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Algebraic Geometry. – *Singularities of the infinitesimal invariant of normal functions*, by Mark Green and Phillip Griffiths, accepted on 1 July 2025.

To Enrico Bombieri, esteemed colleague and friend.

Abstract. – Normal functions ν provide a method for studying algebraic cycles $Z_t \subset X_t$ varying in a family of smooth projective varieties. Associated with ν is an infinitesimal invariant $\delta\nu$ that reflects the first-order variation of pairs (X_t, Z_t). Over the years, $\delta\nu$ has been widely used in the study of various geometric questions. We note that whereas ν is a transcendental invariant, like periods of algebraic integrals, $\delta\nu$ has a natural filtration whose associated graded gives algebraic sections of coherent sheaves. In a number of interesting cases, these sections have had geometric interpretations. In this paper, we will discuss an identification between the singularities ν and $\delta\nu$. The formal proof of this result will be given in a separate work.

Keywords. – singularities, infinitesimal invariants, normal functions.

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1. Introduction

Recently, there has been interest in the analysis of the singularities $\operatorname{sing}(\nu)$ that ν acquires as the pair (X_t, Z_t) degenerates. In this work, we will define the singularity $\operatorname{sing}(\delta\nu)$ of the infinitesimal invariant. Our main result will be a sketch of the proof of an identification

$$\operatorname{sing}(\delta\nu) = \operatorname{sing}(\nu). \tag{1.1}$$

Such a result was suggested in [11], cf. Section I.3.1 there. The full details of the argument in a more general setting will be given in a separate work. It is well known [18] that $\operatorname{sing}(\nu)$ is constructed from the local monodromy of ν in a punctured neighborhood of the singular locus, typically taken as $(\Delta^*)^r$. Informally stated, the basic idea behind (1.1) is that the residues of $\delta\nu$ along the singular locus, in this case the coordinate hyperplanes, give the logarithms of monodromy of the extension class. In order to carry out the calculations, a main step will be to associate with an admissible normal function defined over $(\Delta^*)^r$ an analogue of the nilpotent orbit that is used to describe the singular part of a variation of Hodge structure on $(\Delta^*)^r$. It is not the same as an admissible normal function defined on a nilpotent orbit as in [18]. The construction is different as built into it is the result that non-torsion singularities of admissible normal functions occur only in codimension at least 2. It explicitly uses the way in which the codimension 1 singularities of ν vanish for a multiple $m\nu$ as given in [12, pp. 299–302]. In fact, the argument here may be viewed as what one obtains when trying to extend the argument in loc. cit. to the several-parameter case.

There is a vast literature on singularities of normal functions and on infinitesimal invariants of normal functions. As a reference for the first of these, we suggest [5, 11], and especially [18]. In fact, the expository parts of this paper may be thought of as an invitation to [18] which gives background, history, examples, formulations and at least indications of the proofs of the main results in the theory. For infinitesimal invariants of normal functions, there are relevant chapters in [13] and e.g. the papers [7, 15] which discuss the use of infinitesimal invariants in the interesting and important example of the Ceresa cycle [4, 8].

2. Background on variations of Hodge structure

A *variation of Hodge structure* (VHS) is given by the data $(\mathcal{H}, \mathfrak{F}^\bullet, \nabla, B)$ where

- B is a smooth variety with smooth partial completion $\overline{B}$ where $B = \overline{B}\backslash D$ with $D = \bigcup D_i$ a normal crossing divisor. We shall mainly be concerned with the *semi-global case* where $B = (\Delta^*)^r$ and $\overline{B} = \Delta^r$. Everything may be done with parameters, e.g. with $B = (\Delta^*)^r \times \Delta^s$.
- $\mathcal{H} \to B$ is a holomorphic vector bundle with integrable connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_B$ and with $\mathbb{H} = \ker \nabla$ a local system defined over $\mathbb{Z}$; i.e., $\mathbb{H} = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{H}_{\mathbb{Z}}$.
- $\mathfrak{F}^\bullet$ is a decreasing filtration by sub-bundles $\mathfrak{F}^p$, $0 \le p \le n$, of $\mathcal{H}$ satisfying the transversality condition

$$\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_B. \tag{2.1}$$

- H_b and F^p_b denote the fibers at $b \in B$ of $\mathcal{H}$ and $\mathfrak{F}^p$.
- At each $b \in B$, the opposite condition

$$F^p_b \oplus \overline{F}^{n-p+1}_b \xrightarrow{\sim} H_b, \quad 0 \le p \le n,$$

  is satisfied so that together with $H_{b,\mathbb{Q}} = \mathbb{H}_{\mathbb{Q},b}$ there is a Hodge structure of weight n. We shall always assume these Hodge structures are polarized by a horizontal bilinear form.

Usually we shall assume the weight $n = 2m - 1$ and H or H^{2m-1} will denote a typical fiber.

The *geometric case* is when the VHS arises from the cohomology along the fibers of a smooth family $\mathfrak{X} \xrightarrow{f} B$ of projective varieties $X_b = f^{-1}(b)$ where

$$\mathcal{H} = R^n_f \mathbb{C}_{\mathfrak{X}} \otimes_{\mathbb{C}} \mathcal{O}_B \cong \mathbb{R}^n_f \Omega^\bullet_{\mathfrak{X}/B}.$$

We denote by X a typical fiber so that using the above notation

$$H = H^n(X, \mathbb{C}).$$

In the semi-global case, we assume that the monodromy generators $T_i \in \mathrm{Aut}(H_{\mathbb{Z}})$ are unipotent with $\log T_i = N_i \in \mathrm{End}(H_{\mathbb{Q}})$, a nilpotent operator.

There are canonical Deligne extensions $\mathcal{H}_e$, $\mathfrak{F}^\bullet_e$ where

$$\nabla : \mathcal{H}_e \to \mathcal{H}_e \otimes \Omega^1_{\overline{B}}(\log D). \tag{2.2}$$

The fiber $\mathcal{H}_{e,\{0\}} := H_{\lim}$ has a limiting mixed Hodge structure $(H_{\lim}, \mathcal{W}_\bullet(N), F^\bullet_{\lim})$ with the monodromy weight filtration $\mathcal{W}_\bullet(N)$ defined by any $N = \sum \lambda_i N_i$, $\lambda_i \in \mathbb{Q}^{>0}$. We will further discuss $H_{\lim}$, $W_\bullet(N)$ and $F^\bullet_{\lim}$ below.

In the case when $\dim B = 1$, we have

$$\begin{array}{ccc} \mathfrak{X} & \subset & \overline{\mathfrak{X}} \\ \big\downarrow & & \big\downarrow f \\ B & \subset & \overline{B} \end{array}$$

where $\overline{\mathfrak{X}}$ is a smooth partial completion of $\mathfrak{X}$ with $X_0 = f^{-1}(\{0\})$, a reduced normal crossing divisor. In this case,

$$\mathcal{H}_e \simeq \mathbb{R}^n_f \Omega^\bullet_{\overline{\mathfrak{X}}/\overline{B}}(\log X_0).$$

In the semi-global situation, we denote by $t = (t_1, \ldots, t_r)$ coordinates in Δ^r. We will be particularly interested in the $r \geq 2$ case. Then as t approaches each boundary component $D^*_i := \{t_i = 0,\ t_j \neq 0 \text{ for } i \neq j\}$ over $\mathbb{Q}$, there will be vanishing cycles $\delta_{i,\alpha} \in \mathrm{Im}(T_i - I)$. The $\delta_{i,\alpha}$'s may not be independent, and their relations will be reflected in the cohomology of Koszul-type complexes constructed from the commuting operators $T_i - I$ (cf. [11, Section 5.2.4]).

3. Abel–Jacobi maps and normal functions

For $(H^{2m-1}, F^\bullet)$, a polarized Hodge structure of weight $2m-1$, the corresponding *intermediate Jacobian* is

$$J = F^m \backslash H^{2m-1} / H^{2m-1}_{\mathbb{Z}} \cong \check{F}^m / \check{H}^{2m-1}_{\mathbb{Z}}. \tag{3.1}$$

Here, the dualities are induced by the assumed unimodular polarizing form. In the geometric case with X a smooth projective variety of dimension $2m-1$ and $H^{2m-1} = H^{2m-1}(X, \mathbb{C})$, we set $J = J(X)$. For $Z \in Z^m_{\mathrm{hom}}(X)$, a codimension m algebraic cycle whose fundamental class $[Z] \in H^{2m}(X, \mathbb{Z})$ is zero, the *Abel–Jacobi map* is

$$\langle AJ_X(Z), \omega\rangle = \int_\Gamma \omega \quad \text{mod periods,} \tag{3.2}$$

where Γ is a real $(2m-1)$-dimensional chain with $\partial\Gamma = Z$ and $\omega \in F^m H^{2m-1}_{\mathrm{DR}}(X)$.

To make this well defined, one may use the quasi-isomorphism

$$\{A^\bullet_{\bar\partial}(X), \mathrm{d}\} \hookrightarrow \{A^\bullet(X), \mathrm{d}\}, \tag{3.3}$$

where $A^\bullet(X)$ are the C^∞ forms on X and $A^\bullet_{\bar\partial}(X)$ are the $\bar\partial$-closed forms. Then,

$$H^*_{\mathrm{DR}}(X) \cong H^*\big(A^\bullet_{\bar\partial}(X)\big)$$

and ω is defined up to adding a form $\mathrm{d}\phi$ where $\phi \in F^m A^{2m-2}_{\bar\partial}(X)$ so that

$$\int_\Gamma \mathrm{d}\phi = \int_Z \phi = 0$$

by type.

Given a variation of Hodge structure $(\mathcal{H}^{2m-1}, \mathfrak{F}^\bullet, \nabla; B)$, there is a corresponding family of intermediate Jacobians whose sheaf of sections is

$$\mathfrak{J} = \mathfrak{F}^m \backslash \mathcal{H}^{2m-1}/\mathbb{H}^{2m-1}_{\mathbb{Z}} \cong \check{\mathfrak{F}}^m/\check{\mathbb{H}}^{2m-1}_{\mathbb{Z}}. \tag{3.4}$$

For ν, a local section of $\mathfrak{J}$ for any local lifting $\hat\nu$ of ν to a section of $\mathcal{H}^{2m-1}$ from (2.2) the condition

$$\nabla\hat\nu \in \mathfrak{F}^{m-1} \tag{3.5}$$

is well defined. The sections ν of $\mathfrak{J}$ that satisfy (3.5) will be said to be *quasi-horizontal*, and the sheaf $\mathfrak{J}_h \subset \mathfrak{J}$ of such sections are *normal functions*.

There is a subtlety here due to the fact that although $\int_\Gamma$ defines a linear function on $F^m H^{2m-1}_{\mathrm{DR}}(X, \mathbb{C})$, the "obvious" extension $\int_\Gamma \phi$ to all $\phi \in H^{2m-1}_{\mathrm{DR}}(X, \mathbb{C})$ is not well defined. If $\phi = \mathrm{d}\gamma$ where $\bar\partial\gamma = 0$, then

$$\int_\Gamma \mathrm{d}\phi = \int_Z \gamma^{(m-1,m-1)}$$

may be non-zero. One may correct the current $T_\Gamma = \int_\Gamma$ by subtracting from T_Γ the current T_η defined by a singular differential form η having the following properties:

- $\eta \in F^m A^{2m-1}_{\mathrm{loc}}(X)$ is a locally L^1 form and therefore defines a current T_η;
- η is smooth in $X\backslash|Z|$ and along Z has a *residue current* [9, 14, 17, 19] with $T_{\mathrm{Res}\,\eta}(\alpha) = \int_Z \alpha$;
- $\mathrm{d}(\eta|_{X\backslash|Z|}) = 0$ and thus defines a class $[\eta]$ in $H^{2m-1}(X\backslash|Z|)$ such that $\Gamma - [\eta] \in \mathrm{Im}\{H^{2m-1}(X) \hookrightarrow H^{2m-1}(X\backslash|Z|)\}$.

The current

$$\widehat{\Gamma}_\eta := T_\Gamma - T_\eta \tag{3.6}$$

is then a lifting to $H^{2m-1}_{\mathrm{DR}}(X,\mathbb{C})$ of $\int_\Gamma$.

In the geometric case, we assume given an algebraic cycle $\mathfrak{Z} \in Z^m(\mathfrak{X})$ with proper intersections

$$\mathfrak{Z}\cdot X_t := Z_t \in Z^m_{\mathrm{hom}}(X_t).$$

The quasi-horizontality condition (3.5) is satisfied and so the $AJ_{X_t}(Z_t) \in J(X_t)$ define a normal function

$$\nu_{\mathfrak{z}} : B \to \mathfrak{J}_h.$$

Although not explicitly stated there, the method used in [14] to construct T_η may be adapted to have holomorphic dependence on parameters (cf. also [17]). This gives local liftings $\hat{\nu}$ to $\mathcal{H}$ of the section ν of $\mathfrak{F}^m\backslash\mathcal{H}/\mathbb{H}_{\mathbb{Z}}$.

A very interesting normal function is given by the Ceresa cycle [4]. Given a smooth, non-hyperelliptic genus $g \geq 3$ curve C, the Jacobian variety $J(C)$ is a g-dimensional abelian variety. Upon choosing a base point, there is the standard embedding $u : C \to J(C)$ with image C. Then, denoting by $-_J$ the group law in $J(C)$,

$$Z := \big\{u(p) -_J u(p) : p \in C\big\} \in Z^{g-1}_{\mathrm{alg}}\big(J(C)\big)$$

is independent of the choice of base point and defines the *Ceresa cycle* with image

$$AJ_{J(H^3_{\mathrm{prim}}(J(C)))}(Z) \in J\big(H^3_{\mathrm{prim}}\big(J(C)\big)\big),$$

where $J(H^3_{\mathrm{prim}}(J(C)))$ is the primitive part of the intermediate Jacobian variety of the Jacobian variety $J(C)$. For a family of curves $\mathcal{C} \to B$, the corresponding normal function is the subject of continuing interest (e.g. [7, 15]).

In the geometric case described above, given a holomorphic section

$$\omega_t \in F^m H^{2m-1}(X_t)$$

upon choice of holomorphically varying Γ_t with $\partial\Gamma_t = Z_t$,

$$\langle \nu_{\mathfrak{z},t}, \omega_t\rangle = \int_{\Gamma_t} \omega_t \tag{3.7}$$

is well defined and we may ask for its differential. Taking the case $\dim B = 1$ and thinking of Γ_t as representing a class in $H_{2m-1}(X_t, |Z_t|)$, one might hope for a formula of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \nu_{\mathfrak{Z},t}, \omega_t\rangle = \int_{\Gamma_t} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}\omega_t + \int_{\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}\Gamma_t} \omega_t. \tag{3.8}$$

There are subtleties here; for example, $\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}\omega_t \in F^{m-1}H^{2m-1}(X_t)$ and as noted above $\int_{\Gamma_t}$ is not well defined as a linear function on that vector space. To remedy this, one needs considerations similar to (3.6) above. Regarding the second term in (3.8) for geometric reasons, we might expect $\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}\Gamma_t$ to be concentrated along Z_t. But this is not well defined since we may add to Γ_t a cycle representing a class in $H_{2m-1}(X_t, \mathbb{Z})$. An underlying issue is to have a Kodaira–Spencer formula for computing the differential of the mapping

$$T\,\mathrm{Def}(X_t, Z_t) \to T\big\{\text{MHS's of the type of } H^{2m-1}(X_t, Z_t)\big\}.$$

Such a formula exists and its use in leading to a cohomological formula for $\delta\nu$ will be discussed elsewhere.

When $m \geq 2$ so that the weight $2m-1 \geq 3$, there is a *truncated Abel–Jacobi map*

$$AJ_{\tau,X} : Z^m_{\mathrm{hom}}(X) \to J_\tau(X), \tag{3.9}$$

where $J_\tau(X) = (F^{2m-1}H^{2m-1}(X))^\vee / H_{2m-1}(X, \mathbb{Z})$ and (3.9) is defined on $\omega \in F^{2m-1}H^{2m-1}(X)$ by

$$\big\langle AJ_{\tau,X}(Z), \omega\big\rangle = \int_\Gamma \omega. \tag{3.10}$$

For a family $\mathfrak{X} \to B$ and $\mathfrak{Z} \subset \mathfrak{X}$ as above, (3.10) defines a *truncated normal function* $\nu_{\tau,t}$ where we restrict ω_t to be in $F^{2m-1}H^{2m-1}(X_t)$. In this case, because of (2.1), each of the two terms in (3.8) may be unambiguously defined.

4. Infinitesimal invariant of normal functions

Given a VHS of weight $2m-1$ using (2.1) for each k, there is a complex

$$K^\bullet_k = \big\{\mathfrak{F}^k \xrightarrow{\nabla} \mathfrak{F}^{k-1} \otimes \Omega^1_B \xrightarrow{\nabla} \mathfrak{F}^{k-2} \otimes \Omega^2_B \xrightarrow{\nabla} \cdots\big\}.$$

Given a normal function $\nu : B \to \mathfrak{J}_h$, we will define the *infinitesimal invariant* (cf. [13])

$$\delta\nu \in H^1(K^\bullet_m). \tag{4.1}$$

Using $\mathfrak{J}_h \subset \mathfrak{J} = \mathfrak{F}^m \backslash \mathcal{H}^{2m-1}/\mathbb{H}_{\mathbb{Z}}$, we choose a local lifting $\hat{\nu}$ of ν to a section of $\mathcal{H}^{2m-1}$. Then the cohomology class

$$[\nabla\hat{\nu}] \in H^1(\mathcal{K}^\bullet_m)$$

defines $\delta\nu$.

As with periods, normal functions are transcendental quantities. We will define a variant of $\delta\nu$ that is an algebraic section of a coherent sheaf over B, thus giving an algebraic invariant of ν. For this we have an exact sequence

$$0 \to \mathcal{K}^\bullet_{m+1} \to \mathcal{K}^\bullet_m \to \mathrm{Gr}_m(\mathcal{K}^\bullet) \to 0;$$

the image $\mathrm{Gr}(\delta\nu)$ of $\delta\nu$ in the map

$$H^1(\mathcal{K}^\bullet_m) \to H^1\big(\mathrm{Gr}_m(\mathcal{K})\big)$$

defines $\mathrm{Gr}(\delta\nu)$. When we discuss singularities of $\delta\nu$, we will see that $\mathrm{sing}(\delta\nu)$ and $\mathrm{sing}(\mathrm{Gr}_m(\delta\nu))$ are defined over $\mathbb{Q}$ and the induced map

$$\mathrm{sing}(\delta\nu) \xrightarrow{\sim} \mathrm{sing}\left(\mathrm{Gr}_m(\delta\nu)\right)$$

is an isomorphism.

5. Singularities of normal functions

Over $B = (\Delta^*)^r$, we assume given a VHS of weight $2m-1$ with unipotent monodromies and canonical extensions $\mathcal{H}^{2m-1}_e$ and $\mathfrak{F}^\bullet_e$. There is an extension of the sheaf $\mathfrak{J}$ of intermediate Jacobians which is expressed by an exact sequence

$$0 \to \mathfrak{J}^o_e \to \mathfrak{J}_e \xrightarrow{\sigma} \mathcal{G} \to 0, \tag{5.1}$$

where

- $\mathfrak{J}^o_e \to \overline{B}$ is the sheaf of sections of a fiber space of connected abelian complex Lie groups;[1]
- $\mathcal{G}$ is a constructible sheaf of abelian groups supported over ∂B and which on each smooth stratum of ∂B is given modulo torsion by the local system associated with a variation of mixed Hodge structure (VMHS).

We refer to [12] for the case $\dim B = 1$ and to [18] for a general account of the terms in (5.1) together with references to the literature and illustrative examples.

(1) Although not relevant here, the total space $\mathfrak{J}^o_e$ is a slit analytic variety.

For normal functions $\nu : B \to \mathfrak{J}_h$ in [18, 20], there is defined the condition of *admissibility*, one that is satisfied by normal functions arising from geometry. *Admissible normal functions* (ANF's) have canonical extensions to

$$\nu_e : \overline{B} \to \mathfrak{J}_e.$$

DEFINITION 5.1. The singularity sing(ν) of an ANF is the image

$$\sigma(\nu_e) : \partial B \to \mathcal{G}.$$

We will now give an intuitive description of sing(ν). This is based on monodromy, so first we will discuss the *local monodromy of a normal function*.

In the geometric case where we have a codimension m algebraic cycle $\mathfrak{Z}$ with $\mathfrak{Z} \cdot X_t = Z_t$ a cycle with support $|Z_t|$ and fundamental class $[Z_t] = 0$ in $H_{2m-2}(X_t; \mathbb{Z})$, from (3.10) we have

$$\langle \nu_{\mathfrak{Z},t}, \omega_t \rangle = \int_{\Gamma_t} \omega_t,$$

where $\omega_t \in F_t^m$ and Γ_t is a $2m-1$ chain representing a class in $H_{2m-1}(X_t, |Z_t|; \mathbb{Z})$. Under analytic continuation around a loop $\gamma \in \pi_1(B, t_0)$, we assume that the cycle Z_t returns to itself while

$$\Gamma_{t_0} \to \Gamma_{t_0} + \delta.$$

Here, $\delta \in \operatorname{Im}\{H_{2m-1}(X_{t_0}; \mathbb{Z}) \hookrightarrow H_{2m-1}(X_{t_0} \backslash |Z_{t_0}|; \mathbb{Z})\}$. Thus, the monodromy $\widetilde{T_\gamma}$ on $H_{2m-1}(X_{t_0} \backslash |Z_{t_0}|; \mathbb{Z})$ is of the form

$$\widetilde{T_\gamma} = \begin{pmatrix} T_\gamma & E_\gamma \\ 0 & I \end{pmatrix}, \quad E_\gamma \in \operatorname{Hom}_{\mathbb{Z}}\big(\mathbb{Z}(-m), H^{2m-1}(X_t; \mathbb{Z})\big), \tag{5.2}$$

where T_γ is the monodromy on $H_{2m-1}(X_{t_0}; \mathbb{Z})$.[2] What this means is that given a reference point t_0, we choose a splitting of

$$0 \to H^{2m-1}(X_{t_0}; \mathbb{Z}) \to H^{2m-1}(X_{t_0} \backslash |Z_{t_0}|; \mathbb{Z}) \to \mathbb{Z}(-m) \to 0. \tag{5.3}$$

Continuing this splitting around γ gives (5.2). Changing the splitting by $\left(\begin{smallmatrix} I & \lambda \\ 0 & I \end{smallmatrix}\right)$ where $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}(-m), H^{2m-1}(X_{t_0}; \mathbb{Z}))$ gives

$$E_\gamma \to E_\gamma + (T_\gamma - I)\lambda.$$

(2) The monodromy acting on the above correction term T_η is the identity.

Thus, having a monodromy invariant splitting of (5.3) around γ is equivalent to

$$(\widetilde{T_\gamma} - I)\nu = \delta,$$

where

$$\delta = (T_\gamma - I)\lambda.$$

Still in the geometric case we choose a locally defined on B lifting of $\nu \in \mathfrak{F}^m \backslash \mathcal{H}^{2m-1}/H_{\mathbb{Z}}$ to $\widehat{\nu} \in \mathcal{H}^{2m-1}/H_{\mathbb{Z}}$. Then, analytic continuation of $\widehat{\nu}$ around γ leads to the same equation (5.2) where now

$$\delta \in \operatorname{Im}\left\{H^{2m-1}(X_{t_0};\mathbb{Z}) \hookrightarrow H^{2m-1}\big(X_{t_0}\backslash|Z_{t_0}|;\mathbb{Z}\big)\right\}.$$

When $B = \Delta^*$ and γ is a generator of $\pi_1(B)$, a basic result in [12, 18] is that δ is a vanishing cycle, and then by the local invariant cycle theorem [6], we have $\delta \in \operatorname{Im}(T_\gamma - I)_{\mathbb{Q}}$; i.e., over $\mathbb{Q}$,

$$(\widetilde{T_\gamma} - I)\nu = (T_\gamma - I)\lambda, \tag{5.4}$$

where $\lambda \in H^{2m-1}(X_{t_0};\mathbb{Q}) \cong H_{2m-1}(X_{t_0};\mathbb{Q})$. Then, we can subtract λ from the lift $\widehat{\nu}$, or subtract it from Γ_{t_0} in the geometric case, to conclude that the monodromy of the normal function is torsion. Thus, a multiple of ν extends to a section of $\mathfrak{J}_e^o$ in (5.1); i.e., $\operatorname{sing}(\nu)$ *is torsion.* These considerations in the geometric case apply as well to the general case [18].

In the semi-global case $B = (\Delta^*)^r$, we denote by $T_i \in \operatorname{Aut}(H_{\mathbb{Z}})$ the monodromy generator around $t_i = 0$, $t_j \neq 0$ for $j \neq i$. From (5.2) and (5.4), we have

$$\widetilde{T_i} = \begin{pmatrix} T_i & (T_i - I)\lambda_i \\ 0 & I \end{pmatrix}. \tag{5.5}$$

Then,

$$\begin{aligned}\widetilde{T}_i\widetilde{T}_j &= \begin{pmatrix} T_i & (T_i - I)\lambda_i \\ 0 & I \end{pmatrix}\begin{pmatrix} T_j & (T_j - I)\lambda_j \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} T_iT_j & T_i(T_j - I)\lambda_j + (T_i - I)\lambda_i \\ 0 & I \end{pmatrix}.\end{aligned}$$

Commutativity of $\pi_1((\Delta^*)^r)$ leads to the relation

$$(T_i - I)(T_j - I)\lambda_i = (T_j - I)(T_i - I)\lambda_j \tag{5.6}$$

which will be a critical ingredient in the construction of the nilpotent-orbit-like approximation $P(z)$ to an admissible normal function ν in the proof of Proposition (7.1) below.

In (5.5), we can add to each λ_i the same fixed λ. Then, over $\mathbb{Q}$, the monodromy of ν is described by the $(T_i - I)\lambda_i$'s taken modulo $(T_i - I)\lambda$. We note the subtlety that in this description of the monodromy of ν, the *way* in which over $\mathbb{Q}$ the codimension 1 singularities of ν are removed appears. This is why $\operatorname{sing}(\nu)$ will be non-zero only in codimension ≥ 2.

For the remainder of this section, we will work over $\mathbb{Q}$. The operators $T_i - I$, $T_j - I$ commute and this suggests a Koszul complex

$$(M^\bullet)^{3} \quad H \xrightarrow{\oplus(T_i - I)} \bigoplus_i (T_i - I)H \xrightarrow{\oplus(T_j - I)} \bigoplus_{i<j} (T_i - I)(T_j - I)H \to \cdots .$$

Definition 5.2 (Provisional definition). For an ANF ν defined over $B = (\Delta^*)^r$, the *singularity* $\operatorname{sing}(\nu)\{0\}$ at the origin $\{0\} \in B$ is defined to be the element in $H^1(M^\bullet)$ defined by the equivalence class of the $(T_i - I)\lambda_i$.

In general, we may define $\operatorname{sing}(\nu)$ by taking the above construction along the non-singular strata in ∂B. This defines a section $\sigma(\nu)$ of a constructible sheaf $\mathcal{G} \to \overline{B}$. Then,

$$\sigma(\nu) = 0 \iff \sigma \text{ extends to a section of } \mathfrak{J}_e^o \text{ in (5.1).} \tag{5.7}$$

Recalling that $H = \mathcal{H}_{t_0}^{2m-1}$ for a reference point t_0, the operator $T_i - I$ acts on H but does not interact with the Hodge filtration $\mathfrak{F}_{t_0}^\bullet$. Nonetheless, we may define the algebraic operator

$$N_i = \log T_i,$$

on H, and we have

$$T_i - I = \sum_{k=1}^{\infty} \frac{N_i^k}{k!} = N_i \cdot \sum_{k=1}^{\infty} \frac{N_i^{k-1}}{k!} := N_i \cdot A_i.$$

We note that $\ker(T_i - I) = \ker N_i$ and $\operatorname{Im}(T_i - I) = \operatorname{Im} N_i$. The motivation for making the switch is that $N_i : H_{\lim} \to H_{\lim}(-1)$ is a morphism of MHS whereas $T_i - I$ is not.

Then, the N_i and A_i all commute and there is a quasi-isomorphism of complexes

$$\begin{array}{ccccccc} H & \xrightarrow{\oplus N_i} & \bigoplus_i N_i H & \xrightarrow{\oplus N_j} & \bigoplus_{i<j} N_i N_j H & \longrightarrow \\ \Big\downarrow & & \Big\downarrow \oplus A_i & & \Big\downarrow \oplus A_i A_j & \\ H & \xrightarrow{\oplus T_i - I} & \bigoplus_i (T_i - I)H & \xrightarrow{\oplus T_j - I} & \bigoplus_{i<j} (T_i - I)(T_j - I)H & \longrightarrow \end{array} \tag{5.8}$$

(3) M stands for monodromy.

A basic fact from Hodge theory [21], one which will be elaborated on below, is that there is an isomorphism of vector spaces

$$H \cong H^{2m-1}_{\lim} \cong \mathcal{H}_{e,\{0\}} \tag{5.9}$$

such that N_i becomes the usual monodromy operator; $N = \sum N_i$ defines the monodromy weight filtration and $(H^{2m-1}_{\lim}, W_\bullet(N), F^\bullet_{\lim})$ defines the limiting mixed Hodge structure (LMHS) at $\{0\}$.

Definition 5.3. We define the complex $(\mathrm{LM}^\bullet)$ to be the Koszul complex associated with $\{H^{2m-1}_{\lim}; N_1, \ldots, N_r\}$.[4]

Using (5.9), the complex $(\mathrm{LM}^\bullet)$ is the top row in (5.8) with H replaced by $H^{2m-1}_{\lim}$. Since the vertical arrows in (5.8) induce a quasi-isomorphism of complexes, we have

$$H^1(M^\bullet) \cong H^1(\mathrm{LM}^\bullet). \tag{5.10}$$

Definition 5.4. The singularity $\mathrm{sing}(\nu)\{0\}$ is defined to be the class in $H^1(\mathrm{LM}^\bullet)$ corresponding to the provisional definition (Definition 5.2) using the identification (5.10).

This is the standard definition [18]. It measures the obstruction to finding over B a monodromy invariant lift $\hat{\nu}$ of the normal function ν.

6. Singularities of the infinitesimal invariant of normal functions and statement of the main result

Over $\overline{B}$ we have the complex

$$0 \to \mathfrak{F}^m_e \xrightarrow{\nabla} \mathfrak{F}^{m-1}_e \otimes \Omega^1_{\overline{B}}(\log D) \xrightarrow{\nabla} \mathfrak{F}^{m-2}_e \otimes \Omega^2_{\overline{B}}(\log D) \to \cdots .$$

Restricting to the stalk at the origin, this is

$$0 \to \mathfrak{F}^m_{e,\{0\}} \xrightarrow{\nabla} \mathfrak{F}^{m-1}_{e,\{0\}} \otimes \Omega^1_{\overline{B}}(\log D)_{\{0\}} \xrightarrow{\nabla} \mathfrak{F}^{m-2}_{e,\{0\}} \otimes \Omega^2_{\overline{B}}(\log D)_{\{0\}} \to \cdots , \tag{6.1}$$

where ∇ is an integrable connection with connection matrix

$$\sum_i N_i^* \otimes \frac{-dt_i/t_i}{2\pi\sqrt{-1}} + \big(\text{terms holomorphic at } \{0\}\big).$$

Taking residues in (6.1) and using that the fiber at the origin of $\mathfrak{F}^p_{e,\{0\}}$ is $F^p_{\lim}$, we obtain the *residue complex*

$$(R^\bullet) \quad 0 \to F^m_{\lim} \to F^{m-1}_{\lim} \otimes T^* \to F^{m-2}_{\lim} \otimes \wedge^2 T^* \to,$$

(4) LM stands for “log of monodromy”.

where $T^* \subset \mathrm{End}^{-1}_{F^\bullet}(H^{2m-1}_{\lim}) \cap \mathrm{End}^{-2}_{W_\bullet}(H^{2m-1}_{\lim,\mathbb{Q}})$ has the basis $N_1^*, \dots, N_m^*$. We note that

$$(R^\bullet) \text{ is a sub-complex of the Koszul complex } \mathrm{LM}^\bullet. \tag{6.2}$$

To define $\mathrm{sing}(\delta\nu)$, we will use that

- $\delta\nu$ is the class in $H^1(\mathfrak{F}^{m-\bullet} \otimes \Omega^\bullet_B, \nabla)$ given by $\nabla\hat{\nu}$ where $\hat{\nu}$ is a local lifting to $\mathcal{H}$ of $\nu \in \mathfrak{F}^m \backslash \mathcal{H}/\mathbb{H}_\mathbb{Z}$;
- by admissibility, we may choose the lifting to have a section

$$\nabla\hat{\nu}_e \in \mathfrak{F}^{m-1}_e \otimes \Omega^1_{\bar{B}}(\log D).$$

Definition 6.1. $\mathrm{sing}(\delta\nu)$ is the class of $\mathrm{Res}(\nabla\hat{\nu}_e) := \sum_i \mathrm{Res}_{z_i=0}(\nabla\nu_e) \otimes N_i^*$ in the residue complex.

Theorem 6.1. *There is a natural inclusion* $i : R^\bullet \to \mathrm{LM}^\bullet$ *of complexes and under the induced map on cohomology*

$$i_* \,\mathrm{sing}(\delta\nu) = \mathrm{sing}(\nu). \tag{6.3}$$

We will give a complete proof in a subsequent work; below we will sketch a special case of the argument. An intuitive reason is that if the local lifting $\hat{\nu}$ is defined using the liftings $\hat{\Gamma}_{\eta,t}$ with holomorphic dependence on t as discussed above, then

- $\mathrm{sing}(\nu)$ is the obstruction over $B = (\Delta^*)^r$ being able to obtain a monodromy invariant lifting $\hat{\nu}$ of ν;
- any lifting $\hat{\nu}$ has an expression as a polynomial in the $l(t_j) = (\frac{1}{2\pi i})\log t_j$ with holomorphic coefficients and the residue terms in $\nabla\hat{\nu}$ vanish if, and only if, there are no $l(t_j)$ terms in $\hat{\nu}$ (this is a crucial part of the proof);
- $\mathrm{sing}(\delta\nu)$ is then the obstruction to finding a lifting $\hat{\nu}$ such that $\nabla\hat{\nu}$ has no residue terms.

Put all together, this implies that the vanishing of $\mathrm{sing}(\nu)$ and $\mathrm{sing}(\delta\nu)$ are equivalent.

We recall that $H^{2m-1}_{\lim}$ has a $\mathbb{Q}$-structure, a weight filtration $W_\bullet(N)$ and a Hodge filtration defined up to scaling by the action of $\exp(\sum_i \mu_i N_i)$ where $\mu_i \in \mathbb{C}$. This data defines a mixed Hodge structure where the induced pure Hodge structures on $\mathrm{Gr}^{W(N)}(H^{2m-1})$ are independent of the parametrization $(t_1, \dots, t_r)$. There are induced mixed Hodge structures on the $H^q(\mathrm{LM}^\bullet)$, and a central result [3] is

$$\text{the weights of } H^1(\mathrm{LM}^\bullet) \text{ are } \le 2m-2.\text{[5]} \tag{6.4}$$

([5]) Our indexing is adapted to the situation at hand; it is different from that in [3].

In particular, $\mathrm{Gr}^{W(N)}_{2m-2}(F^{m-1}H^{2m-1}_{\lim})$ is of Hodge type $(m-1, m-1)$; it is defined over $\mathbb{Q}$ and we denote it by

$$Hg^{m-1}_{\lim}.$$

Turning to the residue complex $R^\bullet \subset \mathrm{LM}^\bullet$, we note that by strictness, the induced maps $H^q(R^\bullet) \to H^q(\mathrm{LM}^\bullet)$ are injective and the image defines $F^{m-q}H^q(\mathrm{LM}^\bullet)$. The above inclusion maps then imply that $H^1(R^\bullet)$ has a $\mathbb{Q}$-structure and there is an inclusion

$$H^1(R^\bullet)_{\mathbb{Q}} \hookrightarrow Hg^{m-1}_{\lim}. \tag{6.5}$$

7. Asymptotics of normal functions and sketch of the proof of Theorem 6.1

A fundamental result in Hodge theory is Schmid's result that the singular part of the asymptotic expansion of a period matrix is given by a nilpotent orbit (cf. [18] for an exposition and references). The strategy adopted here is to use a variant of the proof of the nilpotent orbit theorem applied to the lift $\hat{\nu}$ of an ANF and infer the result through an analysis of the resulting formulas. Just as nilpotent orbits are constructed from monodromy, a similar result will hold here for ANF's. The argument here is necessarily somewhat subtle in that the approximating polynomial in the $l(t_i)$ will depend on the λ_i taken modulo a fixed λ and can only be constructed when the conditions (5.6) are satisfied.

Denoting by $U = \{z : \mathrm{Im}\, z > 0\}$ with $U \to \Delta^*$ given by $t = \exp(2\pi i z)$, as in the proof of the usual nilpotent orbit theorem given a locally defined lift $\hat{\nu}(t) \in \mathcal{H}^{2m-1}$ of an ANF $\nu(t)$, it is convenient to pull $\hat{\nu}(t)$ up to U^r to have $\hat{\nu}(z) \in \mathcal{H}^{2m-1}_z$ with

$$\hat{\nu}(z + e_i) = \widetilde{T}_i\, \hat{\nu}(z).$$

Then, over U^r this gives (i) in

$$\begin{cases} \hat{\nu}(z+e_i) - \widetilde{T}_i \hat{\nu}(z) = (T_i - I)\lambda_i, \ \lambda_i \in H_{\mathbb{Q}}, & \text{(i)} \\ (T_i - I)(T_j - I)\lambda_i = (T_j - I)(T_i - I)\lambda_j. & \text{(ii)} \end{cases} \tag{7.1}$$

Here, $\hat{\nu}(z)$ is a section of $\mathcal{O}(U^r) \otimes H$ where H is a nearby fiber; $\widetilde{T}_i$ acts on H (monodromy of flat sections) and $z \mapsto z + e_i$ acts on the coefficients.

The main step in the proof is the following.

Proposition 7.1. *There exists a unique H-valued polynomial $P(z)$ such that*

$$P(z + e_i) - T_i P(z) = (T_i - I)\lambda_i. \tag{7.2}$$

Proof. We will do the first non-trivial case when $r = 2$. The method used to derive (7.6) below will extend to the general case. This will be done in a separate work in which we will give examples and discuss implications of Theorem 6.1.

Setting

$$\begin{cases} g(z) = \exp(z_1 N_1 + z_2 N_2), \\ g_i(z) = \exp(z_i N_i), \quad i = 1, 2, \end{cases} \tag{7.3}$$

the formula for $P(z)$ is

$$\begin{aligned} P(z_1, z_2) = g(z)\lambda &+ \big(g_1(z_1) - I\big)\big(g_2(z_2) - I\big)\lambda_i \\ &+ \big(g_1(z_1) - I\big)\lambda_1 + \big(g_2(z_2) - I\big)\lambda_2. \end{aligned} \tag{7.4}$$

Here, $\lambda \in H$ and in the second term i may be taken to be 1 or 2. One then checks that (7.2) holds. ∎

REMARK. To derive the formula (7.4) for $P(z_1, z_2)$, one sets

$$q_k(z) = \begin{cases} 1 & k = 0, \\ \frac{z(z-1)\cdots(z-(k-1))}{k!} & k \geq 1. \end{cases}$$

Then,

$$q_{k+1}(z+1) - q_k(z) = q_{k+1}(z).$$

Setting

$$P(z_1, z_2) = \sum a_{k,l} q_k(z_1) q_l(z_2), \tag{7.5}$$

one then uses (7.1) and (7.2) to recursively determine the $a_{k,l}$, $k + l > 0$, in terms of a_{00}. The result is

$$\begin{aligned} P(z_1, z_2) = a_{00} &+ \sum_{k>0} q_k(z_1)(T_1 - I)^k (a_{00} + \lambda_1) \\ &+ \sum_{l>0} q_l(z_2)(T_2 - I)^l (a_{00} + \lambda_2) \\ &+ \sum_{\substack{k>0 \\ l>0}} q_k(z_1) q_l(z_2)(T_1 - I)^k (T_2 - I)^l (a_{00} + \lambda_i), \end{aligned} \tag{7.6}$$

where i may be 1 or 2 in the last term. The a_{00} reflects that we may add to each λ_i a fixed λ.

We note that $P(z_1, z_2)$ depends only on a_{00} and the $(T_i - I)\lambda_i$ for $i = 1, 2$. Having determined the a_{kl} in (7.5) in terms of a_{00}, one then checks that (7.6) is summarized in (7.4).

From (7.2), it follows that

$$(P - \widehat{\nu})(z + e_i) - T_i(P - \widehat{\nu})(z) = 0.$$

Thus, $P(z) - \widehat{\nu}(z)$ descends to $B = (\Delta^*)^r$; we write

$$P(z) = \widehat{\nu}(z) + Q(t).$$

Since ν is an admissible normal function, it follows that $Q(t)$ is holomorphic on $\overline{B} = \Delta^r$. We will write $\equiv$ to mean congruence modulo holomorphic terms on $\overline{B}$.

From the formula for P, we obtain

$$g(z)^{-1}\mathrm{d}P(z) \equiv \sum_i N_i\lambda_i \mathrm{d}z_i.$$

It follows that

$$g(z)^{-1}\mathrm{d}\widehat{\nu}(z) \equiv \sum_j N_j\lambda_j \mathrm{d}z_i = \left(\frac{1}{2\pi i}\right)\sum (N_j\lambda_j)\frac{\mathrm{d}t_j}{t_j}. \tag{7.7}$$

Since

$$\widehat{\nu}(z+e_i) - \widehat{\nu}(z) = (T_i - I)\widehat{\nu}(z) = (T_i - I)\lambda_i,$$

it follows that $\mathrm{d}\widehat{\nu}(z)$ descends to $\nabla\widehat{\nu}(t)$. Taking $g(z)^{-1}\mathrm{d}\widehat{\nu}(z)$ and letting $\operatorname{Im} z \to 0$ pick out the residue term in $\nabla\widehat{\nu}$. This argument gives that

$$\operatorname{sing}_{\{0\}}(\delta\nu) := \left(\operatorname{Res}_{z_1=0}(\delta\nu), \ldots, \operatorname{Res}_{z_r=0}(\delta\nu)\right)$$

is equal to

$$\operatorname{sing}_{\{0\}}(\nu) := (N_1\lambda_1, \ldots, N_r\lambda_r)$$

which is what was to be proved. ∎

As remarked above, a subtlety is that in the usual construction of nilpotent orbits one uses $g(z)$ in (7.3) above. For the construction of $P(z)$, it is necessary to have a power series in the $(T_i - I)$'s, not one in the N_i's. The reason for this is the different λ_i's that appear on the right-hand side of (7.6).

Another subtlety is that for $F^\bullet(z)$, the filtration on H given by the VHS on B pulled back to U^r, the approximating nilpotent orbit is obtained from

$$G^\bullet(z) = g(z)^{-1}F^\bullet(z).$$

Then, $G^\bullet(z)$ descends to B and the resulting $G^\bullet(t)$ extends across $t = 0$, and setting $F^\bullet_\infty = G^\bullet(0)$, the nilpotent orbit is $g(z)\cdot F^\bullet_\infty$. Using that

$$\nabla\widehat{\nu}(z) \in F^{m-1}(z) \otimes \Omega^1_{U^r},$$

we may infer that all

$$N_i\lambda_i \in F^{m-1}_{\lim}. \tag{7.8}$$

Finally, the $\lambda_i \in H_{\mathbb{Q}}$ so that

$$N_i \lambda_i \in Hg^{m-1}_{\lim}$$

as has been noted above.

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