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Number Theory. – *Height limits, equidistribution, and dilogarithms*, by David Masser, accepted on 1 July 2025.

At Carampane we'd dine,
With a carafe or three of the wine.
It's a place of some fame,
But in spite of its name,
The tarts are exceedingly fine.

Abstract. – When R is a rational (or even algebraic) function, it is well known how the height of $R(\alpha)$ behaves asymptotically as the height of α tends to ∞. Here we prove some analogous versions as the height of α tends instead to 0, also with explicit error terms. Our most general results, some with a trigonometrical flavour, are for roots of unity of large order. For example, the height of $\tan 2\pi/n$ tends to $2G/\pi$ where G is the Catalan constant.

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1. Introduction

This note is concerned with the standard (logarithmic) height function h on the set $\bar{\mathbf{Q}}$ of all algebraic numbers. Perhaps one of the most interesting of its many properties is the functorial aspect. Without being too precise right now, we take a function R in the algebraic closure $\overline{\mathbf{Q}(X)}$. Then as $h(\alpha) \to \infty$, the height $h(R(\alpha))$ is asymptotically $Dh(\alpha)$ for an associated "degree" D. In fact, there is $c = c(R)$ such that

$$|h(R(\alpha)) - Dh(\alpha)| \leq c\sqrt{h(\alpha)+1} \tag{1.1}$$

for all reasonable α. For this see Bombieri and Gubler [9, pp. 289–293] or Lang [21, pp. 113–115] or Hindry and Silverman [20, pp. 183–210, especially p. 190 and p. 209], with some alternative viewpoints in Bombieri [6], one of them based on Bombieri [5]. See also Habegger [18]. At a lower level one can see an example worked out in the appendix of [23].

As $h(\alpha) \to 0$, this says only that $h(R(\alpha))$ is bounded. The object of this note is to show that there can be asymptotics here too.

Here the naive definition

$$h(\alpha) = \frac{1}{d}\log a_0 + \frac{1}{d}\sum_\sigma \log\max\{1, |\alpha^\sigma|\} \geq 0 \tag{1.2}$$

will suffice, where d is the degree of α, $a_0 \geq 1$ is the leading coefficient of the minimal polynomial in $\mathbf{Z}[X]$ vanishing at α, and σ runs over all complex embeddings of $\mathbf{Q}(\alpha)$ into $\mathbf{C}$.

A simple example comes from taking α as a root of unity ζ; of course then $h(\zeta) = 0$. With $R(X) = 1 - X$ it is easy to see from (1.2) that $h(1-\zeta) \leq \log 2$. But if the order of ζ is n, it is known (see also below) that

$$h(1-\zeta) \to \frac{1}{2\pi}\int_0^{2\pi} \log\max\{1, |1-e^{i\theta}|\}\mathrm{d}\theta = 0.32306594721945051409363651072 38\ldots \tag{1.3}$$

as $n \to \infty$.

In this note, we will generalize (1.3) to $h(R(\zeta))$ for any R in $\mathbf{Q}(X)$.

We can write $R = P/Q$ for P, Q in $\mathbf{Z}[X]$ and coprime there. Then the lengths $L(P), L(Q)$, or the sums of the absolute values of the coefficients, are uniquely defined, and we may write

$$L(R) = L(P) + L(Q) \geq 1.$$

Similarly we may define

$$\mathbf{m}(R) = \frac{1}{2\pi}\int_0^{2\pi} \log\max\{|P(e^{i\theta})|, |Q(e^{i\theta})|\}\mathrm{d}\theta \tag{1.4}$$

as in (1.3). This may look a bit like a Mahler measure (and in Section 9 we will see that in some sense it really is) – but it is not to be confused with the Mahler measure that is sometimes (see for example [25, p. 225]) defined as the difference of those of P and Q (that can easily be negative, whereas our $\mathbf{m}(R)$ cannot be – see the comments just below). Our main result is the following.

Theorem. *There is an effective absolute constant c, and an effective constant C depending only on the degree of R, such that for any root of unity ζ of order n with $Q(\zeta) \neq 0$, we have*

$$|h(R(\zeta)) - \mathbf{m}(R)| \leq cL(R)^C\left(\frac{\log 2n}{\phi(n)}\right)^{1/2}$$

for the Euler function ϕ.

This superficially resembles a recent deep result of Dimitrov and Habegger [16] about $\log |P(\underline{\zeta}^{\sigma})|$, where P is a (Laurent) polynomial in several variables (over $\bar{\mathbf{Q}}$) and $\underline{\zeta}$ is a point whose coordinates are roots of unity. Under suitable conditions they show that the average over σ (as in (1.2) above) tends to the Mahler measure of P. An earlier version for one variable can be deduced from the work [2] of Baker, Ih, and Rumely (see the remarks around their equation (6), p. 221). Here too we consider only one variable; on the other hand, we treat a rational function (over just $\mathbf{Q}$) and we relate the whole height to our look-alike Mahler measure.

Then $h(R(\zeta)) \to \mathbf{m}(R)$ as $n \to \infty$.

It follows of course that $\mathbf{m}(R) \geq 0$, which is not very obvious from the definition. In fact, this sort of thing (and more) is known independently, as we will see in Section 9.

We will find that

$$c = 10^{10}, \quad C = 4D^2 \tag{1.5}$$

are permissible, for the degree

$$D = \max\{\deg P, \deg Q\}$$

of R.

We pause here to dispose of a couple of easy special cases.

If $D = 0$, then we see at once that $h(R(\alpha)) = \mathbf{m}(R)$, even for any α whatsoever.

And if $L(R) = 1$, then $P = 0$ and $Q = \pm 1$, so $D = 0$ and even $h(R(\alpha)) = \mathbf{m}(R) = 0$.

Thus, henceforth, we will assume that

$$D \geq 1, \quad L(R) \geq 2.$$

The arrangement of the rest of this note is as follows; all is worked out for general α except in Section 5.

In Section 2, we decompose $h(R(\alpha))$ into three parts, one involving a resultant. Then in Section 3, we record for resultants some estimates of a somewhat familiar type, taking some trouble with the numerical constants. A by-product is

$$\big|h\big(R(\alpha)\big) - Dh(\alpha)\big| \leq (2D-1)\log L(R) \tag{1.6}$$

(when $D \geq 1$), an explicit version of (1.1). It is well known that the square root there can here be avoided.

In Section 4, we use the work [3] of Roger Baker and the author on equidistribution to deal with a part of $h(R(\alpha))$ involving the conjugates of α.

Then in Section 5, we restrict ourselves to roots of unity and deal with the part involving a resultant. Here we seem to need $\mathcal{P}$-adic linear forms in logarithms. We did not explore the possibility of using instead the work of Bombieri [7] or its extensions by Bombieri and Cohen [8].

After that the very short Section 6 completes the proof of our Theorem. We add a short Section 7 to show that linear forms in logarithms can be avoided when $D = 1$.

And in Section 8, we escape from roots of unity by showing how [3] can be used to get the asymptotics for $h(1-\alpha)$ when α is arbitrary with $h(\alpha) \to 0$. In fact, our result

$$\big|h(1-\alpha) - h(\alpha) - \mathbf{m}_0\big| \le 19\Big(h(\alpha) + \frac{\log 2d}{d}\Big)^{1/2}, \tag{1.7}$$

with $\mathbf{m}_0$ as in (1.3) and d the degree of α, gives the asymptotics simultaneously even for $h(\alpha) \to \infty$ as well as for $h(\alpha) \to 0$ (recall that then $d \to \infty$ if α is not a root of unity as in (1.3) above). But (1.6) gives (the easy) $|h(1-\alpha) - h(\alpha)| \le \log 2$, even better as $h(\alpha) \to \infty$.

And in the last section (Section 9), we point out that the quantities $\mathbf{m}(R)$ in the theorem can be evaluated in terms of the classical dilogarithm. This leads to more explicit examples of the theorem, like

$$\lim_{n\to\infty} h\Big(\tan\frac{2\pi}{n}\Big) = \frac{2}{\pi}G = 0.58312180806163756027676891293 6789\ldots$$

for Catalan's constant

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \cdots. \tag{1.8}$$

It might be interesting to interpret our results in terms of the canonical height h_R as defined in Silverman [26, p. 99] and considered also by Szpiro and Tucker in [27] (note too that Everest and Ní Fhlathúin [17] also used linear forms in logarithms, but elliptic and ∞-adic).

2. A resultant

With R, P, Q and $D \ge 1$ as above, let α be any algebraic number of degree say d with minimal polynomial Φ in $\mathbf{Z}[X]$ and leading coefficient $a_0 \ge 1$. We define Ψ as the resultant (with respect to X)

$$\Psi(Y) = \Psi_R(Y) = \mathrm{res}_X\big(Q(X)Y - P(X), \Phi(X)\big),$$

non-zero in $\mathbf{Z}[Y]$ because $Q(X)Y - P(X)$ is irreducible. It has a "content" $c(\Psi) \ge 1$, the highest common factor of the coefficients.

Proposition 2.1. *If $Q(\alpha) \ne 0$, we have*

$$h\big(R(\alpha)\big) = h_0(R,\alpha) + h_{\mathrm{fin}}(R,\alpha) + h_{\mathrm{inf}}(R,\alpha) \tag{2.1}$$

with

$$h_0(R,\alpha) = \frac{D}{d}\log a_0, \quad h_{\mathrm{fin}}(R,\alpha) = -\frac{1}{d}\log c(\Psi),$$
$$h_{\mathrm{inf}}(R,\alpha) = \frac{1}{d}\sum_\sigma \log\max\left\{\left|P(\alpha^\sigma)\right|, \left|Q(\alpha^\sigma)\right|\right\}$$

for the sum over all complex embeddings σ of $\mathbf{Q}(\alpha)$.

Proof. As the zeroes of Φ are just the α^σ, we have

$$\Psi(Y) = (-1)^{dD} a_0^D \prod_\sigma \left(Q(\alpha^\sigma)Y - P(\alpha^\sigma)\right) = a\prod_\sigma \left(Y - R(\alpha^\sigma)\right)$$

with $a = (-1)^{dD} a_0^D \prod_\sigma Q(\alpha^\sigma)$ in $\mathbf{Z}$. The degree d_α of $R(\alpha)$ over $\mathbf{Q}$ divides d; write $e = d/d_\alpha$ so that $\Psi(Y) = a\prod_\tau (Y - R(\alpha)^\tau)^e$ taken over all complex embeddings τ of $\mathbf{Q}(R(\alpha))$. Now the minimal polynomial of $R(\alpha)$ is $\Theta(Y) = b\prod_\tau (Y - R(\alpha)^\tau)$ for some $b \geq 1$ in $\mathbf{Z}$. Thus, $b^e\Psi = a\Theta^e$. Comparing contents, we get $b^e c(\Psi) = |a|$. Using Θ to calculate $h(R(\alpha))$ as in (1.2) we get

$$h\big(R(\alpha)\big) = \frac{1}{d_\alpha}\log b + \frac{1}{d_\alpha}\sum_\tau \log\max\left\{1, \left|R(\alpha)^\tau\right|\right\}.$$

The first term on the right is

$$\frac{1}{d}\log b^e = \frac{1}{d}\big(-\log c(\Psi) + \log|a|\big)$$

and the second term is

$$\frac{1}{d}\sum_\sigma \log\max\left\{1, \left|R(\alpha)^\sigma\right|\right\}.$$

Here of course

$$\log|a| = D\log a_0 + \sum_\sigma \log\left|Q(\alpha)^\sigma\right|$$

and so we get the required result. ∎

In the following, we proceed to treat each part on the right-hand side of (2.1) in turn. Clearly,

$$\left|h_0(R,\alpha)\right| \leq Dh(\alpha) \tag{2.2}$$

will be small as $h(\alpha) \to 0$.

3. More on resultants

Let R, P, Q and $D \geq 1$ be as above.

Lemma 3.1. *There are $A_0, B_0, A_\infty, B_\infty$ in $\mathbf{Z}[X]$ of degrees at most $D-1$, and $r \neq 0$ in $\mathbf{Z}$, such that*

$$A_0 P + B_0 Q = r, \quad A_\infty P + B_\infty Q = r X^{2D-1} \tag{3.1}$$

and

$$L(A_0)+L(B_0) \leq L(R)^{2D-1}, \; L(A_\infty)+L(B_\infty) \leq L(R)^{2D-1}, \; |r| \leq L(R)^{2D}. \tag{3.2}$$

Proof. We shall see that r is often the resultant of P, Q; but in some interesting cases not. Suppose P, Q have degrees p, q, respectively. We use the "reciprocal" polynomials

$$\widetilde{P}(X) = X^p P(1/X), \quad \widetilde{Q} = X^q Q(1/X).$$

A "non-degenerate" situation is

$$p \geq 1, \quad q \geq 1, \quad P(0) \neq 0, \quad Q(0) \neq 0. \tag{3.3}$$

Then using the product formula for resultants we see easily that

$$\operatorname{res}(\widetilde{P}, \widetilde{Q}) = (-1)^{pq} \operatorname{res}(P, Q). \tag{3.4}$$

Then the Sylvester determinant gives the first of (3.1) for $r = \operatorname{res}(P, Q)$, with A_0, B_0 of degrees at most $q-1, p-1$, respectively. We see that

$$|r| \leq L(P)^q L(Q)^p \leq L(R)^{p+q} \leq L(R)^{2D}$$

as in the third of (3.2) – see [23, p. 53] for a picture. And the coefficients of A_0 are in absolute value at most $L(P)^{q-1} L(Q)^p$ – see [23, p. 54]. So $L(A_0) \leq q L(P)^{q-1} L(Q)^p$. Similarly, $L(B_0) \leq p L(P)^q L(Q)^{p-1}$, and so

$$L(A_0) + L(B_0) \leq \big(L(P) + L(Q)\big)^{p+q-1}. \tag{3.5}$$

This is because the binomial coefficient

$$\binom{p+q-1}{q-1} \geq q$$

(even if $q = 1$), and likewise

$$\binom{p+q-1}{q} \geq p$$

(even if $p = 1$). Now (3.5) implies the first of (3.2).

To deal with A_∞, B_∞ we temporarily assume $p = q$ and we apply the above to $\widetilde{Q}$ and $\widetilde{P}$. We get A_1, B_1 with $A_1\widetilde{Q} + B_1\widetilde{P} = r$. We replace X by $1/X$ and multiply by X^{2D-1}. We get the second of (3.1) with

$$A_\infty(X) = X^{D-1}A_1(1/X), \quad B_\infty(X) = X^{D-1}B_1(1/X).$$

Let us now relax $p = q$, without loss of generality to $p > q$ (so $D = p$), still with (3.3). Then the Sylvester determinant gives $A, B, r_0 = \mathrm{res}(P, Q)$, even with A, B of degrees at most $q - 1, p - 1$, respectively, with

$$AP + BQ = r_0 \tag{3.6}$$

and $L(A) + L(B) \le L(R)^{p+q-1}$ as in (3.5). Also $\widetilde{P}, X^{D-q}\widetilde{Q}$ are coprime and so the same procedure gives

$$A_1\widetilde{P} + B_1X^{D-q}\widetilde{Q} = r_1 \tag{3.7}$$

with A_1, B_1 of degree at most $D - 1$,

$$L(A_1) + L(B_1) \le L(R)^{2D-1},$$

and r_1 for the new resultant satisfying $|r_1| \le L(R)^{2D}$. Now

$$r_1 = \mathrm{res}(\widetilde{P}, X^{D-q}\widetilde{Q}) = \pm\widetilde{P}(0)^{D-q}\,\mathrm{res}(\widetilde{P}, \widetilde{Q}).$$

We replace X by $1/X$ in (3.7) and multiply by X^{2D-1}. We get

$$X^{D-1}A_1(1/X)P(X) + X^{D-1}B_1(1/X)Q(X) = r_1X^{2D-1}$$

which looks like the second of (3.1) except that r_1 is not the same as $r_0 = \mathrm{res}(P, Q)$ in (3.6). As the latter resultant is $\pm\,\mathrm{res}(\widetilde{P}, \widetilde{Q})$, we can just multiply (3.6) by an extra $\widetilde{P}(0)^{D-q}$. We get $A_0P + B_0Q = r_1$ with

$$L(A_0)+L(B_0) \le \left|\widetilde{P}(0)\right|^{D-q}\big(L(A)+L(B)\big) \le L(R)^{D-q}L(R)^{p+q-1} \le L(R)^{2D-1}.$$

So $r = r_1$ works.

Next we relax $P(0) \ne 0$, $Q(0) \ne 0$ in (3.3), without loss of generality to $Q(0) = 0$ (so $P(0) \ne 0$, and still $p \ge 1, q \ge 1$). We write $Q(X) = X^mQ_0(X)$ with $1 \le m \le q$ and $Q_0(0) \ne 0$. We assume for now that $m < q$, so Q_0 has degree $q - m \ge 1$. Now P, Q_0 are non-degenerate in the sense of (3.3).

Again we get (3.6), with

$$r_0 = \mathrm{res}(P, Q) = \mathrm{res}(P, X^mQ_0) = \pm P(0)^m\,\mathrm{res}(P, Q_0).$$

Now $X^{D-p}\widetilde{P}, X^{D-q}\widetilde{Q}_0$ are coprime and we get

$$A_1 X^{D-p}\widetilde{P} + B_1 X^{D-q}\widetilde{Q}_0 = r_1 \tag{3.8}$$

with $r_1 = \mathrm{res}(X^{D-p}\widetilde{P}, X^{D-q}\widetilde{Q}_0)$. We can no longer assume $p > q$, but certainly $D = p$ or $D = q$.

If $D = p$, then

$$r_1 = \mathrm{res}(\widetilde{P}, X^{D-q}\widetilde{Q}_0) = \pm\widetilde{P}(0)^{D-q}\,\mathrm{res}(\widetilde{P}, \widetilde{Q}_0) = \pm\widetilde{P}(0)^{D-q}\,\mathrm{res}(P, Q_0).$$

So now we have to multiply (3.6) by $\widetilde{P}(0)^{D-q}$ and (3.8) by $P(0)^m$ to get

$$r = \widetilde{P}(0)^{D-q} P(0)^m\,\mathrm{res}(P, Q_0)$$

with

$$|r| \le L(P)^{D-q+m} L(R)^{p+q-m} \le L(R)^{2D}.$$

And if $D = q$, a similar argument works, getting $r = \widetilde{Q}_0(0)^{D-p} P(0)^m\,\mathrm{res}(P, Q_0)$.

What if $m = q$ in the above? Then $\widetilde{Q}(X) = \widetilde{Q}(0)$ is constant and there is no Sylvester determinant for the left-hand side of (3.4). But for the right-hand side,

$$\mathrm{res}(P, Q) = \mathrm{res}\left(P, \widetilde{Q}(0)X^q\right) = \pm P(0)^q\widetilde{Q}(0)^p.$$

If $D = p > q$, then we get $A_1 P + B_1 Q = r_0$ for $r_0 = P(0)^q\widetilde{Q}(0)^p$ and A_1, B_1 of degrees at most $p-1$. Also $A_2\widetilde{P} + B_2\widetilde{Q}(0)X^{p-q} = r_\infty$ for

$$r_\infty = \mathrm{res}\left(\widetilde{P}, \widetilde{Q}(0)X^{p-q}\right) = \pm\widetilde{P}(0)^{p-q}\widetilde{Q}(0)^p$$

and A_2, B_2 of degrees at most $p-1$. Replacing X by $1/X$ and multiplying by $X^{2p-1} = X^{2D-1}$ gives $A_3 P + B_3 Q = r_\infty X^{2D-1}$ for

$$A_3(X) = X^{p-1}A_2(1/X), \quad B_3(X) = X^{p-1}B_2(1/X).$$

So we can take $r = \widetilde{P}(0)^{p-q}P(0)^q\widetilde{Q}(0)^p$ and all checks.

If $D = q \ge p$, we get $A_0 P + B_0 Q = r_0$ for $r_0 = P(0)^q\widetilde{Q}(0)^p$. And now after all these exertions, we can afford a “silly” solution $A_\infty P + B_\infty Q = r_\infty X^{2D-1}$ with

$$A_\infty = 0, \quad B_\infty = X^{D-1}, \quad r_\infty = \widetilde{Q}(0).$$

So $r = r_0$ suffices.

Finally, we relax the conditions $p \ge 1, q \ge 1$ in (3.3). By symmetry, we can assume $p \ge 1$ and $q = 0$. So now there is no Sylvester determinant for the right-hand side of (3.4). This “super-degenerate” situation is actually quite significant because now R is a polynomial.

But here we can start off with the even sillier $A_0P + B_0Q = r_0$ with

$$A_0 = 0, \quad B_0 = 1, \quad r_0 = Q(0) = \widetilde{Q}(0).$$

Also $A_1\widetilde{P} + B_1Q(0)X^p = r_\infty$ for

$$r_\infty = \operatorname{res}\bigl(\widetilde{P}, Q(0)X^p\bigr) = \pm\widetilde{P}(0)^p Q(0)^p$$

which leads to $A_\infty P + B_\infty Q = r_\infty X^{2D-1}$ with

$$A_\infty(X) = X^{p-1}A_1(1/X), \quad B_\infty(X) = X^{p-1}B_1(1/X).$$

So in this situation, $r = r_\infty$ in the usual way. This completes the proof. ∎

Now the usual game with valuations can be played with (3.1); in fact, Lemma 3.1 has been laboriously set up for a direct application of Lemma 13 of Habegger, Jones and the author [19, p. 465]. Using also (3.2), we get a lower bound for $h(R(\alpha))$. Combined with the "practically obvious" upper bound cited in [19, p. 466], we deduce (1.6). Note that when $D = 0$, then trivially we get the bound $\log L(R)$.

4. Equidistribution

Here we investigate the average

$$h_{\inf}(R,\alpha) = \frac{1}{d}\sum_{\sigma} \log\max\bigl\{\bigl|P(\alpha^\sigma)\bigr|, \bigl|Q(\alpha^\sigma)\bigr|\bigr\}$$

in Proposition 2.1. In the usual way, we relate it to the average

$$\mathbf{m}(R) = \frac{1}{2\pi}\int_0^{2\pi} \log\max\bigl\{\bigl|P(e^{i\theta})\bigr|, \bigl|Q(e^{i\theta})\bigr|\bigr\}d\theta$$

in (1.4). Thus, we should study the "test function"

$$f(z) = \log\max\bigl\{\bigl|P(z)\bigr|, \bigl|Q(z)\bigr|\bigr\},$$

and for this purpose, Theorem 1.6 of Baker and the author [3, p. 13011] seems best suited. Note that f is well defined on all of $\mathbf{C}$ because P, Q have no common zero (thus there is no danger of Autissier-type counterexamples).

This result contains the parameters M, M_∞, M_0, L; and we proceed to estimate these for the above f.

As for M (an upper bound for $|f(z)|$ with $1/2 \le |z| \le 2$), it is clear that

$$\bigl|P(z)\bigr| \le 2^D L(P), \quad \bigl|Q(z)\bigr| \le 2^D L(Q) \quad \bigl(|z| \le 2\bigr)$$

and so

$$f(z) \le D \log 2 + \log L(R). \tag{4.1}$$

By the first equation in (3.1) above, we have

$$\begin{aligned} 1 \le |r| &= \big|A_0(z)P(z) + B_0(z)Q(z)\big| \\ &\le 2^D\big(L(A_0) + L(B_0)\big) \max\big\{|P(z)|, |Q(z)|\big\} \quad \big(|z| \le 2\big). \end{aligned}$$

So by the first inequality in (3.2), we get

$$\max\big\{|P(z)|, |Q(z)|\big\} \ge 2^{-D} L(R)^{-(2D-1)}. \tag{4.2}$$

Therefore,

$$f(z) \ge -D \log 2 - (2D-1) \log L(R).$$

Together with (4.1) this shows that we can take

$$M = D \log 2 + (2D-1) \log L(R) \le 3DL(R) \tag{4.3}$$

(estimating crudely). As for M_∞ (an upper bound for $|f(z)|/\log|z|$ with $|z| \ge 2$), we use

$$\big|P(z)\big| \le |z|^D L(P), \quad \big|Q(z)\big| \le |z|^D L(Q) \quad \big(|z| \ge 2\big)$$

to get first

$$f(z) \le D \log|z| + \log L(R) \le \left(D + \frac{\log L(R)}{\log 2}\right) \log|z|;$$

and similarly

$$f(z) \ge -D \log|z| - (2D-1) \log L(R) \ge -\left(D + (2D-1)\frac{\log L(R)}{\log 2}\right) \log|z|$$

using (3.1) and (3.2). Thus, we can take

$$M_\infty = D + (2D-1)\frac{\log L(R)}{\log 2} \le 5DL(R). \tag{4.4}$$

For M_0 (an upper bound for $|f(z)|/\log|z|^{-1}$ with $0 < |z| \le 1/2$), we use

$$\big|P(z)\big| \le L(P), \quad \big|Q(z)\big| \le L(Q) \quad \big(|z| \le 1/2\big)$$

to get $f(z) \le \log L(R)$; and similarly, $f(z) \ge -(2D-1) \log L(R)$ as in (4.2), allowing

$$M_0 = (2D-1) \log L(R) \le 2DL(R). \tag{4.5}$$

The "Lipschitz" parameter L needs a bit more attention; we shall find L such that

$$\left|f(z)-f(w)\right| \le L|z-w| \quad \big(|z| \le 2,\ |w| \le 2\big).$$

We use the simple inequality

$$|\log x - \log y| \le \frac{|x-y|}{\min\{x,y\}}$$

for $x > 0$, $y > 0$, proved for example by the Mean Value Theorem.

Taking $x = |z-\lambda|$, $y = |w-\lambda|$, we get

$$\big|\log|z-\lambda| - \log|w-\lambda|\big| \le \frac{|z-w|}{\min\big\{|z-\lambda|, |w-\lambda|\big\}}.$$

Writing $P(z) = p_0 \prod(z-\lambda)$, we deduce

$$\big|\log|P(z)| - \log|P(w)|\big| \le \frac{D}{\delta}|z-w| \tag{4.6}$$

provided

$$|z-\lambda| \ge \delta, \quad |w-\lambda| \ge \delta \tag{4.7}$$

for all λ (and some $\delta > 0$).

Similarly, with $Q(z) = q_0 \prod(z-\mu)$, we deduce

$$\big|\log|Q(z)| - \log|Q(w)|\big| \le \frac{D}{\delta}|z-w| \tag{4.8}$$

provided

$$|z-\mu| \ge \delta, \quad |w-\mu| \ge \delta \tag{4.9}$$

for all μ.

Next we use the inequality

$$\big|\max\{x_1,y_1\} - \max\{x_2,y_2\}\big| \le \max\big\{|x_1-x_2|, |y_1-y_2|\big\}$$

for all real x_1, y_1, x_2, y_2. With

$$x_1 = \log\big|P(z)\big|, \quad y_1 = \log\big|Q(z)\big|, \quad x_2 = \log\big|P(w)\big|, \quad y_2 = \log\big|Q(w)\big|$$

we find using (4.6) and (4.8)

$$\big|f(z)-f(w)\big| \le \frac{D}{\delta}|z-w| \tag{4.10}$$

provided (4.7) holds for all λ and (4.9) holds for all μ. We have in effect removed from $|z| \leq 2$ the "P-holes" $|z - \lambda| < \delta$ and the "Q-holes" $|z - \mu| < \delta$, to leave an Emmentaler Cheese E, so (4.10) holds for z, w in E.

What about the holes? We will choose $\delta \leq 1$ so small that inside each hole it will be clear which of $|P(z)|, |Q(z)|$ is biggest.

Suppose $|z - \lambda_0| < \delta$ for some λ_0 (so z is in a P-hole). Then

$$\left|P(z)\right| \leq |p_0|\delta \widehat{\prod}|z-\lambda| \leq |p_0|\delta \prod 3\max\left\{1, |\lambda|\right\} = 3^D \delta M(P),$$

where the first product omits $|z - \lambda_0|$, and now we have the standard Mahler measure (non-logarithmic). This is at most $3^D\delta L(P) < 3^D \delta L(R)$ (see for example [25, Corollary 11, p. 248]). Thus, if $3^D\delta L(R) = 2^{-D} L(R)^{-(2D-1)}$, we see from (4.2) that $|P(z)| < |Q(z)|$ so $f(z) = \log|Q(z)|$. This shows also that a P-hole cannot intersect a Q-hole, for otherwise the same argument would give $|Q(z)| < |P(z)|$.

If simultaneously $|w - \lambda_0| < \delta$ (so w is in the same P-hole), we have $f(w) = \log|Q(w)|$. These z, w lie in no Q-hole, so that (4.9) holds for all μ. Thus, (4.8) follows, which is now the same as (4.10).

Similarly, (4.10) holds for z, w in the same Q-hole.

Now let z, w be arbitrary in the whole disc, that is, $|z| \leq 2$, $|w| \leq 2$. We join z, w by a straight line. If this line is entirely in E, then we get a "Lipschitz constant" D/δ by the original (4.10). But if parts of the line go through a hole, then inside that hole again we get D/δ (it does not matter if two P-holes intersect or two Q-holes intersect). Now we appeal to a "Straight Line Remark" which is an analogue (more "straight" forward) of the Great Circle Remark of [3, p. 13024]. It shows that (4.10) holds on the whole disc. Thus, we can take

$$L = \frac{D}{\delta} < D\big(3L(R)\big)^{2D}, \tag{4.11}$$

by far our largest parameter.

Now Theorem 1.6 of [3], with our parameters in (4.3), (4.4), (4.5), (4.11), shows that

$$\left|h_{\inf}(R,\alpha) - \mathbf{m}(R)\right| \leq \big(25L(R)\big)^{2D} h_d^{1/2} \tag{4.12}$$

for

$$h_d = h(\alpha) + \frac{\log 2d}{d}$$

provided $\alpha \neq 0$ and $h(\alpha) \leq 1$.

5. Contents

We next investigate

$$h_{\mathrm{fin}}(R,\alpha) = -\frac{1}{d}\log c(\Psi)$$

in Proposition 2.1, where $c(\Psi)$ is the content of

$$\Psi(Y) = \mathrm{res}_X\left(Q(X)Y - P(X), \Phi(X)\right)$$

for the minimal polynomial Φ of α.

From now on α will be a root of unity. This enables us to use $\mathcal{P}$-adic linear forms in logarithms according to the following comparatively simple version.

Lemma 5.1. *Let ξ be an algebraic number of degree $d(\xi)$ and let $n \geq 2$ be an integer with $\xi^n \neq 1$. Then for any prime ideal $\mathcal{P}$ in $\mathbf{Q}(\xi)$ of norm N we have*

$$\mathrm{ord}_{\mathcal{P}}(\xi^n - 1) \leq 10^7 d(\xi)^5 N \max\left\{1, h(\xi)\right\}\log n.$$

Proof. See the first displayed equation of Yu [28, p. 30] for a single α_1, and use

$$\log(e^5 d) \leq 5d, \quad h_1 \leq \frac{\log p}{\log 2}\max\left\{1, h(\xi)\right\}, \quad \log B \leq 3\log n$$

to get the result. ∎

For definiteness we take $\alpha = \zeta_n$ a root of unity of order $n \geq 2$, so that $\Phi = \Phi_n$ the corresponding cyclotomic polynomial of degree $d = \phi(n)$.

First we show that the prime factors of $c = c(\Psi)$ divide the quantity r in (3.1).

As Φ_n is monic, we have

$$\Psi(Y) = (-1)^{dD}\prod_{\Phi_n(\zeta)=0}\left(Q(\zeta)Y - P(\zeta)\right).$$

The content ideals in $\mathbf{Z}[\zeta_n]$ satisfy

$$(c) = \prod_{\zeta}\left(Q(\zeta), P(\zeta)\right).$$

Now r is in each ideal factor on the right, so c divides $r^{\phi(n)}$ which implies what we want.

Thus, to bound c it remains to estimate $\mathrm{ord}_p\, c$ for each prime p dividing r.

We write

$$Q = c(Q)Q_0Q_1, \quad P = c(P)P_0P_1$$

with the contents exhibited, where all zeroes of Q_1, P_1 are roots of unity, and no zero of P_0, Q_0 is a root of unity. Thus, Q_1 is the product of the cyclotomic Φ_m dividing Q (with multiplicity), and similarly with P_1. Of course, Q_0, Q_1, P_0, P_1 are all in $\mathbf{Z}[X]$.

Pick any p dividing r. As $c(Q), c(P)$ are coprime, either p does not divide $c(Q)$ or p does not divide $c(P)$.

Assume first that p does not divide $c(Q)$. As c divides the leading coefficient

$$c(Q)^{\phi(n)} \operatorname{res}(Q_0, \Phi_n) \operatorname{res}(Q_1, \Phi_n) \tag{5.1}$$

of Ψ, it suffices just to consider these two resultants.

Writing $Q_1 = \prod \Phi_m^{e_m}$, of course with $m \neq n$ as $Q(\zeta_n) \neq 0$, we see that $\sum_m e_m \phi(m) \leq D$ and the second resultant is $\prod_m \operatorname{res}(\Phi_m, \Phi_n)^{e_m}$.

For m with $n > m > 1$ we see from Theorems 3 and 4 of Apostol [1, p. 460] that $\operatorname{res}(\Phi_m, \Phi_n) = \pm 1$ unless m divides n and n/m is a power of a prime l, in which case it is $\pm l^{\phi(m)}$. And if $n > m = 1$, then Theorem 1 of [1, p. 459] gives the same conclusion. So whenever $n > m$, we get

$$\left| \operatorname{res}(\Phi_m, \Phi_n)^{e_m} \right| \leq l^{e_m \phi(m)} \leq n^{e_m \phi(m)}.$$

Thus, the product over these m is at most n^D.

For m with $m > n$ we get $l^{\phi(n)}$ instead of $l^{\phi(m)}$, and the corresponding product is at most

$$\prod_m m^{e_m \phi(n)} \leq \prod_m m^{e_m n} < \prod_m m^{e_m m}$$

which is $\exp(\sum_m e_m m \log m)$. Now it is not hard to verify that $m \log m / \phi(m)^2$ attains its maximum $c_0 = (3/2) \log 6$ (at $m = 6$). Thus,

$$\sum_m e_m m \log m \leq c_0 \sum_m e_m \phi(m)^2 \leq c_0 D^2.$$

Therefore, in (5.1), we get

$$\left| \operatorname{res}(Q_1, \Phi_n) \right| \leq e^{c_0 D^2} n^D. \tag{5.2}$$

To deal with $\operatorname{res}(Q_0, \Phi_n)$ is less elementary. Write $Q_0(X) = q_0 \prod_{i=1}^k (X - \xi_i)$. So this resultant divides

$$\operatorname{res}(Q_0, X^n - 1) = q_0^n \prod_{i=1}^k (\xi_i^n - 1). \tag{5.3}$$

Assume next that p does not divide q_0. For each i pick $\mathcal{P}_i$ dividing p in $\mathbf{Q}(\xi_i)$. Then by Lemma 5.1,

$$\operatorname{ord}_{\mathcal{P}_i}(\xi_i^n - 1) \leq 10^7 d_i^5 p^D \max\left\{1, h(\xi_i)\right\} \log n,$$

with $d_i = [\mathbf{Q}(\xi_i) : \mathbf{Q}]$. If $Q^{(i)}$ is the irreducible factor of Q_0 vanishing at ξ_i, we have

$$d_i h(\xi_i) = \log M(Q^{(i)}) \le \log M(Q_0) \le \log M(Q) \le \log L(Q) \le \log L(R).$$

This leads to

$$\mathrm{ord}_{\mathcal{P}_i}(\xi_i^n - 1) \le 10^7 D^4 p^D \mathcal{L} \log n$$

where $\mathcal{L} = \max\{D, \log L(R)\}$.

Now pick $\mathcal{P}$ dividing $\mathcal{P}_i$ in $\mathbf{Q}(\xi_1, \dots, \xi_k)$. Then

$$\mathrm{ord}_{\mathcal{P}}(\xi_i^n - 1) = E_i \,\mathrm{ord}_{\mathcal{P}_i}(\xi_i^n - 1) \le 10^7 E_i D^4 p^D \mathcal{L} \log n$$

for the corresponding ramification index E_i. Thus,

$$\mathrm{ord}_p \prod_{i=1}^{k} (\xi_i^n - 1) = E^{-1} \mathrm{ord}_{\mathcal{P}} \prod_{i=1}^{k} (\xi_i^n - 1) \le 10^7 k D^4 p^D \mathcal{L} \log n$$

with ramification index $E \ge E_i$. As $k \le D$, we conclude

$$\mathrm{ord}_p \,\mathrm{res}(Q_0, \Phi_n) \le 10^7 D^5 p^D \mathcal{L} \log n \tag{5.4}$$

in this case that p does not divide q_0 (and still assuming p does not divide $c(Q)$ as above).

If p does divide q_0, then we have to work a little bit harder, to avoid the q_0^n in (5.3). As Q_0 is primitive, we have

$$1 = |q_0|_{\mathcal{P}} \prod_{i=1}^{k} \max\left\{1, |\xi_i|_{\mathcal{P}}\right\}$$

for any normalization of the valuation corresponding to $\mathcal{P}$. Choosing $|p|_{\mathcal{P}} = p^{-1}$ (so that $|\;|_{\mathcal{P}}$ extends $|\;|_p$), we write $\max\{1, |\xi_i|_{\mathcal{P}}\} = p^{\varepsilon_i}$ for rational $\varepsilon_i \ge 0$. Then $|q_0|_p = |q_0|_{\mathcal{P}} = \prod_{i=1}^{k} p^{-\varepsilon_i}$, so by (5.3),

$$\mathrm{res}(Q_0, X^n - 1) = q_0' \prod_{i=1}^{k} \left((p^{\varepsilon_i}\xi_i)^n - p^{\varepsilon_i n}\right)$$

for $q_0' = q_0^n |q_0^n|_p$ in $\mathbf{Z}$ not divisible by p. If some $\varepsilon_i = 0$, we can again use Lemma 5.1 as before, while if some $\varepsilon_i > 0$, then

$$|p^{\varepsilon_i}\xi_i|_{\mathcal{P}} = p^{-\varepsilon_i}|\xi_i|_{\mathcal{P}} = p^{-\varepsilon_i} p^{\varepsilon_i} = 1$$

so the $(p^{\varepsilon_i}\xi_i)^n - p^{\varepsilon_i n}$ is not divisible by $\mathcal{P}$. We obtain again (5.4).

Now (5.1) gives

$$\operatorname{ord}_p c \le \operatorname{ord}_p \operatorname{res}(Q_0, \Phi_n) + \operatorname{ord}_p \operatorname{res}(Q_1, \Phi_n). \tag{5.5}$$

All this was on the assumption that p does not divide $c(Q)$. If that fails, then as remarked p does not divide $c(P)$. Now, as long as $P(\zeta_n) \neq 0$, we can repeat the whole argument using the constant coefficient of Ψ, which is

$$\pm \operatorname{res}(P, \Phi_n) = \pm c(P)^{\phi(n)} \operatorname{res}(P_0, \Phi_n) \operatorname{res}(P_1, \Phi_n).$$

We thus obtain

$$\operatorname{ord}_p c \le \operatorname{ord}_p \operatorname{res}(P_0, \Phi_n) + \operatorname{ord}_p \operatorname{res}(P_1, \Phi_n). \tag{5.6}$$

We combine the two cases simply by addition. We get $\log c = \sum_{p|r} (\operatorname{ord}_p c) \log p$ bounded by a sum of four quantities corresponding to the right-hand sides of (5.5) and (5.6). As

$$\sum_{p|r} p^D \log p \le |r|^D \log |r| \sum_{p|r} 1 \le |r|^D \frac{\left(\log |r|\right)^2}{\log 2},$$

we get from (5.4) the upper bound

$$10^7 D^5 \mathcal{L}(\log n) |r|^D \frac{\left(\log |r|\right)^2}{\log 2} \tag{5.7}$$

corresponding to the first in (5.5). We can check that

$$D^8 (\log L)^2 \le c_1 L^{D^2}, \quad D^7 (\log L)^3 \le c_2 L^{D^2}$$

for any $D \ge 1, L \ge 2$, where

$$c_1 = \frac{256}{e^4 (\log 2)^2} > c_2 = \frac{343\sqrt{14}}{16 e^{7/2} \sqrt{\log 2}}.$$

As $|r| \le L(R)^{2D}$ from Lemma 3.1, we see that (5.7) is at most

$$6.10^8 L(R)^{3D^2} \log n.$$

The same holds for the first of (5.6).

For the second in (5.5) we get

$$\sum_p \operatorname{ord}_p \operatorname{res}(Q_1, \Phi_n) \log p = \log \left| \operatorname{res}(Q_1, \Phi_n) \right| \le c_0 D^2 + D \log n$$

as in (5.2). And the same holds for the second in (5.6). We conclude that

$$\log c \leq 8.10^8 L(R)^{4D^2} \log n. \tag{5.8}$$

Thus, finally,

$$\left|h_{\text{fin}}(R, \zeta_n)\right| \leq 10^9 L(R)^{4D^2} \frac{\log n}{\phi(n)}. \tag{5.9}$$

But if it happens that $P(\zeta_n) = 0$ above, then $\phi(n) \leq D$ and our Theorem with (1.5) reduces to

$$\left|\mathbf{m}(R)\right| \leq 10^{10} L(R)^{4D^2} \left(\frac{\log 2n}{\phi(n)}\right)^{1/2}.$$

As observed, $\mathbf{m}(R) \geq 0$ (see (9.3) below); and clearly, $\mathbf{m}(R) \leq \log L(R)$, so it suffices to check that

$$D^{1/2} \log L \leq 10^{10} L^{4D^2} (\log 2)^{1/2}$$

for all $D \geq 1$ and $L \geq 2$, a piece of cake.

6. Proof of Theorem

This is immediate, with constants (1.5), from Proposition 2.1 and (2.2), (4.12), (5.9); recall that $\alpha = \zeta_n$.

7. Eliminating linear forms in logarithms

Above we estimated $c(\Psi)$ using only the first and last coefficients of Ψ. The method now to be described uses in some way all the coefficients, at least when $D = 1$.

Then

$$P = p_0 X + p_1, \quad Q = q_0 X + q_1$$

and $c = c(\Psi)$ divides

$$S(Y) = (-1)^n \operatorname{res}_X(QY - P, X^n - 1) = \prod_{\zeta^n = 1} \big((q_0\zeta + q_1)Y - (p_0\zeta + p_1)\big)$$

in $\mathbf{Z}[Y]$. This is

$$\prod_{\zeta} \big((-p_1 + q_1 Y) - \zeta(p_0 - q_0 Y)\big) = (-p_1 + q_1 Y)^n - (p_0 - q_0 Y)^n.$$

Taking a hint from "abc", we note that c then divides the derivative

$$S'(Y) = nq_1(-p_1 + q_1 Y)^{n-1} + nq_0(p_0 - q_0 Y)^{n-1}.$$

It therefore divides

$$nq_0 S(Y) + (p_0 - q_0 Y)S'(Y) = n(-p_1 + q_1 Y)^{n-1}(p_0 q_1 - p_1 q_0)$$

as well as

$$nq_1 S(Y) - (-p_1 + q_1 Y)S'(Y) = -n(p_0 - q_0 Y)^{n-1}(p_0 q_1 - p_1 q_0).$$

Now $-p_1 + q_1 Y$, $p_0 - q_0 Y$ are coprime in $\mathbf{Z}[Y]$ because P, Q are. Thus, c divides $n(p_0 q_1 - p_1 q_0)$, giving a great improvement on (5.8); for example, $\log c \leq \log n + 2 \log L(R)$.

This leads to the slightly better $c = 500$, $C = 2$ in our theorem for $D = 1$.

But we could not make this method work even for $D = 2$.

8. Small height

As indicated in Section 1, there is hope that our Theorem can be extended from roots of unity ζ to α with suitably small height. Here we do this for $R = 1 - X$ in accordance with (1.3).

Suppose as above $\Phi(\alpha) = 0$ with Φ of leading coefficient $a_0 \geq 1$. Then $\pm a_0$ is also the leading coefficient of the minimal polynomial of $1 - \alpha$. Thus,

$$h(1-\alpha) = \frac{\log a_0}{d} + S_1, \quad h(\alpha) = \frac{\log a_0}{d} + S_0$$

where

$$S_1 = \frac{1}{d} \sum_{\sigma} \log \max \{1, |1 - \alpha^\sigma|\}, \quad S_0 = \frac{1}{d} \sum_{\sigma} \log \max \{1, |\alpha^\sigma|\}.$$

By [3, Corollary 1.4, p. 13010], S_1 differs from the constant $\mathbf{m}_0$ in (1.3) by at most $18 h_d^{1/2}$, provided $h(\alpha) \leq 1$; and then clearly $|S_0| \leq h(\alpha) \leq h_d^{1/2}$. This implies (1.7); and that is valid even if $h(\alpha) > 1$, for then the left-hand side is at most

$$\log 2 + \mathbf{m}_0 \leq (\log 2 + \mathbf{m}_0) h_d^{1/2}.$$

9. Dilogarithms and examples

It is classical that the $\mathbf{m}_0$ just above is the two-variable Mahler measure

$$m(X+Y-1) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|e^{i\theta}+e^{i\varphi}-1|\mathrm{d}\theta\mathrm{d}\varphi,$$

and that this can be expressed in terms of the even more classical dilogarithm

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2} \quad \big(|z|\le 1\big)$$

(we will not need any fancy analytic continuations here) as

$$\frac{1}{\pi}\Im\,\mathrm{Li}_2(e^{\pi i/3}) = \frac{1}{2\pi i}\big(\mathrm{Li}_2(e^{\pi i/3}) - \mathrm{Li}_2(e^{-\pi i/3})\big) \tag{9.1}$$

involving the imaginary part.

More generally,

$$m\big(Q(X)Y-P(X)\big) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log\big|Q(e^{i\theta})e^{i\varphi}-P(e^{i\theta})\big|\mathrm{d}\theta\mathrm{d}\varphi \tag{9.2}$$

which is

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(\log\big|Q(e^{i\theta})\big| + \log\big|e^{i\varphi}-R(e^{i\theta})\big|\big)\mathrm{d}\theta\mathrm{d}\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\big(\log\big|Q(e^{i\theta})\big| + \log\max\big\{1,\big|R(e^{i\theta})\big|\big\}\big)\mathrm{d}\theta$$

and so what we called $\mathbf{m}(R)$ in (1.4). Thus,

$$\mathbf{m}(R) = m(Q) + \frac{1}{2\pi}\int_0^{2\pi}\log\max\big\{1,\big|R(e^{i\theta})\big|\big\}\mathrm{d}\theta \ge m(Q)\ge 0 \tag{9.3}$$

as mentioned in Section 1 (this follows the sketch given by Boyd [10, p. 117]).

The expressions in (9.2) can also be evaluated with dilogarithms. Boyd and Rodriguez-Villegas observed this for cyclotomic P, Q in [12, Proposition 1, p. 470] – see also their [13]. The experts surely know this for general P, Q in $\mathbf{Z}[X]$. Here for completeness we give a short proof for arbitrary coprime P, Q in $\mathbf{R}[X]$. See also Maillot [22, Proposition 7.3.1, p. 107] for P of degree 1 and Q of degree 0 in $\mathbf{C}[X]$.

We use a function

$$\mathrm{Lc}_t(\Theta) = \int_0^{\Theta}\log(t^2+1-2t\cos\theta)\mathrm{d}\theta$$

for real t, Θ. Note that

$$t^2 + 1 - 2t\cos\theta \geq t^2 + 1 - 2|t| = \big(|t| - 1\big)^2 \geq 0.$$

By relating $\mathrm{Li}_2(z)$ to the integral of $\log(1-z)/z$, it is not difficult to see that

(9.4) $$\mathrm{Lc}_t(\Theta) = -2\Im\,\mathrm{Li}_2(te^{i\Theta}) \quad \big(|t| \leq 1\big),$$

(9.5) $$\mathrm{Lc}_t(\Theta) = 2\Theta\log|t| - 2\Im\,\mathrm{Li}_2(t^{-1}e^{i\Theta}) \quad \big(|t| \geq 1\big).$$

Now in (1.4) it can happen that $|Q(e^{i\theta})| = |P(e^{i\theta})|$ identically. Then we get just $\log|P(e^{i\theta})|$ in (1.4). We treat each linear factor of P separately. If P has a real zero t, then

$$\log|e^{i\theta} - t| = \frac{1}{2}\log(t^2 + 1 - 2t\cos\theta)$$

and so we see just $(1/4\pi)\,\mathrm{Lc}_t(2\pi)$. In general, a zero at $te^{i\mu}$ leads to

(9.6) $$\log|e^{i\theta} - te^{i\mu}| = \log|e^{i(\theta-\mu)} - t|$$

and so we see

$$\frac{1}{4\pi}\big(\mathrm{Lc}_t(2\pi - \mu) - \mathrm{Lc}_t(-\mu)\big).$$

If $|Q(e^{i\theta})| \neq |P(e^{i\theta})|$ identically, then they are equal for at most finitely many θ in the interval $[0, 2\pi)$; if any at all, order them as $(0 \leq)\ \theta_1 < \cdots < \theta_k\ (< 2\pi)$. On each subinterval $[\theta', \theta'')$, the maximum is $|Q|$ or $|P|$, and we may argue as above. We get for $te^{i\mu}$ in (9.6)

$$\frac{1}{2\pi}\int_{\theta'}^{\theta''}\log|e^{i(\theta-\mu)} - t|\,d\theta = \frac{1}{4\pi}\big(\mathrm{Lc}_t(\theta'' - \mu) - \mathrm{Lc}_t(\theta' - \mu)\big).$$

By the way, dilogarithms can also be more reliable numerically, as Maple seems to have strange difficulties in calculating $\mathbf{m}(R)$ as an integral.

We present now some simple examples.

(a) $P(X) = 1 - X$, $Q(X) = 1$. These are by now old friends (and both cyclotomic), and

$$\mathbf{m}_0 = m(QY - P) = \frac{1}{2\pi}\int_{\pi/3}^{5\pi/3}\log|1 - e^{i\theta}|\,d\theta = \frac{1}{4\pi}\big(\mathrm{Lc}_1(5\pi/3) - \mathrm{Lc}_1(\pi/3)\big)$$

which works out as (9.1). This is therefore the limit of $h(1-\alpha)$ as $h(\alpha) \to 0$. Taking $\alpha = e^{2\pi i/n}$, we have $1 - \alpha = -2ie^{\pi i/n}\sin(\pi/n)$ and so this is $\lim_{n\to\infty} h(2\sin(\pi/n))$. Or $\alpha = e^{4\pi i/n}$ gives the slightly stronger

$$\lim_{n\to\infty} h\left(2\sin\frac{2\pi}{n}\right) = \mathbf{m}_0.$$

This adds one more property to Boyd's list in [11, p. 346]. Also

$$\frac{4\pi\sqrt{3}}{9}\mathbf{m}_0 = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \cdots$$

and the very recent work of Calegari, Dimitrov, Tang [14] shows among other things that this is irrational. We still cannot prove that G in (1.8) is irrational.

(b) $P(X) = 1 - X$, $Q(X) = 2$. Now (not quite cyclotomic) we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log\max\left\{|1 - e^{i\theta}|, 2\right\} d\theta = \log 2$$

the limit of $h((1-\zeta)/2)$ as the order of the root of unity ζ tends to ∞. This statement corresponds to a question of Su-ion Ih (see below). The limit is $\lim_{n\to\infty} h(\sin(\pi/n))$ or even better

$$\lim_{n\to\infty} h\left(\sin\frac{2\pi}{n}\right) = \log 2.$$

(c) $P(X) = X^2 + 1$, $Q(X) = X$. As in (a) but slightly more complicated, we find for example

$$\lim_{n\to\infty} h\left(2\cos\frac{2\pi}{n}\right) = \frac{1}{2}\mathbf{m}_0.$$

(d) $P(X) = X^2 + 1$, $Q(X) = 2X$. As in (b), we get

$$\lim_{n\to\infty} h\left(\cos\frac{2\pi}{n}\right) = \log 2.$$

(e) $P(X) = 1 - X$, $Q(X) = 1 + X$. Now $m(QY - P)$ is known to be $2G/\pi$ for the Catalan constant G as in (1.8); see for example Boyd [11, p. 347] after changing the signs of his x and y. So we get as mentioned in Section 1

$$\lim_{n\to\infty} h\left(\tan\frac{2\pi}{n}\right) = \frac{2}{\pi}G,$$

thus completing the list of our "trigonometric" examples. See also [11, p. 342] for the relevance of this number for Gelfond-type results on heights of products of polynomials (compare also the remarkable [4] of Beauzamy, Bombieri, Enflo, Montgomery).

(f) $P(X) = (X^2+1)^2$, $Q(X) = 4(X^3 - X)$. The more difficult (and still conjectural) aspects of dilogarithms concern elliptic curves. Here we add as a joke the Lattès map

$$R(X) = \frac{(X^2+1)^2}{4(X^3 - X)}$$

corresponding to $y^2 = x^3 - x$ (where by the way the canonical height h_R is just Néron–Tate). We find the limit

$$\mathbf{m}(R) = \frac{1}{\pi}\big(\Im\,\mathrm{Li}_2(\gamma) - 2\Im\,\mathrm{Li}_2(-\gamma) + 2(\log 2)\big(\pi - 2\sin^{-1}(\sqrt{2}-1)\big)\big)$$

where

$$\gamma = 4\sqrt{2} - 5 + i(2\sqrt{2}-2)\sqrt{2\sqrt{2}-2}.$$

This limit is also

$$\lim_{n\to\infty} h\left(\frac{\cos^2(2\pi/n)}{2\sin(2\pi/n)}\right) = 1.68696364419\ldots$$

sort of trigonometric.

(g) $P(X) = X^3 - X - 1$, $Q(X) = 1$. This is far from cyclotomic. With $\xi > 1$, the real zero of P (the smallest PV-number, Smyth's constant, etc.), and ξ', ξ'' the other complex conjugates (inside the unit disc), and $\omega = e^{2\pi i/3}$, we find that $\lim_{n\to\infty} h(1 + \zeta_n - \zeta_n^3)$ (not very trigonometric) is

$$\frac{2}{3}\log\xi - \frac{1}{\pi}\big(\Im\,\mathrm{Li}_2(\omega\xi^{-1}) + \Im\,\mathrm{Li}_2(\omega\xi') + \Im\,\mathrm{Li}_2(\omega\xi'')\big) = 0.42728783735\ldots.$$

(h) In other contexts, things like

$$\frac{X - w}{1 - \bar{w}X}$$

are of interest. That is the simplest version among Blaschke-type functions, mapping the unit circle into itself. They have played a role in clarifying aspects of the former Manin-Mumford conjecture – see for example Corvaja, Zannier and the author [15, p. 228] or more recently Ostafe and Shparlinski [24, p. 775]. In our context, they provide families of examples for our Theorem.

We fix w in $\mathbf{Q}$. Then with

$$R(X) = \frac{X - w}{1 - wX}$$

we have

$$\lim_{n\to\infty} h\left(\frac{\zeta_n - w}{1 - w\zeta_n}\right) = h(w). \tag{9.7}$$

To see this, write $w = a/b$ for coprime integers a and $b \geq 1$, so that

$$P(X) = bX - a, \quad Q(X) = b - aX$$

and of course $|Q(e^{i\theta})| = |P(e^{i\theta})|$ identically. Thus, (1.4) gives

$$\mathbf{m}(R) = \frac{1}{2\pi}\int_0^{2\pi} \log|be^{i\theta} - a|\mathrm{d}\theta = \log b + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - w|\mathrm{d}\theta$$

and the second integral is well known to be $2\pi \log\max\{1, |w|\}$ (see also (9.4), (9.5) above), leading to (9.7).

It seems amusing also to examine separately the numerator $\zeta_n - w$ and denominator $1 - w\zeta_n$ in (9.7). As

$$h(\zeta_n - w) = h(1 - w/\zeta_n) = h(1 - w\overline{\zeta_n}) = h(1 - w\zeta_n),$$

there is a common height limit. If $|w| \geq 2$, this works out as $h(w)$ as in (9.7). But if $|w| \leq 2$, then we get

$$h(w) + f\big(|w|\big)$$

where

$$f(x)=\left(-\frac{1}{2}+\frac{1}{\pi}\sin^{-1}\left(\frac{x}{2}\right)\right)\log\xi + \frac{1}{\pi}\Im\,\mathrm{Li}_2\left(\frac{x}{\xi^2}\left(\frac{x}{2}+i\sqrt{1-\frac{x^2}{4}}\right)\right) \quad (0\leq x\leq 2)$$

for $\xi = \max\{1, x\}$, the inverse sine between 0 and $\pi/2$, and the non-negative square root. For example, $h(1/2 - \zeta)$ (not quite as in (b) above) tends to

$$\log 2 + \frac{1}{\pi}\Im\,\mathrm{Li}_2\left(\frac{1 + i\sqrt{15}}{8}\right) = 0.85286432679\ldots.$$

Thus, whenever $f(|w|) > 0$ (as seems likely for all $|w| \neq 2$ – and indeed $f(1)$ is the number in (1.3) above), then there is some sort of cancellation between $\zeta_n - w$ and $1 - w\zeta_n$ (compare with $h(\alpha_1/\alpha_2) = \max\{h(\alpha_1), h(\alpha_2)\}$ for coprime integers α_1, α_2).

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