

Optimal L^1 -Control in Coefficients for Dirichlet Elliptic Problems: H -Optimal Solutions

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Abstract. In this paper we study a Dirichlet optimal control problem associated with a linear elliptic equation the coefficients of which we take as controls in $L^1(\Omega)$. In particular, when the coefficient matrix is taken to satisfy the decomposition $B(x) = \rho(x)A(x)$ with a scalar function ρ , we allow the ρ to degenerate. Such problems are related to various applications in mechanics, conductivity and to an approach in topology optimization, the SIMP-method. Since equations of this type can exhibit the Lavrentieff phenomenon and non-uniqueness of weak solutions, we show that the optimal control problem in the coefficients can be stated in different forms depending on the choice of the class of admissible solutions. Using the direct method in the Calculus of variations, we discuss the solvability of the above optimal control problems in the so-called class of H -admissible solutions.

Keywords. Degenerate elliptic equations, control in coefficients, weighted Sobolev spaces, Lavrentieff phenomenon, direct method in the Calculus of variations.

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1. Introduction

The aim of this work is to study optimal control problems associated with a linear elliptic equation and homogeneous Dirichlet boundary conditions. The control variable is the matrix of L^1 -coefficients in the main part of elliptic operator. Existence or non-existence of L^1 -optimal solutions heavily depends on the class of admissible controls. The main questions concern the appropriate space-setting for the optimal control problem with L^1 -controls in the coefficients and the right choice of class of admissible solutions. Using the direct method

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in the Calculus of variations, we discuss the solvability of the above optimal control problems in the class of H -admissible solutions.

Note that optimal control problems in coefficients for elliptic equations are not new in the literature. As François Murat showed in 1970 (see [22, 23]), in general, such problems have no solution even if the original elliptic equation is non-degenerate. It turns out that this feature is typical for the majority of problems for optimal control in coefficients. Note that this topic has been widely studied by many authors in the case of non-degenerate weight function. We mainly could mention Allaire [1], Calvo-Jurado & Casado-Díaz [9], Haslinger & Neittaanmaki [13], Kapustyan & Kogut [14], Lions [18], Litvinov [19], Lurie [20], Murat [23], Murat & Tartar [25], Pironneau [28], Raytum [29], Sokolowski & Zolesio [30].

In this paper we deal with an optimal control problem in coefficients for the boundary value problem

$$\begin{cases} -\operatorname{div} B(x)\nabla y + y = f & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $f \in L^2(\Omega)$ is a given function and B is a non negative invertible matrix such that $B+B^{-1} \in L^1(\Omega; \mathbb{R}^{N \times N})$. Several physical phenomena related to equilibrium of continuous media are modeled by this elliptic problem. While the scalar situation discussed in this paper relates e.g. to conductivity problems, where $B(x)$ may represent a perfect conductor or a perfect insulator (see [11]), vector-valued analogues, which are under investigation, refer to problems in elasticity, where, in turn, $B(x)$ represents the elasticity tensor which may vanish for voids or damaged regions. In order to be able to handle such situations, we allow the matrix B to vanish on thin sets in Ω or to be unbounded there.

In Section 4 we will further concentrate on matrices $B(x)$ that admit a decomposition $B(x) = \rho(x)A(x)$ where the scalar coefficient $\rho(x)$ may degenerate, but satisfies certain bound-constraints almost everywhere as well as a “volume-type” constraint. The classical SIMP-approach to topology optimization [2] is reminiscent of the optimal control problem handled in this paper. In the SIMP-approach the function $\rho(\cdot)$ is taken as a so-called pseudo-density. However, in contrast to the modeling in this paper, the pseudo-density is taken to satisfy a positive lower bound, which is not assumed here. In a sense, the problem handled here is more general. In the context of image registration degenerate problems of the kind discussed in this paper occur, if one considers an optimal masking of thin features represented by ρ .

Even though numerous papers (see, for instance, [8, 26, 27, 34] and references there) are devoted to variational and non variational approaches to problems related to (1), only few papers deal with optimal control problems for degenerate partial differential equations (see, for example, [4, 6, 7]). This can be explained by several reasons. Firstly, boundary value problem (1) for every locally in-

tegrable matrix B exhibit the Lavrentieff phenomenon, the non-uniqueness of weak solutions, as well as other surprising consequences. So, in general, the mapping $B \mapsto y(B)$ can be multi-valued. Besides, the characteristic feature of this problem is the fact that for different admissible controls B with properties prescribed above, the corresponding weak solutions of (1) belong to different weighted Sobolev spaces. In addition, even if the original elliptic equation is non-degenerate, i.e., admissible controls B are such that

$$B(x) \geq \alpha I, \quad (B(x))^{-1} \geq \beta^{-1} I, \quad \text{a.e. in } \Omega,$$

the majority of optimal control problems in coefficients have no solution.

Our paper is organized as follow: at the beginning we state the problem of optimal control in the coefficients and prescribe the class of admissible controls which includes some div-like conditions in weighted spaces. After that we discuss the classification of admissible solutions to the above optimal control problem. We show that one of the characteristic features of this problem is the following fact: for every admissible L^1 -control the corresponding H -solution to the boundary value problem belongs to a weighted Sobolev space which essentially depends on the original control. So, the set of the so-called H -admissible solutions to the above problem can be viewed as a collection of pairs "control-state" in variable spaces each of which is embedded into $L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega)$.

Further we deal with the existence of optimal solutions to the original problem. We begin with a refinement of the celebrated div-curl lemma of F. Murat and L.C. Tartar [24] to the case of variable weighted Sobolev spaces. After that we study the topological properties of the class of H -admissible solutions and show that this set possesses some compactness properties with respect to the appropriate convergence in variable spaces. In conclusion, using the direct method in the Calculus of variations, we prove the existence of the H -optimal solutions to the original problem.

2. Notation and Preliminaries

In this section we introduce some notation and preliminaries that will be useful later on.

Let Ω be a bounded open subset of \mathbb{R}^N ($N \geq 1$) with a Lipschitz boundary. Let χ_E be the characteristic function of a subset $E \subseteq \Omega$, i.e., $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$. The space $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,1}(\Omega)$. For any subset $E \subset \Omega$ we denote by $|E|$ its N -dimensional Lebesgue measure $\mathcal{L}^N(E)$. Let $M_\alpha^\beta(\Omega)$ be the set of all matrices $A = [a_{ij}]$ in $L^\infty(\Omega; \mathbb{R}^{N \times N})$ such that

$$A(x) \geq \alpha I, \quad (A(x))^{-1} \geq \beta^{-1} I, \quad \text{a.e. in } \Omega \tag{2}$$

for two fixed constants α and β with $0 < \alpha \leq \beta < +\infty$. Here I is the identity matrix in $\mathbb{R}^{N \times N}$, and inequalities (2) should be considered in the sense of the quadratic forms defined by $(A\xi, \xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$. Note that (2) implies the inequality $|A(x)| \leq \beta$ a.e. in Ω .

Hereinafter by a weight we mean a locally integrable function ρ on \mathbb{R}^N such that $\rho(x) > 0$ for a.e. $x \in \mathbb{R}^N$. As a matter of fact, every weight ρ gives rise to a measure on the measurable subsets of \mathbb{R}^N through integration. This measure will also be denoted by ρ . Thus $\rho(E) = \int_E \rho dx$ for measurable sets $E \subset \mathbb{R}^N$. We will use the standard notation $L^2(\Omega, \rho dx)$ for the set of measurable functions f on Ω such that

$$\|f\|_{L^2(\Omega, \rho dx)} = \left(\int_{\Omega} f^2 \rho dx \right)^{\frac{1}{2}} < +\infty.$$

Definition 2.1. We say that a weight function $\rho : \mathbb{R}^N \rightarrow \mathbb{R}_+$ is degenerate on Ω if

$$\rho + \rho^{-1} \in L^1_{loc}(\mathbb{R}^N),$$

and the sum $\rho + \rho^{-1}$ does not belong to $L^\infty(\Omega)$.

With each of the degenerate weight functions ρ we will associate two weighted Sobolev spaces $W_\rho = W(\Omega, \rho dx)$ and $H_\rho = H(\Omega, \rho dx)$, where W_ρ is the set of functions $y \in W_0^{1,1}(\Omega)$ for which the norm

$$\|y\|_\rho = \left(\int_{\Omega} (y^2 + \rho |\nabla y|^2) dx \right)^{\frac{1}{2}}$$

is finite, and H_ρ is the closure of $C_0^\infty(\Omega)$ in the W_ρ -norm. Note that due to the compact embedding $W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ and estimates

$$\int_{\Omega} |y| dx \leq |\Omega|^{\frac{1}{2}} \left(\int_{\Omega} |y|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{|\Omega|} \|y\|_\rho, \quad (3)$$

$$\int_{\Omega} |\nabla y| dx \leq \left(\int_{\Omega} |\nabla y|^2 \rho dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \rho^{-1} dx \right)^{\frac{1}{2}} \leq C \|y\|_\rho, \quad (4)$$

we come to the following result (we refer to [34] for the details):

Theorem 2.2. *Let $\rho : \mathbb{R}^N \rightarrow \mathbb{R}_+$ be a degenerate weight on Ω . Then*

- (i) *the spaces H_ρ and W_ρ are complete with respect to the norm $\|\cdot\|_\rho$*
- (ii) *$H_\rho \subseteq W_\rho$, and W_ρ, H_ρ are Hilbert spaces*
- (iii) *$H_\rho \subset W_0^{1,1}(\Omega)$, $W_\rho \subset W_0^{1,1}(\Omega)$, and the estimate*

$$\|v\|_{W_0^{1,1}(\Omega)} \leq \left(\sqrt{|\Omega|} + \left(\int_{\Omega} \rho^{-1} dx \right)^{\frac{1}{2}} \right) \|v\|_\rho$$

is valid for every element $v \in W_\rho$

- (iv) *the embeddings $H_\rho \hookrightarrow L^1(\Omega)$ and $W_\rho \hookrightarrow L^1(\Omega)$ are compact.*

If ρ is a non-degenerate weight function, that is, ρ is bounded between two positive constants, then it is easy to verify that $W_\rho = H_\rho$. Note also that in the case when the weight function ρ belongs to the class of A_2 weights that was introduced by B. Muckenhoupt in the early 1970's (see [21]), then $W_\rho = H_\rho$ as well. However, for a "typical" degenerate weight ρ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in W_ρ . Hence the identity $W_\rho = H_\rho$ is not always valid (for the corresponding examples we refer to [10, 31]).

3. Radon measures and convergence in variable spaces

We recall here the definition and main properties of convergence in variable L^2 -spaces with respect to Radon measures (see, for instance, [33]).

By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω . The space of all nonnegative Radon measures on Ω will be denoted by $\mathcal{M}_+(\Omega)$. If μ is a nonnegative Radon measure on Ω , we will use $L^r(\Omega, d\mu)$, $1 \leq r \leq \infty$, to denote the usual Lebesgue space with respect to the measure μ with the corresponding norm

$$\|f\|_{L^r(\Omega, d\mu)} = \left(\int_{\Omega} |f(x)|^r d\mu \right)^{\frac{1}{r}}.$$

Let $\{\mu_k\}_{k \in \mathbb{N}}$, μ be Radon measures such that $\mu_k \xrightarrow{*} \mu$ in $\mathcal{M}_+(\Omega)$, i.e.,

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi d\mu_k = \int_{\Omega} \varphi d\mu \quad \forall \varphi \in C_0(\mathbb{R}^N), \quad (5)$$

where $C_0(\mathbb{R}^N)$ is the space of all compactly supported continuous functions. The typical example of such measures is

$$d\mu_k = \rho_k(x) dx, \quad d\mu = \rho(x) dx, \quad \text{where } 0 \leq \rho_k \rightarrow \rho \text{ in } L^1(\Omega).$$

1. A sequence $\{v_k \in L^2(\Omega, d\mu_k)\}_{k \in \mathbb{N}}$ is called bounded if

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |v_k|^2 d\mu_k < +\infty.$$

2. A bounded sequence $\{v_k \in L^2(\Omega, d\mu_k)\}_{k \in \mathbb{N}}$ converges weakly to an element $v \in L^2(\Omega, d\mu)$ if

$$\lim_{k \rightarrow \infty} \int_{\Omega} v_k \varphi d\mu_k = \int_{\Omega} v \varphi d\mu \quad \text{for any } \varphi \in C_0^\infty(\Omega),$$

which is denoted as $v_k \rightharpoonup v$ in $L^2(\Omega, d\mu_k)$.

3. Strong convergence $v_k \rightarrow v$ in $L^2(\Omega, d\mu_k)$ means that $v \in L^2(\Omega, d\mu)$ and

$$\lim_{k \rightarrow \infty} \int_{\Omega} v_k z_k d\mu_k = \int_{\Omega} v z d\mu \quad \text{as } z_k \rightarrow z \text{ in } L^2(\Omega, d\mu_k).$$

The following convergence properties in variable spaces are well known:

- (a) *Compactness criterium*: If the sequence is bounded in $L^2(\Omega, d\mu_k)$, then this sequence is compact w.r.t. the weak convergence in $L^2(\Omega, d\mu_k)$.
- (b) *Property of lower semicontinuity*: If $v_k \rightharpoonup v$ in $L^2(\Omega, d\mu_k)$, then

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |v_k|^2 d\mu_k \geq \int_{\Omega} v^2 d\mu.$$

- (c) *Criterium of strong convergence*: $v_k \rightarrow v$ if and only if $v_k \rightharpoonup v$ in $L^2(\Omega, d\mu_k)$ and

$$\lim_{k \rightarrow \infty} \int_{\Omega} |v_k|^2 d\mu_k = \int_{\Omega} v^2 d\mu.$$

In what follows, we make use the following results concerning the convergence in the variable space $L^2(\Omega, \rho_k dx)$.

Lemma 3.1 ([33]). *Let $\{\rho_k\}_{k \in \mathbb{N}}$ be a sequence of non-negative functions of $L^1(\Omega)$ such that $\rho_k \rightarrow \rho$ in $L^1(\Omega)$. Then the following statements hold true:*

- (B₁) *If a sequence $\{v_k \in L^2(\Omega, \rho_k dx)\}_{k \in \mathbb{N}}$ is bounded, then weak convergence $v_k \rightharpoonup v$ in $L^2(\Omega, \rho_k dx)$ is equivalent to weak convergence $\rho_k v_k \rightharpoonup \rho v$ in $L^1(\Omega)$.*
- (B₂) *If $a \in L^\infty(\Omega)$ and $v_k \rightharpoonup v$ in $L^2(\Omega, \rho_k dx)$, then $av_k \rightharpoonup av$ in $L^2(\Omega, \rho_k dx)$.*

Throughout the paper we will often use the concepts of weak and strong convergence in $L^1(\Omega)$. Recall also several definitions and facts about convergence in the classical L^1 -space. Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence in $L^1(\Omega)$. We recall that $\{a_k\}_{k \in \mathbb{N}}$ is called equi-integrable if for any $\delta > 0$ there is $\tau = \tau(\delta)$ such that $\int_S |a_k| dx < \delta$ for every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. Then the following assertions are equivalent:

- (i) a sequence $\{a_k\}_{k \in \mathbb{N}}$ is weakly compact in $L^1(\Omega)$;
- (ii) the sequence $\{a_k\}_{k \in \mathbb{N}}$ is equi-integrable;
- (iii) given $\delta > 0$ there exists $\lambda = \lambda(\delta)$ such that $\sup_{k \in \mathbb{N}} \int_{\{|a_k| > \lambda\}} |a_k| dx < \delta$.

Theorem 3.2 (Lebesgue's Theorem). *If a sequence $\{a_k\}_{k \in \mathbb{N}} \subset L^1(\Omega)$ is equi-integrable and $a_k \rightarrow a$ almost everywhere in Ω then $a_k \rightarrow a$ in $L^1(\Omega)$.*

4. Setting of the Optimal Control Problem

Let ξ_1, ξ_2 be given elements of $L^1(\Omega)$ satisfying the conditions

$$0 < \xi_1(x) \leq \xi_2(x) \text{ a.e. in } \Omega, \quad \xi_1^{-1} \in L^1(\Omega). \quad (6)$$

Let $m \in \mathbb{R}_+$ be a positive value such that $\|\xi_1\|_{L^1(\Omega)} \leq m \leq \|\xi_2\|_{L^1(\Omega)}$. Let S be a nonempty compact subset of $L^1(\Omega)$ satisfying

$$S \cap \{g \in L^1(\Omega) : \xi_1(x) \leq g(x) \leq \xi_2(x) \text{ a.e. in } \Omega\} \neq \emptyset. \quad (7)$$

In order to introduce the class of admissible L^1 -controls, we adopt the following concept:

Definition 4.1. For given $\rho \in L^1(\Omega)$ and $\vec{v} \in [L^2(\Omega, \rho dx)]^N$ we say that an element $g \in L^2(\Omega, d\mu)$ is the divergence of the vector field \vec{v} with respect to the weight ρ (in symbols $g(x) = \operatorname{div}_\rho \vec{v}(x)$), if \vec{v} and g are related by the formula

$$\int_{\mathbb{R}^n} g(x)\varphi(x) \rho(x) dx = - \int_{\mathbb{R}^n} (\vec{v}(x), \nabla\varphi(x))_{\mathbb{R}^N} \rho(x) dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (8)$$

Definition 4.2. We say that a matrix $B \in L^1(\Omega; \mathbb{R}^{N \times N})$ is an admissible control to the Dirichlet problem

$$-\operatorname{div} B(x)\nabla y + y = f \quad \text{in } \Omega \quad (9)$$

$$y = 0 \quad \text{on } \partial\Omega. \quad (10)$$

(it is written as $B \in \mathcal{B}_{ad}$) if there is a matrix $A = [\vec{a}_1, \dots, \vec{a}_N] \in L^\infty(\Omega; \mathbb{R}^{N \times N})$ and a weight $\rho \in L^1(\Omega)$ such that

$$B(x) = A(x)\rho(x), \quad A \in M_\alpha^\beta(\Omega) \quad (11)$$

$$|\operatorname{div}_\rho \vec{a}_i| \leq \gamma_i \quad \rho\text{-a.e. in } \Omega, \quad \forall i = 1, \dots, N \quad (12)$$

$$\rho \in \mathcal{R}_{ad}. \quad (13)$$

Here

$$\mathcal{R}_{ad} = \left\{ \rho \in S : \int_\Omega \rho dx = m, \quad \xi_1(x) \leq \rho(x) \leq \xi_2(x) \text{ a.e. in } \Omega \right\}, \quad (14)$$

$f \in L^2(\Omega)$, $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ is a given positive vector, and elements $\operatorname{div}_\rho \vec{a}_i \in L^2(\Omega, \rho dx)$ are defined by (8).

Remark 4.3. As an example of a compact subset S of $L^1(\Omega)$, we have (see [12])

$$S = \left\{ f \in L^1(\Omega) : \|f\|_{L^1(\Omega)} + \int_\Omega |Df| \leq C \right\},$$

where the variation $\int_\Omega |Df|$ of a measure Df is defined as follows

$$\int_\Omega |Df| = \sup \left\{ \int_\Omega f \operatorname{div} \varphi dx : \varphi = (\varphi_1, \dots, \varphi_N) \in C_0^1(\Omega; \mathbb{R}^N), |\varphi(x)| \leq 1 \text{ for } x \in \Omega \right\}$$

and $\operatorname{div} \varphi = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial x_i}$.

Remark 4.4. As follows from Definition 4.2 and properties (6)–(7), for every admissible control $B \in L^1(\Omega; \mathbb{R}^{N \times N})$ we deal with the boundary value problem for the degenerate elliptic equation

$$-\operatorname{div}(\rho A(x) \nabla y) + y = f \quad \text{in } \Omega, \quad y \in W_0^{1,1}(\Omega). \quad (15)$$

It means that for some admissible matrices of coefficients $B \in \mathcal{B}_{ad}$ the boundary value problem (9)–(10) can exhibit the Lavrentieff phenomenon [31] as well as other surprising consequences.

The optimal control problem we consider in this paper is to minimize the difference between a given distribution $y_d \in L^2(\Omega)$ and the solution of the Dirichlet problem (9)–(10) by choosing an appropriate matrix of coefficients $B \in \mathcal{B}_{ad}$. More precisely, we are concerned with the following optimal control problem

$$\text{Minimize } \left\{ I(B, y) = \zeta \int_{\Omega} |y(x) - y_d(x)|^2 dx + \int_{\Omega} |\nabla y(x)|_{\mathbb{R}^N}^2 \rho dx \right\} \quad (16)$$

subject to the constraints (11)–(13).

Here $\zeta > 0$ is a penalization parameter.

For our further analysis we make use the following observation. Let $B = A\rho \in \mathcal{B}_{ad}$ be an admissible control, and let A be a symmetric matrix. Then the quadratic form $\Phi(y) = \int_{\Omega} A(x) \nabla y \cdot \nabla y \rho dx$ with domain $W_{\rho} \subset L^2(\Omega)$ is closed and corresponds to a non-negative self-adjoint operator $\mathcal{A}_W = -\operatorname{div} \rho A \nabla$ in $L^2(\Omega)$. At the same time this form will also be closed in $H_{\rho} \subset L^2(\Omega)$, which leads to another non-negative self-adjoint operator $\mathcal{A}_H = -\operatorname{div} \rho A \nabla$ in $L^2(\Omega)$. Thus, there exist at least two different problems

$$\mathcal{A}_W y + y = f \quad \text{and} \quad \mathcal{A}_H y + y = f, \quad (17)$$

relating to boundary value problem (9)–(10). As we will see later, each of the problem (17) is uniquely solvable. So, the mapping $B \mapsto y(B, f)$, where $y(B, f)$ is a solution to problem (9)–(10), is multivalued, in general.

5. Classification of optimal solutions

In view of the observation given above, we adopt the classification of the solutions to the boundary valued problem (9)–(10) following Pastukhova & Zhikov [34] (for more details and other types of solutions we refer to [3, 16, 33]).

Definition 5.1. We say that a function $y = y(B, f) = y(A, \rho, f) \in W_{\rho}$ is a weak solution to the boundary value problem (9)–(10) for a fixed control $B = A\rho \in \mathcal{B}_{ad}$ and a given function $f \in L^2(\Omega)$, if the integral identity

$$\int_{\Omega} \left(\left(A(x) \nabla y, \nabla \varphi \right)_{\mathbb{R}^N} \rho(x) + y \varphi \right) dx = \int_{\Omega} f \varphi dx \quad (18)$$

holds for any $\varphi \in C_0^{\infty}(\Omega)$.

Definition 5.2. Let V_ρ be some intermediate space with $H_\rho \subseteq V_\rho \subseteq W_\rho$. We say that a function $y = y(B, f) = y(A, \rho, f) \in V_\rho$ is a V_ρ -solution or a variational solution to the boundary value problem (9)–(10) if the integral identity (18) holds for every test function $\varphi \in V_\rho$.

Remark 5.3. Note that for every fixed $B = A\rho \in \mathcal{B}_{ad}$ the existence and uniqueness of a V_ρ -solution are the direct consequence of the Riesz Representation Theorem. At the same time, the variational solutions do not exhaust the entire set of the weak solutions to the above boundary value problem. Indeed, as follows from [34], a weak solution $y = y(B, f) \in W_\rho$ is a variational one if and only if the energy equality

$$\int_{\Omega} \left(\left(A(x) \nabla y, \nabla y \right)_{\mathbb{R}^N} \rho + y^2 \right) dx = \int_{\Omega} f y dx \quad (19)$$

holds true. Therefore, if $y_1(B, f), y_2(B, f) \in W$ are variational solutions with $y_1(B, f) \neq y_2(B, f)$ (hence they belong to the different intermediate spaces $V_{1,\rho}$ and $V_{2,\rho}$), then

$$y = \frac{1}{2} (y_1(B, f) + y_2(B, f))$$

is a weak solution to (9)–(10) but not variational one. Moreover, as follows from Definition 5.1 the set of weak solutions to the boundary value problem (9)–(10) for a fixed control $B = A\rho \in \mathcal{B}_{ad}$ is convex and closed. Hence if $y_1(B, f), y_2(B, f) \in W$ are variational solutions such that $y_1(B, f) \neq y_2(B, f)$ then the corresponding set of the weak solutions is infinite.

It is obvious that for every fixed $B \in \mathcal{B}_{ad}$, $f \in C_0^\infty(\mathbb{R}^N)$, and $V_\rho(H_\rho \subseteq V_\rho \subseteq W_\rho)$ a variational solution is also a weak solution to the problem (9)–(10). However, the inverse assertion is not true in general. For a "typical" degenerate weight function ρ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in W_ρ , and hence there is no uniqueness of the weak solutions (see, for instance, [17,33]). However, we can describe a case when the weak solution is unique.

Lemma 5.4 ([27]). *Assume that the set $C_0^\infty(\Omega)$ is dense in W_ρ . Then the two concepts of a solution to the boundary value problem (9)–(10), given by Definitions 5.1 and 5.2, coincide and hence the weak solution $y = y(B, f) = y(A, \rho, f) \in W_\rho$ is unique.*

Now it is clear that the mapping $B \mapsto y(B, f)$ can be viewed as multivalued in general, and this depends on the choice of the corresponding solutions space V_ρ . As a result, the variational formulation of the optimal control problem (11)–(13), (16) can be stated in different forms. Taking this fact into account, we restrict of our analysis to the two sets of admissible solutions for the original

optimal control problem. Namely, we indicate the following sets

$$\begin{aligned}\Xi_H &= \{(B, y) \mid B = A\rho \in \mathcal{B}_{ad}, y \in H_\rho, (B, y) \text{ are related by (19)}\}, \\ \Xi_W &= \{(B, y) \mid B = A\rho \in \mathcal{B}_{ad}, y \in W_\rho, (B, y) \text{ are related by (19)}\}.\end{aligned}$$

As was mentioned above (see Remark 5.3), the sets Ξ_H and Ξ_W are always nonempty. Hence the corresponding minimization problems

$$\left\langle \inf_{(B,y) \in \Xi_H} I(B, y) \right\rangle \quad \text{and} \quad \left\langle \inf_{(B,y) \in \Xi_W} I(B, y) \right\rangle \quad (20)$$

are regular. However, because of the Lavrentieff effect, it may happen that for some fixed control $B = A\rho \in \mathcal{B}_{ad}$ and a given $f \in C_0^\infty(\mathbb{R}^N)$ the corresponding H_ρ -solution $y_H(A, \rho, f)$ and W_ρ -solution $y_W(A, \rho, f)$ to the boundary value problem (15) are not the same. This implies that the variational problems (20) are essentially different, in general. Hence, the minimizers to (20) can be different as well as $\inf_{(B,y) \in \Xi_H} I(B, y) \neq \inf_{(B,y) \in \Xi_W} I(B, y)$.

Note that due to the estimates (3)–(4), we have the obvious inclusions

$$\Xi_H \subset L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega), \quad \Xi_W \subset L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega).$$

Taking this fact into account, we adopt the following concept:

Definition 5.5. We say that a pair $(B^0, y^0) \in L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega)$ is an H -optimal solution to the problem (11)–(13), (16) if

$$(B^0, y^0) \in \Xi_H \quad \text{and} \quad I(B^0, y^0) = \inf_{(B,y) \in \Xi_H} I(B, y).$$

Definition 5.6. We say that a pair $(B^0, y^0) \in L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega)$ is an W -optimal solution to the problem (11)–(13), (16) if

$$(B^0, y^0) \in \Xi_W \quad \text{and} \quad I(B^0, y^0) = \inf_{(B,y) \in \Xi_W} I(B, y).$$

The main question for the optimal control problem (11)–(13), (16) to be answered in this paper is about its solvability in the class of H -solutions. It should be noted that to the best knowledge of the authors, the existence of optimal pairs to the above problem in the sense of Definition 5.5 has not been studied in the literature.

6. On Compensated Compactness in Weighted Sobolev Spaces

We begin this section with some auxiliary results that will be useful later. Let $\{(B_k, y_k) = (A_k \rho_k, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$ be any sequence of H -admissible solutions. With every function $\rho_k \in \mathcal{R}_{ad} \subset L^1(\Omega)$ we associate the space

$$X(\Omega, \rho_k dx) = \left\{ \vec{f} \in L^2(\Omega, \rho_k dx)^N \mid \operatorname{div}_{\rho_k} \vec{f} \in L^2(\Omega, \rho_k dx) \right\} \quad \forall k \in \mathbb{N}$$

and endow it with the norm

$$\|\vec{f}\|_{X(\Omega, \rho_k dx)} = \left(\|\vec{f}\|_{L^2(\Omega, \rho_k dx)^N}^2 + \|\operatorname{div}_{\rho_k} \vec{f}\|_{L^2(\Omega, \rho_k dx)}^2 \right)^{\frac{1}{2}}.$$

We say that a sequence $\left\{ \vec{f}_k \in X(\Omega, \rho_k dx) \right\}_{k \in \mathbb{N}}$ is bounded if

$$\sup_{k \rightarrow \infty} \|\vec{f}_k\|_{X(\Omega, \rho_k dx)} < +\infty.$$

Lemma 6.1. *Let $\{\rho_k\}_{k \in \mathbb{N}}$ be a sequence in \mathcal{R}_{ad} such that $\rho_k \rightarrow \rho$ in $L^1(\Omega)$ as $k \rightarrow \infty$. Then $\rho \in \mathcal{R}_{ad}$ and $(\rho_k)^{-1} \rightarrow \rho^{-1}$ in the variable space $L^2(\Omega, \rho_k dx)$.*

Proof. Let $\rho \in L^1(\Omega)$ be a strong L^1 -limit of the sequence $\{\rho_k\}_{k \in \mathbb{N}}$. Then the properties (6)–(7) and definition of the set \mathcal{R}_{ad} (see (14)) immediately lead to the conclusion $\rho \in \mathcal{R}_{ad}$ and

$$\int_{\Omega} |\rho_k^{-1}| dx \leq \int_{\Omega} |\xi_1^{-1}| dx \quad \forall k \in \mathbb{N},$$

i.e., the sequence $\{\rho_k^{-1}\}_{k \in \mathbb{N}}$ is equi-integrable on Ω . Since $\xi_2^{-1} \leq \rho_k^{-1} \leq \xi_1^{-1}$ and $\rho_k \rightarrow \rho$ a.e. in Ω , Lebesgue Theorem implies $\rho_k^{-1} \rightarrow \rho^{-1}$ in $L^1(\Omega)$. For the remaining part of the proof of this lemma we make use of the ideas of the paper [34]. Let $\varphi \in C_0^\infty(\Omega)$ be a fixed function. Then the equality

$$\int_{\Omega} \rho_k^{-1} \varphi \rho_k dx \equiv \int_{\Omega} \varphi dx = \int_{\Omega} \rho^{-1} \varphi \rho dx \quad \forall k \in \mathbb{N}$$

leads us to the weak convergence $\rho_k^{-1} \rightharpoonup \rho^{-1}$ in $L^2(\Omega, \rho_k dx)$. It should be stressed here that $\rho_k dx \xrightarrow{*} \rho dx$ in the space of Radon measures $\mathcal{M}_+(\Omega)$ (see (5)). However, taking into account the strong convergence $\rho_k^{-1} \rightarrow \rho^{-1}$ in $L^1(\Omega)$ and the fact that Ω is a bounded domain, we get

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\rho_k|^{-2} \rho_k dx \equiv \lim_{k \rightarrow \infty} \int_{\Omega} \rho_k^{-1} dx = \int_{\Omega} \rho^{-1} dx \equiv \int_{\Omega} |\rho|^{-2} \rho dx.$$

Hence, by the criterium of the strong convergence in $L^2(\Omega, \rho_k dx)$, we come to the required conclusion. The proof is complete. \square

Further, for every $k > 0$ we define a cut-off operator $T_k : \mathbb{R} \rightarrow \mathbb{R}$ as follows $T_k(s) = \max\{\min\{s, k\}, -k\}$. By analogy with the well-known results for the classical Sobolev spaces (see [15]), it is easy to verify the following assertion:

Proposition 6.2. *Let y be an arbitrary element of H_ρ . Then*

- (i) $T_k(y) \in H_\rho$ for every $k > 0$
- (ii) $\nabla T_k(y) = \chi_{\{|y| < k\}} \nabla y$ almost everywhere in Ω
- (iii) $T_k(y) \rightarrow y$ almost everywhere in Ω and strongly in H_ρ as $k \rightarrow \infty$.

Taking these facts into account, we have

Proposition 6.3. *Let $\{\rho_k\}_{k \in \mathbb{N}}$ be a sequence in \mathcal{R}_{ad} such that $\rho_k \rightarrow \rho$ in $L^1(\Omega)$ as $k \rightarrow \infty$. Let $\{g_k \in H_{\rho_k}\}_{k \in \mathbb{N}}$ be a bounded sequence and let $g_k \rightarrow g$ in the variable space $H_{\rho_k} = H(\Omega, \rho_k dx)$, i.e.,*

$$g_k \rightarrow g \text{ in } L^2(\Omega), \quad \text{and} \quad \nabla g_k \rightarrow \nabla g \text{ in } L^2(\Omega, \rho_k dx)^N \quad \text{as } k \rightarrow \infty. \quad (21)$$

Then there exists a decreasing sequence of positive numbers $\{\ell_k\}_{k \in \mathbb{N}}$ such that $\ell_k \rightarrow +\infty$ as $k \rightarrow \infty$, and

$$T_{\ell_k}(g_k) \rightarrow g \quad \text{strongly in } L^1(\Omega) \quad \text{as } k \rightarrow \infty. \quad (22)$$

Proof. The key point of the proof is to show that up to a subsequence the element $g \in L^2(\Omega, \rho dx)$ is the strong limit of $\{g_k \in H_{\rho_k}\}_{k \in \mathbb{N}}$ in $L^1(\Omega)$ -topology. Indeed, properties (21) and estimates

$$\begin{aligned} \int_{\Omega} |g_k| dx &\leq \left(\int_{\Omega} g_k^2 dx \right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \leq C |\Omega|^{\frac{1}{2}}, \\ \int_{\Omega} |\nabla g_k|_{\mathbb{R}^N} dx &\leq \left(\int_{\Omega} |\nabla g_k|_{\mathbb{R}^N}^2 \rho_k dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \rho_k^{-1} dx \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega} \xi_1^{-1} dx \right)^{\frac{1}{2}}, \end{aligned}$$

imply that the family $\{g_k\}_{k \in \mathbb{N}}$ is equi-integrable on Ω and bounded in $W^{1,1}(\Omega)$. Hence, by compact embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ this sequence is compact in $L^1(\Omega)$ with respect to the strong topology. Passing to a subsequence if necessary, we can assume that there exists an element $g^* \in L^1(\Omega)$ such that $g_k \rightarrow g^*$ in $L^1(\Omega)$ as $k \rightarrow \infty$. Hence $\lim_{k \rightarrow \infty} \int_{\Omega} \varphi g_k dx = \int_{\Omega} \varphi g^* dx$ for all $\varphi \in C_0^\infty(\Omega)$. On the other side the condition (21) means that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi g_k dx = \int_{\Omega} \varphi g dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Hence $g = g^*$ almost everywhere in Ω . As a result, we have

$$g_k \rightarrow g \quad \text{strongly in } L^1(\Omega) \quad \text{as } k \rightarrow \infty$$

and by the compactness embedding $H_{\rho_k} \hookrightarrow L^1(\Omega)$ and property (iii) of Proposition 6.2,

$$T_{\ell}(g_k) \rightarrow g_k \quad \text{strongly in } L^1(\Omega) \quad \text{as } \ell \rightarrow +\infty \quad \forall k \in \mathbb{N}.$$

Hence, having used the diagonal trick, we just to the required conclusion. \square

Now we are in the position to give the main result of this section (for comparison we refer to the Compensated Compactness Lemma in [5, 24]).

Theorem 6.4. *Let $\{\rho_k\}_{k \in \mathbb{N}}$ be a sequence of weights with properties indicated in Lemma 6.1. Let $\{\vec{f}_k \in L^2(\Omega, \rho_k dx)^N\}_{k \in \mathbb{N}}$, $\{g_k \in H_{\rho_k}\}_{k \in \mathbb{N}}$, $\vec{f} \in L^2(\Omega, \rho dx)^N$, and $g \in H_\rho$ be such that*

- (i) $\{\vec{f}_k\}_{k \in \mathbb{N}}$ is bounded in the variable space $X(\Omega, \rho_k dx)$, and $\vec{f}_k \rightharpoonup \vec{f}$ in $L^2(\Omega, \rho_k dx)^N$ as $k \rightarrow \infty$;
- (ii) the sequence $\{g_k\}_{k \in \mathbb{N}}$ is bounded in the variable spaces $H(\Omega, \rho_k dx)$ and $g_k \rightarrow g$ in $L^2(\Omega)$, and $\nabla g_k \rightarrow \nabla g$ in $L^2(\Omega, \rho_k dx)^N$ as $k \rightarrow \infty$.

Then

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi \left(\vec{f}_k, \nabla g_k \right)_{\mathbb{R}^N} \rho_k dx = \int_{\Omega} \varphi \left(\vec{f}, \nabla g \right)_{\mathbb{R}^N} \rho dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Proof. We divide our proof into several steps. Our first step is to prove that

$$\operatorname{div}_{\rho_k} \vec{f}_k \rightharpoonup \operatorname{div}_{\rho} \vec{f} \quad \text{in } L^2(\Omega, \rho_k dx) \quad \text{as } k \rightarrow \infty. \quad (23)$$

Indeed, since the sequence $\{\operatorname{div}_{\rho_k} \vec{f}_k \in L^2(\Omega, \rho_k dx)\}_{k \in \mathbb{N}}$ is bounded, by the compactness criterium in the variable spaces, we can suppose that there exists an element $\phi \in L^2(\Omega, \rho dx)$ such that $\operatorname{div}_{\rho_k} \vec{f}_k \rightharpoonup \phi$ in $L^2(\Omega, \rho_k dx)$ as $k \rightarrow \infty$. Then passing to the limit in the relation

$$\int_{\Omega} \left(\vec{f}_k, \nabla \varphi \right)_{\mathbb{R}^N} \rho_k dx = - \int_{\Omega} \varphi \left(\operatorname{div}_{\rho_k} \vec{f}_k \right) \rho_k dx \quad \forall \varphi \in C_0^\infty(\Omega) \quad (24)$$

as $k \rightarrow \infty$, we obtain

$$\int_{\Omega} \left(\vec{f}, \nabla \varphi \right)_{\mathbb{R}^N} \rho dx = - \int_{\Omega} \varphi \phi \rho dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Therefore (see Definition 4.1), the element ϕ is the anisotropic divergence of the vector field $\vec{f} \in L^2(\Omega, \rho dx)^N$ with respect to the weight ρ , i.e., $\phi = \operatorname{div}_{\rho} \vec{f} \in L^2(\Omega, \rho dx)$. So, (23) is valid.

The next step is to study the asymptotic behavior as $k \rightarrow +\infty$ of the following numerical sequence $\left\{ \int_{\Omega} \varphi \left(\vec{f}_k, \nabla g_k \right)_{\mathbb{R}^N} \rho_k dx \right\}_{k \in \mathbb{N}}$.

To begin with, we note that as was shown in the proof of Proposition 6.3, up to a subsequence the element $g \in L^2(\Omega, \rho dx)$ is the strong limit of $\{g_k \in H_{\rho_k}\}_{k \in \mathbb{N}}$ in $L^1(\Omega)$ -topology. So, we can suppose that

$$g_k \rightarrow g \quad \text{a.e. in } \Omega. \quad (25)$$

In view of the estimates

$$\begin{aligned} \left| \int_{\Omega} \left(\vec{f}_k, \nabla \varphi \right)_{\mathbb{R}^N} \rho_k dx \right| &\leq \left(\int_{\Omega} \left\| \vec{f}_k \right\|_{\mathbb{R}^N}^2 \rho_k dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left\| \nabla \varphi \right\|_{\mathbb{R}^N}^2 \rho_k dx \right)^{\frac{1}{2}}, \\ \left| \int_{\Omega} \varphi \left(\operatorname{div}_{\rho_k} \vec{f}_k \right) \rho_k dx \right| &\leq \|\varphi\|_{L^\infty(\Omega)} \|\xi_2\|_{L^1(\Omega)}^{\frac{1}{2}} \left(\int_{\Omega} \left(\operatorname{div}_{\rho_k} \vec{f}_k \right)^2 \rho_k dx \right)^{\frac{1}{2}} \end{aligned}$$

and by the density of $C_0^\infty(\Omega)$ in H_{ρ_k} for every $k \in \mathbb{N}$, the relation (24) can be extended to the functions φ of $H_{\rho_k} \cap L^\infty(\Omega)$. Since $T_\ell(g_k) \in H_{\rho_k} \cap L^\infty(\Omega)$ for every $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$, it follows that

$$\int_{\Omega} \left(\vec{f}_k, \nabla (T_\ell(g_k)\varphi) \right)_{\mathbb{R}^N} \rho_k dx = - \int_{\Omega} \left(\operatorname{div}_{\rho_k} \vec{f}_k \right) \varphi T_\ell(g_k) \rho_k dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Due to this relation, we make use of the following equality

$$\begin{aligned} \int_{\Omega} \varphi \left(\vec{f}_k, \nabla T_\ell(g_k) \right)_{\mathbb{R}^N} \rho_k dx &= \int_{\Omega} \left(\vec{f}_k, \nabla (T_\ell(g_k)\varphi) \right)_{\mathbb{R}^N} \rho_k dx \\ - \int_{\Omega} T_\ell(g_k) \left(\vec{f}_k, \nabla \varphi \right)_{\mathbb{R}^N} \rho_k dx &= - \int_{\Omega} \left(\operatorname{div}_{\rho_k} \vec{f}_k \right) \varphi T_\ell(g_k) \rho_k dx \\ - \int_{\Omega} T_\ell(g_k) \left(\vec{f}_k, \nabla \varphi \right)_{\mathbb{R}^N} \rho_k dx &= -I_{1,\ell}^k - I_{2,\ell}^k \quad \forall \varphi \in C_0^\infty(\Omega). \end{aligned} \quad (26)$$

Our next intention is to study the asymptotic behavior of the integrals $I_{1,\ell}^k$ and $I_{2,\ell}^k$ as $k \rightarrow \infty$. Since the sequence $\left\{ \operatorname{div}_{\rho_k} \vec{f}_k \in L^2(\Omega, \rho_k dx) \right\}_{k \in \mathbb{N}}$ is bounded, the property (23) and Lemma 3.1 imply that

$$\rho_k \operatorname{div}_{\rho_k} \vec{f}_k \rightharpoonup \rho \operatorname{div}_{\rho} \vec{f} \quad \text{in } L^1(\Omega). \quad (27)$$

Hence the family $\left\{ \rho_k \operatorname{div}_{\rho_k} \vec{f}_k \right\}_{k \in \mathbb{N}}$ is equi-integrable on Ω . Therefore, because of the boundedness of $\{T_\ell(g_k) - T_\ell(g)\}$ the sequence $\left\{ \rho_k (T_\ell(g_k) - T_\ell(g)) \operatorname{div}_{\rho_k} \vec{f}_k \right\}_{k \in \mathbb{N}}$ is equi-integrable on Ω as well. Using the property (25), we have $T_\ell(g_k) \rightarrow T_\ell(g)$ a.e. in Ω for every $\ell \in \mathbb{N}$. Then Lebesgue's Theorem implies

$$\rho_k (T_\ell(g_k) - T_\ell(g)) \operatorname{div}_{\rho_k} \vec{f}_k \rightarrow 0 \quad \text{in } L^1(\Omega) \quad \text{as } k \rightarrow \infty.$$

Moreover, by (27), we get

$$T_\ell(g) \rho_k \operatorname{div}_{\rho_k} \vec{f}_k \rightharpoonup T_\ell(g) \rho \operatorname{div}_{\rho} \vec{f} \quad \text{in } L^1(\Omega) \quad \text{as } k \rightarrow \infty.$$

Combining these results, we obtain

$$\begin{aligned} \rho_k T_\ell(g_k) \operatorname{div}_{\rho_k} \vec{f}_k &= \rho_k (T_\ell(g_k) - T_\ell(g)) \operatorname{div}_{\rho_k} \vec{f}_k + \rho_k T_\ell(g) \operatorname{div}_{\rho_k} \vec{f}_k \\ &\rightarrow \rho T_\ell(g) \operatorname{div}_{\rho} \vec{f} \quad \text{in } L^1(\Omega). \end{aligned} \quad (28)$$

On the other hand, the inequality

$$\left\| T_\ell(g_k) \operatorname{div}_{\rho_k} \vec{f}_k \right\|_{L^2(\Omega, \rho_k dx)} \leq \|T_\ell(g_k)\|_{L^\infty(\Omega)} \left\| \operatorname{div}_{\rho_k} \vec{f}_k \right\|_{L^2(\Omega, \rho_k dx)} \leq C,$$

immediately yields that the sequence $\left\{ T_\ell(g_k) \operatorname{div}_{\rho_k} \vec{f}_k \right\}_{k \in \mathbb{N}}$ is bounded in variable space $L^2(\Omega, \rho_k dx)$ for every $\ell \in \mathbb{N}$. Hence, by the compactness criterion there

exists an element $\eta^\ell \in L^2(\Omega, \rho dx)$ such that $T_\ell(g_k) \operatorname{div}_{\rho_k} \vec{f}_k \rightharpoonup \eta^\ell$ in $L^2(\Omega, \rho_k dx)$, that is, $T_\ell(g_k) \rho_k \operatorname{div}_{\rho_k} \vec{f}_k \rightharpoonup \eta^\ell \rho$ in $L^1(\Omega)$ (by Lemma 3.1). Then, in view of (28), we get $\eta^\ell = T_\ell(g) \operatorname{div}_\rho \vec{f}$ (ρ -almost everywhere in Ω). As a result, we come to the relation

$$\lim_{k \rightarrow \infty} I_{1,\ell}^k = \int_{\Omega} T_\ell(g) \varphi \operatorname{div}_\rho \vec{f} \rho dx.$$

Using similar arguments, we can prove that

$$\lim_{k \rightarrow \infty} I_{2,\ell}^k = \int_{\Omega} T_\ell(g) \left(\vec{f}, \nabla \varphi \right)_{\mathbb{R}^N} \rho dx.$$

Thus, the passage to the limit in (26) leads us to the relation

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{\Omega} \varphi \left(\vec{f}_k, \nabla T_\ell(g_k) \right)_{\mathbb{R}^N} \rho_k dx \\ &= - \int_{\Omega} T_\ell(g) \varphi \operatorname{div}_\rho \vec{f} \rho dx - \int_{\Omega} T_\ell(g) \left(\vec{f}, \nabla \varphi \right)_{\mathbb{R}^N} \rho dx \\ &= \int_{\Omega} \left(\vec{f}, \nabla (T_\ell(g) \varphi) \right)_{\mathbb{R}^N} \rho dx - \int_{\Omega} T_\ell(g) \left(\vec{f}, \nabla \varphi \right)_{\mathbb{R}^N} \rho dx \\ &= \int_{\Omega} \varphi \left(\vec{f}, \nabla T_\ell(g) \right)_{\mathbb{R}^N} \rho dx \quad \forall \varphi \in C_0^\infty(\Omega), \end{aligned} \quad (29)$$

which holds true for every $\ell \in \mathbb{N}$.

Let $\{T_{\ell_k}(g_k) \in H_{\rho_k}\}_{k \in \mathbb{N}}$ be a sequence with property (22) which is ensured by Proposition 6.3. Then for any $\delta > 0$ there exists a value $k^* \in \mathbb{N}$ such that

$$\|T_{\ell_k}(g_k) - g_k\|_{\rho_k} \leq \delta \quad \forall k > k^* \quad (\text{by Proposition 6.2}).$$

By Cauchy-Bunyakovskii inequality we have the estimate

$$L = \sup_{k \in \mathbb{N}} \left| \int_{\Omega} \varphi \left(\vec{f}_k, \nabla T_{\ell_k}(g_k) - g_k \right)_{\mathbb{R}^N} \rho_k dx \right| \leq \delta \|\varphi\|_{C(\bar{\Omega})} \|\vec{f}_k\|_{L^2(\Omega, \rho_k dx)^N} \leq C\delta. \quad (30)$$

Taking into account that $\chi_{\{|g_k| < \ell_k\}} \rightarrow \chi_\Omega$ strongly in $L^\infty(\Omega)$, it finally follows that

$$\begin{aligned} & \left| \lim_{k \rightarrow \infty} \int_{\Omega} \varphi \left(\vec{f}_k, \nabla g_k \right)_{\mathbb{R}^N} \rho_k dx - \int_{\Omega} \varphi \left(\vec{f}, \nabla g \right)_{\mathbb{R}^N} \rho dx \right| \\ & \leq \left| \lim_{k \rightarrow \infty} \int_{\Omega} \varphi \left(\vec{f}_k, \nabla T_{\ell_k}(g_k) \right)_{\mathbb{R}^N} \rho_k dx - \int_{\Omega} \varphi \left(\vec{f}, \nabla g \right)_{\mathbb{R}^N} \rho dx \right| + C\delta \quad (\text{by (30)}) \\ & \leq \left| \lim_{k \rightarrow \infty} \int_{\Omega} \chi_{\{|g_k| < \ell_k\}} \varphi \left(\vec{f}, \nabla g \right)_{\mathbb{R}^N} \rho dx - \int_{\Omega} \varphi \left(\vec{f}, \nabla g \right)_{\mathbb{R}^N} \rho dx \right| + C\delta \quad (\text{by (29)}) \\ & = C\delta. \quad (\text{by Proposition 6.2}) \end{aligned}$$

Since $\delta > 0$ is arbitrary, this concludes the proof. \square

Remark 6.5. The key point of the proof of this lemma is the fact that the space of smooth functions $C_0^\infty(\Omega)$ is dense in the weighted spaces $H_{\rho_k} = H(\Omega, \rho_k dx)$ for every $k \in \mathbb{N}$. So, in general, Lemma 6.4 does not hold for the case when $\{g_k\}_{k \in \mathbb{N}}$ is a bounded sequence in the variable space W_{ρ_k} .

7. Existence Theorem for H -optimal solutions

Our prime interest in this section deals with the solvability of optimal control problem (11)–(13), (16) in the class of H -solutions. To begin with, we consider the topological properties of the set of H -admissible solutions Ξ_H to the problem (11)–(13), (16). To do so, we introduce the following concepts:

Definition 7.1. We say that a sequence $\{(B_k, y_k) = (A_k \rho_k, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$ is bounded if

$$\sup_{k \in \mathbb{N}} \left[\|A_k\|_{L^\infty(\Omega; \mathbb{R}^{N \times N})} + \|\rho_k\|_{L^1(\Omega)} + \|y_k\|_{L^2(\Omega)} + \|\nabla y_k\|_{L^2(\Omega, \rho_k dx)^N} \right] < +\infty.$$

Definition 7.2. We say that a bounded sequence of H -admissible solutions $\{(B_k, y_k) = (A_k \rho_k, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$ τ -converges to a pair $(B, y) \in L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega)$ if

- (a) $B = A\rho$, where $A \in L^\infty(\Omega; \mathbb{R}^{N \times N})$ and $\rho \in L^1(\Omega)$
- (b) $A_k \xrightarrow{*} A$ in $L^\infty(\Omega; \mathbb{R}^{N \times N})$
- (c) $\rho_k \rightarrow \rho$ in $L^1(\Omega)$
- (d) $y_k \rightharpoonup y$ in $L^2(\Omega)$
- (e) $\nabla y_k \rightharpoonup \nabla y \ni L^2(\Omega, \rho dx)^N$ in the variable space $L^2(\Omega, \rho_k dx)^N$.

Theorem 7.3. For every $f \in C_0^\infty(\mathbb{R}^N)$ the set Ξ_H is closed with respect to the τ -convergence.

Proof. Let $\{(B_k, y_k)\}_{k \in \mathbb{N}} \subset \Xi_H$ be a bounded τ -convergent sequence of H -admissible pairs to the optimal control problem (11)–(13), (16). Let $(B_0, y_0) = (A_0 \rho_0, y_0)$ be its τ -limit. Our aim is to prove that $(B_0, y_0) \in \Xi_H$.

In view of the initial assumptions (11)–(12) we have:

$$A_k = [\vec{a}_{1k}, \dots, \vec{a}_{Nk}] \in M_\alpha^\beta(\Omega) \quad \text{and} \\ |\operatorname{div}_{\rho_k} \vec{a}_{ik}| \leq \gamma_i \quad \rho_k dx\text{-a.e. in } \Omega \quad \forall i = 1, \dots, N, \forall k \in \mathbb{N}.$$

Hence, the sequences $\{\operatorname{div}_{\rho_k} \vec{a}_{ik} \in L^2(\Omega, \rho_k dx)\}_{k \in \mathbb{N}}$ for all $i = 1, \dots, N$ are uniformly bounded. The compactness criterium in variable $L^2(\Omega, \rho_k dx)$ -spaces and the fact that $\rho_k \rightarrow \rho_0$ in $L^1(\Omega)$ imply the existence of $\{\phi_i \in L^2(\Omega, \rho_0 dx)\}_{i=1}^N$ such that

$$\operatorname{div}_{\rho_k} \vec{a}_{ik} \rightharpoonup \phi_i \quad \text{in } L^2(\Omega, \rho_k dx) \quad \text{as } k \rightarrow \infty \quad \forall i = 1, \dots, N.$$

Then passing to the limit as $k \rightarrow \infty$ in the relations

$$\begin{aligned} \int_{\Omega} (\vec{a}_{ik}, \nabla \varphi)_{\mathbb{R}^N} \rho_k dx &= - \int_{\Omega} \varphi \operatorname{div}_{\rho_k} \vec{a}_{ik} \rho_k dx \\ &\quad \forall \varphi \in C_0^\infty(\Omega), \forall i \in \{1, \dots, N\}, \forall k \in \mathbb{N}, \\ -\gamma_i \int_{\Omega} \varphi \rho_k dx &\leq \int_{\Omega} \varphi \operatorname{div}_{\rho_k} \vec{a}_{ik} \rho dx \leq \gamma_i \int_{\Omega} \varphi \rho_k dx \\ &\quad \forall i \in \{1, \dots, N\}, \forall k \in \mathbb{N}, \forall \varphi \geq 0, \\ A_k &= [\vec{a}_{1k}, \dots, \vec{a}_{Nk}] \in M_\alpha^\beta(\Omega), \end{aligned}$$

we come to the conclusion:

$$\operatorname{div}_{\rho_k} \vec{a}_{ik} \rightharpoonup \phi_i = \operatorname{div}_\rho \vec{a}_{i0} \quad \text{in } L^2(\Omega, \rho_k dx) \quad \text{as } k \rightarrow \infty, \quad (31)$$

$$|\operatorname{div}_\rho \vec{a}_{i0}| \leq \gamma_i \quad \rho\text{-a.e. in } \Omega \quad \forall i \in \{1, \dots, N\}, \quad (32)$$

$$A_k \xrightarrow{*} A_0 = [\vec{a}_{10}, \dots, \vec{a}_{N0}] \in M_\alpha^\beta(\Omega). \quad (33)$$

Combining these results with the property $\rho_k \rightarrow \rho_0$ in $L^1(\Omega)$, we deduce

$$\rho_0 \in S : \int_{\Omega} \rho_0 dx = m, \quad \xi_1(x) \leq \rho_0(x) \leq \xi_2(x) \quad \text{a.e. in } \Omega,$$

i.e., $\rho_0 \in \mathcal{R}_{ad}$ and hence the limit matrix $B_0 = A_0 \rho_0$ is an admissible control to the problem (11)–(13), (16).

It remains to show that the pair (B_0, y_0) is related by the energy equality (19). We will do it in several steps.

Step 1. To begin with, we note that, by the initial assumptions there exists of a constant $C > 0$ such that

$$\|y_k\|_{L^2(\Omega)} \leq C, \quad \|\nabla y_k\|_{L^2(\Omega, \rho_k dx)^N} \leq C \quad \forall k \in \mathbb{N}.$$

Hence

$$\begin{aligned} y_k &\rightharpoonup y_0 \quad \text{weakly in the variable Sobolev space } H_{\rho_k}, \\ y_k &\rightharpoonup y_0 \quad \text{in } W_0^{1,1}(\Omega) \quad \text{and} \quad y_k \rightarrow y_0 \quad \text{in } L^1(\Omega) \end{aligned}$$

(for the details see the proof of Proposition 6.3). Further, we note that the sequence $\{A_k \nabla y_k\}_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega, \rho_k dx)^N$. Hence passing to a subsequence if necessary, we may assume that there exists a vector-function $\vec{\eta} \in \mathbf{L}^2(\Omega, \rho_0 dx)^N$ such that

$$A_k \nabla y_k =: \vec{\eta}_k \rightharpoonup \vec{\eta} \quad \text{in } \mathbf{L}^2(\Omega, \rho_k dx)^N. \quad (34)$$

Taking these facts into account, we can pass to the limit in the integral identity

$$\int_{\Omega} \left((A_k \nabla y_k, \nabla \varphi)_{\mathbb{R}^N} \rho_k + y_k \varphi \right) dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega) \quad (35)$$

as $k \rightarrow \infty$. As a result, we get

$$\int_{\Omega} ((\vec{\eta}, \nabla \varphi)_{\mathbb{R}^N} \rho_0 + y_0 \varphi) dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega) \quad (36)$$

or $-\operatorname{div}(\rho_0 \vec{\eta}) = f - y_0$ in the sense of distributions.

Step 2. Here we show that $\vec{\eta} = A_0 \nabla y_0$. To do so, we introduce the following scalar function

$$v(x) = (\vec{z}, x)_{\mathbb{R}^N}, \quad (37)$$

where \vec{z} is a fixed element of \mathbb{R}^N . By the initial assumptions, we have

$$\int_{\Omega} \varphi \left(A_k (\nabla y_k - \nabla v), \nabla y_k - \nabla v \right)_{\mathbb{R}^N} \rho_k dx \geq 0, \quad \forall \varphi \geq 0$$

or, in view of (37), this inequality can be rewritten as

$$\int_{\Omega} \varphi \left(A_k (\nabla y_k - \vec{z}), \nabla y_k - \vec{z} \right)_{\mathbb{R}^N} \rho_k dx \geq 0. \quad (38)$$

Our next intention is to pass to the limit in (38) as $k \rightarrow \infty$ using Theorem 6.4. As follows from the initial assumptions, the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ is admissible for Theorem 6.4. Having put in the statement of this lemma: $\vec{f}_k = A_k \nabla (y_k - v)$, and $g_k = y_k - v$ for all $k \in \mathbb{N}$, we see that the sequence $\{g_k = y_k - v\}_{k \in \mathbb{N}}$ satisfies all assumptions of Theorem 6.4.

In view of (34) and (33), we have

$$\vec{f}_k = A_k \nabla (y_k - v) = A_k (\nabla y_k - \vec{z}) \rightharpoonup \vec{\eta} - A_0 \vec{z} \quad \text{in } \mathbf{L}^2(\Omega, \rho_k dx)^N. \quad (39)$$

It remains to show that the sequence $\{\vec{f}_k = A_k \nabla (y_k - v)\}_{k \in \mathbb{N}}$ is bounded in $X(\Omega, \rho_k dx)$. Indeed, from integral identity (35), we get

$$-\int_{\Omega} \operatorname{div}_{\rho_k} (A_k \nabla y_k) \varphi \rho_k dx = \int_{\Omega} \varphi (f - y_k) dx \quad \forall k \in \mathbb{N}.$$

Since $(f - y_k) \rightharpoonup (f - y_0) = \rho_0^{-1} (f - y_0) \rho_0$ in $L^2(\Omega)$, it follows that the sequence $\{\operatorname{div}_{\rho_k} (A_k \nabla y_k)\}_{k \in \mathbb{N}}$ is weakly compact in $L^2(\Omega, \rho_k dx)$, and

$$\operatorname{div}_{\rho_k} (A_k \nabla y_k) \rightharpoonup \rho_0^{-1} y_0 - \rho_0^{-1} f \quad \text{in } L^2(\Omega, \rho_k dx). \quad (40)$$

To apply Theorem 6.4 we have to show that the sequence $\{\operatorname{div}_{\rho_k} (A_k \vec{z})\}_{k \in \mathbb{N}}$ is also weakly convergent in $L^2(\Omega, \rho_k dx)$, where the elements $\operatorname{div}_{\rho_k} (A_k \vec{z})$ are defined as

$$\int_{\Omega} (A_k \vec{z}, \nabla \varphi)_{\mathbb{R}^N} \rho_k dx = - \int_{\Omega} \varphi \operatorname{div}_{\rho_k} (A_k \vec{z}) \rho_k dx \quad \forall \varphi \in C_0^\infty(\Omega), \quad \forall k \in \mathbb{N}.$$

Indeed, for every test function $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
 \int_{\Omega} (A_k \vec{z}, \nabla \varphi)_{\mathbb{R}^N} \rho_k dx &= \int_{\Omega} \sum_{i=1}^N \sum_{j=1}^N a_{i,j}^k(x) \frac{\partial \varphi}{\partial x_i} z_j \rho_k dx \\
 &= \sum_{j=1}^N z_j \int_{\Omega} (\vec{a}_{j,k}(x), \nabla \varphi)_{\mathbb{R}^N} \rho_k dx \\
 &= - \sum_{j=1}^N z_j \int_{\Omega} \varphi \operatorname{div}_{\rho_k} \vec{a}_{j,k} \rho_k dx \\
 &= J_k.
 \end{aligned} \tag{41}$$

Then using (31), we get

$$\lim_{k \rightarrow \infty} J_k = - \sum_{j=1}^N z_j \lim_{k \rightarrow \infty} \int_{\Omega} \varphi \operatorname{div}_{\rho_k} \vec{a}_{j,k} \rho_k dx = - \sum_{j=1}^N z_j \int_{\Omega} \varphi \operatorname{div}_{\rho_0} \vec{a}_{j,0} \rho_0 dx. \tag{42}$$

Applying the converse transformations with (42) as we did it in (41), we arrive at

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_{\Omega} \varphi \operatorname{div}_{\rho_k} (A_k \vec{z}) \rho_k dx &= - \lim_{k \rightarrow \infty} \int_{\Omega} (A_k \vec{z}, \nabla \varphi)_{\mathbb{R}^N} \rho_k dx \\
 &= - \int_{\Omega} (A_0 \vec{z}, \nabla \varphi)_{\mathbb{R}^N} \rho_0 dx \\
 &= \int_{\Omega} \varphi \operatorname{div}_{\rho_0} (A_0 \vec{z}) \rho_0 dx \quad \forall \varphi \in C_0^\infty(\Omega).
 \end{aligned} \tag{43}$$

Thus, from (40) and (43) it finally follows that

$$\operatorname{div}_{\rho_k} (A_k (\nabla y_k - \vec{z})) \rightharpoonup \rho_0^{-1} (y_0 - f) - \operatorname{div}_{\rho_0} (A_0 \vec{z}) \quad \text{in } L^2(\Omega, \rho_k dx). \tag{44}$$

As a result, combining properties (34), (44), (39) and the fact that $\nabla(y_k - v) \rightharpoonup \nabla(y_0 - v)$ in $\mathbf{L}^2(\Omega, \rho_k dx)^N$, we see that all suppositions of Theorem 6.4 are fulfilled. So, passing to the limit in inequality (38) as $k \rightarrow \infty$, we get

$$\int_{\Omega} \varphi(x) (\vec{\eta} - A_0 \vec{z}, \nabla y_0 - \vec{z})_{\mathbb{R}^N} \rho_0 dx \geq 0, \quad \forall \vec{z} \in \mathbb{R}^N$$

for all positive $\varphi \in C_0^\infty(\Omega)$. After localization, we have $\rho_0(\vec{\eta} - A_0 \vec{z}, \nabla y_0 - \vec{z})_{\mathbb{R}^N} \geq 0$ for all $\vec{z} \in \mathbb{R}^N$. Hence

$$\vec{\eta} = A_0 \nabla y_0 \quad \rho_0 dx\text{-almost everywhere in } \Omega. \tag{45}$$

Step 3. Taking (45) into account, we can represent the integral identity (36) in the form

$$\int_{\Omega} ((A_0 \nabla y_0, \nabla \varphi)_{\mathbb{R}^N} \rho_0 + y_0 \varphi) dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega), \quad (46)$$

or $-\operatorname{div}(\rho_0 A_0 \nabla y_0) + y_0 = f$ in the sense of distributions. Since $C_0^\infty(\Omega)$ dense in H_{ρ_0} , this relation remains true for all $\varphi \in H_{\rho_0}$. Hence, taking $\varphi = y_0$ as a test function in (46), we arrive at the energy equality

$$\int_{\Omega} ((A_0 \nabla y_0, \nabla y_0)_{\mathbb{R}^N} \rho_0 + y_0^2) dx = \int_{\Omega} f y_0 dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

Thus, the τ -limit pair (B_0, y_0) belongs to Ξ_H , and this concludes the proof. \square

Now we are in a position to state the existence of H -optimal pairs to the problem (11)–(13), (16).

Theorem 7.4. *Let ξ_1, ξ_2 be given elements of $L^1(\Omega)$ satisfying the conditions (6) and*

$$\int_{\Omega} \xi_1 dx \leq m \leq \int_{\Omega} \xi_2 dx.$$

Let S be a compact subset of $L^1(\Omega)$ with the property (7), and let also $f \in L^2(\Omega)$ and $y_d \in L^2(\Omega)$ be given functions. Then the optimal control problem (11)–(13), (16) admits at least one H -solution

$$(B^{opt}, y^{opt}) \in \Xi_H \subset L^1(\Omega; \mathbb{R}^{N \times N}) \times W_0^{1,1}(\Omega).$$

Proof. First of all we note that for the given function $f \in L^2(\Omega)$ and every admissible control $B = A\rho \in \mathcal{B}_{ad}$, the Riesz Representation Theorem ensures the existence and uniqueness of an H -solution $y = y(B, f) \in H_\rho$ such that energy equality (19) holds true. Let $\{(B_k, y_k) = (A_k \rho_k, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$ be an H -minimizing sequence to the problem (11)–(13), (16). Then as follows from the inequality

$$\inf_{(B, y) \in \Xi_H} I(B, y) = \lim_{k \rightarrow \infty} \left[\zeta \int_{\Omega} |y_k(x) - v y_d(x)|^2 dx + \int_{\Omega} |\nabla y_k(x)|_{\mathbb{R}^N}^2 \rho_k dx \right] < +\infty,$$

there is a constant $C > 0$ such that

$$\sup_{k \in \mathbb{N}} \|y_k\|_{L^2(\Omega)} \leq C, \quad \sup_{k \in \mathbb{N}} \|\nabla y_k\|_{L^2(\Omega, \rho_k dx)^N} \leq C.$$

Hence, in view of the definition of the class of admissible controls \mathcal{B}_{ad} , we may assume that, within a subsequence, there exist functions $\rho^* \in S$, $y^* \in L^2(\Omega)$, $\vec{g} \in L^2(\Omega, \rho^* dx)^N$, and a matrix $A^* \in L^\infty(\Omega; \mathbb{R}^{N \times N})$ such that

$$A_k \xrightarrow{*} A^* \text{ in } L^\infty(\Omega, \mathbb{R}^{N \times N}), \quad \rho_k \rightarrow \rho^* \text{ in } L^1(\Omega), \quad (47)$$

$$y_k \rightharpoonup y^* \text{ in } L^2(\Omega), \quad \nabla y_k \rightharpoonup \vec{g} \text{ in } L^2(\Omega, \rho_k dx)^N. \quad (48)$$

Using the arguments of the proof of Theorem 7.3, it can be shown that the matrix $B^* = A^* \rho^* \in L^1(\Omega, R^{N \times N})$ is admissible control to the problem (11)–(13), (16). Let us prove that the equality $\nabla y^* = \vec{g}$ holds true. To do so, it is enough to show that

$$y_k \rightharpoonup y^* \text{ in } L^1(\Omega) \quad \text{and} \quad \nabla y_k \rightharpoonup \vec{g} \text{ in } L^1(\Omega)^N. \quad (49)$$

The validity of the first assertion in (49) immediately follows from the first relation in (48). Further we note that by estimate

$$\int_{\Omega} |\nabla y_k|_{\mathbb{R}^N} dx \leq \left(\int_{\Omega} |\nabla y_k|_{\mathbb{R}^N}^2 \rho_k dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \rho_k^{-1} dx \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega} \rho_k^{-1} dx \right)^{\frac{1}{2}}$$

the sequence $\{\nabla y_k \in L^2(\Omega, \rho_k dx)^N\}$ is equi-integrable, and hence this one is weakly compact in $L^1(\Omega)^N$. By Lemma 6.1 and properties of the strong convergence in $L^2(\Omega, \rho_k dx)^N$, we immediately obtain $\lim_{k \rightarrow \infty} \int_{\Omega} (\nabla y_k, \vec{\varphi})_{\mathbb{R}^N} dx = \lim_{k \rightarrow \infty} \int_{\Omega} \rho_k^{-1} (\nabla y_k, \vec{\varphi})_{\mathbb{R}^N} \rho_k dx = \int_{\Omega} (\rho^*)^{-1} (\vec{g}, \vec{\varphi})_{\mathbb{R}^N} \rho^* dx = \int_{\Omega} (\vec{g}, \vec{\varphi})_{\mathbb{R}^N} dx$. Thus $\nabla y_k \rightharpoonup \vec{g}$ in $L^1(\Omega)$ and $y_k \rightharpoonup y^*$ in $L^1(\Omega)$. As a result, the equality $\nabla y^* = \vec{g}$ follows from the completeness of normed space $W_0^{1,1}(\Omega)$.

Combining these results, we obtain: the pair (B^*, y^*) is the τ -limit of the H -minimizing sequence $\{(B_k, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$. Then, by Theorem 7.3, this pair is an H -admissible to the problem (11)–(13), (16). Since the cost functional I is lower τ -semicontinuous, we get

$$I(B^*, y^*) \leq \liminf_{k \rightarrow \infty} I(B_k, y_k) = \inf_{(B, y) \in \Xi_H} I(B, y).$$

Hence (B^*, y^*) is an H -optimal pair, and this concludes the proof. \square

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