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Number Theory. – *On the Bombieri–Davenport large sieve inequalities*, by John Friedlander and Henryk Iwaniec, accepted on 1 July 2025.

To Enrico, just because!

Abstract. – The large sieve inequality, although best possible in general, is subject to improvements for some basic special sequences. We exemplify how the ideas of Bombieri and Davenport can be exploited to this end.

Keywords. – large sieve, characters, primes, residue classes.

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1. Introduction: A bit of history

We are grateful to Umberto Zannier for having invited us to contribute a paper in celebration of Enrico Bombieri's eighty-fifth birthday. As we began the project we noted that it was just fifty years ago at the Vancouver ICM that Enrico had received the Fields Medal. These anniversaries provoked us to refresh the memory of some of Enrico's early achievements in Analytic Number Theory. We choose to write about the large sieve inequalities for Dirichlet characters, in part because they are fundamental to the theory of L-functions and in part because Enrico's role was so central to their development.

The modern version of the LSI asserts that

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{M<n\leqslant M+N}a_n\chi(n)\Big|^2\leqslant(N+Q^2)\sum_{M<n\leqslant M+N}|a_n|^2, \tag{1.1}$$

the asterisk restricting the summation to primitive characters $\chi\,(\mathrm{mod}\,q)$. This holds for all complex numbers a_n supported on any interval of length $N\geqslant 1$. In numerous applications to prime numbers, the LSI serves as a substitute for the Riemann hypothesis for $L(s,\chi)$. In fact, the LSI is found to be quite robust. Using (1.1), one can recover territories beyond the threshold of the GRH.

The first ideas of the large sieve appear in the short paper of Ju. V. Linnik [6] followed by that of A. Rényi [8]. Great innovations began in the 1960s with the works of K. Roth [9] and E. Bombieri [1]. Various conceptual interpretations of the LSI have been gradually revealed, the common ground of all lying in the super-orthogonality of primitive characters, the pseudo-characters of A. Selberg [10] included.

Many results, with $N+Q^2$ in (1.1) replaced by $c_1N+c_2Q^2$ or by $\max(c_1N, c_2Q^2)$ for specific constants c_1, c_2, have been produced by different authors, among them Davenport, Halberstam, Gallagher and Elliott. Ultimately, the inequality (1.1), best possible in general, was established independently by Montgomery and Vaughan [7] and by Selberg [11] using different arguments.

For our considerations in this work, especially in Section 3, the value of $c_1 = 1$ is crucial while that of c_2 is less important. The first LSI of that strength in the literature is the following.

THEOREM 1.1 (Bombieri–Davenport 1969). *For all complex numbers a_n supported on an interval of length $N \geqslant 1$, we have*

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_n a_n\chi(n)\Big|^2 \leqslant (\sqrt{N}+Q)^2\sum_n |a_n|^2. \tag{1.2}$$

Because, in these notes, Q will be much smaller than $\sqrt{N}$, the BD inequality (1.2) is, for our purposes, as good as the MV-S inequality (1.1). They are essentially sharp in general. However, for special coefficients a_n, some improvements are possible. We have the following.

THEOREM 1.2 (Bombieri–Davenport 1969). *Let $\mathcal{Q}$ be a set of positive integers $\leqslant Q$ which have no prime divisors in a set $\mathcal{P}$. Then,*

$$\sum_{q\in\mathcal{Q}}\sum_{\chi\,(\mathrm{mod}\,q)}\frac{|\tau(\chi)|^2}{\varphi(q)}\Big|\sum_n a_n\chi(n)\Big|^2 \leqslant (\sqrt{N}+Q)^2\sum_n |a_n|^2 \tag{1.3}$$

for all complex numbers a_n supported on an interval of length $N \geqslant 1$ with n being free of prime divisors in $\mathcal{P}$.

Note that in the BD inequality (1.3) we have the Gauss sum

$$\tau(\chi) = \sum_{m\,(\mathrm{mod}\,q)}\chi(m)e(m/q)$$

and χ ranges over all characters in $\mathcal{Q}$, not only the primitive characters as in (1.2). However, every $\chi\,(\mathrm{mod}\,q)$ with $1 \leqslant q \leqslant Q$ is induced by a unique primitive character of conductor q_1 with $q = q_1 r \leqslant Q$. Taking all these characters into account, one obtains

$$\mathop{\sum\sum}_{\substack{qr\leqslant Q\\(q,r)=1}}\frac{q\mu^2(r)}{\varphi(qr)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_n a_n\chi(n)\Big|^2 \leqslant (\sqrt{N}+Q)^2\sum_n |a_n|^2 \tag{1.4}$$

for all complex numbers a_n supported on an interval of length $N \geqslant 1$ with n being free of prime divisors $\leqslant Q$. Because

$$\sum_{\substack{r \leqslant X \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \geqslant \frac{\varphi(q)}{q} \log X, \quad \text{if } X \geqslant 1, \tag{1.5}$$

we have

$$\sum_{q \leqslant Q} \left(\log \frac{Q}{q}\right) \sum_{\chi \,(\mathrm{mod}\, q)}^{*} \left|\sum_n a_n \chi(n)\right|^2 \leqslant (\sqrt{N} + Q)^2 \sum_n |a_n|^2. \tag{1.6}$$

See the arguments on p. 21 of [3] and a slightly stronger inequality in [2, Theorem 8], wherein the right-hand side is the same as the right-hand side of (1.1).

Discarding all but the principal character in (1.6) and taking $a_p = 1$ for $M < p \leqslant M + N$ and zero elsewhere, Bombieri and Davenport show that (choose $Q = \sqrt{N}/\log N$)

$$\pi(M+N) - \pi(M) \leqslant \frac{2N}{\log N}\left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)$$

if $M > \sqrt{N}$. Recall that the linear sieve (whether Selberg or beta) produces the same interesting constant 2 and similar error terms.

Our task in these notes is to further exploit the Bombieri–Davenport large sieve arguments, first in Section 2 for coefficients which have restricted support. Then, in Section 3, we show how an exceptional real character interacts with all of the others. Our point of departure for these results is the following specific inequality which occurs with a nice proof as Theorem 7A in [2].

THEOREM 1.3. *For all complex numbers a_n supported on an interval of length $N \geqslant 1$, we have*

$$\sum\sum_{\substack{qr \leqslant Q \\ (q,r)=1}} \frac{q}{\varphi(qr)} \sum_{\chi \,(\mathrm{mod}\, q)}^{*} \left|\sum_n a_n \chi(n) c_r(n)\right|^2 \leqslant (N + Q^2) \sum_n |a_n|^2, \tag{1.7}$$

where $c_r(n)$ is the Ramanujan sum

$$c_r(n) = \sum_{u \,(\mathrm{mod}\, r)}^{*} e(un/r) = \sum_{d \mid (n,r)} d\mu(r/d). \tag{1.8}$$

2. THE LARGE SIEVE OF RESTRICTED SUPPORT

Our first observation is that (1.7) implies the following.

PROPOSITION 2.1. *Let $\mathcal{R}$ be a set of numbers $r \leqslant R$. Put*

$$\mathcal{L}_q(\mathcal{R}) = \frac{q}{\varphi(q)} \sum_{\substack{r \in \mathcal{R} \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)}, \tag{2.1}$$

$$\mathcal{L}(Q, R) = \min_{q \leqslant Q} \mathcal{L}_q(\mathcal{R}). \tag{2.2}$$

Then,

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \Big| \sum_{M < n \leqslant M+N} a_n \chi(n) \Big|^2 \leqslant \frac{N + Q^2 R^2}{\mathcal{L}(Q, R)} \sum_{M < n \leqslant M+N} |a_n|^2 \tag{2.3}$$

for sequences a_n having $(n, r) = 1$ for every $r \in \mathcal{R}$.

PROOF. This follows from the fact that $c_r(n) = \mu(r)$ in this case. ∎

EXAMPLE 2.1. Taking $\mathcal{R}$ to be the set of all numbers $r \leqslant R$ with $R \geqslant 2$, we get

$$\mathcal{L}(Q, R) \geqslant \log R. \tag{2.4}$$

The number of primitive characters $\chi \bmod q$ with $q \leqslant Q$ is about Q^2 so we have lost the factor R^2 in the second term of $N + Q^2$ in order to gain $\log R$ in both terms. Although at first glance this seems wasteful, it could be only a minor complication in some important applications when the character sums $\sum a_n \chi(n)$ are quite a bit longer than Q^2.

The previous example covers the situation where the a_n are supported on the primes in an interval. With more work we consider what might be regarded as the next natural case for study.

THEOREM 2.1. *Let $r(n)$ denote the number of representations of n as the sum of two co-prime squares. For all complex numbers a_n supported on odd integers n, we have*

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \Big| \sum_{M < n \leqslant M+N} r(n) a_n \chi(n) \Big|^2 \leqslant \left(1 + \frac{1}{\alpha}\right) \frac{\lambda N}{\sqrt{\log N/Q^2}} \sum_{M < n \leqslant M+N} r^2(n) |a_n|^2 \tag{2.5}$$

if $N \geqslant Q^2 \exp(\alpha \log\log Q)^3$, where α is any sufficiently large absolute constant and

$$\lambda = \frac{\pi}{4\omega}, \quad \omega = \prod_{p \equiv 3 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)^{-1/2}. \tag{2.6}$$

REMARK 2.1. For n odd we have $r(n) = 0$ unless every prime divisor of n satisfies $p \equiv 1 \pmod 4$.

Let $\nu(n) = 1$ if n is the product of distinct primes $p \not\equiv 1 \pmod 4$ and $\nu(n) = 0$ otherwise. We take $\mathcal{R}$ to be the set of integers $r \leqslant R$, $\nu(r) = 1$. To prove Theorem 2.1 using Proposition 2.1 we want to show that $\mathcal{L}(Q, R) \gg (\log R)^{1/2}$ for an appropriate choice of R. For this we need estimates for the sum

$$T_q(x) = \sum_{\substack{n \leqslant x \\ (n,q)=1}} \frac{\nu(n)}{\varphi(n)}. \tag{2.7}$$

LEMMA 2.1. *For every $q \geqslant 1$ and $x \geqslant 1$ we have*

$$\frac{q}{\varphi(q)} T_q(x) = \mu_q (\log x)^{\frac{1}{2}} + O\big((\log\log 3q)^3\big), \tag{2.8}$$

with

$$\mu_q = \frac{4\sqrt{2}}{\pi} \omega \prod_{\substack{p \mid q \\ p \equiv 1 \pmod 4}} \Big(1 - \frac{1}{p}\Big)^{-1}. \tag{2.9}$$

PROOF. We apply Theorem A.5 from the appendix in [4] for the multiplicative function $g(m)$ supported on square-free numbers with

$$g(p) = \frac{1}{p-1} \quad \text{if } p \nmid q,\ p \not\equiv 1 \pmod 4$$

and $g(p) = 0$ otherwise. The formula (A.25) in [4] yields[1]

$$T_q(x) = c_g (\log x)^{\frac{1}{2}} + O\big((\log\log 3q)^3\big), \tag{2.10}$$

where

$$c_g = \frac{1}{\Gamma(3/2)} \prod_p \Big(1 - \frac{1}{p}\Big)^{\frac{1}{2}} (1 + g(p)).$$

We borrow and return the factor $L^{-\frac{1}{2}}(1, \chi)$ where $\chi = \chi_4$ is the Dirchlet character of conductor 4 so $L(1, \chi) = \pi/4$ by Leibniz's formula. We have $\Gamma(3/2) = \pi/\sqrt{2}$. Therefore,

$$c_g = \frac{4}{\pi} \prod_p \Big(1 - \frac{\chi(p)}{p}\Big)^{-\frac{1}{2}} \Big(1 - \frac{1}{p}\Big)^{\frac{1}{2}} (1 + g(p)).$$

(1) Actually, formula (A.25) of [4] is insufficient to give this explicit estimate for the error term. We supplement it in the appendix here.

The product over all $p|q$ is equal to

$$\prod_{p|q}\left(1-\frac{\chi(p)}{p}\right)^{-\frac{1}{2}}\left(1-\frac{1}{p}\right)=(2,q)^{-\frac{1}{2}}\prod_{\substack{p|q\\ p\equiv 3\ (\mathrm{mod}\ 4)}}\left(1-\frac{1}{p}\right)^{-\frac{1}{2}}\left(1-\frac{1}{p}\right).$$

The product over all $p\nmid q$, $p\equiv 1\pmod 4$ is equal to 1. The product over all $p\nmid q$, $p\equiv 3\pmod 4$ is equal to

$$\prod_{\substack{p\nmid q\\ p\equiv 3\ (\mathrm{mod}\ 4)}}\left(1+\frac{\chi(p)}{p}\right)^{-\frac{1}{2}}\left(1-\frac{1}{p}\right)^{\frac{1}{2}}\left(1-\frac{1}{p}\right)^{-1}=\prod_{\substack{p\nmid q\\ p\equiv 3\ (\mathrm{mod}\ 4)}}\left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

The factor at $p=2$, $p\nmid q$ is equal to $\sqrt{2}$.

From the above computations we get

$$c_g=\frac{4\sqrt{2}\omega}{(2,q)\pi}\prod_{\substack{p|q\\ p\equiv 3\ (\mathrm{mod}\ 4)}}\left(1-\frac{1}{p}\right),$$

so

$$\frac{q}{\varphi(q)}c_g=\frac{4\sqrt{2}\omega}{\pi}\prod_{\substack{p|q\\ p\equiv 1\ (\mathrm{mod}\ 4)}}\left(1-\frac{1}{p}\right)^{-1}.$$

This completes the proof of Lemma 2.1. ∎

We can ignore the product from (2.9), after which the formula (2.7) with the choice $x=R=(\sqrt{N}/Q)\log N$ implies

$$\mathcal{L}(Q,R)>\frac{4\omega}{\pi}\left(\log\frac{N}{Q^2}\right)^{1/2}+O\big((\log\log 3Q)^3\big).$$

Thus, Proposition 2.1 implies Theorem 2.1. The latter has the following corollaries. The first of these is the following.

COROLLARY 2.1. *Let* $b_n=1$ *if every prime divisor of* n *satisfies* $p\equiv 1\pmod 4$ *and* $b_n=0$ *otherwise. Then,*

$$\sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\Big|\sum_{M<n\leqslant M+N}b_n\chi(n)\Big|^2 \leqslant\left(1+\frac{1}{\alpha}\right)\frac{\pi}{4\omega}\frac{N}{\sqrt{\log N/Q^2}}\sum_{M<n\leqslant M+N}b_n \tag{2.11}$$

if $N>Q^2\exp(\alpha\log\log 3Q)^3$ *where* α *is any sufficiently large constant.*

PROOF. Apply (2.5) for $r(n)a_n = b_n$. ■

The second of these is a Brun–Titchmarsh type inequality.

COROLLARY 2.2. *Let $\varepsilon > 0$. For all N sufficiently large in terms of ε, we have*

$$\sum_{M<n\leqslant M+N} b_n < (1+\varepsilon)\frac{\pi}{8\omega}\frac{N}{\sqrt{\log N}}. \tag{2.12}$$

PROOF. Take $Q = 4$. The sum in (2.12) appears (squared) twice on the left-hand side of (2.11), once for $\chi = 1$ and once for χ_4 while the other term, for χ_3, can be ignored by positivity. ■

It is interesting to compare (2.12) with the asymptotic formula for

$$B(x) = \sum_{n\leqslant x} b_n. \tag{2.13}$$

E. Landau [5] showed that

$$C(x) = \sum_{n\leqslant x} c(n) \sim \frac{\omega}{\sqrt{2}}\frac{x}{\sqrt{\log x}}, \tag{2.14}$$

where $c(n) = 1$ if n is the sum of two squares and $c(n) = 0$ otherwise. Because $2n$ is the sum of two squares equivalent to n being the sum of two squares, we see from (2.14) that

$$C_1(x) = \sum_{\substack{n\leqslant x\\ 2\nmid n}} c(n) = C(x) - C(x/2) \sim \frac{\omega}{2\sqrt{2}}\frac{x}{\sqrt{\log x}}. \tag{2.15}$$

Moreover, every odd n which is the sum of two squares factors uniquely into $n = m\ell^2$ where m has only prime factors $p \equiv 1 \pmod 4$ and ℓ has only prime factors $p \equiv 3 \pmod 4$. Therefore, $C_1(x)$ differs from $B(x)$ asymptotically by the factor

$$\sum_{\ell} \ell^{-2} = \prod_{p\equiv 3 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)^{-1} = \omega^2.$$

Hence, $C_1(x) \sim \omega^2 B(x)$ and Landau's formula for $C_1(x)$ is equivalent to

$$B(x) \sim \frac{1}{2\sqrt{2}\omega}\frac{x}{\sqrt{\log x}}. \tag{2.16}$$

This asymptotic formula can also be proved by the beta sieve of dimension $\kappa = 1/2$, see [4, Theorem 14.2].

Note that the upper bound (2.12) obtained by LSI is worse than the true asymptotic value in (2.16) by the ratio $\pi/2\sqrt{2} = 1.11072\ldots$, which is better than the factor 2 obtained in the case of primes, the classical Brun–Titchmarsh inequality.

Next, we consider coefficients a_n with $a_n\sqrt{n}$ small whenever n has a relatively small prime divisor. One may think of n as being almost a prime.

THEOREM 2.2. *Suppose that a_n with $M < n < M + N$ satisfy*

$$\sum_n |a_n|^2 \leqslant 1, \qquad \sum_{n\equiv 0\,(\mathrm{mod}\,p)} |a_n|^2 \leqslant \frac{1}{p}\left(\frac{\log p}{\log N}\right)^2 \tag{2.17}$$

for every prime p. We have

$$\sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \Bigl|\sum_{M<n\leqslant M+N} a_n\chi(n)\Bigr|^2 \leqslant 24N(\log N/Q^2)^{-1} \tag{2.18}$$

if $8Q^2 \leqslant N$.

PROOF. Let $R \geqslant 2$ and $Q^2R^3 < N$. For the character sums $\sum a_n\chi(n)$ running over n having no prime divisors $\leqslant R$ we apply (2.3) and (2.4), obtaining the bound

$$S_1 = \frac{2N}{\log R}\sum_n |a_n|^2 \leqslant \frac{2N}{\log R}.$$

Then, we estimate the character sums $\sum a_n\chi(n)$ over n having a prime divisor $\leqslant R$ by

$$\sum_{p\leqslant R}\Bigl|\sum_{(n,P(p))=1} a_{np}\chi(n)\Bigr|,$$

where $P(z)$ denotes the product of all primes $< z$. This part of the sum (2.18) is bounded by $S_2 = \sum_{p_1}\sum_{p_2} S(p_1, p_2)$ with $p_1, p_2 \leqslant R$ and

$$\begin{aligned} S(p_1, p_2) &= \sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \Bigl|\sum_{(n,P(p_1))=1} a_{np_1}\chi(n)\Bigr|\Bigl|\sum_{(n,P(p_2))=1} a_{np_2}\chi(n)\Bigr| \\ &\leqslant S(p_1, p_1)^{1/2} S(p_2, p_2)^{1/2}. \end{aligned}$$

Applying (2.3), we get

$$S(p, p) \leqslant \frac{N/p + Q^2p^2}{\log p}\sum_n |a_{np}|^2 \leqslant \frac{2N\log p}{(p\log N)^2}.$$

Hence,

$$S_2 \leqslant \frac{2N}{(\log N)^2}\Bigl(\sum_{p\leqslant R}\frac{1}{p}\sqrt{\log p}\Bigr)^2 \leqslant 18N(\log N)^{-2}\log R \leqslant 2N(\log R)^{-1}.$$

This completes the proof of Theorem 2.2 on choosing $R = (N/Q^2)^{1/3}$. ■

REMARK 2.2. We could assume the somewhat stronger bound

$$a_n \ll \frac{\rho(n)}{\sqrt{n}}, \quad 1 \leqslant n \leqslant N \tag{2.19}$$

with

$$\rho(n) = \prod_{p|n} \rho(p), \quad \rho(p) = \frac{\log p}{\log N}. \tag{2.20}$$

These coefficients satisfy (2.17) up to a constant. Indeed, we have

$$\sum_{n\leqslant N} \frac{\rho(n)^2}{n} \ll \prod_{p\leqslant N}\Big(1+\frac{\rho(p)^2}{p}\Big) \leqslant \exp\Big(\sum_{p\leqslant N}\frac{\rho(p)^2}{p}\Big),$$
$$\sum_{p\leqslant N}\frac{\rho(p)^2}{p} \ll \sum_{p\leqslant N}\frac{1}{p}\Big(\frac{\log p}{\log N}\Big)^2 \ll 1.$$

The above estimates verify both conditions in (2.17).

EXAMPLE 2.2. The coefficients

$$a_n = n^{-1/2}\Lambda_k(n)(\log N)^{-k}, \quad \text{with } 1 \leqslant n \leqslant N,$$

satisfy (2.17). Here, Λ_k is the generalized von Mangoldt function $\Lambda_k = \mu * \log^k$. Apply induction in k using $\Lambda_{k+1} = \Lambda_k \cdot \log + \Lambda_k * \Lambda$.

3. THE LARGE SIEVE WITH AN EXCEPTIONAL CHARACTER

Recall that taking only the one term of (1.6) coming from the principal character over primes in a segment Bombieri and Davenport obtained a strong Brun–Titchmarsh upper bound for the number of primes in an interval. One may additionally extract the contribution of a second real character, say χ_D of conductor D, one which amounts to

$$\Big(\log\frac{Q}{D}\Big)\Big(\sum_{M<p\leqslant M+N}\chi_D(p)\Big)^2.$$

In this section, we study some of the effects of doing so. No surprise that this will be of greatest interest if the character is exceptional.

We put $\lambda(p) = 1 + \chi_D(p)$ and write

$$\Big(\sum_p \chi_D(p)\Big)^2 = \Big(-\sum_p 1 + \sum_p \lambda(p)\Big)^2 \geqslant \Big(\sum_p 1\Big)^2 - 2\Big(\sum_p 1\Big)\Big(\sum_p \lambda(p)\Big).$$

Hence, Theorem 8 of [2] yields

$$\sum_{1<q\leqslant Q}\Big(\log\frac{Q}{q}\Big)\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_D}}\Big|\sum_p \chi(p)\Big|^2 \leqslant (N+Q^2)\Big(\sum_p 1\Big)-\Big(\log\frac{Q^2}{D}\Big)\Big(\sum_p 1\Big)^2+2\Big(\log\frac{Q}{D}\Big)\Big(\sum_p 1\Big)\Big(\sum_p \lambda(p)\Big). \tag{3.1}$$

Note that the last part of (3.1) is negligible if $\chi_D(p)=-1$ very frequently. We are going to use the above arrangement for the coefficients in (1.6) given by

$$a_n=\Lambda(n)f(n/N), \tag{3.2}$$

where $f(u)$ is a nice function supported on $0\leqslant u\leqslant 1$ and with $0\leqslant f(u)\leqslant 1$. Let $3\leqslant Q\leqslant\sqrt{N}$. We get

$$\sum_{q\leqslant Q}\Big(\log\frac{Q}{q}\Big)\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\Big|\sum_n \chi(n)\Lambda(n)f(n/N)\Big|^2 \leqslant (N+Q^2)\sum_n |a_n|^2+O(N^{7/4}\log N), \tag{3.3}$$

where the error term on the right side comes from an estimation of the contribution of prime powers $p^2, p^3,\ldots\leqslant N$ and primes $p\leqslant Q$.

For $q=1$ we have the principal character whose contribution is

$$(\log Q)\Big(\sum_n a_n\Big)^2. \tag{3.4}$$

Let χ_D be a primitive real character of conductor $D\leqslant Q$. Its contribution is

$$\Big(\log\frac{Q}{D}\Big)\Big(\sum_n \chi_D(n)a_n\Big)^2. \tag{3.5}$$

Put $\lambda=1*\chi_D$ so $\chi_D=\mu*\lambda$. Note that $\lambda(p)=1+\chi_D(p)$. We think of λ as being lacunary. We write

$$\sum_n \chi_D(n)a_n=-\sum_n a_n+\sum_n \rho(n)a_n, \tag{3.6}$$

where $\rho(n)=1+\chi_D(n)$. Hence,

$$\Big(\sum_n \chi_D(n)a_n\Big)^2\geqslant\Big(\sum_n a_n\Big)^2-2\Big(\sum_n a_n\Big)\Big(\sum_n \rho(n)a_n\Big). \tag{3.7}$$

Summing up the contributions of χ_0, χ_D, we obtain from (3.3)

$$\sum_{1<q\leqslant Q}\left(\log\frac{Q}{q}\right)\sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_D}}\left|\sum_n \chi(n)\Lambda(n) f(n/N)\right|^2 \leqslant (N+Q^2)\sum_n a_n^2 - \left(\log\frac{Q^2}{D}\right)\left(\sum_n a_n\right)^2 + 2\left(\log\frac{Q}{D}\right)\left(\sum_n a_n\right)\left(\sum_n \lambda(n)a_n\right) + O(N^{7/4}\log N). \tag{3.8}$$

Note that we have here replaced $\rho(n)$ by $\lambda(n)$ by estimating trivially the terms at higher prime powers. Next, we estimate as follows:

$$\sum_n \lambda(n)a_n \leqslant \sum_{n\leqslant N}\lambda(n)\log N = L(1,\chi_D)N\log N + O(N^{1/2}D^{1/4}\log N),$$

where $L(1,\chi_D)$ is the residue of $\zeta(s)L(s,\chi_D)$ at $s=1$.

Moreover, we have $\sum_n a_n \ll N$, but we need more precise estimations for the leading terms. Using the prime number theorem, we get

$$\sum_n a_n = A_1 N + O(N/\log N), \quad A_1 = \int_0^1 f(u)du; \tag{3.9}$$

$$\sum_n a_n^2 = A_2 N\log N + O(N), \quad A_2 = \int_0^1 f^2(u)du. \tag{3.10}$$

Introducing these estimates into (3.8) and choosing $Q=\sqrt{N}/\log N$, we obtain the following.

Lemma 3.1. *Let* $3\leqslant D\leqslant Q=\sqrt{N}/\log N$. *We have*

$$\sum_{1<q\leqslant Q}\left(\log\frac{Q}{q}\right)\sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_D}}\left|\sum_n \chi(n)\Lambda(n) f(n/N)\right|^2 \leqslant (A_2\log N - A_1^2\log N/D)N^2 + 2A_1(\log Q/D)N^2L(1,\chi_D)\log N + O(N^2). \tag{3.11}$$

Choosing $f(u)=1$, we find $A_1=A_2=1$. Hence, we have proved the following.

Proposition 3.1. *Let* $3\leqslant D\leqslant Q=\sqrt{N}/\log N$. *We have*

$$\sum_{1<q\leqslant Q}\left(\log\frac{Q}{q}\right)\sideset{}{^*}\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_D}}\left|\sum_n \chi(n)\Lambda(n)\right|^2 \leqslant N^2\big(\log cD + L(1,\chi_D)(\log N)^2\big), \tag{3.12}$$

where $c\geqslant 1$ *is an absolute constant.*

Proposition 3.2. *Let D be larger than a suitable absolute constant. Suppose*

$$L(1,\chi_D)\log D\leqslant\varepsilon^5. \tag{3.13}$$

Then, for every primitive character $\chi \pmod q$, $\chi\neq\chi_0,\chi_D$ *and every* $N\geqslant q^4$ *with* $D^{1/\varepsilon^2}<N<D^{1/\varepsilon^3}$, *we have*

$$\Big|\sum_{n\leqslant N}\chi(n)\Lambda(n)\Big|\leqslant 3\varepsilon N. \tag{3.14}$$

Proof. On the left side of (3.12) we have $Q/q\geqslant N^{1/4}/\log N$. Hence,

$$\Big|\sum_{n\leqslant N}\chi(n)\Lambda(n)\Big|^2<\frac{9}{2}\left(\frac{\log D}{\log N}+\varepsilon^5\frac{\log N}{\log D}\right)N^2<(3\varepsilon N)^2$$

because D,N are large and $4<9/2$. ■

Appendix: An appendage to the Opera

The following result, a corollary to Theorem A.5 of [4], gives a more explicit error term for the sum considered there, as required in the proof of Lemma 2.1 in the current paper. For ease of reference we first re-state that theorem here.

Theorem A.1. *Suppose g is a multiplicative function supported on square-free numbers and satisfying the conditions:*

$$\sum_{p\leqslant x}g(p)\log p=k\log x+\delta(x), \tag{A.1}$$

where $k>-\frac{1}{2}$ *and* $\delta(x)$ *is bounded for all* $x\geqslant 2$,

$$\prod_{w\leqslant p<z}\big(1+g(p)\big)\ll\left(\frac{\log z}{\log w}\right)^{|k|} \tag{A.2}$$

for $z>w\geqslant 2$,

$$\sum_p g(p)^2\log p<\infty. \tag{A.3}$$

Then,

$$M_g(x):=\sum_{m\leqslant x}g(m)=c_g(x)(\log x)^k+O\big((\log x)^{|k|-1}\big) \tag{A.4}$$

for every $x \geqslant 2$, *where*

$$c_g = \frac{1}{\Gamma(k+1)} \prod_p \Big(1 - \frac{1}{p}\Big)^k \big(1 + g(p)\big) \tag{A.5}$$

and the implied constant depends only on k *and the constants implied in* (A.1), (A.2) *and* (A.3).

Now, we have the following.

Corollary A.1. *Suppose* g *satisfies the conditions* (A.1), (A.2) *and* (A.3) *of Theorem* A.1. *Let* $h(m) = g(m)$ *if* $(m, q) = 1$ *and* $h(m) = 0$ *otherwise. Then, for* $x \geqslant 2$ *we have*

$$\mathcal{M}_h(x) = c_h (\log x)^k + O\big(A(q)(\log x)^{|k|-1}\big),$$

where

$$A(q) = \Big(1 + \sum_{p|q} |g(p)| \log p\Big) \prod_{p|q} \big(1 + |g(p)|\big)$$

and c_h *is given by* (A.5). *The implied constant depends only on* k *and the constants implied in* (A.1), (A.2) *and* (A.3).

Proof. Let $f(d)$ be the multiplicative function defined by $f(p^r) = (-g(p))^r$ if $p|q$ and $f(p^r) = 0$ if $p \nmid q$ for $r \geqslant 1$. Then, we have $h = f * g$. Hence,

$$\begin{aligned} \mathcal{M}_h(x) &= \sum\sum_{dm \leqslant x} f(d) g(m) \\ &= \sum_{d \leqslant x} f(d) \big(c_g (\log x/d)^k + O\big((\log 2x/d)^{|k|-1}\big)\big). \end{aligned}$$

We have

$$\Big(\log \frac{x}{d}\Big)^k = (\log x)^k + O\big((\log 2d)(\log x)^{k-1}\big)$$

and

$$\Big(\log \frac{2x}{d}\Big)(\log 2d) \gg \log x.$$

Hence,

$$\mathcal{M}_h(x) = c_g \Big(\sum_d f(d)\Big)(\log x)^k + O\big(B(q)(\log x)^{k-1}\big),$$

where

$$\sum_d f(d) = \prod_{p|q} \big(1 + g(p)\big)^{-1}$$

and

$$B(q) = \sum_d |f(d)| \log 2d \leqslant \sum_d |f(d)| \Big(\log 2 + \sum_{\ell | d} \Lambda(\ell) \Big).$$

Thus, $B(q) \ll A(q)$, completing the proof of Corollary A.1. ∎

Note that if $0 \leqslant g(p)p \leqslant 1 + O(1/p)$, then $0 \leqslant k \leqslant 1$ and $A(q) \ll (\log\log 3q)^2$. In this case, we get (A.4) with the stronger error term

$$\sum_{\substack{m \leqslant x \\ (m,q)=1}} g(m) = c(\log x)^k + O\big((\log\log 3q)^2\big),$$

where again $c = c_g$ is given by (A.5).

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John Friedlander
Department of Mathematics, University of Toronto
Toronto, Ontario M5S 2E4, Canada
frdlndr@math.toronto.edu

Henryk Iwaniec
Department of Mathematics, Rutgers University
Piscataway, NJ 08903, USA
iwaniec@comcast.net