Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 31 (2012), 75–91
DOI: 10.4171/ZAA/1449

On an Extension of the Space of Bounded Deformations

Martin Kružík and Johannes Zimmer

Abstract. As for functions of bounded variations (BV), the space of functions of bounded deformations (BD) can be equipped with different topologies. We propose an extension of the traditional space of BD functions equipped with the weak-$\star$ topology. It is shown that in this fine extension, both compactness and weak-$\star$ continuity of the trace hold under weak assumptions. This is relevant for applications, where the specimen is typically exposed to forces on the boundary. We give an application to a functional with a linearly growing energy density.

Keywords. Bounded sequences of symmetrised gradients, bounded deformation, concentrations and oscillations, quasiconvexity

Mathematics Subject Classification (2010). Primary 46E35, secondary 26B30, 46E40, 54C35

1. Introduction

Various variational problems are characterised by linear growth of the functional; examples can be found in plasticity and minimal surfaces. Those problems (e.g., the minimal surface functional $\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$) have a natural setting in a fine extension of $W^{1,1}(\Omega)$ or, if the functional depends on the linearised gradient $\frac{1}{2}\left(\nabla u + \nabla u^\top\right)$, the equivalent space for symmetrised gradients. For example, perfect plasticity naturally leads to functionals involving the expression $\frac{1}{2}\left(\nabla u + \nabla u^\top\right)$, so there is a natural interest in a suitable functional analytic framework. In this article, we develop such a functional setting for linearised gradients tailored for the calculus of variations. The setting introduced here offers both convenient compactness properties and continuity and

M. Kružík: Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic; & Faculty of Civil Engineering, Czech Technical University, Thákurova 7, CZ-166 29 Praha 6, Czech Republic; kruzik@utia.cas.cz

J. Zimmer: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom; zimmer@maths.bath.ac.uk

weak-$\star$ continuity of the trace. This combination is required in applications in mechanics where boundary forces are present.

One extension of $W^{1,1}(\Omega)$ is the space of bounded variations, $BV(\Omega)$ (see, e.g., the accounts in [1, 5]). Apart from the norm topology, two topologies are common for $BV(\Omega)$: the weak-$\star$ topology [1, Definition 3.11] and the strict convergence [1, Definition 3.14]. The former is useful for its compactness properties (see, e.g., [1, Theorem 3.23]), while the latter ensures continuity of the trace operator [1, Theorem 3.88]. In applications, one commonly would like to have both to pass to the limit of a minimising sequence, to obtain a limit and preserve the boundary data. As far as the existence of a limit is concerned, the weak-$\star$ topology is a natural choice. Unfortunately, the trace operator in BV is not continuous when BV is endowed with the weak-$\star$ topology, as can be seen in the simplest toy model. Namely, consider $\Omega := (0, 1)$, and $u_k : \Omega \to \mathbb{R}$ given by $u_k(x) = kx$ for $x \in \left[0, \frac{1}{k}\right]$, and $u_k(x) = 1$ for $x \in \left[\frac{1}{k}, 1\right]$. It is then easy to verify that $u_k \stackrel{\star}{\rightharpoonup} 1$ in $BV(\Omega)$, so the boundary data at 0 is lost in the limit procedure. The ultimate reason is that the test functions introduced in the definition of the weak-$\star$ convergence vanish on the boundary. Souček [12] has introduced a fine extension of $BV(\Omega)$ where the test functions do not have to vanish on the boundary, and both compactness and continuity of the trace can be obtained. See [8] for a brief summary.

The purpose of this article is to introduce an analogous fine extension of the space of bounded deformations (BD). The space $BD(\Omega)$ is composed of vector-valued functions u in $L^1(\Omega)$ for which the symmetrised gradients $\frac{1}{2}\left(Du + Du^\top\right)$ are (component-wise) bounded measures. The one-dimensional example given above for BV shows that BD suffers from the same inconvenience: while a weak-$\star$ convergence leads naturally to (weak-$\star$) compactness, the trace operator is not continuous in that topology. Stronger topologies, such as the norm topology in BD, make the trace operator continuous, but compactness becomes hard or impossible to prove. This is a serious obstacle for applications: indeed, BD is a natural function space for problems in plasticity, for which the direct methods from the Calculus of Variations are powerful tools. However, it is natural for applications in plasticity to prescribe boundary conditions. One would then like to work in a setting where the boundary data of a minimising sequence are preserved in the limit, while such a sequence should converge to a limit function in the same topology. We develop a fine extension where these two aims can be achieved simultaneously, unlike for existing topologies. Specifically, the main result of this article is a fine extension of DB where weak-$\star$ compactness holds (Theorem 2.6), *and* the trace operator is both continuous and weak-$\star$ continuous (Theorem 2.5). While some of the development of the theory presented here naturally resembles that of Souček's extension of the space of bounded variation, various arguments differ from those in [12] since we cannot rely on Sobolev embeddings available in the situation studied there. The result given here refines the main result of Temam's and Strang's paper [14] in a specific way: they show that the boundary values of a function in $BD(\Omega)$ are integrable, and the trace operator is continuous in the norm topology [14, Theorem 1.1]. The trace we define in this article is the trace introduced by Temam and Strang augmented by a term to handle possible concentration on the boundary (see Equation (6)). An application of the framework developed in this article is given in Section 4.

Basic notation. In this article, $\Omega \subset \mathbb{R}^n$ is always a bounded domain with smooth boundary. $C_0(\Omega)$ stands for the space of continuous functions $f : \Omega \to \mathbb{R}$ such that $\left\{x \in \Omega \;\middle|\; |f(x)| \geq \epsilon\right\}$ is compact for every $\epsilon > 0$. If Γ_D is a part of the boundary $\partial\Omega$ with positive $n-1$-dimensional Hausdorff measure, $W^{1,1}_{u_D}(\Omega; \mathbb{R}^m)$ stands for the set of functions $u \in W^{1,1}(\Omega; \mathbb{R}^m)$ with $u = u_D$ on Γ_D. We denote the (signed) Radon measures with finite mass on a locally compact Hausdorff space X by $M(X)$; the cone of non-negative Radon measures with finite mass is denoted $M^+(X)$, and $\mathrm{Prob}(X)$ is the set of probability measures. The space of Radon measures with compact support on $\bar{\Omega}$ is denoted $\left(M\left(\bar{\Omega}\right), \|\cdot\|\right)$. The Jordan decomposition for signed measures $\mu = \mu^+ - \mu^-$ gives rise to the total variation $|\mu|$, which is the measure $|\mu| := \mu^+ + \mu^-$. Endowed with the total variation $\|\mu\| := |\mu|(X)$ as a norm, $M(X)$ is a Banach space. By the Riesz representation theorem, $(M(X), \|\cdot\|)$ is isometrically isomorphic to the dual of $(C_0(X), \|\cdot\|_\infty)$ via the pairing

$$\langle f, \mu \rangle := \int_\Omega f(x)\mu(\mathrm{d}x).$$

The weak-$\star$ topology on $M(X)$ is defined by this duality. Weak convergence respectively weak-$\star$ convergence is expressed as $u_k \rightharpoonup u$ respectively $u_k \overset{\star}{\rightharpoonup} u$, while $u_n \to u$ denotes strong convergence. We write w-lim for the weak limit, and analogously w-$\star$ lim for the weak-$\star$ limit.

We follow the convention of writing C for a generic constant, whose value may change from line to line.

2. Fine extensions of $W^{1,1}(\Omega; \mathbb{R}^m)$ with symmetrised gradients

In this section we consider extensions of $W^{1,1}(\Omega; \mathbb{R}^n)$ with symmetrised gradients. If $u \in W^{1,1}(\Omega; \mathbb{R}^n)$, we set $\varepsilon(u) := \frac{1}{2}\left(\nabla u + \nabla^\top u\right)$. Where the symmetrised gradients exist only in the sense of measures, we write $Eu = \frac{1}{2}\left(Du + D^\top u\right)$.

2.1. Functions of bounded deformation. We now turn our attention to the space of functions with bounded deformation $BD(\Omega; \mathbb{R}^n)$ (see, e.g., [1, 5]).

The space $BD\left(\Omega;\mathbb{R}^n\right)$ stands for the linear subspace of $L^1\left(\Omega;\mathbb{R}^n\right)$ containing maps with bounded deformation in Ω, i.e., $u \in BD\left(\Omega;\mathbb{R}^n\right)$ if its symmetrised gradient, $Eu = \frac{1}{2}\left(Du + D^\top u\right)$ is a measure in $M\left(\Omega;\mathbb{R}^{n\times n}\right)$. We define its norm as

$$\|u\|_{BV(\Omega;\mathbb{R}^m)} = \|u\|_{L^1(\Omega;\mathbb{R}^m)} + \|Eu\|_{M(\Omega;\mathbb{R}^{n\times n})} < +\infty \ . \tag{1}$$

Temam and Strang [14, Theorem 1.1] showed that there is a unique continuous (in the norm topology) linear operator $T_{BD}\colon BD\left(\Omega;\mathbb{R}^n\right) \to L^1\left(\partial\Omega\right)$ such that $T_{BD}u = u|_{\partial\Omega}$ if $u \in BD\left(\Omega;\mathbb{R}^n\right) \cap C\left(\bar{\Omega};\mathbb{R}^n\right)$. Moreover, for every $i,j \in \{1,2,\ldots,n\}$ and for every $\varphi \in C^1\left(\bar{\Omega}\right)$

$$\frac{1}{2}\int_{\partial\Omega} \varphi(x)[\nu \otimes T_{BD}u]\mathrm{d}S = \int_\Omega \varphi(x)Eu(\mathrm{d}x) + \frac{1}{2}\int_\Omega u(x) \otimes \nabla\varphi(x)\,\mathrm{d}x \ , \tag{2}$$

where $[\nu \otimes T_{BD}u] \in L^1\left(\partial\Omega;\mathbb{R}^{n\times n}\right)$ is defined as

$$\int_{\partial\Omega} [\nu \otimes T_{BD}u]_{ij}(\mathrm{d}S) := \int_{\partial\Omega} \nu_i(x)T_{BD}u_j(\mathrm{d}S) + \int_{\partial\Omega} \nu_j(x)T_{BD}u_i(\mathrm{d}S) \ ,$$

with $i,j \in \{1,2,\ldots,n\}$, and $\nu\colon \partial\Omega \to \mathbb{R}^n$ is again the outer unit normal to $\partial\Omega$. The main result in Temam's and Strang's article [14] is this trace theorem for $BD\left(\Omega\right)$; to distinguish their notion of a trace from the notion develop here, we call the trace defined in [14] occasionally the BD-trace.

2.2. The space $BD\mu(\Omega;\mathbb{R}^n)$. One central aim of this article is to establish a finer extension of $BD\left(\Omega;\mathbb{R}^n\right)$, in the spirit of Souček's work [12]. We start with the definition, where we introduce μ in the notation to stress the analogy to the Souček space $W^{1,\mu}(\Omega;\mathbb{R}^n)$ introduced in [12],

$$W^{1,\mu}(\Omega;\mathbb{R}^m) = \big\{(u,\bar{D}u) \in L^1(\Omega;\mathbb{R}^n) \times M(\bar{\Omega});\ \text{there exists } \{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^n) \\ \text{s.t. } u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^n),\ \nabla u_k \to \bar{D}u \text{ weakly}\star \text{ in } M(\bar{\Omega};\mathbb{R}^{n\times n})\big\}$$

(see also [8] for properties of $W^{1,\mu}\left(\Omega;\mathbb{R}^m\right)$).

Definition 2.1. We say that $\left(u,\bar{E}u\right) \in BD\mu\left(\Omega;\mathbb{R}^n\right)$ if there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}\left(\Omega;\mathbb{R}^m\right)$ such that $u_k \to u$ in $L^1\left(\Omega;\mathbb{R}^n\right)$ and $\varepsilon\left(u_k\right) \stackrel{\star}{\rightharpoonup} \bar{E}u$ in $M\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)$. We norm $BD\mu(\Omega;\mathbb{R}^n)$ by

$$\left\|(u,\bar{E}u)\right\|_{BD\mu(\Omega;\mathbb{R}^n)} := \|u\|_{L^1(\Omega;\mathbb{R}^n)} + \left\|\bar{E}u\right\|_{M\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)} \ . \tag{3}$$

Remark 2.2. 1. In particular, in the setting of Definition 2.1, $\{\varepsilon(u_k)\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega;\mathbb{R}^{n\times n})$.
2. Obviously $W^{1,\mu}(\Omega;\mathbb{R}^n) \subset BD\mu(\Omega;\mathbb{R}^n)$.

Theorem 2.3. *Let* $\left(u, \bar{E}u\right) \in BD\mu\left(\Omega; \mathbb{R}^n\right)$. *Then there is a uniquely defined measure* $\beta \in M\left(\partial\Omega; \mathbb{R}^n\right)$ *such that the following integration by parts formula holds for every* $\varphi \in C^1\left(\bar{\Omega}\right)$*:*

$$\frac{1}{2}\int_{\partial\Omega} \varphi(x)[\nu \otimes \beta](\mathrm{d}S) = \int_{\bar{\Omega}} \varphi(x)\bar{E}u(\mathrm{d}x) + \frac{1}{2}\int_{\Omega} u(x) \otimes \nabla\varphi(x)\,\mathrm{d}x\ , \qquad (4)$$

where $[\nu \otimes \beta] \in M\left(\partial\Omega; \mathbb{R}^{n\times n}\right)$ *is defined as*

$$\int_{\partial\Omega} \phi(x)[\nu \otimes \beta]_{ij}(\mathrm{d}S) := \int_{\partial\Omega} \phi(x)\nu_i(x)\beta_j(\mathrm{d}S) + \int_{\partial\Omega} \phi(x)\nu_j(x)\beta_i(\mathrm{d}S)$$

for every $\phi \in C\left(\partial\Omega\right)$*, again with* $i, j \in \{1, 2, \ldots, n\}$*, and* $\nu \colon \partial\Omega \to \mathbb{R}^n$ *is the outer unit normal to* $\partial\Omega$ *as before.*

Proof. Initially, we follow the proof of [12, Theorem 1]. According to the definition, there is a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}\left(\Omega; \mathbb{R}^n\right)$ such that $u_k \to u$ in $L^1\left(\Omega; \mathbb{R}^n\right)$ and $\varepsilon\left(u_k\right) \overset{\star}{\rightharpoonup} \bar{E}u$. We have by Green's theorem

$$\int_{\partial\Omega} [\mathrm{tr}\ u_k^i](x)\varphi(x)\nu_j(x)\,\mathrm{d}S = \int_{\Omega} \frac{\partial u_k^i(x)}{\partial x_j}\varphi(x)\,\mathrm{d}x + \int_{\Omega} u_k^i(x)\frac{\partial\varphi(x)}{\partial x_j}\,\mathrm{d}x\ .$$

Similarly, interchanging i and j, we obtain

$$\int_{\partial\Omega} [\mathrm{tr}\ u_k^j](x)\varphi(x)\nu_i(x)\,\mathrm{d}S = \int_{\Omega} \frac{\partial u_k^j(x)}{\partial x_i}\varphi(x)\,\mathrm{d}x + \int_{\Omega} u_k^j(x)\frac{\partial\varphi(x)}{\partial x_i}\,\mathrm{d}x\ .$$

Summing up these two equalities, we get

$$\int_{\partial\Omega} ([\mathrm{tr}\ u_k] \otimes \nu)(x)\varphi(x)\,\mathrm{d}S = 2\int_{\Omega} \varepsilon(u_k(x))\varphi(x)\,\mathrm{d}x + \int_{\Omega} u_k(x) \otimes \nabla\varphi(x)\,\mathrm{d}x\ . \quad (5)$$

Passing to the limit for $k \to \infty$ and using (2) in the fourth equality, we get

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\partial\Omega} ([\mathrm{tr}\ u_k] \otimes \nu)(x)\varphi(x)\,\mathrm{d}S \\
&= 2\int_{\bar{\Omega}} \varphi(x)\bar{E}u(\mathrm{d}x) + \int_{\Omega} u(x) \otimes \nabla\varphi(x)\,\mathrm{d}x \\
&= 2\int_{\Omega} \varphi(x)\bar{E}u(\mathrm{d}x) + 2\int_{\partial\Omega} \varphi(x)\bar{E}u(\mathrm{d}x) + \int_{\Omega} u(x) \otimes \nabla\varphi(x)\,\mathrm{d}x \qquad (6)\\
&= 2\int_{\Omega} \varphi(x)Eu(\mathrm{d}x) + 2\int_{\partial\Omega} \varphi(x)\bar{E}u(\mathrm{d}x) + \int_{\Omega} u(x) \otimes \nabla\varphi(x)\,\mathrm{d}x \\
&= \int_{\partial\Omega} \varphi(x)[\nu \otimes T_{BD}u]\mathrm{d}S + 2\int_{\partial\Omega} \varphi(x)\bar{E}u(\mathrm{d}x).
\end{aligned}$$

To show that β is a measure, we argue differently from the case of the full gradient [12]. Namely, we define a continuous linear functional on the set $\mathcal{S} := \{\nu\varphi|_{\partial\Omega} \mid \varphi \in C^1(\bar{\Omega})\}$ by setting

$$
\begin{aligned}
\langle \beta, \nu\varphi \rangle &:= \lim_{k\to\infty} \int_{\partial\Omega} ([\mathrm{tr}\ u_k] \otimes \nu)(x)\varphi(x)\,\mathrm{d}S \\
&= \int_{\partial\Omega} \varphi(x)[\nu \otimes T_{BD}u]\mathrm{d}S + 2\int_{\partial\Omega} \varphi(x)\bar{E}u(\mathrm{d}x)\ .
\end{aligned}
\tag{7}
$$

However, the linear hull of $\mathcal{S}$ is dense in $C(\partial\Omega)$ [12, page 19], so β can be uniquely extended to a functional (denoted again by β) on the whole $C(\partial\Omega)$. □

As for the full gradient, where the BV-trace coincides with the $W^{1,\mu}$-trace if $\bar{D}u$ does not concentrate on $\partial\Omega$, Equation (7) shows that the BD-trace introduced in [14] agrees with the $BD\mu$-trace defined here if $\bar{E}$ does does not concentrate on $\partial\Omega$.

Definition 2.4. We call the measure β constructed in Theorem 2.3 the *trace* or occasionally *$BD\mu$-trace* of $(u, \bar{E}u) \in BD\mu(\Omega;\mathbb{R}^n)$ and we write $\bar{T}(u, \bar{E}u) := \beta$.

Theorem 2.5. *The mapping*

$$BD\mu(\Omega;\mathbb{R}^n) \to M(\partial\Omega;\mathbb{R}^n) :\ (u, \bar{E}u) \mapsto \bar{T}(u, \bar{E}u)$$

is continuous and sequentially weakly$\star$ continuous.

Proof. The mapping in question is linear and bounded. Indeed, it follows from the proof of Theorem 2.3 that there is $C > 0$ such that for all $(u, \bar{E}u) \in BD\mu(\Omega;\mathbb{R}^n)$ we have $\|\beta\|_{M(\partial\Omega;\mathbb{R}^n)} \le C\left\|(u, \bar{E}u)\right\|_{BD\mu(\Omega;\mathbb{R}^n)}$, where β is the trace of $(u, \bar{E}u)$. This can be deduced from (7) and the continuity of the trace operator T_{BD} in $BD(\Omega;\mathbb{R}^n)$ [14, Theorem 1.1] as follows. Consider in (7) a sequence $(u_k, \bar{E}u_k) \overset{\star}{\rightharpoonup} (u, \bar{E}u)$ as $k \to \infty$. The statement of the theorem follows easily from the limit passage for $k \to \infty$ in (4) and from the fact that the linear hull of $\mathcal{S} := \{\nu\varphi|_{\partial\Omega} \mid \varphi \in C^1(\bar{\Omega})\}$ is dense in $C(\partial\Omega)$. □

Theorem 2.6 (weak-$\star$ compactness)**.** *Let $\{(u_k, \bar{E}u_k)\}_{k\in\mathbb{N}} \subset BD\mu(\Omega;\mathbb{R}^n)$ be uniformly bounded. Then there exists a subsequence (not relabeled) and $(u, \bar{E}u) \in BD\mu(\Omega;\mathbb{R}^n)$ such that* w-$\star\lim_{k\to\infty}(u_k, \bar{E}u_k) = (u, \bar{E}u)$.

Proof. Notice that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $BD(\Omega;\mathbb{R}^n)$ because there is an obvious continuous projection of $BD\mu(\Omega;\mathbb{R}^n)$ on $BD(\Omega;\mathbb{R}^n)$ defined as $(v, \bar{E}v) \mapsto (v, Ev)$, where Ev is the restriction of the measure $\bar{E}v \in M(\bar{\Omega};\mathbb{R}^{n\times n})$ on Ω. Due to a result of Suquet [13], there is a compact embedding of $BD(\Omega;\mathbb{R}^n)$ in $L^p(\Omega;\mathbb{R}^n)$ for all $1 \le p < \frac{n}{n-1}$. Hence, there is a subsequence of $\{(u_k, \bar{E}u_k)\}_{k\in\mathbb{N}} \subset BD\mu(\Omega;\mathbb{R}^n)$ converging strongly to some

$u \in L^p(\Omega;\mathbb{R}^n)$. By the definition of $BD\mu(\Omega;\mathbb{R}^n)$, for every $\left(u_k, \bar{E}u_k\right)$ there is a sequence $\left\{u_k^j\right\}_{j\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^n)$ such that $u_k^j \to u_k$ in $L^1(\Omega;\mathbb{R}^n)$. Thus, we have $\lim_{k\to\infty}\lim_{j\to\infty} u_k^j = u$ in $L^1(\Omega;\mathbb{R}^n)$. We apply the standard diagonalisation argument to extract a sequence $\{\hat{u}_m\}_{m\in\mathbb{N}} \subset \left\{u_k^j\right\}_{j,k\in\mathbb{N}}$ such that $\lim_{m\to\infty}\hat{u}_m = u$ in $L^1(\Omega;\mathbb{R}^n)$.

Moreover, $\left\{\bar{E}\hat{u}_m\right\}_{m\in\mathbb{N}}$ is uniformly bounded in $M\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)$, so for a further subsequence we have a measure $\mu \in M\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)$ such that $\bar{E}\hat{u}_m \overset{\star}{\rightharpoonup} \mu$. By the definition of $BD(\Omega;\mathbb{R}^n)$, $\mu = \bar{E}u$. □

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ a be bounded domain, with $\partial\Omega$ belonging to class C^1. Let $\Gamma_D \subset \partial\Omega$ be open and of positive $(n-1)$-dimensional Lebesgue measure; suppose further that $z \in M(\Gamma_D;\mathbb{R}^m)$. Then there is $C > 0$ such that the estimate*

$$\|u\|_{L^1(\Omega;\mathbb{R}^m)} \le C\left(\left\|\bar{E}u\right\|_{M\left(\bar{\Omega};\mathbb{R}^{m\times n}\right)} + \|z\|_{M(\Gamma_D;\mathbb{R}^m)}\right) \tag{8}$$

holds for all $\left(u,\bar{E}u\right) \in BD\mu(\Omega;\mathbb{R}^n)$ with $\bar{T}\left(u,\bar{E}u\right) = z$ on Γ_D.

Proof. Suppose that (8) does not hold. This means that for all $k \in \mathbb{N}$ there is $\left(u_k,\bar{E}u_k\right) \in BD\mu(\Omega;\mathbb{R}^m)$ with $\bar{T}\left(u_k,\bar{E}u_k\right) = z$ on Γ_D such that

$$\|u_k\|_{L^1(\Omega;\mathbb{R}^m)} > k\left(\left\|\bar{E}u_k\right\|_{M\left(\bar{\Omega};\mathbb{R}^{m\times n}\right)} + \|z\|_{M(\Gamma_D;\mathbb{R}^m)}\right).$$

Let us put $v_k := \frac{u_k}{\|u_k\|_{L^1(\Omega;\mathbb{R}^m)}}$ and $\bar{E}v_k := \frac{\bar{E}u_k}{\|u_k\|_{L^1(\Omega;\mathbb{R}^m)}}$. Then the last inequality implies

$$1 > k\left(\left\|\bar{E}v_k\right\|_{M\left(\bar{\Omega};\mathbb{R}^{m\times n}\right)} + \|u_k\|^{-1}_{L^1(\Omega;\mathbb{R}^m)}\|z\|_{M(\Gamma_D;\mathbb{R}^m)}\right).$$

In particular, we have $\|v_k\|_{L^1(\Omega;\mathbb{R}^m)} = 1$ and $\left\|\bar{E}v_k\right\|_{M\left(\bar{\Omega}\right)} \le \frac{1}{k}$. Consequently, for all $k \in \mathbb{N}$, $\left\|\left(v_k,\bar{E}v_k\right)\right\|_{BD\mu(\Omega;\mathbb{R}^m)} \le 2$. The weak-$\star$ compactness of balls in $BD\mu(\Omega;\mathbb{R}^m)$ implies that there is $\left(v,\bar{E}v\right) \in BD\mu(\Omega;\mathbb{R}^m)$ such that for a subsequence (not relabelled) w-$\star\lim_{k\to\infty}\left(v_k,\bar{E}v_k\right) = \left(v,\bar{E}v\right)$. Moreover, $\|v\|_{L^1(\Omega;\mathbb{R}^m)} = 1$ and $\bar{E}v = 0$. Finally, the sequential weak-$\star$ continuity of the trace operator (Theorem 2.5) and the fact that $\|u_k\|_{L^1(\Omega;\mathbb{R}^m)} \to \infty$ imply that $\bar{T}\left(v,\bar{D}v\right) = 0$ on Γ_D. As $\bar{E}v = 0$, we have that $v \in BD\mu(\Omega;\mathbb{R}^m)$ and that $v(x) = a + Rx$, where $a \in \mathbb{R}^n$ and $R \in \mathbb{R}^{n\times n}$ is skew-symmetric; see, e.g., [14]. In other words, v is a rigid motion, that is, a translation and/or an infinitesimal rotation. On the other hand, $\bar{T}\left(v,\bar{E}v\right) = 0$, i.e., $v = 0$. This, however, contradicts the fact that $\|v\|_{L^1(\Omega;\mathbb{R}^m)} = 1$. □

Remark 2.8. For the validity of Lemma 2.7, it is important that Γ_D is open in $\partial\Omega$. Namely, the sequential weak-$\star$ continuity of $\bar{T}$ implies that $\bar{T}\left(v_k,\bar{E}v_k\right) \to \bar{T}\left(v,\bar{D}v\right)$ weakly$\star$ in $M(\partial\Omega;\mathbb{R}^m)$. Clearly, this does not necessarily imply that $\bar{T}\left(v_k,\bar{E}v_k\right) \to \bar{T}\left(v,\bar{E}v\right)$ weakly$\star$ in $M(\Gamma_D;\mathbb{R}^m)$ for Γ_D not open in $\partial\Omega$.

3. Example: crystal plasticity with infinite late hardening

As an example, we will investigate the model for single-crystal plasticity with infinite latent hardening proposed by Conti and Ortiz [3]. As shown there, the plastic stored energy has to have linear growth at infinity except on the trace.

We sketch the problem that motivates our investigation. Crystalline materials can often be characterised via energy minimisation; for plastically deformed crystals, Ortiz and Repetto [9] provide a setting in which dislocation structures can be described by a nonconvex minimisation problem. The nature of this variational model is incremental, to reflect the irreversible nature of plastic deformations [9]. Against this background, Conti and Ortiz [3] investigated a static problem recalled below; here, we prove here existence of a minimiser with prescribed boundary data. As discussed in the introduction, prescribed Dirichlet conditions are natural for these mechanical problems (for example, they correspond to a mechanically clamped specimen). The setting is that of infinite latent hardening without self-hardening. This leads to a microscopic energy W that is linear along single-slip orbits. Conti and Ortiz [3] have shown that the macroscopic energy in this setting has linear growth on traceless symmetric matrices, and quadratic on trace part. They consider linearised kinematics, so if $u\colon \Omega \to \mathbb{R}^3$ is the displacement, then $\beta^{\rm sym} := \varepsilon(u) := \frac{1}{2}\left(\nabla u + \nabla^\top u\right)$. The plastic strain in single crystals for monotonic deformations is $\varepsilon^p(u) := \frac{1}{2}\left(\beta^p + \beta^{pT}\right)$, where

$$\beta^p(\gamma) = \sum_{j=1}^{J} \gamma_j s_j \otimes m_j,$$

with γ_j being the *slip strain*, s_j the *slip direction* and m_j *plane normal*. For the microscopic energy, the assumptions of infinite latent hardening and the absence of self-hardening lead to a microscopic energy W^{**} that is linear along single-slip orbits. Conti and Ortiz show that convex envelope in this situation has linear growth on traceless symmetric matrices, and quadratic on trace part,

$$c\left(|\beta^{\rm sym}| + |{\rm Tr}(\beta)|^2 - 1\right) \le W^{**}(\beta) \le C\left(|\beta^{\rm sym}| + |{\rm Tr}(\beta)|^2 + 1\right) . \tag{9}$$

Thus, the macroscopic energy is *linear* except for the trace.

The variational problem is then

$$\text{minimise } \; I(u) := \int_\Omega W^{**}(x, \beta(u(x)) \, {\rm d}x \;\; \text{among} \;\; u \in W^{1,1}_{u_D}(\Omega; \mathbb{R}^m) . \tag{10}$$

It is natural to use direct methods to prove existence of a solution to (11); the central task is to find a setting in which a minimising sequence converges *and* retains the boundary in the limit. As discussed earlier, the space $BD\mu(\Omega; \mathbb{R}^n)$ is a natural choice; we restrict it here to obtain a norm in correspondance to the growth condition (9),

$$BD\mu_{\rm tr}(\Omega) = \left\{u \in BD\mu\left(\Omega, \mathbb{R}^3\right) \;\middle|\; {\rm tr}(Eu) \in L^2(\Omega)\right\} . \tag{11}$$

Proposition 3.1. *For prescribed Dirichlet data, the variational problem* (10) *has a unique minimiser in* $BD\mu_{\mathrm{tr}}(\Omega)$.

Proof. The argument is elementary as a minimising sequence $\{u_k\}_{k\in\mathbb{N}}$ in the space $BD\mu(\Omega)$ is weak-$\star$ compact by the growth condition (9) and Theorem 2.6; the subspace $BD\mu_{\mathrm{tr}}(\Omega)$ is closed in $BD\mu(\Omega)$ with respect to the weak-$\star$ topology; since the trace operator is weak-$\star$ continuous (Theorem 2.5), the boundary data is preserved in the limit. □

4. Example: simultaneous relaxation in u and Du

As an application, we consider the relaxation of the functional

$$\text{Minimise } I(u) := \int_\Omega W(x, u(x), \varepsilon(u(x)))\,\mathrm{d}x - \int_\Omega f(x)\cdot u(x)\,\mathrm{d}x \tag{12}$$

among $u \in W^{1,1}_{u_D}(\Omega;\mathbb{R}^n)$. We recall that $W^{1,1}_{u_D}(\Omega;\mathbb{R}^n)$ is the set of functions $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ with $u = u_D$ on Γ_D in the sense of traces. In (12), $W\colon \bar\Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R}$ is assumed to be continuous. To demonstrate the functional analytic framework laid out in this article, we assume that W has linear growth in the third component, as it is the case for applications in plasticity (or minimal surfaces). That is, we assume that there exist constants $\beta \geq \alpha > 0$ with

$$\alpha\left(|u| + |s|\right) - \beta \leq W(x,u,s) \leq \beta\left(1 + |u| + |s|\right) \text{ for every } x \in \bar\Omega\,. \tag{13}$$

Further, to demonstrate the treatment of boundary conditions, we include a forcing term in the analysis,

$$f \in L^p(\Omega;\mathbb{R}^n) \tag{14}$$

with $p > n$; precise assumptions on the smallness of this forcing are stated later in this section.

We do not assume that W is quasiconvex in s and thus have to resort to a relaxed formulation of (12) in the space of DiPerna-Majda measures; see Appendix B for a brief survey.

Before stating the relaxed version of the static problem (12), we have to collect an auxiliary statement that permits us to recover information regarding a function u whose measure derivative, Du, is the first moment of a gradient DiPerna-Majda measure.

Definition 4.1. We say that $(\sigma,\hat\nu) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{n\times(n\times n)})$ (a *joint DiPerna-Majda measure*) if there is $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}(\Omega;\mathbb{R}^n)$ such that $\{u_k,\varepsilon(u_k)\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$.

In what follows, we are going to norm $\mathbb{R}^{n\times(n\times n)}$ by $\|\cdot\|_{\mathbb{R}^{n\times(n\times n)}} := \|\cdot\|_{\mathbb{R}^n} + \|\cdot\|_{\mathbb{R}^{n\times n}}$, where the norms on the right-hand side are Euclidean ones.

Example 4.2. We note that the mere knowledge of a DiPerna Majda measure generated by $\{\varepsilon(u_k)\}_{k\in\mathbb{N}}$ does not provide us with sufficient information on $(\sigma,\hat{\nu}) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{n\times(n\times n)})$ even if $\{u_k\}_{k\in\mathbb{N}}$ converges strongly in $L^1(\Omega;\mathbb{R}^n)$. Indeed, consider the following toy problems for $n = 1$ and $\Omega = (-1,1)$, i.e., $\varepsilon(u) = u'$, and the one-point compactification of $\mathbb{R}$:

$$u_k(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ kx & \text{if } 0 \le x \le \frac{1}{k} \\ 1 & \text{otherwise.} \end{cases} \tag{15}$$

Thus $u_k \to u$ in $L^1(-1,1)$ where

$$u(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ 1 & \text{otherwise.} \end{cases} \tag{16}$$

In this case, $\sigma(\mathrm{d}x) = (1 + |u(x)|)\mathrm{d}x + \delta_0$ and

$$\hat{\nu}_x = \begin{cases} \delta_{(0,0)} & \text{if } -1 \le x < 0 \\ \delta_{(0,\infty)} & \text{if } x = 0 \\ \delta_{(1,0)} & \text{otherwise.} \end{cases} \tag{17}$$

On the other hand, if we slightly modify $\{u_k\}_{k\in\mathbb{N}}$ to get

$$u_k(x) = \begin{cases} 0 & \text{if } -1 \le x \le -\frac{1}{k} \\ kx + 1 & \text{if } -\frac{1}{k} \le x \le 0 \\ 1 & \text{otherwise.} \end{cases} \tag{18}$$

we have again $u_k \to u$ in $L^1(-1,1)$ as $k \to \infty$, but the DiPerna-Majda measure generated by $\{(u_k, u_k')\}_{k\in\mathbb{N}}$ is now $(\tilde{\sigma},\tilde{\nu}) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{n\times(n\times n)})$, where $\tilde{\sigma}(\mathrm{d}x) = (1 + |u(x)|)\mathrm{d}x + \delta_0$ and

$$\tilde{\nu}_x = \begin{cases} \delta_{(0,0)} & \text{if } -1 \le x < 0 \\ \delta_{(1,\infty)} & \text{if } x = 0 \\ \delta_{(1,0)} & \text{otherwise.} \end{cases} \tag{19}$$

Nevertheless, in both examples the DiPerna-Majda measure $(\hat{\gamma},\pi)$ generated by $\{u_k'\}_{k\in\mathbb{N}}$ is the same, namely $\pi(\mathrm{d}x) = \mathrm{d}x + \delta_0$ and

$$\hat{\gamma}_x = \begin{cases} \delta_0 & \text{if } x \ne 0 \\ \delta_\infty & \text{if } x = 0. \end{cases} \tag{20}$$

Lemma 4.3. *Let* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}(\Omega;\mathbb{R}^m)$ *be such that* $\{u_k, \varepsilon(u_k)\}_{k\in\mathbb{N}}$ *generates* $(\sigma,\hat{\nu}) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{n\times(n\times n)})$. *Then there is* $(u,\bar{E}u) \in BD\mu(\Omega;\mathbb{R}^n)$ *and a subsequence (not relabelled) such that* $u_k \to u$ *in* $L^1(\Omega;\mathbb{R}^n)$. *Furthermore,* $(u,\bar{E}u)$ *satisfies* $\bar{T}(u,\bar{E}u) = u_D$ *on* Γ_D*, and* u *is a unique solution to*

$$\int_{\bar{\Omega}} \phi(x)\,(u,\bar{E}u)\,(\,\mathrm{d}x) = \int_{\bar{\Omega}} \phi(x) \int_{\beta_{\mathcal{F}}\mathbb{R}^{n\times(n\times n)}} \frac{s}{1+|s|}\,\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \tag{21}$$

for every $\phi \in C(\bar{\Omega})$*, i.e.,*

$$u = \int_{\beta_{\mathcal{F}}\mathbb{R}^{n\times(n\times n)}} \frac{s_1}{1+|s|}\,\hat{\nu}_x(\mathrm{d}s)\sigma \quad \textit{and} \quad \bar{E}u = \int_{\beta_{\mathcal{F}}\mathbb{R}^{n\times n}} \frac{s_2}{1+|s|}\,\hat{\nu}_x(\mathrm{d}s)\sigma$$

in the sense of measures on $\bar{\Omega}$*, where we wrote* $s = (s_1,s_2) \in \mathbb{R}^{n\times(n\times n)}$.

Proof. As $\left\{\frac{1}{2}\left(\nabla u_k + \nabla u_k^\top\right)\right\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat{\nu}) \in \mathcal{GDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{n\times n})$, it is bounded in $L^1(\Omega,\mathbb{R}^{n\times n})$. The Dirichlet boundary condition on Γ_D permits an application of the Poincaré inequality (8),

$$\|u\|_{L^1(\Omega;\mathbb{R}^m)} \le C\left(\|\bar{E}u\|_{M\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)} + \|u_D\|_{M(\Gamma_D;\mathbb{R}^n)}\right) \tag{22}$$

and thus yields that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega;\mathbb{R}^n)$ and therefore in $BD\mu(\Omega;\mathbb{R}^n)$. Hence, there is a subsequence (not relabelled) converging weakly $\star$ in $BD\mu(\Omega;\mathbb{R}^n)$ to some $u \in BD\mu(\Omega;\mathbb{R}^n)$ by Theorem 2.6. By definition of weak-$\star$ convergence in $BD\mu(\Omega;\mathbb{R}^n)$, this means that $u_k \to u$ strongly in $L^1(\Omega;\mathbb{R}^n)$ and $\frac{1}{2}\left(\nabla u_k + \nabla u_k^\top\right) \to \bar{E}u$ weakly$\star$ in $M\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)$. Formula (21) then follows by comparing (31) and the definition of weak-$\star$ convergence,

$$\lim_{k\to\infty}\int_{\Omega} \phi(x)\bar{E}u_k(\mathrm{d}x) = \int_{\Omega} \phi(x)\bar{E}u(\mathrm{d}x) \tag{23}$$

for test functions $\phi \in C\left(\bar{\Omega}\right))$ component-wise for $s = \{s_{jk}\}$, with $j,k = 1,\ldots,n$. The fact that $u = u_D$ on Γ_D follows from the weak-$\star$ continuity of the trace operator $\bar{T}$ (Theorem 2.5). □

Now let us discuss a suitable relaxation of the problem (12). One would expect that a compactification involves only the gradient part, and strong convergence is inferred for the second argument, u, of W. However, as demonstrated in Example 4.2, we need to consider a compactification both in the second and the third argument. Accepting this for the moment, we take a subalgebra $\mathcal{F}$ of bounded continuous functions on $\mathbb{R}^n \times \mathbb{R}^{n\times n}$ such that for every $s_1 \in \mathbb{R}^n$ and every $s_2 \in \mathbb{R}^{n\times n}$

$$\tilde{W}(x,s_1,s_2) \in \mathcal{F}; \tag{24}$$

we recall that $\mathcal{F}$ contains all functions where all radial limits exist, as a compactification by a sphere or finer is considered. We extend the previous notation slightly to accommodate for spatially inhomogeneous functions by writing $\tilde{W}(x, s_1, s_2) := \frac{W(x,s_1,s_2)}{1+|s_1|+|s_2|}$. We write $s := (s_1, s_2)$.

The relaxed problem then reads as follows:

$$\begin{aligned} &\text{minimise } \bar{I}(u, \bar{E}u, \sigma, \hat{\nu}) := \int_{\bar{\Omega}} \int_{\beta_{\mathcal{F}}\mathbb{R}^{n\times n}} \tilde{W}(x, s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) - \int_{\Omega} f(x)\cdot u(x)\,\mathrm{d}x \\ &\qquad \text{among } (u, \bar{E}u) \in BD\mu(\Omega; \mathbb{R}^n),\ \bar{T}(u, \bar{E}u) = u_D \text{ on } \Gamma_D, \\ &\qquad \text{and } (\sigma, \hat{\nu}) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega; \mathbb{R}^{n\times(n\times n)}),\ (u, \bar{E}u) \text{ satisfies (21).} \end{aligned} \tag{25}$$

Proposition 4.4. *If* $\|f\|_{L^\infty(\Omega;\mathbb{R}^n)} < \alpha$, *then a minimiser of* (25) *exists. Furthermore, the minimum of* (25) *equals the infimum of* (12). *If* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}(\Omega; \mathbb{R}^n)$ *is an infimising sequence of* (12), *then a subsequence generates* (*in the sense* (32)) *a minimiser of* (25). *Moreover, any minimiser of* (25) *is generated by an infimising sequence of* (12).

Proof. We first show that $\inf I \geq \inf \bar{I}$. Let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}(\Omega; \mathbb{R}^m)$ be an infimising sequence of (12). Obviously $\inf I < \infty$. Thus, there exists $K > 0$ so that the following estimate holds (we employ the coercivity assumption (13) on W).

$$\begin{aligned} K &> \int_\Omega W(x, u_k(x), \varepsilon(u_k(x)))\,\mathrm{d}x - \int_\Omega f(x)\cdot u_k(x)\,\mathrm{d}x \\ &\geq \alpha \int_\Omega [|\varepsilon(u_k(x))| + |u_k(x)|]\,\mathrm{d}x - \beta|\Omega| - \|u_k\|_{L^1(\Omega;\mathbb{R}^n)}\|f\|_{L^\infty(\Omega;\mathbb{R}^n)} \\ &\geq \alpha \int_\Omega [|\varepsilon(u_k(x))| + |u_k(x)|]\,\mathrm{d}x - \beta|\Omega| - \|u_k\|_{L^1(\Omega;\mathbb{R}^n)}\|f\|_{L^\infty(\Omega;\mathbb{R}^n)}\,. \end{aligned}$$

Hence, since $\alpha > \|f\|_{L^\infty(\Omega;\mathbb{R}^n)}$ we have that $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega; \mathbb{R}^n)$ is bounded in $BD(\Omega; \mathbb{R}^n)$. By the DiPerna-Majda result (31), $\{u_k, \varepsilon(u_k)\}_{k\in\mathbb{N}}$ then generates (up to a subsequence) $(\sigma, \hat{\nu}) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega; \mathbb{R}^{n\times(n\times n)})$. At the same time we may suppose that $u_k \to u$ strongly in $L^1(\Omega; \mathbb{R}^n)$ by compact embedding. Since $\{u_k\}_{k\in\mathbb{N}}$ is an infimising sequence, and the map $u \mapsto \int_\Omega f(x)\cdot u(x)\,\mathrm{d}x$ is sequentially continuous, (31) shows that $\inf I = \lim_{k\to\infty} I(u_k) = \bar{I}(u, \bar{E}u, \sigma, \hat{\nu})$. Suppose that $\inf \bar{I} < \inf I$. Then there is $\bar{I}\left(u, \bar{E}u, \sigma, \hat{\nu}\right) < \inf I$ for some $\left(u, \bar{E}u, \sigma, \hat{\nu}\right)$. As $(\sigma, \hat{\nu}) \in \mathcal{JDM}^{u_D}_{\mathcal{F}}(\Omega; \mathbb{R}^{n\times(n\times n)})$, I converges along a generating sequence of $(\sigma, \hat{\nu})$ to $\bar{I}\left(u, \bar{E}u, \sigma, \hat{\nu}\right)$. Thus, $\inf I = \inf \bar{I}$. The fact that $\inf \bar{I}$ is attained follows immediately. □

Remark 4.5. Notice that we did not need the Poincaré-type inequality (8) because we control u by means of the growth conditions of W. We would need inequality (8) if W is independent of u.

A. Young measures

We briefly recall the concept of Young measures [15] and follow the presentation in [8]. Young measures describe the limit of a sequence $\{u_k\}_{k\in\mathbb{N}}$ of functions $u_k\colon \Omega \to \mathbb{R}^m$ which converges weakly in $L^q(\Omega;\mathbb{R}^m)$ for $1 \le q < \infty$ or weakly$\star$ if $q = \infty$. The precise concept is as follows. A *Young measure* on a bounded domain $\Omega \subset \mathbb{R}^n$ is a weakly$\star$ measurable mapping

$$\Omega \to \operatorname{Prob}(\mathbb{R}^m), \qquad x \mapsto \hat{\nu}_x,$$

with values in the probability measures. We recall that a mapping with values in the Radon measures is *weakly$\star$ measurable* if for any $f \in C_0(\mathbb{R}^m)$, the mapping

$$\Omega \to \mathbb{R}, \qquad x \mapsto \langle f, \hat{\nu}_x\rangle := \int_{\mathbb{R}^m} f(s)\hat{\nu}_x(\mathrm{d}s)$$

is measurable in the usual sense. We write $\hat{\nu}$ for the Young measure to distinguish it from the normal ν introduced in the previous sections. We denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^m)$.

It is known that $\mathcal{Y}(\Omega;\mathbb{R}^m)$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega; M(\mathbb{R}^m)) \cong L^1(\Omega; C_0(\mathbb{R}^m))^*$, where $L^\infty_{\mathrm{w}}(\Omega; M(\mathbb{R}^m))$ is the space of weakly$\star$ measurable bounded functions. The *parametrised Young measure theorem* [11] states that for every sequence $\{u_k\}_{k\in\mathbb{N}}$ which is bounded in $L^\infty(\Omega;\mathbb{R}^m)$, there exists a subsequence (denoted by the same indices for notational simplicity) and a Young measure $\hat{\nu} = \{\hat{\nu}_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ such that for every continuous function $f\colon \mathbb{R}^m \to \mathbb{R}$,

$$f(u_k) \overset{\star}{\rightharpoonup} x \mapsto \langle f, \hat{\nu}_x\rangle \text{ weakly}\star \text{ in } L^\infty(\Omega), \tag{26}$$

where

$$\langle f, \hat{\nu}_x\rangle := \int_{\mathbb{R}^m} f(s)\hat{\nu}_x(\mathrm{d}s) \tag{27}$$

is the *expectation* of f. Let $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$ denote set of all Young measures which are generated by taking all bounded sequences $\{u_k\}_{k\in\mathbb{N}}$ in $L^\infty(\Omega;\mathbb{R}^m)$.

The above concept is applicable if $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega;\mathbb{R}^m)$. If in addition to the uniform bound in $L^\infty(\Omega;\mathbb{R}^m)$, $u_k \rightharpoonup y$ in $L^q(\Omega;\mathbb{R}^m)$ with $1 \le q < \infty$, then $u_k \to u$ if and only if the corresponding Young measure is a Dirac mass, $\hat{\nu}_x = \delta_{u(x)}$. Non-Dirac Young measures thus record possible oscillations in the limit process.

The assumption that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ can be relaxed to the assumption of such a bound in $L^q(\Omega;\mathbb{R}^m)$ with $1 < q < \infty$. The parametrised Young measure theorem is then valid under stronger growth conditions on the nonlinearity f. The precise formulation has been given by Schonbek [11, Theorem 2.2] (see also [2] for a general formulation of the parametrised Young measure theorem). Namely, for every sequence $\{u_k\}_{k\in\mathbb{N}}$ which is uniformly bounded in $L^q(\Omega;\mathbb{R}^m)$ for some $q > 1$, there exists a subsequence, still indexed by k for notational convenience, and a Young measure $\hat{\nu} = \{\hat{\nu}_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ such that for every $f \in C(\mathbb{R}^m)$ with

$$f(x) = o(|x|^q) \text{ for } |x| \to \infty, \tag{28}$$

the following holds in $L^1(\Omega;\mathbb{R}^m)$:

$$f(u_k) \rightharpoonup \langle f, \hat{\nu}_x \rangle. \tag{29}$$

We say that $\{u_k\}_{k\in\mathbb{N}}$ generates $\hat{\nu}$ if (29) holds; we denote the set of all Young measures obtained as limits of bounded sequences in $L^q(\Omega;\mathbb{R}^m)$ by $\mathcal{Y}^q(\Omega;\mathbb{R}^m)$. If $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega;\mathbb{R}^m)$, then the set of Young measures generated by subsequences of $\{\nabla u_k\}_{k\in\mathbb{N}}$ will be denoted $\mathbb{G}^q(\Omega;\mathbb{R}^{m\times m})$. In the spirit of (26), we extend the energy V to $\bar{V}(\hat{\nu}) := \int_\Omega \int_{\mathbb{R}^{m\times m}} \varphi(s)\hat{\nu}_x(\mathrm{d}s)\,\mathrm{d}x$ for $\hat{\nu} \in \mathbb{G}^q(\Omega;\mathbb{R}^{m\times m})$.

B. DiPerna-Majda measures

In the situation under consideration, unlike that of Appendix A, no bound in $L^\infty(\Omega;\mathbb{R}^m)$ is available, and even the extension to bounds in $L^p(\Omega;\mathbb{R}^m)$ for $1 < p < \infty$ is not sufficient. Namely, the energy density W will be a test function f in the sense of (26). Obviously, a linearly growing energy density does not satisfy (28) even for $p = 1$, and it is not hard to see that the bound (28) on the growth of the nonlinearity f is sharp [11, Example 2.1]. DiPerna-Majda measures are an extension of Young measures to describe concentration effects, which may occur due to the non-reflexivity of $L^1(\Omega;\mathbb{R}^m)$. That is, let f be a function $\mathbb{R}^m \to \mathbb{R}$ with p-growth at infinity. DiPerna-Majda measures then describe the limit of a sequence $\{f(u_k)\}_{k\in\mathbb{N}}$, where the functions $u_k\colon \Omega \to \mathbb{R}^m$ converge weakly in $L^p(\Omega;\mathbb{R}^m)$ for $1 \le p < \infty$, but are not uniformly bounded in $L^\infty(\Omega;\mathbb{R}^m)$.

The definition of DiPerna-Majda measures involves a compactification; we refer to Appendix of [8] for details and some intuition. There, a motivation is given as to why one considers a completely regular subalgebra $\mathcal{F}$ of the space of bounded continuous functions $BC(\mathbb{R}^m)$.

We consider compactifications $\beta_{\mathcal{F}}\mathbb{R}^m$ by a sphere or finer. That is, $\mathcal{F}$ contains all functions $\tilde{f}$ for which the radial limit $\lim_{r\to\infty} \tilde{f}(rs)$ exists for arbitrary $s \in \mathbb{R}^m$. We note that $\mathcal{F}$ also may contain functions $\tilde{f}$ which have no well-defined radial limits. To deal with functions f with linear growth at infinity in a convenient manner, we set $\tilde{f}(s) := \frac{f(s)}{1+|s|}$, with $\tilde{f} \in \mathcal{F}$.

The motivation for the construction of DiPerna-Majda measures can be described as follows. We are given a sequence $\{u_k\}_{k\in\mathbb{N}}$, uniformly bounded in $L^p(\Omega,\mathbb{R}^m)$. For the application discussed below, it suffices to consider the case $p = 1$. The goal is to describe the weak limit $\lim_{k\to\infty} \int_\Omega \phi(x) f(u_k(x))\,\mathrm{d}x$, with

$\phi \in C_0(\Omega)$ and $f(s) := \tilde{f}(s)\,(1+|s|)$, where $\tilde{f} \in BC\,(\mathbb{R}^m)$. A canonical norm for f of this form is $|f|_\infty := \max_{s\in\mathbb{R}^m} \tilde{f}(s) = \left|\tilde{f}\right|_\infty$.

DiPerna and Majda have shown the following results for open domains Ω and test functions $\phi \in C_0(\Omega)$. We state the results for $\bar{\Omega}$ and test functions $\phi \in C\left(\bar{\Omega}\right)$. The proofs remain the same, except that the isomorphism between the dual space of $\left(C_0\left(\Omega\right), \|\cdot\|\right)$ and the space $(M(\Omega), \|\cdot\|)$ of Radon measures with finite mass has to be replaced by the isomorphism of $\left(C\left(\bar{\Omega}\right), \|\cdot\|\right)$ and the space of Radon measures with compact support $\left(M\left(\bar{\Omega}\right), \|\cdot\|\right)$.

For a bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^1\left(\bar{\Omega};\mathbb{R}^m\right)$, there exists a non-negative Radon measure $\sigma \in M^+(\bar{\Omega})$ such that

$$(1+|u_k(x)|)\;\mathrm{d}x \overset{\star}{\rightharpoonup} \sigma \text{ in } M\left(\bar{\Omega}\right); \tag{30}$$

see [4, Theorem 4.1]. Furthermore, for a separable completely regular subalgebra $\mathcal{F}$ of $BC\,(\mathbb{R}^m)$, there exist a σ-measurable map $\hat{\nu}\colon \Omega \to \mathrm{Prob}\,(\beta_{\mathcal{F}}\mathbb{R}^m)$, $x \mapsto \hat{\nu}_x$, and a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ (not relabelled) such that for every $\tilde{f} \in \mathcal{F}$

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \phi(x) f\left(u_k(x)\right)\,\mathrm{d}x = \int_{\bar{\Omega}} \phi(x) \int_{\beta_{\mathcal{F}}\mathbb{R}^m} \tilde{f}(s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) \tag{31}$$

holds for every $\phi \in C\left(\bar{\Omega}\right)$ [4, Theorem 4.3]. We say that $\{u_k\}_{k\in\mathbb{N}}$ *generates* the pair $(\sigma, \hat{\nu})$ if Equation (31) holds. A pair $(\sigma,\hat{\nu}) \in M^+\left(\bar{\Omega}\right) \times L^\infty_{\mathrm{w}}\left(\bar{\Omega}, \sigma; \mathrm{Prob}\,(\beta_{\mathcal{F}}\mathbb{R}^m)\right)$ attainable by sequences in $L^1\,(\Omega;\mathbb{R}^m)$ is called a *DiPerna-Majda measure*. The set of all DiPerna-Majda measures is denoted $\mathcal{DM}_{\mathcal{F}}\,(\Omega;\mathbb{R}^m)$.

The explicit description of the elements of $\mathcal{DM}_{\mathcal{F}}\,(\Omega;\mathbb{R}^m)$ for unconstrained sequences is given in [7, Theorem 2]. The characterisation of DiPerna-Majda measures generated by gradients of Sobolev maps in $W^{1,p}\,(\Omega;\mathbb{R}^m)$ for $p > 1$ can be found in [6].

It is sometimes convenient to consider an alternative representation of DiPerna-Majda measures. Specifically, in analogy to the proof of Theorem 4.1 in [4], we consider measures in $M\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^m\right)$. We say that $\{u_k\}_{k\in\mathbb{N}} \subset L^1\left(\bar{\Omega};\mathbb{R}^m\right)$ *generates* the measure $\eta \in M\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^m\right)$ if, for every $\tilde{h} \in C\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^m\right)$,

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \tilde{h}\left(x, u_k(x)\right)(1+|u_k(x)|)\;\mathrm{d}x = \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^m} \tilde{h}(x,s)\eta(\mathrm{d}s\mathrm{d}x)$$

holds. The set of all measures generated in this way will be denoted $DM_{\mathcal{F}}\,(\Omega;\mathbb{R}^m)$. Since $\phi(x)\tilde{f}(u)$ with $\phi \in C\left(\bar{\Omega}\right)$ and $\tilde{f} \in BC\,(\beta_{\mathcal{F}}\mathbb{R}^m)$ is dense in $C\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^m\right)$, one can say that $\eta \cong (\sigma,\hat{\nu})$ for $\eta \in DM_{\mathcal{F}}\,(\Omega;\mathbb{R}^m)$ and $(\sigma,\hat{\nu}) \in \mathcal{DM}_{\mathcal{F}}\,(\Omega;\mathbb{R}^m)$ if

$$\langle\tilde{h},\eta\rangle := \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^m} \tilde{h}(x,s)\eta(\mathrm{d}x\mathrm{d}s) = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{F}}\mathbb{R}^m} \tilde{h}(x,s)\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)$$

for any $\tilde{h} \in C\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^m\right)$. Consequently, the elements of $DM_{\mathcal{F}}(\Omega; \mathbb{R}^m)$ will be addressed as DiPerna-Majda measures as well.

It is known [10, Chapter 3] that $DM_{\mathcal{F}}(\Omega; \mathbb{R}^m)$ is a closed, convex, non-compact but locally compact and locally sequentially compact subset of the locally convex space $M\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^m\right)$ considered in its weak-$\star$ topology.

We denote by $\mathcal{GDM}_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ the subset of $\mathcal{DM}_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ of those measures which are generated by gradients of mappings in $W^{1,1}(\Omega; \mathbb{R}^m)$. Expressed differently, $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ if there is $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega; \mathbb{R}^m)$ such that for all $\phi \in C\left(\bar{\Omega}\right)$ and all $\tilde{f} \in \mathcal{F}$

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \phi(x) f(\nabla u_k(x))\, \mathrm{d}x = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \phi(x)\tilde{f}(s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x). \qquad (32)$$

Similarly we write $\eta \in GDM_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ if $\eta \in DM_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ is generated by gradients. Finally, $\mathcal{GDM}^{u_D}_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ denotes elements $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$ with the property that $(\sigma, \hat{\nu})$ is generated by $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}(\Omega; \mathbb{R}^m)$, with $u_D \in W^{1,1}(\Omega; \mathbb{R}^m)$.

Acknowledgements. This work was initiated when M. K. visited Department of Mathematical Sciences at the University of Bath. The hospitality and support of the department is gratefully acknowledged.

Both authors were funded by the Royal Society through an International Joint Project, JP080789. M. K. was supported by the grants A 100750802 (GA AV ČR), VZ6840770021 (MŠMT ČR). J. Z. gratefully acknowledges the financial support of the EPSRC through an Advanced Research Fellowship (GR / S99037 / 1).

References

[1] Ambrosio, L., Fusco, N. and Pallara, D., *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Math. Monogr. New York: Oxford Univ. Press 2000.

[2] Ball, J. M., A version of the fundamental theorem for Young measures. In: *PDEs and Continuum Models of Phase Transitions* (Nice, 1988; eds.: M. Rascle et al.). Berlin: Springer 1989, pp. 207 – 215.

[3] Conti, S. and Ortiz, M., Dislocation microstructures and the effective behavior of single crystals. *Arch. Ration. Mech. Anal.* 176 (2005), 103 – 147.

[4] DiPerna, R. J. and Majda, A. J., Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.* 108 (1987), 667 – 689.

[5] Evans, L. C. and Gariepy, R. F., *Measure Theory and Fine Properties of Functions.* Stud. Adv. Math. Boca Raton (FL): CRC Press 1992.

[6] Kałamajska, A. and Kružík, M., Oscillations and concentrations in sequences of gradients. *ESAIM Control Optim. Calc. Var.* 14 (2008), 71 – 104.

[7] Kružík, M. and Roubíček, T., On the measures of DiPerna and Majda. *Math. Bohem.* 122 (1997), 383 – 399.

[8] Kružík, M. and Zimmer, J., Evolutionary problems in nonreflexive spaces. *ESAIM Control Optim. Calc. Var.* (2010), 1 – 22.

[9] Ortiz, M. and Repetto, E. A., Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids* 47 (1999), 397 – 462.

[10] Roubíček, T., *Relaxation in Optimization Theory and Variational Calculus.* de Gruyter Ser. Nonlinear Anal. Appl. 4. Berlin: de Gruyter 1997.

[11] Schonbek, M. E., Convergence of solutions to nonlinear dispersive equations. *Comm. Part. Diff. Equ.* 7 (1982), 959 – 1000.

[12] Souček, J., Spaces of functions on domain Ω, whose k-th derivatives are measures defined on $\bar{\Omega}$. *Časopis Pěst. Mat.* 97 (1972), 10 – 46, 94.

[13] Suquet, P.-M., Existence et régularité des solutions des équations de la plasticité. *C. R. Acad. Sci. Paris, Sér. A-B* 286 (1978), A1201 – A1204.

[14] Temam, R. and Strang, G., Functions of bounded deformation. *Arch. Ration. Mech. Anal.* 75 (1980/81), 7 – 21.

[15] Young, L. C., Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Classe III* (1937), 212 – 234.

Received April 27, 2010