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Number Theory. – *Combining the conjectures of Schanuel and Zilber–Pink*, by JONATHAN PILA, accepted on 1 July 2025.

To Enrico Bombieri, on his 85th birthday.

ABSTRACT. – We formulate a conjecture that is strictly equivalent to the conjunction of Schanuel's conjecture and the multiplicative Zilber–Pink conjecture.

KEYWORDS. – Schanuel's conjecture, Zilber–Pink conjecture.

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1. INTRODUCTION

The Zilber–Pink conjecture (ZP; see Conjecture 2.3) is a finiteness conjecture in diophantine geometry. It is a descendant of the Mordell conjecture ([19]; theorem of Faltings [12]), includes the Mordell–Lang conjecture [21] and the André–Oort conjecture [2, 20, 23] as very special cases, and is widely open.

The conjecture has three independent sources and was arrived at in somewhat different forms and by different paths. Bombieri–Masser–Zannier [6, 7, 9] studied certain problems of Schinzel and were led to formulate the conjecture [8] in the setting of $\mathbb{G}_{\mathrm{m}}^n$. Zilber [33], studying the model theory of complex exponentiation and Schanuel's conjecture ([17]; see Conjecture 3.1), the over-arching conjecture in the transcendence theory of the exponential function, formulated it in the setting of semi-abelian varieties, while Pink [24, 25] formulated the conjecture now known as ZP in the general setting of a mixed Shimura variety. ZP has subsequently been generalized to variations of mixed Hodge structure (see [16] and [5, Conjecture 4.9]; see also [4] mentioning unpublished notes [11] of de Jong). Looking back, one finds precursor conjectures and results on "unlikely intersections" in the work of Schinzel [26], André ([1]; see also [2]), and Zhang [32]. On ZP, generally see [22, 31].

Here we consider the context in which Zilber came to his formulation. In [33], Zilber formulated his "Conjecture on Intersections with Tori" (CIT) for multiplicative groups, which we will call Multiplicative Zilber–Pink (MZP; Conjecture 2.2), and more generally for semi-abelian varieties. He showed that, in combination with Schanuel's conjecture, MZP implies a stronger "uniform" version of Schanuel's conjecture (USC;

Conjecture 6.1), capable of expression in a suitable first-order model-theoretic setting for the complex exponential function. Kirby and Zilber [15] further explore the model-theoretic implications of MZP and show that, while SC is not first-order expressible, a variant Schanuel's conjecture "over the kernel" (see Conjecture 3.2) is first-order expressible if and only if MZP is true. On the other hand, Zilber [33] already establishes that USC implies statements of a similar flavour to MZP (CIT "with parameters").

The purpose of this paper is to formulate a conjecture that is strictly equivalent to the conjunction of SC and MZP.

Ax [3] proved the analogue of SC in differential fields and related settings. This result, in its various forms, goes by the name "Ax–Schanuel" (AS; see Theorems 3.4 and 3.5). Since Ax–Schanuel is a theorem, including (a weak version of) Ax–Schanuel as well does not alter the strength of our statement as a conjecture.

Our conjecture is formulated in terms of the function $e(z) = \exp(2\pi i z)$, rather than in terms of exp, for reasons explained below in Section 4. We let $\omega = 2\pi i$ and define $E(z) = (e(z), e'(z)) = (e(z), \omega e(z))$. We will use exp, e, and E also to refer to their cartesian powers:

$$\exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n, \quad e : \mathbb{C}^n \to (\mathbb{C}^\times)^n, \quad E : \mathbb{C}^n \to (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n.$$

By a *special subvariety* of $\mathbb{C}^n$, we will mean a linear subvariety defined over $\mathbb{Q}$. For a special subvariety $L \subset \mathbb{C}^n$, we denote by $\langle\langle L \rangle\rangle$ the $\overline{\mathbb{Q}}$-Zariski closure of $E(L)$, that is, the smallest subvariety of $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$ that contains $E(L)$ and is defined over $\overline{\mathbb{Q}}$. Then $\langle\langle L \rangle\rangle$ is a *torsion coset* in $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$, meaning that it is a coset of a subtorus by a torsion point, and we have $\dim\langle\langle L \rangle\rangle = \dim L + 1$ (we always assume that varieties are non-empty, and we identify them with their sets of complex points).

Let π_{a} be the projection of $\mathbb{C}^n \times \langle\langle \mathbb{C}^n \rangle\rangle$ onto $\mathbb{C}^n$, and π_{m} the projection onto $\langle\langle \mathbb{C}^n \rangle\rangle$. Let $\Gamma \subset \mathbb{C}^n \times \langle\langle \mathbb{C}^n \rangle\rangle$ denote the graph of E. We will also write $\Gamma = \Gamma_L$ for the graph of the restriction of E to $L \to \langle\langle L \rangle\rangle$, for any special $L \subset \mathbb{C}^n$.

1.1 Conjecture (Combined Schanuel, multiplicative Zilber–Pink (CSMZP)). *Let $M \subset \mathbb{C}^n$ be a special subvariety. Let $W \subset \langle\langle M \rangle\rangle$ be a subvariety defined over $\overline{\mathbb{Q}}$, not contained in $\langle\langle L \rangle\rangle$ for any proper special subvariety $L \subset M$. There is a finite set $\mathcal{L}(W)$ of proper special subvarieties $L \subset M$ with the following property. Let $V \subset M \times \langle\langle M \rangle\rangle$ be an algebraic subvariety, defined over $\overline{\mathbb{Q}}$, with $\pi_{\mathrm{m}}(V) = W$. Let $V' \subset V$ be the proper subvariety where the fibres of $V \to W$ have dimension exceeding* $\dim V - \dim W$. *Let $A \subset V \cap \Gamma$ be complex analytically irreducible component with $A \not\subset V'$. Then*

$$\dim A = \dim V + \dim \Gamma - \dim M \times \langle\langle M \rangle\rangle = \dim V - \dim\langle\langle M \rangle\rangle$$

unless $\pi_{\mathrm{m}}(A) \subset \langle\langle L \rangle\rangle$ for some $L \in \mathcal{L}(W)$.

The dimension of A cannot be smaller [18, III.4.6], so the conjecture expresses that atypically large intersections between Γ and algebraic varieties over $\overline{\mathbb{Q}}$ are accounted for by special subvarieties. We will prove the following result.

1.2 THEOREM. *The conjunction SC + MZP implies, and is implied by, CSMZP.*

The paper is organized as follows. In Section 2, we state MZP and recall some known results about it. In Section 3, we state Schanuel's conjecture and Ax–Schanuel in various forms. In Section 4, we explicate the corresponding formulations involving $e(z)$, and then in Section 5, we prove Theorem 1.2. Finally, in Section 6, we state USC and make some observations.

It is a pleasure for me to dedicate this paper to Enrico Bombieri. Enrico is an inspiring figure whose profound work has shaped many areas of mathematics. In diophantine geometry, his influence is deep and pervasive: highlights include his work on diophantine approximation, on the Mordell conjecture, his immediate-classic book with Gubler, his deep work on G-functions which completes the programme sketched by Siegel [29] and forms a bridge to the work of André [2], and his wonderful papers with David Masser and Umberto Zannier on "unlikely intersections".

I was fortunate to collaborate with Enrico very early in my career. Our 1989 paper [10] set my work in a direction that flourished following its connections with model theory and its application to the "unlikely intersection" problems pioneered by Enrico. I am further fortunate to have had his guidance over the years, so for me he has been an inspiration at a personal level as well. I dedicate this paper to Enrico with admiration, with appreciation, and with affection.

2. MULTIPLICATIVE ZILBER–PINK

Zilber originally formulated this conjecture for varieties $W \subset (\mathbb{C}^\times)^n$ defined over $\overline{\mathbb{Q}}$ (even over $\mathbb{Q}$), this being the statement required to deduce USC from SC. A similar conjecture was formulated a little later by Bombieri–Masser–Zannier [8]. In their version W, is defined over $\mathbb{C}$. They later showed [9] that the version over $\overline{\mathbb{Q}}$ implies the version over $\mathbb{C}$, indeed in a very precise sense.

So whether we restrict to W defined over $\overline{\mathbb{Q}}$ or not is immaterial, but we do so as this is the statement that follows readily from CSMZP.

2.1 DEFINITION. Let $T \subset (\mathbb{C}^\times)^n$ be a torsion coset and $W \subset T$ an irreducible algebraic subvariety defined over $\mathbb{C}$. A subvariety $A \subset V$ is called *atypical* for W in T if there is a torsion coset $S \subset T$ such that $A \subset S \cap W$ and

$$\dim A > \dim W + \dim S - \dim T.$$

The union of all atypical subvarieties of W in T we call $\mathrm{Atyp}_T(W)$ or just $\mathrm{Atyp}(W)$ or W^{atyp} if T is implicit.

2.2 Conjecture (Multiplicative Zilber–Pink (MZP)). *Let $T \subset (\mathbb{C}^\times)^n$ be a torsion coset and $W \subset T$ an irreducible algebraic subvariety defined over $\bar{\mathbb{Q}}$. Then $\mathrm{Atyp}_T(W)$ is a finite union of atypical subvarieties.*

If $A \subset W$ is an atypical subvariety, then so is the component of the relevant $W \cap S$ containing it. So $\mathrm{Atyp}_T(W)$ is a union of such components. The point is that there are countably infinitely many possibilities for such components, but, conjecturally, finitely many suffice to give the union of all of them. An equivalent formulation is that there are only finitely many maximal atypical subvarieties (for W in T). If W is contained in a torsion coset S properly contained in T, then W itself is atypical for W in T so that $\mathrm{Atyp}_T(W) = W$ and the conjecture holds trivially.

In general, the Zilber–Pink conjecture is framed in the setting of mixed Shimura varieties, each of which (if of positive dimension) is equipped with a countably infinite collection of special subvarieties.

2.3 Conjecture (Zilber–Pink (ZP)). *Let S be a mixed Shimura variety and $W \subset S$ an irreducible algebraic subvariety defined over $\mathbb{C}$. Call a subvariety $A \subset W$* atypical *if there is a special subvariety T of S such that $A \subset T \cap W$ and* $\dim A > \dim W + \dim B - \dim S$. *Denote by $\mathrm{Atyp}_S(W)$ the union of all atypical subvarieties $A \subset W$. Then $\mathrm{Atyp}_S(W)$ is a finite union of atypical subvarieties.*

2.4 Proposition. *MZP is equiavlent to the statement that, for $W \subset T$, the atypical subvarieties of W in T are contained in the union of $W \cap S$ for S running over some* finite *set of torsion cosets S properly contained in T.*

Proof. Over $\mathbb{C}$ this is proved in [9]. It easy to check that it goes through over $\bar{\mathbb{Q}}$. ∎

We will make use of an equiavlent formulation of MZP from [13]. This is formulated in terms of "optimal" subvarieties of a given $W \subset (\mathbb{C}^\times)^n$. The same formulation can be made for any mixed Shimura variety (and any abelian variety; see [13]).

For a subvariety $A \subset (\mathbb{C}^\times)^n$, we let $\langle A\rangle$ denote the smallest torsion coset containing A and denote by $\delta(A) = \dim\langle A\rangle - \dim A$ the *defect* of A.

2.5 Definition. Let $V \subset (\mathbb{C}^\times)^n$. A subvariety $A \subset V$ is called *optimal* for V if there is no subvariety $B \subset V$ with $A \subset B$, $A \neq B$, and $\delta(B) \le \delta(A)$.

It is shown in [13] that, taken over all subvarieties of $(\mathbb{C}^\times)^n$ for a given n (and indeed for all mixed Shimura varieties), ZP is equiavlent to the following conjecture.

2.6 Conjecture. *For $V \subset (\mathbb{C}^\times)^n$, the set* $\mathrm{Opt}(V)$ *of optimal subvarieties is finite.*

We note that V itself is always optimal as a subvariety of itself, and that all other optimal subvarieties must be atypical. We let $\mathrm{Opt}^0(V)$ denote the set of optimal points of V, i.e., individual points which are optimal in the above sense. Embedded in the proofs in [13] is a proof of a further equivalence.

2.7 Proposition. *For given n, ZP for $(\mathbb{C}^\times)^m$ for all $m \leq n$ (and respectively for $Y(1)^m$ for all $m \leq n$) is equivalent to the statement that* $\mathrm{Opt}^0(V)$ *is a finite set for any subvariety $V \subset (\mathbb{C}^\times)^m$ for $m \leq n$ (respectively for $V \subset Y(1)^m$ for $m \leq n$).*

Proof. This is established though not stated in [13] in the case of $Y(1)^n$ for subvarieties defined over $\mathbb{C}$, but the equivalence holds also for subvarieties defined over $\overline{\mathbb{Q}}$, and also for $(\mathbb{C}^\times)^n$. ■

2.8 Proposition. *For $(\mathbb{C}^\times)^m$ (and also for $Y(1)^m$) for all $m \leq n$, ZP is equivalent to the following statement: for a subvariety $W \subset (\mathbb{C}^\times)^n$, defined over $\overline{\mathbb{Q}}$, there are finitely many proper torsion cosets $T_i \subset (\mathbb{C}^\times)^n$ such that every $P \in$* $\mathrm{Opt}^0(V)$ *is contained in one of the T_i.*

Proof. The finiteness $\mathrm{Opt}^0(V)$ clearly implies the other statement. For the converse implication, suppose first that W is not contained in any proper torsion coset. Then we find that each $P \in \mathrm{Opt}^0(V)$ is contained in some $W_i = W \cap T_i$. Suppose that W is contained in some proper torsion coset. Using a monomial transformation, W is transformed to some $W' \subset (\mathbb{C}^\times)^k$, $k < n$, and then the finite number of proper torsion cosets transforms back to proper torsion cosets $T_i \subset T$. So we may complete the proof by induction. ■

For other equivalences, see [9]. It is shown in [9] that MZP as above for W defined over $\overline{\mathbb{Q}}$ implies the same statement for W defined over $\mathbb{C}$.

3. Schanuel and Ax–Schanuel

If A is a set contained in some field containing $\mathbb{Q}$, then by $\mathrm{tr\,deg}(A)$ we mean $\mathrm{tr\,deg}_{\mathbb{Q}}\, \mathbb{Q}(A)$. If $z = (z_1, \ldots, z_n)$, then $e(z)$ means $(e(z_1), \ldots, e(z_n))$, and similarly for exp and E.

3.1 Conjecture (Schanuel (SC)). *Suppose $z_1, \ldots, z_n \in \mathbb{C}$. Then*

$$\mathrm{tr\,deg}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n$$

unless $z_1, \ldots, z_n$ are linearly dependent over $\mathbb{Q}$.

For a set $A \subset \mathbb{C}^n$, we denote by $\overline{\mathbb{Q}}$-dim(A) the dimension of the smallest algebraic set V which is defined over $\overline{\mathbb{Q}}$ and contains A. Such V need not be irreducible and we take its dimension to be the maximum dimension of its components. We define $\mathrm{lin\,dim}(A)$ to be the dimension of the smallest rational subspace of $\mathbb{C}^n$ containing A.

If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a point, then $\overline{\mathbb{Q}}$-dim$(z) = \mathrm{tr\,deg}(z)$ and an equiavlent statement of SC is as follows: *If $z \in \mathbb{C}^n$, then $\overline{\mathbb{Q}}$-dim$(z, \exp(z)) \geq \mathrm{lin\,dim}(z)$.*

We now give the statement of Schanuel's conjecture over the kernel following [15]. Here, $\mathrm{tr\,deg}(A/B) = \mathrm{tr\,deg}_{\mathbb{Q}(B)} \mathbb{Q}(A \cup B)$, and $\mathrm{lin\,dim}(A/B)$ is the $\mathbb{Q}$-dimension of the quotient of the $\mathbb{Q}$-vector space spanned by $A \cup B$ by the subspace spanned by B.

3.2 Conjecture (Schanuel over the Kernel). *Let $z_1, \ldots, z_n \in \mathbb{C}$. Then*

$$\mathrm{tr\,deg}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}/2\pi i) \geq \mathrm{lin\,dim}(z_1, \ldots, z_n/2\pi i).$$

We now state the differential version of Ax–Schanuel. Let K be a differential field of characteristic zero with commuting derivations $D_1, \ldots, D_m$. Let $C = \bigcap_j \mathrm{Ker}\, D_j$. By "rank" below we mean rank over K.

3.3 Definition. Elements $x_1, \ldots, x_n \in K$ are called *linearly independent over $\mathbb{Q}$ modulo C* if there is no non-trivial relation $\sum_i q_i x_i = c$ where $q_i \in \mathbb{Q}$ (the non-triviality meaning they are not all zero), and $c \in C$.

3.4 Theorem (Differential Ax–Schanuel; Ax [3]). *Let $x_i, y_i \in K^\times$, $i = 1, \ldots, n$, with $D_j y_i = y_i D_j x_i$ for all i, j. Then*

$$\mathrm{tr\,deg}_C\, C(x_1, \ldots, y_1, \ldots, y_n) \geq n + \mathrm{rank}(D_j x_i)$$

unless the x_i are linearly independent over $\mathbb{Q}$ modulo C (or, equivalently, the y_i are multiplicatively independent over C). ∎

We consider the uniformisation $\exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n$. We let $\Gamma \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ denote the graph of exp. We let $\pi_{\mathrm{a}} : \mathbb{C}^n \times (\mathbb{C}^\times)^n \to \mathbb{C}^n$ be the projection onto the first factor, and π_{m} the projection onto the second factor.

Via the Seidenberg embedding theorem [27, 28], the above is equivalent to the following version (see [14, 30]).

3.5 Theorem (Complex Ax–Schanuel [30, Theorem 1.3]). *Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety, and let A be a complex analytically irreducible component of $V \cap \Gamma$. Then*

$$\dim V = \dim A + n$$

unless $\pi_{\mathrm{a}}(A)$ is contained in a translate of a proper rational subspace of $\mathbb{C}^n$ (or equivalently $\pi_{\mathrm{m}}(A)$ is contained in a coset of a proper subtorus of $(\mathbb{C}^\times)^n$). ∎

Note that the dimension of such a component A can never be less than the "expected" dimension

$$\dim V + \dim \Gamma - \dim(\mathbb{C}^n \times \big(\mathbb{C}^\times)^n\big) = \dim V - n$$

(see [18, III.4.6]), and the burden of the theorem is that it can never be bigger except in the case described.

3.6 Conjecture (Schanuel plus $\overline{\mathbb{Q}}$-Ax–Schanuel (S$\overline{\mathbb{Q}}$AS)). *Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety defined over $\overline{\mathbb{Q}}$, and let A be a complex analytically irreducible component of $V \cap \Gamma$. Then*

$$\dim V = \dim A + n$$

unless $\pi_{\mathrm{a}}(A)$ is contained in a proper rational subspace of $\mathbb{C}^n$.

An equivalent formulation (cf. [30, Theorem 1.2]): *Let Γ be the graph of some cartesian power of* exp, *and $A \subset \Gamma$ an irreducible complex analytic subvariety of $V \cap \Gamma$. Then* $\overline{\mathbb{Q}}$-dim$(A) \geq \dim A + \operatorname{lin dim}(\pi_{\mathrm{a}}(A))$.

3.7 Remarks. (1) One could formulate a version for an arbitraty rational subspace $M \subset \mathbb{C}^n$.

(2) It is unclear whether S$\overline{\mathbb{Q}}$AS formally implies Ax–Schanuel. Since the latter is a theorem, this does not affect the strength of the former as a conjecture.

3.8 Proposition. *S$\overline{\mathbb{Q}}$AS is equivalent to SC.*

Proof. This is the same as the corresponding result for formal power series [3, Theorem 2]. Taking A to be a point and V its $\overline{\mathbb{Q}}$-Zariski closure, we see that S$\overline{\mathbb{Q}}$AS implies SC. For the other implication, assume SC. Suppose that the functions $z_1, \dots, z_n$ on A are linearly independent over $\mathbb{Q}$. We may assume that $d = \dim A > 0$, or the conclusion is just SC. We can assume (after possible re-indexing) that $z_1, \dots, z_p$ are a $\mathbb{Q}$-basis for $z_1, \dots, z_n$ modulo $\mathbb{C}$. That is, we have

$$z_i = \sum_{j=1}^{p} q_{ij} z_j + c_i, \quad q_{ij} \in \mathbb{Q}, \quad c_i \in \mathbb{C}, \quad i = 1, \dots, n,$$

but $z_1, \dots, z_p$ are independent over $\mathbb{Q}$ modulo $\mathbb{C}$ on A. We let D_j be partial differentiation with respect to z_{i_j}, $j = 1, \dots, d$, for some maximal algebraically independent set z_{i_j} among $z_1, \dots, z_p$. Let $C = \mathbb{Q}(c_i, e^{c_i}, i = p+1, \dots, n)$. Let $F = \{z_1, \dots, z_p, e^{z_1}, \dots, e^{z_p}\}$, considered as a set of functions on A. By Theorems 3.4 (Differential Ax–Schanuel), we have

$$\operatorname{tr deg}_{\mathbb{C}} \mathbb{C}(F) \geq p + d$$

and hence the same holds for $\operatorname{tr deg}_C C(F)$. We observe that $C(F) = \mathbb{Q}(F)$. By our assumptions, $c_{p+1}, \ldots, c_n$ are linearly independent over $\mathbb{Q}$. Hence, by SC $\operatorname{tr deg}_{\mathbb{Q}} C \geq n - p$, and the conclusion follows. ∎

4. Schanuel and Ax–Schanuel for $e(z)$

The fact that the pre-images under exp of special subvarieties in $(\mathbb{C}^\times)^n$ are not defined over $\overline{\mathbb{Q}}$ causes some problems with framing a uniform version of SC which directly implies MZP (see Section 6 below).

A clearer picture results from working with the function $e(z) = \exp(2\pi i z)$ rather than exp because special subvarieties in $\mathbb{C}^n$ are defined over $\overline{\mathbb{Q}}$.

The price for this seems to be to have to work with the derivative as well (note that the lowest order algebraic differential equation over $\overline{\mathbb{Q}}$ satisfied by $e(z)$ is of second order).

Thus, we set $\omega = 2\pi i$ and consider the map

$$E : \mathbb{C}^n \to (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n, \quad E(z) = \big(e(z), \omega e(z)\big)$$

where $z = (z_1, \ldots, z_n), e(z) = (e(z_1), \ldots, e(z_n))$.

We have the following natural analogue of Schanuel's conjecture, which we show is equivalent to SC.

4.1 Conjecture (Schanuel for $e(z)$ (SC_e)). *Let L be a special subvariety of $\mathbb{C}^n$ and $z \in L$. Then*

$$\operatorname{tr deg}\big(z, e(z), \omega e(z)\big) \geq \dim L + 1$$

unless z lies in a proper special subvariety of L.

Denote by $\langle B\rangle$ the smallest special subvariety of $\mathbb{C}^n$ containing $B \subset \mathbb{C}^n$. Then SC_e may be equivalently formulated: *If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then* $\overline{\mathbb{Q}}$-$\dim\{(z, E(z))\} \geq \dim\langle\{z\}\rangle + 1$.

4.2 Proposition. *SC implies* SC_e.

Proof. We assume SC, let $L \subset \mathbb{C}^n$ be a special subvariety, and $z \in L$. We put $y = \omega z$. If $\operatorname{lin dim}(z, 1) < \dim L + 1$, then z lies in a proper special subvariety. So we assume that $\operatorname{lin dim}(z, 1) = \dim L + 1$. Then also $\operatorname{lin dim}(y, \omega) = \dim L + 1$ and by SC we have

$$\dim L + 1 \leq \operatorname{tr deg}\big(y, \omega, e^y, 1\big) = \operatorname{tr deg}\big(\omega z, e(z), \omega\big) = \operatorname{tr deg}\big(z, e(z), \omega e(z)\big).$$

This gives SC_e. ∎

4.3 Proposition. SC_e *implies SC.*

Proof. We assume SC_e and suppose that $z_1, \dots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$. Then so are $y_i = z_i/\omega$. We have two cases.

In the first case, $y_1, \dots, y_n, 1$ are l.i. over $\mathbb{Q}$. Then y does not lie in any proper special subvariety of $\mathbb{C}^n$, and by SC_e we have

$$n + 1 \le \operatorname{tr deg}\big(y, e(y), \omega\big) = \operatorname{tr deg}(z/\omega, e^z, \omega) = \operatorname{tr deg}(z, e^z, \omega).$$

Then $\operatorname{tr deg}(z, e^z) \ge n$.

Otherwise, $y_1, \dots, y_n, 1$ are linearly dependent over $\mathbb{Q}$ though the y_i are independent. By SC_e, we have

$$n \le \operatorname{tr deg}\big(y, e(y), \omega\big) = \operatorname{tr deg}(z, e^z, \omega).$$

But ω is linearly dependent, hence algebraically dependent on $z_1, \dots, z_n$, and so $n \le \operatorname{tr deg}(z, e^z)$. ■

4.4 Remark. There is another SC variant for e which is equivalent to SC: *For* $z \in \mathbb{C}^n$, $\operatorname{tr deg}(z, e(z), \omega e(z)) \ge \operatorname{lin dim}(z) + 1$.

We now turn to Ax–Schanuel for $e(z)$. The reformulations are very straightforward. If we fix $\lambda \in \mathbb{C}^\times$ in our differential field, we have an obvious analogue of Differential Ax–Schanuel for the corresponding equation $Dy = \lambda y Dx$, and the other hypotheses and the conclusion are exactly the same (and the proof is simply by applying Differential Ax–Schanuel to λx_i and y_i).

A first version of Complex Ax–Schanuel for $e(z)$ (or any function $\exp(\lambda z)$) is exactly the same as for $\exp(z)$, that is, Theorem 3.5, with Γ taken to be the graph of $e(z)$. However, we want to have a version in which the ambient is $\mathbb{C}^n \times \langle\langle \mathbb{C}^n \rangle\rangle$ rather than $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, and then there is one new exceptional case that must be observed.

In the following, we let $\overline{\Gamma}$ be the graph of E in $\mathbb{C}^n \times \langle\langle \mathbb{C}^n \rangle\rangle$, to distinguish it from the graph of e, and use $x_i, y_i, i = 1, \dots, n$, for the variables in $\langle\langle \mathbb{C}^n \rangle\rangle \subset (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$.

4.5 Proposition (Complex Ax–Schanuel for $e(z)$). *Let* $\overline{V} \subset \mathbb{C}^n \times \langle\langle \mathbb{C}^n \rangle\rangle$ *be an irreducible algebraic subvariety, and let* $\overline{A}$ *be a complex analytically irreducible component of* $\overline{V} \cap \overline{\Gamma}$. *Then*

$$\dim \overline{V} = \dim \overline{A} + (n + 1)$$

unless $\pi_a(\overline{A})$ *is contained in a translate of a proper rational subspace of* $\mathbb{C}^n$, *or the equations* $y_i = \omega x_i$, $i = 1, \dots, n$, *hold identically on* $\overline{V}$.

PROOF. Let V be the image of $\overline{V}$ under projection to $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, and A the image of $\overline{A}$. Note that Γ is the image of $\overline{\Gamma}$ under the same projection. We suppose that $\pi_a(\overline{A})$ is not contained in a translate of a proper rational subspace of $\mathbb{C}^n$. Then neither is $\pi_a(A) = \pi_a(\overline{A})$.

Suppose that $\dim V = \dim \overline{V} - 1$, so that the projection has fibres of dimension 1 (generically and hence everywhere). We have $A \subset V \cap \Gamma$. Also, $\dim A = \dim \overline{A}$ as the two sets are analytically isomorphic. Thus, using Complex Ax–Schanuel in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, we have

$$\dim \overline{A} = \dim A \le \dim V - n = \dim \overline{V} - (n+1)$$

and hence we have equality (and A is a component of $V \cap \Gamma$).

Suppose that $\dim V = \dim \overline{V}$, so that the fibres of $\overline{V} \to V$ are generically finite. Let $B \subset V \cap \Gamma$ be a component containing A. We claim that B is Zariski dense in V. Otherwise, if its Zariski closure W has $\dim W < \dim V$, then B would have atypically large dimension as a component of $V \cap \Gamma$, and then $\pi_a(B)$ would be contained in a translate of a proper rational subspace, and then this would also hold for A, contrary to our assumptions.

If A is not a component, then $\dim A < \dim B = \dim V - n$, and we get the result we want. So suppose that $A = B$ is a component of $V \cap \Gamma$. Then $\overline{V}$, which is generically finite over V, satisfies, generically over B, the equations $y_i = \omega x_i$. But B is Zariski dense in V, so these equations are satisfied identically. ∎

We now remove the overlines; in particular, we revert to using Γ to denote the graph of E.

4.6 CONJECTURE (Schanuel plus $\overline{\mathbb{Q}}$-Ax–Schanuel for $e(z)$. (S$\overline{\mathbb{Q}}$AS$_e$)). *Let $V \subset \mathbb{C}^n \times \langle\langle \mathbb{C}^n \rangle\rangle$ be an irreducible algebraic subvariety defined over $\overline{\mathbb{Q}}$, and let A be a complex analytically irreducible component of $V \cap \Gamma$. Then*

$$\dim V = \dim A + (n+1)$$

unless $\pi_a(A)$ is contained in a proper special subvariety of $\mathbb{C}^n$.

Thus, S$\overline{\mathbb{Q}}$AS$_e$ may be equivalently formulated: *If $A \subset \Gamma_e$ is an irreducible complex analytic subvariety of Γ, then* $\overline{\mathbb{Q}}$-dim$(A) \ge \dim A + \dim\langle\langle \pi_a(A) \rangle\rangle$.

4.7 PROPOSITION. *S$\overline{\mathbb{Q}}$AS$_e$ is equiavlent to* SC$_e$ *(and hence to SC).*

PROOF. This is just like the proof of Proposition 3.8, after we observe that no V defined over $\overline{\mathbb{Q}}$ can satisfy identically the equations $y_i = \omega x_i$, for any i. Taking A to be a point and V its $\overline{\mathbb{Q}}$-Zariski closure one deduces SC_e. For the other implication, we assume SC_e. Suppose that $\pi_{\mathrm{a}}(A)$ is not contained in a proper special subvariety of $\mathbb{C}^n$, that is, that the function $z_1, \dots, z_n$ on A are linearly independent over $\mathbb{Q}$ modulo $\mathbb{Q}$. We may assume that $d = \dim A > 0$ or the statement is just SC_e. We may suppose (up to possible re-indexing) that $z_1, \dots, z_p$ are a $\mathbb{Q}$-basis for $z_1, \dots, z_n$ modulo $\mathbb{C}$, so we have

$$z_i = \sum_{j=1}^{p} q_{ij} z_j + c_i, \quad q_{ij} \in \mathbb{Q}, \quad c_i \in \mathbb{C}, \quad i = p+1, \dots, n,$$

with the c_i linearly independent over $\mathbb{Q}$ modulo $\mathbb{Q}$, while $z_1, \dots, z_p$ are linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$. The rest of the proof is the same as that of Proposition 3.8, as SC_e contributes a "+1" to transcendence degree compared to SC. ∎

5. Atypical intersections with the graph of e

Now suppose that $M \subset \mathbb{C}^n$ is special, and let $\Gamma = \Gamma_M \subset M \times \langle\langle M \rangle\rangle$ be the graph of E restricted to M.

Suppose $V \subset M \times \langle\langle M \rangle\rangle$ is an algebraic variety of dimension $\dim V \le \dim M$, defined over $\overline{\mathbb{Q}}$. Suppose $(z, E(z)) \in V \cap \Gamma$. Then, according to SC_e, the point z lies in a proper special subvariety of M.

We let W be the image of V under projection to $\langle\langle M \rangle\rangle$. We suppose that the generic fibre dimension of the projection is $d = \dim V - \dim W$, and that the fibre over $w \in W$ has this dimension outside the proper (closed) subvariety W', and that V' is the pre-image of W'. Then $V' \subset V$ is also proper (by the fibre dimension theorem, see e.g. [8]).

We emulate the argument (in [33, Proof of Proposition 5]) by which Zilber formulated his uniform Schanuel conjecture. He shows that a point $(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$ in the intersection of a variety $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ defined over $\overline{\mathbb{Q}}$ and of dimension $\dim V < n$ with the graph of exponentiation leads (under SC) to an atypical intersection of the projection of V with a subtorus of $(\mathbb{C}^\times)^n$ containing $(e^{z_1}, \dots, e^{z_n})$. We show that, similarly, under SC, atypical intersections $V \cap \Gamma$ lead to atypical intersections of W with $\langle\langle M \rangle\rangle$.

5.1 Proposition. *Assume $\mathrm{S}\overline{\mathbb{Q}}\mathrm{AS}_e$. Let $M \subset \mathbb{C}^n$ be a special subvariety. Suppose that $V \subset M \times \langle\langle M \rangle\rangle$ is a subvariety defined over $\overline{\mathbb{Q}}$. Let $A \subset V \cap \Gamma$ be a complex analytically irreducible component with $A \not\subset V'$ and*

$$\dim A > \dim V - \dim\langle\langle M \rangle\rangle.$$

Then $\pi_{\mathrm{m}}(A) \subset B$ for some atypical component B of $W \cap \langle\langle L \rangle\rangle$ for some proper special subvariety $L \subset M$.

Proof. Let $L = \langle \pi_{\mathrm{a}}(A) \rangle$. By $\mathrm{S}\overline{\mathbb{Q}}\mathrm{AS}_e$, L is a proper special subvariety of M. Moreover, also by $\mathrm{S}\overline{\mathbb{Q}}\mathrm{AS}_e$, $\overline{\mathbb{Q}}$-dim $A = \dim A + \dim\langle\langle L \rangle\rangle$. Now $E(A) \subset B$ for some component of $W \cap \langle\langle L \rangle\rangle$ and so $\overline{\mathbb{Q}}$-dim $A \le \dim B + d$. Combining, we find

$$\dim\langle\langle L \rangle\rangle \le \dim B + \dim V - \dim W - \dim A$$

and since $\dim V - \dim A - \dim\langle\langle M \rangle\rangle < 0$ we get

$$\dim B > \dim W + \dim\langle\langle L \rangle\rangle - \dim\langle\langle M \rangle\rangle$$

and the intersection is atypical. ■

5.2 Proposition. *MZP + SC implies CSMZP.*

Proof. Let $M \subset \mathbb{C}^n$ be a special subvariety, and let $W \subset \langle\langle M \rangle\rangle$ be a subvariety defined over $\overline{\mathbb{Q}}$, not contained in $\langle\langle L \rangle\rangle$ for any proper special $L \subset M$.

First suppose that W is not contained in any proper torsion coset in $\langle\langle M \rangle\rangle$. By MZP, there are finitely many proper torsion cosets T_i with the property that every atypical component $B \subset W \cap T$ with a torsion coset T is contained in an atypical component of $W \cap T_i$ for some T_i. Moreover, since W is not contained in a proper torsion coset, the T_i have codimension at least 2. Thus, each T_i is defined by at least 2 independent multiplicative conditions on $x_1, \ldots, x_n, y_1, \ldots, y_n$.

However, the equations $x_i y_j = x_j y_i$, for all i, j, hold in $\langle\langle L \rangle\rangle$. With these we can eliminate all the y_i to find that T_i imposes a new multiplicative condition $x_1^{c_1} \ldots x_n^{c_n} = \zeta$ on $x_1, \ldots, x_n$.

Let L_i be a special subvariety of M formed by imposing a corresponding condition $\sum c_i z_i = q$ where $e(q) = \zeta$, and let $\mathcal{L}(W) = \{L_i\}$.

Now suppose that V and A are as in the statement of CSMZP, with the genericity condition, and that $\dim A > \dim V - \dim\langle\langle M \rangle\rangle$. By Proposition 5.1, $\pi_{\mathrm{m}}(A)$ is contained in an atypical intersection of W with $\langle\langle L \rangle\rangle$.

Hence, $\pi_{\mathrm{m}}(A)$ lies in $\langle\langle L_i \rangle\rangle$ for some L_i.

Now suppose that W is contained in a proper torsion coset $T \subset \langle\langle M \rangle\rangle$. Then T must be of codimension 1; otherwise, we could eliminate the dependencies on the y_i and find $W \subset \langle\langle L \rangle\rangle$ for some proper special $L \subset M$. By MZP, atypical intersections of W with torsion cosets of T lie in some finitely many proper torsion cosets Q_i, again of codimension at least 2. Each of these is contained in $\langle\langle L_i \rangle\rangle$ for some proper special $L_i \subset M$, and the rest of the proof is analogous to the above, except it is now Zilber's version [33, Proof of Proposition 5] of Proposition 5.1 assuming SC that gives the conclusion that $\pi_{\mathrm{m}}(A)$ is atypical and so $A \subset Q_i$ for some i. ■

5.3 Proposition. *CSMZP implies* SC_e.

Proof. We will assume CSMZP and deduce S$\overline{\mathbb{Q}}$AS$_e$. Let then $V \subset \mathbb{C}^n \times \langle\langle\mathbb{C}^n\rangle\rangle$ be defined over $\overline{\mathbb{Q}}$, and $W = \pi_{\mathrm{m}}(V)$. Let A be a component of $V \cap \Gamma$. We may assume that $A \not\subset V'$; otherwise, we replace V by V'. Suppose

$$\dim A > \dim V - (n+1).$$

By CSMZP, $\pi_{\mathrm{m}}(A) \subset \langle\langle L\rangle\rangle$ for some proper special $L \subset \mathbb{C}^n$. Thus, $\pi_{\mathrm{a}}(A) \subset L + k$ for some $k \in \mathbb{Z}$. ∎

5.4 Proposition. *CSMZP implies MZP.*

Proof. Suppose $W \subset (\mathbb{C}^\times)^n$ is defined over $\overline{\mathbb{Q}}$, and not contained in any proper torsion coset. Define $\overline{W} \subset \langle\langle\mathbb{C}^n\rangle\rangle$ as the subset defined by the equations for W combined with the equations

$$x_i y_j = x_j y_i, \quad i, j = 1, \ldots, n.$$

Then $\dim \overline{W} = \dim W + 1$, and $\overline{W}$ is not contained in $\langle\langle M\rangle\rangle$ for any proper special $M \subset \mathbb{C}^n$. Let $\mathcal{L}(\overline{W})$ be the set of proper special subvarieties of $\langle\langle\mathbb{C}^n\rangle\rangle$ afforded by CSMZP.

Suppose $B \subset W \cap T$ is an atypical component of $W \cap T$, where T is a torsion coset. Let $L \subset \mathbb{C}^n$ be a special subvariety with $e(L) = T$. Let $V = L \times \overline{W} \subset \mathbb{C}^n \times \langle\langle\mathbb{C}^n\rangle\rangle$ so that $V' = \emptyset$.

Let A be an irreducible component of $\{(z, e(z), \omega e(z)) : z \in e^{-1}(B) \cap L\}$, and so a component of $V \cap \Gamma$ with

$$\dim A = \dim B > \dim W + \dim T - n = \dim V - (n+1).$$

By CSMZP, $\pi_{\mathrm{m}}(A) \subset \langle\langle H\rangle\rangle$ for some $H \in \mathcal{L}(W)$. Then $B \subset e(H)$.

This proves MZP for such W. Now MZP for any $W \subset (\mathbb{C}^\times)^n$ defined over $\overline{\mathbb{Q}}$ follows from this case (and as remarked MZP for all W defined over $\mathbb{C}$ follows from MZP over $\overline{\mathbb{Q}}$). ∎

6. Uniform Schanuel conjecture

Here we consider Zilber's "Uniform Schanuel conjecture", and his deduction of it from the conjunction of SC and MZP. In fact, our statement is a slight variant of the one given by Zilber [33], and our proof is organised a bit differently, our purpose being to make some remarks and to isolate one interesting lemma.

6.1 Conjecture (Uniform Schanuel (USC)). *Consider $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$, defined over $\overline{\mathbb{Q}}$ and of dimension at most $n-1$. There exists a finite set of proper rational subspaces $L \subset \mathbb{C}^n$, and a finite set of torsion cosets $Q \subset (\mathbb{C}^\times)^n$, with each Q of codimension at least 2, with the following property.*

If $(z, e^z) \in V$ but not in the exceptional set V' for the projection $V \to W = \pi_{\mathrm{m}}(V)$, then either $z \in L$ for one of the L, or $e^z \in Q$ for some Q.

6.2 PROPOSITION. *USC implies SC.*

PROOF. Let (z, e^z) be a "co-example" for SC, i.e., $\overline{\mathbb{Q}}$-dim$(z, e^z) < n$. Let V be the $\overline{\mathbb{Q}}$-Zariski closure of (z, e^z), and $W = \pi_{\mathrm{m}}(V)$. The exceptional set for $V \to W$ is defined over $\overline{\mathbb{Q}}$, so (z, e^z) does not lie in it.

Then either $z \in L$ or $e^z \in Q$ for some Q. In the latter case, let $x_1^{a_1} \dots x_n^{a_n} = 1$ and $x_1^{b_1} \dots x_n^{b_n} = 1$, where $a_i \in \mathbb{Z}$ not all zero and $b_j \in \mathbb{Z}$ not all zero, be independent equations satisfied by Q. That is, the exponent vectors are linearly independent over $\mathbb{Q}$. Then there exist integers a^*, b^* such that

$$a_1 z_1 + \dots + a_n z_n = 2\pi i a^*, \quad b_1 z_1 + \dots + b_n z_n = 2\pi i b^*.$$

Eliminate the "constant term"; since the exponent vectors are linearly independent over $\mathbb{Q}$, we find that $c_1 z_1 + \dots + c_n z_n = 0$ for rational numbers c_k, not all zero. ∎

6.3 PROPOSITION. *SC + MZP implies USC.*

PROOF. Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ defined over $\overline{\mathbb{Q}}$ and with $\dim V < n$. Let $W = \pi_{\mathrm{m}}(V) \subset (\mathbb{C}^\times)^n$. Now we want to apply MZP to W to get our finite collection of torsion cosets Q. But there is a small wrinkle.

The dimension conditions exclude $W = (\mathbb{C}^\times)^n$. Generically, W is not contained in any proper torsion coset, and so MZP provides a finite set of proper torsion cosets Q of codimension at least 2, and we are done. If W is contained in a proper torsion coset of codimension at least 2, we are also done. If W is properly contained in a proper torsion coset T of codimension exactly 1, we can apply MZP to W as a subset of T, and with a slight elaboration of the argument, we get our finitely many Q.

There remains the case that $W = T$ is a codimension 1 torsion coset, and now MZP does not offer us any possible choice of a finite set of proper torsion cosets besides T itself. Fortunately, we are saved by the following lemma. With this lemma and the slight elaboration indicated we complete the proof of Proposition 6.3. ∎

6.4 LEMMA. *Assume SC. Suppose that $z_1, \dots, z_n \in \mathbb{C}$ with $e^{a_1 z_1} \dots e^{a_n z_n} = \zeta$ for some rational numbers a_i, not all zero, and root of unity ζ. Suppose*

$$z_i \text{ is algebraic over } \mathbb{Q}(e^{z_1}, \dots, e^{z_n}), \quad i = 1, \dots, n.$$

Then $a_1 z_1 + \dots + a_n z_n = 0$ (and so $\zeta = 1$).

PROOF. Suppose not. Then we have $a_1 z_1 + \cdots + a_n z_n = z^* = 2\pi i q$ for some $q \in \mathbb{Q}^\times$. Now by our hypotheses

$$\operatorname{tr deg}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) = \operatorname{tr deg}(e^{z_1}, \ldots, e^{z_n}) \leq n - 1.$$

For any $n-1$ choices w_i from $z_1, \ldots, z_n$, we have

$$\operatorname{tr deg}(w_1, \ldots, w_{n-1}, z^*, e^{w_1}, \ldots, e^{w_{n-1}}, e^{z^*}) \leq \operatorname{tr deg}(e^{z_1}, \ldots, e^{z_n}) \leq n - 1.$$

By SC we have $b_1 w_1 + \cdots + b_{n-1} w_{n-1} + b^* z^* = 0$ for some rational numbers b_i, b^*, not all zero, and we must have some $b_i \neq 0$, as $z^* \neq 0$. So $e^{w_1}, \ldots, e^{w_{n-1}}$ are algebraically dependent over $\mathbb{Q}$, and so

$$\operatorname{tr deg}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) = \operatorname{tr deg}(e^{z_1}, \ldots, e^{z_n}) \leq n - 2.$$

For any $n-2$ choices w_i from $z_1, \ldots, z_n$, we have

$$\operatorname{tr deg}(w_1, \ldots, w_{n-2}, z^*, e^{w_1}, \ldots, e^{w_{n-2}}, e^{z^*}) \leq \operatorname{tr deg}(e^{w_1}, \ldots, e^{w_{n-1}}) \leq n - 2$$

and so by SC we have $c_1 w_1 + \cdots + c_{n-2} w_{n-2} + c^* z^* = 0$ with rational c_i, c^* and some $c_i \neq 0$ (since $z^* \neq 0$). So $e^{w_1}, \ldots, e^{w_{n-1}}$ are algebraically dependent over $\mathbb{Q}$, and

$$\operatorname{tr deg}(e^{z_1}, \ldots, e^{z_n}) \leq n - 3.$$

For any $n-3$ choices w_i from $z_1, \ldots, z_{n-1}$, we have

$$\operatorname{tr deg}(w_1, \ldots, w_{n-3}, z^*, e^{w_1}, \ldots, e^{w_{n-3}}, e^{z^*}) \leq \operatorname{tr deg}(e^{w_1}, \ldots, e^{w_{n-1}}) \leq n - 3$$

and by SC we have $d_1 w_1 + \cdots + d_{n-3} w_{n-3} + d^* z^* = 0$ for some rational d_i, d^* with some $d_i \neq 0$.

Eventually, we find that all $z_i, e^{z_i} \in \overline{\mathbb{Q}}$, whence $z^* \in \overline{\mathbb{Q}}$, which is a contradiction as z^* is a non-zero rational multiple of $2\pi i$. So we must have had $z^* = 0$ to begin with. ■

ALTERNATIVE PROOF. Suppose $(z, e^z) \in V$ with W as in the lemma, and (z, e^z) not in the exceptional set for $V \to W$. So z is algebraic over $\mathbb{Q}(e^z)$. By SC, z lies in some proper rational subspace L and so, by the analogue of Proposition 5.1 for exponentiation (i.e., Zilber's original argument in [33, Proof of Proposition 5]) e^z lies in an atypical intersection of $T = \exp(L)$ with W. But if W is a codimension 1 torsion coset, this can only happen if $\exp(L) = W$, i.e., if W is a subtorus, and z is in the unique rational subspace which maps onto it. ■

6.5 REMARK. Given V defined over $\mathbb{Q}$, the exceptional locus V' is also defined over $\mathbb{Q}$. We can also find subspaces/torsion cosets applicable to the exceptional set V', outside its exceptional locus. In this way, we generate a finite set of L and Q which work for every $(z, e^z) \in V$. This gives USC. The corresponding set of axioms is then a first-order system which implies SC. In fact, the proof shows we can be more precise. A point (z, e^z) is always outside the non-generic subvariety of its $\mathbb{Q}$-closure. So an axiom scheme, indexed by such V, can be precise about the linear condition (if W is a codimension one torsion coset), or the finitely many torsion cosets Q_i (otherwise). CSMZP leads to a similar system of axioms (involving a language for e) asserting a uniform version of SC_e.

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