal setting for noncommutative topology, as the Mayer–Vietoris technology still works while new types of examples are within the scope of the theory.

As mentioned before, the contravariant induction was already studied for injective graph homomorphisms. It starts in [1, Definition 2.4.11] (quotient graphs), followed by [17] (admissible and strongly admissible inclusions), which was recently generalized in [6] (breaking vertices allowed). Our motivation comes from noncommutative topology, which includes the theory of  $q$ -deformations of algebras of functions on certain compact topological spaces. In [21], it was shown that pushouts (unions) of graphs lead to pullback structures of the  $C^*$ -algebras of even quantum spheres. This remarkable feature was explored in [6, 17], where more general pushout-to-pullback theorems were proved. Similar results can be found in the context of higher-rank graph  $C^*$ -algebras [26] and Cuntz–Pimsner algebras [29]. However, when restricted to graph  $C^*$ -algebras, these results have limited scope as they assume graphs to be without sinks [26] or to be row finite [29].

In Section 2, we consider three conditions on graph homomorphisms: properness, target bijectivity, and regularity. They turn out to be the discrete-topology versions of Katsura’s conditions [22, Definitions 2.1 and 2.6]. We prove that they define subcategories that in Section 4 are domains of contravariant functors for path algebras and Leavitt path algebras, respectively. On the way, in Section 3, we systematically study pushouts of graphs proving many needed technical results. In Section 5, we unravel new types of

admissible graph homomorphism focusing on non-injective admissible graph homomorphism. Section 6 crowns the paper with pushout-to-pullback theorems for path algebras, Leavitt path algebras, and graph  $C^*$ -algebras. The pushout-to-pullback result for graph  $C^*$ -algebras is obtained as a corollary of its Leavitt counterpart using a beautiful theorem of Chirvasitu [7]. Finally, we end the paper with Section 7 devoted to applications in noncommutative topology, which involve multichamber even quantum spheres, the Cuntz algebra  $\mathcal{O}_2$ , and the boundary quantum spheres of even Hong–Szymański quantum balls.

## 2. Graphs and morphisms

### 2.1. Directed graphs

A graph (directed graph, quiver) is a quadruple  $E := (E^0, E^1, s_E, t_E)$ , where

- $E^0$  is the set of *vertices*,
- $E^1$  is the set of *edges* (arrows),
- $E^1 \xrightarrow{s_E} E^0$  is the *source* map assigning to each edge its beginning,
- $E^1 \xrightarrow{t_E} E^0$  is the *target* (range) map assigning to each edge its end.

Let  $v$  be a vertex in a graph  $E$ . It is called a *sink* iff  $s_E^{-1}(v) = \emptyset$ , it is called a *source* iff  $t_E^{-1}(v) = \emptyset$ , and it is called *regular* iff it is not a sink and  $|s_E^{-1}(v)| < \infty$ . The subset of regular vertices of a graph  $E$  is denoted by  $\text{reg}(E)$ . A *finite path* in  $E$  is a vertex or a finite sequence  $(e_1, \dots, e_n)$  of edges satisfying

$$t_E(e_1) = s_E(e_2), \quad t_E(e_2) = s_E(e_3), \quad \dots, \quad t_E(e_{n-1}) = s_E(e_n). \quad (2.1)$$

We denote the set of all finite paths in  $E$  by  $\text{FP}(E)$ . The beginning  $s_E(p_n)$  of  $p_n$  is  $s_E(e_1)$  and the end  $t_E(p_n)$  of  $p_n$  is  $t_E(e_n)$ . The beginning and the end of a vertex is the vertex itself. Thus, we extend the source and target maps to  $s_{PE}, t_{PE}: \text{FP}(E) \rightarrow E^0$ . Vertices are considered finite paths of length 0. The *length* of a finite path that is not a vertex is the number of elements in the sequence. In particular, every edge is a path of length 1. We denote the set of all paths of length  $n$  by  $\text{FP}_n(E)$ .

### 2.2. Categories of graphs

Let  $E := (E^0, E^1, s_E, t_E)$  and  $F := (F^0, F^1, s_F, t_F)$  be graphs. A *homomorphism* from  $E$  to  $F$  is a pair of maps

$$(f^0 : E^0 \rightarrow F^0, f^1 : E^1 \rightarrow F^1) \quad (2.2)$$

satisfying the conditions:

$$s_F \circ f^1 = f^0 \circ s_E, \quad t_F \circ f^1 = f^0 \circ t_E. \quad (2.3)$$

We denote the category of graphs and graph homomorphisms by  $\text{OG}$  and call it the *standard category of graphs*. We call a graph homomorphism  $(f^0, f^1)$  *injective* or *surjective* iff both  $f^0$  and  $f^1$  are injective or surjective, respectively.

If  $(f^0, f^1): E \rightarrow F$  is a homomorphism of graphs, then we define  $f: \text{FP}(E) \rightarrow \text{FP}(F)$  as follows:

$$\begin{aligned} \forall v \in E^0: f(v) &:= f^0(v), \quad \forall e \in E^1: f(e) := f^1(e), \\ \forall (e_1, \dots, e_n) \in \text{FP}(E): f((e_1, \dots, e_n)) &:= (f^1(e_1), \dots, f^1(e_n)) \in \text{FP}(F). \end{aligned} \tag{2.4}$$

If  $(f^0, f^1)$  is injective or surjective, then so is  $f$ . Note also that now we can think of  $\text{FP}$  as a covariant functor from the category  $\text{OG}$  of graphs and graph homomorphisms to the category of sets and maps.

**Definition 2.1.** A *proper homomorphism* of graphs  $f: E \rightarrow F$  is a homomorphism of graphs whose both maps are finite-to-one, i.e.

$$\forall v \in E^0: |(f^0)^{-1}(v)| < \infty, \quad \forall e \in E^1: |(f^1)^{-1}(e)| < \infty.$$

We denote the category of graphs and proper graph homomorphisms by  $\text{POG}$ .

First, observe that  $\text{POG}$  is indeed a subcategory of  $\text{OG}$  due to the fact that the composition of finite-to-one maps is again a finite-to-one map. Moreover, if  $(f^0, f^1): E \rightarrow F$  is a proper homomorphism of graphs, then the induced map  $f: \text{FP}(E) \rightarrow \text{FP}(F)$  is finite to one. Indeed, if  $p \in \text{FP}(E)$  is a vertex, then  $f^{-1}(p) = (f^0)^{-1}(p)$  is a finite set. Next, let  $p = (p_1, \dots, p_n)$ ,  $p_i \in F^1$  for all  $1 \leq i \leq n$ , and  $q \in f^{-1}(p)$ . Then we can write  $q = (q_1, \dots, q_n)$ ,  $q_i \in E^1$  for all  $1 \leq i \leq n$ , and  $f(q) = (f^1(q_1), \dots, f^1(q_n)) = (p_1, \dots, p_n)$ . Hence,  $q_i \in (f^1)^{-1}(p_i)$  for all  $1 \leq i \leq n$ , so the number of elements in  $f^{-1}(p)$  is limited by the number of elements in

$$(f^1)^{-1}(p_1) \times \dots \times (f^1)^{-1}(p_i) \times \dots \times (f^1)^{-1}(p_n), \tag{2.5}$$

which is a finite set. Finally, observe also that, much as before, we can view  $\text{FP}$  as a covariant functor from the category  $\text{POG}$  to the category of sets and finite-to-one maps.

**Definition 2.2.** We say that a graph homomorphism  $(f^0, f^1): E \rightarrow F$  satisfies the *target-injectivity (target-surjectivity) condition* if

$$\forall x \in F^1: (f^1)^{-1}(x) \ni e \mapsto t_E(e) \in (f^0)^{-1}(t_F(x)) \text{ is injective (surjective)}. \tag{2.6}$$

We say that  $(f^0, f^1)$  satisfies the *target-bijection condition* if it satisfies both the target-injectivity condition and the target-surjectivity condition.

Note that the bijectivity of  $(f^0, f^1)$  implies the target bijectivity of  $(f^0, f^1)$ , so (2.6) is satisfied for  $(\text{id}_{E^0}, \text{id}_{E^1})$ . However, an injective homomorphism of graphs need not satisfy the target-bijection condition. Indeed, mapping the one-vertex graph into the one-loop graph by assigning the vertex to the base of the loop is an injective graph homomorphism but the target-bijection condition fails. Next, a graph homomorphism  $(f^0, f^1): E \rightarrow F$  that is injective on vertices and satisfies the target-bijection condition is injective: if  $e_1$  and  $e_2$  are edges such that  $f^1(e_1) = f^1(e_2)$ , then  $e_1, e_2 \in (f^1)^{-1}(f^1(e_1))$  and  $|(f^0)^{-1}(t_F(f^1(e_1)))| = 1$ , so  $e_1 = e_2$ .

Next, we present a more conceptual version of the target-bijection condition.

**Proposition 2.3.** *A graph homomorphism  $(f^0, f^1) : E \rightarrow F$  satisfies the target-bijectivity condition if and only if the commutative diagram*

$$\begin{array}{ccc}
 & E^1 & \\
 t_E \swarrow & & \searrow f^1 \\
 E^0 & & F^1 \\
 f^0 \searrow & & \swarrow t_F \\
 & F^0 &
 \end{array} \tag{2.7}$$

given by (2.3) is a pullback diagram in the category of sets and maps.

*Proof.* Recall that the pullback of  $f^0 : E^0 \rightarrow F^0$  and  $t_F : F^1 \rightarrow F^0$  in the category of sets and maps is the fibered product

$$E^0 \times_{F^0} F^1 := \{(v, x) \in E^0 \times F^1 \mid f^0(v) = t_F(x)\} \tag{2.8}$$

together with the projections onto each component. Since the diagram (2.7) is commutative, the universal property of the pullback manifests itself in the existence of the map

$$\Phi : E^1 \longrightarrow E^0 \times_{F^0} F^1, \quad e \longmapsto (t_E(e), f^1(e)). \tag{2.9}$$

We have to prove that  $\Phi$  is a bijection  $\Leftrightarrow (f^0, f^1)$  satisfies the target-bijectivity condition. In fact, we will prove that  $\Phi$  is injective if and only if  $(f^0, f^1)$  satisfies the target-injectivity condition, and that  $\Phi$  is surjective if and only if  $(f^0, f^1)$  satisfies the target-surjectivity condition.

First, note that (2.6) defines a family of maps labeled by  $x \in F^1$ :

$$E^1 \supseteq (f^1)^{-1}(x) \ni e \xrightarrow{\Phi_x} (t_E(e), f^1(e)) \in ((f^0)^{-1}(t_F(x)), x) \subseteq E^0 \times_{F^0} F^1. \tag{2.10}$$

It is clear that the target-injectivity of  $(f^0, f^1)$  is equivalent to the injectivity of  $\Phi_x$  for all  $x \in F^1$ , and that the target-surjectivity of  $(f^0, f^1)$  is tantamount to the surjectivity of  $\Phi_x$  for all  $x \in F^1$ . Observe also that if  $x \neq y$ , then  $\Phi_x$  and  $\Phi_y$  have disjoint domains and counterdomains, and the union of all domains is  $E^1$ . Now, since  $\Phi$  agrees with  $\Phi_x$  on the domain of the latter for any  $x \in F^1$ , it follows immediately that the target-injectivity condition of  $(f^0, f^1)$  is equivalent to the injectivity of  $\Phi$ .

Assume next the target-surjectivity of  $(f^0, f^1)$ . This implies that the union of the counterdomains of  $\Phi_x$  is  $E^0 \times_{F^0} F^1$ . Indeed, take any  $(v, x) \in E^0 \times_{F^0} F^1$ . Then  $v \in (f^0)^{-1}(t_F(x))$ , and there exists  $e \in E^1$  such that  $t_E(e) = v$  and  $f^1(e) = x$ , so  $(v, x) \in \Phi_x((f^1)^{-1}(x))$ . Now one can see that

$$\Phi = \bigsqcup_{x \in F^1} \Phi_x, \tag{2.11}$$

so the target-surjectivity of  $(f^0, f^1)$  implies the surjectivity of  $\Phi$ . Vice versa, since the image of  $\Phi$  contains the union of the counterdomains of  $\Phi_x$ , it is immediate that the surjectivity of  $\Phi$  implies the target-surjectivity of  $(f^0, f^1)$ . ■

We can now easily claim the desired composability of the target-bijection condition.

**Lemma 2.4.** *Restricting morphisms of the category POG to the morphisms satisfying the target-bijection condition yields a subcategory of POG.*

*Proof.* Let  $(f^0, f^1) : E \rightarrow F$  and  $(g^0, g^1) : F \rightarrow G$  be morphisms in POG. We already know that  $(g^1 \circ f^1, g^0 \circ f^0) \in \text{Mor}(\text{POG})$ . Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccc}
 E^1 & \xrightarrow{f^1} & F^1 & \xrightarrow{g^1} & G^1 \\
 \downarrow t_E & & \downarrow t_F & & \downarrow t_G \\
 E^0 & \xrightarrow{f^0} & F^0 & \xrightarrow{g^0} & G^0.
 \end{array} \tag{2.12}$$

Now it follows from standard category theory (e.g., see [3, Proposition 11.10]) that if both squares are pullback diagrams, then the outer rectangle is also a pullback diagram, which ends the proof by Proposition 2.3. ■

This subcategory enjoys the following useful result.

**Lemma 2.5.** *Let  $f : E \rightarrow F$  be a morphism in TBPOG and  $\alpha \in \text{FP}(E)$ . Then the map*

$$f^{-1}(f(\alpha)) \ni \beta \mapsto t_{PE}(\beta) \in f^{-1}(f(t_{PE}(\alpha))) \tag{2.13}$$

*is bijective.*

*Proof.* Note first that the statement is clearly true when  $\alpha$  is a vertex. Assume now that  $\alpha := e_1 \dots e_n$ , where all  $e_i$  are edges. Then

$$\forall v \in (f^0)^{-1}(f^0(t_E(e_n))) \exists! x_n \in (f^1)^{-1}(f^1(e_n)) : t_E(x_n) = v \tag{2.14}$$

by the target-bijection condition. Much in the same way,

$$\forall x_i \in (f^1)^{-1}(e_i) \exists! x_{i-1} \in (f^1)^{-1}(e_{i-1}) : t_E(x_{i-1}) = s_E(x_i), i \in \{2, \dots, n\}. \tag{2.15}$$

Thus, every vertex  $v \in (f^0)^{-1}(f^0(t_E(e_i)))$  generates a unique path in  $f^{-1}(f(\alpha))$  ending at  $v$ , i.e. the map (2.13) is bijective. ■

**Definition 2.6.** *A regular homomorphism of graphs  $(f^0, f^1) : E \rightarrow F$  is a homomorphism of graphs satisfying the condition*

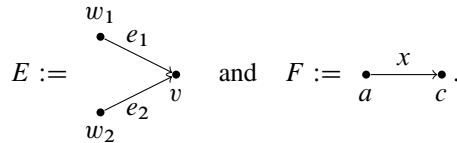
$$f^0(E^0 \setminus \text{reg}(E)) \subseteq F^0 \setminus \text{reg}(F). \tag{2.16}$$

Note that (2.16) can be equivalently written as

$$(f^0)^{-1}(\text{reg}(F)) \subseteq \text{reg}(E). \tag{2.17}$$

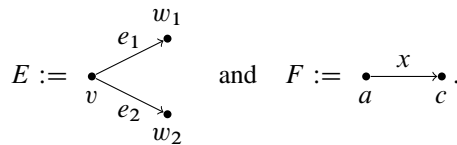
Moreover, it is clear that the identity is a regular homomorphism and that a composition of regular homomorphisms is regular. Thus, there exists a subcategory of TBPOG given by restricting morphisms therein to regular graph homomorphisms. We denote the category of graphs and regular proper homomorphisms of graphs satisfying the target-bijection condition by CRTBPOG, and call it the *admissible category of graphs*. Morphisms in this category are called *admissible graph homomorphisms*.

**Example 2.7.** Let



Then mapping the vertices  $w_1$  and  $w_2$  to  $a$ , the edges  $e_1$  and  $e_2$  to  $x$ , and the vertex  $v$  to  $c$  defines a regular proper graph homomorphism that does not satisfy the target-bijection condition.

**Example 2.8.** Let



Then mapping the vertex  $v$  to  $a$ , the vertices  $w_1$  and  $w_2$  to  $c$ , and the edges  $e_1$  and  $e_2$  to  $x$  defines an admissible graph homomorphism. This is an elementary example of an unlabeled Stallings folding [32].

### 3. Pushouts of graphs

We refer the reader to [11] for an extensive study of pushouts of directed graphs.

#### 3.1. Unions of graphs

We begin with unions of graphs, which are the simplest examples of pushouts of graphs. Let  $E$  and  $F$  be directed graphs. If there is an injective graph homomorphism  $(f^0, f^1) : E \hookrightarrow F$  given by inclusions, then we say that  $E$  is a *subgraph* of  $F$ , which we write  $E \subseteq F$ . Next, let  $F$  and  $G$  be graphs. Assume that  $s_F$  and  $t_F$  agree, respectively, with  $s_G$  and  $t_G$  on  $F^1 \cap G^1$ . Then we can define the *intersection graph*

$$F \cap G := (F^0 \cap G^0, F^1 \cap G^1, s_\cap, t_\cap), \tag{3.1}$$

where  $s_\cap, t_\cap : F^1 \cap G^1 \rightarrow F^0 \cap G^0$  are given by

$$\forall e \in F^1 \cap G^1 : \quad s_\cap(e) := s_G(e) = s_F(e), \quad t_\cap(e) := t_G(e) = t_F(e). \tag{3.2}$$

Next, we can define the *union* graph

$$F \cup G := (F^0 \cup G^0, F^1 \cup G^1, s_{\cup}, t_{\cup}), \tag{3.3}$$

where  $s_{\cup}, t_{\cup} : F^1 \cup G^1 \rightarrow F^0 \cup G^0$  are given by

$$\forall e \in F^1 \cup G^1 : \quad s_{\cup}(e) := \begin{cases} s_F(e) & \text{for } e \in F^1, \\ s_G(e) & \text{for } e \in G^1, \end{cases} \quad \text{and} \quad t_{\cup}(e) := \begin{cases} t_F(e) & \text{for } e \in F^1, \\ t_G(e) & \text{for } e \in G^1. \end{cases}$$

The intersection graph  $F \cap G$  is a subgraph of both  $F$  and  $G$ , and both  $F$  and  $G$  are subgraphs of the union graph  $F \cup G$ . The intersection graph  $F \cap G$  exists if and only if the union graph  $F \cup G$  exists.

Now, we recall the concept of hereditary and saturated subsets of the set of vertices in a graph. Let  $E$  be a graph. A subset  $H \subseteq E^0$  is called *hereditary* if any edge starting at  $v \in H$  ends at  $w \in H$ , and it is called *saturated* if there does not exist a regular vertex  $v \in E^0 \setminus H$  such that  $t_E(s_E^{-1}(v)) \subseteq H$ . Note that in the above definition of a hereditary subset one can replace the word “edge” by the phrase “finite path”. Observe also that the formulas  $s_F(e) := s_E(e)$ ,  $t_F(e) := t_E(e)$ ,  $e \in F^1 := E^1 \setminus t_E^{-1}(H)$ , define a subgraph  $F$  of  $E$  with  $F^0 := E^0 \setminus H$  if and only if  $H$  is hereditary.

Furthermore, we say that  $v \in E^0$  is a *breaking vertex* for  $H$  iff

$$v \in E^0 \setminus H, \quad |s_E^{-1}(v)| = \infty, \quad \text{and} \quad 0 < |s_E^{-1}(v) \cap t_E^{-1}(E^0 \setminus H)| < \infty. \tag{3.4}$$

We denote the set of all breaking vertices for  $H$  by

$$B_H := \{v \in E^0 \setminus H \mid v \text{ is a breaking vertex for } H\}. \tag{3.5}$$

A subset  $H$  of  $E^0$  is called *unbroken* if and only if  $B_H = \emptyset$ . Note that a breaking vertex of  $H$  becomes regular in the subgraph obtained by removing all vertices in  $H$  and all edges ending in  $H$ .

We are ready now to bundle up the three properties of being hereditary, saturated, and unbroken to restrict subgraphs to these that played a crucial role in [17].

**Definition 3.1.** An injective graph homomorphism  $(f^0, f^1) : E \hookrightarrow F$  is called  *$\cup$ -admissible* iff it satisfies the following conditions:

- (A1)  $F^0 \setminus f^0(E^0)$  is saturated,
- (A2)  $t_F^{-1}(f^0(E^0)) \subseteq f^1(E^1)$ .

We call a  $\cup$ -admissible injective graph homomorphism *strongly  $\cup$ -admissible* iff, in addition, the subset  $F^0 \setminus f^0(E^0)$  is unbroken. In the case the maps defining a (strongly)  $\cup$ -admissible injective graph homomorphism  $(f^0, f^1)$  are inclusions, we call  $E$  a (strongly) *admissible subgraph* of  $F$ . Furthermore, we call intersecting graphs  $F$  and  $G$  (strongly) *admissible* if both inclusions  $F \cap G \subseteq F$  and  $F \cap G \subseteq G$  are (strongly)  $\cup$ -admissible. Much in the same way, we call taking the union of graphs  $F$  and  $G$  (strongly) *admissible* if both inclusions  $F \subseteq F \cup G$  and  $G \subseteq F \cup G$  are (strongly)  $\cup$ -admissible.

The above definition already appeared in [8, Definition 3.1] (see also [17, Definition 2.1]), where it is also assumed that  $f^1(E^1) \subseteq t_F^{-1}(f^0(E^0))$  and  $F^0 \setminus f^0(E^0)$  is hereditary. However, the first condition is always true for any graph homomorphism and the heredity follows from the condition (A2).

**Proposition 3.2.** *Let  $(f^0, f^1) : E \rightarrow F$  be any graph homomorphism satisfying (A2). Then  $F^0 \setminus f^0(E^0)$  is hereditary.*

*Proof.* Suppose that  $F^0 \setminus f^0(E^0)$  is not hereditary, i.e. there is  $x \in F^1$  such that  $s_F(x) \in F^0 \setminus f^0(E^0)$  and  $t_F(x) \in f^0(E^0)$ . Since  $t_F^{-1}(f^0(E^0)) \subseteq f^1(E^1)$ , there is an edge  $e \in E^1$  such that  $f^1(e) = x$ . Then  $s_F(x) = s_F(f^1(e)) = f^0(s_E(e))$ , which gives a contradiction. ■

Next, we turn to unbroken subsets. Our next result shows that the assumption of strong admissibility of taking the union in [17, Theorem 3.1] is superfluous.

**Lemma 3.3.** *Let  $F$  and  $G$  be arbitrary graphs whose source and target maps agree, respectively, on  $F^1 \cap G^1$ . Then, if intersecting  $F$  and  $G$  is strongly admissible, so is taking the union of  $F$  and  $G$ .*

*Proof.* Assume that intersecting  $F$  and  $G$  is strongly admissible. Then, to prove that also taking the union of  $F$  and  $G$  is strongly admissible, it suffices to show that both  $(F^0 \cup G^0) \setminus F^0$  and  $(F^0 \cup G^0) \setminus G^0$  are unbroken in  $F \cup G$ . To this end, suppose that  $v \in F^0$  is a breaking vertex for  $(F^0 \cup G^0) \setminus F^0$  in  $F \cup G$ . Then  $v$  emits infinitely many, whence not zero, edges in  $G^0 \setminus F^0$ , so  $v \in F^0 \cap G^0$ . Also, all these edges are from  $G^1$ . Furthermore,  $v$  emits at least one and at most finitely many edges ending in  $F^0$ . If all of them end in  $F^0 \setminus G^0$ , then they are all from  $F^1$ , and they render  $F^0 \setminus G^0$  not saturated in  $F$ , which is not allowed by the  $\cup$ -admissibility of  $(F \cap G) \subseteq F$ . Therefore,  $v$  emits at least one edge  $e$  ending in  $F^0 \cap G^0$ . Now, from the  $\cup$ -admissibility of  $(F \cap G) \subseteq F$  and  $(F \cap G) \subseteq G$ , we obtain

$$e \in t_U^{-1}(F^0 \cap G^0) = t_F^{-1}(F^0 \cap G^0) \cup t_G^{-1}(F^0 \cap G^0) = F^1 \cap G^1, \tag{3.6}$$

so  $e \in G^1$ . Also, since  $v$  emits only finitely many edges into  $F^0$ , in particular, it emits only finitely many edges from  $G^1$  ending in  $F^0 \cap G^0$ . All this makes  $v$  a breaking vertex for  $G^0 \setminus F^0$  in  $G$ , which contradicts the strong  $\cup$ -admissibility of  $(F \cap G) \subseteq G$ . Hence,  $(F^0 \cup G^0) \setminus F^0$  is unbroken in  $F \cup G$ . Finally, the symmetric argument shows that  $(F^0 \cup G^0) \setminus G^0$  is unbroken in  $F \cup G$ . ■

Now, let us show that  $\cup$ -admissible injective graph homomorphisms are special cases of morphisms in the category CRTBPOG. This is why we call CRTBPOG the admissible category of graphs.

**Proposition 3.4.** *Let  $(f^0, f^1) : E \hookrightarrow F$  be an injective graph homomorphism. Then  $(f^0, f^1)$  is  $\cup$ -admissible if and only if it is admissible.*

*Proof.* ( $\Rightarrow$ ) Since  $(f^0, f^1)$  is injective, it is clearly proper. Next, we check the regularity of  $(f^0, f^1)$ . Since infinite emitters in the subgraph remain infinite emitters in the graph,

it suffices to consider the images of sinks. Assume that  $v \in E^0$  is a sink and suppose that  $f^0(v)$  is regular in  $F$ . First, note that  $t_F(s_F^{-1}(f^0(v))) \subseteq F^0 \setminus f^0(E^0)$ . Indeed, take an edge  $y \in F^1$  such that  $s_F(y) = f^0(v)$  and suppose that  $t_F(y) \in f^0(E^0)$ . Then, by (A2), there is an edge  $e \in E^1$  such that  $f^1(e) = y$ . However, since  $f^0(s_E(e)) = s_F(f^1(e)) = f^0(v)$ , by the injectivity of  $f^0$ , we obtain that  $s_E(e) = v$ , which contradicts the assumption that  $v$  is a sink. Consequently, what we have just proved contradicts the fact that  $F^0 \setminus f^0(E^0)$  is saturated (the condition (A1)). Finally, we have to show that the target-bijection condition is satisfied. Due to the injectivity of  $(f^0, f^1)$ , we know that, for any  $x \in F^1$ , the sets  $(f^1)^{-1}(x)$  and  $(f^0)^{-1}(t_F(x))$  are either empty or consist of a single element. To prove the claim, it suffices to exclude the possibility in which one of these sets is empty and the other one is not. First, if  $(f^1)^{-1}(x) = \{e\}$ , then  $t_E(e) \in (f^0)^{-1}(t_F(x))$ . Next, since  $t_F^{-1}(f^0(E^0)) \subseteq f^1(E^1)$ , if  $(f^0)^{-1}(t_F(x)) = \{v\}$ , then  $(f^1)^{-1}(x) \neq \emptyset$ .

( $\Leftarrow$ ) Assume that  $(f^0, f^1) : E \hookrightarrow F$  is a morphism in CRTBPOG. First, let us prove that  $t_F^{-1}(f^0(E^0)) \subseteq f^1(E^1)$ . Let  $x \in F^1$  be such that  $t_F(x) \in f^0(E^0)$ . This implies that there is  $v \in E^0$  such that  $f^0(v) = t_F(x)$ . In turn, from the target-bijection condition, we infer that there exists  $e \in (f^1)^{-1}(x)$  such that  $t_E(e) = v$ . Now it suffices to prove that  $F^0 \setminus f^0(E^0)$  is saturated. To this end, suppose that there is a regular vertex  $w \in f^0(E^0)$  such that  $t_F(s_F^{-1}(w)) \subseteq F^0 \setminus f^0(E^0)$ . It follows that  $s_F^{-1}(w) \cap f^1(E^1) = \emptyset$ . Indeed, suppose that there is an edge  $y \in F^1$  such that  $s_F(y) = w$  and  $f^1(e) = y$  for some  $e \in E^1$ . Then  $t_F(y) = t_F(f^1(e)) = f^0(t_E(e))$ , which contradicts the assumption. Therefore, if  $v \in E^0$  is a vertex such that  $f^0(v) = w$ , then  $s_E^{-1}(v) = \emptyset$ , which contradicts the regularity of  $(f^0, f^1)$ . ■

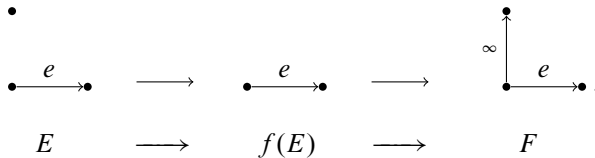
**Remark 3.5.** Note that, if  $f := (f^0, f^1) : E \rightarrow F$  is an admissible graph homomorphism, then  $f(E)$  is an admissible subgraph of  $F$ . Indeed, suppose that  $f^0(v) \in \text{reg}(F)$  and

$$t_F(s_F^{-1}(f^0(v))) \subseteq F^0 \setminus f^0(E^0).$$

Then  $v \in \text{reg}(E)$ , so there exists  $e \in s_E^{-1}(v)$ . Furthermore,  $f^1(e) \in s_F^{-1}(f^0(v))$  but

$$t_F(f^1(e)) = f^0(t_E(e)) \in f^0(E^0),$$

which gives a contradiction, and proves Definition 3.1 (A1). To prove Definition 3.1 (A2), take any  $x \in t_F^{-1}(f^0(E^0))$ . (If  $t_F^{-1}(f^0(E^0)) = \emptyset$ , we are done.) Then  $(f^0)^{-1}(t_F(x)) \neq \emptyset$ , so, by the target-surjectivity of  $f$ , there exists  $e \in (f^1)^{-1}(x)$ . Hence,  $x \in f^1(E^1)$ , as needed. However, restricting the codomain of  $f$  to its image might yield a graph homomorphism  $E \rightarrow f(E)$  that is not admissible:



**3.2. Pushouts of graphs in different categories**

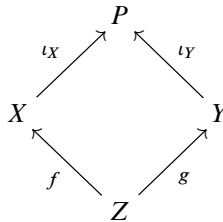
In the category of sets and maps, the pushout of  $X \xleftarrow{f} Z \xrightarrow{g} Y$  is

$$X \xrightarrow{\iota_X} X \amalg_Z Y \xleftarrow{\iota_Y} Y, \quad X \amalg_Z Y := (X \amalg Y)/R_Z, \tag{3.7}$$

where  $R_Z$  is the minimal equivalence relation generated by  $f(z)R_Zg(z)$ ,  $z \in Z$ , and  $\iota_X$  and  $\iota_Y$  are the obvious induced maps. We call a pushout diagram *one-injective* whenever at least one of the defining maps is injective.

The above pushout construction does not always yield a pushout in the category of sets and finite-to-one maps. Therefore, we need the following elementary result.

**Lemma 3.6.** *Let*



*be a pushout diagram in the category of sets and maps. If one of the maps  $f$  and  $g$  is injective and the other one is finite to one, then the above diagram is a pushout diagram in the category of sets and finite-to-one maps.*

*Proof.* Assume without the loss of generality that  $f$  is injective and  $g$  is finite to one. Then the canonical maps of their pushout

$$X \xrightarrow{\iota_X} X \amalg_Z Y \xleftarrow{\iota_Y} Y \tag{3.8}$$

are also finite to one. Indeed, if  $\iota_Y(y) = \iota_Y(y')$ , then  $y = y'$  or there exists a finite sequence  $(z_1, \dots, z_{2n}) \in Z^{2n}$  such that

$$\begin{aligned} y &= g(z_1) \quad \text{and} \quad f(z_1) = x_1 = f(z_2), \\ g(z_2) &= y_1 = g(z_3) \quad \text{and} \quad f(z_3) = x_2 = f(z_4), \\ &\vdots \\ g(z_{2n-2}) &= y_{n-1} = g(z_{2n-1}) \quad \text{and} \quad f(z_{2n-1}) = x_n = f(z_{2n}), \\ g(z_{2n}) &= y'. \end{aligned} \tag{3.9}$$

In the latter case, from the injectivity of  $f$ , we conclude that  $z_{2k-1} = z_{2k}$  for all  $k \in \{1, \dots, n\}$ , so

$$y = g(z_1) = g(z_2) = y_1 = \dots = y_{n-1} = g(z_{2n-1}) = g(z_{2n}) = y'. \tag{3.10}$$

Hence,  $\iota_Y$  is injective. Next, if  $\iota_X(x) = \iota_X(x')$ , then  $x = x'$  or there exists a finite sequence  $(z_1, \dots, z_{2m}) \in Z^{2m}$  such that

$$\begin{aligned}
 x &= f(z_1) \quad \text{and} \quad g(z_1) = y_1 = g(z_2), \\
 f(z_2) &= x_1 = f(z_3) \quad \text{and} \quad g(z_3) = y_2 = g(z_4), \\
 &\vdots \\
 f(z_{2m-2}) &= x_{m-1} = f(z_{2m-1}) \quad \text{and} \quad g(z_{2m-1}) = y_m = g(z_{2m}), \\
 f(z_{2m}) &= x'.
 \end{aligned} \tag{3.11}$$

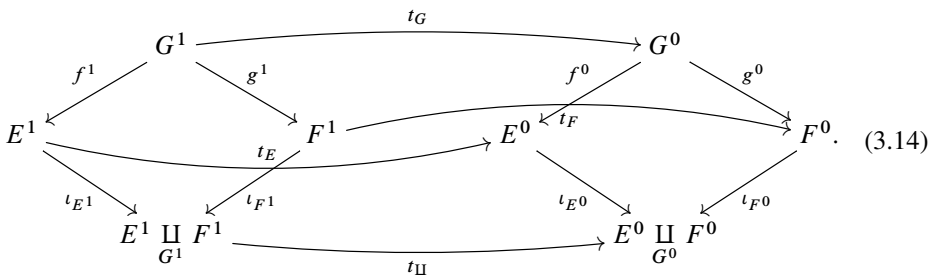
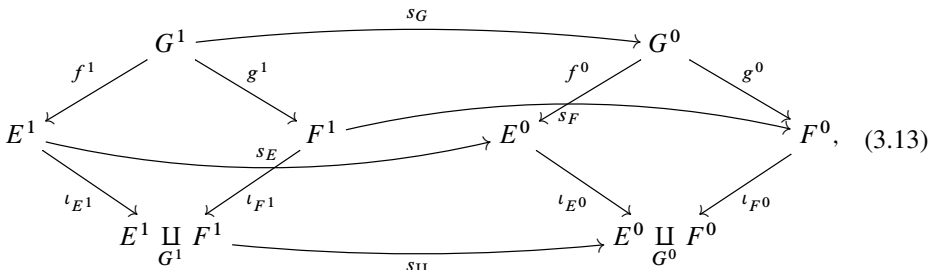
It follows from the injectivity of  $f$  that  $z_{2k} = z_{2k+1}$  for all  $k \in \{1, \dots, m - 1\}$ , so

$$g(z_1) = y_1 = g(z_2) = g(z_3) = y_2 = \dots = g(z_{2m-2}) = g(z_{2m-1}) = y_m. \tag{3.12}$$

Therefore, all  $y_i$  are equal to  $g(z_1)$ , where  $z_1$  is uniquely determined by  $x$ . Hence, also  $y_m$  is uniquely determined by  $x$ . Consequently, as  $x' \in f(g^{-1}(y_m))$ , which is a finite set by assumption, there are only finitely many  $x'$  such that  $\iota_X(x) = \iota_X(x')$ , so the map  $\iota_X$  is finite to one.

Finally, if  $j_X: X \rightarrow Q$  and  $j_Y: Y \rightarrow Q$  are finite-to-one maps such that  $j_X \circ f = j_Y \circ g$ , then the universal-property map  $h: P \rightarrow Q$  is also finite to one. Indeed, suppose that the set  $h^{-1}(q)$  is infinite for some  $q \in Q$ . Then, as  $P = \iota_X(X) \cup \iota_Y(Y)$ , one of the sets  $h^{-1}(q) \cap \iota_X(X)$  and  $h^{-1}(q) \cap \iota_Y(Y)$  is infinite, which contradicts the assumption that both  $j_X = h \circ \iota_X$  and  $j_Y = h \circ \iota_Y$  are finite to one. ■

Let  $E \xleftarrow{(f^0, f^1)} G \xrightarrow{(g^0, g^1)} F$  be graph homomorphisms and let  $E^i \amalg_{G^i} F^i$  be the corresponding pushout of  $E^i \xleftarrow{f^i} G^i \xrightarrow{g^i} F^i$ ,  $i = 0, 1$ , in the category of sets. Then we have the following commutative diagrams:



Here the left and right square subdiagrams commute by the definition of a pushout, and the top subdiagrams commute by the definition of a graph homomorphism. Moreover,  $s_{\sqcup}$  and  $t_{\sqcup}$  exist by the universal property of the pushout  $E^1 \sqcup_{G^1} F^1$ , which applies due to the equalities

$$\begin{aligned} \iota_{E^0} \circ s_E \circ f^1 &= \iota_{E^0} \circ f^0 \circ s_G = \iota_{F^0} \circ g^0 \circ s_G = \iota_{F^0} \circ s_F \circ g^1, \\ \iota_{E^0} \circ t_E \circ f^1 &= \iota_{E^0} \circ f^0 \circ t_G = \iota_{F^0} \circ g^0 \circ t_G = \iota_{F^0} \circ t_F \circ g^1, \end{aligned} \tag{3.15}$$

which in turn follow, respectively, from the aforementioned commutativities in the above diagrams. Better still, the universal property of the pushout guarantees that both  $s_{\sqcup}$  and  $t_{\sqcup}$  are uniquely determined, and they render, respectively, the bottom subdiagrams of both diagrams commutative.

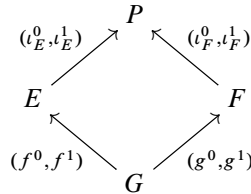
**Definition 3.7.** We call the graph

$$E \sqcup_G F := (E^0 \sqcup_{G^0} F^0, E^1 \sqcup_{G^1} F^1, s_{\sqcup}, t_{\sqcup})$$

the *pushout graph* of  $E \xleftarrow{(f^0, f^1)} G \xrightarrow{(g^0, g^1)} F$ .

It is straightforward to show that  $E \sqcup_G F$  is indeed a pushout in the category OG of graphs and graph homomorphisms. However, a pushout in the category OG might not be a pushout in the admissible category of graphs. Therefore, we need the following result.

**Lemma 3.8.** *Let the diagram*



be a one-injective pushout diagram in the category OG of graphs and graph homomorphisms. If  $(f^0, f^1)$  and  $(g^0, g^1)$  are proper, regular, and satisfy the target-bijectivity condition, then the same is true, respectively, for  $(\iota_E^0, \iota_E^1)$  and  $(\iota_F^0, \iota_F^1)$ .

*Proof.* Let  $(f^0, f^1)$  be injective and regular and let  $(g^0, g^1)$  be proper and regular. Then, by Lemma 3.6, both  $(\iota_E^0, \iota_E^1)$  and  $(\iota_F^0, \iota_F^1)$  are proper, and, by the pushout property,  $(\iota_F^0, \iota_F^1)$  is injective. Next, to prove the regularity of  $(\iota_F^0, \iota_F^1)$ , note that every infinite emitter in  $F^0$  stays an infinite emitter in its image under  $\iota_F^0$ . Now, consider a sink  $v \in F^0$  and suppose that  $w := \iota_F^0(v)$  is regular. As  $\iota_F^0$  is injective and  $v$  is a sink, any edge emitted from  $w$  must be in the image of  $\iota_E^1$ . Hence,  $w$  is in the image of  $\iota_E^0$ , so  $v = g^0(u)$  for some  $u \in G^0$ . Then  $u$  has to be a sink because if there is an edge  $a \in G^1$  such that  $s_G(a) = u$ , then

$$v = g^0(u) = g^0(s_G(a)) = s_E(g^1(a)), \tag{3.16}$$

which is impossible. Furthermore, observe that  $w = \iota_F^0(g^0(u)) = \iota_E^0(f^0(u))$ . The vertex  $f^0(u)$  cannot be an infinite emitter because  $\iota_E^1$  is proper and  $w$  is regular. So suppose that  $f^0(u)$  is a sink. Then, since  $g^0(u) = v$  is also a sink, this would again contradict the regularity of  $w$ , so  $f^0(u)$  is regular, which contradicts the regularity of  $(f^0, f^1)$  because  $u$  is a sink.

Next, suppose that  $(\iota_E^0, \iota_E^1)$  is not regular, i.e. that there is a vertex  $v \in E^0 \setminus \text{reg}(E)$  such that  $w := \iota_E(v) \in \text{reg}(P)$ . Since  $\iota_E$  is proper, the vertex  $v$  cannot be an infinite emitter. Suppose that  $v$  is a sink. If  $v \notin f^0(G^0)$ , then  $w \notin \iota_F^0(F^0)$ , so we get a contradiction because  $\iota_E^0$  is the identity when restricted to  $E^0 \setminus f^0(G^0)$ . If there is a vertex  $u \in G^0$  such that  $f^0(u) = v$ , then, arguing as before,  $u$  is a sink. Hence, as  $w = \iota_E^0(f^0(u)) = \iota_F^0(g^0(u))$ , we get a contradiction with the regularity of  $(\iota_F^0 \circ g^0, \iota_F^1 \circ g^1)$ .

Let us now prove that  $(\iota_F^0, \iota_F^1)$  and  $(\iota_E^0, \iota_E^1)$  satisfy the target-bijectivity condition. Take  $x \in P^1$ . Since  $(\iota_F^0, \iota_F^1)$  is injective, it suffices to exclude the two possibilities:

$$(T1) \quad (\iota_F^1)^{-1}(x) = \emptyset \text{ and } (\iota_F^0)^{-1}(t_P(x)) = \{v\} \text{ for some } v \in F^0,$$

$$(T2) \quad (\iota_F^1)^{-1}(x) = \{e\} \text{ for some } e \in F^1 \text{ and } (\iota_F^0)^{-1}(t_P(x)) = \emptyset.$$

Suppose that the condition (T1) is satisfied, so  $\iota_F^0(v) = t_P(x)$ . Since  $P^1 = \iota_E^1(E^1) \cup \iota_F^1(F^1)$ , we infer that  $x \in \iota_E^1(E^1) \setminus \iota_F^1(F^1)$ . Hence, there is an edge  $y \in E^1 \setminus f^1(G^1)$  such that  $\iota_E^1(y) = x$ . Note that  $\iota_F^0(v) = t_P(x) = t_P(\iota_E^1(y)) = \iota_E^0(t_E(y))$ . Therefore, there is a vertex  $u \in G^0$  such that  $f^0(u) = t_E(y)$ . Due to the target-surjectivity of  $(f^0, f^1)$ , we get an edge  $a \in (f^1)^{-1}(y)$  such that  $t_G(a) = u$ . However,  $\iota_F^1(g^1(a)) = \iota_E^1(f^1(a)) = \iota_E^1(y) = x$ , which contradicts  $(\iota_F^1)^{-1}(x) = \emptyset$ . Next, suppose that the condition (T2) is satisfied, so  $\iota_F^1(e) = x$  and  $\iota_F^0(t_F(e)) = t_P(\iota_F^1(e)) = t_P(x)$ , which contradicts  $(\iota_F^0)^{-1}(t_P(x)) = \emptyset$ .

Finally, we prove that  $(\iota_E^0, \iota_E^1)$  satisfies the target-bijectivity condition. Take any  $x \in P^1$  and consider the following three cases.

**Case 1.** If  $x \in \iota_F^1(F^1) \setminus \iota_E^1(E^1)$ , then  $(\iota_E^1)^{-1}(x) = \emptyset$  and there is an edge  $e \in F^1 \setminus g^1(G^1)$  such that  $\iota_F^1(e) = x$ . Suppose that there is a vertex  $v \in E^0$  such that  $\iota_E^0(v) = t_P(x)$ . Then

$$\iota_F^0(t_F(e)) = t_P(\iota_F^1(e)) = t_P(x) = \iota_E^0(v), \tag{3.17}$$

which implies that there is a vertex  $u \in G^0$  such that  $g^0(u) = t_F(e)$ . Due to the target-surjectivity of  $(g^0, g^1)$ , there is an edge  $a \in (g^1)^{-1}(e)$  such that  $t_G(a) = u$ . In turn, this implies that

$$\iota_E^1(f^1(a)) = \iota_F^1(g^1(a)) = \iota_F^1(e) = x, \tag{3.18}$$

which contradicts  $(\iota_E^1)^{-1}(x) = \emptyset$ .

**Case 2.** Let  $x \in \iota_E^1(E^1) \cap \iota_F^1(F^1)$ . Then we have the following commutative diagram:

$$\begin{array}{ccc} (\iota_E^1)^{-1}(x) & \xrightarrow{\iota_E} & (\iota_E^0)^{-1}(t_P(x)) \\ f^1 \uparrow \cong & & f^0 \uparrow \cong \\ (f^1)^{-1}((\iota_E^1)^{-1}(x)) & \xrightarrow[\iota_G]{\cong} & (f^0)^{-1}((\iota_E^0)^{-1}(t_P(x))). \end{array} \tag{3.19}$$

Indeed, since  $(t_E^1)^{-1}(x) \subseteq f^1(G^1)$  and  $(t_E^0)^{-1}(t_P(x)) \subseteq f^0(G^0)$ , the two arrows going upwards are bijections. To see that the bottom arrow is also a bijection, observe that

$$\begin{aligned} (f^1)^{-1}((t_E^1)^{-1}(x)) &= (t_E^1 \circ f^1)^{-1}(x) = (t_F^1 \circ g^1)^{-1}(x), \\ (f^0)^{-1}((t_E^0)^{-1}(t_P(x))) &= (t_E^0 \circ f^0)^{-1}(t_P(x)) = (t_F^0 \circ g^0)^{-1}(t_P(x)). \end{aligned} \tag{3.20}$$

Now, as both  $(t_F^0, t_F^1)$  and  $(g^0, g^1)$  satisfy that target-bijectivity condition, so does their composition, whence we infer the bijectivity of the bottom arrow. The desired bijectivity of the top arrow follows now from the commutativity of the diagram.

**Case 3.** If  $x \in t_E^1(E^1) \setminus t_F^1(F^1)$ , then  $(t_E^1)^{-1}(x) = \{y\}$  for some  $y \in E^1$  because  $t_E^1$  is injective when restricted to  $E^1 \setminus f^1(G^1)$ . Consequently,  $t_E(y) \in (t_E^0)^{-1}(t_P(x))$ . Suppose that  $(t_F^0)^{-1}(t_P(x)) \neq \emptyset$ . Then there is a vertex  $w \in F^0$  such that  $t_F^0(w) = t_P(x)$ . It follows that

$$t_E^0(t_E(y)) = t_P(t_E^1(y)) = t_P(x) = t_F^0(w), \tag{3.21}$$

which means that there is a vertex  $u \in G^0$  such that  $f^0(u) = t_E(y)$ . By the target-surjectivity of  $(f^0, f^1)$ , there exists an edge  $a \in (f^1)^{-1}(y)$ . Therefore,

$$x = t_E^1(f^1(a)) = t_F^1(g^1(a)), \tag{3.22}$$

which contradicts our assumption. ■

### 3.3. The covariant FP functor

For the purposes of our study of path algebras, we consider the pushout diagram in the category of sets and maps:

$$\begin{array}{ccc} & \text{FP}(E) \amalg_{\text{FP}(G)} \text{FP}(F) & \\ & \swarrow \text{ } \text{ } \searrow & \\ \text{FP}(E) & & \text{FP}(F) \\ & \nwarrow \text{ } \text{ } \nearrow & \\ & \text{FP}(G) & \end{array} \tag{3.23}$$

This leads to the following natural question: Under which assumptions does the covariant functor FP from the category OG of graphs and graph homomorphisms to the category of sets and maps commute with pushouts:

$$\text{FP}(E) \amalg_{\text{FP}(G)} \text{FP}(F) = \text{FP}\left(E \amalg_G F\right)? \tag{3.24}$$

The answer is as follows:

**Lemma 3.9.** *Let  $E \xleftarrow{(f^0, f^1)} G \xrightarrow{(g^0, g^1)} F$  be graph homomorphisms. Then there exists a natural map*

$$h: \text{FP}(E) \amalg_{\text{FP}(G)} \text{FP}(F) \longrightarrow \text{FP}(E \amalg_G F).$$

Moreover, if the graph homomorphisms are such that

- (1) both  $f^0$  and  $g^0$  are injective (vertex injectivity),
- (2)  $t_{\amalg}(x) = s_{\amalg}(y) \Rightarrow (x, y \in \iota_{E^1}(E^1) \text{ or } x, y \in \iota_{F^1}(F^1))$  (one color),

then the natural map  $h$  is bijective.

*Proof.* The natural map  $\text{FP}(E) \amalg \text{FP}(F) \rightarrow \text{FP}(E \amalg_G F)$  exists because  $\text{FP}$  is a functor. This map descends to  $h$  by the universal property of pushouts in the category of sets and maps. Since graph homomorphisms preserve the length of paths, we obtain the decomposition

$$\text{FP}(E) \amalg_{\text{FP}(G)} \text{FP}(F) = \coprod_{n \in \mathbb{N}} \text{FP}_n(E) \amalg_{\text{FP}_n(G)} \text{FP}_n(F). \tag{3.25}$$

Hence, we can write  $h$  on elements as follows:

$$h([p]) := \begin{cases} [p] & \text{for } p \in (E^0 \amalg F^0) \cup (E^1 \amalg F^1), \\ ([a_1], \dots, [a_n]) & \text{for } p := (a_1, \dots, a_n) \in \text{FP}(E) \amalg \text{FP}(F), \\ & a_i \in E^1 \amalg F^1, 1 \leq i \leq n, n \in \mathbb{N} \setminus \{0, 1\}. \end{cases} \tag{3.26}$$

Now, using the two assumptions, we will define the inverse of  $h$ . For starters, we put

$$h^{-1}([p]) := [p] \text{ when } p \in (E^0 \amalg F^0) \cup (E^1 \amalg F^1). \tag{3.27}$$

Next, let us take  $([a_1], \dots, [a_n]) \in \text{FP}(E \amalg_G F)$  with

$$[a_i] \in E^1 \amalg_{G^1} F^1, \quad 1 \leq i \leq n, n \in \mathbb{N} \setminus \{0, 1\}. \tag{3.28}$$

Since  $t_{\amalg}([a_i]) = s_{\amalg}([a_{i+1}])$  for all  $1 \leq i \leq n - 1$ , from the one-color condition, we conclude that there exist  $b_i \in E^1 \amalg F^1, 1 \leq i \leq n$ , such that  $[b_i] = [a_i]$  for all  $i$ , and

$$\forall 1 \leq i \leq n: b_i \in E^1 \text{ or } \forall 1 \leq i \leq n: b_i \in F^1. \tag{3.29}$$

Hence,  $[t_E(b_i)] = [s_E(b_{i+1})]$  or  $[t_F(b_i)] = [s_F(b_{i+1})]$ . We can now apply the vertex-injectivity condition to infer that  $t_E(b_i) = s_E(b_{i+1})$  or  $t_F(b_i) = s_F(b_{i+1})$ .

Furthermore, if  $(b'_1, \dots, b'_n) \in \text{FP}(E)$  or  $(b'_1, \dots, b'_n) \in \text{FP}(F)$ , then

$$(\forall 1 \leq i \leq n: [b_i] = [b'_i]) \implies [(b_1, \dots, b_n)] = [(b'_1, \dots, b'_n)]. \tag{3.30}$$

Indeed, let  $c^1_i, \dots, c^m_i \in G^1$  be sequences such that

$$\forall 1 \leq i \leq n: \begin{cases} g^1(c^m_i) =: b_i^{m-1} \text{ and } f^1(c^m_i) = b'_i & \text{if } m \text{ is even,} \\ f^1(c^1_i) = b_i \text{ and } g^1(c^1_i) =: b_i^1, \dots, \\ f^1(c^m_i) =: b_i^{m-1} \text{ and } g^1(c^m_i) = b'_i & \text{if } m > 1 \text{ is odd,} \end{cases}$$

or

$$\forall 1 \leq i \leq n: \begin{cases} g^1(c_i^m) =: b_i^{m-1} \text{ and } f^1(c_i^m) = b'_i & \text{if } m > 1 \text{ is odd,} \\ g^1(c_i^1) = b_i \text{ and } f^1(c_i^1) =: b_i^1, \dots & \left\{ \begin{array}{l} f^1(c_i^m) =: b_i^{m-1} \text{ and } g^1(c_i^m) = b'_i \\ \text{if } m \text{ is even.} \end{array} \right. \end{cases}$$

Note that, if the primed and unprimed edges are in the same graph, then the parity of all  $c$ -sequences is always even, and if they are in different graphs, then the parity is always odd. Therefore, although for each  $i$  the length of the sequence  $c_i^1, \dots, c_i^m$  might be different, we can always choose the longest such a sequence and extend shorter sequences by the constant extrapolation by an even number of elements. Next, we need to prove that  $t_G(c_i^j) = s_G(c_{i+1}^j)$ . Depending on the parity of  $j$  and the above alternative between  $E$  and  $F$ , we either have

$$\begin{aligned} [f^1(c_i^j)] &= [b_i] \quad \text{and} \quad [f^1(c_{i+1}^j)] = [b_{i+1}] \\ &\text{or} \\ [g^1(c_i^j)] &= [b_i] \quad \text{and} \quad [g^1(c_{i+1}^j)] = [b_{i+1}]. \end{aligned} \tag{3.31}$$

In the former case, by the vertex injectivity, we obtain

$$t_E(f^1(c_i^j)) = t(b_i) \quad \text{and} \quad s_E(f^1(c_{i+1}^j)) = s(b_{i+1}), \tag{3.32}$$

where  $t$  and  $s$  are, respectively,  $t_E$  and  $s_E$ , or  $t_F$  and  $s_F$ , depending on whether  $b_i \in E^1$  or  $b_i \in F^1$ . It follows from (3.32) that

$$f^0(t_G(c_i^j)) = t_E(f^1(c_i^j)) = t(b_i) = s(b_{i+1}) = s_E(f^1(c_{i+1}^j)) = f^0(s_G(c_{i+1}^j)), \tag{3.33}$$

so, from the injectivity of  $f^0$ , we get  $t_G(c_i^j) = s_G(c_{i+1}^j)$ , as needed. In the latter case, the reasoning is completely analogous but uses the injectivity of  $g^0$  instead of the injectivity of  $f^0$ . Thus, we have shown that  $(c_1^j, \dots, c_n^j) \in \text{FP}(G)$ ,  $1 \leq j \leq m$ , is a sequence of paths implementing the desired equivalence relation between  $(b_1, \dots, b_n)$  and  $(b'_1, \dots, b'_n)$ .

Finally, it follows from (3.30) that we can define

$$h^{-1}([a_1], \dots, [a_n]) := [(b_1, \dots, b_n)]. \tag{3.34}$$

Combining it with (3.27) gives us a map

$$h^{-1} : \text{FP}(E \amalg_G F) \longrightarrow \text{FP}(E) \amalg_{\text{FP}(G)} \text{FP}(F), \tag{3.35}$$

which is, clearly, the inverse of  $h$ . ■

### 3.4. From pushouts to pullbacks

Let us consider the contravariant functors  $\text{Map}(\cdot, K)$  and  $\text{Map}_f(\cdot, K)$ , where  $K$  is a non-empty set with a chosen element  $0 \in K$ ,  $\text{Map}$  stands for all maps, and  $\text{Map}_f$  denotes finitely supported maps (all but finitely many elements are mapped to 0). The first functor

is a contravariant functor from the category of sets and maps to the category of sets and maps:

$$X \mapsto \text{Map}(X, K), \quad (X \xrightarrow{f} Y) \mapsto (\text{Map}(Y, K) \xrightarrow{f^*} \text{Map}(X, K)), \quad f^*(F) := F \circ f.$$

Much in the same way, the second functor is a contravariant functor from the category of sets and finite-to-one maps to the category of sets and maps:

$$X \mapsto \text{Map}_f(X, K), \quad (X \xrightarrow{f} Y) \mapsto (\text{Map}_f(Y, K) \xrightarrow{f^*} \text{Map}_f(X, K)), \quad f^*(F) := F \circ f.$$

Here  $\text{Map}_f(X, K) := \{F \in \text{Map}(X, K) \mid |F^{-1}(K \setminus \{0\})| < \infty\}$ , and

$$|(F \circ f)^{-1}(K \setminus \{0\})| = |f^{-1}(F^{-1}(K \setminus \{0\}))| < \infty \tag{3.36}$$

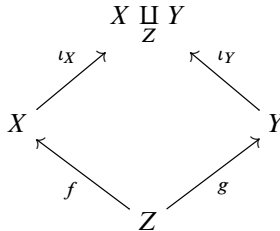
because a finite union of finite sets is a finite set.

We have the following elementary lemmas whose routine proofs we omit.

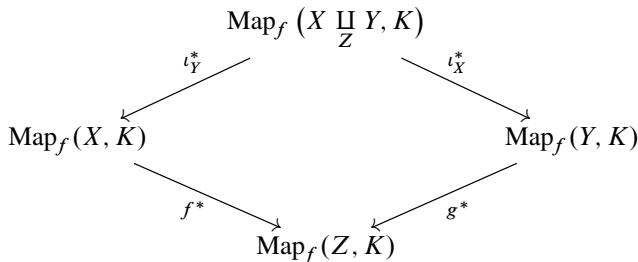
**Lemma 3.10.** *Let  $K$  be any set. Then  $\text{Map}(\cdot, K)$  is a contravariant functor from the category of sets and maps to the category of sets and maps transforming pushouts to pullbacks.*

If we restrict to finite-to-one maps, we get a contravariant functor  $\text{Map}_f(\cdot, K)$ . Furthermore, using Lemma 3.6, we have the following result.

**Lemma 3.11.** *Let  $K$  be a non-empty set with a chosen element  $0 \in K$ , let  $f$  be an injective map, and let*



*be a pushout diagram in the category of sets and finite-to-one maps. Then the contravariant functor  $\text{Map}_f(\cdot, K)$  transforms the above pushout diagram into the pullback diagram*



*in the category of sets and maps such that its left defining morphism is surjective.*

### 4. Graph algebras as contravariant functors

To alleviate the notation, from now on, we will write  $e_1 \dots e_n$  for a path  $(e_1, \dots, e_n)$ . Furthermore, if  $p = e_1 \dots e_n, q = f_1 \dots f_m$ , and the end of  $e_n$  is the beginning of  $f_1$ , then  $pq = e_1 \dots e_n f_1 \dots f_m$ .

#### 4.1. Path algebras

Let  $k$  be a field,  $E$  any non-empty graph, and  $\text{FP}(E)$  the set of all its finite paths in  $E$ . Consider the vector space

$$kE := \text{Map}_f(\text{FP}(E), k), \tag{4.1}$$

where  $\text{Map}_f(\text{FP}(E), k)$  is the vector space of all finitely supported functions from  $\text{FP}(E)$  to  $k$  in which the addition and scalar multiplication are pointwise. Then the set of functions  $\{\chi_p\}_{p \in \text{FP}(E)}$  given by

$$\chi_p(q) = \begin{cases} 1 & \text{for } p = q, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

is a linear basis of  $kE$ . By checking the associativity, one can prove that the formulas

$$m : kE \times kE \longrightarrow kE, \quad m(\chi_p, \chi_q) := \begin{cases} \chi_{pq} & \text{if } t(p) = s(q), \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

define a multiplication on  $kE$  rendering it a  $k$ -algebra.

**Definition 4.1** ([5, Definition II.1.2]). Let  $E$  be a non-empty graph. The above constructed algebra  $(kE, +, 0, m)$  is called the *path algebra* of  $E$ . If  $E = \emptyset$ , then  $kE := 0$ .

Let  $\text{KA}$  denote the category of algebras over  $k$  together with algebra homomorphisms, and let  $\text{UKA}$  denote the category of unital algebras over  $k$  together with unital algebra homomorphisms.

**Lemma 4.2.** *The assignment*

$$\begin{aligned} \text{Obj}(\text{POG}) \ni E &\xrightarrow{\text{Map}_f} kE \in \text{Obj}(\text{KA}), \\ \text{Mor}(\text{POG}) \ni ((f^0, f^1): E \rightarrow F) &\xrightarrow{\text{Map}_f} (f^*: kF \rightarrow kE) \in \text{Mor}(\text{KA}), \\ kF \ni \chi_p &\xrightarrow{f^*} \sum_{q \in f^{-1}(p)} \chi_q \in kE, \end{aligned} \tag{4.4}$$

where  $f: \text{FP}(E) \rightarrow \text{FP}(F)$  is the map induced by  $(f^0, f^1)$ , defines a contravariant functor. Furthermore, the same assignment restricted to the subcategory given by graphs with finitely many vertices yields a contravariant functor to the category  $\text{UKA}$ .

*Proof.* Since  $f: \text{FP}(E) \rightarrow \text{FP}(F)$  is finite to one, the sum in (4.4) is well defined. Furthermore, for any  $r \in \text{FP}(E)$  and  $p \in \text{FP}(F)$ , we have

$$f^*(\chi_p)(r) := \left( \sum_{q \in f^{-1}(p)} \chi_q \right)(r) = \left( \sum_{q \in f^{-1}(p)} \delta_{q,r} \right) = \delta_{p,f(r)} = \chi_p(f(r)) = (\chi_p \circ f)(r).$$

Hence,  $f^*$  is the pullback linear map

$$f^*: \text{Map}_f(\text{FP}(F), k) \ni \alpha \mapsto \alpha \circ f \in \text{Map}_f(\text{FP}(E), k). \quad (4.5)$$

The contravariance is obvious because  $(f \circ g)^* = g^* \circ f^*$ .

To check that it is an algebra homomorphism, using the fact that  $f$  preserves the length of paths, we compute

$$\begin{aligned} f^*(\chi_p \chi_q) &= \delta_{t_F(p), s_F(q)} f^*(\chi_{pq}) \\ &= \delta_{t_F(p), s_F(q)} \sum_{r \in f^{-1}(pq)} \chi_r \\ &= \sum_{\substack{r_1 \in f^{-1}(p) \\ r_2 \in f^{-1}(q)}} \delta_{t_E(r_1), s_E(r_2)} \chi_{r_1 r_2} \\ &= \sum_{r_1 \in f^{-1}(p)} \chi_{r_1} \sum_{r_2 \in f^{-1}(q)} \chi_{r_2} \\ &= f^*(\chi_p) f^*(\chi_q). \end{aligned} \quad (4.6)$$

Here, in the third step, we used the implication

$$t_F(p) \neq s_F(q) \implies \forall r_1 \in f^{-1}(p), r_2 \in f^{-1}(q): t_E(r_1) \neq s_E(r_2). \quad (4.7)$$

Finally, the unitality of  $f^*$  for unital path algebras follows from  $f^{-1}(F^0) = E^0$ :

$$f^*(1) = f^*\left(\sum_{v \in F^0} \chi_v\right) = \sum_{w \in f^{-1}(F^0)} \chi_w = \sum_{w \in E^0} \chi_w = 1. \quad \blacksquare$$

## 4.2. Leavitt path algebras

Let  $E = (E^0, E^1, s_E, t_E)$  be a graph. The *extended graph*  $\bar{E} := (\bar{E}^0, \bar{E}^1, s_{\bar{E}}, t_{\bar{E}})$  of the graph  $E$  is given as follows:

$$\begin{aligned} \bar{E}^0 &:= E^0, & \bar{E}^1 &:= E^1 \sqcup (E^1)^*, & (E^1)^* &:= \{e^* \mid e \in E^1\}, \\ \forall e \in E^1 &: s_{\bar{E}}(e) &:= s_E(e), & t_{\bar{E}}(e) &:= t_E(e), \\ \forall e^* \in (E^1)^* &: s_{\bar{E}}(e^*) &:= t_E(e), & t_{\bar{E}}(e^*) &:= s_E(e). \end{aligned} \quad (4.8)$$

Observe that every graph homomorphism  $(f^0, f^1): E \rightarrow F$  can be extended to a graph homomorphism  $(\bar{f}^0, \bar{f}^1): \bar{E} \rightarrow \bar{F}$  in the following way:

$$\begin{aligned} \bar{f}^0(v) &:= f^0(v), & v \in E^0 = \bar{E}^0, \\ \bar{f}^1(e) &:= f^1(e), & e \in E^0, \\ \bar{f}^1(e^*) &:= f^1(e)^*, & e^* \in (E^1)^*. \end{aligned} \tag{4.9}$$

Also, if  $e_1 \dots e_n \in \text{FP}_n(E)$ , we define

$$(e_1 \dots e_n)^* := e_n^* \dots e_1^* \in \text{FP}(\bar{E}). \tag{4.10}$$

We state the following straightforward result without a proof.

**Lemma 4.3.** *The assignment*

$$E \mapsto \bar{E}, \quad (f^0, f^1) \mapsto (\bar{f}^0, \bar{f}^1),$$

*defines a covariant endofunctor of the category OG of graphs and graph homomorphisms. Furthermore, it restricts to an endofunctor of the category POG of graphs and proper graph homomorphisms.*

**Definition 4.4.** The *Leavitt path algebra*  $L_k(E)$  of a graph  $E$  is the quotient of the path algebra  $k\bar{E}$  of the extended graph  $\bar{E}$  by the ideal generated by the union of the following sets:

$$\begin{aligned} \text{(CK1)} & \{ \chi_{e^*} \chi_f - \delta_{e,f} \chi_{t_E(e)} \mid e, f \in E^1 \}, \\ \text{(CK2)} & \{ \chi_v - \sum_{e \in s_E^{-1}(v)} \chi_e \chi_{e^*} \mid v \in \text{reg}(E) \}. \end{aligned}$$

The algebra  $L_k(E)$  is isomorphic with the universal algebra generated by the elements  $\chi_v, \chi_e, \chi_{e^*}, v \in E^0, e \in E^1$ , subject to the relations (CK1) and (CK2) and the standard path-algebraic relations  $\chi_v \chi_w = \delta_{v,w} \chi_v$  and  $\chi_{s_E(e)} \chi_e = \chi_e \chi_{t_E(e)}$ . The algebra  $L_k(E)$  is  $\mathbb{Z}$ -graded by the lengths of paths, where edges from  $(E^1)^*$  have length  $-1$ . For  $k = \mathbb{C}$ , the  $\mathbb{Z}$ -grading is equivalent to the  $U(1)$ -action  $\gamma$  on  $L_k(E)$ , called the *gauge action*, given by

$$\gamma_z([\chi_v]) := [\chi_v], \quad \gamma_z([\chi_e]) := z[\chi_e], \quad \gamma_z([\chi_{e^*}]) := \bar{z}[\chi_{e^*}], \quad z \in U(1), \quad v \in E^0, \quad e \in E^1. \tag{4.11}$$

Let ZKA denote the category of  $\mathbb{Z}$ -graded algebras over  $k$  together with algebra homomorphisms preserving the  $\mathbb{Z}$ -grading, and let ZUKA denote its subcategory of unital  $\mathbb{Z}$ -graded algebras and unital homomorphisms preserving the  $\mathbb{Z}$ -grading. The next theorem exploits the contravariant functoriality of the Leavitt-path-algebra construction when restricted to the admissible category of graphs.

**Theorem 4.5.** *The assignment*

$$\begin{aligned} \text{Obj}(\text{CRTBPOG}) \ni E &\mapsto L_k(E) \in \text{Obj}(\text{ZKA}), \\ \text{Mor}(\text{CRTBPOG}) \ni ((f^0, f^1): E \rightarrow F) &\mapsto (f_L^*: L_k(F) \rightarrow L_k(E)) \in \text{Mor}(\text{ZKA}), \\ L_k(F) \ni [\chi_p] &\xrightarrow{f_L^*} \sum_{q \in \bar{f}^{-1}(p)} [\chi_q] \in L_k(E), \end{aligned}$$

where  $\tilde{f}: \text{FP}(\bar{E}) \rightarrow \text{FP}(\bar{F})$  is the map induced by  $(f^0, f^1)$ , defines a contravariant functor. Furthermore, the same assignment restricted to the subcategory given by graphs with finitely many vertices yields a contravariant functor to the category ZUKA.

*Proof.* It follows from Lemma 4.2 combined with Lemma 4.3 that we have a contravariant functor

$$\begin{aligned} \text{Obj}(\text{POG}) \ni E &\longmapsto k\bar{E} \in \text{Obj}(KA), \\ \text{Mor}(\text{POG}) \ni ((f^0, f^1): E \rightarrow F) &\longmapsto (\tilde{f}^*: k\bar{F} \rightarrow k\bar{E}) \in \text{Mor}(KA), \end{aligned} \tag{4.12}$$

$$k\bar{F} \ni \chi_p \xrightarrow{\tilde{f}^*} \sum_{q \in \tilde{f}^{-1}(p)} \chi_q \in k\bar{E}.$$

To complete the proof of the first statement, it suffices to show that  $\tilde{f}^*$  descends to a  $\mathbb{Z}$ -graded algebra homomorphism  $L_k(F) \rightarrow L_k(E)$  when we restrict to the admissible category of graphs.

First, we have to demonstrate that

$$\forall f \in \text{Mor}(\text{CRTBPOG}) \forall x, y \in F^1: [\tilde{f}^*(\chi_{x^*})][\tilde{f}^*(\chi_y)] = \delta_{x,y}[\tilde{f}^*(\chi_{t_F(x)})], \tag{4.13}$$

which is equivalent to

$$\forall f \in \text{Mor}(\text{CRTBPOG}) \forall x, y \in F^1: \sum_{\substack{e_x \in f^{-1}(x) \\ e_y \in f^{-1}(y)}} [\chi_{e_x^*}][\chi_{e_y}] = \delta_{x,y} \sum_{v \in f^{-1}(t_F(x))} [\chi_v]. \tag{4.14}$$

For  $x \neq y$ , we have  $f^{-1}(x) \cap f^{-1}(y) = \emptyset$ , so (4.14) clearly holds. For  $x = y$ , using the target-bijection condition (2.6), we compute

$$\sum_{e_1, e_2 \in f^{-1}(x)} [\chi_{e_1^*}][\chi_{e_2}] = \sum_{e \in f^{-1}(x)} [\chi_{e^*}][\chi_e] = \sum_{e \in f^{-1}(x)} [\chi_{t_E(e)}] = \sum_{v \in f^{-1}(t_F(x))} [\chi_v]. \tag{4.15}$$

Next, much as above, we need to show that

$$\forall w \in \text{reg}(F): \sum_{\substack{x \in s_F^{-1}(w) \\ e_1, e_2 \in (f^1)^{-1}(x)}} [\chi_{e_1}][\chi_{e_2^*}] = \sum_{v \in (f^0)^{-1}(w)} [\chi_v]. \tag{4.16}$$

For starters, using (2.17), we obtain

$$\begin{aligned} \sum_{v \in (f^0)^{-1}(w)} [\chi_v] &= \sum_{\substack{v \in (f^0)^{-1}(w) \\ e \in s_E^{-1}(v)}} [\chi_e][\chi_{e^*}] = \sum_{e \in (f^0 \circ s_E)^{-1}(w)} [\chi_e][\chi_{e^*}] \\ &= \sum_{e \in (s_F \circ f^1)^{-1}(w)} [\chi_e][\chi_{e^*}] = \sum_{\substack{x \in s_F^{-1}(w) \\ e \in (f^1)^{-1}(x)}} [\chi_e][\chi_{e^*}]. \end{aligned} \tag{4.17}$$

Next, using the target-injectivity condition (2.6), we compute the left-hand side of (4.16):

$$\sum_{\substack{x \in s_F^{-1}(w) \\ e_1, e_2 \in (f^1)^{-1}(x)}} [\chi_{e_1}][\chi_{e_2^*}] = \sum_{\substack{x \in s_F^{-1}(w) \\ e \in (f^1)^{-1}(x)}} [\chi_e][\chi_{e^*}]. \tag{4.18}$$

Summarizing, we have proved that  $\bar{f}^*$  descends to an algebra homomorphism

$$f_L^* : L_k(F) \longrightarrow L_k(E), \tag{4.19}$$

which obviously preserves the  $\mathbb{Z}$ -grading because  $\bar{f}$  preserves the lengths of paths. Finally, the functoriality of the assignment

$$\text{Mor}(\text{CRTBPOG}) \ni (f^0, f^1) \longmapsto f_L^* \in \text{Mor}(\text{ZKA}) \tag{4.20}$$

is immediate, and the unitality of  $f_L^*$  for graphs with finitely many vertices follows from the unitality of  $\bar{f}^*$  under the same restriction. ■

**Corollary 4.6.** *If  $f : E \rightarrow F$  is an injective (surjective) morphism in the admissible category of graphs, then  $f_L^*$  is surjective (injective).*

*Proof.* By Theorem 4.5, the admissibility of  $f$  implies the existence of a  $\mathbb{Z}$ -graded algebra homomorphism  $f_L^* : L_k(F) \rightarrow L_k(E)$ . If  $f : E \rightarrow F$  is injective, then  $[\chi_v] = f_L^*([\chi_{f(v)}])$  for all  $v \in E^0$  and  $[\chi_e] = f_L^*([\chi_{f(e)}])$  and  $[\chi_{e^*}] = f_L^*([\chi_{f(e)^*}])$  for all  $e \in E^1$ . Consequently,  $f_L^*$  is surjective because  $L_k(E)$  is generated by  $[\chi_v]$ ,  $v \in E^0$ ,  $[\chi_e]$  and  $[\chi_{e^*}]$ ,  $e \in E^1$ . Vice versa, if  $f : E \rightarrow F$  is surjective, then  $f^{-1}(w) \neq \emptyset$  for all  $w \in F^0$ , so

$$\forall w \in F^0 : f_L^*([\chi_w]) = \sum_{v \in (f^0)^{-1}(w)} [\chi_v] \neq 0. \tag{4.21}$$

Here the last step follows from the basis theorem [1, Corollary 1.5.12]. Now, from the graded uniqueness theorem for Leavitt path algebras [1, Theorem 2.2.15], we conclude that  $f_L^*$  is injective. ■

### 4.3. Graph $C^*$ -algebras

For basic facts about  $C^*$ -algebras, we refer the reader to [10]. Let us now consider the Leavitt path algebra construction in the case  $k = \mathbb{C}$ . One defines an anti-homomorphism  $*$  :  $L_{\mathbb{C}}(E) \rightarrow L_{\mathbb{C}}(E)$  by mapping the generators as follows:

$$([\chi_v])^* := [\chi_v], \quad ([\chi_e])^* := [\chi_{e^*}], \quad ([\chi_{e^*})^* := [\chi_e], \quad v \in E^0, e \in E^1. \tag{4.22}$$

The above defined  $*$  operation turns  $L_{\mathbb{C}}(E)$  into a complex  $*$ -algebra. Thus, we arrive at the key definition.

**Definition 4.7** ([1, Definition 5.2.1]). Let  $E$  be a graph. The *graph  $C^*$ -algebra*  $C^*(E)$  of  $E$  is the universal  $C^*$ -envelope of the complex  $*$ -algebra  $L_{\mathbb{C}}(E)$ .

It is worth noting that, unlike for general universal  $C^*$ -envelopes, for the graph  $C^*$ -algebras, the canonical  $*$ -homomorphism  $L_{\mathbb{C}}(E) \rightarrow C^*(E)$  is injective (e.g., see [1, Theorem 5.2.9]). Better still, the gauge action (4.11) extends to graph  $C^*$ -algebras by universality and continuity. Note also that Definition 4.7 is equivalent to [12, Definition 1] defining  $C^*(E)$  as the universal  $C^*$ -algebra generated by mutually orthogonal projections

$P_v, v \in E^0$ , and partial isometries  $S_e, e \in E^1$ , with mutually orthogonal ranges, satisfying

- (1)  $S_e^* S_e = P_{t(e)}$  for all  $e \in E^1$ ,
- (2)  $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$  for all  $v \in \text{reg}(E)$ ,
- (3)  $S_e S_e^* \leq P_{s(e)}$  for all  $e \in E^1$ .

In what follows, we will need the notation  $S_p := S_{e_1} S_{e_2} \cdots S_{e_n}$  and  $p^* := e_n^* \cdots e_1^*$  for a positive-length path  $p = e_1 \cdots e_n$ , and  $S_v := P_v$  for a 0-length path  $v$ . Note that this notation allows us to replace  $[\chi_p]$  by  $S_p$ .

Let  $\text{GC}^*\text{A}$  denote the category of  $U(1)$ - $C^*$ -algebras together with  $U(1)$ -equivariant  $*$ -homomorphisms, and let  $\text{GUC}^*\text{A}$  denote the category of unital  $U(1)$ - $C^*$ -algebras and unital  $U(1)$ -equivariant  $*$ -homomorphisms. The following  $C^*$ -algebraic counterpart of Theorem 4.5 is the discrete-topology case of Katsura’s [22, Proposition 2.9]. As we were unaware of the just cited result prior to obtaining our own version, we retain its complete and self-contained original proof, which is routed via the algebraic constructions of path algebras and Leavitt path algebras absent in Katsura’s work.

**Corollary 4.8.** *The assignment*

$$\begin{aligned} \text{Obj}(\text{CRTBPOG}) \ni E &\longmapsto C^*(E) \in \text{Obj}(\text{GC}^*\text{A}), \\ \text{Mor}(\text{CRTBPOG}) \ni ((f^0, f^1): E \rightarrow F) &\longmapsto (f_{C^*}^*: C^*(F) \rightarrow C^*(E)) \in \text{Mor}(\text{GC}^*\text{A}), \\ C^*(F) \ni S_p &\xrightarrow{f_{C^*}^*} \sum_{q \in \bar{f}^{-1}(p)} S_q \in C^*(E), \end{aligned}$$

where  $\bar{f}: \text{FP}(\bar{E}) \rightarrow \text{FP}(\bar{F})$  is the map induced by  $(f^0, f^1)$ , defines a contravariant functor. Furthermore, the same assignment restricted to the subcategory given by graphs with finitely many vertices yields a contravariant functor to the category  $\text{GUC}^*\text{A}$ .

*Proof.* From Theorem 4.5, in the case  $k = \mathbb{C}$ , we know that every

$$(f^0, f^1) \in \text{Mor}(\text{CRTBPOG}) \tag{4.23}$$

gives rise to a homomorphism  $f_L^*: L_{\mathbb{C}}(F) \rightarrow L_{\mathbb{C}}(E)$  that preserves the  $\mathbb{Z}$ -grading coming from the path lengths. It is automatically a  $*$ -homomorphism because

$$f_L^*([\chi_p]^*) = f_L^*([\chi_{p^*}]) = \sum_{q \in \bar{f}^{-1}(p^*)} [\chi_q] = \left( \sum_{q \in \bar{f}^{-1}(p)} [\chi_q] \right)^* = f_L^*([\chi_p])^*. \tag{4.24}$$

Since the grading is equivalent to the gauge action,  $f_L^*$  is gauge equivariant. From [2, Theorem 4.4], we infer that  $f_L^*$  extends to a unique  $*$ -homomorphism  $f_{C^*}^*: C^*(F) \rightarrow C^*(E)$ , which is also gauge equivariant by the continuity of the gauge action and  $*$ -homomorphisms between  $C^*$ -algebras. Finally, the unitality of  $f_{C^*}^*$  for graphs with finitely many vertices follows from the unitality of  $f_L^*$  under the same restriction. ■

Furthermore, note that using [1, Theorem 5.2.12] instead of [1, Theorem 2.2.15], we obtain a  $C^*$ -algebraic version of Corollary 4.6.

**Corollary 4.9.** *If  $f : E \rightarrow F$  is an injective (surjective) morphism in the admissible category of graphs, then  $f_{C^*}$  is surjective (injective).*

## 5. New types of admissible graph homomorphisms

### 5.1. Generalized foldings

For starters, let us observe that mapping the disjoint union of a graph with its copy into the graph by identifying the same elements in two different copies is an admissible morphism inducing the diagonal map:

$$f : E \sqcup E \longrightarrow E \quad \rightsquigarrow \quad f_{C^*} : C^*(E) \ni a \longmapsto (a, a) \in C^*(E) \oplus C^*(E). \quad (5.1)$$

In this section, we replace disjoint unions of graphs by pushouts over an admissible subgraph.

**Definition 5.1.** Let  $G$  be an admissible subgraph of a graph  $F$ . A *generalized folding* is the graph homomorphism

$$g : F \amalg_G F \longrightarrow F \quad (5.2)$$

given by the universal property of the pushout applied to the identity maps  $F \rightarrow F$ .

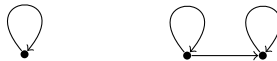
It is clear that generalized foldings are morphisms in the admissible category of graphs and that Definition 5.1 generalizes Example 2.8. It also includes (5.1) as a special case obtained by taking  $G$  to be the empty graph. Let us now further exemplify Definition 5.1.

**Example 5.2.** For any  $n \in \mathbb{N}$  and  $q \in [0, 1)$ , the  $C^*$ -algebra  $C(S_q^{2n})$  of the Hong–Szymański even quantum sphere  $S_q^{2n}$  is isomorphic to the graph  $C^*$ -algebra of the graph  $L_{2n}$  (see [20, Section 5.1]).



**Figure 1.** The graph  $L_{2n}$  for  $n = 1$  and  $n = 2$ .

Similarly, the  $C^*$ -algebra  $C(S_q^{2n-1})$  of the Vaksman–Soibelman odd quantum sphere  $S_q^{2n-1}$  [33] is isomorphic to the graph  $C^*$ -algebra of the graph  $L_{2n-1}$  (see [20, Section 4.1]).



**Figure 2.** The graph  $L_{2n-1}$  for  $n = 1$  and  $n = 2$ .

Finally, the  $C^*$ -algebra  $C(B_q^{2n})$  of the Hong–Szymański even quantum ball  $B_q^{2n}$  is isomorphic to the graph  $C^*$ -algebra of  $M_n$  (see [21, Section 3.1] and [20]).



**Figure 3.** The graph  $M_n$  for  $n = 1$  and  $n = 2$ .

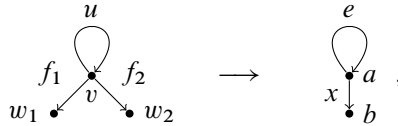
It is clear that  $L_{2n-1}$  is an admissible subgraph of  $L_{2n}$  and that  $L_{2n} = M_n \sqcup_{L_{2n-1}} M_n$ , for every  $n \in \mathbb{N}$  (e.g., see [17, Section 4.1]). Therefore, we can consider the generalized folding

$$L_{2n} = M_n \amalg_{L_{2n-1}} M_n \longrightarrow M_n. \tag{5.3}$$

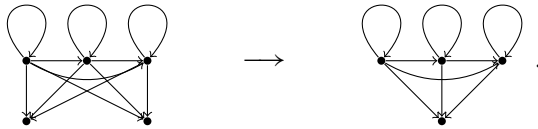
Note that the admissible graph homomorphism (5.3) induces a gauge-equivariant unital  $*$ -homomorphism

$$C(B_q^{2n}) \longrightarrow C(S_q^{2n}),$$

which is an analog of flattening an even-dimensional sphere to an even-dimensional ball of the same dimension. For instance, in the case  $n = 1$ , we have a graph homomorphism



where  $v \mapsto a$ ,  $w_1, w_2 \mapsto b$ ,  $u \mapsto e$ , and  $f_1, f_2 \mapsto x$ . Much in the same way, in the case  $n = 3$ , we have the following graph homomorphism:



Next, if  $F$  is an admissible subgraph of  $E$ , then we have the injective graph homomorphism

$$f : F \amalg_G F \longrightarrow F \amalg_G E \tag{5.4}$$

given by the universal property of the pushout applied to the obvious maps. Similarly, from the universal property of the pushout applied to the obvious maps, we obtain the graph homomorphism

$$\iota_g : F \amalg_G E \longrightarrow E. \tag{5.5}$$

**Proposition 5.3.** *The injective graph homomorphism (5.4) is admissible.*

*Proof.* We have the following commutative diagram of graph homomorphisms:

$$\begin{array}{ccccc}
 F & \longrightarrow & F \amalg_G F & \longrightarrow & F \amalg_G E. \\
 \uparrow & & \uparrow & & \uparrow \\
 G & \longrightarrow & F & \longrightarrow & E
 \end{array} \tag{5.6}$$

It is clear that both the left square and the outer rectangle are pushout diagrams in the category OG. It follows by standard categorical arguments that the right square is a pushout diagram (see [3, Proposition 11.10] for the dual result for pullbacks). The admissibility of  $G \hookrightarrow F$  implies the admissibility of  $F \hookrightarrow F \amalg_G F$  by Lemma 3.8. We use Lemma 3.8 again to infer the admissibility of  $F \amalg_G F \hookrightarrow F \amalg_G E$  from the admissibility of  $F \hookrightarrow E$  and  $F \hookrightarrow F \amalg_G F$ . ■

### 5.2. Line graphs

Now we explore a class of non-injective admissible graph homomorphisms whose induced  $*$ -homomorphisms are surjective. For starters, let us recall the notion of a line graph [19]. The *line graph*  $LE = (LE^0, LE^1, s_{LE}, r_{LE})$  of a graph  $E$  is defined as follows:

$$LE^0 := E^1, \quad LE^1 := \text{FP}_2(E), \quad s_{LE}(ee') = e, \quad t_{LE}(ee') = e', \quad e, e' \in E^1. \quad (5.7)$$

Next, consider the graph homomorphism

$$f^0 : LE^0 \longrightarrow E^0, \quad f^0(e) := s_E(e), \quad f^1 : LE^1 \longrightarrow E^1, \quad f^1(ee') := e. \quad (5.8)$$

The following result is the discrete-topology version of [23, Proposition 2.6].

**Proposition 5.4.** *Let  $E$  be a row-finite graph without sinks. Then the graph homomorphism (5.8) is admissible and surjective.*

*Proof.* Since  $E$  is row-finite,  $f$  is automatically proper. Next, as for every  $e \in E^1$  the map

$$(f^1)^{-1}(e) \ni ee' \longmapsto t_{LE}(ee') = e' \in (f^0)^{-1}(t_E(e)) \quad (5.9)$$

is clearly bijective, we conclude that  $f$  satisfies the target-bijectivity condition. Furthermore,  $f$  is automatically regular because all vertices in  $E$  and  $LE$  are regular. Finally, the surjectivity of  $f$  follows from the assumption that there are no sinks. ■

By Corollary 4.9,  $f : LE \rightarrow E$  induces an injective  $*$ -homomorphism  $f_{C^*}^* : C^*(E) \rightarrow C^*(LE)$ . However, it is known that this  $*$ -homomorphism is also surjective (e.g., see [28, Corollary 2.6]). Indeed, the inverse  $*$ -homomorphism  $(f_{C^*}^*)^{-1} : C^*(LE) \rightarrow C^*(E)$  is given by

$$P_e \longmapsto S_e S_e^*, \quad S_{ee'} \longmapsto S_e S_{e'} S_{e'}^*. \quad (5.10)$$

Thus, we obtain a class of examples of non-injective admissible graph homomorphisms inducing surjective  $*$ -homomorphisms of graph  $C^*$ -algebras.

### 5.3. Locally derived graphs

We end the section by discussing another class of non-injective admissible graph homomorphisms coming from finite group actions. A *base graph* (or a *voltage graph*) [14, Section 2.1] is a graph  $(E^0, E^1, s_E, t_E)$  along with a function  $L : E^1 \rightarrow \Gamma$ , where  $\Gamma$  is a

group. Given a base graph, one can construct a *derived graph*  $(E_L^0, E_L^1, s_L, t_L)$  [14, Section 2.1.1] (or a *skew-product graph* [25, Definition 2.1]) as follows:

$$\begin{aligned}
 E_L^0 &:= E^0 \times \Gamma, & E_L^1 &:= E^1 \times \Gamma, \\
 s_L((e, g)) &:= (s_E(e), g), & t_L((e, g)) &:= (t_E(e), L(e)g), \quad e \in E^1, g \in \Gamma.
 \end{aligned}
 \tag{5.11}$$

There is a natural surjective graph homomorphism  $\pi : E_L \rightarrow E$ , called the *covering projection*, given by

$$\pi^0((v, g)) := v, \quad \pi^1((e, g)) := e, \quad v \in E^0, e \in E^1, g \in \Gamma.
 \tag{5.12}$$

**Proposition 5.5.** *If  $\Gamma$  is finite, then the covering projection  $\pi : E_L \rightarrow E$  is admissible.*

*Proof.* Since  $\Gamma$  is finite,  $\pi$  is proper. Next, for any  $e \in E^1$ , the map

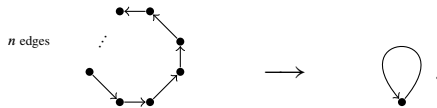
$$(\pi^1)^{-1}(e) \ni (e, g) \mapsto (t_E(e), L(e)g) \in (\pi^0)^{-1}(t_E(e))
 \tag{5.13}$$

has an inverse given by  $(t_E(e), h) \mapsto (e, L(e)^{-1}h)$ , so  $\pi$  satisfies the target-bijectivity condition. Finally, let  $v \in E^0$  be a regular vertex and  $g \in \Gamma$ . Then, by the regularity of  $v$ , there is  $e \in E^1$  such that  $s_L((e, g)) = (v, g)$ , so  $(v, g)$  is not a sink. Also, since  $v$  is not an infinite emitter, neither is  $(v, g)$ . We conclude thus that  $(v, g)$  is a regular vertex for any  $g \in \Gamma$ . Hence,  $\pi$  is regular, so we infer that  $\pi$  is admissible, as claimed. ■

It follows from Corollary 4.8 that  $\pi_{C^*} : C^*(E) \rightarrow C^*(E_L)$  is determined by the assignments:

$$P_v \mapsto \sum_{g \in \Gamma} P_{(v,g)}, \quad S_e \mapsto \sum_{g \in \Gamma} S_{(e,g)}.
 \tag{5.14}$$

**Example 5.6.** Consider the surjective admissible graph homomorphism  $\pi : A_n \rightarrow A_1$  given by collapsing all edges to one edge and all vertices to one vertex:



Note that  $\pi$  is the projection from a derived graph to its base graph. Indeed, let  $\Gamma := \mathbb{Z}/n\mathbb{Z}$  be the cyclic group of order  $n$ , and let  $L : A_1^1 \rightarrow \Gamma$  map the single edge in  $A_1$  to  $[1] \in \mathbb{Z}/n\mathbb{Z}$ . Then it is clear that  $A_n$  is isomorphic to  $(A_1)_L$  and that  $\pi$  is the covering projection. The morphism  $\pi$  induces an injective  $*$ -homomorphism  $\pi_{C^*} : C^*(A_1) \rightarrow C^*(A_n)$ , which, combined with the standard identification  $C^*(A_n) \cong C(S^1) \otimes M_n(\mathbb{C})$  (e.g., see [28, Example 2.14]), yields

$$C(S^1) \longrightarrow C(S^1) \otimes M_n(\mathbb{C}), \quad u \mapsto \sum_{i=1}^{n-1} (1 \otimes E_{i(i+1)}) + u \otimes E_{n1}.
 \tag{5.15}$$

Here,  $u$  is the unitary generator of  $C(S^1)$  and  $E_{ij}, i, j = 1, \dots, n$ , are the matrix units of  $M_n(\mathbb{C})$ . To end with, observe that precomposing the map (5.15) with  $C(S^1) \ni u \mapsto u^n \in C(S^1)$  produces the standard tensorial inclusion

$$C(S^1) \longrightarrow C(S^1) \otimes M_n(\mathbb{C}), \quad u \longmapsto u \otimes 1.$$

Now, let us generalize the construction of a derived graph. For a graph  $E$ , let  $\{E_i\}_{i \in I}$  be a family of pairwise-disjoint subgraphs of  $E$ , and let  $\{\Gamma_i\}_{i \in I}$  be a family of groups. Assume that there is a labeling map  $L_i : t_E^{-1}(E_i^0) \rightarrow \Gamma_i$  for every  $i \in I$ , and combine them to a map

$$\mathcal{E}^1 := \bigsqcup_{i \in I} t_E^{-1}(E_i^0) \xrightarrow{\mathcal{L}} \bigsqcup_{i \in I} \Gamma_i =: \mathcal{G}. \tag{5.16}$$

(Note that we can view  $\mathcal{G}$  as a groupoid in the obvious way.) We call the pair  $(E, \mathcal{L})$  a *base graph*. The idea of constructing a locally derived graph is that in the base graph we replace every subgraph  $E_i$  by its derived graph, and unfold (keeping the source fixed) every edge that does not belong to any of the subgraphs but ends in a subgraph. More precisely, we have the following:

**Definition 5.7.** The *locally derived graph*  $(E_{\mathcal{L}}^0, E_{\mathcal{L}}^1, s_{\mathcal{L}}, t_{\mathcal{L}})$  of a base graph  $(E, \mathcal{L})$  is given by

$$\begin{aligned} E_{\mathcal{L}}^0 &:= \left( E^0 \setminus \bigcup_{i \in I} E_i^0 \right) \sqcup \bigcup_{i \in I} (E_i^0 \times \Gamma_i), \\ E_{\mathcal{L}}^1 &:= \left( E^1 \setminus \bigcup_{i \in I} t_E^{-1}(E_i^0) \right) \sqcup \bigcup_{i \in I} (t_E^{-1}(E_i^0) \times \Gamma_i), \\ \begin{cases} s_{\mathcal{L}}((e, g)) := (s_E(e), g) & \text{for } e \in E_i^1, g \in \Gamma_i, i \in I, \\ s_{\mathcal{L}}((e, g)) := (s_E(e), 1_j) & \text{for } e \in t_E^{-1}(E_i^0) \setminus E_i^1, g \in \Gamma_i, s_E(e) \in E_j^0, i, j \in I, \\ s_{\mathcal{L}}((e, g)) := s_E(e) & \text{for } e \in t_E^{-1}(E_i^0) \setminus E_i^1, g \in \Gamma_i, i \in I, s_E(e) \notin \bigcup_{j \in I} E_j^0, \\ s_{\mathcal{L}}(e) := (s_E(e), 1_i) & \text{for } e \notin \mathcal{E}^1, s_E(e) \in E_i^0, i \in I, \\ s_{\mathcal{L}}(e) := s_E(e) & \text{for } e \notin \mathcal{E}^1, s_E(e) \notin \bigcup_{i \in I} E_i^0, \end{cases} \\ \begin{cases} t_{\mathcal{L}}((e, g)) := (t_E(e), \mathcal{L}(e)g) & \text{for } e \in t_E^{-1}(E_i^0), g \in \Gamma_i, i \in I, \\ t_{\mathcal{L}}(e) := t_E(e) & \text{for } e \notin \mathcal{E}^1. \end{cases} \end{aligned} \tag{5.17}$$

Here,  $1_i, i \in I$ , is the neutral element of  $\Gamma_i$ .

**Lemma 5.8.** *The following assignments*

$$\begin{cases} \pi_E^0((v, g)) := v & \text{for } v \in E_i^0, g \in \Gamma_i, i \in I, \\ \pi_E^0(v) := v & \text{for } v \notin \bigcup_{i \in I} E_i^0, \\ \pi_E^1((e, g)) := e & \text{for } e \in t_E^{-1}(E_i^0), g \in \Gamma_i, i \in I, \\ \pi_E^1(e) := e & \text{for } e \notin \mathcal{E}^1, \end{cases}$$

define a surjective graph homomorphism  $\pi_E : E_{\mathcal{L}} \rightarrow E$ .

*Proof.* First, note that, for all  $e \in E_i^1$ ,  $g \in \Gamma_i$ ,  $i \in I$ , we have

$$\pi_E^0(s_{\mathcal{X}}((e, g))) = \pi_E^0((s_E(e), g)) = s_E(e) = s_E(\pi_E^1((e, g))). \quad (5.18)$$

Much in the same way, if  $e \in t_E^{-1}(E_i^0) \setminus E_i^1$ ,  $g \in \Gamma_i$ , and  $s_E(e) \in E_j^0$  for some  $i, j \in I$ , then

$$\pi_E^0(s_{\mathcal{X}}((e, g))) = \pi_E^0((s_E(e), 1_j)) = s_E(e) = s_E(\pi_E^1((e, g))). \quad (5.19)$$

Next, if  $e \in t_E^{-1}(E_i^0) \setminus E_i^1$ ,  $s_E(e) \notin \bigcup_{j \in I} E_j^0$ , and  $g \in \Gamma_i$  for some  $i \in I$ , then

$$\pi_E^0(s_{\mathcal{X}}((e, g))) = \pi_E^0(s_E(e)) = s_E(e) = s_E(\pi_E^1((e, g))). \quad (5.20)$$

Now, if  $e \notin \mathcal{E}^1$  and  $s_E(e) \in E_i^0$  for some  $i \in I$ , we obtain

$$\pi_E^0(s_{\mathcal{X}}(e)) = \pi_E^0((s_E(e), 1_i)) = s_E(e) = s_E(\pi_E^1(e)). \quad (5.21)$$

Finally, if  $e \notin \mathcal{E}^1$  and  $s_E(e) \notin \bigcup_{i \in I} E_i^0$ , then

$$\pi_E^0(s_{\mathcal{X}}(e)) = \pi_E^0(s_E(e)) = s_E(e) = s_E(\pi_E^1(e)). \quad (5.22)$$

The calculations for the target map are much simpler and analogous to (5.18) and (5.22).  $\blacksquare$

We call the graph homomorphism  $\pi_E : E_{\mathcal{X}} \rightarrow E$  a *projection folding*. Under additional assumptions, we can prove that this graph homomorphism is admissible.

**Proposition 5.9.** *If, for all  $i \in I$ , the group  $\Gamma_i$  is finite and the inclusion  $E_i \hookrightarrow E$  is regular, then the projection folding  $\pi_E : E_{\mathcal{X}} \rightarrow E$  is admissible.*

*Proof.* First, since all  $\Gamma_i$  are finite,  $\pi_E$  is proper. To prove target bijectivity, we consider two cases. If  $e \in t_E^{-1}(E_i^0)$  for some  $i \in I$ , then we obtain a map

$$(\pi_E^1)^{-1}(e) \ni (e, g) \mapsto t_{\mathcal{X}}((e, g)) = (t_E(e), \mathcal{L}(e)g) \in (\pi_E^0)^{-1}(t_E(e)) \quad (5.23)$$

whose inverse is given by  $(t_E(e), g) \mapsto (e, \mathcal{L}(e)^{-1}g)$ . Next, if  $e \notin \mathcal{E}^1$ , then the map

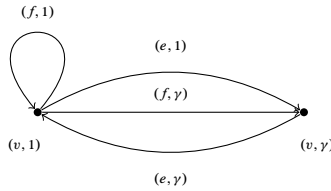
$$(\pi_E^1)^{-1}(e) = \{e\} \longrightarrow \{t_{\mathcal{X}}(e)\} = \{t_E(e)\} = (\pi_E^0)^{-1}(t_E(e)) \quad (5.24)$$

is, clearly, a bijection. Furthermore, the regularity of  $\pi_E$  at  $v \in E_i^0$ ,  $i \in I$ , follows from the regularity of the inclusions  $E_i \hookrightarrow E$  and the reasoning as in the proof of Proposition 5.5. Finally, the regularity at other vertices  $v \in E^0 \setminus \bigcup_{i \in I} E_i^0$  is immediate.  $\blacksquare$

**Example 5.10.** Consider the graph  $R_2$  of the Cuntz algebra  $\mathcal{O}_2$ , i.e.  $R_2^0 := \{v\}$ ,  $R_2^1 := \{e, f\}$ . Take the subgraph  $E$  of  $R_2$  given by  $E^0 := \{v\}$  and  $E^1 := \{e\}$ , and consider

$$\mathcal{L} : t_{R_2}^{-1}(v) = R_2^1 \longrightarrow \mathbb{Z}/2\mathbb{Z}, \quad e \mapsto \gamma, \quad f \mapsto 1,$$

where  $\gamma$  is the generator of  $\mathbb{Z}/2\mathbb{Z}$ . The graph  $(R_2)_{\mathcal{X}}$  is presented in the picture below.



It is clear that  $C^*((R_2)_{\mathcal{X}})$  is simple (e.g., see [28, Proposition 4.2]). Next, by Corollary 4.8, the admissible graph homomorphism  $\pi_{R_2} : (R_2)_{\mathcal{X}} \rightarrow R_2$  induces a unital  $*$ -homomorphism  $(\pi_{R_2})_{C^*}^* : \mathcal{O}_2 \rightarrow C^*((R_2)_{\mathcal{X}})$  given on generators by

$$S_e \mapsto S_{(e,1)} + S_{(e,\gamma)}, \quad S_f \mapsto S_{(f,1)} + S_{(f,\gamma)}.$$

The above  $*$ -homomorphism is injective because  $\mathcal{O}_2$  is simple. However, since

$$K_0(C^*((R_2)_{\mathcal{X}})) = \mathbb{Z}/2\mathbb{Z}$$

by [28, Theorem 7.16], and  $K_0(\mathcal{O}_2) = 0$ , it cannot be surjective. Hence,  $\mathcal{O}_2$  is a proper subalgebra of  $C^*((R_2)_{\mathcal{X}})$  and the  $K_0$ -class of  $1 \in C^*((R_2)_{\mathcal{X}})$  is zero. On the other hand, it follows from [30, Theorem 6.5] that  $C^*((R_2)_{\mathcal{X}})$  is stably isomorphic with  $\mathcal{O}_3$ . Better still, as the  $K_0$ -class of  $1 \in M_2(\mathcal{O}_3)$  is zero, using again [30, Theorem 6.5], one can conclude that  $C^*((R_2)_{\mathcal{X}}) \cong M_2(\mathcal{O}_3)$ .<sup>1</sup>

Let  $F$  be a subgraph of a graph  $E$  such that  $t_E^{-1}(F^0) \subseteq F^1$  and let  $(F, \mathcal{L})$  be a base graph. Then any family  $\{F_i\}_{i \in I}$  of pairwise-disjoint subgraphs of  $F$  is a family of pairwise-disjoint subgraphs of  $E$  and  $t_E^{-1}(F_i^0) = t_F^{-1}(F_i^0)$  for all  $i \in I$ . Therefore, given  $\mathcal{L} : \mathcal{F}^1 \rightarrow \mathcal{G}$ , we can construct both locally derived graphs  $F_{\mathcal{X}}$  and  $E_{\mathcal{X}}$ .

**Proposition 5.11.** *If  $F$  is an admissible subgraph of a graph  $E$  and  $(F, \mathcal{L})$  is a base graph, then  $F_{\mathcal{X}}$  is an admissible subgraph of  $E_{\mathcal{X}}$ .*

*Proof.* We need to check the conditions (A1) and (A2) of Definition 3.1. To prove (A1), suppose that  $w \in F_{\mathcal{X}}^0 \cap \text{reg}(E_{\mathcal{X}})$  is such that  $t_{E_{\mathcal{X}}}(s_{E_{\mathcal{X}}}^{-1}(w)) \subseteq E_{\mathcal{X}}^0 \setminus F_{\mathcal{X}}^0$ . First, note that, by the construction of  $E_{\mathcal{X}}$ ,  $w$  cannot be of the form  $(v, g) \in F_i^0 \times \Gamma_i$ , where  $g \neq 1_i, i \in I$ . Therefore,  $w$  has to be of the form  $(v, 1_i) \in F_i^0 \times \Gamma_i$  for some  $i \in I$ , or  $v \in F^0 \setminus \bigcup_{i \in I} F_i^0$ . In both cases, it follows immediately from (5.17) that the condition  $t_{E_{\mathcal{X}}}(s_{E_{\mathcal{X}}}^{-1}(w)) \subseteq E_{\mathcal{X}}^0 \setminus F_{\mathcal{X}}^0$  implies that  $t_E(s_E^{-1}(v)) \subseteq E^0 \setminus F^0$ , which contradicts the fact that  $E^0 \setminus F^0$  is saturated. Hence,  $E_{\mathcal{X}}^0 \setminus F_{\mathcal{X}}^0$  is saturated. Finally, using again (5.17) combined with the condition (A2) for  $F \subseteq E$ , we conclude the condition (A2) for  $F_{\mathcal{X}} \subseteq E_{\mathcal{X}}$ . ■

<sup>1</sup>We owe this argument to Jack Spielberg.

### 6. Pullbacks of graph algebras from pushouts of graphs

In this section, we prove our main theorems stating under which conditions the two contravariant functors given by Lemma 4.2 and Theorem 4.5 turn pushouts of directed graphs into pullbacks of algebras.

#### 6.1. Path algebras

We begin with the pushout-to-pullback theorem for path algebras.

**Theorem 6.1.** *Let the diagram*

$$\begin{array}{ccc}
 & E \amalg F & \\
 \begin{matrix} \xrightarrow{(\iota_E^0, \iota_E^1)} \\ \end{matrix} & \downarrow G & \begin{matrix} \xleftarrow{(\iota_F^0, \iota_F^1)} \\ \end{matrix} \\
 E & & F \\
 \begin{matrix} \xleftarrow{(f^0, f^1)} \\ \end{matrix} & & \begin{matrix} \xrightarrow{(g^0, g^1)} \\ \end{matrix} \\
 & G &
 \end{array} \tag{6.1}$$

be a pushout diagram in the category of graphs and proper graph homomorphisms such that

- (1) both  $f^0$  and  $g^0$  are injective (vertex injectivity),
- (2)  $\iota_{\amalg}(x) = s_{\amalg}(y) \Rightarrow (x, y \in \iota_{E^1}(E^1) \text{ or } x, y \in \iota_{F^1}(F^1))$  (one color),
- (3)  $f^1$  or  $g^1$  is injective (one-sided injectivity).

Then, for any field  $k$ , the contravariant functor  $\text{Map}_f(\cdot, k)$  transforms the above pushout diagram to the following one-surjective pullback diagram in the category of algebras over  $k$ :

$$\begin{array}{ccc}
 & k(E \amalg F) & \\
 \begin{matrix} \xleftarrow{\iota_E^*} \\ \end{matrix} & \downarrow kG & \begin{matrix} \xrightarrow{\iota_F^*} \\ \end{matrix} \\
 kE & & kF \\
 \begin{matrix} \xrightarrow{f^*} \\ \end{matrix} & & \begin{matrix} \xleftarrow{g^*} \\ \end{matrix} \\
 & kG &
 \end{array}$$

Here,  $f$  and  $g$  are the maps induced by the graph homomorphisms  $E \xleftarrow{(f^0, f^1)} G \xrightarrow{(g^0, g^1)} F$ , respectively. Furthermore, if  $E^0$  and  $F^0$  are finite, then all algebras in the above diagram and the homomorphisms between them are unital, and the diagram is a pullback diagram in the category of unital algebras.

*Proof.* To begin with, Lemma 3.6 guarantees that (6.1) is a pushout diagram in the category of graphs and proper graph homomorphisms for any proper graph homomorphisms

$$E \xleftarrow{(f^0, f^1)} G \xrightarrow{(g^0, g^1)} F \tag{6.2}$$

satisfying the assumptions (1) through (3). Next, as proper graph homomorphisms induce finite-to-one maps between the path spaces, and injective graph homomorphisms induce

injective maps between the path spaces, Lemma 3.9 and Lemma 3.6 yield the following pushout diagram in the category of sets and finite-to-one maps:

$$\begin{array}{ccc}
 & \text{FP}(E) \amalg \text{FP}(F) \\
 & \text{FP}(G) \\
 \iota_E \nearrow & & \nwarrow \iota_F \\
 \text{FP}(E) & & \text{FP}(F) \\
 f \searrow & & \nearrow g \\
 & \text{FP}(G) &
 \end{array} \tag{6.3}$$

Now, Lemma 4.2 turns this diagram into the commutative diagram in the category of algebras:

$$\begin{array}{ccc}
 & k(E \amalg F) \\
 & k(G) \\
 \iota_E^* \nearrow & & \nwarrow \iota_F^* \\
 kE & & kF \\
 f^* \searrow & & \nearrow g^* \\
 & kG &
 \end{array} \tag{6.4}$$

Furthermore, setting  $K = k$  in Lemma 3.11, we conclude that the above diagram is a pullback diagram in the category of sets and maps. We can combine these two facts to conclude that the above diagram is a pullback diagram in the category of algebras.

To prove the last part of the theorem, assume that both  $E^0$  and  $F^0$  are finite. As graph homomorphisms are assumed to be proper, not only the finiteness of  $E^0 \amalg_{G^0} F^0$  but also the finiteness of  $G^0$  follow from the finiteness of  $E^0$  and  $F^0$ , so all algebras are unital, as claimed. Next, the unitality of all homomorphisms follows from Lemma 4.2. Finally, under these circumstances, the maps witnessing the universality in the category of unital algebras are evidently unital, so the above diagram is a pullback diagram in the category of unital algebras. ■

### 6.2. Leavitt path algebras and graph $C^*$ -algebras

For starters, if  $f: G \rightarrow E$  is an injective admissible graph homomorphism, then it follows from Remark 3.5 and [1, Corollary 2.4.13 (ii)] that  $\ker f_L^*$  is the ideal generated by

$$\{[\chi_v] \mid v \in E^0 \setminus f^0(G^0)\} \cup \left\{ [\chi_w] - \sum_{e \in s_E^{-1}(w) \cap t_E^{-1}(f^0(G^0))} [\chi_e][\chi_{e^*}] \mid w \in B_{E^0 \setminus f^0(G^0)} \right\}. \tag{6.5}$$

As  $E^0 \setminus f^0(G^0)$  is hereditary by Proposition 3.2, we can combine the above with [1, Lemma 2.4.6] to obtain:

$$\begin{aligned}
 \ker f_L^* = & \text{span} \left\{ [\chi_\alpha][\chi_v][\chi_{\beta^*}] \mid \begin{array}{l} \alpha, \beta \in \text{FP}(E), v \in E^0 \setminus f^0(G^0), \\ t_E(\alpha) = t_E(\beta) = v \end{array} \right\} \\
 & + \text{span} \left\{ [\chi_\mu] \left( [\chi_w] - \sum_{e \in s_E^{-1}(w) \cap t_E^{-1}(f^0(F^0))} [\chi_e][\chi_{e^*}] \right) [\chi_{v^*}] \mid \begin{array}{l} \mu, v \in \text{FP}(E), w \in B_{E^0 \setminus f^0(G^0)}, \\ t_E(\mu) = t_E(v) = w \end{array} \right\}. \tag{6.6}
 \end{aligned}$$

Before we consider our pushout-to-pullback theorem for Leavitt path algebras, we need further technical results.

**Lemma 6.2.** *Let  $G$  and  $E$  be arbitrary graphs, and let  $(f^0, f^1): G \rightarrow E$  be an admissible graph homomorphism. Then*

$$\forall v \in E^0: [\chi_v] \in \ker f_L^* \iff v \in E^0 \setminus f^0(G^0).$$

*Proof.* The claim of the lemma is equivalent to the following statement:

$$\forall v \in E^0: \sum_{w \in (f^0)^{-1}(v)} [\chi_w] = 0 \iff (f^0)^{-1}(v) = \emptyset. \tag{6.7}$$

The implication  $(\Leftarrow)$  is true due to the convention that the sum over the empty set equals 0. The other implication follows from the fact that the elements  $[\chi_w]$ ,  $w \in (f^0)^{-1}(v)$ , are linearly independent by [1, Corollary 1.5.12]. ■

**Lemma 6.3.** *Let  $(\iota_E^0, \iota_E^1): E \rightarrow P$  and  $(\iota_F^0, \iota_F^1): F \rightarrow P$  be admissible graph homomorphisms, and let  $(\iota_E)_L^*$  and  $(\iota_F)_L^*$  be the induced  $\mathbb{Z}$ -graded algebra homomorphisms. Then*

$$\ker(\iota_E)_L^* \cap \ker(\iota_F)_L^* = \{0\} \iff P^0 = \iota_E^0(E^0) \cup \iota_F^0(F^0).$$

*Proof.* Since  $\ker(\iota_E)_L^* \cap \ker(\iota_F)_L^*$  is a  $\mathbb{Z}$ -graded ideal, by [1, Theorem 2.5.8],

$$\ker(\iota_E)_L^* \cap \ker(\iota_F)_L^* = \{0\} \iff \{v \in P^0 \mid [\chi_v] \in \ker(\iota_E)_L^* \cap \ker(\iota_F)_L^*\} = \emptyset. \tag{6.8}$$

Finally, by Lemma 6.2, the above set equals  $P^0 \setminus (\iota_E^0(E^0) \cup \iota_F^0(F^0))$ , which ends the proof. ■

We are now ready for our main result.

**Theorem 6.4.** *Let  $(f^0, f^1)$  and  $(g^0, g^1)$  be admissible graph homomorphisms and let*

$$\begin{array}{ccc}
 & E \amalg F & \\
 (\iota_E^0, \iota_E^1) \nearrow & & \nwarrow (\iota_F^0, \iota_F^1) \\
 E & & F \\
 (f^0, f^1) \searrow & G & \nearrow (g^0, g^1)
 \end{array} \tag{6.9}$$

*be a pushout diagram in the category  $\text{OG}$  of graphs and graph homomorphisms. Assume also that*

- (P1)  $(f^0, f^1)$  is injective,
- (P2)  $g^0$  restricted to  $(f^0)^{-1}(B_{E^0 \setminus f^0(G^0)})$  is injective,
- (P3)  $\iota_E^0(B_{E^0 \setminus f^0(G^0)}) \subseteq B_{(E \amalg F)^0 \setminus \iota_F^0(F^0)}$ .

Then, for any field  $k$ , there exists the commutative diagram of the induced  $\mathbb{Z}$ -graded algebra homomorphisms

$$\begin{array}{ccc}
 & L_k(E \amalg_G F) & \\
 (\iota_E)_L^* \swarrow & & \searrow (\iota_F)_L^* \\
 L_k(E) & & L_k(F) \\
 f_L^* \searrow & & \swarrow g_L^* \\
 & L_k(G) &
 \end{array} \tag{6.10}$$

Moreover, it is a left-surjective pullback diagram in the category  $\text{ZKA}$  of  $\mathbb{Z}$ -graded algebras and  $\mathbb{Z}$ -graded algebra homomorphisms. Finally, if  $E^0$  and  $F^0$  are finite, then all algebras in the above diagram and homomorphisms between them are unital, and the diagram is a pullback diagram in the category  $\text{ZUKA}$  of unital  $\mathbb{Z}$ -graded algebras and unital  $\mathbb{Z}$ -graded algebra homomorphisms.

*Proof.* Throughout the proof, let  $P := E \amalg_G F$ . The existence of the diagram (6.10) follows from Lemma 3.8 and Theorem 4.5. Its commutativity is due to the commutativity of (6.9), the surjectivity of  $f_L^*$  is due to (P1) and Corollary 4.6. To show that (6.10) is a pullback, by [27, Proposition 3.1] (cf. [15, Lemma 4.1]), we have to prove that

- (1)  $\ker(\iota_E)_L^* \cap \ker(\iota_F)_L^* = \{0\}$ ,
- (2)  $(g_L^*)^{-1}(f_L^*(L_k(E))) = (\iota_F)_L^*(L_k(P))$ ,
- (3)  $\ker f_L^* = (\iota_E)_L^*(\ker(\iota_F)_L^*)$ .

Since (6.9) is a pushout, we infer that  $P^0 = \iota_E^0(E^0) \cup \iota_F^0(F^0)$ . Therefore, the condition (1) follows from Lemma 6.3. Furthermore, as (P1) implies the injectivity of  $(\iota_F^0, \iota_E^0)$ , we conclude that both  $f_L^*$  and  $(\iota_F)_L^*$  are surjective, which proves the condition (2). Finally, as the diagram (6.10) is commutative, to prove the condition (3), it suffices to show that  $\ker f_L^* \subseteq (\iota_E)_L^*(\ker(\iota_F)_L^*)$ . Note that  $(\iota_E)_L^*(\ker(\iota_F)_L^*)$  is a vector subspace, so it is enough to prove that all the elements that span  $\ker f_L^*$  by (6.6) belong to  $(\iota_E)_L^*(\ker(\iota_F)_L^*)$ .

First, let us prove that  $(\iota_E)_L^*(\ker(\iota_F)_L^*)$  contains the generators of  $\ker f_L^*$ . For starters, since (6.9) is a pushout,  $\iota_E^0(E^0 \setminus f^0(G^0)) \subseteq P^0 \setminus \iota_F^0(F^0)$  and  $(\iota_E^0)^{-1}(\iota_E^0(v)) = \{v\}$  for any  $v \in E^0 \setminus f^0(G^0)$ . In turn, this implies that  $(\iota_E)_L^*([\chi_{\iota_E^0(v)}]) = [\chi_v]$ , where  $[\chi_{\iota_E^0(v)}] \in \ker(\iota_F)_L^*$  by Lemma 6.2.

Now we have to take care of breaking vertices, i.e. to show that

$$\forall w \in B_{E^0 \setminus f^0(G^0)}: \Omega_w := [\chi_w] - \sum_{e \in s_E^{-1}(w) \cap f^1(G^1)} [\chi_e][\chi_{e^*}] \in (\iota_E)_L^*(\ker(\iota_F)_L^*). \tag{6.11}$$

To this end, let us take  $v := (f^0)^{-1}(w)$  and note that  $v \in \text{reg}(G)$  by the injectivity of  $f^0$ . Next, the commutativity of the pushout diagram and the condition (P3) imply that

$$\iota_F^0(g^0(v)) = \iota_E^0(f^0(v)) \in B_{P^0 \setminus \iota_F^0(F^0)}, \tag{6.12}$$

so  $g^0(v)$  cannot be an infinite emitter. Also,  $g^0(v)$  cannot be a sink because  $v$  is regular. Consequently,  $g^0(v) \in \text{reg}(F)$ . Furthermore, note that  $\iota_E^0(w) = \iota_E^0(w')$  implies that  $w' \in f^0(G^0)$ . Better still,

$$(f^0)^{-1}((\iota_E^0)^{-1}(\iota_E^0(w))) = (g^0)^{-1}((\iota_F^0)^{-1}(\iota_F^0(f^0(v)))) = (g^0)^{-1}(g^0(v)) \subseteq \text{reg}(G) \tag{6.13}$$

because  $g$  is regular and  $g^0(v) \in \text{reg}(F)$ .

Now, we need to argue that  $w$  is the only element of

$$V_w := (\iota_E^0)^{-1}(\iota_E^0(w)) \tag{6.14}$$

that is an infinite emitter. Indeed, suppose that  $|s_E^{-1}(w')| = \infty$  for  $w' \in V_w$ . Then, as  $(f^0)^{-1}(w') \in \text{reg}(G)$  by (6.13), we conclude that  $w' \in \text{reg}(f(G))$  by the injectivity of  $f^0$ . Hence,  $w' \in B_{E^0 \setminus f^0(G^0)}$ , so  $w' = w$  by the condition (P2).

We are now ready to define the following element of  $L_k(P)$ :

$$\begin{aligned} \tilde{\Omega}_w &:= [\chi_{\iota_F^0(g^0(v))}] - \sum_{e \in s_F^{-1}(g^0(v))} [\chi_{\iota_F^1(e)}][\chi_{\iota_F^1(e)^*}] \\ &\quad - \sum_{e' \in s_E^{-1}(V_w \setminus \{w\}) \setminus f^1(G^1)} [\chi_{\iota_E^1(e')}] [\chi_{\iota_E^1(e')^*}]. \end{aligned} \tag{6.15}$$

First, using the injectivity of  $\iota_F$  and the fact that every  $\iota_E^1(e') \notin \iota_F^1(F^1)$ , we check that

$$(\iota_F)_L^*(\tilde{\Omega}_w) = [\chi_{g^0(v)}] - \sum_{e \in s_F^{-1}(g^0(v))} [\chi_e][\chi_{e^*}] = 0, \tag{6.16}$$

so  $\tilde{\Omega}_w \in \ker(\iota_F)_L^*$ .

Next, to prove that  $(\iota_E)_L^*(\tilde{\Omega}_w) = \Omega_w$ , first we show that

$$(\iota_E^1)^{-1}(\iota_F^1(s_F^{-1}(g^0(v)))) = s_E^{-1}(V_w) \cap f^1(G^1). \tag{6.17}$$

If  $x$  belongs to the left-hand side, then  $x \in f^1(G^1)$ . Furthermore, as  $\iota_E^1(x) = \iota_F^1(y)$  for  $y \in s_F^{-1}(g^0(v))$ , we can compute

$$\begin{aligned} \iota_E^0(s_E(x)) &= s_P(\iota_E^1(x)) = s_P(\iota_F^1(y)) = \iota_F^0(s_F(y)) \\ &= \iota_F^0(g^0(v)) = \iota_E^0(f^0(v)) = \iota_E^0(w). \end{aligned} \tag{6.18}$$

Hence, the left-hand side is contained in the right-hand side. To prove the opposite inclusion, take  $f^1(z) \in s_E^{-1}(V_w)$ . Then

$$\iota_E^0(s_E(f^1(z))) = \iota_E^0(w) = \iota_E^0(f^0(v)) = \iota_F^0(g^0(v)). \tag{6.19}$$

On the other hand,

$$\iota_E^0(s_E(f^1(z))) = \iota_E^0(f^0(s_G(z))) = \iota_F^0(g^0(s_G(z))). \tag{6.20}$$

Combining these two calculations with the injectivity of  $\iota_F^0$ , we infer that  $g^0(s_G(z)) = g^0(v)$ , so

$$s_F(g^1(z)) = g^0(s_G(z)) = g^0(v). \tag{6.21}$$

Hence,

$$\iota_E^1(f^1(z)) = \iota_F^1(g^1(z)) \in \iota_F^1(s_F^{-1}(g^0(v))), \tag{6.22}$$

which proves the desired inclusion.

Now, using the injectivity of  $\iota_E^1$  on  $E^1 \setminus f^1(G^1)$  and (6.17), we can compute:

$$\begin{aligned} (\iota_E)_L^*(\tilde{\Omega}_w) &= (\iota_E)_L^*([\chi_{\iota_E^0(w)}]) - \sum_{e \in (\iota_E^1)^{-1}(\iota_F^1(s_F^{-1}(g^0(v))))} [\chi_e][\chi_{e^*}] \\ &\quad - \sum_{e' \in s_E^{-1}(V_w \setminus \{w\}) \setminus f^1(G^1)} [\chi_{e'}][\chi_{e'^*}] \\ &= \sum_{w' \in V_w} [\chi_{w'}] - \sum_{e \in s_E^{-1}(V_w) \cap f^1(G^1)} [\chi_e][\chi_{e^*}] - \sum_{e' \in s_E^{-1}(V_w \setminus \{w\}) \setminus f^1(G^1)} [\chi_{e'}][\chi_{e'^*}] \\ &= [\chi_w] - \sum_{e \in s_E^{-1}(w) \cap f^1(G^1)} [\chi_e][\chi_{e^*}] + \sum_{w' \in V_w \setminus \{w\}} [\chi_{w'}] \\ &\quad - \sum_{e \in s_E^{-1}(V_w \setminus \{w\}) \cap f^1(G^1)} [\chi_e][\chi_{e^*}] - \sum_{e' \in s_E^{-1}(V_w \setminus \{w\}) \setminus f^1(G^1)} [\chi_{e'}][\chi_{e'^*}] \\ &= \Omega_w + \sum_{w' \in V_w \setminus \{w\}} \left( [\chi_{w'}] - \sum_{e \in s_E^{-1}(w')} [\chi_e][\chi_{e^*}] \right) \\ &= \Omega_w. \end{aligned} \tag{6.23}$$

Here the last step follows from  $V_w \setminus \{w\} \subseteq \text{reg}(E)$ .

We can now turn to lifting positive-length paths  $\alpha$  in  $E$  ending in

$$(E^0 \setminus f^0(G^0)) \cup B_{E^0 \setminus f^0(G^0)}. \tag{6.24}$$

To begin with, if  $t_{PE}(\alpha) \notin f^0(G^0)$ , then  $(\iota_E^0)^{-1}(\iota_E^0(t_{PE}(\alpha))) = \{t_{PE}(\alpha)\}$ , so

$$(\iota_E)_L^*([\chi_{\iota_E(\alpha)}]) = [\chi_\alpha] \tag{6.25}$$

by Lemma 2.5. Hence, the first spanning set of (6.6) is contained in  $(\iota_E)_L^*(L_k(P))$  because  $[\chi_{\alpha^*}] = (\iota_E)_L^*([\chi_{\iota_E(\alpha^*)}])$ .

Next, if  $t_{PE}(\mu) \in B_{E^0 \setminus f^0(G^0)}$ , then

$$(\iota_E)_L^*([\chi_{\iota_E(\mu)}]) = \sum_{w' \in (\iota_E^0)^{-1}(\iota_E^0(t_{PE}(\mu)))} [\chi_{v_{w'}}] \tag{6.26}$$

and

$$(\iota_E)_L^*([\chi_{\iota_E(\mu^*)}]) = \sum_{w' \in (\iota_E^0)^{-1}(\iota_E^0(t_{PE}(\mu)))} [\chi_{v_{w'}^*}] \tag{6.27}$$

by Lemma 2.5, where  $t_{PE}(v_{w'}) = w'$  and  $v_{t_{PE}(\mu)} = \mu$ . Now,

$$\sum_{w' \in (t_E^0)^{-1}(t_E^0(t_{PE}(\mu)))} [\chi_{v_{w'}}] \Omega_{t_{PE}(\mu)} = [\chi_\mu] \Omega_{t_{PE}(\mu)} \tag{6.28}$$

and

$$\Omega_{t_{PE}(\mu)} \sum_{w' \in (t_E^0)^{-1}(t_E^0(t_{PE}(\mu)))} [\chi_{v_{w'}^*}] = \Omega_{t_{PE}(\mu)} [\chi_{\mu^*}]. \tag{6.29}$$

Hence, the second spanning set of (6.6) is contained in  $(t_E)_L^*(L_k(P))$ .

Finally, as the generators  $[\chi_v]$ ,  $v \in E^0 \setminus f^0(G^0)$ , and  $\Omega_w$ ,  $w \in B_{E^0 \setminus f^0(G^0)}$ , are elements of  $(t_E)_L^*(\ker(t_F)_L^*)$ , and kernels are ideals, we infer that all the spanning elements in (6.6) belong to  $(t_E)_L^*(\ker(t_F)_L^*)$ , which proves the desired inclusion  $\ker f_L^* \subseteq (t_E)_L^*(\ker(t_F)_L^*)$ . To end with, note that the last part of the theorem regarding unitality is immediate. ■

**Theorem 6.5.** *Under the assumptions of Theorem 6.4, there exists the commutative diagram of the induced gauge-equivariant \*-homomorphisms*

$$\begin{array}{ccc}
 & C^*(E \amalg_G F) & \\
 (t_E)_{C^*}^* \swarrow & & \searrow (t_F)_{C^*}^* \\
 C^*(E) & & C^*(F) \\
 f_{C^*}^* \searrow & & \swarrow g_{C^*}^* \\
 & C^*(G) &
 \end{array} \tag{6.30}$$

Moreover, it is a left-surjective pullback diagram in the category  $\text{GC}^*\text{A}$ . Finally, if  $E^0$  and  $F^0$  are finite, then all  $C^*$ -algebras in the above diagram and homomorphisms between them are unital, and the diagram is a pullback diagram in the category  $\text{GUC}^*\text{A}$ .

*Proof.* The existence of the above commutative diagram in the category  $\text{GC}^*\text{A}$  follows from Lemma 3.8 and Corollary 4.8. The surjectivity of  $f_{C^*}^*$  is implied by Corollary 4.9. Next, recall that a \*-algebra is called AF if it is the union of a directed family of finite-dimensional \*-subalgebras (e.g., see [7, Definition 2.2]). For any Leavitt path algebra  $L_C(E)$ , its degree-0 component is a \*-subalgebra which is AF (e.g., see [1, Proposition 2.1.14]). Therefore, using the surjectivity of  $f_L^*$  (see Corollary 4.6), we can apply [7, Theorem 2.6] to conclude that (6.30) is a pullback diagram in the category  $\text{GC}^*\text{A}$ . Again, the last part of the theorem regarding unitality is immediate. ■

### 7. Applications in noncommutative topology

The plethora of applications of Theorem 6.5 in noncommutative topology goes beyond the strictly contravariant setup that is the scope of this paper. In particular, it is very promising to combine Theorem 6.5 with the mixed-pullback theorem [18, Theorem 5.2]. Herein, we

will focus on three applications: generalized Stalling’s folding, collapsing line graphs to initial graphs, and projecting folding locally derived graphs onto their base graphs.

**7.1. Generalized foldings and multichamber even quantum spheres**

Herein, we consider the following special case of Theorem 6.5.

**Corollary 7.1.** *Let  $G$  be an admissible subgraph of  $F$  and let  $F$  be an admissible subgraph of  $E$ . Then the following diagram*

$$\begin{array}{ccc}
 & E & \\
 \iota_g \nearrow & & \nwarrow \iota_f \\
 F \amalg_G E & & F, \\
 \nwarrow f & & \nearrow g \\
 & F \amalg_G F &
 \end{array} \tag{7.1}$$

where the map  $\iota_f$  is the inclusion  $F \subseteq E$  and the maps  $g$ ,  $f$ , and  $\iota_g$  are given by (5.2), (5.4), and (5.5), respectively, is a pushout diagram in the category of graphs. Furthermore, if we assume that  $B_{E^0 \setminus F^0} \subseteq G^0$ , then the induced diagram of gauge-equivariant  $*$ -homomorphisms

$$\begin{array}{ccc}
 & C^*(E) & \\
 \swarrow & & \searrow \\
 C^*(F \amalg_G E) & & C^*(F) \\
 \swarrow & & \searrow \\
 & C^*(F \amalg_G F) &
 \end{array}$$

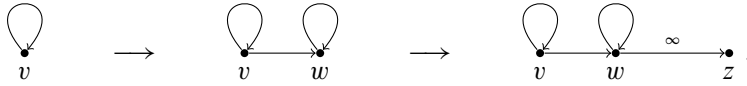
exists and is a pullback diagram in the category of  $C^*$ -algebras and  $U(1)$ -equivariant  $*$ -homomorphisms.

*Proof.* It is straightforward to show that the diagram (7.1) is a pushout diagram in the category OG by verifying the universal property. To prove the second claim, we have to check whether the assumptions of Theorem 6.5 are satisfied. For starters, we already know that  $g$  is admissible, and the admissibility of  $f$  is established in Proposition 5.3. Next, the condition (P1) is immediate because  $f$  is injective. Furthermore, note that  $B_{(F \amalg_G E)^0 \setminus (F \amalg_G F)^0} = B_{E^0 \setminus F^0}$ . Now, since  $g^0$  is injective when restricted to  $G^0$ , the assumption  $B_{E^0 \setminus F^0} \subseteq G^0$  ensures that (P2) holds true. Finally, it also follows that

$$\iota_g^0(B_{(F \amalg_G E)^0 \setminus (F \amalg_G F)^0}) = B_{E^0 \setminus F^0}, \tag{7.2}$$

so (P3) holds. ■

**Example 7.2.** Observe that the assumption  $B_{E^0 \setminus F^0} \subseteq G^0$  is not vacuous. Indeed, consider the following admissible inclusions of graphs  $G \hookrightarrow F \hookrightarrow E$ :



Here, the symbol  $\infty$  above the arrow means that there are infinitely many edges from  $w$  to  $z$ . It is clear that  $w \in B_{E^0 \setminus F^0}$  but  $w \notin G^0 = \{v\}$ .

Let  $n, k \in \mathbb{N}, n \geq 1$ . We inductively define the *multichamber sphere*  $S_k^n$  via the following pushout diagram:

$$\begin{array}{ccc}
 & S_{k+1}^n & \\
 \nearrow & & \nwarrow \\
 S_k^n & & B^n \\
 \nwarrow & & \nearrow \\
 & S^{n-1} &
 \end{array} \tag{7.3}$$

Here,  $S_0^n := B^n$  and  $S^{n-1}$  is embedded in  $B^n$  as the boundary sphere. Note that  $S_1^n = S^n$ . Next, consider the following pushout diagram of spaces representing the process of collapsing a chamber in a multichamber sphere ( $k \geq 1$ ):

$$\begin{array}{ccc}
 & S_{k-1}^n & \\
 \nearrow & & \nwarrow \\
 S_k^n & & B^n \\
 \nwarrow & & \nearrow \\
 & S^n &
 \end{array} \tag{7.4}$$

Here, the map

$$S^n = B^n \amalg_{S^{n-1}} B^n \longrightarrow B^n \tag{7.5}$$

is the flattening of a sphere defined as in (5.3), and the map  $S^n \hookrightarrow S_k^n$  is the inclusion of  $S^n$  as a chamber in a multichamber sphere.

The concept of a multichamber sphere admits a straightforward generalization to the realm of noncommutative topology. Since odd quantum balls do not have a graph  $C^*$ -algebraic presentation, in what follows we focus on the even case. Now, recall from Example 5.2 that graphs corresponding to the  $C^*$ -algebras of quantum spheres  $S_q^n$  and even quantum balls  $B_q^{2n}$  are denoted by  $L_n$  and  $M_n$ , respectively. Using Lemma 3.8, we inductively define the graph  $C_k^{2n}$  of a *multichamber even quantum sphere* via the following pushout diagram:

$$\begin{array}{ccc}
 & C_{k+1}^{2n} & \\
 \nearrow & & \nwarrow \\
 C_k^{2n} & & M_n \\
 \nwarrow & & \nearrow \\
 & L_{2n-1} &
 \end{array} \tag{7.6}$$

Here,  $C_0^{2n} := M_n$  and  $L_{2n-1} \hookrightarrow M_n$  is the admissible inclusion of graphs corresponding to the dual boundary map  $C(B_q^{2n}) \rightarrow C(S_q^{2n-1})$ . Note that  $C_1^{2n} = L_{2n}$ .

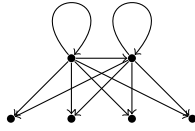
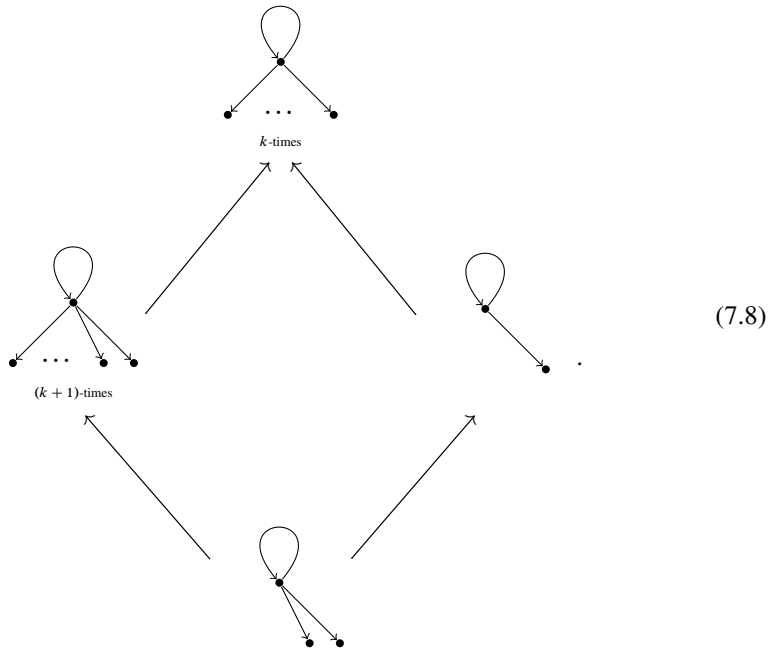


Figure 4. The graph  $C_3^4$ .

We call  $C(S_{k,q}^{2n}) := C^*(C_k^{2n})$  the  $C^*$ -algebra of the multichamber even quantum sphere  $S_{k,q}^{2n}$ . Next, we consider an analog of the diagram (7.4):

$$\begin{array}{ccc}
 & C_{k-1}^{2n} & \\
 & \nearrow & \nwarrow \\
 C_k^{2n} & & M_n \\
 & \nwarrow & \nearrow \\
 & L_{2n} &
 \end{array} \tag{7.7}$$

Here, the graph homomorphism  $L_{2n} \rightarrow M_n$  is a generalized folding (5.3) and  $L_{2n} \rightarrow C_k^{2n}$  is the admissible inclusion mapping  $L_{2n}$  to its rightmost copy inside of  $C_k^{2n}$ . For instance, in the case  $n = 1$ , we get the following pushout:



Finally, taking  $G := L_{2n-1}$ ,  $F := M_n$ , and  $E := C_{k-1}^{2n}$ , we apply Corollary 7.1 to obtain the following pullback diagram of unital gauge-equivariant  $*$ -homomorphisms in the category of unital  $U(1)$ - $C^*$ -algebras ( $k \geq 1$ ):

$$\begin{array}{ccc}
 & C^*(C_{k-1}^{2n}) & \\
 \swarrow & & \searrow \\
 C^*(C_k^{2n}) & & C^*(M_n) \\
 \searrow & & \swarrow \\
 & C^*(L_{2n}) &
 \end{array} \tag{7.9}$$

**7.2. Line graphs and the Cuntz algebra  $\mathcal{O}_2$**

Consider the graph  $R_2$  with one vertex and two edges and the surjective admissible graph homomorphism  $LR_2 \rightarrow R_2$  (see (5.8)):

$$\tag{7.10}$$

Here,  $e, f \mapsto v$ ,  $ee, ef \mapsto e$ , and  $ff, fe \mapsto f$ .

Now we are ready to present an application of Theorem 6.5 that is beyond Corollary 7.1 of the previous section. Consider the following pushout diagram in the category of graphs and graph homomorphisms:

$$\tag{7.11}$$

Denote the left graph and the upper graph in the above diagram by  $\widetilde{LR}_2$  and  $\widetilde{R}_2$ , respectively. Here, the right-graph homomorphism  $LR_2 \rightarrow R_2$  is given by (7.10) and the

left-bottom graph homomorphism  $LR_2 \rightarrow \widetilde{LR}_2$  is an admissible inclusion. As the assumptions of Theorem 6.5 are clearly satisfied, we obtain the following pullback diagram in the category of unital  $U(1)$ - $C^*$ -algebras and unital  $U(1)$ -equivariant  $*$ -homomorphisms:

$$\begin{array}{ccc}
 & C^*(\widetilde{R}_2) & \\
 \swarrow & & \searrow \\
 C^*(\widetilde{LR}_2) & & C^*(R_2) \cong \mathcal{O}_2 \\
 \searrow & & \swarrow \\
 & C^*(LR_2) \cong \mathcal{O}_2 &
 \end{array} \tag{7.12}$$

### 7.3. Locally derived graphs and quantum balls and spheres

Proposition 5.11 and Proposition 5.9 allow us to formulate the following:

**Corollary 7.3.** *Let  $(F, \mathcal{L})$  be a base graph such that, for all  $i \in I$ , the group  $\Gamma_i$  is finite and the inclusion  $F_i \hookrightarrow F$  is regular, and let  $F$  be an admissible subgraph of a graph  $E$ . Then the diagram*

$$\begin{array}{ccc}
 & E & \\
 \nearrow \pi_E & & \nwarrow \\
 E_{\mathcal{L}} & & F \\
 \nwarrow & & \nearrow \pi_F \\
 & F_{\mathcal{L}} &
 \end{array} \tag{7.13}$$

of admissible inclusions (the left-bottom arrow and the right-top arrow) and projection foldings (the right-bottom arrow and the left-top arrow) is a pushout diagram in the category  $\text{OG}$  of graphs and graph homomorphisms. Moreover, if

$$B_{E^0 \setminus F^0} \cap \bigcup_{i \in I} F_i^0 = \emptyset, \tag{7.14}$$

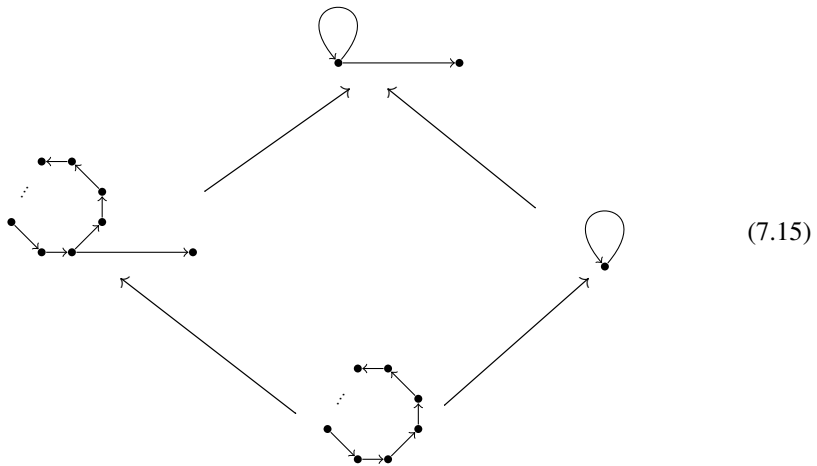
the induced diagram of gauge-equivariant  $*$ -homomorphisms

$$\begin{array}{ccc}
 & C^*(E) & \\
 \swarrow & & \searrow \\
 C^*(E_{\mathcal{L}}) & & C^*(F) \\
 \searrow & & \swarrow \\
 & C^*(F_{\mathcal{L}}) &
 \end{array}$$

is a pullback diagram in the category of  $C^*$ -algebras and  $U(1)$ -equivariant  $*$ -homomorphisms.

*Proof.* From the definitions of the maps involved, it is straightforward to show that the diagram (7.13) is a pushout diagram in the category OG by checking the universal property. It remains to check the conditions (P1)–(P3) of Theorem 6.4. The condition (P1) is immediately satisfied because the inclusion map  $F_{\mathcal{X}} \hookrightarrow E_{\mathcal{X}}$  is injective. The condition (P2) follows from the assumption (7.14). Indeed, as only the vertices of the form  $(v, g)$  can be identified, suppose that  $(v, g) \in B_{E_{\mathcal{X}} \setminus F_{\mathcal{X}}^0}$ . For starters, since  $(v, g)$  is an infinite emitter, so is  $v$  by the finiteness of all  $\Gamma_i$ . Better still,  $v$  emits infinitely many edges beyond  $F^0$ . Furthermore, if  $v$  would emit infinitely many or zero edges into  $F^0$ , then  $(v, g)$  would emit infinitely many or zero edges into  $F_{\mathcal{X}}^0$ , which contradicts the assumption that  $(v, g)$  is a breaking vertex. We thus conclude that  $v \in B_{E^0 \setminus F^0} \cap \bigcup_{i \in I} F_i^0$ , which contradicts (7.14). Hence, an element of  $B_{E_{\mathcal{X}} \setminus F_{\mathcal{X}}^0}$  cannot be of the form  $(v, g)$ , which proves (P2). Finally, let  $v \in B_{E_{\mathcal{X}} \setminus F_{\mathcal{X}}^0}$ . Then  $\pi_F^0(v) = \pi_E^0(v)$  is an infinite emitter in  $E$  because  $\pi_E^1$  is proper. On the other hand,  $v \in \text{reg}(F_{\mathcal{X}})$ ,  $(\pi_F^0)^{-1}(\pi_F^0(v)) = \{v\}$  and  $\pi_F^1$  is surjective, so  $\pi_F^0(v) \in \text{reg}(F)$ . Therefore,  $\pi_E^0(v) \in B_{E^0 \setminus F^0}$ , which proves (P3). ■

We end the paper by exemplifying Corollary 7.3. We begin with a very simple example



of a pushout diagram of graphs in the category OG that induces a pullback diagram in the category of unital  $U(1)$ - $C^*$ -algebras. The right-top  $*$ -homomorphism of this pullback diagram is the symbol map  $\sigma$  giving rise to the standard short exact sequence for the Toeplitz  $C^*$ -algebra  $\mathcal{T}$ :

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0. \tag{7.16}$$

Here,  $\mathcal{K}$  stands for the  $C^*$ -algebra of compact operators. Tensoring this short exact sequence with the matrix algebra  $M_n(\mathbb{C})$ , we obtain

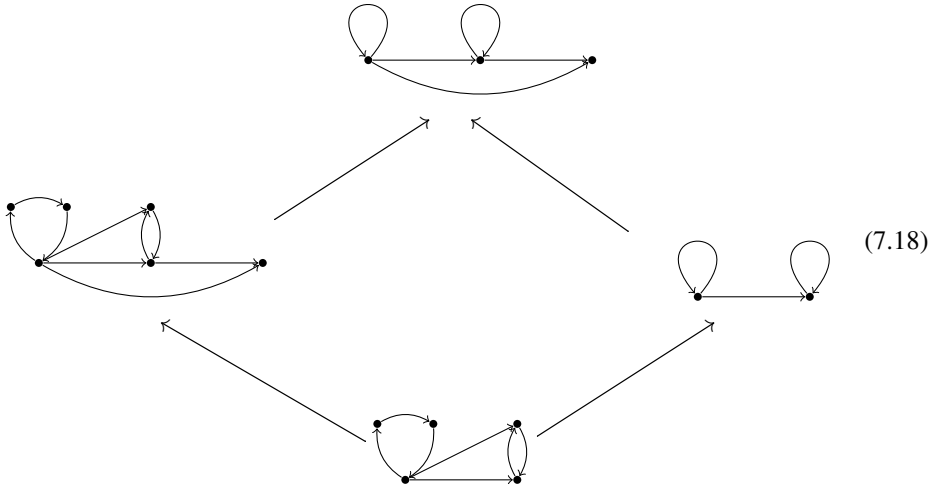
$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \otimes M_n(\mathbb{C}) \longrightarrow C(S^1) \otimes M_n(\mathbb{C}) \longrightarrow 0. \tag{7.17}$$

Now, possibly except for the middle term, the above short exact sequence coincides with the short exact sequence obtained from the left-bottom  $*$ -homomorphism of the induced pullback diagram.

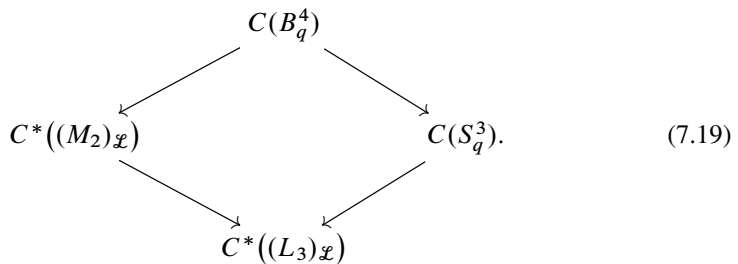
To go beyond the simple setting of the above example, consider the graphs  $M_2$  and  $L_3$  of the  $C^*$ -algebra  $C(B_q^4)$  of the even quantum ball  $B_q^4$  and the  $C^*$ -algebra  $C(S_q^3)$  of the boundary quantum sphere  $S_q^3$ , respectively. Recall that  $L_3^0 := \{v_1, v_2\}$ ,  $L_3^1 := \{e_{11}, e_{12}, e_{22}\}$ , and  $s_{L_3}(e_{ij}) := v_i, t_{L_3}(e_{ij}) := v_j, 1 \leq i \leq j \leq 2$ . Next, let  $\Gamma_1 := \mathbb{Z}/3\mathbb{Z}$ ,  $\Gamma_2 := \mathbb{Z}/2\mathbb{Z}$ , and let  $F_i^0 := \{v_i\}, F_i^1 := \{e_{ii}\}$ , define subgraphs  $F_i \subseteq L_3, i = 1, 2$ . Finally, put  $\gamma_n$  for the generator of  $\mathbb{Z}/n\mathbb{Z}$  and define

$$\mathcal{L} : t_{L_3}^{-1}(v_1) \cup t_{L_3}^{-1}(v_2) \longrightarrow \mathbb{Z}/3\mathbb{Z} \sqcup \mathbb{Z}/2\mathbb{Z}, \quad e_{11} \mapsto \gamma_3, \quad e_{12} \mapsto \gamma_2, \quad e_{22} \mapsto \gamma_2.$$

Now the pushout of Corollary 7.3 takes the form



and gives rise to the pullback diagram in the category of unital  $U(1)$ - $C^*$ -algebras and unital  $U(1)$ -equivariant  $*$ -homomorphisms



It is straightforward to generalize the above example to any even quantum ball and its boundary quantum sphere and any finite cyclic group.

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