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Partial Differential Equations. – *On saddle solutions of the Allen–Cahn equation I: Pointwise estimates*, by Yong Liu, Kelei Wang and Juncheng Wei, accepted on 1 July 2025.

Dedicated to Professor Enrico Bombieri on the occasion of his 85th birthday.

Abstract. – Saddle solutions of the Allen–Cahn equation in $\mathbb{R}^{2m}$ are characterized by the property that they vanish precisely on the Simons cones, a family of classical minimal surfaces with one singularity at the origin. Their existence and uniqueness are known, by results of Cabré–Terra (2009–2012) and Dang (1992). Schatzman (1995) proved that the saddle solution is unstable for $m = 1$. Cabré–Terra (2009–2012) showed the instability for $m = 2, 3$ and stability for $m \geq 7$. This left open the case of $m = 4, 5, 6$, which is conjectured to be stable (even energy minimizing). Towards this conjecture, here we establish some pointwise estimates for the saddle solutions.

Keywords. – Allen–Cahn equation, saddle solution, Simons cone, stability.

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1. Introduction

Allen–Cahn type equation in the entire space has the form

$$-\Delta u = F'(u) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where F is a "double well" potential. Typical examples of F include $\frac{1}{4}(1-u^2)^2$ or $1 + \cos u$. The first case corresponds to the classical Allen–Cahn nonlinearity, which arises from phase transition theory. The second case is also called elliptic sine-Gordon equation, originating from integrable system theory. In this paper, we will focus on the Allen–Cahn nonlinearity.

Properties of solutions for the equation (1.1) depend quite delicately on the dimension n. In the simplest case of dimension one, all the solutions can be completely understood, thanks to the phase plane analysis technique. In this case, (1.1) has a heteroclinic solution H. It is monotone increasing and plays an important role in the De Giorgi conjecture. Recall that the De Giorgi conjecture, which originates from the Γ-convergence theory [11], states that monotone bounded solutions of (1.1) have to be one-dimensional if $n \le 8$. This conjecture has been proved to be true in dimension $n = 2$ (Ghoussoub–Gui [21]), $n = 3$ (Ambrosio–Cabré [1,3]). In dimension $4 \le n \le 8$, Savin [35] proved it under an additional limiting condition:

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Up to now, it is still not known whether this condition can be removed. Counter examples in dimension $n \ge 9$ have been constructed by Del Pino–Kowalczyk–Wei in [13] using Lyapunov–Schmidt reduction. The zero level set of these solutions resemble the famous Bombieri–De Giorgi–Giusti minimal graphs [4] dating back to 1969, which constitute first counter-examples of the Bernstein conjecture. Later on, we also constructed new nontrivial monotone solutions of the Allen–Cahn equation in [31] using Jerison–Monneau's program [25]. We refer to [14–19, 26, 34] and the references therein for related results on this subject.

It is by now commonly accepted that the theory of Allen–Cahn equation has deep relations with the minimal surface theory. The above mentioned De Giorgi conjecture is such an example. To mention more recent developments in this direction, let us point out the references [8, 9, 20, 22], where minimax of the Allen–Cahn and minimal surfaces are discussed, with deep application to the minimal surface theory. The saddle solution to be analyzed in this paper also has analogy in the minimal surface theory. To explain this, let us now recall some results from the minimal surface theory.

It is well known that in $\mathbb{R}^8$, the Simons cone is a minimal hypersurface. It is singular at the origin and minimizes the area. Explicitly, the Simons cone is given by

$$\left\{(x_1, \ldots, x_8) \in \mathbb{R}^8 : x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\right\}.$$

The minimality of the Simons cone is proved in [4,12] and the construction of Bombieri–De Giorgi–Giusti minimal graph is essentially based on this property. More generally, if we consider the so-called Lawson's cone in $\mathbb{R}^{i+j}$ $(2 \le i \le j)$:

$$C_{i,j} = \left\{(x, y) \in \mathbb{R}^i \oplus \mathbb{R}^j : |x|^2 = \frac{i-1}{j-1}|y|^2\right\},$$

then for $i + j \le 7$, $C_{i,j}$ is an unstable minimal cone (Simons [38]). It is also known that for $i + j \ge 8$, and $(i, j) \ne (2, 6)$, $C_{i,j}$ are area minimizing, and $C_{2,6}$ is not area minimizing but it is one-sided minimizer. See [2, 12, 24, 27–29, 37] for more detailed discussion on this subject.

For each $m \in \mathbb{N}$, the analogous object of the Simons cone $C_{m,m}$ in the theory of Allen–Cahn equation is the so-called saddle-shaped solution defined in $\mathbb{R}^{2m}$, which vanishes precisely on the cone $C_{m,m}$. In this paper, it will simply be called saddle solution and denoted by U_m. The properties of this solution have been analyzed in [10, 23, 36] for $m = 1$. In the higher-dimensional case, it was studied in a series of important papers: Cabré–Terra [6,7] and Cabré [5]. In particular, it has been proved in [5] that saddle solution is unique (up to a sign) in the class of symmetric functions.

In view of the stability of the Simons cone, an important question is whether saddle solution is stable or energy minimizing. Recall that by definition, U_m is stable if and only if

$$\int_{\mathbb{R}^{2m}} \left[-\Delta\eta + (3U_m^2 - 1)\eta \right] \eta \geq 0, \text{ for any } \eta \in C_0^\infty(\mathbb{R}^{2m}).$$

In [6,7], it is proved that for $2 \leq m \leq 3$, the saddle solution is unstable, while for $m \geq 7$, it is stable [5]. It is then conjectured in [5] that for each $m \geq 4$, U_m should be stable. Indeed, it is even expected that U_m should be a global minimizer of the corresponding energy functional for all $m \geq 4$.

The saddle solution depends only on the variables $s := \sqrt{x_1^2 + \cdots + x_m^2}$ and $t := \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2}$. In this case, (1.1) reduces to

$$-\partial_s^2 u - \partial_t^2 u - \frac{m-1}{s}\partial_s u - \frac{m-1}{t}\partial_t u = u - u^3.$$

U_m can be regarded as a function of the variables s and t. In the (s,t) plane, we define the domain

$$\Omega := \{(s,t) : s > t > 0\}. \tag{1.2}$$

Recall that saddle solution vanishes on the Simons cone. This means that $U_m(s,s) = 0$ for all $s \geq 0$. Moreover, $U_m(s,t) = -U_m(t,s)$. Without loss of generality, we may assume $U_m > 0$ in Ω. It is also worth pointing out that U_m is positive on the s axis, that is, $U_m(s,0) > 0$, for $s > 0$.

We will use L_m to denote the linearized Allen–Cahn operator around U_m:

$$L_m\eta := \Delta_m\eta + (1 - 3U_m^2)\eta. \tag{1.3}$$

Here Δ_m is the Laplacian operator in $\mathbb{R}^{2m}$, which can also be written in terms of (s,t) variables.

Proving the stability of the saddle solution amounts to constructing a positive function Φ satisfying

$$L_m\Phi \leq 0 \quad \text{in } \mathbb{R}^{2m}. \tag{1.4}$$

Consider the function

$$f := \left(\tanh\left(\frac{s}{t}\right)\frac{s}{\sqrt{s^2+t^2}}\right)(s+t)^{-2.5},$$

$$h := -\left(\tanh\left(\frac{t}{s}\right)\frac{t}{\sqrt{s^2+t^2}}\right)(s+t)^{-2.5}.$$

In [30], we would like to prove that the function Φ of the form $f\partial_s U_m + h\partial_t U_m$ (plus some small modification terms) satisfies (1.4). One can see that after applying the linearized operator L_m to this function, we obtain derivatives of U_m up to order 2. This means that to prove an inequality like (1.4), very precise pointwise information of the first and second derivatives of U_m is needed. This is the main motivation of this paper.

It will also be natural to ask that whether the stability of the saddle solution depends on the double well potential F. Note that for elliptic sine-Gordon equation in the plane

$$-\Delta u = \sin u \quad \text{in } \mathbb{R}^2, \tag{1.5}$$

it can be checked [32] that the explicit function

$$4\arctan\left(\frac{\cosh\left(\frac{y}{\sqrt{2}}\right)}{\cosh\left(\frac{x}{\sqrt{2}}\right)}\right) - \pi$$

is a saddle solution to (1.5). However, in dimension $2m$ with $m > 1$, the saddle solution of (1.5) does not have explicit expression. This makes precise estimation of U_m as well as its derivatives a nontrivial task.

Our main results in this paper include pointwise estimate of U_m itself, both from below and above, and some estimates of its first- and second-order derivatives. Some of these estimates are not sharp and can be significantly improved in a future work [30]. To state our main results, let us recall that in [5], it has already been proven that in Ω,

$$\partial_s U_m + \partial_t U_m > 0, \quad \partial_s U_m > 0, \quad \partial_t U_m < 0, \quad \partial_s\partial_t U_m > 0.$$

Following [5], we introduce new variables

$$y = \frac{s+t}{\sqrt{2}}, \quad z = \frac{s-t}{\sqrt{2}}.$$

We collect some of those estimates derived in this paper in the following.

THEOREM 1.1. *In Ω, the first-order derivatives of U_m satisfy*

$$t\partial_s U_m + s\partial_t U_m \le 0,$$
$$\left(\frac{1}{y} + \frac{1}{z}\right)U_m - \partial_y U_m - \partial_z U_m \ge 0.$$

The second-order derivatives satisfy $\partial_t^2 U_m \le 0$, $\partial_s^2 U_m$ changes sign, and

$$\frac{1}{s}\partial_s U_m - \partial_s^2 U_m \ge 0,$$
$$\sqrt{2}U_m\partial_s U_m + \partial_s^2 U_m \ge 0, \quad \sqrt{2}U_m\partial_t U_m + \partial_s\partial_t U_m \le 0,$$
$$\frac{1}{s}\partial_s U_m + \frac{1}{t}\partial_t U_m - \partial_s^2 U_m - \partial_t^2 U_m \ge 0, \quad -\frac{1}{t}\partial_t U_m + \partial_s\partial_t U_m + \partial_t^2 U_m \ge 0.$$

Moreover, $U_m \geq H(a_m y)H(a_m z)$*, where the constant* a_m *is given by* (2.5)*. We also have*

$$\partial_s U_4 \leq 2\left(e^{\frac{3}{4}t} + \frac{2}{\sqrt{t}}\right)e^{-\frac{3}{4}s} \quad \textit{for } s - t \geq 4,$$
$$\partial_s U_4 + \partial_t U_4 \leq \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left[4(\partial_s U_4 - \partial_t U_4) + \sqrt{\partial_s U_4 - \partial_t U_4}\right].$$

Note that the last two estimates are only stated for U_4, but similar estimates hold for U_5, U_6, possibly with different coefficients.

During the proof of this theorem, we heavily use the Maximum Principle (Theorem 2.1), due to Cabré [5], and apply the linearized Allen–Cahn operator on various carefully designed functions. The rest of the paper is devoted to the proof of these pointwise estimates.

2. Pointwise estimates for the saddle solution and its derivatives

The area-minimizing property of the Simons cone can be proved by using explicitly constructed calibrations. Saddle solution can be regarded as an object which is one-dimensional higher than the Simons cone. To prove its stability directly by constructing suitable test functions, it seems to be necessary to obtain some a priori information of the solution, especially in the region away from the Simons cone. For instance, in dimension $n = 2m \geq 14$, monotonicity and the asymptotic behavior are used in an essential way [5]. In dimensions 8, 10, 12, much more precise estimates on the saddle solution and its derivatives are needed. Note that although the saddle solution is unique in each dimension, we believe it does not have a closed form expression when the dimension is larger than 2. As a consequence, we should perform a lot of careful analysis to obtain these estimates, which will be the main task of this section.

Our analysis relies on the special structure of the equation and also crucially on the maximum principle of Cabré [5], which will be recalled in the following.

Theorem 2.1 ([5, Proposition 2.2]). *Let* L_m *be the operator defined in* (1.3)*. Let* Θ *be an open set contained in the domain*

$$\mathcal{O} := \left\{(x_1, \ldots, x_{2m}) : s > t \geq 0\right\}.$$

Suppose $c(\cdot)$ *is a nonpositive continuous function in* Θ*. Then the maximum principle holds for the operator* $L_m + c$ *in the following sense: Suppose* $v \in C^2(\Theta) \cap C(\overline{\Theta})$ *satisfies*

$$L_m v + cv \geq 0 \textit{ in } \Theta; \quad v \leq 0 \textit{ on } \partial\Theta$$

and

$$\limsup_{x\in\Theta,|x|\to+\infty} v(x) \le 0. \tag{2.1}$$

Then necessarily $v \le 0$ *in* Θ.

Remark that the domain $\Omega \subset \mathcal{O}$. Hence, in the statement of Theorem 2.1, $\mathcal{O}$ can be replaced by Ω. Note that condition (2.1) is important. In the application below, various functions we considered do satisfy this condition although sometimes we will not mention them explicitly.

We will also need the next lemma, which essentially follows from the monotone iteration arguments of [7, Lemma 3.4], and Theorem 2.1 stated above.

Lemma 2.2. *Suppose* $\bar{w} \in C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ *satisfies*

$$\bar{w} > 0 \text{ in } \mathcal{O}; \quad \bar{w} = 0 \text{ on } \partial\mathcal{O}.$$

Moreover, we assume

$$-\Delta\bar{w} - \bar{w} + \bar{w}^3 \ge 0 \quad \text{in } \mathcal{O}. \tag{2.2}$$

Then $U_m \le \bar{w}$. *Similarly, suppose* Ξ *is a subdomain (with piecewise* C^1 *boundary) of* $\mathcal{O}$ *and* $\underline{w} \in C^2(\Xi) \cap C(\bar{\Xi})$ *satisfies*

$$\underline{w} > 0 \text{ in } \Xi; \quad \underline{w} = 0 \text{ on } \partial\Xi,$$

and

$$-\Delta\underline{w} - \underline{w} + \underline{w}^3 \le 0 \quad \text{in } \Xi;$$

then $U_m \ge \underline{w}$ *in* Ξ.

Proof. Starting with the supersolution $\bar{w}$, we can use monotone iteration technique as that of [7, Lemma 3.4] to obtain a sequence of monotone decreasing functions which converges to a positive solution of the Allen–Cahn equation w in $\mathcal{O}$ with $w = 0$ on $\partial\mathcal{O}$. By the uniqueness of saddle solution, we obtain $w = U_m \le \bar{w}$.

On the other hand, if we start with $\underline{w}$, we can use monotone iteration to find a sequence of monotone increasing functions which converges to a positive solution v (may not be the saddle solution) of the Allen–Cahn equation in Ξ, with $v = 0$ on $\partial\Xi$. To see that $v \le U_m$, we assume to the contrary that $v > U_m$ in a subdomain $\Gamma \subset \Xi$, and $v = U_m$ on $\partial\Gamma$. Then

$$L_m(v - U_m) \ge 0, \quad \text{in } \Gamma.$$

This contradicts with the maximum principle stated in Theorem 2.1. ∎

2.1. *Lower and upper bound of the saddle solution*

To begin with, we would like to estimate the saddle solution U_m itself both from below and above.

The saddle solution satisfies

$$\partial_s^2 U_m + \partial_t^2 U_m + \frac{m-1}{s}\partial_s U_m + \frac{m-1}{t}\partial_t U_m = U_m^3 - U_m. \tag{2.3}$$

In the (s,t) coordinate, the linearized operator L_m around U_m takes the form

$$L_m\eta = \partial_s^2\eta + \partial_t^2\eta + \frac{m-1}{s}\partial_s\eta + \frac{m-1}{t}\partial_t\eta + (1-3U_m^2)\eta.$$

Differentiating equation (2.3) with respect to s and t, we get

$$L_m(\partial_s U_m) = \frac{m-1}{s^2}\partial_s U_m, \quad L_m(\partial_t U_m) = \frac{m-1}{t^2}\partial_t U_m. \tag{2.4}$$

These two identities have already been used in [5] to obtain important information of saddle solution and will be frequently used in this paper.

Recall that (see [5]) in the cone Ω, $\partial_s U_m \geq 0$ and $\partial_t U_m \leq 0$; moreover,

$$y = \frac{s+t}{\sqrt{2}}, \quad z = \frac{s-t}{\sqrt{2}} > 0.$$

Then the derivatives of U_m are related by

$$\partial_s U_m = \frac{1}{\sqrt{2}}(\partial_y U_m + \partial_z U_m), \quad \partial_t U_m = \frac{1}{\sqrt{2}}(\partial_y U_m - \partial_z U_m).$$

Since U_m is also monotone increasing in the y direction in Ω, we have $\partial_s U_m + \partial_t U_m \geq 0$.

The next result provides more information on the absolute value of $\partial_s U_m$ and $\partial_t U_m$, in terms of s and t.

LEMMA 2.3. *For any fixed $m \in \mathbb{N}$,*

$$t\partial_s U_m + s\partial_t U_m \leq 0 \quad \text{in } \Omega.$$

PROOF. Let k be a parameter and define

$$\eta := t^k\partial_s U_m + s^k\partial_t U_m.$$

Using (2.4), we compute

$$L_m\eta = t^k\left(\frac{m-1}{s^2} + \frac{k(k+m-2)}{t^2}\right)\partial_s U_m + s^k\left(\frac{m-1}{t^2} + \frac{k(k+m-2)}{s^2}\right)\partial_t U_m$$
$$+ 2k(s^{k-1} + t^{k-1})\partial_s\partial_t U_m.$$

We write this equation as

$$\begin{aligned} L_m\eta - \left(\frac{m-1}{t^2}+\frac{k(k+m-2)}{s^2}\right)\eta \\ = t^k\left(\frac{m-1}{s^2}+\frac{k(k+m-2)}{t^2}-\left(\frac{m-1}{t^2}+\frac{k(k+m-2)}{s^2}\right)\right)\partial_s U_m \\ + 2k(s^{k-1}+t^{k-1})\partial_s\partial_t U_m. \end{aligned}$$

We may choose $k = 1$. For this choice, we have

$$m-1-k(k+m-2)=0.$$

This combined with the fact that $\partial_s\partial_t U_m \geq 0$ led to

$$L_m\eta - \left(\frac{m-1}{t^2}+\frac{k(k+m-2)}{s^2}\right)\eta = 2k(s^{k-1}+t^{k-1})\partial_s\partial_t U_m \geq 0.$$

Observe that $t\partial_s U_m + s\partial_t U_m = 0$ on $\partial\Omega$. Indeed, this follows from the facts that $\partial_s U_m = -\partial_t U_m$ if $s = t \geq 0$; $\partial_t U_m = 0$ on the s axis (this is a consequence of the evenness of u in each x_j; see [5, equation (4.4)]). On the other hand, we have

$$t\partial_s U_m + s\partial_t U_m = \frac{t+s}{\sqrt{2}}\partial_y U_m + \frac{t-s}{\sqrt{2}}\partial_z U_m,$$

which implies

$$\limsup_{x\in\Omega,|x|\to+\infty} (t\partial_s U_m + s\partial_t U_m)(x) \leq 0.$$

Hence, from the maximum principle (Theorem 2.1), we deduce that

$$t\partial_s U_m + s\partial_t U_m \leq 0 \quad \text{in } \Omega.$$

This finishes the proof. ∎

Let $H(x) = \tanh(\frac{x}{\sqrt{2}})$ be the monotone increasing one-dimensional heteroclinic solution:

$$-H'' = H - H^3 \text{ with } H(0)=0, \quad H(\pm\infty)=\pm 1.$$

Note that H converges exponentially fast to 1 at infinity. As we will see from a computational perspective, the fact that this heteroclinic solution has an explicit form will help us in later analysis.

In the (y, z) coordinate system, the Allen–Cahn equation in $\mathbb{R}^{2m}$ can be written as

$$-\partial_y^2 u - \partial_z^2 u - \frac{2(m-1)}{y^2-z^2}(y\partial_y u - z\partial_z u) = u - u^3.$$

The following lemma states that saddle solutions in different dimensions are actually ordered.

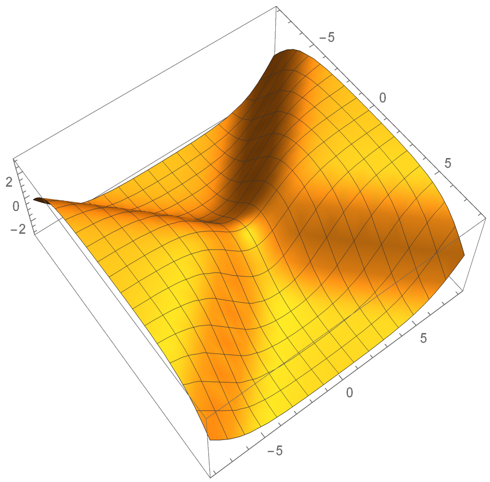


FIGURE 1. 2D saddle solution of the elliptic sine-Gordon equation.

LEMMA 2.4. *For any* $m \in \mathbb{N}$, $U_{m+1} \le U_m \le H(y)H(z)$ *in* $\mathcal{O}$.

PROOF. Using Lemma 2.3, we can compute

$$-\Delta_{m+1}\big(U_m(s,t)\big) - U_m(s,t) + U_m^3(s,t) = -\frac{1}{s}\partial_s U_m - \frac{1}{t}\partial_t U_m \ge 0.$$

It then follows from Lemma 2.2 that $U_{m+1} \le U_m$.

The fact that $U_1(y,z) \le H(y)H(z)$ has already been proven in [36]. The proof of the lemma is then completed. ∎

With upper bound at hand, we proceed to establish a lower bound of U_m. In $\mathbb{R}^2$, we know from [36] that $U_1(y,z) \ge H(\frac{y}{\sqrt{2}})H(\frac{z}{\sqrt{2}})$. For the sine-Gordon nonlinearity, one even has an explicit expression for U_1. See Figure 1 for its picture.

However, in higher dimensions, the lower bound of U_m turns out to be more delicate. We can prove the following.

LEMMA 2.5. *For* $m = 4, 5, 6$, *there holds* $U_m \ge \underline{u} := H(a_m y)H(a_m z)$ *in* $\mathcal{O}$, *where*

$$a_m = \sqrt{\frac{3}{2m+4}}. \tag{2.5}$$

PROOF. Let $a = a_m$ be a parameter and denote the function $H(ay)H(az)$ by η. Let us write $\tilde{y} = ay$, $\tilde{z} = az$. We find that $-\Delta\eta - \eta + \eta^3$ is equal to

$$\begin{aligned} &H(\tilde{y})H(\tilde{z})\big(2a^2 - 1 - a^2H^2(\tilde{y}) - a^2H^2(\tilde{z}) + H^2(\tilde{y})H^2(\tilde{z})\big) \\ &\quad - \frac{2(m-1)a^2}{\tilde{y}^2 - \tilde{z}^2}\big(\tilde{y}H'(\tilde{y})H(\tilde{z}) - \tilde{z}H(\tilde{y})H'(\tilde{z})\big). \end{aligned} \tag{2.6}$$

Let us consider the explicit function

$$\sigma_m(\tilde{y},\tilde{z}) := \frac{2 - H^2(\tilde{y}) - H^2(\tilde{z}) - \frac{2(m-1)}{\tilde{y}^2-\tilde{z}^2}\left(\frac{\tilde{y}H'(\tilde{y})}{H(\tilde{y})} - \frac{\tilde{z}H'(\tilde{z})}{H(\tilde{z})}\right)}{1 - H^2(\tilde{y})H^2(\tilde{z})}.$$

With this definition, we have

$$-\Delta\eta - \eta + \eta^3 = H(\tilde{y})H(\tilde{z})\big(1 - H^2(\tilde{y})H^2(\tilde{z})\big)(a^2\sigma_m - 1).$$

It turns out that

$$\sup_{0<\tilde{z}<\tilde{y}} \sigma_m(\tilde{y},\tilde{z}) = \frac{2m+4}{3},$$

and σ_m tends to this supremum as $(\tilde{y},\tilde{z}) \to 0$. Indeed, for $\tilde{y}$, $\tilde{z}$ close to 0, the Taylor expansion of σ_m (up to order 2) takes the form

$$\frac{2m+4}{3} - \frac{14m+31}{90}(\tilde{y}^2 + \tilde{z}^2).$$

Note that for $\tilde{z} = 0$, the function σ_m reduces to

$$\sigma_m(\tilde{y},0) = 2 - H^2(\tilde{y}) + \frac{2(m-1)}{\tilde{y}^2}\left(1 - \frac{\tilde{y}H'(\tilde{y})}{H(\tilde{y})}\right).$$

This function is monotone decreasing in $\tilde{y}$ and hence bounded from above by $\frac{2m+4}{3}$.

On the other hand, for fixed $\tilde{z}$, as $\tilde{y}$ tends to $+\infty$, σ_m tends to 1, which is obviously less than $\frac{2m+4}{3}$. Moreover, when $\tilde{z} < \tilde{y}$ are both large enough, since $H(z) \sim 1 - 2e^{-\sqrt{2}z}$, σ_m will be close to the function

$$1 - \frac{\sqrt{2}(m-1)}{\tilde{y}^2 - \tilde{z}^2}\frac{(\tilde{y}e^{-\sqrt{2}\tilde{y}} - \tilde{z}e^{-\sqrt{2}\tilde{z}})}{e^{-\sqrt{2}\tilde{y}} + e^{-\sqrt{2}\tilde{z}}}.$$

Denoting $\tilde{y} - \tilde{z}$ by α, we find that this function is equal to

$$\begin{aligned} 1 - \frac{\sqrt{2}(m-1)}{(2\tilde{z}+\alpha)\alpha}\frac{\big((\tilde{z}+\alpha)e^{-\sqrt{2}\alpha} - \tilde{z}\big)}{e^{-\sqrt{2}\alpha}+1} &\le 1 + \frac{(m-1)}{\sqrt{2}\alpha}\frac{(e^{\sqrt{2}\alpha}-1)}{e^{\sqrt{2}\alpha}+1} \\ &\le \frac{m+1}{2} < \frac{2m+4}{3}. \end{aligned}$$

Finally, as $\tilde{y} \to \tilde{z}$, σ_m tends to

$$\frac{2 - 2H^2(\tilde{z}) - \frac{(m-1)}{\tilde{z}}\left(\frac{\tilde{z}H'(\tilde{z})}{H(\tilde{z})}\right)'}{1 - H^4(\tilde{z})}.$$

This function is not monotone decreasing, but it is bounded from above by $\frac{2m+4}{3}$. Now with all this information at hand and observing that the functions are even in the t variable, we can use *Mathematica* (see the appendix for the code) to estimate σ_m in suitable *bounded* region to show that if a_m is chosen to be the number defined in (2.5), then $-\Delta\eta - \eta + \eta^3 \le 0$ in $\mathcal{O}$ and the conclusion of the lemma then readily follows from Lemma 2.2. ∎

This lower bound is far from being optimal because along the Simons cone, as y tends to $+\infty$, U_m should converge to $H(z)$.

In the sequel, to simplify the notation, we shall write $H(y)H(z)$ as $\bar{u}$, and write U_m as u. We also set $\phi = \bar{u} - u \ge 0$. Then by Lemma 2.5,

$$\phi \le \bar{u} - \underline{u} \quad \text{in } \mathcal{O}. \tag{2.7}$$

Note that actually $\underline{u}$ depends on m, while $\bar{u}$ is independent of m. We observe that $\bar{u} - \underline{u}$ does not have decay along the y-axis. Our next purpose is to improve the estimate (2.7). Using similar idea as that of Lemma 2.5, we can prove the following decay estimate.

PROPOSITION 2.6. *In $\mathcal{O}$, we have*

$$\phi \le \frac{H(y)\big(\alpha_m H(z) + \beta_m z H'(z)\big) + H(z)\big(\alpha_m H(y) + \beta_m y H'(y)\big)}{y^2 + z^2}, \tag{2.8}$$

where the constants α_m, β_m are given by

$$(\alpha_m, \beta_m) = \begin{cases} (1,3), & m = 4, \\ (1,4), & m = 5, \\ (2,5), & m = 6. \end{cases} \tag{2.9}$$

PROOF. The "error" of the "approximate solution" $\bar{u}$ can be directly computed:

$$\begin{aligned} -\Delta\bar{u} + \bar{u}^3 - \bar{u} = {} & H(y)H(z)\big(1 - H^2(y)\big)\big(1 - H^2(z)\big) \\ & - \frac{2(m-1)yH'(y)H(z)}{y^2 - z^2} + \frac{2(m-1)zH(y)H'(z)}{y^2 - z^2}. \end{aligned}$$

We have in mind that in $\mathcal{O}$, for y large, the (positive) main-order term in the error should be

$$\frac{2(m-1)zH(y)H'(z)}{y^2 - z^2}. \tag{2.10}$$

In view of this, we introduce the function

$$g(z) := \alpha H(z) + \beta z H'(z),$$

where α, β are two positive constants to be determined later on. Before proceeding, let us explain the reason of choosing such a function g. Observe that H and zH' satisfy

$$-H'' + (3H^2 - 1)H = 2H^3, \tag{2.11}$$

$$-(zH')'' + (3H^2 - 1)zH' = -2H'' = 2(1 - H^2)H. \tag{2.12}$$

Intuitively, we see from these two identities that after applying the linearized Allen–Cahn operator, the function H is well behaved for z large; but for z close to 0, since H^3 is of the order $O(z^3)$, it will not be good enough to control the error term (2.10). On the other hand, applying the linearized Allen–Cahn operator to the function zH', we obtain $2(1 - H^2)H$. This can be used to control (2.10) near $z = 0$, but not for z large.

Consider the function

$$\Lambda(y, z) := \bar{u}(y, z) - \frac{H(y)g(z) + H(z)g(y)}{y^2 + z^2}. \tag{2.13}$$

Let us choose (α, β) to be the numbers (α_m, β_m) defined in (2.9). We would like to show that in the region where $\Lambda > 0$, there holds

$$-\Delta\Lambda - \Lambda + \Lambda^3 \leq 0. \tag{2.14}$$

To see this, we first observe that in the region where y is large, due to (2.11) and (2.12), the main-order term of the left-hand side is

$$J := \frac{2(m-1)}{y^2 - z^2} zH'(z) - \frac{2\alpha_m\big(2H^3(z)\big) + \beta_m\big(2\big(1 - H^2(z)\big)H(z)\big)}{y^2 + z^2}.$$

In the region where y is large with $0 < z < 3$, since $\beta_m = m - 1$, we have the estimate

$$2(m-1)zH'(z) \leq 3\alpha_m H^3(z) + \beta_m\big(2\big(1 - H^2(z)\big)H(z)\big).$$

Therefore, J is nonpositive here. On the other hand, when y is large with $3 \leq z < 0.8y$,

$$J \leq \frac{2(m-1)}{y^2 - z^2} zH'(z) - \frac{2\alpha_m\big(2H^3(z)\big)}{y^2 + z^2} < 0.$$

In the case when y is large and $z \geq 0.8y$, the main-order term in $-\Delta\Lambda - \Lambda + \Lambda^3$ is

$$\frac{2(m-1)}{y^2 - z^2}\big[zH'(z) - yH'(y)\big] - \frac{4\alpha_m}{y^2 + z^2}.$$

This is negative due to the exponential decay of H'.

Next let us consider the region where $y \le 1$. By the definition (2.13) of Λ, and observing that $1/(y^2+z^2)$ blows up at origin, one verifies

$$\Lambda \le 0, \quad \text{for } y \le 1.$$

Define

$$\Xi := \{p \in \mathcal{O} : \Lambda > 0 \text{ at } p\}.$$

Then in Ξ, we necessarily have $y \ge 1$. Now in a bounded sub-region of Ξ (that is, when $y \ge 1$ but not too large), we can use *Mathematica* to verify (see the code in the appendix) that (2.14) holds. Therefore, applying Lemma 2.2, we find that the saddle solution u is bounded from below by Λ in Ξ. Therefore, in Ξ, we have

$$\phi = \bar{u} - u \le \frac{H(y)g(z) + H(z)g(y)}{y^2+z^2}. \tag{2.15}$$

We also observe that again by the definition of Λ, if $\Lambda \le 0$, then

$$\bar{u}(y,z) - \frac{H(y)g(z) + H(z)g(y)}{y^2+z^2} \le 0.$$

This means that in $\mathcal{O}\backslash\Xi$, there holds

$$\phi = \bar{u} - u \le \bar{u} \le \frac{H(y)g(z) + H(z)g(y)}{y^2+z^2}. \tag{2.16}$$

Combining (2.15) and (2.16), we get the desired estimate of ϕ. ■

We emphasize that the constants α_m, β_m are not optimal. One can also obtain an estimate similar to Proposition 2.6 using other functions instead of H and zH'. For instance, we may introduce the function

$$\rho(z) := H'(z)\int_0^z \left(H'^{-2}\int_s^{+\infty} H'^2\right)ds. \tag{2.17}$$

It satisfies

$$-\rho'' + (3H^2-1)\rho = H'. \tag{2.18}$$

This function can be explicitly integrated, and

$$\rho(z) = \frac{6z + \frac{9\sqrt{2}}{2} - 4\sqrt{2}\exp(-\sqrt{2}z) - \frac{\sqrt{2}}{2}\exp(-2\sqrt{2}z)}{24\cosh^2(\frac{\sqrt{2}z}{2})}.$$

We have $\lim_{z\to+\infty}\frac{\rho(z)}{zH'(z)} = \frac{\sqrt{2}}{4}$. We also have

$$-(2\rho + z^2H')'' + (3H^2-1)(2\rho + z^2H') = 4\sqrt{2}zHH'.$$

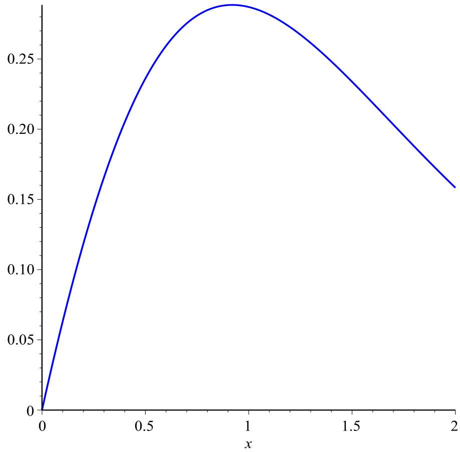


FIGURE 2. The function ρ.

We point out that if the right-hand side of (2.18) is replaced by $zH'(z)$, which is the case we are truly interested in, then the solution does not have a closed form.

Using similar arguments as that of Proposition 2.6, we can prove that for $m = 4, 5, 6$, there holds

$$\phi \le \frac{H(y)\big(8\rho(z) + 3z^2H'(z)\big) - H(z)\big(8\rho(y) + 3y^2H'(y)\big)}{y^2 - z^2} + \frac{1}{2}\frac{H(y)H(z)(e^{-\frac{y}{2}} + e^{-\frac{z}{2}})}{y^2 + z^2}. \tag{2.19}$$

See Figure 2 for its picture. Note that the right-hand side is always positive. Indeed, although ρ and z^2H' are not monotone, the functions $\frac{\rho}{H}$ and $\frac{2\rho+z^2H'}{H}$ are both monotone decreasing in $(0, +\infty)$.

2.2. *Estimates of the first-order derivatives*

In view of the sign of u_s and u_t, we see that Lemma 2.3 already yields a lower bound of $|u_t|$ in terms of u_s. That is,

$$|u_t| \ge \frac{t}{s}u_s \quad \text{in } \Omega.$$

This inequality also implies that u_y has the following decaying property along the y axis (that is, along the Simons cone):

$$u_y = \frac{u_s + u_t}{\sqrt{2}} \le \frac{\sqrt{2}z}{y + z}u_s. \tag{2.20}$$

On the other hand, it is well known that bounded solutions of Allen–Cahn equation automatically satisfy the Modica estimate (see [33, Theorem II]):

$$\frac{1}{2}|\nabla u|^2 \leq F(u) = \frac{(1-u^2)^2}{4}. \tag{2.21}$$

This inequality immediately yields an upper bound for u_s, provided that we have some lower bound on u, which in turn provides a bound for u_y through (2.20). That is,

$$u_y \leq \frac{z}{y+z}(1-u^2).$$

However, this estimate is also far from optimal because the computation of Proposition 2.6 intuitively tells us that u_y should decay like $O(y^{-3})$ along the Simons cone.

The next result gives us an exponential decay estimate of u_s away from the Simons cone. Note that by (2.21) and Lemma 2.5 or estimate (2.19), we can already deduce an exponential decay of u_s. However, the method of monotone iteration used there can never provide the optimal decay rate. In the following proposition, we shall derive a better decay rate (but still not optimal). For simplicity, we will only consider the case of $m = 4$. The case of $m = 5, 6$ can be similarly analyzed, but with different choices of parameters.

Proposition 2.7. *Let $m = 4$. Then in Ω, we have*

$$u_s \leq 2\Big(e^{\frac{3}{4}t} + \frac{2}{\sqrt{t}}\Big)e^{-\frac{3}{4}s} \quad \textit{for } s - t \geq 4.$$

Proof. Let a, b be two positive parameters to be determined later on, and define

$$\eta = \Big(e^{at} + \frac{b}{\sqrt{t}}\Big)e^{-as}.$$

Note that $u = \bar{u} - \phi$ in Ω, where ϕ satisfies the upper bounds (2.7) and (2.8). We remark that (2.8) provides better estimate than (2.7) near the Simons cone, when y is large. It follows that $u \geq A$, where the function A is given by

$$\max\left\{\bar{u} - \frac{H(y)\big(H(z) + 3zH'(z)\big) + H(z)\big(H(y) + 3yH'(y)\big)}{y^2 + z^2}, H\Big(\frac{y}{2}\Big)H\Big(\frac{z}{2}\Big)\right\}.$$

We have the following upper bound of u_s ensured by the Modica estimate:

$$u_s \leq \frac{s}{\sqrt{s^2+t^2}}\frac{1-u^2}{\sqrt{2}}. \tag{2.22}$$

Taking advantage of the estimates of u, we get

$$u_s \leq \frac{s}{\sqrt{s^2+t^2}} \frac{1-A^2}{\sqrt{2}}.$$

Now let us consider the region

$$\Omega_1 := \{(s,t) \in \Omega : s-t \geq 4\}.$$

Choose $a = \frac{3}{4}$ (note that we cannot expect a decay rate greater than 1), $b = 2$. We can use (2.22) to directly verify (use *Mathematica*; see the code in the appendix) that on $\partial\Omega_1$,

$$u_s \leq 2\eta.$$

Recall that u_s satisfies

$$L_m u_s - \frac{(m-1)u_s}{s^2} = 0.$$

We then have

$$\left(L_m - \frac{(m-1)}{s^2}\right)\left(\eta - \frac{u_s}{2}\right) = \Delta\eta + \left(1 - 3u^2 - \frac{(m-1)}{s^2}\right)\eta.$$

For s large, the main-order terms of the right-hand side equal

$$\begin{aligned} J := &\left(2a^2 + \frac{a(m-1)}{t} + (1-3u^2)\right)e^{at-as} \\ &+ b\left(a^2 + \left(\frac{3}{4} - \frac{m-1}{2}\right)\frac{1}{t^2} + 1 - 3u^2\right)\frac{1}{\sqrt{t}}e^{-as}. \end{aligned}$$

Under our choice of a and b, we have, when $m = 4$,

$$J \leq 0 \quad \text{for } t > 0, \tag{2.23}$$

provided that u is close to 1. Now

$$\left(L_m - \frac{(m-1)}{s^2}\right)\left(\eta - \frac{u_s}{2}\right) \leq \Delta\eta + \left(1 - 3A^2 - \frac{(m-1)}{s^2}\right)\eta.$$

One then checks using *Mathematica* that the right-hand side is also negative in a large bounded region Ω_1 for $m = 4$; see the code in the appendix.

Note that η tends to $+\infty$ as t tends to 0. This implies that $2\eta > u_s$ near the s axis. Now suppose to the contrary that there were a subregion of Ω_1 where $u_s \leq 2\eta$; then applying the maximum principle in this subregion yields that actually $u_s \leq 2\eta$ in Ω_1. This completes the proof. ∎

The following theorem gives another estimate of u_s in terms of $\frac{1}{y}+\frac{1}{z}$ throughout the whole region $\mathcal{O}$, which seems to be optimal.

THEOREM 2.8. *In the region $\mathcal{O}$, there holds*

$$\frac{u}{y}+\frac{u}{z}-u_y-u_z\geq 0.$$

PROOF. Let $\eta=\frac{u}{y}+\frac{u}{z}-u_y-u_z$. Then when $s=t$, we have $u=u_y=z=0$ and

$$\lim_{z\to 0}\frac{u}{z}=u_z.$$

Hence, $\eta=0$ there. Moreover, along the Simons cone, u tends to H in the normal direction (the z direction); hence,

$$\lim_{|x|\to+\infty}\eta(x)\geq 0.$$

It is also worth pointing out that the s axis is in the interior of $\mathcal{O}$. On the s axis,

$$\eta=\sqrt{2}\left(\frac{2u}{s}-u_s\right).$$

At this moment, we still do not know whether η is nonnegative on the s axis, although this is true if s is sufficiently large, due to the exponential decay of u_s.

Using the Laplacian in (y,z) coordinate and the identities

$$u_y+u_z=\sqrt{2}u_s,\quad L_m u_s=\frac{m-1}{s^2}u_s,$$

we can compute

$$\begin{aligned}L_m\eta=&-\frac{2u^3}{y}-2\frac{u_y}{y^2}+\left(\frac{2}{y^3}-\frac{2(m-1)}{y(y^2-z^2)}\right)u\\&-\frac{2u^3}{z}-2\frac{u_z}{z^2}+\left(\frac{2}{z^3}+\frac{2(m-1)}{z(y^2-z^2)}\right)u\\&-\frac{2(m-1)}{(y+z)^2}(u_y+u_z).\end{aligned}$$

We have

$$\begin{aligned}L_m\eta-\frac{2(m-1)}{(y+z)^2}\eta=&-\frac{2u^3}{y}-\frac{2u_y}{y^2}+\left(\frac{2}{y^3}-\frac{2(m-1)}{y(y^2-z^2)}\right)u\\&-\frac{2u^3}{z}-\frac{2u_z}{z^2}+\left(\frac{2}{z^3}+\frac{2(m-1)}{z(y^2-z^2)}\right)u-\frac{2(m-1)}{(y+z)^2}\left(\frac{u}{y}+\frac{u}{z}\right)\\=&-2u^3\left(\frac{1}{y}+\frac{1}{z}\right)-\frac{2u_y}{y^2}-\frac{2u_z}{z^2}+\frac{2u}{y^3}+\frac{2u}{z^3}.\end{aligned}$$

On the other hand, we know that in $\mathcal{O}$,

$$yu_y - zu_z = \frac{1}{2}(s+t)(u_s+u_t) - \frac{1}{2}(s-t)(u_s-u_t) = su_t + tu_s \leq 0. \tag{2.24}$$

To proceed, assume to the contrary that $\eta < 0$ somewhere in $\mathcal{O}$. Define the domain

$$\Gamma := \left\{(x_1,\dots,x_{2m}) : \eta = \frac{u}{y} + \frac{u}{z} - u_y - u_z < 0\right\}. \tag{2.25}$$

By (2.24), $u - yu_y \geq u - zu_z$. It then follows from definition (2.25) that

$$\frac{u}{z} - u_z \leq 0 \text{ in } \Gamma.$$

We then deduce that in Γ,

$$\begin{aligned} &L_m\eta - \frac{2(m-1)}{(y+z)^2}\eta - \frac{2}{y^2}\eta \\ &\leq -\frac{2u_y}{y^2} - \frac{2u_z}{z^2} + \frac{2u}{y^3} + \frac{2u}{z^3} - \frac{2}{y^2}\left(\frac{u}{y} + \frac{u}{z} - u_y - u_z\right) \\ &= \left(\frac{2}{z^2} - \frac{2}{y^2}\right)\left(\frac{u}{z} - u_z\right) \leq 0. \end{aligned} \tag{2.26}$$

This contradicts with the maximum principle and the result is thus proved. ■

Due to (2.24), Theorem 2.8 in particular implies that

$$u \geq yu_y \quad \text{in } \Omega. \tag{2.27}$$

Observe that $u = yu_y = 0$ for $z = 0$. It then follows from (2.27) that

$$u_z - yu_{yz} \geq 0 \quad \text{if } z = 0. \tag{2.28}$$

This inequality can also be written as

$$\frac{u_s - u_t}{\sqrt{2}} - \frac{s+t}{\sqrt{2}}\frac{u_{ss} - u_{tt}}{2} \geq 0 \quad \text{if } z = 0.$$

Then using the fact that $u_{ss} + u_{tt} = 0$ if $z = 0$, we get

$$u_s - su_{ss} \geq 0 \quad \text{if } z = 0. \tag{2.29}$$

This inequality will be used in the next subsection.

2.3. *Estimate the second-order derivatives*

In this subsection, we would like to analyze the second-order derivatives of the saddle solution. By a result of [5], it is known that $u_{st} > 0$ in Ω. Later on we will show that $u_{tt} < 0$ in Ω. However, u_{ss} may change sign. Indeed, on the Simons cone, $u_{yy} = u_{zz} = 0$. Moreover, by the identity (see (2.37))

$$L_m u_y - \frac{m-1}{2}\left(\frac{1}{s^2} + \frac{1}{t^2}\right)u_y \le 0$$

and the Hopf Lemma, we have $u_{yz} > 0$ on the Simons cone. Therefore,

$$u_{ss} = \frac{1}{2}(u_{yy} + 2u_{yz} + u_{zz}) > 0, \text{ near the Simons cone.}$$

Additionally, the fact that u tends to H at infinity implies that $\lim_{y\to+\infty} u_{ss} \le 0$. This also tells us that u_{ss} necessarily changes sign in Ω.

At this moment, it is worth pointing out that u_{yz} also changes sign in Ω. This is one of the main differences between the saddle solution u and the supersolution $H(y)H(z)$.

The following result tells us that u_{ss} can be estimated from above in terms of u_s. This is useful when u_{ss} is positive.

PROPOSITION 2.9. *$\frac{u_s}{s} - u_{ss} \ge 0$ in $\mathcal{O}$.*

PROOF. Let $\eta = u_s s^{-1} - u_{ss}$. By (2.29), $\eta \ge 0$ on $\partial\mathcal{O}$. We compute

$$L_m u_{ss} = -\frac{2(m-1)u_s}{s^3} + \frac{2(m-1)u_{ss}}{s^2} + 6u_s^2 u. \tag{2.30}$$

It follows that

$$L_m \eta = \frac{2mu_s}{s^3} - \frac{2mu_{ss}}{s^2} - 6u_s^2 u.$$

This implies

$$L_m \eta - \frac{2m}{s^2}\eta = -6u_s^2 u \le 0.$$

By the maximum principle, $\eta \ge 0$. This finishes the proof. ■

The next theorem tells us that although u_{ss} is positive somewhere, $u_{ss} + u_{tt}$ will nevertheless always be negative.

THEOREM 2.10. *In $\mathcal{O}$, we have*

$$\frac{u_s}{s} + \frac{u_t}{t} - u_{ss} - u_{tt} \ge 0.$$

PROOF. Let $\eta = \frac{u_s}{s} + \frac{u_t}{t} - u_{ss} - u_{tt}$. This function is even in t. We have $\eta = 0$ on $\partial\mathcal{O}$. We have

$$\begin{aligned} L_m\eta &= \frac{2mu_s}{s^3} + \frac{2mu_t}{t^3} - \frac{2mu_{ss}}{s^2} - \frac{2mu_{tt}}{t^2} - 6(u_s^2 + u_t^2)u \\ &= \frac{2m}{t^2}\eta + 2m\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(u_{ss} - \frac{u_s}{s}\right) - 6(u_s^2 + u_t^2)u. \end{aligned}$$

By Lemma 2.9, $u_{ss} - \frac{u_s}{s} \le 0$; therefore, $L_m\eta - \frac{2m}{t^2}\eta \le 0$. Assume to the contrary that η were negative in a subdomain Γ. Then Γ does not intersect with the s-axis. Then applying the maximum principle in Γ, we can conclude $\eta \ge 0$ in $\mathcal{O}$. ■

In a similar spirit, we can estimate $u_{st} + u_{tt}$ in the following way.

THEOREM 2.11. *In Ω, we have $u_{tt} < 0$ and*

$$-\frac{u_t}{t} + u_{st} + u_{tt} \ge 0.$$

PROOF. We compute

$$\begin{aligned} L_m u_{st} &= \left(\frac{m-1}{s^2} + \frac{m-1}{t^2}\right)u_{st} + 6u_s u_t u, \\ L_m u_{tt} &= \frac{2(m-1)}{t^2}u_{tt} - \frac{2(m-1)}{t^3}u_t + 6u_t^2 u. \end{aligned}$$

In particular, u_{tt} satisfies

$$L_m u_{tt} - \frac{2(m-1)}{t^2}u_{tt} \ge 0, \quad \text{in } \Omega.$$

Note that when $s = t$, $u_{tt} = -u_{ss} < 0$. Therefore, by the maximum principle,

$$u_{tt} < 0 \quad \text{in } \Omega.$$

To proceed, let us define $\eta = -\frac{u_t}{t} + u_{st} + u_{tt}$. Then

$$\begin{aligned} L_m\eta &= -\frac{2}{t^3}u_t + \frac{2}{t^2}u_{tt} + \left(\frac{m-1}{s^2} + \frac{m-1}{t^2}\right)u_{st} + 6u_s u_t u \\ &\quad + \frac{2(m-1)}{t^2}u_{tt} - \frac{2(m-1)}{t^3}u_t + 6u_t^2 u \\ &= \frac{2m}{t^2}\eta + \left(\frac{m-1}{s^2} - \frac{m+1}{t^2}\right)u_{st} + 6u_t u(u_s + u_t). \end{aligned}$$

Since $u_{st} > 0$ and $u_y > 0$ in Ω, we find that in Ω,

$$L_m\eta - \frac{2m}{t^2}\eta \le 0.$$

If $z = 0$, then $u_{st} = 0$ and $u_{tt} = -u_{yz}$. Hence, using (2.28), we infer that

$$-u_t + tu_{tt} = \frac{u_z}{\sqrt{2}} + \frac{y}{\sqrt{2}}(-u_{yz}) = \frac{1}{\sqrt{2}}(u_z - yu_{yz}) \geq 0, \quad \text{for } z = 0.$$

We also observe that $-\frac{u_t}{t} + u_{st} + u_{tt} \to 0$ as $t \to 0$. Then $\eta \geq 0$ on $\partial\Omega$. The desired result then follows from the maximum principle. ∎

From Theorem 2.10, we see that in Ω, $2u_{zz} = u_{ss} + u_{tt} - 2u_{st} \leq 0$. As a consequence,

$$(zu_z - u)' = u_{zz} \leq 0,$$

which readily implies that

$$u_s - u_t = \sqrt{2}u_z \leq \frac{\sqrt{2}u}{z}.$$

On the other hand, by the Modica estimate (2.21),

$$1 - u^2 \geq \sqrt{2}|\nabla u| \geq u_s - u_t.$$

Therefore, the following estimate holds in Ω:

$$u_s - u_t \leq \min\left\{1 - u^2, \frac{\sqrt{2}u}{z}\right\}.$$

Now we proceed to estimate u_{ss} in the region where it is negative.

LEMMA 2.12. *u satisfies*

$$u - u^3 + u_{ss} \geq 0 \quad \textit{in } \Omega. \tag{2.31}$$

PROOF. This inequality follows directly from the equation

$$u_{ss} + u_{tt} + \frac{m-1}{s}u_s + \frac{m-1}{t}u_t + u - u^3 = 0,$$

and the fact that in Ω,

$$\frac{m-1}{s}u_s + \frac{m-1}{t}u_t \leq 0, \quad u_{tt} \leq 0.$$ ∎

We point out that Lemma 2.12 can also be proved by using the maximum principle.

ANOTHER PROOF OF LEMMA 2.12. We have, for any $\sigma > 0$,

$$\Delta(u^\sigma) = \sigma u^{\sigma-1}\Delta u + \sigma(\sigma - 1)u^{\sigma-2}|\nabla u|^2,$$

which implies

$$\begin{aligned} L_m u^\sigma &= \sigma u^{\sigma-1}(u^3-u) + (1-3u^2)u^\sigma + \sigma(\sigma-1)u^{\sigma-2}|\nabla u|^2 \\ &= (1-\sigma)u^\sigma + (\sigma-3)u^{\sigma+2} + \sigma(\sigma-1)u^{\sigma-2}|\nabla u|^2. \end{aligned}$$

In particular,

$$L_m u = -2u^3 \quad \text{and} \quad L u^3 = -2u^3 + 6u|\nabla u|^2.$$

Hence,

$$L_m(u-u^3) = -6u|\nabla u|^2. \tag{2.32}$$

Let us define $\eta = u - u^3 + u_{ss}$. Applying (2.30) and (2.32), we then obtain

$$L_m\eta = -6u|\nabla u|^2 + \frac{2(m-1)}{s^2}u_{ss} - \frac{2(m-1)}{s^3}u_s + 6u_s^2 u.$$

Therefore, in Ω^*,

$$L_m\eta - \frac{2(m-1)}{s^2}\eta = -\frac{2(m-1)}{s^2}(u-u^3) - \frac{2(m-1)}{s^3}u_s - 6u_t^2 u \le 0.$$

By the maximum principle, $\eta \ge 0$. The proof is thus completed. ∎

It turns out that the estimate of Lemma 2.12 is not optimal. Indeed, u_{ss} can be directly estimated by u_s, and a very similar estimate holds for the mixed derivative u_{st}. This is the content of the following.

THEOREM 2.13. *In Ω, u satisfies*

$$\begin{aligned} \sqrt{2}u_s u + u_{ss} &\ge 0, \\ \sqrt{2}u_t u + u_{st} &\le 0. \end{aligned}$$

PROOF. We compute

$$\begin{aligned} L_m(u_s u) &= L(u_s)u + 2\nabla u \cdot \nabla(u_s) + u_s \Delta u \\ &= \frac{m-1}{s^2}u_s u + 2u_{ss}u_s + 2u_{st}u_t + u_s(u^3-u). \end{aligned}$$

Let $\eta = a u_s u + u_{ss}$, where $a > 0$ is a constant to be chosen later on. We have

$$\begin{aligned} L_m\eta = &\frac{(m-1)a}{s^2}u_s u + 2a u_{ss}u_s + 2a u_{st}u_t + a u_s(u^3-u) \\ &+ \frac{2(m-1)}{s^2}u_{ss} - \frac{2(m-1)}{s^3}u_s + 6u_s^2 u. \end{aligned}$$

Observe that the right hand contains terms involving u_{ss}. Since u_{ss} does not have a fixed sign, we would like to put it into the left-hand side. That is, we shall write the equation in the following form:

$$\begin{aligned} L_m\eta - \left(2au_s + \frac{2(m-1)}{s^2}\right)\eta &\\ = \frac{(m-1)a}{s^2}u_s u + 2au_{st}u_t + au_s(u^3-u)&\\ - \frac{2(m-1)}{s^3}u_s + 6u_s^2 u - \left(2au_s + \frac{2(m-1)}{s^2}\right)(au_s u).& \end{aligned}$$

Now the right-hand side does not contain u_{ss} and can be simplified into

$$\begin{aligned} L_m\eta - \left(2au_s + \frac{2(m-1)}{s^2}\right)\eta = -\frac{(m-1)a}{s^2}u_s u + 2au_{st}u_t - \frac{2(m-1)}{s^3}u_s \\ + (6-2a^2)u_s^2 u + au_s(u^3-u). \end{aligned}$$

By the Modica estimate (2.21), $1-u^2 \geq \sqrt{2}|\nabla u| \geq \sqrt{2}u_s$. Hence, using the fact that $u_{st}u_t \leq 0$, we obtain

$$\begin{aligned} L_m\eta - \left(2au_s + \frac{2(m-1)}{s^2}\right)\eta &\leq (6-2a^2)u_s^2 u - \sqrt{2}au_s^2 u \\ &= (6-2a^2-\sqrt{2}a)u_s^2 u. \end{aligned}$$

In particular, if we choose a to be $\sqrt{2}$, then $L\eta - (2au_s + \frac{2(m-1)}{s^2})\,\eta \leq 0$. It follows from the maximum principle that $\sqrt{2}u_s u + u_{ss} \geq 0$.

To prove the second inequality in this theorem, consider the function $\eta = \sqrt{2}u_t u + u_{st}$. We compute

$$\begin{aligned} L_m\eta = \frac{(m-1)\sqrt{2}}{t^2}u_t u + 2\sqrt{2}u_{st}u_s + 2\sqrt{2}u_{tt}u_t + \sqrt{2}u_t(u^3-u) \\ + \left(\frac{m-1}{t^2} + \frac{m-1}{s^2}\right)u_{st} + 6u_s u_t u. \end{aligned}$$

This can be written as

$$\begin{aligned} L_m\eta - \left(2\sqrt{2}u_s + \frac{m-1}{t^2} + \frac{m-1}{s^2}\right)\eta &\\ = \frac{(m-1)\sqrt{2}}{t^2}u_t u + 2\sqrt{2}u_{tt}u_t + \sqrt{2}u_t(u^3-u)&\\ + 6u_s u_t u - \sqrt{2}u_t u\left(2\sqrt{2}u_s + \frac{m-1}{t^2} + \frac{m-1}{s^2}\right)&\\ = \sqrt{2}u_{tt}u_t + \sqrt{2}u_t(u^3-u) + 2u_s u_t u - \frac{(m-1)\sqrt{2}u_t u}{s^2}.& \end{aligned} \tag{2.33}$$

From Theorem 2.11, we know $u_{tt} \le 0$. Recall also that $u_t \le 0$ in Ω. We then use the Modica estimate to infer that

$$L_m\eta - \left(2\sqrt{2}u_s + \frac{m-1}{t^2} + \frac{m-1}{s^2}\right)\eta \ge 0 \quad \text{in } \Omega.$$

By the maximum principle, $\sqrt{2}u_t u + u_{st} \le 0$. This finishes the proof. ∎

REMARK 2.14. The identity (2.33) can also be written as

$$\begin{aligned} L_m\eta &- \left(2\sqrt{2}u_s + \frac{m-1}{t^2}\right)\eta \\ &= \sqrt{2}u_{tt}u_t + \sqrt{2}u_t(u^3-u) + 2u_s u_t u + \frac{(m-1)u_{st}}{s^2} \ge 0. \end{aligned}$$

Then an application of the maximum principle also gives the desired result.

We point out that Theorem 2.13 together with the Modica estimate (2.21) also implies the estimate (2.31).

Let σ be a parameter. The following identity will be used later:

$$\begin{aligned} &L_m\left[(u_s-u_t)^\sigma\right] \\ &= (u_s-u_t)^{\sigma-1}\left[\frac{(m-1)\sigma u_s}{s^2} - \frac{(m-1)\sigma u_t}{t^2} + (1-\sigma)(1-3u^2)(u_s-u_t)\right] \\ &\quad + \sigma(\sigma-1)(u_s-u_t)^{\sigma-2}\left|\nabla(u_s-u_t)\right|^2. \end{aligned} \tag{2.34}$$

As we have already mentioned before, one can actually use the asymptotic behavior at infinity of u to show that for y large, the function u_y decays at the rate $O(y^{-3})$ along the Simons cone. We would like to get a *quantitative* description of this decay. In the next result, we consider the case of $m = 4$ although the same arguments can be well adapted to treat the other cases.

THEOREM 2.15. *Let $m = 4$. In Ω, we have*

$$u_s + u_t \le \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left[4(u_s-u_t) + \sqrt{u_s-u_t}\right]. \tag{2.35}$$

PROOF. Throughout the proof, we assume $m = 4$. In this case, the function $u_s + u_t$ satisfies

$$L_4(u_s+u_t) = \frac{3u_s}{s^2} + \frac{3u_t}{t^2} \le 0. \tag{2.36}$$

We write equation (2.36) as

$$L_4(u_s+u_t) - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)(u_s+u_t) = \frac{3}{2}\left(\frac{1}{s^2}-\frac{1}{t^2}\right)(u_s-u_t). \tag{2.37}$$

Note that the right-hand side is negative.

Let $\sigma = \frac{1}{2}$ and define the function

$$\xi_\sigma(s,t) := \left(\frac{1}{t^2} - \frac{1}{s^2}\right)(u_s - u_t)^\sigma.$$

Using the fact that in dimension 8, $\Delta(t^{-2}) = \Delta(s^{-2}) = 0$, we compute

$$\begin{aligned}(2.38)\qquad & L_4\xi_\sigma - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)\xi_\sigma \\ &= \sigma(u_s-u_t)^{\sigma-1}\left[\frac{4}{s^3}(u_{ss}-u_{st}) - \frac{4}{t^3}(u_{st}-u_{tt})\right] \\ &\quad + \left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left[L\big[(u_s-u_t)^\sigma\big] - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)(u_s-u_t)^\sigma\right].\end{aligned}$$

Let α, β be two positive parameters, and introduce the function

$$\eta := \alpha\xi_1 + \beta\xi_\sigma - (u_s + u_t).$$

Then $\eta \geq 0$ on $\partial\Omega$. We would like to use the maximum principle to show that $\eta \geq 0$ in Ω, for suitable choice of α, β.

Using (2.38), we compute

$$\begin{aligned}L_4\eta - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)\eta &= \left(\alpha + \beta\sigma(u_s-u_t)^{\sigma-1}\right)\left[\frac{4}{s^3}(u_{ss}-u_{st}) - \frac{4}{t^3}(u_{st}-u_{tt})\right] \\ &\quad + \alpha\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left[L(u_s-u_t) - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)(u_s-u_t)\right] \\ &\quad + \beta\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left[L\big[(u_s-u_t)^\sigma\big] - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)(u_s-u_t)^\sigma\right] \\ &\quad + \frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t).\end{aligned}$$

From (2.34), we infer

$$\begin{aligned}(2.39)\quad & L_4\eta - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)\eta \\ &= \underbrace{\left(\alpha + \beta\sigma(u_s-u_t)^{\sigma-1}\right)\left[\frac{4}{s^3}(u_{ss}-u_{st}) - \frac{4}{t^3}(u_{st}-u_{tt})\right]}_{I_1} \\ &\quad + \beta(1-\sigma)\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left[(u_s-u_t)^\sigma(1-3u^2) - \sigma(u_s-u_t)^{\sigma-2}\left|\nabla(u_s-u_t)\right|^2\right] \\ &\quad + \frac{3\beta}{2}(u_s-u_t)^{-\frac{1}{2}}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(\frac{u_t}{s^2}-\frac{u_s}{t^2}\right) \\ &\quad - \frac{3\alpha}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)^2(u_s+u_t) + \frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t).\end{aligned}$$

Now suppose to the contrary that the inequality (2.35) were not true. Consider the region Γ, which is defined to be the subregion of Ω where

$$u_s + u_t > \alpha\xi_1 + \beta\xi_\sigma. \tag{2.40}$$

Observe that this region does not intersect with the s axis.

By definition, in Γ, there holds

$$u_s + u_t \geq \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left[\alpha(u_s - u_t) + \beta\sqrt{u_s - u_t}\right].$$

This implies

$$\begin{aligned}
0 &\geq \frac{u_s}{s} + \frac{u_t}{t} = \frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right)(u_s + u_t) + \frac{1}{2}\left(\frac{1}{s} - \frac{1}{t}\right)(u_s - u_t) \\
&\geq \frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right)\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left[\alpha(u_s - u_t) + \beta\sqrt{u_s - u_t}\right] + \frac{1}{2}\left(\frac{1}{s} - \frac{1}{t}\right)(u_s - u_t).
\end{aligned}$$

We then deduce

$$\beta \leq \left(\left(\frac{1}{t} + \frac{1}{s}\right)^{-2} - \alpha\right)\sqrt{u_s - u_t}. \tag{2.41}$$

In particular, since $\beta > 0$, the above inequality tells us that in Γ, necessarily

$$\left(\frac{1}{t} + \frac{1}{s}\right)^{-2} - \alpha > 0.$$

We would like to show that in Γ, $L_4\eta - \frac{3}{2}(\frac{1}{s^2} + \frac{1}{t^2})\eta \leq 0$. To achieve this, we need to estimate the terms in (2.39).

We first estimate I_1. Taking advantage of the Allen–Cahn equation, we have

$$\begin{aligned}
&\frac{1}{s^3}(u_{ss} - u_{st}) - \frac{1}{t^3}(u_{st} - u_{tt}) \\
&\quad = \frac{u_{ss}}{s^3} + \frac{u_{tt}}{t^3} - \left(\frac{1}{s^3} + \frac{1}{t^3}\right)u_{st} \\
&\quad = \frac{u^3 - u}{s^3} - \frac{3}{s^3}\left(\frac{u_s}{s} + \frac{u_t}{t}\right) + \left(\frac{1}{t^3} - \frac{1}{s^3}\right)u_{tt} - \left(\frac{1}{s^3} + \frac{1}{t^3}\right)u_{st}.
\end{aligned}$$

Recall that $u_{tt} < 0$ and $u_{st} > 0$. We then deduce

$$\begin{aligned}
\frac{1}{s^3}(u_{ss} - u_{st}) - \frac{1}{t^3}(u_{st} - u_{tt}) &\leq \frac{u^3 - u}{s^3} - \frac{3}{s^3}\left(\frac{u_s}{s} + \frac{u_t}{t}\right) \\
&\leq \frac{u^3 - u}{s^3} + \frac{3}{2s^3}(u_s - u_t)\left(\frac{1}{t} - \frac{1}{s}\right).
\end{aligned}$$

As a consequence,

$$I_1 \le \left(4\alpha + \frac{2\beta}{\sqrt{u_s - u_t}}\right)\left[\frac{u^3 - u}{s^3} + \frac{3}{2s^3}(u_s - u_t)\left(\frac{1}{t} - \frac{1}{s}\right)\right]. \tag{2.42}$$

We proceed to estimate the remaining terms in (2.39). Using the fact that $u_{ss} + u_{tt}$ is negative (this follows from Lemma 2.3 and Theorem 2.10), we infer

$$\begin{aligned} \left|\nabla(u_s - u_t)\right|^2 &= (u_{ss} - u_{st})^2 + (u_{st} - u_{tt})^2 \ge \frac{1}{2}(u_{ss} + u_{tt} - 2u_{st})^2 \\ &\ge \frac{1}{2}\left[u^3 - u - 3\left(\frac{u_s}{s} + \frac{u_t}{t}\right)\right]^2. \end{aligned} \tag{2.43}$$

Using the facts that $u_s \ge |u_t|$ and $tu_s + su_t \le 0$, we also obtain

$$\begin{aligned} &\left(\frac{u_s}{t^2} - \frac{u_t}{s^2}\right)(u_s - u_t) - \left(\frac{u_s}{s} + \frac{u_t}{t}\right)^2 - \frac{1}{4}\left(\frac{1}{t} + \frac{1}{s}\right)^2 (u_s - u_t)^2 \\ &\quad = \left(\frac{1}{t^2} - \frac{1}{s^2}\right)(u_s^2 - u_t^2) - \frac{1}{4}\left(\frac{1}{t} + \frac{1}{s}\right)^2 (u_s + u_t)^2 \\ &\quad \ge \frac{3}{4s}\left(\frac{1}{t} + \frac{1}{s}\right)(u_s + u_t)\left(\frac{s}{t} - \frac{t}{s}\right)u_s \ge 0. \end{aligned} \tag{2.44}$$

Applying (2.43) and (2.44), we can control the following two terms appearing in the right-hand side of (2.39):

$$\begin{aligned} &-(1-\sigma)\sigma\left|\nabla(u_s - u_t)\right|^2 - \frac{3}{2}\left(\frac{u_s}{t^2} - \frac{u_t}{s^2}\right)(u_s - u_t) \\ &\quad \le -\frac{1}{8}\left[u^3 - u - 3\left(\frac{u_s}{s} + \frac{u_t}{t}\right)\right]^2 - \frac{3}{2}\left(\frac{u_s}{s} + \frac{u_t}{t}\right)^2 - \frac{3}{8}\left(\frac{1}{t} + \frac{1}{s}\right)^2 (u_s - u_t)^2 \\ &\quad \le -\frac{1}{14}(u^3 - u)^2 - \frac{3}{8}\left(\frac{1}{t} + \frac{1}{s}\right)^2 (u_s - u_t)^2. \end{aligned} \tag{2.45}$$

From (2.40), (2.42), and (2.45), we obtain

$$\begin{aligned} &\frac{L_4\eta - \frac{3}{2}(\frac{1}{s^2} + \frac{1}{t^2})\eta}{\sqrt{u_s - u_t}} \\ &\le \left(\frac{4\alpha}{\sqrt{u_s - u_t}} + \frac{2\beta}{u_s - u_t}\right)\left(\frac{u^3 - u}{s^3} + \frac{3}{2s^3}(u_s - u_t)\left(\frac{1}{t} - \frac{1}{s}\right)\right) \\ &\quad - \frac{\beta}{(u_s - u_t)^2}\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(\frac{1}{14}(u^3 - u)^2 + \frac{3}{8}\left(\frac{1}{t} + \frac{1}{s}\right)^2 (u_s - u_t)^2\right) \\ &\quad + \frac{\beta}{2}\left(\frac{1}{t^2} - \frac{1}{s^2}\right)(1 - 3u^2) - \frac{3\alpha}{2}\left(\frac{1}{t^2} - \frac{1}{s^2}\right)^3 (\alpha\sqrt{u_s - u_t} + \beta) + \frac{3}{2}\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\sqrt{u_s - u_t}. \end{aligned}$$

The right-hand side still contains terms of u_s, u_t. But keeping track on the sign of each term, we can use Modica estimate (2.21) again and (2.41) to infer that in Γ,

$$\begin{aligned}\frac{L_4\eta - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)\eta}{\sqrt{u_s-u_t}} \leq{}& (4\alpha\sqrt{1-u^2}+2\beta)\left(-\frac{u}{s^3}+\frac{3}{2s^3}\left(\frac{1}{t}-\frac{1}{s}\right)\right)\\ &-\beta\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(\frac{1}{14}u^2+\frac{3}{8}\left(\frac{1}{t}+\frac{1}{s}\right)^2\right)\\ &+\frac{\beta}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(1-3u^2)+\frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\sqrt{1-u^2}\\ &-\frac{3\alpha}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)^3\left(\frac{\alpha\beta}{(\frac{1}{t}+\frac{1}{s})^{-2}-\alpha}+\beta\right).\end{aligned}$$

Let us denote the right-hand side of this inequality by J. Now J only involves s, t, u, which will be easier to be estimated, since we already have a lower bound of u.

Recall that in dimension 8, by Lemma 2.5 and Proposition 2.6, the saddle function satisfies $u \geq H(\frac{y}{2})H(\frac{z}{2})$ and

$$0 \leq H(y)H(z)-u \leq \frac{H(y)\big(H(z)+3zH'(z)\big)+H(z)\big(H(y)+3yH'(y)\big)}{y^2+z^2}. \tag{2.46}$$

Let us choose $\alpha = 4$, $\beta = 1$. Appealing to (2.41), one can show that Γ is included in the region of Ω where $t > 3.7$. Note that we want J to be negative. For points near the Simons cone with $t \geq 6$ and $u < 0.9$, using the lower bound of u, we find that the main-order term of J is bounded from above by

$$\begin{aligned}&-\frac{4\alpha u}{s^3}\sqrt{1-u^2}+\frac{\beta}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(1-3u^2)+\frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\sqrt{1-u^2}\\ &\quad< \sqrt{1-u^2}\left[-\frac{16u}{s^3}+\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(\frac{3}{2}+\frac{1-3u^2}{2\sqrt{1-u^2}}\right)\right] < 0.\end{aligned}$$

Moreover, for points away from the origin with $u \geq 0.9$,

$$\begin{aligned}J &\leq -\beta\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\frac{1}{14}u^2+\frac{\beta}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(1-3u^2)+\frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\sqrt{1-u^2}\\ &\leq \left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(-\frac{1}{14}(0.8)+\frac{1}{2}(1-2.4)+\frac{3}{2}\sqrt{0.2}\right) < 0.\end{aligned}$$

Using this information, we directly check with the help of *Mathematica* that

$$L_4\eta - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)\eta \leq 0 \quad \text{in } \Gamma.$$

Maximum principle then ensures that $\eta \geq 0$. This finishes the proof. ■

REMARK 2.16. In the case of $m \neq 4$, the functions t^{-2} and s^{-2} are no longer harmonic functions. However, if $m > 4$, they are even better behaved and indeed super-harmonic in the sense that in Ω,

$$-\Delta(t^{-2} - s^{-2}) = 2(m-4)(t^{-4} - s^{-4}) > 0.$$

Therefore, a similar argument as that of Theorem 2.15 also applies, with suitable modification of the coefficients.

Results obtained in this section do have provided useful information of the saddle solution u. However, to prove the stability, we still need to have much more precise pointwise estimates in the whole domain Ω. For instance, Theorem 2.15 can help us to improve the constant $\sqrt{2}$ that appeared in the inequality $\sqrt{2}u_s u + u_{ss} \geq 0$. Since we do not know the precise location where u_{ss} changes sign, the estimate of $u_{st} + u_{ss}$ turns out to be crucial for us to control u_{ss} in terms of u_{st}. We will need inequalities of the form

$$a(u_s + u_t) + u_{st} + u_{ss} \geq 0,$$

as well as

$$b\left(\frac{1}{t} - \frac{1}{s}\right)u_s + u_{st} + u_{ss} \geq 0,$$

where a, b are suitable positive constants. While these should be intuitively clear in the region far away from the origin, in places between the origin and far field region, the optimal choice of constants a, b would play an important role in the stability analysis. Note that the behavior of saddle solution in this “intermediate region” should constitute the main difference between the analysis of the Simons cone and the Allen–Cahn equation. In this regard, u_{tt} seems to be the most difficult term. These problems will be pursued in a future work [30] in which we will prove that the saddle solution is stable. It will also be interesting if one can numerically compute the saddle solution as accurately as possible.

3. APPENDIX

In this appendix, we put the *Mathematica* code of the numerical computation. Some of the parameters that appear here can be adjusted.

Code for Lemma 2.5.

```
Clear["Global‘*"];
m=6;H[x_]:=Tanh[x/Sqrt[2]]; G[x_]:=x H′[x]/H[x];
sigma[y_,z_]=(2-(H[y])^2-(H[z])^2-(2(m-1)/(y^2-z^2)) (G[y]-G[z]))/(1-(H[y] H[z])^2);
Normal[Series[sigma[y,z],{y,0,2},{z,0,2}]]//FullSimplify;
```

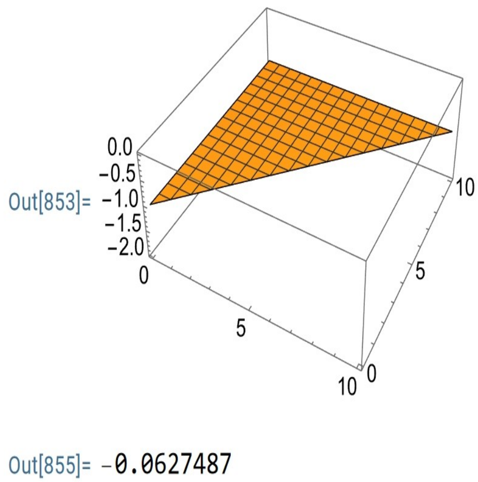


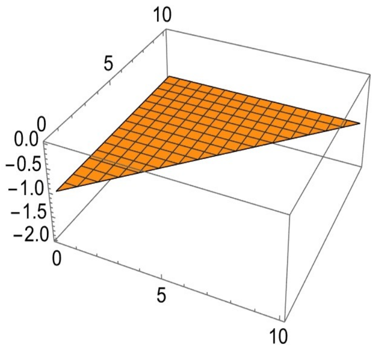
FIGURE 3. Output for Lemma 2.5.

```
Plot3D[Sign[sigma[y,z]-(2m+4)/3],{z,0.01,10},{y,z+0.01,10}]
p[y_]:=MaxValue[{sigma[y,z]-(2m+4)/3,z>=0.1&&y>=z+0.1},{z}];
MaxValue[{p[y],y>=0.2&&y<=10},{y}]
```

Code for Proposition 2.6.

```
Clear["Global‘*"];
m=4;H[x_]:=Tanh[x/Sqrt[2]];
Lapla=D[#,{y,2}]+D[#,{z,2}]+2(m-1)/(y^2-z^2) (y D[#,y]-z D[#,z])&;
AC=-Lapla[#]-#+#^3&;
alpha={1,1,2};beta={3,4,5};
g[z_]=alpha[[m-3]] H[z]+beta[[m-3]] z H′[z];
rho[z_]=(6z+9/Sqrt[2]-4Sqrt[2]Exp[-Sqrt[2]z]-Sqrt[2]Exp[-2Sqrt[2]z]/2)/(24Cosh[z/Sqrt[2]]^2);
F[1]=(H[y] g[z]+H[z] g[y])/(y^2+z^2);
F[2]=(H[y] (8rho[z]+3z^2 H’[z])-H[z] (8rho[y]+3 y^2H’[y]))/(y^2-z^2)+1/2(H[y]H[z] (Exp[-
y/2]+Exp[-z/2]))/(y^2+z^2);
ubar[y_,z_]:=H[y] H[z];
Table[k[i]=AC[ubar[y,z]-F[i]],{i,1,2}];
Plot3D[Max[Sign[k[1]],Sign[k[2]]],{z,0.1,10},{y,z+0.1,10.1}]
MaxValue[{k[1],z>=0.1&&y>=z+0.1&&y<=10.1},{y,z}]
MaxValue[{k[2],z>=0.1&&y>=z+0.1&&y<=10.1},{y,z}]
```

REMARK 3.1. The numerical algorithm of the operator “MaxValue” may fail to capture the *global* maximum in some cases.



Out[868]= -0.000610246

Out[869]= -0.0118899

FIGURE 4. Output for Proposition 2.6.

Code for Proposition 2.7.

```
Clear["Global‘*"];
H[x_]:=Tanh[x/Sqrt[2]];alp=Sqrt[3/(2#+4)]&;
g=H[#]+3 # H′[#]&;
F[y_,z_]=(H[y] g[z]+H[z] g[y])/(y^2+z^2);
ubar[y_,z_]=H[y] H[z];
ulow[y_,z_]=H[alp[m] y] H[alp[m] z];
subsolution[y_,z_]:=ubar[y,z]-F[y,z];
Lapla:=D[#,{s,2}]+D[#,{t,2}]+(m-1)(1/s D[#,s]+1/t D[#,t])&;
m=4;a=0.75;b=2;
eta[s_,t_]=2(Exp[a t]+b/t^(1/2)) Exp[-a s];
A[s_,t_]=Max[ulow[(s+t)/Sqrt[2],(s-t)/Sqrt[2]],subsolution[(s+t)/Sqrt[2],(s-t)/Sqrt[2]]];
B[s_,t_]=eta[s,t]-s (1-(A[s,t])^2)/Sqrt[2 s^2+2 t^2];
p[s_,t_]=Lapla[eta[s,t]]+(1-3(A[s,t])^2-(m-1)/(s^2)) eta[s,t];
Plot3D[Sign[p[s,t]],{t,0.05,10},{s,t+4,14}]
q[s_]:=MaxValue[{p[s,t],t>=0.1&&s>=t+4},{t}];
MinValue[{B[t+4,t],t>=0.1&&t<=10},{t}]
MaxValue[{q[s],s>=4.1&&s<=10},{s}]
```

Code for Theorem 2.15.

```
Clear["Global‘*"];
H[x_]=Tanh[x/Sqrt[2]];
g[z_]=H[z]+3 z H′[z];
```

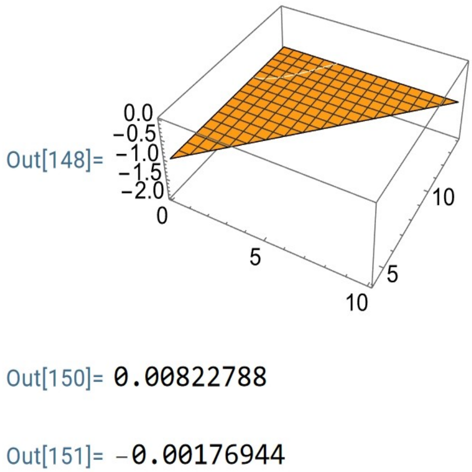
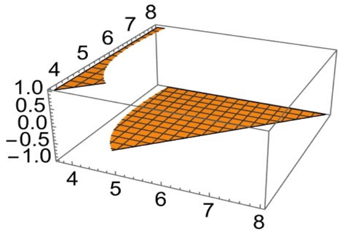


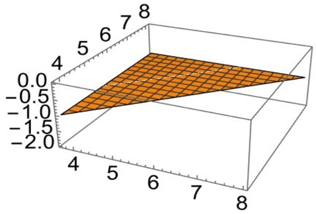
FIGURE 5. Output for Proposition 2.7.

```
F[y_,z_]=(H[y] g[z]+H[z] g[y])/(y^2+z^2);
ubar[y_,z_]=H[y] H[z];
ulow[y_,z_]=H[y/2] H[z/2];
subs[y_,z_]=ubar[y,z]-F[y,z];
a=4;b=1;
A1[s_,t_]= Max[ulow[(s+t)/Sqrt[2],(s-t)/Sqrt[2]],subs[(s+t)/Sqrt[2],(s-t)/Sqrt[2]]];
A2[s_,t_]=ubar[(s+t)/Sqrt[2],(s-t)/Sqrt[2]];
f1[s_,t_,u_]=-4a Sqrt[1-u^2] u/(s^3);
f2[s_,t_,u_]=4a Sqrt[1-u^2] (3/(2 s^3)) (1/t-1/s);
f3[s_,t_,u_]=2 b /(s^3)(-u+3/2 (1/t-1/s));
f4[s_,t_,u_]=(1/t^2-1/s^2) (-b/14 u^2-3 b/8 (1/t+1/s)^2+b/2(1-3u^2)+3/2 Sqrt[1-u^2]);
f5[s_,t_]=-3/2 a (1/t^2-1/s^2)^3 (a b/((1/t-1/s)^(-2)-a)+b);
g[1]=Max[f1[s,t,A1[s,t]],f1[s,t,A2[s,t]]];
g[2]=f2[s,t,A1[s,t]];
g[3]=f3[s,t,A1[s,t]];
g[4]=f4[s,t,A1[s,t]];
g[5]=f5[s,t];
k[s_,t_]=Sum[g[i],{i,1,5}]/(1/t^2-1/s^2);
gammatest[s_,t_]=b-((1/t+1/s)^(-2)-a) Sqrt[1-(A1[s,t])^2];
Plot3D[Sign[gammatest[s,t]],{t,3.7,8},{s,t+0.1,8.1}]
Plot3D[Sign[k[s,t]],{t,3.7,8},{s,t+0.1,8.1}]
MaxValue[{k[s,t],t>=3.7&&s-t>=0.1&&s<=8.1},{s,t}]
Max[k@@@RandomPoint[ImplicitRegion[t>3.7&&s-t>0.1&&s<8.1,{s,t}],100000]]
```

Out[58]=



Out[59]=



Out[60]= −0.220727

Out[61]= −0.221455

FIGURE 6. Output for Theorem 2.15.

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