

# A series of Nash resolutions of a singular foliation

Ruben Louis

**Abstract.** We construct a series of blowups  $(\tilde{M}_i, \pi_i)_{i \in \mathbb{N}_0}$  of a singular foliation by applying to the universal Lie  $\infty$ -algebroid of a singular foliation the so-called Nash modification. For  $i = 0$ , we recover a blowup introduced by Sinan Sertöz, and for  $i = 1$ , we recover a notion due to Omar Mohsen. One of the important features is that any singular foliation becomes a Debord foliation (= projective singular foliation) after one blowup. Examples are also given.

## Introduction

Singular foliations generalize the notion of regular foliations by allowing leaves of different dimensions. They arise frequently in differential or algebraic geometry. Here, as in [24], we unify [1, 2, 5, 7, 19, 21] in smooth differential geometry and [3, 41] in holomorphic differential geometry by defining a singular foliation on a smooth, complex, algebraic or real analytic manifold  $M$ , with sheaf of functions  $\mathcal{O}$ , to be a subsheaf  $\mathfrak{F}: U \rightarrow \mathfrak{F}(U)$  of the sheaf of vector fields  $\mathfrak{X}$ , which is closed under the Lie bracket and locally finitely generated as an  $\mathcal{O}$ -module. By Hermann’s theorem [19], this is enough to induce a partition of the manifold  $M$  into immersed submanifolds of possibly different dimensions, called *leaves* of the singular foliation. Singular foliations appear for instance as orbits of Lie group actions, with possibly different dimensions. In the realm of Poisson geometry, we encounter a particularly intricate class of singular foliations known as “the symplectic leaves of a Poisson structure” (see [6, 25]). When all the leaves have the same dimension, we recover the usual “regular foliations” [18, 24]. We refer to [24, Section 4] for a detailed list of examples.

In this paper, we address blowups of a singular foliation  $\mathfrak{F}$  on  $M$ , that is, pairs  $(\tilde{M}, \pi)$  such that

- (1)  $\pi: \tilde{M} \rightarrow M$  is onto and proper,
- (2) the restriction  $\pi|_{M_{\text{reg}, \mathfrak{F}}}: \pi^{-1}(M_{\text{reg}, \mathfrak{F}}) \rightarrow M_{\text{reg}, \mathfrak{F}}$  to the regular points  $M_{\text{reg}, \mathfrak{F}}$  of  $\mathfrak{F}$  is one-to-one,
- (3) the pullback  $\pi^! \mathfrak{F}$  of  $\mathfrak{F}$  on  $\tilde{M}$  exists and satisfies  $\pi^! \mathfrak{F}|_{\pi^{-1}(M_{\text{reg}, \mathfrak{F}})} \simeq \mathfrak{F}|_{M_{\text{reg}, \mathfrak{F}}}$ .

This notion has been studied in various forms by many authors [8, 33, 37]. We consider singular foliations, which admit (locally) geometric resolutions, that is, those for which there exists an anchored complex of vector bundles

$$(E, d, \rho): \begin{array}{ccccccc} \longrightarrow & E_{-i-1} & \xrightarrow{d^{(i+1)}} & E_{-i} & \xrightarrow{d^{(i)}} & E_{-i+1} & \longrightarrow \cdots \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \longleftarrow & M & \longleftarrow & M & \longleftarrow & M & \longleftarrow & M & \longleftarrow & M \end{array}$$

such that the following complex of sheaves

$$\longrightarrow \Gamma(E_{-i-1}) \xrightarrow{d^{(i+1)}} \Gamma(E_{-i}) \xrightarrow{d^{(i)}} \Gamma(E_{-i+1}) \longrightarrow \cdots \longrightarrow \Gamma(E_{-1}) \xrightarrow{\rho} \mathfrak{F} \quad (1)$$

is exact. Those singular foliations were introduced and studied by Laurent-Gengoux, Lavau and Strobl [21]. It is quite natural to work with this class of singular foliation, as it contains the class of (locally) real analytic singular foliations. It is also a natural object in the holomorphic setting, since  $\mathfrak{F}$  is then a coherent sheaf and such geometric resolutions always exist locally.

We will use geometric resolutions to construct a sequence of blowups  $(\tilde{M}_i, \pi_i)_{i \in \mathbb{N}_0}$ . There is a very long story behind our construction. Let us be precise on the relations with other works: we were first inspired by O. Mohsen [33], who introduced a notion of blowup of a smooth manifold along the singular leaves of a singular foliation which does not consist of blowing up along a singular leaf as in [39] or [8] or gluing Lie groupoids as in [37]. The construction of Mohsen extends an older idea that consists in replacing every singular point of a singular foliation by the limiting positions of the tangent spaces of the nearly regular leaves. This method goes back to the mathematician J. Nash [38] and is mainly used in algebraic geometry for desingularization of affine varieties or schemes. To the best of my knowledge, Sinan Sertöz [41] was the first to apply this method in his PhD dissertation to compute the Baum–Bott residues of singular holomorphic foliations. A more general construction for generic coherent sheaves was done earlier by [40]. For further details, see also [4]. In fact, Sinan Sertöz went further by applying the Nash construction to coherent subsheaves of locally free sheaves, thereby generalizing the work of Nobile [38].

In the complex setting, therefore, our blowups  $(\tilde{M}_i, \pi_i)$  coincide for  $i = 0, 1$  with blowup of  $\mathfrak{F}$ , seen as a holomorphic coherent sheaf as in [4, 40, 41]. Also, in the smooth setting, we will see that  $\tilde{M}_1$  is the blowup space of the singular foliation  $(M, \mathfrak{F})$  defined by O. Mohsen [33]. In general, the blowup spaces  $(\tilde{M}_i)_{i \geq 0}$  are Nash blowups (also called Nash modification) of coherent sheaves. This coincidence has practical consequences: for instance, the smoothness of the  $\tilde{M}_i$ 's can be studied using classical results on Nash modifications, as in [41].

Now, when dealing with a singular foliation, rather than just a coherent sheaf or a sheaf that admits a geometric resolution, additional structures emerge. Specifically, this context gives rise to Lie algebroids or Lie  $n$ -algebroid structures. For instance, we will see

that  $\pi_1^! \mathfrak{F}$  is always a Debord singular foliation on  $\tilde{M}_1$ , meaning it is the image of a Lie algebroid whose anchor map is injective on an open dense subset. It is shown in [21, 22] that “behind” any singular foliation admitting a geometric resolution, there is a Lie  $\infty$ -algebroid constructed over a geometric resolution  $(E, d, \rho)$  of  $\mathfrak{F}$ , which is unique up to homotopy. The latter is referred to as a universal Lie  $\infty$ -algebroid of  $\mathfrak{F}$ . For instance, it was employed by S. Lavau in [27] to define the modular class of a singular foliation or in [30] to study symmetries of singular foliations (see also [26, 42] for other applications).

The existence of a structure of a universal Lie  $\infty$ -algebroid on geometric resolutions has consequences for Nash modifications. In fact, this structure is necessary to check that the blowups  $(\tilde{M}_i, \pi_i)$  satisfy the third condition in the definition of a blowup, namely that the pullback of  $\mathfrak{F}$  exists and is a singular foliation. More precisely, we do not apply the Nash modification idea directly on the tangent space of our singular foliation  $\mathfrak{F} \subseteq \mathfrak{X}(M)$  but on the images of the differential maps  $d^{(i+1)}: E_{-i-1} \rightarrow E_{-i}$ ,  $i \geq 1$  and of the anchor map  $\rho: E_{-1} \rightarrow TM$  of a geometric resolution  $(E, d, \rho)$ . The choice of considering the images of all the  $d^{(i+1)}$ 's allows generalizing the Nash construction to the whole universal Lie  $\infty$ -algebroid of the singular foliation built over a geometric resolution. As we said, we then recover several notions of resolution of singularities for  $i = 0, 1$ . But for  $i \geq 2$ , these resolutions have never been introduced before to our knowledge. A consequence of our construction for  $i = 1$  is that a resolution of any singular foliation can be constructed, which is given by an action of a Lie algebroid whose anchor map is injective on a dense open subset (a result implicit in [33] but not stated as such there). For generic  $i$ , one obtains a singular foliation which is the image of the anchor map of a Lie  $i$ -algebroid.

In general, we must admit that very often the blowup spaces  $\tilde{M}_i$  are not smooth manifolds. However, our singular foliation on  $\tilde{M}_i$  makes sense and admits leaves that are smooth submanifolds. Also, the blowup spaces  $\tilde{M}_i$  are analytic varieties if the initial singular foliation admits real analytic generators. Last, we are able to decide when  $\tilde{M}_i$  is smooth: it suffices to study the properties of some ideal of functions that we describe in the text.

The paper is structured as follows. In Section 1, we revisit the concept of singular foliations and their universal Lie  $\infty$ -algebroids. Section 2 presents the Nash blowup construction for vector bundle morphisms and discusses smoothness. In Section 2.2, we introduce a series of Nash blowups of a singular foliation, indexed by  $i \in \mathbb{N}_0$ , followed by the main theorems. In Section 3, we prove the results of Section 2.2. In Section 4, we provide examples of our constructions and demonstrate how the usual notions of blowups for affine varieties can be recovered. Finally, in order to fix notations, we review in Appendix A the definition and properties of Grassmann bundles.

### 1. Preliminaries: Singular foliations and Lie $\infty$ -algebroids

**Convention 1.1.** Throughout the article,  $\mathcal{O}$  stands for the sheaf of (smooth, polynomial, real analytic or holomorphic) functions on (a manifold, affine variety—depending on the

context)  $M$  and, for a vector bundle  $E \rightarrow M$  of constant rank,  $\Gamma(E)$  stands for the sheaf of sections of  $E$ . Also,  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . The results of this paper apply to the smooth, algebraic, real analytic and holomorphic contexts, with some adaptations. However, for simplicity, we will primarily work in the smooth setting.

We refer the reader to [1, 2, 5, 7, 21, 24] for the topic of singular foliations, in particular to [21, 22] for the notion of universal Lie  $\infty$ -algebroids. For Lie algebroids, see [32].

### 1.1. Singular foliations

We recall some basic definitions and properties on singular foliations.

- (1) A *singular foliation* on a manifold  $M$  is a subsheaf  $\mathfrak{F} \subseteq \mathfrak{X}(M)$  that fulfills the following conditions:
  - (a) *Stability under Lie bracket*:  $[\mathfrak{F}, \mathfrak{F}] \subseteq \mathfrak{F}$ .
  - (b)  $\mathfrak{F}$  is a *module* over its respective relevant sheaf of functions.
  - (c) *Locally finitely generateness*<sup>1</sup>: every  $m \in M$  admits an open neighborhood  $\mathcal{U}$  together with a finite number of vector fields  $X_1, \dots, X_k \in \mathfrak{X}(\mathcal{U})$  such that for every open subset  $\mathcal{V} \subseteq \mathcal{U}$ , the vector fields  $X_1|_{\mathcal{V}}, \dots, X_k|_{\mathcal{V}}$  generate  $\mathfrak{F}$  on  $\mathcal{V}$  as a module over functions on  $\mathcal{V}$ .

We are particularly interested in three specific classes of singular foliations, which we now define:

- A *locally polynomial/analytic singular foliation* is a singular foliation over a smooth or complex manifold which admits, around each point, generators with polynomial/analytic coefficients in some local chart.
  - A *globally finitely generated singular foliation*  $\mathfrak{F} \subseteq \mathfrak{X}(M)$  is a singular foliation which is generated as an  $\mathcal{O}$ -submodule of  $\mathfrak{X}(M)$  by finitely many vector fields on  $M$ .
  - A singular foliation  $\mathfrak{F}$  is *Debord* if it is projective as a module over functions on  $M$ , equivalently if and only if there exists a Lie algebroid  $(A, [\cdot, \cdot]_A, \rho)$  such that  $\rho(\Gamma(A)) = \mathfrak{F}$  whose anchor is injective on an open dense subset. In particular, Debord foliations are globally finitely generated.
- (2) Here are some important features of the above definition in the smooth/real analytic/complex cases (see [24, Section 7]):
    - Singular foliation admits leaves: there exists a partition of  $M$  into immersed submanifolds called *leaves* such that for all  $m \in M$ , the image of the evaluation map  $\mathfrak{F} \rightarrow T_m M$  is the tangent space of the leaf through  $m$ .

---

<sup>1</sup>In the holomorphic case, this condition can be equivalently replaced by the notion of coherent sheaf [3, 41].

- *Singular foliations are self-preserving*: the flow  $\phi_t^X$  of vector fields  $X \in \mathfrak{F}$ , whenever defined, preserves  $\mathfrak{F}$  [1, 13, 19], that is,

$$\forall m \in M, \exists \varepsilon > 0 \text{ such that } \forall t \in ]-\varepsilon, \varepsilon[, \quad (\phi_t^X)_*(\mathfrak{F}) = \mathfrak{F}.$$

**1.1.1. Nagano–Sussmann theorem.** We introduce the following definitions, which are particular cases of a more general notion applied to singular spaces studied in [43]. Let  $S$  be a closed subset of a manifold  $N$ .

- (1) A *vector field on  $S \subseteq N$*  is the restriction to  $S$  of a vector field  $Z \in \mathfrak{X}(N)$  whose flow preserves  $S$ , that is,  $\phi_t^Z(S) \subseteq S$  whenever it makes sense. In that case, we shall say that such a  $Z$  is *tangent to  $S$* . The set of vector fields on  $S$  forms a Lie algebra that we denote as in the usual case by  $\mathfrak{X}(S)$ .
- (2) The *tangent space  $T_s S$*  of  $S$  at  $s \in S$  is the evaluation at  $s$  of the vector fields on  $S$ .
- (3) We also make sense of the notion of Lie algebroid on a closed subset  $S \subseteq M$  as follows: a *Lie algebroid over  $S$*  is a locally finitely generated projective Lie–Rinehart algebra over  $\mathcal{O}/\mathcal{I}_S$ . Here,  $\mathcal{I}_S$  is the ideal of vanishing functions on  $S$ .

**Remark 1.2.** Notice that when  $S$  is a submanifold or an analytic subvariety of a complex or real analytic manifold, this notion of vector field or Lie algebroid on  $S$  agrees to the usual case.

Now, we recall a crucial theorem that allows us to define singular foliations correctly on a closed subset  $S$  of a manifold  $M$ .

**Definition 1.3** ([21, 24]). Let  $S$  be a closed subset of  $N$ .

- (1) A *singular foliation on  $S$*  is an involutive<sup>2</sup> locally finitely generated  $\mathcal{O}_S$ -submodule  $\mathfrak{F} \subseteq \mathfrak{X}(S)$ .
- (2) For  $s \in S$ , the *leaf* of a singular foliation  $\mathfrak{F}$  on  $S$  through  $s$  is the set

$$L_s := \{ \phi_{t_1}^{Z_1} \circ \phi_{t_2}^{Z_2} \circ \dots \circ \phi_{t_k}^{Z_k}(s), t_1, \dots, t_k \in \mathbb{R} \}. \tag{1.1}$$

Above,  $Z_1, \dots, Z_k$  are vector fields on  $N$  whose restrictions are in  $\mathfrak{F}$ . We implicitly assume that the flows are defined.

**Remark 1.4.** From Definition 1.3, it is easily checked that being in the same leaf is an equivalence relation on  $S$ , hence the leaves induce a partition of  $S$ .

The notion of leaves of singular foliation on  $S \subseteq N$  is justified by the following theorem that generalizes the Stefan–Sussmann theorem [44, 45] that says the leaves are smooth manifolds.

---

<sup>2</sup>Notice that  $\mathfrak{F}$  is generated by the restrictions to  $S$  of vector fields on  $N$ ; they are required to be involutive only after restrictions to  $S$ .

**Theorem 1.5.** *Let  $\mathfrak{F}$  be a singular foliation on a closed subset  $S \subseteq N$ . The leaves  $\mathfrak{F}$  form a partition of  $S$  into connected manifolds, immersed as submanifolds of  $N$ .*

The explanation of this result is based on a very strong theorem known as the Nagano–Sussmann theorem [36]. This theorem, widely used in control theory, provides a very strong result regarding the smoothness of the orbits of a finite number of vector fields on a manifold without any assumptions.

**Theorem 1.6** (Nagano–Sussmann). *Let  $\mathcal{V} \subseteq \mathfrak{X}(N)$  be a locally finitely generated  $\mathcal{O}_N$ -submodule of vector fields on a manifold  $N$ . For every  $\ell \in N$ , the set*

$$\{\phi_{t_1}^{Z_1} \circ \phi_{t_2}^{Z_2} \circ \dots \circ \phi_{t_n}^{Z_n}(\ell), t_1, \dots, t_n \in \mathbb{R}, Z_1, \dots, Z_n \in \mathcal{V}, n \in \mathbb{N}\}$$

*is a connected immersed submanifold of  $N$ .*

*Proof (of Theorem 1.5).* For simplicity, let us assume that  $\mathfrak{F} \subseteq \mathfrak{X}(S)$  is globally finitely generated (the general case is left to the reader). Let  $\xi_1, \dots, \xi_k$  be generators for  $\mathfrak{F}$ . By definition, the  $\xi_i$ 's are the restrictions to  $S$  of vector fields  $Z_i$ 's on  $N$  whose flows  $\phi_t^{Z_i}$  preserve  $S$ , that is,  $\phi_t^{Z_i}(S) \subseteq S$  where the flows are defined. By the Nagano–Sussmann theorem, the orbits generated by the vector fields  $Z_1, \dots, Z_k \in \mathfrak{X}(N)$  are immersed submanifolds of  $N$ . By assumption, the orbits through a point of  $S$  of the  $\xi_i$ 's coincide with the orbits of the  $Z_i$ 's and are included in  $S$ . This completes the proof. ■

**1.2. Universal Lie  $\infty$ -algebroid of a singular foliation**

Let us recall the notion of universal Lie  $\infty$ -algebroid of a singular foliation. Let  $\mathfrak{F} \subseteq \mathfrak{X}(M)$  be a submodule.

- (1) A complex of vector bundles  $(E, d, \rho)$

$$\begin{array}{ccccccc} \longrightarrow & E_{-i-1} & \xrightarrow{d^{(i+1)}} & E_{-i} & \xrightarrow{d^{(i)}} & E_{-i+1} & \longrightarrow \dots \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \longleftarrow & M & \longleftarrow & M & \longleftarrow & M & \longleftarrow & M & \longleftarrow & M \end{array}$$

is said to be a *geometric resolution of  $\mathfrak{F}$*  if the following complex is an exact sequence of sheaves:

$$\longrightarrow \Gamma(E_{-i-1}) \xrightarrow{d^{(i+1)}} \Gamma(E_{-i}) \xrightarrow{d^{(i)}} \Gamma(E_{-i+1}) \longrightarrow \dots \longrightarrow \Gamma(E_{-1}) \xrightarrow{\rho} \mathfrak{F}. \tag{1.2}$$

A geometric resolution is said to be of *finite length*  $n \in \mathbb{N}_0$  if  $E_{-i} = 0$  for all  $i \geq n + 1$ . Also,  $(E, d, \rho)$  is said to be *minimal* at a point  $x \in M$  if the linear maps  $d^{(i)}|_x: E_{-i}|_x \rightarrow E_{-i+1}|_x$  vanish for all  $i \geq 2$ .

- (2) A *graded almost Lie algebroid over M* is the datum of a complex  $(E, d = \ell_1, \rho)$  of vector bundles over  $M$  equipped with a graded symmetric degree +1  $\mathbb{K}$ -bilinear bracket

$$\ell_2: \Gamma(E) \odot \Gamma(E) \rightarrow \Gamma(E)$$

such that

- (a)  $\ell_2$  satisfies the *Leibniz identity* with respect to  $\rho: \Gamma(E_{-1}) \rightarrow \mathfrak{X}(M)$ , that is,

$$\ell_2(x, fy) = f\ell_2(x, y) + \rho(x)[f]y$$

for all  $x \in \Gamma(E_{-1}), y \in \Gamma(E)$  and  $f \in \mathcal{O}$ ,

- (b)  $\ell_1$  is degree +1-derivation of  $\ell_2$ , that is, for all  $x \in \Gamma(E_{-i}), y \in \Gamma(E)$ ,

$$\ell_1(\ell_2(x, y)) + \ell_2(\ell_1(x), y) + (-1)^i \ell_2(x, \ell_1(y)) = 0,$$

- (c)  $\rho$  is a morphism, that is, for all  $x, y \in \Gamma(E_{-1})$ ,

$$\rho(\ell_2(x, y)) = [\rho(x), \rho(y)].$$

The  $\mathcal{O}$ -linear map  $\rho$  is called the *anchor map* and  $\ell_1$  the *differential*.

- (3) A *Lie  $\infty$ -algebroid over M* is the datum of a sequence  $E = (E_{-i}), 1 \leq i < \infty$  of vector bundles over  $M$  together with a structure of Lie  $\infty$ -algebra  $(\ell_k)_{k \geq 1}$  on the sheaf of sections of  $E$  and a vector bundle morphism,  $\rho: E_{-1} \rightarrow TM$ , called *anchor map* such that the  $k$ -ary brackets  $\ell_k, k \neq 2$  are  $\mathcal{O}$ -multilinear and such that

$$\ell_2(e_1, fe_2) = \rho(e_1)[f]e_2 + f\ell_2(e_1, e_2) \tag{1.3}$$

for all  $e_1 \in \Gamma(E_{-1}), e_2 \in \Gamma(E_\bullet)$  and  $f \in \mathcal{O}$ . The sequence

$$\dots \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} E_{-1} \xrightarrow{\rho} TM \tag{1.4}$$

is a complex called the *linear part* of the Lie  $\infty$ -algebroid.

Notice that given a Lie  $\infty$ -algebroid  $(E, (\ell_k)_{k \geq 1}, \rho)$ , the quadruple  $(E, d = \ell_1, \ell_2, \rho)$  is a graded almost Lie algebroid over  $M$ .

The following theorem is important (see [21, Section 2] or [22] for more details).

**Theorem 1.7.** *Let  $\mathfrak{F}$  be a singular foliation over M. Any geometric resolution of  $\mathfrak{F}$*

$$\dots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} E_{-1} \xrightarrow{\rho} TM \tag{1.5}$$

*comes equipped with a Lie  $\infty$ -algebroid structure whose unary bracket is  $d$  and whose anchor map is  $\rho$  (in particular,  $\rho(\Gamma(E_{-1})) = \mathfrak{F}$ ). Such a Lie  $\infty$ -algebroid structure is unique up to homotopy and is called a universal Lie  $\infty$ -algebroid of  $\mathfrak{F}$ .*

In particular, this Lie  $\infty$ -algebroid structure can be truncated to a graded almost Lie algebroid for  $\mathfrak{F}$ .

- (4) Let  $(E_\bullet, \ell_\bullet, \rho)$  be a universal Lie  $\infty$ -algebroid of a singular foliation  $\mathfrak{F}$ . For every point  $x \in M$ ,
- (a) We let  $H^\bullet(\mathfrak{F}, x) = \bigoplus_{i \geq 1} H^{-i}(\mathfrak{F}, x)$  be the cohomology of the complex (1.5). The cohomology groups  $H^\bullet(\mathfrak{F}, x)$  do not depend on the choice of a geometric resolution of  $\mathfrak{F}$ . Notice that when the complex (1.5) is minimal at  $x$ ,  $H^{-i}(\mathfrak{F}, x) \simeq E_{-i}|_x$  for every  $i \geq 1$ .
  - (b) The 1-ary and the 2-ary brackets restrict to the graded vector space

$$\left( \bigoplus_{i \geq 2} E_{-i}|_x \right) \oplus \ker(\rho_x)$$

and equip the latter with a graded almost Lie  $\infty$ -algebra structure as follows: for every  $k \in \{1, 2\}$ ,

$$\{x_1, \dots, x_k\}_k := \ell_k(s_1, \dots, s_k)|_x$$

for all  $x_1, \dots, x_k \in \text{ev}(E, x)$  and  $s_1, \dots, s_k \in \Gamma(E)$  sections of  $E$  such that  $s_i(x) = x_i$  with  $i = 1, \dots, k$ .

The bracket  $\{\cdot, \cdot\}_2$  induces a graded Lie algebra on  $H^\bullet(\mathfrak{F}, x)$ . In particular, the 2-ary bracket  $\{\cdot, \cdot\}_2$  satisfies the Jacobi identity on  $H^{-1}(\mathfrak{F}, x) = \frac{\ker(\rho_x)}{\text{im}(d_x^{(2)})}$  and equips the latter with a Lie algebra structure.

- (5) Let  $(M, \mathfrak{F})$  be a singular foliation, and let  $\mathcal{I}_x := \{f \in C^\infty(M) \mid f(x) = 0\}$  and  $\mathfrak{F}(x) := \{X \in \mathfrak{F} \mid X(x) = 0\}$ . The quotient  $\mathfrak{g}_x = \frac{\mathfrak{F}(x)}{\mathcal{I}_x \mathfrak{F}}$  is a Lie algebra and is called the isotropy Lie algebra of  $\mathfrak{F}$  at  $x$ . A point  $x \in M$  is said to be a *regular point* of  $\mathfrak{F}$  if  $\mathfrak{g}_x = \{0\}$ ; otherwise, we say that  $x$  is a *singular point*. The set of regular points of  $\mathfrak{F}$  is denoted by  $M_{\text{reg}, \mathfrak{F}}$ .

**Lemma 1.8** ([21]). *Let  $(E, \ell_\bullet, \rho)$  be a universal Lie  $\infty$ -algebroid of  $\mathfrak{F}$ . Consider its underlying geometric resolution*

$$(E, d, \rho) : \quad \dots \xrightarrow{\ell_1=d^{(4)}} E_{-3} \xrightarrow{\ell_1=d^{(3)}} E_{-2} \xrightarrow{\ell_1=d^{(2)}} E_{-1} \xrightarrow{\rho=d^{(1)}} TM.$$

Then

- (a) for all  $x \in M$ , we have  $H^{-1}(\mathfrak{F}, x) \simeq \mathfrak{g}_x$  as Lie algebras;
- (b) the subset of regular points of  $\mathfrak{F}$  in  $M$  satisfies

$$\begin{aligned} M_{\text{reg}, \mathfrak{F}} &= \{x \in M \mid \text{rk}(d_x^{(2)}) = \dim(\ker \rho_x)\} \\ &= \{x \in M \mid H^{-i}(\mathfrak{F}, x) = 0, \forall i \geq 1\}, \end{aligned}$$

$M_{\text{reg}, \mathfrak{F}}$  is open and dense in  $M$ ;

- (c) the restriction of the foliation  $\mathfrak{F}$  to  $M_{\text{reg}, \mathfrak{F}}$  is the set of sections of a subbundle of  $TM$ , that is, is a regular foliation;

- (d) for every  $i \geq 0$ , the dimension of  $\text{im}(d^{(i+1)})$  is locally constant on  $M_{\text{reg}, \mathfrak{F}}$ . Moreover, if  $r$  is the dimension of a regular leaf, then  $\text{im}(d^{(i+1)})$  is of codimension

$$r_i = \sum_{j=1}^{i-1} (-1)^{j+1} \text{rk}(E_{-j}) + (-1)^{i+1} r \quad \text{for } i \geq 1$$

in  $E_{-i}$  or  $r_0 = \dim M - r$ , with  $E_0 := TM$ ;

- (e) if  $(E, d, \rho)$  is of finite length, then all the regular leaves have the same dimension.

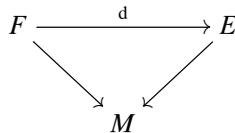
In the sequel, we assume that a geometric resolution of finite length exists. Under these assumptions, all the regular leaves have the same dimension. We denote by  $r$  the common dimension of the regular leaves.

## 2. Blowup procedures

### 2.1. Blowup of vector bundle morphisms

Firstly, let us explain a general construction on morphisms of vector bundles that we refer as Nash blowup. For an open subset  $\mathcal{V} \subseteq M$ , we shall denote by  $\mathcal{V}^{\mathbb{N}}$  the  $\mathcal{V}$ -valued sequences of points  $(x_n)$  indexed by  $\mathbb{N}$ . We direct the reader to Appendix A for conventions and notations regarding Grassmannians.

**2.1.1. The Nash blowup of a vector bundle morphism.** Let  $E, F$  be vector bundles over  $M$  and



a morphism of vector bundles over the identity. In the smooth case, we assume that  $d$  is of constant rank on an open dense subset  $M_{\text{reg}, d} \subset M$ , that is, the dimensions of  $\text{im}(d_x)$  or  $\text{ker}(d_x)$  are constant for  $x \in M_{\text{reg}, d}$ , called the *regular part*. Let  $q$  be the codimension of  $\text{im}(d_x) \subseteq E_x$  for a point  $x \in M_{\text{reg}, d}$ . Notice that for every  $x \in M_{\text{reg}, d}$ ,  $\text{im}(d_x)$  is a point of the Grassmannian  $\text{Gr}_{-q}(E_x)$  of vector subspaces of  $E_x$  of codimension  $q$ . Also,  $\text{ker}(d_x)$  is a point of the Grassmannian  $\text{Gr}_{-\text{rk}(F)-q}(F_x)$  of vector subspaces of  $F_x$  of codimension  $\text{rk}(F) - q$ . We consider the natural section of the Grassmann bundle  $\Pi: \text{Gr}_{-q}(E) \rightarrow M$  which is defined on  $M_{\text{reg}, d}$  by

$$\sigma: M_{\text{reg}, d} \longrightarrow \text{Gr}_{-q}(E), \quad x \longmapsto \text{im}(d_x). \tag{2.1}$$

Then we define the *Nash blowup space of  $M$  along  $d$*  to be the closure  $\tilde{M} := \overline{\sigma(M_{\text{reg}, d})}$  of the image of the section  $\sigma$  in  $\text{Gr}_{-q}(E)$ . It comes together with the projection  $\pi: \tilde{M} \rightarrow M$ , where  $\pi$  denotes the restriction of  $\Pi: \text{Gr}_{-q}(E) \rightarrow M$  to  $\tilde{M}$ .

**Remark 2.1.** A detailed line-by-line comparison provides the construction of Nash as presented in [4, 41], for a coherent sheaf of modules  $\mathcal{K}$  over  $\mathcal{O}$ , that is, a sheaf of  $\mathcal{O}$ -modules  $\mathcal{K}$  such that for every  $m \in M$ , there is an open neighborhood  $\mathcal{U}$  of  $m$  and an exact sequence

$$\mathcal{O}_{\mathcal{U}}^n \rightarrow \mathcal{O}_{\mathcal{U}}^k \rightarrow \mathcal{K}_{\mathcal{U}} \rightarrow 0$$

for some integers  $n$  and  $k$ . In Section 2.2, we apply the construction (2.1) to the case where  $\ker d \subseteq \Gamma(F)$  is locally finitely generated, that is to say when the sheafification of the image  $\text{im}(d) \subseteq \Gamma(E)$  is a coherent sheaf.

**Remark 2.2.** Intuitively, for  $x \in M$ ,  $\pi^{-1}(x) = \tilde{M} \cap \Pi^{-1}(x)$  is the set of all possible limits  $\text{Gr}_{-q}(E)$  of the images  $\text{im}(d_y)$  when  $y \in M_{\text{reg},d}$  converges to  $x$ .

**Remark 2.3.** One can make a similar construction with the kernel of  $d$ .

Here is an immediate property of that construction.

**Proposition 2.4.** *Let  $F \xrightarrow{d} E$  be a vector bundle morphism over  $M$ . The projection  $\pi: \tilde{M} \rightarrow M$  has the following properties:*

- (1)  $\pi$  is proper and surjective. In particular, for each point  $x \in M$ , the fiber  $\pi^{-1}(x)$  is non-empty.
- (2) For every  $x \in M$  and  $V \in \pi^{-1}(x)$ , one has  $\text{im}(d_x) \subseteq V$ .
- (3) For every  $x \in M_{\text{reg},d}$ ,  $\pi^{-1}(x) = \text{im}(d_x)$  is reduced to a point in  $\text{Gr}_{-q}(E)$ . Also,  $\pi^{-1}(M_{\text{reg},d})$  is a manifold<sup>3</sup>, and the restriction  $\pi: \pi^{-1}(M_{\text{reg},d}) \rightarrow M_{\text{reg},d}$  is invertible<sup>4</sup> in the smooth and holomorphic contexts.

*Proof.* Properness derives from the fact that the projection  $\Pi$  admits compact fibers. For any  $x \in M$ , choose  $\mathcal{U} \subset M$  an open neighborhood of  $x$  that trivializes  $E \rightarrow M$  over  $\mathcal{U}$ . Then  $\text{Gr}_{-q}(E) \simeq \mathcal{U} \times \text{Gr}_{-r}(\mathbb{K}^{\text{rk}(E)})$ . Notice that

$$\pi^{-1}(x) = \left\{ V \subset E_x \mid \exists (x_n) \in M_{\text{reg},d}^{\mathbb{N}}, \text{ such that } \text{im}(d_{x_n}) \xrightarrow[n \rightarrow +\infty]{} V \text{ as } x_n \xrightarrow[n \rightarrow +\infty]{} x \right\}.$$

For any sequence  $(x_n)$  in  $(M_{\text{reg},d} \cap \mathcal{U})^{\mathbb{N}}$  that converges to  $x$ , we can extract a sequence  $(x_{\varphi(n)})$  such that  $n \mapsto \text{im}(d_{x_{\varphi(n)}}) \in \text{Gr}_{-r}(\mathbb{K}^{\text{rk}(E)})$  has a limit  $V$ , since the Grassmannian manifold  $\text{Gr}_{-r}(\mathbb{K}^{\text{rk}(E)})$  is compact. Hence,  $\pi^{-1}(x) \neq \emptyset$ , and  $\pi$  is onto. This proves item (1).

Let us show item (2). Let  $V \in \pi^{-1}(x)$  and  $(x_n) \in (M_{\text{reg},d})^{\mathbb{N}}$  such that  $x_n \xrightarrow[n \rightarrow +\infty]{} x$  and  $\text{im}(d_{x_n}) \xrightarrow[n \rightarrow +\infty]{} V$ . Let  $v \in \text{im}(d_x)$ . We have  $v = d_x u$  for some  $u \in F_x$ . Choose a

<sup>3</sup>Manifold is to be understood as quasi-projective when  $M$  is quasi-projective.

<sup>4</sup>Invertible here means diffeomorphism in the smooth case, bi-holomorphism in the complex case.

(local) section  $\tilde{u}$  of  $F$  through  $u$ . By continuity,  $d_{x_n}\tilde{u}(x_n) \xrightarrow{n \rightarrow +\infty} d_x u$ , hence  $d_x u \in V$ . Thus,  $\text{im}(d_x) \subseteq V$ .

In particular, if  $x \in M_{\text{reg},d}$  and  $V \in \pi^{-1}(x)$ , one has  $\text{im}(d_x) = V$  since  $\dim V = \dim(\text{im}(d_x))$ . Therefore,  $\pi^{-1}(M_{\text{reg},d})$  is the image of the map  $\sigma$  on  $M_{\text{reg},d}$ , and it is isomorphic/bi-holomorphic to  $M_{\text{reg},d}$ . This proves item (3). ■

**Remark 2.5.** Let  $A, B, C, E, F$  be vector bundles over  $M$ . The Nash blowup of  $M$  along a vector bundle morphism  $d : F \rightarrow E$  coincides with the Nash blowup of  $M$  along the vector bundle morphism

$$A \oplus B \oplus F \rightarrow B \oplus E \oplus C$$

$$(a, b, f) \mapsto (b, d(f), 0)$$

The result is left to the reader.

In Section 2.2, we apply the constructions above to a sequence of vector bundle morphisms which are all of constant rank on an open dense subset.

**2.1.2. On the smoothness of the Nash blowup and monoidal transformations.**

We warn the reader not to confuse two uses of the word “smooth”. An analytic subset of  $\mathbb{C}^N$  or  $\mathbb{R}^N$  is said to be smooth when it admits no singular point. The word “smooth” is also used to say that we work within the context of smooth differential geometry, using smooth manifolds on which functions are of class  $\mathcal{C}^\infty$ . Notice that a smooth analytic variety is also a smooth manifold. The context should, however, prevent all confusions.

The “blowup” Nash construction in the previous section is of a type that algebraic geometers call monoidal transformations, also known as Hironaka blowups [20]. Several authors [38,41] or [15] have used this point of view to study the smoothness of the blowup of a singular foliation and to compute explicitly the blowup space  $\tilde{M}$ .

Let us recall what monoidal transformations are. Denote by  $\mathcal{O}$  the sheaf of holomorphic or real analytic functions on a holomorphic or real analytic manifold  $M$ . Let  $\mathcal{I} \subset \mathcal{O}$  be a locally finitely generated subsheaf of  $\mathcal{O}$ . Denote by  $Z(\mathcal{I}) \subset M$  its zero locus (= the subset of all points where all functions in  $\mathcal{I}$  vanish). The subset  $Z(\mathcal{I})$  is a closed subset for the usual topology, and  $M \setminus Z(\mathcal{I})$  is an open dense subset of  $M$ .

We call *monoidal transformation of  $M$  with respect to  $\mathcal{I}$*  (in the sense of [20]) the pair  $(\text{Bl}_{\mathcal{I}}(M), \pi)$  constructed as follows. Let  $\mathcal{U} \subset M$  be an open subset such that  $\mathcal{I}_{\mathcal{U}}$  is generated by a finite family  $\varphi_1, \dots, \varphi_n$ .

- (1) Define a map

$$H: \mathcal{U} \setminus Z(\mathcal{I}) \longrightarrow \mathbb{P}^{n-1}, \quad x \rightarrow [\varphi_1(x) : \dots : \varphi_n(x)].$$

- (2) Then we consider the ideal  $\mathcal{J}$  of (projective) functions on  $\mathcal{U} \times \mathbb{P}^{n-1}$  generated by

$$((x, [\xi_1 : \dots : \xi_n]) \mapsto \xi_i \varphi_j(x) - \xi_j \varphi_i(x))_{i \neq j=1, \dots, n},$$

where  $[\xi_1 : \dots : \xi_n]$  are the homogeneous coordinates on  $\mathbb{P}^{n-1}$ .

The closure  $\text{Bl}_{\mathcal{I}}(\mathcal{U}) \subset \mathcal{U} \times \mathbb{P}^{n-1}$  of the graph of  $H$  in  $\mathcal{U} \times \mathbb{P}^{n-1}$  is an irreducible component of the zero locus of the ideal  $\mathcal{J}$  and is in particular an irreducible analytic subvariety. The natural projection  $\pi_{\mathcal{U}} : \text{Bl}_{\mathcal{I}}(\mathcal{U}) \rightarrow \mathcal{U}$  is a real analytic or holomorphic proper map and restricts to an invertible map in the relevant category  $\pi^{-1}(\mathcal{U} \setminus Z(\mathcal{I})) \rightarrow \mathcal{U} \setminus Z(\mathcal{I})$ . Last, one can check that the pair  $(\text{Bl}_{\mathcal{I}}(\mathcal{U}), \pi_{\mathcal{U}})$  does not depend on the choice of local generators  $\varphi_1, \dots, \varphi_n$  (see, e.g., [14, Lemma 9.16]). This explains the notation.

Let us explain how the monoidal transformation depends on the ideal. Assume that we are given two ideals  $\mathcal{I}$  and  $\mathcal{J}$  over  $M$ . It is natural to ask what are the conditions that guarantee the existence of the dotted arrow below:

$$\begin{array}{ccc}
 \text{Bl}_{\mathcal{I}}(M) & \cdots \cdots \cdots \rightarrow & \text{Bl}_{\mathcal{J}}(M) \\
 \pi \downarrow & & \downarrow \pi \\
 M & \xrightarrow{=} & M
 \end{array} \tag{2.2}$$

If it exists, then it has to be unique. The theorem proved by Moody in [34] gives a definitive answer by stating that the following two conditions are equivalent:

- (i) The dotted arrow in (2.2) exists.
- (ii) There exist an integer  $n$  and a finitely generated sub- $\mathcal{O}$ -module  $\mathcal{K}$  in the sheaf of the fraction field  $\hat{\mathcal{O}}$  of  $\mathcal{O}$  such that  $\mathcal{K} \cdot \mathcal{J} = \mathcal{I}^n$ .

We will call this equivalence the *Moody criteria*.

**Remark 2.6.** In particular, given an open cover  $(\mathcal{U}_i)_{i \in I}$  of  $M$  and a family  $\mathcal{I}_i \subset \mathcal{O}(\mathcal{U}_i)$  of finitely generated ideals such that, for every  $i, j \in I$ , there exists an  $\phi_{ij} \in \hat{\mathcal{O}}(\mathcal{U}_i \cap \mathcal{U}_j)$  for which  $\mathcal{I}_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \phi_{ij} \mathcal{I}_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$ , the locally defined monoidal transformations  $(\text{Bl}_{\mathcal{I}_i} \mathcal{U}_i, \pi)$  coincide on  $\mathcal{U}_i \cap \mathcal{U}_j$  and define, globally, a pair  $(\tilde{M}, \pi)$ . Moreover, any other family  $(\mathcal{U}'_j, \mathcal{I}'_j)$  such that  $\mathcal{I}_i|_{\mathcal{U}_i \cap \mathcal{U}'_j} = \phi_{ij} \mathcal{I}'_j|_{\mathcal{U}_i \cap \mathcal{U}'_j}$  for some  $\phi_{ij} \in \hat{\mathcal{O}}(\mathcal{U}_i \cap \mathcal{U}'_j)$  will define the same pair  $(\tilde{M}, \pi)$ .

Let us apply these general facts to the situation of a vector bundle morphism  $d: F \rightarrow E$  which can be either holomorphic or real analytic.

There is an open subset  $M_{\text{reg}, d}$  on which  $d$  has constant rank. We denote by  $k$  this rank. The integer  $k$  also admits the following characterization. Any point  $m$  admits a connected neighborhood  $\mathcal{U}$  on which the  $\hat{\mathcal{O}}(\mathcal{U})$ -module  $\hat{\mathcal{O}}(\mathcal{U})\text{im}(d)$  generated by the image of  $d$  is a vector space whose dimension over  $\hat{\mathcal{O}}(\mathcal{U})$  is  $k$ . This characterization has the following consequence: we say that a family  $e_{\bullet} = e_1, \dots, e_k$  of local sections of  $E$  over such an open subset  $\mathcal{U}$  is *admissible* if  $d(e_1), \dots, d(e_k)$  are independent on an open subset of  $\mathcal{U}$ . Equivalently, it means that  $d(e_1), \dots, d(e_k)$  is a basis of  $\hat{\mathcal{O}}(\mathcal{U})\text{im}(d)$ . For any two admissible families  $e_{\bullet} = e_1, \dots, e_k$  and  $e'_{\bullet} = e'_1, \dots, e'_k$ , therefore, there exists a matrix valued in  $\hat{\mathcal{O}}(\mathcal{U})$  such that

$$\begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} = \begin{pmatrix} m_{1,1} & \cdots & m_{k,1} \\ \vdots & & \vdots \\ m_{1,k} & \cdots & m_{k,k} \end{pmatrix} \begin{pmatrix} e'_1 \\ \vdots \\ e'_k \end{pmatrix}. \tag{2.3}$$

Now, given a coordinate open set  $\mathcal{U} \subset M$  and an admissible family  $e_\bullet := e_1, \dots, e_k \in \Gamma(F)$ , one can consider the ideal  $I_{\text{im}(d(e_\bullet))}$  generated by  $\langle \alpha, d(e_1) \wedge \dots \wedge d(e_k) \rangle$  for  $\alpha \in \Gamma(\wedge^k E^*)$ . Equivalently,  $I_{\text{im}(d(e_\bullet))}$  can also be seen as the ideal generated by all  $k \times k$  minors of the  $k \times \text{rk}(F)$ -matrix representing the vectors  $d(e_1), \dots, d(e_k)$  on a given trivialization of  $F$  on  $\mathcal{U}$ . For any two admissible families  $e_\bullet$  and  $e'_\bullet$  defined on two such coordinate open sets  $\mathcal{U}$  and  $\mathcal{U}'$ , respectively, we have on  $\mathcal{U} \cap \mathcal{U}'$

$$I_{\text{im}(d(e_\bullet))} = \phi_{e_\bullet/e'_\bullet} I_{\text{im}(d(e'_\bullet))}, \tag{2.4}$$

where  $\phi_{e_\bullet/e'_\bullet} \in \widehat{\mathcal{O}}(\mathcal{U} \cap \mathcal{U}')$  is the determinant of the  $k \times k$ -matrix as in equation (2.3). One can then cover  $M$  by open coordinate neighborhoods  $(\mathcal{U}_i)_{i \in I}$ , then choose an admissible family on each one of them. The construction in Remark 2.6 applies and yields a pair  $(\widetilde{M}, \pi)$ .

**Lemma 2.7.** *The pair  $(\widetilde{M}, \pi)$  satisfies the following properties:*

- (1) *In a neighborhood  $\mathcal{U}$  of every point of  $M$ , it is given by the monoidal transformation with respect to the ideal generated by the  $k \times k$  minors of  $d(e_1), \dots, d(e_k)$  with  $e_1, \dots, e_k$  being any admissible family.*
- (2) *In particular, each point has a neighborhood near which it is an analytic variety, and  $\pi$  is a holomorphic or real analytic proper map.*
- (3) *Last,  $\pi^{-1}(M_{\text{reg},d})$  admits a natural manifold structure to which the restriction of  $\pi$  is invertible.*

*Proof.* The first and second items hold by construction. The last item follows from the fact that for any  $m \in M_{\text{reg},d}$ , there exists at least one admissible family  $e_\bullet = (e_1, \dots, e_k)$  such that the vectors  $d(e_1), \dots, d(e_k)$  are independent at the point  $m$ . In particular,  $m$  does not belong to the zero locus of the ideal  $I_{\text{im}(d(e_\bullet))}$ . ■

Here is the main result of this discussion, whose proof is delayed to the end of the section. It shares some similarity with [15, Theorem 1] that deals with Nash blowup of affine varieties.

**Proposition 2.8.** *The pair  $(\widetilde{M}, \pi)$  coincides with the Nash blowup.*

A practical consequence of Proposition 2.8 consists in allowing the application of the existing literature about monoidal transformations with respect to ideals, for example, criteria for smoothness for monomial ideals [28] or for tame monomial ideals [11]. Also, notice that Proposition 2.8 still applies in the context of smooth differential geometry, provided that each point of  $M$  admits local coordinates on which  $d$  is given by a matrix whose coefficients are real analytic. These coordinates even do not need to glue in a real analytic manner. Here is an application about the smoothness of  $\widetilde{M}$ .

**Corollary 2.9.** *Let  $M$  be a complex manifold and  $d : E \rightarrow F$  a vector bundle morphism. If  $M_{\text{sing}}$  is a smooth submanifold of  $M$ , and if every point of  $m$  admits local admissible sections  $e_\bullet = e_1, \dots, e_k$  such that there exists  $\chi \in \widehat{\mathcal{O}}$  and  $n \in \mathbb{N}$  that satisfy*

$$I_{\text{im}(d(e_\bullet))} = \chi \mathcal{I}_{\text{sing}}^n,$$

where  $\mathcal{I}_{\text{sing}}$  is the ideal of functions vanishing on the singular locus, then  $\widetilde{M}$  is a smooth manifold.

*Proof.* This follows from the criteria of Moody recalled above and the fact that the monoidal transformation with respect to an ideal and its powers are the same, together with the fact that the monoidal transformation with respect to the ideal of functions vanishing on a smooth submanifold is a smooth manifold. ■

*Proof of Proposition 2.8.* We only give a sketch of the proof, since it is equivalent to the one given in [38, 41]. Let  $\mathcal{U}$  be an open subset of  $M$  that trivializes both  $F$  and  $E$ . Let  $(u_1, \dots, u_d)$  and  $(e_1, \dots, e_{d'})$  be local frames of  $F$  and  $E$ , respectively, on  $\mathcal{U}$ . The sections  $d(u_1), \dots, d(u_d)$  are local real analytic/holomorphic generators of  $\text{im}(d)|_{\mathcal{U}} \subset \Gamma(E)|_{\mathcal{U}}$ . We have  $d \geq k$ , where  $k$  is the rank of  $d$  on regular points. Let  $M_{\text{sing}} := M \setminus M_{\text{reg},d}$ . If  $\mathcal{U} \cap M_{\text{sing}}$  is empty, then there is nothing to prove. Assume that  $\mathcal{U} \cap M_{\text{sing}} \neq \emptyset$ . There exist real analytic/holomorphic functions  $f_i^j \in \mathcal{O}_{\mathcal{U}}$  with  $i = 1, \dots, d'$  and  $j = 1, \dots, d$  such that

$$d(u_j) = \sum_{i=1}^{d'} f_i^j e_i.$$

Now, consider the  $d' \times d$ -matrix  $\mathfrak{M} = (f_i^j)$ . The rank of  $\mathfrak{M}$  is equal to  $k$  on  $M_{\text{reg},d}$  and is less than  $r$  on  $M_{\text{sing}}$ . Let  $q = \text{rk}(E) - k$ . For  $x \in M_{\text{reg},d}$ , denote by  $[\mathfrak{M}(x)]$  the point  $\text{im}(d_x) \in \text{Gr}_{-q}(E)$  generated by the columns of  $\mathfrak{M}(x)$ . In order to pick  $d' \times k$ -matrix that represents  $\text{im}(d_x)$ , consider the following indexing set

$$I_n = \{(n_1, \dots, n_k) \in \mathbb{N}^{d'} \mid 1 \leq n_1 < \dots < n_k \leq n\}.$$

We use  $I_{d'}$  to pick  $k$ -rows and  $I_d$  to pick  $k$ -columns. Pick  $\alpha \in I_{d'}$  and  $\mathfrak{b} \in I_d$  and consider the  $d' \times k$ -matrix  $\mathfrak{M}_{\mathfrak{b}} = (f_i^j)_{i=1, \dots, d', j \in \mathfrak{b}}$  and  $\mathcal{I}_{\mathfrak{b}}$  the ideal generated by all the  $k \times k$ -minors of  $\mathfrak{M}_{\mathfrak{b}}$ , that is, the ideal generated by the determinants  $\Delta_{\alpha\mathfrak{b}} := \det(f_i^j)_{i \in \alpha, j \in \mathfrak{b}}$  with  $\alpha \in I_{d'}$ .

Notice that the zero locus  $Z(\mathcal{I})$  of the ideal  $\mathcal{I}$  is exactly  $\mathcal{U} \cap M_{\text{sing}}$ . There exists  $\mathfrak{b} \in I_d$  such that  $\mathcal{I}_{\mathfrak{b}} \neq 0$ ; let us pick such a  $\mathfrak{b} \in I_d$  and consider the analytic variety which is given by the zero locus  $Z(\mathcal{I}_{\mathfrak{b}})$  of the ideal  $\mathcal{I}_{\mathfrak{b}}$ . We define the following maps:

$$N: \mathcal{U} \setminus Z(\mathcal{I}_{\mathfrak{b}}) \longrightarrow \mathcal{U} \times \text{Gr}_{-q}(\mathbb{K}^{d'}), \quad x \longmapsto (x, [\mathfrak{M}_{\mathfrak{b}}(x)]) \tag{2.5}$$

and

$$H: \mathcal{U} \setminus Z(\mathcal{I}_{\mathfrak{b}}) \longrightarrow \mathcal{U} \times \mathbb{P}^\tau, \quad x \longmapsto (x, [\Delta_{\alpha_0\mathfrak{b}} : \dots : \Delta_{\alpha_\tau\mathfrak{b}}]). \tag{2.6}$$

Here,  $\tau = \binom{d'}{k} - 1$  and  $\alpha_1, \dots, \alpha_\tau \in I_{d'}$ .

- (a) We have  $\overline{N(\mathcal{U} \setminus Z(\mathcal{I}_b))} \simeq \overline{H(\mathcal{U} \setminus Z(\mathcal{I}_b))}$ : to see this, consider the Plücker embedding [49, Chapter 1, Section 5]

$$\text{Pl}: \text{Gr}_{-q}(\mathbb{K}^{d'}) \hookrightarrow \mathbb{P}^\tau$$

and define the map  $\text{id} \times \text{Pl}: \mathcal{U} \times \text{Gr}_{-q}(\mathbb{K}^{d'}) \rightarrow \mathcal{U} \times \mathbb{P}^\tau$ . We have that  $(\text{id}, \text{Pl}) \circ N = H$ . Therefore, the closure of the image of  $N$  and  $H$  is isomorphic.

- (b) The maps  $\sigma: M_{\text{reg},d} \rightarrow \text{Gr}_{-q}(E)$ ,  $x \mapsto \text{im}(d_x)$  of equation (2.1) and  $N$  coincide on  $\mathcal{U} \setminus Z(\mathcal{I}_b)$ . This implies that  $\sigma(\mathcal{U} \setminus \mathcal{U} \cap M_{\text{sing}}) = \overline{N(\mathcal{U} \setminus Z(\mathcal{I}_b))}$ . Also, the closure  $\overline{N(\mathcal{U} \setminus Z(\mathcal{I}_b))}$  is a monoidal transformation of  $\mathcal{U}$  with center  $Z(\mathcal{I}_b)$  and is an analytic variety.

Different open neighborhoods glue together to give an analytic variety by the universal property of monoidal transformations. This proves the statement. ■

**Remark 2.10.** There is another natural sheaf  $\mathcal{I}_{\text{im}(d)} \subset \mathcal{O}$  of ideals that lead to another monoidal transformation. Consider the sheaf of ideals of  $\mathcal{O}$  of all local functions which, near every point, are of the form  $\langle d(e_1) \wedge \dots \wedge d(e_k), \alpha \rangle$  for some local sections  $e_1, \dots, e_k \in \Gamma(F)$  and  $\alpha \in \Gamma(\wedge^k E^*)$ . Equivalently, it can also be seen as the sheaf of ideals of  $\mathcal{O}$  generated by all  $k \times k$  minors of the  $\text{rk}(E) \times \text{rk}(F)$ -matrix that represents  $d$  on given local trivializations of  $F$  and  $E$ . This second monoidal transformation  $(\text{Bl}_{\mathcal{I}_{\text{im}(d)}}(M), \pi)$  and  $(\tilde{M}, \pi)$  enters into a commutative diagram as follows:

$$\begin{array}{ccc} \text{Bl}_{\mathcal{I}_{\text{im}(d)}}(M) & \cdots \cdots \cdots \rightarrow & \tilde{M} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{=} & M \end{array} \tag{2.7}$$

This can be seen as follows. Let  $\mathcal{U}$  be a coordinate neighborhood on which  $E$  and  $F$  are trivial bundles. Let  $e_1, \dots, e_{\text{rk}(E)}$  be a trivialization of  $E$  over  $\mathcal{U}$ . We say that a subset  $w$  of  $k$ -elements  $i_1, \dots, i_k$  in  $\{1, \dots, \text{rk}(E)\}$  is *admissible* if the family  $e_\bullet(w) := e_{i_1}, \dots, e_{i_k}$  is admissible. Then the sheaf of ideals  $\mathcal{I}_d$  is generated by

$$\mathcal{I}_{\text{im}(d)} = \sum_{w \in \text{Adm}} \mathcal{I}_{\text{im}(d(e_\bullet(w)))},$$

where  $\text{Adm}$  is the collection of all admissible subsets in  $\{1, \dots, \text{rk}(E)\}$ . Since  $\text{Adm}$  is not empty, we can select one, say  $w_0$ , and we then have in view of equation (2.4)

$$\mathcal{I}_{\text{im}(d)} = \mathcal{K} \mathcal{I}_{\text{im}(d(e_\bullet(w_0)))},$$

where  $\mathcal{K} \subset \hat{\mathcal{O}}(\mathcal{U})$  is the sub- $\mathcal{O}(\mathcal{U})$ -module generated by the functions  $\phi_{e_\bullet(w)/e_\bullet(w_0)}$  defined as in (2.4). The existence of the dotted arrow is then a consequence of the criteria of Moody.

**Remark 2.11.** Notice that, using the notations of the proof of Proposition 2.8, we have  $\mathcal{U} \cap M_{\text{sing}} \subseteq Z(\mathcal{I}_{\text{im}(d(e, \bullet))})$ . In general, there is no equality. That is, the zero locus of the ideal with respect to which one considers the monoidal transformation does not need to coincide with the singular locus  $M_{\text{sing}}$  of  $d$ . To have equality, S. Sertöz [41] introduced a notion of good generators for  $\text{im}(d) \subseteq \Gamma(E)$  as follows:  $\text{im}(d) \subseteq \Gamma(E)$  admits a *good system of generators* if for any  $x \in M_{\text{sing}}$  there exists an open neighborhood  $\mathcal{U}$  of  $x$  and sections  $s_1, \dots, s_k$  of  $\text{im}(d)$  such that

- (1)  $s_1, \dots, s_k$  span  $\text{im}(d)|_{\mathcal{U} \cap M_{\text{reg}, d}}$ ,
- (2)  $s_1, \dots, s_k$  are linearly dependent on  $\mathcal{U} \cap M_{\text{sing}}$ ,

where  $k$  is the rank of  $d: F \rightarrow E$  on  $M_{\text{sing}}$ . This family is in particular admissible. Also, such a family exists when  $\text{im}(d)$  is a projective submodule of  $\Gamma(E)$ . With such generators, one has

$$Z(\mathcal{I}_{\mathfrak{b}}) = \mathcal{U} \cap M_{\text{sing}}$$

for every neighborhood  $\mathcal{U}$  of a singular point. Nevertheless, the ideal  $\mathcal{I}_{M_{\text{sing}}}$  of vanishing functions on  $M_{\text{sing}}$  does not have to be equal to  $\mathcal{I}_{\mathfrak{b}}$  on  $\mathcal{U}$  but to its radical by the Nullstellensatz theorem [9].

In the sequel, we will not require the Nash blowup space  $\tilde{M}$  to be smooth. The established properties of  $\tilde{M}$  are sufficient to state the results we need.

**2.2. Nash blowups of singular foliations: Main constructions and results**

Let  $\mathfrak{F}$  be a locally finitely generated  $\mathcal{O}$ -submodule of  $\mathfrak{X}(M)$ , that is,  $\mathfrak{F}$  is a subsheaf  $\mathfrak{X}(M)$  such that every point of  $M$  admits an open neighborhood  $\mathcal{U}$  and a finite number of vector fields  $X_1, \dots, X_n \in \mathfrak{X}(\mathcal{U})$  such that for all  $\mathcal{V} \subseteq \mathcal{U}$ ,  $\mathfrak{F}|_{\mathcal{V}} = \sum_{k=1}^n f_k X_k|_{\mathcal{V}}$  for some  $f_k \in \mathcal{O}_{\mathcal{V}}$ . We assume that there exists a *geometric resolution*, that is, a complex of vector bundles  $(E, d, \rho)$  of finite length

$$\begin{array}{ccccccccccc}
 0 \cdots & \longrightarrow & E_{-i-1} & \xrightarrow{d^{(i+1)}} & E_{-i} & \xrightarrow{d^{(i)}} & E_{-i+1} & \longrightarrow & \cdots & \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho=d^{(1)}} & TM \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 M \cdots & \longleftarrow & M & \longleftarrow & M & \longleftarrow & M & \longleftarrow & \cdots & \longleftarrow & M & \longleftarrow & M \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & (2.8)
 \end{array}$$

such that  $\rho(\Gamma(E_{-1})) = \mathfrak{F}$  and which is exact as in equation (1.2). In the smooth case, geometric resolutions exist on every relatively compact open subset of  $M$  such that every point admits local coordinates on which the local generators  $\mathfrak{F}$  are real analytic (see [21] or [23, Section 6]). In the holomorphic case, the existence of a geometric resolution in a neighborhood of each point is a property of coherent sheaves (see [23, Section 6]).

For every  $i \geq 0$ , let  $M_{\text{reg}^i, \mathfrak{F}}$  be the open dense subset of  $M$  made of all points  $m \in M$  such that the image  $\text{im}(d^{(i+1)})$  of the vector bundle morphism  $d^{(i+1)} : E_{-i-1} \rightarrow E_{-i}$  is of constant rank on some neighborhood of  $m$ . For  $i = 0$ , we define it to be the open dense

subset of  $M$  made of all points  $m \in M$  such that the vector bundle morphism  $\rho: E_{-1} \rightarrow TM$  is of constant rank on some open neighborhood of  $m$ . To avoid having to distinguish this case, from now on, we set  $E_0 := TM$  and  $d^{(1)} = \rho$  by convention.

It deserves to be noticed that  $m \in M_{\text{reg}^i, \mathfrak{F}}$  if and only if  $m$  admits a neighborhood on which  $\text{im}(d^{(i+1)}) = \ker(d^{(i)})$ . By Lemma 1.8 (b), we have  $M_{\text{reg}^1, \mathfrak{F}} = M_{\text{reg}^0, \mathfrak{F}}$ . Since any two geometric resolutions of some  $\mathfrak{F}$  homotopy equivalent, and since this property is invariant under homotopy equivalence, the open dense subset  $M_{\text{reg}^i, \mathfrak{F}}$  does not depend on the choice of a geometric resolution of  $\mathfrak{F} \subseteq \mathfrak{X}(M)$ . Also, we will denote  $M_{\text{reg}^1, \mathfrak{F}} = M_{\text{reg}^0, \mathfrak{F}}$  simply by  $M_{\text{reg}, \mathfrak{F}}$  which coincides with the open dense subset of regular points of the singular foliation  $\mathfrak{F}$ . We have a sequence of inclusions of open dense subsets:

$$M_{\text{reg}, \mathfrak{F}} \subset M_{\text{reg}^2, \mathfrak{F}} \subset M_{\text{reg}^3, \mathfrak{F}} \subset \dots \subset M.$$

These points have the following characterizations:

$i = 0, 1:$   $m \in M_{\text{reg}, \mathfrak{F}}$  if and only if there is a neighborhood on which the distribution

$$m' \mapsto T_{m'} \mathfrak{F} = \{X|_{m'} \mid X \in \mathfrak{F}\} \subset TM$$

has constant rank, that is, is a regular foliation.

$i = 2:$   $m \in M_{\text{reg}^2, \mathfrak{F}}$  if and only if there is a neighborhood on which  $\mathfrak{F} \subset \mathfrak{X}(M)$  is a free module over functions.

$i = 3:$   $m \in M_{\text{reg}^3, \mathfrak{F}}$  if and only if there is a neighborhood on which  $\mathfrak{F}$  admits a geometric resolution of length 2.

$\vdots$

$i = n + 1:$   $m \in M_{\text{reg}^{n+1}, \mathfrak{F}}$  if and only if there is a neighborhood on which  $\mathfrak{F}$  admits a geometric resolution of length  $n$ .

**2.2.1. The blowup spaces associated to a singular foliation.** The blowup spaces are constructed as follows. Let  $(M, \mathfrak{F})$  be a singular foliation and  $(E, d, \rho)$  be a geometric resolution of  $\mathfrak{F}$  as in equation (2.8). For every  $i \geq 0$ , we apply the Nash construction to  $d^{(i+1)}: E_{-i-1} \rightarrow E_{-i}$ . By convention, for  $i = 0$ , this means that we apply it to  $\rho: E_{-1} \rightarrow TM$ , because we set  $E_0 = TM$  and  $d^{(1)} = \rho$ . Let us recall this construction.

(a) Let  $\Pi_i: \text{Gr}_{-r_i}(E_{-i}) \rightarrow M$  be the Grassmann bundle of  $E_{-i}$  with  $r_i$  as in Lemma 1.8 (d). Consider the natural section of  $\Pi_i$  on  $M_{\text{reg}^i, \mathfrak{F}}$  defined by

$$\sigma_i: M_{\text{reg}^i, \mathfrak{F}} \longrightarrow \text{Gr}_{-r_i}(E_{-i}), \quad x \longmapsto \text{im}(d_x^{(i+1)}). \tag{2.9}$$

(b) Let  $\tilde{M}_i := \overline{\sigma_i(M_{\text{reg}^i, \mathfrak{F}})}$  be the closure of the image of  $\sigma_i$  in  $\text{Gr}_{-r_i}(E_{-i})$ . Let  $\pi_i: \tilde{M}_i \rightarrow M$  denote the restriction of  $\Pi_i$  to  $\tilde{M}_i$ .

If  $(E, d, \rho)$  is of finite length, we also apply the Nash construction to the vector bundle morphism  $d = \bigoplus_{i \geq 2} d^{(i)}: \bigoplus_{i \geq 2} E_{-i} \rightarrow \bigoplus_{i \geq 1} E_{-i}$  by considering the section

$$\sigma_\infty: M_{\text{reg}, \mathfrak{F}} \longrightarrow \prod_{x \in M} \prod_{i \geq 1} \text{Gr}_{-r_i}(E_{-i}|_x), \quad x \mapsto (\sigma_1(x), \sigma_2(x), \dots, \sigma_i(x), \dots)$$

and define  $\tilde{M}_\infty := \overline{\sigma_\infty(M_{\text{reg}, \mathfrak{F}})}$  which comes with a natural map  $\pi_\infty: \tilde{M}_\infty \rightarrow M$ .

**Remark 2.12.** The space  $\tilde{M}_\infty$  should be understood as the tuples made of elements  $V_1 \in \text{Gr}_{-r_1}(E_{-1}|_x), \dots, V_i \in \text{Gr}_{-r_i}(E_{-i}|_x), \dots$  such that there exists a sequence  $(x_n) \in M_{\text{reg}, \mathfrak{F}}^{\mathbb{N}}$  such that  $\text{im}(d_{x_n}^{(i+1)}) \xrightarrow{n \rightarrow +\infty} V_i$  as  $x_n \xrightarrow{n \rightarrow +\infty} x$  for all  $i \in \mathbb{N}$ . It is important to notice that all the  $V_i$ 's are given by the same sequence  $(x_n) \in M_{\text{reg}, \mathfrak{F}}^{\mathbb{N}}$ . In particular, for every  $i \geq 1$ , there is a natural map

$$\begin{array}{ccc} \tilde{M}_\infty & \cdots \cdots \cdots & \tilde{M}_i \\ \pi_\infty \downarrow & & \downarrow \pi_i \\ M & \xrightarrow{=} & M \end{array}$$

By Proposition 2.4, for each  $i \geq 0$ , the projection  $\pi_i: \tilde{M}_i \rightarrow M$  is invertible on the open dense subset  $M_{\text{reg}^i, \mathfrak{F}}$ ; it is proper and surjective. Moreover, for each point  $x \in M$  and for every  $i \geq 0$ , the fiber  $\pi_i^{-1}(x)$  is non-empty. Also,  $\pi_\infty^{-1}(x)$  is non-empty.

**Definition 2.13.** For each  $i \geq 1$ , the space  $\tilde{M}_i$  together with the map  $\pi_i: \tilde{M}_i \rightarrow M$  is called the  $i$ -th blowup space of  $(M, \mathfrak{F})$ . Likewise,  $\pi_\infty: \tilde{M}_\infty \rightarrow M$  is called the last blowup space of  $(M, \mathfrak{F})$ .

**Remark 2.14.** The map  $\pi_0: \tilde{M}_0 \rightarrow M$  is the Nash blowup of the singular foliation  $\mathfrak{F}$  in the sense of [4]. Also,  $\pi_1: \tilde{M}_1 \rightarrow M$  is the blowup in the sense of [33,40]. While for  $i \geq 2$ , the  $\pi_i: \tilde{M}_i \rightarrow M$ 's do not exist in literature as blowups of the singular foliation  $\mathfrak{F}$  to our knowledge, but they still can be seen as a class of Nash blowups in the sense of [41].

As sets,  $\tilde{M}_i, \tilde{M}_\infty$  do not need to be manifolds. They can be singular (see Section 2.1.2).

**Proposition 2.15.** Let  $\mathfrak{F}$  be a holomorphic singular foliation or a real analytic singular foliation. Then, for every  $i \geq 0$  or  $i = \infty$ ,  $\tilde{M}_i$  is an analytic variety. For  $M = \mathbb{K}^n$  and  $\mathfrak{F}$  a singular foliation with polynomial generators, it is even a quasi-projective variety. Moreover, it is obtained in a neighborhood of every point through a monoidal transformation with respect to an ideal of the sheaf of functions.

The same results hold for  $\tilde{M}_\infty$ .

*Proof.* In all the cases above, there exists a geometric resolution  $(E, d, \rho)$  of  $\mathfrak{F}$  of finite length by trivial vector bundles [21]. Moreover,  $d$  and  $\rho$  are given by holomorphic, real analytic or polynomials depending on the context. Proposition 2.8 applied to  $d^{(i)}: E_{-i} \rightarrow E_{-i+1}$  or  $\rho: E_{-1} \rightarrow TM$  implies that  $\tilde{M}_i$  is an analytic subvariety of the Grassmann bundle, given by a monoidal transformation. ■

The following assertion follows from the existence of homotopy equivalence between any two geometric resolutions.

**Theorem 2.16.** *Let  $i \in \mathbb{N}_0$  or  $i = \infty$ . Let  $\mathfrak{F}$  be a singular foliation on  $M$  that admits a geometric resolution. For any two geometric resolutions of  $\mathfrak{F}$ , the corresponding  $\tilde{M}_i$  are canonically isomorphic.*

Theorem 2.16 may be seen as a consequence of Remark 2.5 since for any two resolutions, the differential map  $d^{(i)}$  differs by transformations as in that remark. However, we prefer to establish it through the following results.

**Proposition 2.17.** *Assume that the sequence (2.8) is a geometric resolution for  $\mathfrak{F}$ . For every  $x \in M$ , for every  $i \geq 1$  and  $V \in \pi_i^{-1}(x)$ , one has*

$$\text{im}(d_x^{(i+1)}) \subseteq V \subseteq \ker(d_x^{(i)}). \tag{2.10}$$

*In particular, for all  $x \in M_{\text{reg}^i, \mathfrak{F}}$  and  $i \geq 1$ ,  $\ker(d_x^{(i)}) = \text{im}(d_x^{(i+1)}) = \pi_i^{-1}(x)$ .*

Let us now equip the geometric resolution with a universal Lie  $\infty$ -structure whose bracket, that we denote by  $(\ell_k)_{k \geq 2}$ , restricts to  $k$ -linear maps on  $\bigoplus_{i \geq 2} E_{-i} + \ker(\rho)$  that we denote by  $(\{\cdot \cdot \cdot\}_k)_{k \geq 1}$ .

**Proposition 2.18.** *Fix a geometric resolution  $(E, d, \rho)$  of  $\mathfrak{F}$  and a universal Lie  $\infty$ -algebroid  $(E, (\ell_k)_{k \geq 1}, \rho)$  of  $\mathfrak{F}$ . The following are satisfied:*

- (1) *For every  $x \in M$  and  $V \in \pi_1^{-1}(x)$ , the 2-ary bracket  $\{\cdot, \cdot\}_2$  on  $\ker \rho_x$  restricts to  $V$ .*
- (2) *For all  $x \in M$ , and  $(V_1 \subset E_{-1}|_x, \dots, V_k \subset E_{-k}|_x, \dots) \in \pi_\infty^{-1}(x)$ , we have  $\{V_i, V_j\}_2 \subset V_{i+j-1}$  for every  $i, j \in \mathbb{N}_0$ .*

In particular, these two items have obvious consequences. Recall that for every  $x \in M$ ,  $H^{-1}(\mathfrak{F}, x) \simeq \mathfrak{g}_x$  is the isotropy Lie algebra and that  $\bigoplus_{i \geq 1} H^{(-i)}(\mathfrak{F}, x)$  comes with a canonical natural graded Lie algebra structure (see discussion of Section 1.2 (4)).

**Corollary 2.19.** *Under the assumption of Proposition 2.18, for every  $x \in M$ ,*

- (1) *The image of  $V \in \pi_1^{-1}(x)$  in  $H^{-1}(\mathfrak{F}, x) \simeq \mathfrak{g}_x$  is a Lie subalgebra of codimension  $r - \dim(L_x)$ , where  $\dim(L_x)$  is the dimension of the leaf through  $x$ .*
- (2) *The image of  $(V_1, \dots, V_k, \dots) \in \pi_\infty^{-1}(x)$  in  $\bigoplus_{i \geq 1} H^{(-i)}(\mathfrak{F}, x)$  is a graded Lie subalgebra.*

**Remark 2.20.** The 3-ary bracket  $\{\cdot, \cdot, \cdot\}_3$  does not restrict to elements of  $\pi_\infty^{-1}(x)$  for  $x \in M$ .

The corollary below is a direct consequence of Proposition 2.17 and is another manner to state that  $M_i$  does not depend on the geometric resolution.

**Corollary 2.21.** *There are inclusions*

$$\tilde{M}_i \hookrightarrow \coprod_{x \in M} \text{Gr}_{-(r_i - \text{rk}(d_x^{(i)}))} (H^{-i}(\mathfrak{F}, x)) \tag{2.11a}$$

and

$$\tilde{M}_\infty \hookrightarrow \coprod_{x \in M} \prod_{i \geq 1} \text{Gr}_{-(r_i - \text{rk}(d_x^{(i)}))} (H^{-i}(\mathfrak{F}, x)). \tag{2.11b}$$

*Proof.* Let  $x \in M$  and  $i \geq 1$ . By Proposition 2.17, elements  $V \in \pi_i^{-1}(x)$  satisfy the inclusions,  $\text{im}(d_x^{(i+1)}) \subseteq V \subseteq \text{ker}(d_x^{(i)})$ , and they correspond injectively to a (unique) sub-vector space of codimension  $r_i - \text{rk}(d^{(i)})$  in  $H^{-i}(\mathfrak{F}, x)$ . In particular, this implies the existence of an inclusion  $\pi_i^{-1}(x) \hookrightarrow \text{Gr}_{-(r_i - \text{rk}(d^{(i)}))} (H^{-i}(\mathfrak{F}, x))$ . ■

We denote by  $\text{GrLie}_{-(r - \dim(L_x))}(\mathfrak{g}_x)$  the sub-Grassmannian of Lie subalgebras of  $\mathfrak{g}_x$  of codimension  $r - \dim(L_x)$ .

**Corollary 2.22.** *The image of the inclusion  $\tilde{M}_1 \hookrightarrow \bigsqcup_{x \in M} \text{GrLie}_{-(r - \dim(L_x))}(\mathfrak{g}_x)$  is the blowup space of  $O$ . Mohsen [33].*

*Proof.* Let  $\mathfrak{F}$  be a singular foliation that admits a geometrical resolution  $(E, d, \rho)$ . For every  $x \in M$ , the fiber  $\text{blup}(\mathfrak{F})_x$  of [33] is constructed out of minimal generators  $X_1, \dots, X_d$  of  $\mathfrak{F}$  in a neighborhood of  $x$  as follows: for  $y \in M_{\text{reg}, \mathfrak{F}}$ , let  $\phi_y$  be the surjective linear map defined by

$$\phi_y: \frac{\mathfrak{F}}{\mathcal{I}_x \mathfrak{F}} \longrightarrow T_y \mathfrak{F}, \quad \phi_y([X_i]_x) = X_i(y) \quad \text{for all } i \in \{1, \dots, d\}, \tag{2.12}$$

where  $T_y \mathfrak{F}$  is the image of the evaluation map  $e_y: \mathfrak{F} \rightarrow T_y M$  at  $y$ . By definition,  $\text{blup}(\mathfrak{F})_x$  is made of subspaces  $V \subseteq \frac{\mathfrak{F}}{\mathcal{I}_x \mathfrak{F}}$  such that there exists a sequence  $x_n \in M_{\text{reg}, \mathfrak{F}}$  such that

$$x_n \longrightarrow x, \quad \phi_{x_n}^{-1}(0) \longrightarrow V \in \text{Gr}_{-r} \left( \frac{\mathfrak{F}}{\mathcal{I}_x \mathfrak{F}} \right). \tag{2.13}$$

We claim that for every  $x \in M$ ,  $\text{blup}(\mathfrak{F})_x \simeq \pi_1^{-1}(x)$ . Indeed, we can assume that  $(E, d, \rho)$  is a minimal geometric resolution at  $x$  such that  $\rho(e_i) = X_i$  for  $i = 1, \dots, d$ , where  $(e_i)_{i=1, \dots, d}$  is a local frame of  $E_{-1}$ . Since  $\frac{\Gamma(E_{-1})}{\mathcal{I}_{x'} \Gamma(E_{-1})} \simeq E_{-1}|_{x'}$  for all  $x' \in M$ , the anchor map defines an isomorphism

$$\bar{\rho}_x: E_{-1}|_x \rightarrow \frac{\mathfrak{F}}{\mathcal{I}_x \mathfrak{F}}$$

such that the diagram

$$\begin{array}{ccc} E_{-1}|_x & \xrightarrow{\bar{\rho}_x} & \frac{\mathfrak{F}}{\mathcal{I}_x \mathfrak{F}} \\ \kappa_y \downarrow \wr & & \downarrow \phi_y \\ E_{-1}|_y & \xrightarrow{\rho_y} & T_y \mathfrak{F} \end{array} \tag{2.14}$$

commutes. The claim follows. ■

**2.2.2. Lift of the singular foliation  $\mathfrak{F}$  to the blowup spaces  $(\tilde{M}_i)_{0 \leq i \leq \infty}$ .** Assume now that  $\mathfrak{F}$  is a singular foliation and that equation (2.8) is a geometric resolution of  $\mathfrak{F}$  of finite length. Notice that the fiber product  $\times_M^{i \geq 1} \text{Gr}_{-r_i}(E_{-i})$  is finite since  $(E, d, \rho)$  is a geometric resolution of finite length. Hence,  $\times_M^{i \geq 1} \text{Gr}_{-r_i}(E_{-i})$  is a smooth manifold.

**Definition 2.23.** Let  $i \geq 0$ . We say that  $X \in \mathfrak{F}$  lifts to  $\tilde{M}_i \subset \text{Gr}_{-r_i}(E_{-i})$ , or  $\tilde{M}_\infty$ , if there exists a vector field  $\tilde{X} \in \mathfrak{X}(\text{Gr}_{-r_i}(E_{-i}))$  or  $\mathfrak{X}(\times_M^{i \geq 1} \text{Gr}_{-r_i}(E_{-i}))$ , projectable to  $X$  and tangent to  $\tilde{M}_i$  in the sense of Section 1.1.1 (1). We denote by  $\tilde{X}_i$  or  $\tilde{X}_\infty$  the restriction of  $\tilde{X}$  to  $\tilde{M}_i$  or  $\tilde{M}_\infty$ , respectively.

We say that an  $\mathfrak{F}$  lifts to  $\tilde{M}_i$  if every vector field  $X \in \mathfrak{F}$  lifts to  $\tilde{M}_i$ .

**Remark 2.24.** The restriction of  $\tilde{X}_i$  to  $\pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}})$  is tangent in the usual sense to the submanifold and projects to  $X$  through  $\pi_i$ . In particular, if a lift exists, its restriction to  $\pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}})$  is unique because  $\pi_i: \pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}}) \xrightarrow{\sim} M_{\text{reg}^i, \mathfrak{F}}$ . Since the other points of  $\tilde{M}_i$  are limits of elements of  $\pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}})$ , its restriction to  $\tilde{M}_i$  is unique.

**Theorem 2.25.** Let  $\mathfrak{F}$  be a singular foliation on  $M$  that admits a geometric resolution. For every  $i \geq 0$ , the following items hold:

- (1) Every vector field  $X \in \mathfrak{F}$  lifts to a unique vector field  $\tilde{X}_i$  on  $\tilde{M}_i$ .
- (2) The map  $X \in \mathfrak{F} \rightarrow \tilde{X}_i \in \mathfrak{X}(\tilde{M}_i)$  does not depend on any choices. In particular, it is a Lie algebra morphism.
- (3) The module  $\tilde{\mathfrak{F}}_i$  over functions on  $\tilde{M}_i$  generated by the  $\tilde{X}_i$ 's for  $X \in \mathfrak{F}$  is a singular foliation.

The same holds for  $\tilde{M}_\infty$ .

The following definition then makes sense.

**Definition 2.26.** For each  $i \geq 1$ , the singular foliation  $\tilde{\mathfrak{F}}_i \subset \mathfrak{X}(\tilde{M}_i)$  of Theorem 2.25 is called the  $i$ -th blowup of  $\mathfrak{F}$  on  $\tilde{M}_i$ . Likewise,  $\tilde{\mathfrak{F}}_\infty$  is called the last blowup of  $\mathfrak{F}$  on  $\tilde{M}_\infty$ .

**Remark 2.27.** Although the closed subset  $\tilde{M}_i$  may have singularities, the singular foliation  $\tilde{\mathfrak{F}}_i \subset \mathfrak{X}(\tilde{M}_i)$  possesses smooth leaves by Theorem 1.5.

**2.2.3. The blowup foliations and their Lie  $k$ -algebroids.** For  $1 \leq k < \infty$ , let  $\pi_k: \tilde{M}_k \rightarrow M$  denote the  $k$ -th blowup of the singular foliation  $(M, \mathfrak{F})$ , and let  $\tilde{\mathfrak{F}}_k$  be the  $k$ -th blowup of  $\mathfrak{F}$  on  $\tilde{M}_k$ . In the following,  $\pi_k^* E_{-i}$  stands for the restriction to  $\tilde{M}_k$  of the pullback vector bundle  $\Pi^* E_{-i} \rightarrow \text{Gr}_{-r_k}(E_{-k})$ . The pullback of a vector bundle map  $d: E_{-i} \rightarrow E_{-i+1}$  on  $\tilde{M}_k$  shall be denoted by  $\pi_k^* d$ .

Here are the main results of this section.

**Theorem 2.28.** Let  $\mathfrak{F} \subseteq \mathfrak{X}(M)$  be a singular foliation on  $M$  that admits universal Lie  $\infty$ -algebroid  $(E, (\ell_k)_{k \geq 1}, \rho)$  built on a geometric resolution  $(E, d, \rho = d^{(1)})$ . For every  $k \geq 1$ , there exists a subvector bundle of  $K \subset \pi_k^* E_{-k}$

- (1) with  $\Gamma(K) \subseteq \ker d^{(k)}$  and  $K|_{\pi_k^{-1}(M_{\text{reg}^k, \mathfrak{F}})} = \ker d^{(k)}|_{M_{\text{reg}^k, \mathfrak{F}}}$ , and
- (2) a vector bundle morphism  $\tilde{\rho}: \pi_k^* E_{-1} \rightarrow T\tilde{M}_k$  such that  $\tilde{\rho}(\Gamma(\pi_k^* E_{-1})) = \tilde{\mathfrak{F}}_k$

so that the complex of vector bundles

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & \pi_k^* E_{-k} & \xrightarrow{\overline{\pi_k^* d^{(k)}}} & E_{-k+1} & \xrightarrow{\pi_k^* d^{(k-1)}} & E_{-i+1} & \longrightarrow & \dots & \xrightarrow{\pi_k^* d^{(2)}} & \pi_k^* E_{-1} & \xrightarrow{\tilde{\rho}} & T\tilde{M}_k \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 \tilde{M}_k & \xlongequal{\quad} & \tilde{M}_k & \xlongequal{\quad} & \tilde{M}_k & \xlongequal{\quad} & \tilde{M}_k & \xlongequal{\quad} & \dots & \xlongequal{\quad} & \tilde{M}_k & \xlongequal{\quad} & \tilde{M}_k
 \end{array}$$

is exact in degree  $k$  and comes equipped with a “natural” Lie  $k$ -algebroid structure<sup>5</sup>. Also,  $\frac{\pi_k^* E_{-k}}{K} \rightarrow \tilde{M}_k$  depends only on the image of  $d^{(k)}$  in  $\Gamma(E_{-k+1})$ , not on  $E_{-k}$ . In the diagram above, the bar  $\overline{\pi_k^* d^{(k)}}$  stands for the quotient of the map  $\pi_k^* d^{(k)} : \pi_k^* E_{-k} \rightarrow E_{-k+1}$ .

Here is a remarkable fact for  $k = 1$ .

**Corollary 2.29.** *Let  $\mathfrak{F}$  be a singular foliation on  $M$  that admits a geometric resolution  $(E, d, \rho)$ .*

- (1) *The singular foliation  $\tilde{\mathfrak{F}}_1 \subset \mathfrak{X}(\tilde{M}_1)$  is Debord, that is, it is the image of a Lie algebroid<sup>6</sup> over  $\tilde{M}_1$  whose anchor map is injective on an open dense subset.*
- (2) *This Lie algebroid is the Lie algebroid<sup>7</sup> of the groupoid of O. Mohsen [33].*

*Proof.* Item (1) follows from Theorem 2.28. For item (2), we also need Corollary 2.22 and a line-by-line comparison with [33]. ■

In Corollary 2.29, we do not need the existence of geometric resolutions of  $\mathfrak{F}$ . Its proof only needs an almost Lie algebroid over  $\mathfrak{F}$ . In the smooth case, the latter always exists as long as  $\mathfrak{F}$  is finitely generated (see [21, Proposition 3.8]).

**Corollary 2.30.** *If  $\tilde{M}_1$  is smooth, then the Lie algebroid of  $\tilde{\mathfrak{F}}_1$  is integrable to a (Debord) Lie groupoid and the groupoid of O. Mohsen is a quotient of the latter.*

*Proof.* By [7], a Lie algebroid  $A \rightarrow \tilde{M}_1$  whose anchor is injective on an open dense subset is integrable to a Lie groupoid referred to as the Debord groupoid. The Debord groupoid is universal among the integrations of  $A$ . Therefore, the groupoid of O. Mohsen is a quotient. ■

<sup>5</sup>We also make sense of the notion of Lie  $k$ -algebroid on a closed subset  $S \subseteq M$ , similarly as in Section 1.1.1 (3).

<sup>6</sup>Lie algebroids in the sense of Definition 1.1.1 (3) are Lie algebroids in the usual sense when  $\tilde{M}_1$  is smooth.

<sup>7</sup>If  $\tilde{M}_1$  is smooth, it is the Lie algebroid in the usual sense. Otherwise, it means that it is the differentiation of Mohsen’s groupoid along the fibers of the source map.

### 3. Proof of the main results

In this section, we prove the results of Section 2.2 whose proofs were delayed.

#### 3.1. Proof of Propositions 2.17 and 2.18

*Proof (of Proposition 2.17).* We know by Proposition 2.4 (2) that, for every  $x \in M$  and  $V \in \pi_i^{-1}(x)$ , one has  $\text{im}(d_x^{(i+1)}) \subseteq V$ . Now, for any element  $v \in V$ , there exists a sequence  $v_n \in \ker(d_{x_n}^{(i)}) = \text{im}(d_{x_n}^{(i+1)})$ ,  $n \in \mathbb{N}$  that converges to  $v$ . In particular,  $d_{x_n}^{(i)}(v_n) = 0$  for all  $n$ . Hence, by continuity, one has  $v \in \ker(d_x^{(i)})$ . Hence,  $V \subseteq \ker d_x^{(i)}$ . This completes the proof. ■

*Proof (of Proposition 2.18).* For all  $i \geq 1$ , choose a local frame  $e_1^{(i)}, \dots, e_{q_i}^{(i)}, \dots, e_{q_i+r_i}^{(i)}$  of  $E_{-i}$  on a neighborhood  $\mathcal{U}$  of  $x$  such that  $e_1^{(i)}(x), \dots, e_{q_i}^{(i)}(x)$  is an orthogonal basis for  $V_i$  for an arbitrary Hermitian structure on  $E_{-i}$ . For  $i, j \geq 1$ , let  $(c_{kl}^{ij,s}) \in \mathcal{O}_{\mathcal{U}}$  be a family of functions over  $\mathcal{U}$  such that for all  $k \leq q_i$  and  $l \leq q_j$ ,

$$\ell_2(e_k^{(i)}, e_l^{(j)}) = \sum_{s \geq 1} c_{kl}^{ij,s} e_s^{(i+j-1)} \in \Gamma_{\mathcal{U}}(E_{-i-j+1}).$$

In particular,

$$\{e_k^{(i)}(x), e_l^{(j)}(x)\}_2 = \sum_{s \geq 1} c_{kl}^{ij,s}(x) e_s^{(i+j-1)}(x). \tag{3.1}$$

The bracket in equation (3.1) is well defined even for  $i = 1$  or  $j = 1$ , although only the 2-ary bracket of local sections is defined in such cases, because even if  $i$  or  $j = 1$ , we are taking the brackets of elements in  $\ker \rho_x$ . Let  $u \in V_i, v \in V_j$ , with  $u = \sum_{s=1}^{q_i} \alpha^s e_s^{(i)}(x)$ , and  $v = \sum_{s=1}^{q_j} \beta^s e_s^{(j)}(x)$ .

Let  $(x_n) \in M_{\text{reg}^i, \mathfrak{F}}^{\mathbb{N}}$  be a sequence of regular points that converges to  $x$  such that  $\text{im}(d_{x_n}^{(i+1)}) \xrightarrow{n \rightarrow +\infty} V_i$  and  $\text{im}(d_{x_n}^{(j+1)}) \xrightarrow{n \rightarrow +\infty} V_j$ . There exist sequences

$$u_n = \sum_{k=1}^{q_i+r_i} \alpha_n^k e_k^{(i)}(x_n) \xrightarrow{n \rightarrow +\infty} u; \quad v_n = \sum_{l=1}^{q_j+r_j} \beta_n^l e_l^{(j)}(x_n) \xrightarrow{n \rightarrow +\infty} v$$

with  $u_n \in \text{im}(d_{x_n}^{(i+1)}) = \ker d_{x_n}^{(i)}$  and  $v_n \in \text{im}(d_{x_n}^{(j+1)}) = \ker d_{x_n}^{(j)}$  for all  $n \in \mathbb{N}$ . In particular, the sequences  $(\alpha_n^k), (\beta_n^l) \in \mathbb{K}^{\mathbb{N}}$  satisfy  $\alpha_n^k \xrightarrow{n \rightarrow +\infty} \alpha^k$  and  $\beta_n^l \xrightarrow{n \rightarrow +\infty} \beta^l$  with  $\alpha^k = \beta^l = 0$  for  $k \geq q_i + 1, l \geq q_j + 1$ . Therefore, for every  $n \in \mathbb{N}$ , we have

$$\sum \alpha_n^k \beta_n^l c_{kl}^{ij,s}(x_n) e_s^{(i+j-1)}(x_n) = \{u_n, v_n\}_2 \in \text{im}(d_{x_n}^{(i+j)}) = \ker(d_{x_n}^{(i+j-1)}). \tag{3.2}$$

We have used in (3.2) the fact that  $\{du_1, du_2\}_2 \in \text{im}(d)$  for all  $u_1, u_2 \in E_{\leq -2}$ . Since

$$\begin{aligned} \sum \alpha_n^k \beta_n^l c_{kl}^{ij,s}(x_n) e_s^{(i+j-1)}(x_n) &\xrightarrow{n \rightarrow +\infty} \sum \alpha^k \beta^l c_{kl}^{ij,s}(x) e_s^{(i+j-1)}(x) \in E_{-i-j+1}|_x \\ &= \{u, v\}_2. \end{aligned} \tag{3.3}$$

As a result,  $\{u, v\}_2 \in V_{i+j-1} \in \pi_{i+j-1}^{-1}(x)$ . So for every point  $(V_1, \dots, V_i, \dots, V_j, \dots) \in \pi_\infty^{-1}(x)$ , one has  $\{V_i, V_j\}_2 \subseteq V_{i+j-1}$ . This proves item (2). By taking  $i = j = 1$  and  $V_i = V_j = V \in \pi_1^{-1}(x)$ , equation (3.3) means that  $\{u, v\}_2 \in V$ . This proves item (1). ■

**3.2. Proof of Theorem 2.16**

In this section, we give a second proof of Theorem 2.16, which is interesting by itself, because it uses a method that we will use in the subsequent proofs. By Corollary 2.21 (whose proof is independent of Theorem 2.16), for every  $i \geq 1$ , we have an inclusion  $\tilde{M}_i \hookrightarrow \coprod_{x \in M} \text{Gr}_{-(r_i - \text{rk}(d_x^{(i)}))}(H^{-i}(\mathfrak{F}, x))$ , where  $r_i$  is defined as in Lemma 1.8 (d). We now need to show this inclusion is canonical, that is, independent of the choice of a geometric resolution  $(E, d, \rho)$ .

**Convention 3.1.** Let  $(E, d, \rho)$  be a geometric resolution of  $\mathfrak{F}$ . We denote by  $\pi_i^E: \tilde{M}_i^E \rightarrow M$  the Nash blowup space constructed out of a geometric resolution  $(E, d, \rho)$  and  $\pi_i^{E'}: \tilde{M}_i^{E'} \rightarrow M$  the Nash blowup space constructed out of a geometric resolution  $(E', d', \rho')$  for  $i \geq 1$ . Also, for  $x \in M$  and  $V \in \pi_i^{-1}(x)$ , we denote by  $\bar{V}$  the image of  $V$  in  $\text{Gr}_{-(r_i - \text{rk}(d_x^{(i)}))}(H^{-i}(\mathfrak{F}, x))$ .

**Remark 3.2.** Let  $x \in M$ . Consider a minimal geometric resolution  $(E', d', \rho')$  of  $\mathfrak{F}$  at  $x$  (see definition (1)). For  $V \in (\pi_1^E)^{-1}(x)$  and  $V' \in (\pi_1^{E'})^{-1}(x)$ , one has that  $\dim V' \leq \dim V$ , because  $\text{rk}(E'_-1) \leq \text{rk}(E_-1)$  by minimality. Hence,  $V, V'$  do not necessarily belong to the same Grassmannian. However,  $\dim \bar{V} = \dim \bar{V}'$ . We prove the latter in the next lemma.

**Lemma 3.3.** *Let  $(E, d, \rho)$  and  $(E', d', \rho')$  be geometric resolutions of  $\mathfrak{F}$ . For all  $i \geq 1$ , and for all  $V \in (\pi_i^E)^{-1}(x)$  and  $V' \in (\pi_i^{E'})^{-1}(x)$ , one has  $\dim \bar{V} = \dim \bar{V}'$ .*

*Proof.* If  $x \in M$  is a regular point, then  $\bar{V} = \bar{V}' = \{0\}$ . Thus, the equality holds. Let  $x \in M$  be a singular point. We prove it only for  $i = 1, 2$ , since  $i = 1$  is a special case and for  $i \geq 3$ , the proof uses a similar argument as for the one of  $i = 2$ . The key point in the latter is, for every  $x \in M$ , the restriction of the complexes  $(E, d, \rho)$  and  $(E', d', \rho')$  at  $x$  is quasi-isomorphic. This implies that the codimension of  $\text{im}(d_x^{(i+1)})$  inside  $\ker d_x^{(i)}$ , respectively,  $\text{im}(d'_x^{(i+1)})$  inside  $\ker d'_x^{(i)}$ , is invariant.

Let  $V \in (\pi_1^E)^{-1}(x)$  and  $V' \in (\pi_1^{E'})^{-1}(x)$ . We have

$$\begin{aligned} \dim \bar{V} &= \dim V - \dim(\text{im}(d_x^{(2)})) \\ &= \dim V - (\dim \ker \rho_x - \dim \ker \rho'_x + \dim(\text{im}(d'_x^{(2)}))) \\ &= \dim V - \text{rk}(E_-1) + \text{rk}(E'_-1) - \dim(\text{im}(d'_x^{(2)})) \\ &= \dim V' - \dim(\text{im}(d'_x^{(2)})) \\ &= \dim \bar{V}'. \end{aligned}$$

We have used the fact that the cohomology groups at degree  $-1$  of both complexes are isomorphic and the rank-nullity theorem.

For  $i = 2$ , let  $V \in (\pi_2^E)^{-1}(x)$  and  $V' \in (\pi_2^{E'})^{-1}(x)$ . Notice that  $\dim V = \text{rk}(E_{-2}) - \text{rk}(E_{-1}) + r$ . We have a similar formula for  $\dim V'$ . By direct computation, we find that

$$\begin{aligned} \dim \bar{V} &= \dim V - \dim(\text{im}(d_x^{(3)})) \\ &= \dim V - \text{rk}(E_{-2}) + \text{rk}(E'_{-2}) + \dim(\text{im}(d_x^{(2)})) \\ &\quad - \dim(\text{im}(d_x'^{(2)})) - \dim(\text{im}(d_x'^{(3)})). \end{aligned} \tag{3.4}$$

We have used the fact that the cohomology groups at degree  $-2$  of both complexes are isomorphic and the rank-nullity theorem. But

$$\dim(\text{im}(d_x^{(2)})) = \text{rk}(E_{-1}) - \dim(\text{im}(\rho_x)) - \dim W,$$

where  $W$  is such that  $\dim(\text{im}(d_x^{(2)})) \oplus W = \ker \rho_x$ . A similar formula holds for  $\dim(\text{im}(d_x'^{(2)}))$  by adding  $'$  everywhere. Substituting them into equation (3.4), we obtain

$$\dim \bar{V} = \dim \bar{V}' + \dim W' - \dim W = \dim \bar{V}',$$

since  $\dim W' = \dim W$ . ■

*Proof of (Theorem 2.16).* For simplicity, we prove it for  $i = 1$ . For  $i \geq 1$ , the same arguments hold.

Let  $(E, d, \rho)$  and  $(E', d', \rho')$  be geometric resolutions of  $\mathfrak{F}$ . There exist chain morphisms  $\varphi: E \rightarrow E'$  and  $\psi: E' \rightarrow E$  whose compositions are homotopic to identity. In particular,  $\varphi, \psi$  induce well-defined isomorphisms  $\bar{\varphi}$  and  $\bar{\psi}$  at the level of cohomology which are inverse to each other. The latter is canonical (see [21, Lemma 4.1]). All we need to show is  $\bar{\varphi}$  sends  $\tilde{M}_1^E$  to  $\tilde{M}_1^{E'}$ .

Let  $x \in M$ . Let  $e_1, \dots, e_k$  be local sections around  $x$  of  $E_{-2}$  such that

$$\text{span}(d^{(2)}e_1|_x, \dots, d^{(2)}e_k|_x) = \text{im}(d_x^{(2)}).$$

There is a neighborhood  $U_x$  of  $x$  such that  $F_y := \text{span}(d^{(2)}e_1|_y, \dots, d^{(2)}e_k|_y) \subseteq \text{im}(d_y^{(2)})$  with  $y \in U_x$  is of constant rank. These sections define a vector bundle  $F$  on  $U_x$  and  $F_x = \text{im}(d_x^{(2)})$ . Likewise, by shrinking  $U_x$  if necessary, one considers the vector bundle  $F' \subseteq \text{im}(d'^{(2)})$  on a neighborhood  $U_x$  of  $x$  such that  $\varphi_y(F_y) \subseteq F'_y$ . Therefore, for every  $y \in U_x$ ,  $\varphi_y$  induces a map  $\hat{\varphi}_y$

$$\frac{\ker(\rho_y)}{F_y} \longrightarrow \frac{\ker(\rho'_y)}{F'_y}$$

which coincides with the isomorphism  $\bar{\varphi}_x: \frac{\ker(\rho_x)}{\text{im}(d_x^{(2)})} \xrightarrow{\cong} \frac{\ker(\rho'_x)}{\text{im}(d_x'^{(2)})}$  at  $x$ . The map  $\hat{\varphi}$  induces a well-defined map

$$\text{Gr}_{-r} \left( \frac{E_{-1}}{F} \right) \rightarrow \text{Gr}_{-r} \left( \frac{E'_{-1}}{F'} \right)$$

on a smaller open neighborhood  $U_x$  of  $x$ . Let  $V \in (\pi_1^E)^{-1}(x)$  and  $V' \in (\pi_1^{E'})^{-1}(x)$ , and let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of regular points in  $U_x$  converging to  $x$  such that  $\text{im}(d_{x_n}^{(2)}) = \ker \rho_{x_n}$  and  $\text{im}(d_{x_n}^{(2)}) = \ker \rho'_{x_n}$  converge to  $V$  and  $V'$ , respectively.

This implies that the sequence  $\frac{\ker \rho_{x_n}}{F_{x_n}}$  converges to

$$[V] = \frac{V}{F_x} = \frac{V}{\text{im}(d_x^{(2)})}$$

in  $\text{Gr}_{-r}(\frac{E-1}{F})$ . Since  $\widehat{\varphi}_{x_n}(\frac{\ker \rho_{x_n}}{F_{x_n}}) \subseteq \frac{\ker \rho'_{x_n}}{F'_{x_n}}$ , it follows that  $\widehat{\varphi}_x([V_x]) \subseteq [V'_x]$ , where  $V'_x$  is the limit of (a subsequence of)  $\ker \rho'_{x_n}$ . By Lemma 3.3,  $[V]$  and  $[V']$  have the same dimension; thus,  $\overline{\varphi}_x([V]) = [V']$ . Also,  $\overline{\psi}_x(V') = \overline{V}$  since  $\overline{\psi}_x$  and  $\overline{\varphi}_x$  are the inverse of each other. This defines the required map and completes the proof. ■

### 3.3. Proof of Theorems 2.25 and 2.28

Theorem 2.25 follows from Lemma 3.5 which itself requires Lemma 3.4. We prove those in the smooth context. Their proofs are similar in the holomorphic context. We recall that for  $p: E \rightarrow M$  a vector bundle over  $M$ , a linear vector field on  $E$  is a pair  $(Z, X) \in \mathfrak{X}(E) \times \mathfrak{X}(M)$  such that

$$\begin{array}{ccc} E & \xrightarrow{Z} & TE \\ p \downarrow & & \downarrow dp \\ M & \xrightarrow{X} & TM \end{array}$$

is a morphism of vector bundles (see, e.g., [32, p. 110]). Equivalently,

- (1)  $Z[C_{\text{lin}}^\infty(E)] \subset C_{\text{lin}}^\infty(E)$  and  $Z[p^*C^\infty(M)] \subset p^*C^\infty(M)$  or
- (2) the flows of  $Z$  on  $E$  are (local) vector bundle isomorphisms  $E \rightarrow E$  over the flows of  $X$  on  $M$ ,

where  $C_{\text{lin}}^\infty(E)$  is the subalgebra of smooth functions on  $E$  which are fiberwise linear. The latter is canonically isomorphic to  $\Gamma(E^*)$  as  $C^\infty(M)$ -modules. Notice in particular that a linear vector field is  $p$ -projectable to  $X$ .

**Lemma 3.4.** *A linear vector field on  $E \rightarrow M$  induces a vector field on  $\Pi: \text{Gr}_{-q}(E) \rightarrow M$  that is  $\Pi$ -projectable on  $M$ .*

*Proof.* Let  $(Z, X)$  be a linear vector field on  $E \rightarrow M$ . Its flow  $\phi_t^Z: E \rightarrow E$  is a vector bundle isomorphism over the flow  $\phi_t^X: M \rightarrow M$  of  $X$  whenever it is defined. Thus,  $\phi_t^Z$  induces a map  $\text{Gr}_{-q}(E) \rightarrow \text{Gr}_{-q}(E)$ ,  $V \mapsto \phi_t^Z(V)$  that we still denote by  $\phi_t^Z$ . Define  $\tilde{Z} \in \mathfrak{X}(\text{Gr}_{-q}(E))$  for all  $V \in \Pi^{-1}(x)$  by

$$\tilde{Z}(V) := \frac{d}{dt} \Big|_{t=0} c(t) \in T_V \text{Gr}_{-q}(E) \tag{3.5}$$

so that the flow  $\phi_t^{\tilde{Z}}: \text{Gr}_{-q}(E) \rightarrow \text{Gr}_{-q}(E)$  of  $\tilde{Z}$  at  $V \in \Pi^{-1}(x)$  is  $\phi_t^Z|_x(V)$ , where  $c(t) = \phi_t^Z|_x(V) \in \Pi^{-1}(\phi_t^X(x))$  for  $t$  in some interval  $I$ . Also,  $\tilde{Z}$  is  $\Pi$ -projectable to  $X$ , by construction. ■

**Lemma 3.5.** *For every  $X \in \mathfrak{F}$ , there exists for all  $i \geq 1$  a linear vector field  $(Z^i, X)$  on the vector bundle  $p_i: E_{-i} \rightarrow M$  and a linear vector field  $(Z^0, X)$  on  $p_0: E_0 := TM \rightarrow M$ ,  $p_i$ -projectable to  $X$ . Their flows are compatible with the complex of vector bundles*

$$\dots \xrightarrow{\ell_1=d(4)} E_{-3} \xrightarrow{\ell_1=d(3)} E_{-2} \xrightarrow{\ell_1=d(2)} E_{-1} \xrightarrow{\rho=d(1)} TM. \tag{3.6}$$

That is, the diagram below commutes for all  $i \geq 1$ :

$$\begin{array}{ccccc}
 & & M & \xrightarrow{\phi_t^X} & M \\
 & \nearrow & \parallel & & \parallel \nearrow \\
 E_{-i} & \xrightarrow{\phi_t^{Z^i}} & E_{-i} & & E_{-i} \\
 \downarrow d^{(i)} & & \parallel & \phi_t^X & \parallel \downarrow d^{(i)} \\
 & \nearrow & M & \xrightarrow{\phi_t^X} & M \\
 E_{-i+1} & \xrightarrow{\phi_t^{Z^{i-1}}} & E_{-i+1} & & E_{-i+1}
 \end{array} \tag{3.7}$$

where  $\phi_t^{Z^i}$  or  $\phi_t^X$  denotes the flow of  $Z^i$  or  $X$ , whenever defined. They induce vector fields  $\tilde{Z}^i$  on  $\text{Gr}_{-r_i}(E_{-i})$  such that

- (1)  $\tilde{Z}^i$  is tangent to  $\tilde{M}_i$ ,
- (2)  $\tilde{Z}^i$  projects onto  $X$ .

*Proof.* Consider  $(E, d = \ell_1, \ell_2, \rho)$  the graded almost Lie algebroid of  $\mathfrak{F}$  induced by a universal Lie  $\infty$ -algebroid  $(E, (\ell_k)_{k \geq 1}, \rho)$  (see Section 1.2 (2)). Let  $X \in \mathfrak{F}$  and  $i \geq 0$ . For  $i \neq 0$ , there exists a section  $v$  of the vector bundle  $p_i: E_{-i} \rightarrow M$  such that  $\rho(v) = X$ . Consider the linear vector field  $Z^i \in \mathfrak{X}(E_{-i})$  defined as follows:

$$Z^i[p_i^* f] := p_i^*(X[f]) \quad \forall f \in C^\infty(M), \tag{3.8}$$

$$Z^i_e[\alpha] := X[\langle \alpha, e \rangle] - \langle \alpha, \ell_2(v, e) \rangle \quad \forall \alpha \in \Gamma(E_{-i}^*), e \in \Gamma(E_{-i}). \tag{3.9}$$

For  $i = 0$ , one replaces  $\ell_2(v, e)$  in (3.9) by  $[X, Y]$  with  $Y \in \Gamma(E_0) = \mathfrak{X}(M)$ . Notice that  $Z^i$  depends on the choice of the graded almost Lie algebroid bracket  $\ell_2$  and  $X$ . The fact that diagram (3.7) commutes follows the exact same lines of the proof given for Proposition 2.2.11 in a preprint version of [24, p. 99]. Let us write it for the sake of completeness<sup>8</sup>. By construction, the vector fields  $(Z^i)_{i \geq 0}$  on  $E_{-i} \xrightarrow{p_i} M$  are  $p_i$ -related

<sup>8</sup>Those arguments could be applied almost word for word to the holomorphic context.

to  $X$ . This implies the vector field  $(Z^i, Z^{i-1}) \in \mathfrak{X}(E_{-i} \times E_{-i+1})$  is tangent to the fiber product  $E_{-i} \times_{p_i, M, p_{i-1}} E_{-i+1}$ . Thus,  $(Z^i, Z^{i-1})$  restricts to a linear vector field on  $E_{-i} \oplus E_{-i+1} \xrightarrow{p} M$  denoted by  $Z_{i,i-1}$ . The latter is given by the formula

$$Z_{i,i-1}[p^* f] := p^*(X[f]) \quad \forall f \in C^\infty(M),$$

$$Z_{i,i-1}[\xi](e \oplus e') := \begin{cases} X[\langle \xi, e \oplus e' \rangle] - \langle \xi, \ell_2(v, e) \oplus \ell_2(v, e') \rangle & \text{for } i \neq 1, \\ X[\langle \xi, e \oplus Y \rangle] - \langle \xi, \ell_2(v, e) \oplus [X, Y] \rangle & \text{for } i = 1, \\ & \text{and } e' = Y \in \mathfrak{X}(M) \end{cases}$$

for all  $\xi \in \Gamma((E_{-i} \oplus E_{-i+1})^*_{-i})$ ,  $e \in \Gamma(E_{-i})$ ,  $e' \in \Gamma(E_{-i+1})$  and is again  $p$ -related to  $X$ . We now consider for  $i \geq 1$  the graph

$$\text{Graph}(d^{(i)}) = \{(e, d^{(i)}(e)) \mid e \in E_{-i}\} \subset E_{-i} \oplus E_{-i+1}$$

of  $d^{(i)}: E_{-i} \rightarrow E_{-i+1}$  which is submanifold of  $E_{-i} \oplus E_{-i+1}$  with  $\rho = d^{(1)}: E_{-1} \rightarrow E_0 = TM$ . Let us check that  $Z_{i,i-1}$  is tangent to the submanifold  $\text{Graph}(d^{(i)})$ . This comes from the following items:

- The submanifold  $\text{Graph}(d^{(i)}) \subset E_{-i} \oplus E_{-i+1}$  is the zero locus of the ideal generated by the functions

$$\begin{aligned} \xi_\alpha: E_{-i} \oplus E_{-i+1} &\rightarrow \mathbb{R} \\ (e, e') &\mapsto \langle \alpha, d^{(i)}(e) - e' \rangle \end{aligned}$$

with  $\alpha \in \Gamma(E_{-i+1}^*)$ .

- For all  $(e, e') \in \text{Graph}(d^{(i)})$ ,

$$\begin{aligned} Z_{i,i-1}[\xi_\alpha](e, e') &= X[\langle \xi_\alpha, e \oplus e' \rangle] - \langle \xi_\alpha, \ell_2(v, e) \oplus \ell_2(v, e') \rangle \\ &= X[\underbrace{\langle \alpha, d^{(i)}(e) - e' \rangle}_{=0}] - \langle \alpha, d^{(i)}\ell_2(v, e) - \ell_2(v, e') \rangle \\ &= -\langle \alpha, \ell_2(v, d^{(i)}(e)) - \ell_2(v, e') \rangle \\ &= -\langle \alpha, \ell_2(v, d^{(i)}(e) - e') \rangle = 0, \end{aligned}$$

where we have used the compatibility condition of the  $d = \ell_1, \ell_2$  brackets of the graded almost Lie algebroid.

Now, let  $\mathcal{U}, \mathcal{V} \subseteq M$  be the open subsets of  $M$  and  $I = (-\varepsilon, \varepsilon) \subseteq \mathbb{R}$  an interval of  $\mathbb{R}$  where the flow  $\phi_t^X: \mathcal{U} \rightarrow \mathcal{V}$  of  $X$  at time  $t$  is defined. Recall that the flow  $\phi_t^{Z^i}: E_{-i}|_{\mathcal{U}} \rightarrow E_{-i+1}|_{\mathcal{V}}$  of  $Z^i$  and the flow  $\phi_t^{Z^{i-1}}: E_{-i+1}|_{\mathcal{U}} \rightarrow E_{-i+1}|_{\mathcal{V}}$  of  $Z^{i-1}$  are vector bundle isomorphisms over  $\phi_t^X: \mathcal{U} \rightarrow \mathcal{V}$ . The flow of  $Z_{i,i-1}$  is also a vector bundle isomorphism given by the formula

$$\begin{aligned} (E_{-i} \oplus E_{-i+1})|_{\mathcal{U}} &\rightarrow (E_{-i} \oplus E_{-i+1})|_{\mathcal{V}} \\ (e, e') &\mapsto (\phi_t^{Z^i}(e), \phi_t^{Z^{i-1}}(e')). \end{aligned}$$

Now, since  $Z_{i,i-1}$  is tangent to  $\text{Graph}(d^{(i)})$ , its flow preserves  $\text{Graph}(d^{(i)})$ , that is, for all  $e \in \Gamma(E_{-i})$ ,

$$(\phi_t^{Z^i}(e), \phi_t^{Z^{i-1}}(d^{(i)}(e))) \in \text{Graph}(d^{(i)}).$$

This implies that diagram (3.7) commutes. Therefore, the family  $(\Phi_t^{Z^i})_{i \geq 0}$  is an isomorphism of complex of vector bundles. This proves the first part of Lemma 3.5.

Now, by Lemma 3.4, the linear vector field  $(Z^i, X)$  induces a vector field  $\tilde{Z}^i$  on the Grassmann bundle  $\text{Gr}_{-r_i}(E_{-i})$ . Let us show item (1),  $\phi_t^{Z^i}$  preserves  $\tilde{M}_i$ : to see this, take  $V \in \pi_i^{-1}(x)$ , let  $x_n \xrightarrow{n \rightarrow +\infty} x$  be such that  $\text{im } d_{x_n}^{(i+1)} \xrightarrow{n \rightarrow +\infty} V$  with  $(x_n) \subset M_{\text{reg}^i, \mathfrak{F}}$ . By using the identity

$$d^{(i+1)} \circ \phi_t^{Z^{i+1}} = \phi_t^{Z^i} \circ d^{(i+1)}$$

for  $i \geq 0$ , one has

$$\phi_t^{Z^i}|_{x_n}(\text{im } d_{x_n}^{(i+1)}) = \text{im } d_{\phi_t^X(x_n)}^{(i+1)} \quad \text{for every } n \in \mathbb{N}_0.$$

Thus,

$$\begin{aligned} \phi_t^{Z^i}|_x(V) &= \lim_{n \rightarrow +\infty} \phi_t^{Z^i}|_{x_n}(\text{im } d_{x_n}^{(i+1)}) \\ &= \lim_{n \rightarrow +\infty} (\text{im } d_{\phi_t^X(x_n)}^{(i+1)}) \in \pi_i^{-1}(\phi_t^X(x)). \end{aligned}$$

Hence, the flow of  $\tilde{Z}_i$  preserves  $\tilde{M}_i$ , that is,  $\tilde{Z}^i$  is tangent to  $\tilde{M}_i$ . ■

*Proof (of Theorem 2.25).* By Lemma 3.5, every vector field  $X \in \mathfrak{F}$  extends to a linear field  $X^i \in \mathfrak{X}(\text{Gr}_{-r_i}(E_{-i}))$  which is tangent to  $\tilde{M}_i$  in the sense of Definition 1.1.1 (1). This proves item (1). Furthermore, the restriction  $\tilde{X}_i$  of  $X^i$  to  $\tilde{M}_i$  is unique, since  $\pi_i|_{\pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}})}: \pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}}) \rightarrow M_{\text{reg}^i, \mathfrak{F}}$  is invertible. In particular, the map  $X \in \mathfrak{F} \rightarrow \tilde{X}|_{\tilde{M}_i}$  does not depend on any choices and is a Lie algebra morphism. The module which is generated by the  $\tilde{X}_i$  is closed under Lie bracket by item (2) of Theorem 2.25. This ends the proof. ■

*Proof (of Theorem 2.28).* Let  $(E, d, \rho)$  be a geometric resolution of  $\mathfrak{F}$ . Fix a universal Lie  $\infty$ -algebroid of  $\mathfrak{F}$  on  $(E, d, \rho)$  and  $k \geq 1$ . Let  $\tau^{E-k}$  and  $A^{E-k}$  be the tautological subbundle and tautological quotient bundle on  $\text{Gr}_{-r}(E_{-k})$  that fit into the exact sequence

$$\begin{array}{ccccc} \tau^{E-k} \hookrightarrow & \Pi_k^* E_{-k} & \twoheadrightarrow & A^{E-k} & \\ \downarrow & \downarrow & & \downarrow & \\ \text{Gr}_{-r_k}(E_{-k}) & = & \text{Gr}_{-r_k}(E_{-k}) & = & \text{Gr}_{-r_k}(E_{-k}) \end{array} \tag{3.10}$$

with  $A^{E-k} \simeq \Pi^* E_{-k} / \tau^{E-k}$ . In particular, for  $k = 1$ ,  $\text{rk}(A^{E-1})$  is the dimension of the regular leaves. One has

(1)  $\tilde{\mathfrak{F}}_k$  the image of an almost Lie algebroid on  $\Pi_k^* E_{-1}|_{\tilde{M}_1}$  through the anchor map

$$\tilde{\rho}: \Gamma(\Pi_k^* E_{-1})|_{\tilde{M}_k} \longrightarrow \mathfrak{X}(\tilde{M}_k)$$

defined by  $\pi_k^* e \mapsto \widetilde{\rho}(e) \in \tilde{\mathfrak{F}}_k$ .

(2) The tautological subbundle  $\tau^{E-k}$  lies in the kernel of the differential map  $d^{(k)}: E_{-k} \rightarrow E_{-k+1}$ : indeed, the fiber of  $\tau^{E-k}$  over a point  $V \in \pi_k^{-1}(x)$  is equal to  $V$  by definition. By Proposition 2.17, the latter is included in  $\ker d_x^{(k)}$  with equality if  $x \in M_{\text{reg}^k, \mathfrak{F}}$ . Also, for  $k = 1$ ,  $\tau^{E-1}$  lies in the kernel of the anchor map  $\rho = d^{(1)}$ .

Therefore, the pullback  $\pi_k^* d^{(k)}: \pi_k^* E_{-k} \rightarrow \pi_k^* E_{-k+1}$  goes to quotient to a well-defined vector bundle morphism

$$\pi_k^* d^{(k)}: \frac{\pi_k^* E_{-k}}{\tau^{E-k}} \rightarrow \pi_k^* E_{-k+1}$$

which is injective on the open dense subset  $\pi_k^{-1}(M_{\text{reg}^k, \mathfrak{F}})$  of  $\tilde{M}_k$ . Denote by  $K \rightarrow \tilde{M}_k$  the restriction of  $\tau^{E-k}$  to  $\tilde{M}_k$ . The  $k$ -th truncation of the pullback of the universal Lie  $\infty$ -algebroid of  $\mathfrak{F}$  to  $\tilde{M}_k$  induces naturally a Lie  $k$ -algebroid on

$$\frac{\pi_k^* E_{-k}}{K} \longrightarrow \pi_k^* E_{-k+1} \longrightarrow \dots \longrightarrow \pi_k^* E_{-1} \longrightarrow T\tilde{M}_k.$$

For  $k = 1$ , the anchor map  $\tilde{\rho}$  goes to quotient

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \pi_1^* E_{-1} & \longrightarrow & A^{E-1}|_{\tilde{M}_1} \longrightarrow 0 \\ & & & & \downarrow \tilde{\rho} & \swarrow & \\ & & & & T\tilde{M}_1 & & \end{array} \tag{3.11}$$

and makes  $\tilde{\mathfrak{F}}_1$  the image of an almost Lie algebroid on  $A^{E-1}|_{\tilde{M}_1}$  whose anchor is injective on the open dense subset  $M_{\text{reg}, \mathfrak{F}}$ . Thus,  $A^{E-1}|_{\tilde{M}_1}$  is a Lie algebroid whose anchor is injective on  $\pi_1^{-1}(M_{\text{reg}, \mathfrak{F}})$ , whose image is  $\tilde{\mathfrak{F}}_1$ . This proves the result. ■

**Remark 3.6.** Notice that in the proof of Corollary 2.29 we do not need the existence of a geometric resolution, we only make use of the anchor map and the bracket of an almost Lie algebroid of  $\mathfrak{F}$ , that is, we only need  $E_{-1}$  and  $\rho: E_{-1} \rightarrow TM$ .

### 4. Examples

Let us start with some examples where our constructions give nothing new, that is,  $\tilde{M}_i \simeq M$  or  $\tilde{M}_\infty \simeq M$ .

**Example 4.1.** If  $\mathfrak{F}$  is a Debord singular foliation (i.e.,  $\mathfrak{F}$  is a projective submodule of  $\mathfrak{X}(M)$ ), then  $\tilde{M}_i \simeq M$  for all  $i \geq 1$  and  $i = +\infty$ . This comes from the fact that there exists a vector bundle  $E_{-1} \rightarrow M$  such that  $\Gamma(E_{-1}) \simeq \mathfrak{F}$  by Serre–Swan theorem [35,46]. This isomorphism is given by a vector bundle morphism,  $E_{-1} \xrightarrow{\rho} TM$  which is injective on the open dense subset  $M_{\text{reg},\mathfrak{F}}$ . As a consequence,  $\dots \rightarrow 0 \rightarrow 0 \rightarrow E_{-1} \xrightarrow{\rho} TM$  is a geometric resolution of  $\mathfrak{F}$ . Therefore,  $\tilde{M}_{i \geq 2} \simeq M$  since  $E_{-i} = 0$  for  $i \geq 2$ . Also, if  $r$  is the dimension of the regular leaves of  $\mathfrak{F}$ , then  $r = \text{rk}(E_{-1})$ . Hence,  $\text{Gr}_{-r}(E_{-1}) \simeq M$ . In particular,  $\tilde{M}_1 \simeq M$ .

**Example 4.2.** If the regular leaves of  $\mathfrak{F}$  are open, then  $\tilde{M}_0 \simeq M$ , since  $\text{Gr}_{-0}(TM) \simeq M$ . For instance, this happens for  $\mathfrak{F}$ , the singular foliation on  $\mathbb{R}^N$  of vector fields vanishing at zero.

**Example 4.3.** If there exists a geometrical resolution  $(E, d, \rho)$  of length  $k$ , then  $\tilde{M}_i \simeq M$  for all  $i \geq k + 1$ . Notice that one also has  $\tilde{M}_k \simeq M$  since the last differential map  $d^{(k)}: E_{-k} \rightarrow E_{-k+1}$  is injective on an open dense subset so that the considered Grassmann bundle is  $\text{Gr}_{-\text{rk}(E_{-k})}(E_{-k}) \simeq M$ .

In contrast with Examples 4.1–4.3, we have other examples where our construction is not trivial.

**Example 4.4.** Let  $(M, \mathfrak{F})$  be a singular foliation admitting a geometric resolution of length 2

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{\rho} TM.$$

Here,  $\text{im}(d^{(2)})|_{M_{\text{reg},\mathfrak{F}}}$  is a vector bundle of rank  $\text{rk}(E_{-2})$ . On the open dense subset of regular points  $M_{\text{reg},\mathfrak{F}}$ , the map  $d^{(2)}: E_{-2} \rightarrow E_{-1}$  is injective, and on  $M_{\text{sing}} = M \setminus M_{\text{reg},\mathfrak{F}}$ , it is not. For simplicity, assume that  $M = \mathbb{R}^N$  or  $\mathbb{C}^N$  and that the vector bundles  $E_{-2}, E_{-1}$  are trivial so that  $d^{(2)}$  becomes an  $\text{rk}(E_{-1}) \times \text{rk}(E_{-2})$ -matrix with coefficient in the algebra of functions on  $M$ . The zero locus of the ideal  $\mathcal{I}_{\mathfrak{b}}$  generated by the minors of this matrix in a basis is exactly  $M_{\text{sing}}$ . By construction, the Nash blowup  $\tilde{M}_1$  is the blowup of  $M$  along the ideal  $\mathcal{I}_{\mathfrak{b}}$ .

For instance, let  $M = \mathfrak{gl}_d(\mathbb{K})$  be the vector space of  $d \times d$ -matrix with coefficient in  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ . Let  $(M, \mathfrak{F})$  be the singular foliations given by the adjoint action of  $\mathfrak{gl}_d(\mathbb{K})$  on  $\mathfrak{gl}_d(\mathbb{K})$ , that is,

$$\text{ad}(x)y = [x, y], \quad x, y \in \mathfrak{gl}_d(\mathbb{K}).$$

The singular foliation  $\mathfrak{F}$  admits a geometric resolution of length 2 (see [21, Example 3.32]) with

$$M \times \mathbb{K}^d \xrightarrow{d^{(2)}} M \times \mathfrak{gl}_d(\mathbb{K}), \quad (x, (\lambda_0, \dots, \lambda_{d-1})) \mapsto \left( x, \sum_{i=0}^{d-1} \lambda_i x^i \right)$$

and

$$M \times \mathfrak{gl}_d(\mathbb{K}) \xrightarrow{\rho} TM \simeq M \times \mathfrak{gl}_d(\mathbb{K}), \quad (x, v) \mapsto (x, [x, v]).$$

The open dense subset of regular points of  $(M, \mathfrak{F})$  is the set of matrices  $x \in M$  whose centralizer  $C(x) := \ker \rho_x$  is of minimal dimension equal to  $N$ . Equivalently,  $M_{\text{reg}, \mathfrak{F}}$  is made of the matrices  $x \in M$  whose characteristic polynomial is equal to the minimal polynomial, also known as non-derogatory matrices [48]. For  $d = 2$ ,  $\tilde{M}_1 \simeq \text{Bl}_{\mathcal{I}_b}(\mathbb{K}^4)$  is the usual blowup of  $\mathbb{K}^4$  along the ideal  $\mathcal{I}_b$  generated by  $\{x_1 - x_4, x_2, x_3\}$ , which is smooth.

For  $d \geq 3$ , computations become complicated, and the singular locus is a cone.

**Example 4.5.** The Nash blowup can be smooth, even if the singular locus is not. In the case of the adjoint action of  $\mathfrak{su}(n)$ , the singular locus is not smooth, but the blowup is smooth (see [31, Example 3.11]).

**Example 4.6.** Consider the projective singular foliation  $\mathfrak{F}$  on  $M = \mathbb{C}^N$  generated by the Euler vector field  $\vec{E} = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}$ . Here,  $M_{\text{reg}, \mathfrak{F}} = \mathbb{C}^N \setminus \{0\}$ . It is easily checked that  $\tilde{M}_0$  is the closure of the graph  $\{(x, [x_1 : \dots : x_N]) \in \mathbb{C}^N \times \mathbb{P}^{N-1}(\mathbb{C}) \mid x \neq 0\}$ . The latter is the blowup of  $\mathbb{C}^N$  at 0. This is an example where  $\mathfrak{F}$  is Debord and  $\tilde{M}_0 \neq M$ . In particular, by Example 4.1,  $\tilde{M}_0 \neq \tilde{M}_1 = M$ .

**Example 4.7.** Let  $\mathfrak{F}$  be the singular foliation of all vector fields vanishing at the origin  $0 \in M = \mathbb{C}^N$ . Here,  $M_{\text{reg}, \mathfrak{F}} = \mathbb{C}^N \setminus \{0\}$ . Let us compute  $\tilde{M}_1$ . A geometric resolution  $(E, d, \rho)$  of  $\mathfrak{F}$  is given in [21, Example 3.34]. Here,  $E_{-1} \simeq \mathbb{C}^N \times \mathfrak{gl}_N(\mathbb{C})$  and the anchor map  $\rho$  is  $E_{ij} \mapsto x_i \frac{\partial}{\partial x_j}$ , where  $\mathfrak{gl}_N(\mathbb{C})$  is the vector space of  $N \times N$  matrix with coefficient in  $\mathbb{C}$  and  $(E_{ij})_{i,j=1,\dots,N}$  its canonical basis.

A direct computation for every  $x \neq 0$  tells that  $\ker \rho_x$  is the subspace of matrices  $M \in \mathfrak{gl}_N(\mathbb{C})$  such that  $Mx = 0$ , where  $x = (x_1, \dots, x_N)$  is seen as a column vector. Equivalently, this kernel can be described as  $N$  copies of  $[x_1 : \dots : x_N]^\perp$ . Hence,  $\tilde{M}_1$  is the blowup of  $\mathbb{C}^N$  at the origin. This is an example of a singular foliation whose regular leaves are open, but such that  $\tilde{M}_1 \neq M$ . In particular, by Example 4.2,  $\tilde{M}_0 \neq \tilde{M}_1$ .

Here is an example related to Poisson manifolds.

**Example 4.8.** Let  $(M, P)$  be a smooth or holomorphic Poisson manifold with  $P \in \Gamma(\wedge^2 TM)$ . Consider the singular foliation generated by the Hamiltonian vector fields associated to  $P$ , that is,  $\mathfrak{F} = P^\#(\Gamma(T^*M))$ , where  $P^\#: T^*M \rightarrow TM, \alpha \mapsto P(\alpha, \cdot)$ . Assume that a geometric resolution exists. By Lemma 3.5, every Hamiltonian vector field lifts to a vector field tangent to  $\tilde{M}_i, i \geq 1$ . It is natural to ask whether the Poisson bivector field  $P$  lifts to  $\tilde{M}_i$ . Assume that  $\tilde{M}_i$  is smooth. Since for every  $i \geq 1, \pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}}) \rightarrow M_{\text{reg}^i, \mathfrak{F}}$  is invertible, the restriction  $P|_U$  lifts to a Poisson bivector field on  $\pi_i^{-1}(M_{\text{reg}^i, \mathfrak{F}})$ . However, it does not lift to  $\tilde{M}_i$  in general, even when  $\tilde{M}_i$  is smooth. Indeed, consider the Poisson manifold  $M = \mathfrak{so}^*(3) \simeq \mathbb{R}^3$  with

$$P = x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}. \tag{4.1}$$

Here,  $\mathfrak{F}$  is generated by the vector fields  $P^\#(dx) = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}$ ,  $P^\#(dy) = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}$ ,  $P^\#(dz) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ . Let us compute  $\tilde{M}_1$ . Given a point  $m \in M_{\text{reg}, \mathfrak{F}} = \mathbb{R}^3 \setminus \{0\}$ , we find that

$$\begin{aligned} \ker P^\#|_m &= \{(a, b, c) \in \mathbb{R}^3 \mid (a, b, c) \in [x(m) : y(m) : z(m)] \in \mathbb{P}^2(\mathbb{R})\} \\ &= [x(m) : y(m) : z(m)]. \end{aligned}$$

Hence,  $\tilde{M}_1$  is the usual blowup  $\text{Bl}_0(\mathbb{R}^3)$  of  $\mathbb{R}^3$  at the origin.

The bivector field  $P$  does not lift to  $\tilde{M}_1$ . Recall that the blowup of  $\mathbb{R}^3$  at the origin  $\text{Bl}_0(\mathbb{R}^3) \subset \mathbb{R}^3 \times \mathbb{P}^2$  is covered by three charts given by  $x \neq 0$ ,  $y \neq 0$  and  $z \neq 0$ . Let us look at the  $x$ -chart where the projection  $\pi_1$  becomes  $(x, y, z) \mapsto (x, xy, xz)$ . In this chart,  $P$  pulls back to

$$y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{1}{x}(1 + y^2 + z^2) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}. \tag{4.2}$$

For  $x = 0$ , equation (4.2) is not defined. In conclusion, the Hamiltonian vector fields of the Poisson structure  $P$  in (4.1) lift to  $\tilde{M}_1$ , but the bivector field  $P$  does not lift to  $\tilde{M}_1$ , although  $\tilde{M}_1$  is smooth.

**Example 4.9.** Let  $(E_{-1}, [\cdot, \cdot], \rho)$  be a Lie algebroid over a manifold  $M$  and denote by  $\mathfrak{F} = \rho(\Gamma(E_{-1}))$  the induced singular foliation. Assume there exist geometric resolutions for  $\mathfrak{F}$ . The Lie algebroid  $E_{-1}$  acts on the spaces  $\tilde{M}_i$  for all  $i \in \mathbb{N}_0$ , and also on  $\tilde{M}_\infty$ , and

$$\begin{array}{ccc} & & \mathfrak{X}(\tilde{M}_i) \\ & \nearrow \bar{\rho} & \uparrow \\ \Gamma(E_{-1}) & \xrightarrow{\rho} & \mathfrak{X}(M) \end{array} \tag{4.3}$$

is a commutative diagram of Lie algebra morphisms, where  $\bar{\rho}$  is defined on a local frame  $(e_k)_k$  of  $E_{-1}$  by  $e_k \mapsto \bar{\rho}(e_k)_i$ . Here,  $\tilde{\cdot}$  is as in Theorem 2.25. In addition, for each  $i \in \mathbb{N}_0$ ,  $\tilde{\mathfrak{F}}_i$  is the image of a Lie algebroid on  $\tilde{M}_i$ , namely the natural pullback of the Lie algebroid  $E_{-1}$  to  $\tilde{M}_i$ . In particular, if  $\mathfrak{F}$  is given by a Lie algebra action of a Lie algebra  $\mathfrak{g}$  on  $M$ , then  $\tilde{\mathfrak{F}}_i$  is given by an action of  $\mathfrak{g}$  on  $\tilde{M}_i$ .

Let us now study some examples related to the notion of an affine variety in  $\mathbb{C}^d$ .

Let  $\mathbb{A}^d$  be an affine space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  with a set of coordinates  $x_1, \dots, x_d$ . Recall that an *affine variety*  $W$  is a subset of the affine space  $\mathbb{A}^d$  given by the zero locus  $Z(\mathcal{I}_W)$  of a radical ideal  $\mathcal{I}_W \subseteq \mathbb{K}[x_1, \dots, x_d]$  and equipped with the induced Zariski topology of  $\mathbb{A}^d$ . The *coordinate ring* of  $W$  is the quotient ring  $\mathcal{O}_W = \mathbb{K}[x_1, \dots, x_d] / \mathcal{I}_W$ . The Lie algebra  $\mathfrak{X}(W)$  of *vector fields* on  $W$  are derivations of  $\mathcal{O}_W$ . We denote by  $W_{\text{reg}}$  the set of regular points of  $W$ . For every  $x \in \mathbb{A}^d$ , we denote by  $\mathfrak{m}_x$  the maximal ideal of vanishing polynomials at  $x$ . See, for instance, [16] for more details on these notions.

**Example 4.10.** Let  $M = \mathbb{C}^d$  and  $\varphi \in \mathbb{C}[x_1, \dots, x_d]$ . Consider the singular foliation  $\mathfrak{F}_\varphi = \{X \in \mathfrak{X}(\mathbb{C}^d) \mid X[\varphi] = 0\}$ . In this case,  $M_{\text{reg}, \mathfrak{F}_\varphi} = \{x \in \mathbb{C}^d, \mid d_x\varphi \neq 0\}$ . For every  $y \in \mathbb{C}^d$ ,  $(T_y\mathfrak{F}_\varphi)^\perp = \langle \nabla_y\varphi \rangle$ . For a convergent sequence of regular points,  $y_n \xrightarrow{n \rightarrow +\infty} y$ , that is, with  $y_n \in M_{\text{reg}, \mathfrak{F}_\varphi}$ , the sequence  $\text{im}(\rho_{y_n}) = T_{y_n}\mathfrak{F}_\varphi$  converges if and only if  $\nabla_{y_n}\varphi$  converges in  $\text{Gr}_{-(d-1)}(\mathbb{C}^d)$ , that is,  $[\frac{\partial\varphi}{\partial x_1}(y_n) : \dots : \frac{\partial\varphi}{\partial x_d}(y_n)]$  converges in the projective space  $\mathbb{P}^{d-1}(\mathbb{C})$ . Therefore,  $\tilde{M}_0$  is the closure of the image of the map,  $y \mapsto (y, [\frac{\partial\varphi}{\partial x_1}(y) : \dots : \frac{\partial\varphi}{\partial x_d}(y)])$  which is the blowup of  $\mathbb{C}^d$  along the singular locus of  $\varphi$ , that is, along the ideal generated by the components of  $d\varphi$ . For instance,

- (1) For  $\varphi(x_1, \dots, x_d) = \sum_{i=1}^d x_i^2$ ,  $\tilde{M}_0$  is the blowup of  $\mathbb{C}^d$  along the ideal  $\langle x_1, \dots, x_d \rangle$ , that is, the blowup of  $\mathbb{C}^d$  at zero, which is smooth.
- (2) For  $\varphi(x_1, \dots, x_d) = \sum_{i=1}^d x_i^3$ ,  $\tilde{M}_0$  is the blowup of  $\mathbb{C}^d$  along the ideal  $\langle x_1^2, \dots, x_d^2 \rangle$ . This is not the blowup of  $\mathbb{C}^d$  at zero, and it is easily seen in the charts that are not smooth with a singularity at the origin.

However, since the ideals  $\langle x_1, \dots, x_d \rangle$  and  $\langle x_1^2, \dots, x_d^2 \rangle$  are related by

$$\langle x_1, \dots, x_d \rangle^{d-1} \langle x_1^2, \dots, x_d^2 \rangle = \langle x_1, \dots, x_d \rangle^{d+1}$$

and since the blowup of  $\mathbb{C}^d$  along the ideals  $\langle x_1, \dots, x_d \rangle$  and  $\langle x_1, \dots, x_d \rangle^{d+1}$  is the same, there is a map

$$\begin{array}{ccc} \text{Bl}_{\langle x_1, \dots, x_d \rangle}(\mathbb{C}^d) & \xrightarrow{\quad} & \tilde{M}_0 = \text{Bl}_{\langle x_1^2, \dots, x_d^2 \rangle}(\mathbb{C}^d) \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{C}^d & \xrightarrow{\quad = \quad} & \mathbb{C}^d \end{array}$$

by Moody’s criteria (see Section 2.1.2).

**Example 4.11** (Nash modification). Let  $M = W$  be an affine irreducible affine variety of dimension  $r$  embedded in  $\mathbb{C}^d$ . Let  $\Sigma$  be its singular locus. Let  $\mathfrak{F} = \text{Der}(\mathcal{O}_W)$  be the singular foliation of vector fields on  $W$  tangent to  $\Sigma$ , where  $\mathcal{I}_\Sigma$  stands for the polynomial functions that vanish on  $\Sigma$ . Here,  $W_{\text{reg}, \mathfrak{F}} = W_{\text{reg}} = W \setminus \Sigma$ . Consider a geometric resolution  $(E_\bullet, d, \rho)$  of  $\mathfrak{F}$  by trivial vector bundles (which exists because  $\mathcal{O}_W$  is Noetherian, see [21, Section 3.3]).

Let us show that for every  $x \in W \setminus \Sigma$ ,  $\text{im}(\rho_x) = T_x\mathfrak{F} = T_xW$ . It is clear that  $\text{im}(\rho_x) \subseteq T_xW$ . Conversely, it is a classical property that  $x \in W$  is a regular point if and only if there exist “local coordinates”  $y_1, \dots, y_d \in \mathcal{O}_x$  such that  $W$  is of the form

$$y_1 = \dots = y_k = 0,$$

that is, the localization of  $\mathcal{I}_W$  is generated by these variables, where  $\mathcal{O}_x$  denotes the local ring at  $x$ . Hence, the tangent space of  $W$  at  $x$  is the vector space,  $\text{span}\{\frac{\partial}{\partial y_i}|_m, i \geq k + 1\}$ . Therefore, for  $v \in T_xW$ , the local vector field

$$X = \sum_{i=1}^{\dim W} v_i \frac{\partial}{\partial y_{k+i}}$$

maps  $\mathcal{O}_x$  to  $\mathcal{O}_x$ ; in particular, it maps  $\mathcal{O}$  to  $\mathcal{O}_x$ , and we have  $X[\mathcal{I}_W] \subset (\mathcal{I}_W)_{m_x}$ . Therefore, for every  $i \in \{1, \dots, d\}$ , there exists a polynomial function  $g_i$  that does not vanish at  $x$  such that  $g_i Y[x_i] \in \mathbb{C}[x_1, \dots, x_d]$ . By construction, the vector field  $\hat{X} = \frac{g_1 \cdots g_r}{g_1(x) \cdots g_r(x)} X$  is tangent to  $W$ , that is,  $\hat{X}[\mathcal{I}_W] \subset \mathcal{I}_W$ , and satisfies  $\hat{X}(x) = v$ .

The map  $\pi_0: W \setminus \Sigma \rightarrow \text{Gr}_{-(d-r)}(\mathbb{C}^d)$   $x \mapsto \text{im}(\rho_x) = T_x W$  is the so-called Gauss map [47]. The Zariski closure  $\tilde{W}_0$  of the image of such a map is by definition the classical Nash blowup of  $W$  along its singular locus  $\Sigma$ .

**Example 4.12 (Monoidal transformation).** Let  $W = \mathbb{R}^d$  or  $\mathbb{C}^d$ . Let  $\mathcal{I} \subseteq \mathcal{O}_W$  be an ideal, and let  $C = Z(\mathcal{I}) \subset \mathbb{C}^d$  be the zero locus of the ideal  $\mathcal{I}$ . Let  $\mathfrak{F} = \mathcal{I}\mathfrak{X}(W)$  be the singular foliation of vector fields vanishing along  $C$ . By Hilbert’s syzygy theorem [10], there exists a free resolution of finite length for the ideal  $\mathcal{I}$  of polynomial functions vanishing on  $C$  of the form

$$\cdots \longrightarrow K_{-2} \xrightarrow{\partial} K_{-1} \xrightarrow{\partial} \mathcal{I} \longrightarrow 0. \tag{4.4}$$

Since  $\mathfrak{X}(W)$  is a flat  $\mathcal{O}_W = \mathbb{C}[x_1, \dots, x_d]$ -module (in fact,  $\mathfrak{X}(W) \simeq \mathcal{O}_W^d$  is a free module), the sequence

$$\cdots \xrightarrow{d=\partial \otimes \text{id}} K_{-2} \otimes_{\mathcal{O}_W} \mathfrak{X}(W) \xrightarrow{d=\partial \otimes \text{id}} K_{-1} \otimes_{\mathcal{O}_W} \mathfrak{X}(W) \xrightarrow{\rho} \mathfrak{F} \tag{4.5}$$

is a free resolution  $\mathbb{K}[W]$  by finitely generated  $\mathbb{K}[W]$ -modules of the singular foliation  $\mathfrak{F} = \mathcal{I}\mathfrak{X}(W)$ , where for  $(\mu_1, \dots, \mu_k)$  a set of generators of  $K_{-1}$ , the anchor map is given by  $\rho(\mu_i \otimes \frac{\partial}{\partial y_j}) = \partial(\mu_i) \frac{\partial}{\partial y_j}$  for  $i = 1, \dots, k$  and  $j = 1, \dots, d$ . By [22, Theorem 2.1],  $\mathfrak{F}$  admits a universal Lie  $\infty$ -algebroid structure over the complex (4.5) whose unary bracket is  $\ell_1 = \partial \otimes \text{id}$  and whose anchor is  $\rho$ .

Here,  $W_{\text{reg}, \mathfrak{F}} = W \setminus C$ . For  $i = 1, \dots, k$ , let  $f_i := \partial(\mu_i) \in \mathcal{I}$ . A direct computation shows that for every  $x \in W \setminus C$ ,  $\ker \rho_x$  is equal to  $d$  copies of  $[f_1(x) : \cdots : f_k(x)]^\perp$ , that is,

$$\ker \rho_x = ([f_1(x) : \cdots : f_k(x)]^\perp)^d,$$

where  $[f_1(x) : \cdots : f_k(x)]$  is a well-defined straight line of  $\mathbb{K}^k$  generated by the vector  $(f_1(x), \dots, f_k(x)) \in \mathbb{K}^k$  seen as a point of the projective space  $\mathbb{P}^{k-1}(\mathbb{C}) = \text{Gr}_{-(k-1)}(\mathbb{C}^k)$ .

One has

$$\pi_1^{-1}(x) = \begin{cases} ([f_1(x) : \cdots : f_k(x)]^\perp)^d & \text{for } x \in W \setminus C, \\ V^d \in (\text{Gr}_{-1}(\mathbb{C}^k))^d & \text{such that } \exists (x_n) \in W_{\text{reg}, \mathfrak{F}}^{\mathbb{N}}, \\ & [f_1(x_n) : \cdots : f_k(x_n)]^\perp \xrightarrow{n \rightarrow +\infty} V, \\ & \text{with } V \in \text{Gr}_{-1}(\mathbb{C}^k) \text{ as } x_n \xrightarrow{n \rightarrow +\infty} x, \\ & \text{for } x \in C. \end{cases}$$

The  $d$  components converge if and only if one of them converges. Since  $[f_1(x_n) : \cdots : f_k(x_n)]^\perp$  converges in  $\text{Gr}_{-1}(\mathbb{K}^k)$  if and only if the straight line  $[f_1(x_n) : \cdots : f_k(x_n)]$

converges in  $\mathbb{P}^{k-1}(\mathbb{K})$ ,  $\tilde{W}_1$  corresponds to the usual monoidal transformation of  $W$  with respect to  $\mathcal{I}$  (see, for instance, [17] or Section 2.1.2). In particular,  $\tilde{W}_1$  does not depend, up to isomorphism over  $W$ , on the choice of the generators  $f_1, \dots, f_k$ .

When  $f_1, \dots, f_k$  form a regular sequence, let us prove that for each  $i \geq 1$ ,  $\tilde{W}_i$  is again the blowup of  $\mathbb{C}^d$  along  $\mathcal{I}$ . The complex in equation (4.4) can then be chosen to be the Koszul complex. Its dual complex is given by the differential map

$$\mathfrak{X}^\bullet(\mathbb{C}^k) \xrightarrow{\partial^*} \mathfrak{X}^{\bullet+1}(\mathbb{C}^k), \quad \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_p} \longmapsto \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_p} \wedge U,$$

where  $U = \sum_{\lambda=1}^k f_\lambda \frac{\partial}{\partial x_\lambda}$ . For a sequence of regular points  $(x_n)$  that converges to  $x \in C$ , it is easily checked that  $\text{im}(d_{x_n}^*)$  converges if and only if  $[f_1(x_n) : \dots : f_k(x_n)]$  does in the projective space. This proves the result. As a consequence,  $\tilde{W}_\infty$  is also the blowup of  $W = \mathbb{C}^d$  along  $\mathcal{I}$ .

### A. Grassmann bundles

For  $E$  a finite-dimensional vector space over a field  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , we denote by  $\text{Gr}_{-r}(E)$  the set of all vector subspaces of  $E$  of codimension  $r \in \mathbb{N}$ . Let us recall a few facts on  $\text{Gr}_{-r}(E)$ .

#### A.1. Topological structure

The set  $\text{Gr}_{-r}(E)$  is metric space, the corresponding metric is defined by

$$\delta(V, V') = \|P_V - P_{V'}\|, \tag{A.1}$$

where  $P_V$  stands for the orthogonal projection of  $E$  onto  $V \subset E$ . It is important to notice that for all  $V, V' \in \text{Gr}_{-r}(E)$ ,

$$\delta(V, V') = \delta(V^\perp, V'^\perp),$$

here  $V^\perp$  stands for the orthogonal space of  $V$ . It is proven (see, e.g., [12]) that  $\text{Gr}_{-r}(E)$  equipped with the topology induced by the so-called “gap” metric (A.1) is equivalent to the Grassmann topology, that is, the topology on  $\text{Gr}_{-r}(E)$  whose open subsets  $\mathcal{W} \subseteq \text{Gr}_{-r}(E)$  are such that  $\tau^{-1}(\mathcal{W})$  is open in  $\text{St}_r(d, \mathbb{K}) := \{A \in M_{d \times r}(\mathbb{K}) \mid \text{rk}(A) = r\}$ , with

$$\tau: \text{St}_r(d, \mathbb{K}) \longrightarrow \text{Gr}_{-r}(E), \quad A \longmapsto \{\text{vector space spanned by the columns of } A\}.$$

Also,  $\text{Gr}_{-r}(E)$  is a compact space.

#### A.2. Manifold structure

Moreover,  $\text{Gr}_{-r}(E)$  is a compact manifold of dimension  $r(d - r)$  and also a projective variety.

- (1) **Coordinates charts:** One manner to define the standard affine coordinates on the Grassmannian  $\text{Gr}_{-r}(E)$  is as follows. Fix a basis  $e_1, \dots, e_{d=\dim E}$  for  $E$ . Let us describe the first chart. Consider

$$\begin{aligned} \psi: M_{r,d-r}(\mathbb{K}) &\longrightarrow M_{d,d-r}(\mathbb{K}) \\ A' &\longmapsto \begin{pmatrix} I_{d-r} \\ A' \end{pmatrix}. \end{aligned}$$

The vector space  $V = \tau\left(\begin{pmatrix} I_{d-r} \\ A' \end{pmatrix}\right)$  admits a basis of the form

$$v_j := e_j + \sum_{k=1}^{\ell} a_{kj} e_k, \quad j = 1, \dots, d - r. \tag{A.2}$$

The vector space  $V$  is completely determined by the matrix  $A'$ . Hence,  $\tau \circ \psi$  is the first chart.

For a permutation  $\sigma \in \mathfrak{S}_d$ , let  $P(\sigma)$  be the permutation matrix of lines associated to  $\sigma$ . We claim that the family  $\tau \circ P(\sigma) \circ \psi(M_{r,d-r}(\mathbb{K}))$ , indexed by  $\sigma \in \mathfrak{S}_d$ , is an atlas of  $\text{Gr}_{-r}(E)$ . Its image consists of (A.2) up to permutation.

- (2) **Grassmann bundle:** Let  $E \rightarrow M$  be a vector bundle of rank  $d$  over a manifold  $M$  (or a quasi-projective variety<sup>9</sup>). Let  $r \leq d$ . The disjoint union

$$\text{Gr}_{-r}(E) := \coprod_{x \in M} \text{Gr}_{-r}(E|_x)$$

comes equipped with a natural manifold structure in the smooth or complex case and a quasi-projective variety structure when  $M$  is a quasi-projective variety. Also,

$$\Pi: \text{Gr}_{-r}(E) \longrightarrow M \tag{A.3}$$

is a fibration. It is called  $(d - r)$ -th *Grassmann bundle*.

For every open subset  $\mathcal{U} \subset M$  on which  $E$  is trivial,  $\Pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \text{Gr}_{-r}(\mathbb{K}^d)$ . An *adapted chart* for  $\text{Gr}_{-r}(E) \rightarrow M$  around a point  $x \in M$  is a set of local coordinates of the form  $(\Pi^*x_1, \dots, \Pi^*x_n, z_1, \dots, z_{r(d-r)})$ , where  $(x_1, \dots, x_n)$  are local coordinates on  $M$  and  $(z_1, \dots, z_{r(d-r)})$  are functions which are standard affine coordinates on an open subset of each fiber of  $\Pi$  as in item (1).

**Convention A.1.** Let  $x \in M$ . Let  $e_1, \dots, e_d$  be local frame for  $E$  in a neighborhood  $\mathcal{U}$  of  $x$ . For  $y \in \mathcal{U}$ , let  $\kappa_y$  be the linear isomorphism defined by

$$\kappa_y: E_x \longrightarrow E_y, \quad \kappa_y(e_i(x)) = e_i(y) \quad \text{for all } i \in \{1, \dots, d\}.$$

---

<sup>9</sup>The intersection inside some projective space of a Zariski-open and a Zariski-closed subset.

Let  $(x_n)$  be a sequence of  $M$  that converges to  $x$ . We will say that a sequence of vector space  $V_{x_n} \in \text{Gr}_{-r}(E)$  with  $V_{x_n} \subset E_{x_n}$  converges to  $V \subset E_x$  and write  $V_{x_n} \xrightarrow[n \rightarrow +\infty]{} V$  if

$$\kappa_{x_n}^{-1}(V_{x_n}) \xrightarrow[n \rightarrow +\infty]{} V \text{ in } \text{Gr}_{-r}(E_x).$$

In the sequel, we will not mention  $\kappa_{x_n}$ , since this notion of convergence does not depend on the chosen local frames of  $E$ .

- (3) **Tautological subbundle:** The Grassmann bundle  $\text{Gr}_{-r}(E)$  comes equipped with two vector bundles  $\tau^E$  and  $A^E$ , called tautological subbundle and tautological quotient bundle, that fit into the exact sequence

$$0 \longrightarrow \tau^E \longrightarrow \Pi^*E \longrightarrow A^E \longrightarrow 0. \tag{A.4}$$

Precisely, the fiber of  $\tau^E$  over the point  $V \in \Pi^{-1}(x)$  is the codimension  $r$  subvector space  $V$  of  $E|_x = E|_{\Pi(V)} = (\Pi^*E)|_V$ . By construction,  $\tau^E$  is a subbundle of the pullback bundle  $\Pi^*E$ . Furthermore,  $A^E \simeq \Pi^*E/\tau^E$ .

This tautological quotient bundle is important for us to express some results of this paper.

**Acknowledgments.** The main results of this paper are taken from Chapter 7 of my PhD thesis [29]. I would like to thank C. Laurent-Gengoux, my PhD supervisor, for supporting me in writing this article and directing me to such questions that originated from Claire Debord and Georges Skandalis. I also would like to thank the University of Lorraine for their financial support through an A.T.E.R. position (2022–2023). I gratefully acknowledge the full financial support for the joint postdoctoral position at Jilin University and University of Göttingen (2023–2025). I also sincerely thank the National Natural Science Foundation of China for awarding me the Research Fund for International Young Scientists (Grant number W2433013). I acknowledge the University of Illinois Urbana-Champaign for awarding me the J.L. Doob Research Assistant Professorship (2025–2028). Finally, I thank the anonymous referee for his/her careful reading of the manuscript.

## References

- [1] I. Androulidakis and G. Skandalis, [The holonomy groupoid of a singular foliation](#). *J. Reine Angew. Math.* **2009** (2009), no. 626, 1–37 Zbl [1161.53020](#) MR [2492988](#)
- [2] I. Androulidakis and M. Zambon, [Holonomy transformations for singular foliations](#). *Adv. Math.* **256** (2014), 348–397 Zbl [1294.53025](#) MR [3177296](#)
- [3] P. Baum and R. Bott, [Singularities of holomorphic foliations](#). *J. Differential Geom.* **7** (1972), no. 3–4, 279–342 Zbl [0268.57011](#) MR [0377923](#)

- [4] J.-P. Brasselet and T. Suwa, [Nash residues of singular holomorphic foliations](#). *Asian J. Math.* **4** (2000), no. 1, 37–50 Zbl [0981.32021](#) MR [1802911](#)
- [5] D. Cerveau, [Distributions involutives singulières](#). *Ann. Inst. Fourier (Grenoble)* **29** (1979), no. 3, 261–294 Zbl [0419.58002](#) MR [0552968](#)
- [6] M. Crainic, R. L. Fernandes, and I. Mărcuț, [Lectures on Poisson geometry](#). Grad. Stud. Math. 217, American Mathematical Society, Providence, RI, 2021 Zbl [1530.53001](#) MR [4328925](#)
- [7] C. Debord, [Holonomy groupoids of singular foliations](#). *J. Differential Geom.* **58** (2001), no. 3, 467–500 Zbl [1034.58017](#) MR [1906783](#)
- [8] C. Debord and G. Skandalis, [Blow-up constructions for Lie groupoids and a Boutet de Monvel type calculus](#). *Münster J. Math.* **14** (2021), no. 1, 1–40 Zbl [1460.58013](#) MR [4300161](#)
- [9] D. Eisenbud, [Commutative algebra with a view toward algebraic geometry](#). Grad. Texts in Math. 150, Springer, New York, NY, 1995 Zbl [0819.13001](#) MR [1322960](#)
- [10] D. Eisenbud, [The geometry of syzygies: a second course in commutative algebra and algebraic geometry](#). Grad. Texts in Math. 229, Springer, New York, NY, 2005 Zbl [1066.14001](#) MR [2103875](#)
- [11] E. Faber and D. B. Westra, [Blowups in tame monomial ideals](#). *J. Pure Appl. Algebra* **215** (2011), no. 8, 1805–1821 Zbl [1217.14011](#) MR [2776424](#)
- [12] J. Ferrer, M. I. García, and F. Puerta, [Differentiable families of subspaces](#). *Linear Algebra Appl.* **199** (1994), no. Suppl. 1, 229–252 (Special issue honoring I. Olkin) Zbl [0803.58010](#) MR [1274417](#)
- [13] A. Garmendia and O. Yudilevich, [On the inner automorphisms of a singular foliation](#). *Math. Z.* **293** (2019), no. 1–2, 725–729 Zbl [1432.53035](#) MR [4002297](#)
- [14] A. Gathmann, [Algebraic geometry](#). Notes for a class taught at the University of Kaiserslautern 2002/2003. 2014, <https://agag-gathmann.math.rptu.de/de/alggeom.php> visited on 19 July 2025
- [15] G. Gonzalez-Sprinberg, [On Nash blow-up of orbifolds](#). In J.-P. Brasselet, S. Ishii, T. Suwa, and M. Vaquie (eds.), *Singularities — Niigata–Toyama 2007*, pp. 133–149, Adv. Stud. Pure Math. 56, Mathematical Society of Japan, Tokyo, 2009 Zbl [1231.14029](#) MR [2604080](#)
- [16] R. Hartshorne, [Algebraic geometry](#). Grad. Texts in Math. 52, Springer, New York, NY, 1977 Zbl [0367.14001](#) MR [0463157](#)
- [17] H. Hauser, [Blowups and resolution](#). In D. Ellwood, H. Hauser, S. Mori, and J. Schicho (eds.), *The resolution of singular algebraic varieties. Proceedings of the Clay Mathematics Institute Summer School 2012, Obergurgl, Tyrolean Alps, Austria, June 3–30, 2012*, pp. 1–80, Clay Math. Proc. 20, American Mathematical Society, Providence, RI, 2014 Zbl [1331.14021](#) MR [3328579](#)
- [18] G. Hector and U. Hirsch, [Introduction to the geometry of foliations, part A: foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy](#). 2nd edn., Asp. Math. 1, Springer, Wiesbaden, 1986 Zbl [0628.57001](#) MR [0881799](#)
- [19] R. Hermann, [The differential geometry of foliations, II](#). *J. Math. Mech.* **11** (1962), no. 2, 303–315 Zbl [0152.20502](#) MR [0142131](#)
- [20] H. Hironaka and H. Rossi, [On the equivalence of imbeddings of exceptional complex spaces](#). *Math. Ann.* **156** (1964), no. 4, 313–333 Zbl [0136.20801](#) MR [0171784](#)
- [21] C. Laurent-Gengoux, S. Lavau, and T. Strobl, [The universal Lie  \$\infty\$ -algebroid of a singular foliation](#). *Doc. Math.* **25** (2020), 1571–1652 Zbl [1453.53033](#) MR [4164730](#)
- [22] C. Laurent-Gengoux and R. Louis, [Lie–Rinehart algebras  \$\simeq\$  acyclic Lie  \$\infty\$ -algebroids](#). *J. Algebra* **594** (2022), 1–53 Zbl [1482.18016](#) MR [4352591](#)

- [23] C. Laurent-Gengoux, R. Louis, and L. Ryvkin, [Canonical geometric and algebraic structures hidden behind a singular foliation](#). In M. Garcia-Fernandez, D. I. Ponte, E. Miranda, C. Oms, and R. Rubio (eds.), *Advances in Poisson Geometry*, pp. 263–390, 1st edn., Adv. Courses Math. CRM Barcelona, Birkhäuser Cham, 2025 MR [4952773](#)
- [24] C. Laurent-Gengoux, R. Louis, and L. Ryvkin, [What is a singular foliation?](#) In M. Garcia-Fernandez, D. I. Ponte, E. Miranda, C. Oms, and R. Rubio (eds.), *Advances in Poisson Geometry*, pp. 123–261, 1st edn., Adv. Courses Math. CRM Barcelona, Birkhäuser Cham, 2025 MR [4952772](#)
- [25] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson structures*. Grundlehren Math. Wiss. 347, Springer, Berlin, Heidelberg, 2013 Zbl [1284.53001](#) MR [2906391](#)
- [26] C. Laurent-Gengoux and L. Ryvkin, [The holonomy of a singular leaf](#). *Selecta Math. (N.S.)* **28** (2022), no. 2, article no. 45, 38 pp. Zbl [1491.53031](#) MR [4376216](#)
- [27] S. Lavau, [The modular class of a singular foliation](#). *J. Geom. Phys.* **192** (2023), article no. 104902, 32 pp. Zbl [1526.53022](#) MR [4626424](#)
- [28] M. Levine, [Blowing up monomial ideals](#). *J. Pure Appl. Algebra* **160** (2001), no. 1, 67–103 Zbl [1064.14014](#) MR [1829313](#)
- [29] R. Louis, *Universal higher Lie algebras of singular spaces and their symmetries*. Ph.D. thesis, Université de Lorraine, 2022
- [30] R. Louis, [On symmetries of singular foliations](#). *J. Geom. Phys.* **189** (2023), article no. 104833, 31 pp. Zbl [1522.53073](#) MR [4575097](#)
- [31] R. Louis, [On Nash resolution of \(singular\) Lie algebroids](#). *Math. Z.* **311** (2025), no. 2, article no. 24, 27 pp. Zbl [08076057](#) MR [4937838](#)
- [32] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*. London Math. Soc. Lecture Note Ser. 213, Cambridge University Press, Cambridge, 2005 Zbl [1078.58011](#) MR [2157566](#)
- [33] O. Mohsen, [Blow-up groupoid of singular foliations](#). [v1] 2021, [v2] 2022, arXiv:[2105.05201v2](#)
- [34] J. A. Moody, [Divisibility of ideals and blowing up](#). *Illinois J. Math.* **45** (2001), no. 1, 163–165 Zbl [0989.13001](#) MR [1849991](#)
- [35] A. S. Morye, [Note on the Serre–Swan theorem](#). *Math. Nachr.* **286** (2013), no. 2–3, 272–278 Zbl [1274.14020](#) MR [3021481](#)
- [36] T. Nagano, [Linear differential systems with singularities and an application to transitive Lie algebras](#). *J. Math. Soc. Japan* **18** (1966), no. 4, 398–404 Zbl [0147.23502](#) MR [0199865](#)
- [37] V. Nistor, [Desingularization of Lie groupoids and pseudodifferential operators on singular spaces](#). *Comm. Anal. Geom.* **27** (2019), no. 1, 161–209 Zbl [1411.35298](#) MR [3951023](#)
- [38] A. Nobile, [Some properties of the Nash blowing-up](#). *Pacific J. Math.* **60** (1975), no. 1, 297–305 Zbl [0324.32012](#) MR [0409462](#)
- [39] H. Posthuma, X. Tang, and K. Wang, [Resolutions of proper Riemannian Lie groupoids](#). *Int. Math. Res. Not. IMRN* **2021** (2021), no. 2, 1249–1287 Zbl [1479.22005](#) MR [4201416](#)
- [40] H. Rossi, [Picard variety of an isolated singular point](#). *Rice Univ. Stud.* **54** (1968), no. 4, 63–73 Zbl [0179.40103](#) MR [0244517](#)
- [41] S. Sertöz, [Residues of singular holomorphic foliations](#). *Compos. Math.* **70** (1989), no. 3, 227–243 Zbl [0683.32008](#) MR [1002044](#)
- [42] K. J. Singh, [On the universal  \$L\_\infty\$ -algebroid of linear foliations](#). *J. Lie Theory* **33** (2023), no. 3, 925–952 Zbl [1525.22008](#) MR [4636915](#)
- [43] J. Śniatycki, [Orbits of families of vector fields on subcartesian spaces](#). *Ann. Inst. Fourier (Grenoble)* **53** (2003), no. 7, 2257–2296 Zbl [1048.53060](#) MR [2044173](#)

- [44] P. Stefan, [Accessibility and foliations with singularities](#). *Bull. Amer. Math. Soc.* **80** (1974), no. 6, 1142–1145 Zbl [0293.57015](#) MR [0353362](#)
- [45] P. Stefan, [Integrability of systems of vector fields](#). *J. Lond. Math. Soc. (2)* **21** (1980), no. 3, 544–556 Zbl [0432.58002](#) MR [0577729](#)
- [46] R. G. Swan, [Vector bundles and projective modules](#). *Trans. Amer. Math. Soc.* **105** (1962), 264–277 Zbl [0109.41601](#) MR [0143225](#)
- [47] L. D. Tráng and T. Urabe, *Geometry of tangents on singular spaces and Chern classes*. Lect. Math. 13, Department of Mathematics, Kyoto University, Kinokuniya, Tokyo, Japan, 1981 Zbl [0524.32004](#) MR [0623255](#)
- [48] T. Wang, Explicit description of centralizers for a matrix. 2019, arXiv:[1910.13666v1](#)
- [49] J. Weyman, [Grassmannians and flag varieties](#). In *Cohomology of vector bundles and syzygies*, pp. 85–109. Cambridge Tracts in Math. 149, Cambridge University Press, 2003 Zbl [1075.13007](#) MR [1988690](#)

Received 7 August 2024.

**Ruben Louis**

Department of Mathematics, University of Illinois Urbana-Champaign, 1409 W. Green Street, Urbana, IL 67801, USA; [rlouis@illinois.edu](mailto:rlouis@illinois.edu), [louisruben96@yahoo.com](mailto:louisruben96@yahoo.com)