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Number Theory. – *Multiplicativity of Fourier coefficients of Maass forms for* $\mathrm{SL}(n,\mathbb{Z})$, by Dorian Goldfeld, Eric Stade and Michael Woodbury, accepted on 1 July 2025.

Dedicated to Enrico Bombieri on the occasion of his 85th birthday.

Abstract. – The Fourier coefficients of a Maass form ϕ for $\mathrm{SL}(n,\mathbb{Z})$ are complex numbers $A_\phi(M)$, where $M = (m_1, m_2, \ldots, m_{n-1})$ and $m_1, m_2, \ldots, m_{n-1}$ are non-zero integers. It is well known that coefficients of the form $A_\phi(m_1, 1, \ldots, 1)$ are eigenvalues of the Hecke algebra and are multiplicative. We prove that the more general Fourier coefficients $A_\phi(m_1, \ldots, m_{n-1})$ are also eigenvalues of the Hecke algebra and satisfy the multiplicativity relations

$$A_\phi(m_1m_1', m_2m_2', \ldots, m_{n-1}m_{n-1}') = A_\phi(m_1, m_2, \ldots, m_{n-1}) \cdot A_\phi(m_1', m_2', \ldots, m_{n-1}')$$

provided the products $\prod_{i=1}^{n-1} m_i$ and $\prod_{i=1}^{n-1} m_i'$ are relatively prime to each other.

Keywords. – Maass forms, Fourier coefficients, Hecke operators.

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1. Introduction

Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}(n,\mathbb{Q})$ for $n \geq 2$. Associated with π we have the Godement–Jacquet L-function [2] given by

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s},$$

where the coefficients $\lambda_\pi(n) \in \mathbb{C}$. In the special case of the group $\mathrm{SL}(n,\mathbb{Z})$, the Godement–Jacquet L-functions can be studied classically in terms of Maass forms on the quotient space $\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n$ where

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R})/\big(\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times\big)$$

is a generalization of the classical upper half-plane. In fact, $\mathfrak{h}^2 := \big\{\big(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\big) \mid y > 0,\ x \in \mathbb{R}\big\}$ is isomorphic to the classical upper half-plane.

For $n \geq 2$, Maass forms are smooth functions $\phi : \mathfrak{h}^n \to \mathbb{C}$ which are automorphic for $\mathrm{SL}(n,\mathbb{Z})$ with moderate growth and which are joint eigenfunctions of the full ring of invariant differential operators on $\mathrm{GL}(n,\mathbb{R})$ as well as joint eigenfunctions of the Hecke algebra. The Fourier expansion of Maass forms on $\mathrm{GL}(n)$ was obtained for the first time by Piatetski-Shapiro [5] and then by Shalika [6, 7] independently.

A classical version of the Fourier coefficients of Maass forms on $\mathrm{SL}(n,\mathbb{Z})$ was announced by Jacquet [4] at the Tata Institute 1979 conference on Automorphic Forms, Representation Theory and Arithmetic. In his book, Bump [1] explicitly worked out Jacquet's classical approach for $\mathrm{GL}(3,\mathbb{R})$. The more general case of $\mathrm{GL}(n,\mathbb{R})$ was first presented in Goldfeld's book [3].

A Maass form ϕ for $\mathrm{SL}(n,\mathbb{Z})$ has a Fourier expansion (see [3, Theorem 9.3.11])

$$\phi(g) = \sum_{\gamma\in U_{n-1}(\mathbb{Z})\backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-2}=1}^{\infty}\sum_{m_{n-1}\neq 0} \frac{A_\phi(m_1,\ldots,m_{n-1})}{\prod_{k=1}^{n-1}|m_k|^{\frac{k(n-k)}{2}}} \cdot W\big(M\left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)\cdot g\big),$$

where $g\in\mathfrak{h}^n$, $M=\begin{pmatrix} m_1\cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1\end{pmatrix}$, and $W:\mathfrak{h}^n\to\mathbb{C}$ is a Whittaker function. Associated with ϕ we have arithmetic Fourier coefficients

$$A_\phi(m_1,m_2,\ldots,m_{n-1})\in\mathbb{C},$$

where $m_1,m_2,\ldots,m_{n-2}\in\mathbb{Z}_{\geq 1}$ while m_{n-1} is a non-zero integer.

It is shown in [3, Proposition 9.2.6] that every Maass form is either even or odd according to whether $A_\phi(m_1,\ldots,m_{n-1}) = \pm A_\phi(m_1,\ldots,-m_{n-1})$. We assume ϕ is normalized so that $A_\phi(1,\ldots,1)=1$.

It is further shown in [3, Theorem 9.3.11] that for each positive integer m there is a Hecke operator T_m acting on the complex vector space of Maass forms of $\mathrm{SL}(n,\mathbb{Z})$ where

$$T_m\phi(g) = A_\phi(m,1,\ldots,1)\cdot\phi(g) \quad (g\in\mathfrak{h}^n),$$

for every Maass form ϕ. Furthermore, $T_{mm'}=T_mT_{m'}$ if m and m' are coprime; i.e., the Hecke operators are multiplicative.

Definition 1.1 (The Hecke algebras $\mathcal{H}$, $\mathcal{H}_p$ and sets of eigenvalues $\mathcal{H}^*$, $\mathcal{H}_p^*$). Fix an integer $n\geq 2$. Let $\mathcal{H}$ denote the (integral) Hecke algebra which is the commutative polynomial ring over $\mathbb{Z}$ generated by the Hecke operators $T_1,T_2,T_3,\ldots$ acting on automorphic forms for $\mathrm{SL}(n,\mathbb{Z})$, i.e.,

$$\mathcal{H} := \mathbb{Z}[T_1,T_2,T_3,\ldots].$$

We also define $\mathcal{H}^*$ to be the set of eigenvalues of the Hecke operators $\mathcal{H}$ acting on ϕ where $A_\phi(m,1,\ldots,1)\in\mathcal{H}^*$ for every positive integer m. For a fixed prime p define

$\mathcal{H}_p$ to be the subalgebra of $\mathcal{H}$ generated by the Hecke operators T_{p^k} with $k \geq 0$. Let $\mathcal{H}_p^*$ denote the set of eigenvalues of the Hecke operators in $\mathcal{H}_p$.

Curiously, the Godement–Jacquet L-function $L(s,\phi)$ associated with a Hecke–Maass form ϕ is only built up with the eigenvalues of the Hecke operators $A_\phi(m,1,\ldots,1)$ with positive integers m and is defined as

$$L(s,\phi) := \sum_{m=1}^{\infty} \frac{A_\phi(m,1,\ldots,1)}{m^s}.$$

Remarkably $L(s,\phi)$ has an Euler product given by (see [3, Definition 9.4.3])

$$L(s,\phi) = \prod_p \left(1 - \frac{A_\phi(p,1,\ldots,1)}{p^s} + \frac{A_\phi(1,p,1,\ldots,1)}{p^{2s}} \cdots + (-1)^{n-1}\frac{A_\phi(1,\ldots,1,p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}}\right)^{-1}.$$

The main aim of this paper is to show that the general Fourier coefficients $A_\phi(m_1, m_2,\ldots,m_{n-1})$ are all eigenvalues of elements in the Hecke algebra and satisfy the multiplicativity relations

$$A_\phi(m_1m_1', m_2m_2',\ldots,m_{n-1}m_{n-1}') = A_\phi(m_1,m_2,\ldots,m_{n-1}) \cdot A_\phi(m_1',m_2',\ldots,m_{n-1}')$$

provided the products $\prod_{i=1}^{n-1} m_i$ and $\prod_{i=1}^{n-1} m_i'$ are relatively prime to each other. This multiplicativity result is stated in (cf. [3, Theorem 9.3.11]) but there is no proof given. Although this is a very well-known result to experts, we were unable to find a proof anywhere else in the literature, so this paper fills a possible gap.

The main results of this paper. The proof that $A_\phi(m_1,\ldots,m_{n-1}) \in \mathcal{H}^*$ is given in (2.5). The fact that the Fourier coefficients $A_\phi(m_1,\ldots,m_{n-1})$ are multiplicative is in the proof of Theorem 2.7. In Sections 3 and 4, we present some explicit examples of constructing Hecke operators whose eigenvalues are not of the form $A_\phi(m,1,\ldots,1)$. We also remark that all these results can be proved for Eisenstein series and residues of Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ with proofs that are essentially the same as the ones we give for Maass forms.

2. Proof of multiplicativity

DEFINITION 2.1. Fix an integer $n \geq 2$ and a prime p. For $M = (m_1,m_2,\ldots,m_{n-1}) \in \mathbb{Z}^{n-1}$, let $A_\phi(M)$ denote the Mth Fourier coefficient of a Maass form for $\mathrm{SL}(n,\mathbb{Z})$.

For a positive integer $1 \le r \le n-1$ let $K_0, K_1, K_2, \dots, K_r \in \mathbb{Z}_{\ge 0}$; assume that $K_0 \ge K_1 + K_2 + \cdots + K_r$. We define

$$\mathcal{A}_p(K_0, K_1, \dots, K_r) := A_\phi(p^{K_0}, 1, \dots, 1) \cdot A_\phi(p^{K_1}, p^{K_2}, \dots, p^{K_r}, 1, \dots, 1).$$

We begin with the following lemma which is a key idea in the proof of multiplicativity of the Fourier coefficients $A_\phi(M)$.

LEMMA 2.2. *Let p be a prime. Fix integers $n \ge 2$, $1 \le r \le n-1$, and $K_0, K_1, K_2, \dots, K_r \in \mathbb{Z}_{\ge 0}$, with $K_0 \ge K_1 + K_2 + \cdots + K_r$. Then*

$$\mathcal{A}_p(K_0, K_1, \dots, K_r) = \sum_{k_1=0}^{K_1} \cdots \sum_{k_r=0}^{K_r} A_\phi(p^L, p^{K_2+k_1-k_2}, p^{K_3+k_2-k_3}, \dots, p^{K_r+k_{r-1}-k_r}, p^{k_r}, 1, \dots, 1),$$

where $L = K_0 + K_1 - 2k_1 - k_2 \cdots - k_r$.

PROOF. The proof of Lemma 2.2 is based on the following identity (cf. [3, p. 277]):

$$A_\phi(m, 1, \dots, 1) A_\phi(m_1, m_2, \dots, m_{n-1}) = \sum_{\substack{c_1 c_2 \cdots c_n = m \\ c_i \mid m_i \ (1 \le i \le n-1)}} A_\phi\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \frac{m_3 c_2}{c_3}, \dots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right). \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned}\mathcal{A}_p(K_0, K_1, \dots, K_r) &= \sum_{\substack{(c_1 c_2 \cdots c_r) c_n = p^{K_0} \\ c_i \mid p^{K_i} \ (1 \le i \le r)}} A_\phi\left(\frac{p^{K_1} c_n}{c_1}, \frac{p^{K_2} c_1}{c_2}, \dots, \frac{p^{K_r} c_{r-1}}{c_r}, c_r, 1, \dots, 1\right) \\ &= \sum_{k_1=0}^{K_1} \cdots \sum_{k_r=0}^{K_r} A_\phi(p^L, p^{K_2+k_1-k_2}, p^{K_3+k_2-k_3}, \dots, p^{K_r+k_{r-1}-k_r}, p^{k_r}, 1, \dots, 1).\end{aligned}$$

We use here that the sum over c_i's in the first line above is equivalent to the sum over k_i's in the second since $c_i \mid p^{K_i}$ implies we can take $c_i = p^{k_i}$ with $0 \le k_i \le K_i$. Also, $c_n = p^{K_0}/p^{k_1+k_2+\cdots+k_r}$, which multiplied by $\frac{m_1}{c_1} = p^{K_1-k_1}$ gives p^L with L as claimed. ∎

Fix a prime p. Next we prove that every Fourier coefficient of a Maass form ϕ for $SL(n, \mathbb{Z})$ of the form $A(p^{K_1}, p^{K_2}, \dots, p^{K_{n-1}})$ is an eigenfunction of an element in the Hecke algebra $\mathcal{H}_p$ as defined in Definition 1.1.

PROPOSITION 2.4. *Let* $K_1, K_2, \dots, K_{n-1} \in \mathbb{Z}_{\geq 0}$. *Then*

$$A_\phi(p^{K_1}, p^{K_2}, \dots, p^{K_{n-1}}) \in \mathcal{H}_p^*.$$

PROOF. We shall prove that for every integer $1 \leq r \leq n-1$

$$A_\phi(p^{K_1}, \dots, p^{K_r}, 1, \dots, 1) \in \mathcal{H}_p^*. \tag{2.5}$$

It is obvious that (2.5) holds for $r = 1$ since $A_\phi(p^{K_1}, 1, \dots, 1)$ is an eigenvalue of $T_{p^{K_1}}$. We complete the proof by using induction on r. Assume (2.5) holds for every $r \leq \mathrm{r}$ with $\mathrm{r} \geq 1$. Then we want to prove that (2.5) holds for $r = \mathrm{r}+1$.

Now it follows from Lemma 2.2 that if $K_0 \geq K_1 + K_2 + \cdots + K_r$, then

$$\begin{aligned}&\mathcal{A}_p(K_0, K_1, \dots, K_\mathrm{r})\\&= \sum_{k_1=0}^{K_1} \cdots \sum_{k_\mathrm{r}=0}^{K_\mathrm{r}} A_\phi(p^L, p^{K_2+k_1-k_2}, p^{K_3+k_2-k_3}, \dots, p^{K_\mathrm{r}+k_{\mathrm{r}-1}-k_\mathrm{r}}, p^{k_\mathrm{r}}, 1, \dots, 1),\end{aligned}$$

where $L = K_0 + K_1 - 2k_1 - k_2 \cdots - k_\mathrm{r}$. It is clear by induction that

$$\mathcal{A}_p(K_0, K_1, \dots, K_\mathrm{r}) \in \mathcal{H}_p^*.$$

Note that if we make the simple change of variables

$$K_0 \to K_0 + 1, \quad K_1 \to K_1 - 1,$$

then $A_\phi(p^L, p^{K_2+k_1-k_2}, p^{K_3+k_2-k_3}, \dots, p^{K_\mathrm{r}+k_{\mathrm{r}-1}-k_\mathrm{r}}, p^{k_\mathrm{r}}, 1, \dots, 1)$ does not change at all. It follows that

$$\begin{aligned}&\mathcal{A}_p(K_0, K_1, \dots, K_\mathrm{r}) - \mathcal{A}_p(K_0+1, K_1-1, \dots, K_\mathrm{r})\\&= \sum_{k_2=0}^{K_2} \cdots \sum_{k_r=0}^{K_\mathrm{r}} A_\phi\big(p^{K_0-K_1-k_2-\cdots-k_\mathrm{r}}, p^{K_2+K_1-k_2}, p^{K_3+k_2-k_3}, \dots,\\&\qquad\qquad p^{K_\mathrm{r}+k_{\mathrm{r}-1}-k_\mathrm{r}}, p^{k_\mathrm{r}}, 1, \dots, 1\big).\end{aligned} \tag{2.6}$$

Note that the entry p^{k_r} occurs in the $(\mathrm{r}+1)$st position.

Next, it is clear that if we make the simple changes

$$K_0 \to K_0 + 1, \quad K_1 \to K_1 + 1, \quad K_2 \to K_2 - 1,$$

then $A_\phi(p^{K_0-K_1-k_2-\cdots-k_\mathrm{r}}, p^{K_2+K_1-k_2}, p^{K_3+k_2-k_3}, \dots, p^{K_\mathrm{r}+k_{\mathrm{r}-1}-k_\mathrm{r}}, p^{k_\mathrm{r}}, 1, \dots, 1)$ does not change.

It follows as before that

$$\big(\mathcal{A}_p(K_0, K_1, K_2, \dots, K_{\mathrm{r}}) - \mathcal{A}_p(K_0+1, K_1-1, K_2, \dots, K_{\mathrm{r}})\big) - \big(\mathcal{A}_p(K_0+1, K_1+1, K_2-1, \dots, K_{\mathrm{r}}) - \mathcal{A}_p(K_0+2, K_1, K_2-1, \dots, K_{\mathrm{r}})\big) = \sum_{k_3=0}^{K_3} \cdots \sum_{k_r=0}^{K_{\mathrm{r}}} A_\phi(p^{K_0-K_1-K_2-k_3-\cdots-k_{\mathrm{r}}}, p^{K_1}, p^{K_3+K_2-k_3}, \dots, p^{K_{\mathrm{r}}+k_{\mathrm{r}-1}-k_{\mathrm{r}}}, p^{k_{\mathrm{r}}}, 1, \dots, 1).$$

This process can be continued inductively until we reach the final conclusion that

$$A_\phi(p^{K_0-K_1-K_2-K_3-\cdots-K_{\mathrm{r}}}, p^{K_1}, p^{K_2}, p^{K_3}, \dots, p^{K_{\mathrm{r}-1}}, p^{K_{\mathrm{r}}}, 1, \dots, 1) \in \mathcal{H}_p^*.$$

Here we simply choose K_0 sufficiently large which allows us to prove this result for every non-negative power of p in the first position of the Fourier coefficient A_ϕ above. ■

It remains to prove our main theorem.

Theorem 2.7 (Multiplicativity of Fourier coefficients). *Let $n \geq 2$ and let ϕ be a Maass form for $\mathrm{SL}(n, \mathbb{Z})$ with arithmetic Fourier coefficients $A_\phi(m_1, \dots, m_{n-1})$, normalized so that*

$$A_\phi(1, 1, \dots, 1) = 1.$$

Then

$$A_\phi(m_1 m_1', m_2 m_2', \dots, m_{n-1} m_{n-1}') = A_\phi(m_1, m_2, \dots, m_{n-1}) \cdot A_\phi(m_1', m_2', \dots, m_{n-1}')$$

if $\gcd(\prod_{i=1}^{n-1} m_i, \prod_{i=1}^{n-1} m_i') = 1$.

Proof. The proof is based on a simple variant of Lemma 2.2 which we now present.

Lemma 2.8. *Let p be a fixed prime where $p \nmid m_1 m_2 \cdots m_{n-1}$. We have*

$$A_\phi(p^{K_0}, 1, \dots, 1) \cdot A_\phi(p^{K_1} m_1, p^{K_2} m_2, \dots, p^{K_r} m_r, m_{r+1}, \dots, m_{n-1}) = \sum_{k_1=0}^{K_1} \cdots \sum_{k_r=0}^{K_r} A_\phi(p^L m_1, p^{K_2+k_1-k_2} m_2, \dots, p^{K_r+k_{r-1}-k_r} m_r, p^{k_r} m_{r+1}, m_{r+2}, \dots, m_{n-1}),$$

where $L = K_0 + K_1 - 2k_1 - k_2 \cdots - k_r$.

PROOF. The proof is exactly the same as the proof of Lemma 2.2. In fact, since $p \nmid m_1 m_2 \cdots m_{n-1}$, it follows that the sum in equation (2.3) which takes the form

$$\sum_{\substack{c_1 c_2 \cdots c_n = p^{K_0} \\ c_i \mid m_i \ (1 \le i \le n-1)}}$$

tells us that each $c_i = p^{k_i}$ as before since all the c_i have to divide p^{K_0}. The proof immediately follows from equation (2.3). ∎

Completion of the proof of Theorem 2.7. It is enough to prove that if p is a fixed prime where $p \nmid m_1 m_2 \cdots m_{n-1}$, then for every $1 \le r \le n-1$ and all $K_1, \ldots, K_r \in \mathbb{Z}_{\ge 0}$ we have

$$\begin{aligned}(2.9)\quad A_\phi(p^{K_1} m_1, \ldots, p^{K_r} m_r, m_{r+1}, \ldots, m_{n-1}) \\ = A_\phi(p^{K_1}, \ldots, p^{K_r}, 1, \ldots, 1) \cdot A_\phi(m_1, \ldots, m_{n-1}).\end{aligned}$$

It follows easily from Lemma 2.8 that (2.9) holds for $r = 1$. We complete the proof of (2.9) by using induction on r. Assume (2.9) holds for every $r \le \mathrm{r}$ with $\mathrm{r} \ge 1$. Then we want to prove that (2.9) holds for $r = \mathrm{r} + 1$. Now it follows from Lemma 2.8 that

$$\begin{aligned}(2.10)\quad & A_\phi(p^{K_0}, 1, \ldots, 1) \cdot A_\phi(p^{K_1} m_1, p^{K_2} m_2, \ldots, p^{K_\mathrm{r}} m_\mathrm{r}, m_{\mathrm{r}+1}, \ldots, m_{n-1}) \\ &= \sum_{k_1=0}^{K_1} \cdots \sum_{k_r=0}^{K_r} A_\phi\big(p^L m_1, p^{K_2+k_1-k_2} m_2, \ldots, p^{K_\mathrm{r}+k_{\mathrm{r}-1}-k_\mathrm{r}} m_\mathrm{r}, \\ &\qquad\qquad p^{k_\mathrm{r}} m_{\mathrm{r}+1}, m_{\mathrm{r}+2}, \ldots, m_{n-1}\big),\end{aligned}$$

where $L = K_0 + K_1 - 2k_1 - k_2 \cdots - k_r$. If we make the change

$$K_0 \to K_0 + 1, \quad K_1 \to K_1 - 1,$$

then the coefficient $A_\phi(*)$ on the right-hand side of (2.10) does not change at all. It follows that

$$\begin{aligned}(2.11)\quad & \big(A_\phi(p^{K_0}, 1, \ldots, 1) \cdot A_\phi(p^{K_1} m_1, p^{K_2} m_2, \ldots, p^{K_\mathrm{r}} m_\mathrm{r}, m_{\mathrm{r}+1}, \ldots, m_{n-1})\big) \\ & - \big(A_\phi(p^{K_0+1}, 1, \ldots, 1) \cdot A_\phi(p^{K_1-1} m_1, p^{K_2} m_2, \ldots, p^{K_\mathrm{r}} m_\mathrm{r}, m_{\mathrm{r}+1}, \ldots, m_{n-1})\big) \\ &= \sum_{k_2=0}^{K_2} \cdots \sum_{k_r=0}^{K_r} A_\phi\big(p^L m_1, p^{K_2+K_1-k_2} m_2, \ldots, p^{K_\mathrm{r}+k_{\mathrm{r}-1}-k_\mathrm{r}} m_\mathrm{r}, \\ &\qquad\qquad p^{k_\mathrm{r}} m_{\mathrm{r}+1}, m_{\mathrm{r}+2}, \ldots, m_{n-1}\big),\end{aligned}$$

where $L = K_0 - K_1 - k_2 \cdots - k_r$.

Now note that by the inductive hypothesis, the left side of (2.11) can be written as

$$\big(A_\phi(p^{K_0},1,\dots,1)\cdot A_\phi(p^{K_1},p^{K_2},\dots,p^{K_\mathfrak{r}},1,\dots,1)$$
$$-A_\phi(p^{K_0},1,\dots,1)\cdot A_\phi(p^{K_1},p^{K_2},\dots,p^{K_\mathfrak{r}},1,\dots,1)\big)\cdot A_\phi(m_1,\dots,m_{n-1}).$$

The induction process can be continued in exactly the same way as the proof of Proposition 2.4 leading to the final result that

$$A_\phi(p^L,p^{K_1},p^{K_2},p^{K_3},\dots,p^{K_{\mathfrak{r}-1}},p^{K_\mathfrak{r}},1,\dots,1)\cdot A_\phi(m_1,\dots,m_{n-1})$$
$$=A_\phi(p^Lm_1,p^{K_1}m_2,\dots,p^{K_{\mathfrak{r}-1}}m_\mathfrak{r},p^{K_\mathfrak{r}}m_{\mathfrak{r}+1},\dots,m_{n-1})$$

with $L=K_0-K_1-K_2-K_3-\cdots-K_\mathfrak{r}$. ∎

REMARK 2.12. If we assume all $m_i=1$ (for $i=1,\dots,n-1$), the first inductive step (2.11) in the above proof is exactly the same as the first inductive step (2.6) in the proof of Proposition 2.4. In fact, all the inductive steps in the proof of the multiplicativity relation (2.9) will exactly match the inductive steps in the proof of Proposition 2.4 if all $m_i=1$. When the m_i are not all equal to 1 (since $p\nmid m_i$ for $i=1,\dots,n-1$), there is really no change in the Hecke identity (2.3) except that the m_i are inserted in the ith place of the Fourier coefficient A_ϕ.

3. THE EXAMPLE $A_\phi(1,\dots,1,p,1,\dots,1)$

Recall that a *composition* of a positive integer ℓ is an ordered tuple $(i_1,i_2,\dots,i_r)$ ($r\in\mathbb{Z}_{\geq 1}$) of positive integers such that

$$i_1+i_2+\cdots+i_r=\ell.$$

We have the following.

PROPOSITION 3.1. *For $n\geq 2$ and $1\leq\ell\leq n-1$, let $\mathcal{C}_\ell$ denote the set of compositions of ℓ. Then*

$$A_\phi(1,\dots,1\underbrace{\ p\ }_{\ell\text{th place}},1,\dots,1)=\sum_{(i_1,i_2,\dots,i_r)\in C_\ell}\prod_{j=1}^{r}(-1)^{i_j+1}A_\phi(p^{i_j},1,\dots,1). \tag{3.2}$$

In other words, the combination of Hecke operators whose eigenvalue equals the left-hand side of (3.2) is $\sum_{(i_1,i_2,\dots,i_r)\in C_\ell}\prod_{j=1}^{r}(-1)^{i_j+1}T_{p^{i_j}}$.

To prove this proposition, we need two lemmas, for which, in turn, we introduce some notation.

DEFINITION 3.3. Let $n \geq 2$; let p be prime; let $j \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$. We define

$$A_{j,k}(p) = \begin{cases} A_\phi(p^j, 1, \ldots, 1) & \text{if } k = 0 \text{ or } k = n; \\ A_\phi(p^{j+1}, 1, \ldots, 1) & \text{if } k = 1; \\ A_\phi(p^j, 1, \ldots, 1, \underbrace{p}_{k\text{th place}}, 1, \ldots, 1) & \text{if } 2 \leq k \leq n-1. \end{cases}$$

That is, $A_{j,k}(p)$ is the result of multiplying the kth coordinate in the argument of $A_\phi(p^j, 1, \ldots, 1)$ by p, where the cases $k = 0$ and $k = n$ correspond to no extra factor of p.

We have the following.

LEMMA 3.4. *For $n \geq 2$, p prime, $j \in \mathbb{Z}_{\geq 1}$, and $1 \leq k \leq n-1$,*

$$A_{j,0}(p) A_{0,k}(p) = A_{j,k}(p) + A_{j-1,k+1}(p). \tag{3.5}$$

PROOF. Into (2.3), we put

$$m = p^j, \quad m_k = p, \quad m_i = 1 \quad (1 \leq i \leq n-1,\ i \neq k),$$

so that the left-hand side of (2.3) equals $A_{j,0}(p) A_{0,k}(p)$. Then the conditions

$$c_1 c_2 \cdots c_n = m; \quad c_i | m_i \quad (1 \leq i \leq n-1)$$

imply that the sum in (2.3) entails two summands: either when $c_n = p^j$ and $c_i = 1$ for $1 \leq i \leq n-1$; or when $c_n = p^{j-1}$, $c_k = p$, and $c_i = 1$ for $1 \leq i \leq n-1$ and $i \neq k$. In the first case, the corresponding summand equals $A_{j,k}(p)$; in the second case, this summand equals $A_{j-1,k+1}(p)$. From this, (3.5) follows. ∎

A consequence of the above lemma is the following.

LEMMA 3.6. *For $2 \leq \ell \leq n-1$, we have*

$$A_\phi(1, \ldots, 1, \underbrace{p}_{\ell\text{th place}}, 1, \ldots, 1) = \sum_{m=1}^{\ell} (-1)^{m+1} A_\phi(p^m, 1, \ldots, 1) A_\phi(1, \ldots, 1, \underbrace{p}_{(\ell-m)\text{th place}}, 1, \ldots, 1), \tag{3.7}$$

with the understanding that, when $\ell = m$,

$$A_\phi(1, \ldots, 1, \underbrace{p}_{(\ell-m)\text{th place}}, 1, \ldots, 1)$$

simply denotes the constant 1.

PROOF. Putting $j = 1$ and $k = \ell - 1$ into (3.5), and rearranging, gives

$$A_{0,\ell}(p) = A_{1,0}(p)A_{0,\ell-1}(p) - A_{1,\ell-1}(p). \tag{3.8}$$

Next, to the term $A_{1,\ell-1}(p)$ in (3.8), we apply (3.5) with $j = 2$ and $k = \ell - 2$. We get

$$A_{0,\ell}(p) = A_{1,0}(p)A_{0,\ell-1}(p) - A_{2,0}(p)A_{0,\ell-2}(p) + A_{2,\ell-2}(p).$$

Iterating this process ultimately yields

$$\begin{aligned} A_{0,\ell}(p) = {} & A_{1,0}(p)A_{0,\ell-1}(p) - A_{2,0}(p)A_{0,\ell-2}(p) \\ & + A_{3,0}(p)A_{0,\ell-3}(p) - \cdots + (-1)^{\ell+1}A_{\ell,0}(p), \end{aligned}$$

which is precisely the statement (3.7). ■

PROOF OF PROPOSITION 3.1. We apply strong induction on ℓ (for fixed n).

The formula (3.2) is clearly true in the case $\ell = 1$. So assume that it holds for any of the integers $1, 2, \ldots, \ell - 1$ in place of ℓ. Then, by Lemma 3.6, we have

$$\begin{aligned} & A_\phi(1, \ldots, 1, \underbrace{p}_{\ell\text{th place}}, 1, \ldots, 1) \\ & = \sum_{m=1}^{\ell} (-1)^{m+1} A_\phi(p^m, 1, \ldots, 1) \cdot \sum_{(i_1, i_2, \ldots, i_r) \in C_{\ell-m}} \prod_{j=1}^{r} (-1)^{i_j+1} A_\phi(p^{i_j}, 1, \ldots, 1) \\ & = \sum_{m=1}^{\ell} \sum_{(i_1, i_2, \ldots, i_r) \in C_{\ell-m}} (-1)^{m+1} A_\phi(p^m, 1, \ldots, 1) \prod_{j=1}^{r} (-1)^{i_j+1} A_\phi(p^{i_j}, 1, \ldots, 1). \end{aligned} \tag{3.9}$$

But, for $1 \le m \le \ell$, $(i_1, i_2, \ldots, i_r)$ is a composition of $\ell - m$ if and only if $(i_1, i_2, \ldots, i_r, m)$ is a composition of ℓ. So, putting $m = i_{r+1}$ into (3.9), we get

$$A_\phi(1, \ldots, 1, \underbrace{p}_{\ell\text{th place}}, 1, \ldots, 1) = \sum_{(i_1, i_2, \ldots, i_r, i_{r+1}) \in C_\ell} \prod_{j=1}^{r+1} (-1)^{i_j+1} A_\phi(p^{i_j}, 1, \ldots, 1). \quad ■$$

4. THE EXAMPLE $A_\phi(1, p^j, 1, \ldots, 1)$

We have the following.

PROPOSITION 4.1. *For $n \ge 3$ and $j \in \mathbb{Z}_{\ge 1}$, we have*

$$A_\phi(1, p^j, 1, \ldots, 1) = A_\phi(p^j, 1, \ldots, 1)^2 - A_\phi(p^{j-1}, 1, \ldots, 1)A_\phi(p^{j+1}, 1, \ldots, 1).$$

In other words, the combination of Hecke operators whose eigenvalue is $A_\phi(1, p^j, 1, \ldots, 1)$ is $(T_{p^j})^2 - T_{p^{j-1}}T_{p^{j+1}}$.

PROOF. Let $n \geq 3$ and $j \in \mathbb{Z}_{\geq 1}$.

We first put

$$m = m_1 = p^j, \quad m_i = 1 \quad (2 \leq i \leq n-1)$$

into (2.3). For such m_i's and m, the set of n-tuples $(c_1, c_2, \dots, c_n)$ such that $c_1 c_2 \cdots c_n = m$ and $c_i | m_i$ for $1 \leq i \leq n-1$ is just the set

$$\{(p^k, 1, \dots, 1, p^{j-k}) \in (\mathbb{Z}_{\geq 1})^n \mid 0 \leq k \leq j\}.$$

So (2.3) yields

$$\begin{aligned}(4.2) \qquad A_\phi(p^j, 1, \dots, 1)^2 &= \sum_{k=0}^{j} A\left(\frac{p^j \cdot p^{j-k}}{p^k}, \frac{1 \cdot p^k}{1}, \frac{1 \cdot 1}{1}, \dots, \frac{1 \cdot 1}{1}\right) \\ &= \sum_{k=0}^{j} A_\phi(p^{2j-2k}, p^k, 1, \dots, 1).\end{aligned}$$

Next, into (2.3), we put

$$m = p^{j-1}, \quad m_1 = p^{j+1}, \quad m_i = 1 \quad (2 \leq i \leq n-1).$$

In this case, the set of n-tuples $(c_1, c_2, \dots, c_n)$ such that $c_1 c_2 \cdots c_n = m$ and $c_i | m_i$ for $1 \leq i \leq n-1$ equals the set

$$\{(p^k, 1, \dots, 1, p^{j-1-k}) \in (\mathbb{Z}_{\geq 1})^n \mid 0 \leq k \leq j-1\}.$$

So (2.3) gives

$$\begin{aligned}(4.3) \qquad & A_\phi(p^{j-1}, 1, \dots, 1) A_\phi(p^{j+1}, 1, \dots, 1) \\ &= \sum_{k=0}^{j-1} A\left(\frac{p^{j+1} \cdot p^{j-1-k}}{p^k}, \frac{1 \cdot p^k}{1}, \frac{1 \cdot 1}{1}, \dots, \frac{1 \cdot 1}{1}\right) \\ &= \sum_{k=0}^{j-1} A_\phi(p^{2j-2k}, p^k, 1, \dots, 1).\end{aligned}$$

Subtracting (4.3) from (4.2) yields

$$\begin{aligned}& A_\phi(p^j, 1, \dots, 1)^2 - A_\phi(p^{j-1}, 1, \dots, 1) A_\phi(p^{j+1}, 1, \dots, 1) \\ &\quad = A_\phi(p^{2j-2j}, p^j, 1, \dots, 1) = A_\phi(1, p^j, 1, \dots, 1),\end{aligned}$$

which is the desired result. ∎

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