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Algebra, Number Theory, Group Theory, and Algebraic Geometry. – *Airy sheaves of Laurent type: An introduction*, by Nicholas M. Katz and Pham Huu Tiep, accepted on 1 July 2025.

To Enrico Bombieri, with the utmost admiration.

Abstract. – We develop the general theory of Airy sheaves of Laurent type, the local systems whose trace functions have a particular "Airy-Laurent" shape. The main goal is to provide tools for the later determination of their monodromy groups. See [Eur. J. Math. 10 (2024), article no. 65] for instances of such determinations.

Keywords. – exponential sums, local systems, monodromy groups.

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1. Introduction

In classical analysis, Airy functions are the Fourier transforms of exponentials $e^{g(x)}$ of polynomials (originally for the polynomial $g(x) := x^3/3$), and Airy differential equations are the linear differential equations $g'(d/dt)y + ty = 0$ they satisfy. These differential equations have an irregular singularity at ∞ and have quite interesting differential Galois groups. In the seminal paper [11] of Such, he introduces their ℓ-adic finite field analogues, the local systems whose trace functions are of the form

$$t \mapsto -\sum_x \psi(g(x) + tx).$$

The local systems we are concerned with here are generalizations of these Airy local systems in several ways. We allow the "t term" tx to be replaced by tx^a, we allow the polynomial $g(x)$ to be replaced by a Laurent polynomial $f(1/x) + g(x)$, and we allow an "outside factor" $\chi(x)$ in the sum. Using the general theory developed in this paper, the monodromy groups of some such local systems were determined in [7]. Here is a more detailed discussion.

We work in characteristic $p > 0$, and denote by $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$. We also fix a prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}_\ell}$-adic cohomology. We fix integers

$$A \geq 1, \quad B \geq 1, \quad a > B$$

about which we assume

$$p \nmid ABa.$$

We fix polynomials

$$f(x) \in k[x], \quad \deg(f) = A, \quad k \text{ some finite subfield of } \overline{\mathbb{F}_p},$$
$$g(x) \in k[x], \quad \deg(f) = B, \quad k \text{ some finite subfield of } \overline{\mathbb{F}_p}.$$

We make the assumption that both $f(x)$ and $g(x)$ are Artin–Schreier reduced: this means that in the expression $f(x) = \sum_i c_i x^i$, $g(x) = \sum_i d_i x^i$ we have $c_i = 0, d_i = 0$ if $p|i$. We define

$$\mathrm{gcd}_{\deg}(f) := \gcd\big(\{i \mid c_i \neq 0\}\big), \quad \mathrm{gcd}_{\deg}(g) := \gcd\big(\{i \mid d_i \neq 0\}\big)$$

the greatest common divisor of the degrees of the monomials appearing in f, respectively, in g. We suppose

$$\gcd\big(a, \mathrm{gcd}_{\deg}(f)\big) = 1, \quad \gcd\big(a, \mathrm{gcd}_{\deg}(g)\big) = 1.$$

We also fix a (possibly trivial) multiplicative character χ of $k^\times$, with the convention that for $\chi \neq \mathbb{1}$, we have $\chi(0) = 0$, but $\mathbb{1}(0) = 1$. We denote by $\mathcal{G}(f, g, a, \chi)$ the lisse sheaf on $\mathbb{G}_m/k$ whose trace function at time $t \in L^\times$, for L/k a finite extension, is

$$t \mapsto -\sum_{x \in L^\times} \psi_L\big(f(1/x) + g(x) + tx^a\big)\chi_L(x).$$

2. Basic facts about $\mathcal{G}(f, g, a, \chi)$

The local system $\mathcal{G}(f, g, a, \chi)$ is lisse of rank $D = A + a$ on $\mathbb{G}_m$, and pure of weight one. We view it as being the Fourier transform

$$\mathrm{FT}_\psi\big([a]_\star\big(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_{\chi}(x)\big)\big).$$

Lemma 2.1. *Given $A \geq 1$, $B \geq 1$, $a > B$, $p \nmid ABa$, f, g both Artin–Schreier reduced, and* $\gcd(a, \mathrm{gcd}_{\deg}(f)) = 1$, $\gcd(a, \mathrm{gcd}_{\deg}(g)) = 1$. *Then the following statements hold for $\mathcal{G}(f, g, a, \chi)$.*

(i) *The $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is irreducible. It has rank $A + a$ and all slopes $A/(A + a)$.*

(ii) *The $I(0)$-representation of $\mathcal{G}(f, g, a, \chi)$ is the direct sum*

$$W(B, a - B) \oplus (\overline{\mathbb{Q}_\ell})^{A+B},$$

with $W(B, a - B)$ an irreducible $I(0)$-representation of rank $a - B$ with all slopes $B/(a - B)$.

PROOF. This is a straightforward application of Laumon's results on the local monodromy of FT_ψ. The input sheaf to FT_ψ is lisse on $\mathbb{G}_m$ of rank a, with $I(0)$-slopes A/a and $I(\infty)$ slopes B/a. The hypotheses $\gcd(a, \gcd_{\deg}(g)) = 1, \gcd(a, \gcd_{\deg}(f)) = 1$ imply, respectively, that the $I(0)$- and $I(\infty)$-representations of the input sheaf are irreducible, cf. [8, the proof of Lemma 2.1].

Then the $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is $\mathrm{FTloc}(0, \infty)(\mathrm{rank}\, a, \mathrm{slopes}\ A/a)$, which has rank $A + a$ and all slopes $A/(A + a)$, cf. [3, Theorem 7.4.4 (4)]. The $I(0)$-representation of $\mathcal{G}(f, g, a, \chi)$, modulo its subspace of $I(0)$-invariants, cf. [3, Corollary 7.4.3.1], is $\mathrm{FTloc}(\infty, 0)(\mathrm{rank}\, a, \mathrm{slopes}\ B/a)$, which is the asserted $W(B, a - B)$. The asserted irreducibilities result from the irreducibilities of the input and the fact that $\mathrm{FTloc}(0, \infty)$ and $\mathrm{FTloc}(\infty, 0)$ are suitable equivalences of categories. ∎

COROLLARY 2.2. *Hypotheses as in Lemma 2.1 suppose in addition that $a > 2B$. Then the determinant of $\mathcal{G}(f, g, a, \chi)$ is tame, so geometrically some Kummer sheaf $\mathcal{L}_\Lambda$. Moreover, if χ has odd order N, then Λ has order dividing $2N$, while if χ has even order N, then Λ has order dividing N.*

PROOF. The slopes of $\mathcal{G}(f, g, a, \chi)$ at ∞ are all < 1, and the slopes at 0 are $B/(a-B) < 1$. Therefore, the determinant of $\mathcal{G}(f, g, a, \chi)$, a priori of finite order by Grothendieck's local monodromy theorem, is tame, hence geometrically some Kummer sheaf $\mathcal{L}_\Lambda$. Then the arithmetic determinant of $\mathcal{G}(f, g, a, \chi)$ is some constant field twist $\mathcal{L}_\Lambda \otimes \alpha^{\deg}$ of $\mathcal{L}_\Lambda$. Denote by M the order of Λ. Over any finite extension L/k containing μ_M, the trace of $\mathsf{Frob}_{t,L}|\mathcal{L}_\Lambda$, as t runs over $L^\times$, attains all values in μ_M. Then we recover M as the ratios of these Frobenius traces at various points $s, t \in L^\times$. Thus, we also recover M as the same ratios of Frobenius trace on the constant field twist $\mathcal{L}_\Lambda \otimes \alpha^{\deg}$ of $\mathcal{L}_\Lambda$. Each $\mathsf{Frob}_{t,L}|\mathcal{G}(f, g, a, \chi)$ and all its powers have traces in $\mathbb{Q}(\zeta_p, \zeta_N)$ for N the order of χ. So each Frobenius determinant lies in $\mathbb{Q}(\zeta_p, \zeta_N)$. Therefore, the geometric determinant takes values both in μ_M and in $\mathbb{Q}(\zeta_p, \zeta_N)$. But the only roots of unity in $\mathbb{Q}(\zeta_p, \zeta_N)$ lie in $\pm\mu_{pN}$. Thus, $\mu_M \le \pm\mu_N$. ∎

LEMMA 2.3. *Given $A \ge 1$, $B \ge 1$, $a > B$, $p \nmid ABa$, f, g both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Then $\mathcal{G}(f, g, a, \chi)$ is geometrically self-dual if and only if ABa is odd, both $f(x)$ and $g(x)$ are odd polynomials, and $\chi^2 = \mathbb{1}$.*

PROOF. The oddness conditions, and χ^2 trivial, imply autoduality. For $p = 2$, over even degree extensions $k/\mathbb{F}_2(\text{coef's of } f, g)$, after the constant field twist by $1/\sqrt{\#k}$, the traces are real (in fact in $\mathbb{Q}$). And when p is odd, after the constant field twist by $1/\mathsf{Gauss}(\psi, \chi_2)$ and over even degree extensions of $\mathbb{F}_p(\text{coef's of } f, g)$, the traces are real.

To prove the converse, we argue as follows. Since $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$ is geometrically irreducible, it is self-dual if and only if $H^2_c(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{G} \otimes \mathcal{G})$ is nonzero (and in fact has dimension 1). We compute this dimension as the limsup, over extensions $k/\mathbb{F}_p$(coef's of f, g), of the sums

$$\frac{1}{\#k(\#k-1)} \sum_{t \in k^\times, x, y \in k} \psi_k\big(f(1/x) + f(1/y) + g(x) + g(y) + t(x^a + y^a)\big)\chi_k(xy).$$

The $t = 0$ "missing" summand is

$$\frac{1}{\#k(\#k-1)} \sum_{x,y \in k} \psi_k\big(f(1/x) + f(1/y) + g(x) + g(y)\big)\chi_k(xy) = \frac{1}{\#k(\#k-1)}\Big(\sum_{x \in k} \psi_k\big(f(1/x) + g(x)\big)\chi_k(x)\Big)^2,$$

which is $O(1/(\#k-1))$, because the sum being squared is, by the Weil bound, of absolute value $\leq (A+B)\sqrt{\#k}$.

So the limsup does not change if we add this term.Then we have the limsup of

$$1/(\#k-1) \sum_{x,y \in k,\, x^a + y^a = 0} \psi_k\big(f(1/x) + f(1/y) + g(x) + g(y)\big)\chi_k(xy).$$

This is then the limsup of the sum of the a sums, one for each ζ with $\zeta^a = -1$,

$$S_\zeta := 1/(\#k-1) \sum_{x \in k} \psi_k\big(f(1/x) + f(1/\zeta x) + g(x) + g(\zeta x)\big)\chi_k(\zeta x^2).$$

Both $f(x)$ and $g(x)$ are Artin–Schreier reduced and

$$\gcd\big(a, \gcd_{\deg}(f)\big) = \gcd\big(a, \gcd_{\deg}(g)\big) = 1.$$

Then the two sums $f(x) + f(x/\zeta)$ and $g(x) + g(\zeta x)$ are each Artin–Schreier reduced. Unless both sums vanish, the S_ζ summand is $O(1/\sqrt{\#k})$, again by the Weil estimate. And even if both sums do vanish, then the sum still vanishes unless χ^2 is the trivial character. Thus, in all cases, we must have χ^2 trivial if we are to have self-duality.

Suppose now that χ^2 is trivial and $\mathcal{G}$ is self-dual. Then for at least one ζ with $\zeta^a = -1$, both $f(x) + f(x/\zeta) = 0$ and $g(x) + g(\zeta x) = 0$. In the case $p = 2$, both f, g are odd polynomials (because both are Artin–Schreier reduced), so there is nothing to prove.

Thus, it remains to treat the case when p is odd. Suppose first that a is even. Then we claim that for any ζ with $\zeta^a = -1$, $f(x) + f(x/\zeta) \neq 0$. To see this, write $f(x) = \sum_n a_n x^n$, and define $\mathcal{E}_f := \{n \mid a_n \neq 0\}$, the set of exponents which occur in f. By hypothesis, we have

$$\gcd(a, \text{all } n \in \mathcal{E}_f) = 1.$$

We rewrite this as

$$\gcd(a, \text{all } n-a \text{ with } n \in \mathscr{E}_f) = 1.$$

If $f(x) + f(x/\zeta) = 0$, then $a_n(1 + 1/\zeta^n) = 0$ for all $n \in \mathscr{E}_f$, i.e., $\zeta^n = -1$ for all $n \in \mathscr{E}_f$, i.e., $\zeta^n = \zeta^a$ for all $n \in \mathscr{E}_f$, and finally $\zeta^{n-a} = 1$ for all $n \in \mathscr{E}_f$. Define

$$D := \gcd(\text{all } n-a \text{ with } n \in \mathscr{E}_f).$$

Then $\zeta^D = 1$. But $\gcd(a, D) = 1$, so there exist integers u, v with $au + Dv = 1$. It follows that $\zeta = (\zeta^a)^u(\zeta^D)^v = (-1)^u$. Thus, ζ is ± 1, neither of which has $\zeta^a = -1$ if a is even.

Suppose next that a is odd. Then the above argument shows that ζ is ± 1. But of these two choices, only $\zeta = -1$ has $\zeta^a = -1$. For this $\zeta = -1$, we have $f(x) + f(-x) = 0$, which means precisely that f is an odd polynomial. The same argument applied to g, using the fact that $\gcd(a, \text{all } n \in \mathscr{E}_g) = 1$, shows that $\zeta = -1$, hence that g is an odd polynomial. ∎

LEMMA 2.4. *Given $A \geq 1$, $B \geq 1$, $a > B$, $p \nmid ABa$, f, g both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that χ and ρ are multiplicative characters of $k^\times$ for $k/\mathbb{F}_p$ a finite extension containing the coefficients of both f and g. If $\chi \neq \rho$, then $\mathscr{G}(f, g, a, \chi)$ and $\mathscr{G}(f, g, a, \rho)$ are not geometrically isomorphic.*

PROOF. We argue by contradiction. Suppose that $\mathscr{G}(f, g, a, \chi)$ and $\mathscr{G}(f, g, a, \rho)$ are geometrically isomorphic. As each is geometrically irreducible, the cohomology group

$$H^2_c\big(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathscr{G}(f, g, a, \chi) \otimes \mathscr{G}(f, g, a, \rho)^\vee\big)$$

is pure of weight 2 and of dimension one. The dual $\mathscr{G}(f, g, a, \rho)^\vee$ is the (-1)-Tate twist of its complex conjugate: its trace function at $t \in L^\times$ is

$$t \in L^\times \mapsto \frac{-1}{\#L} \sum_{x \in L^\times} \psi_L\big(-f(1/x) - g(x) - tx^a\big)\overline{\rho}(x).$$

So the trace function of $\mathscr{G}(f, g, a, \chi) \otimes \mathscr{G}(f, g, a, \rho)^\vee$ is

$$t \in L^\times \mapsto \frac{-1}{\#L} \sum_{x,y \in L^\times} \psi_L\big(f(1/x) - f(1/y) + g(x) - g(y) + t(x^a - y^a)\big)\chi(x)\overline{\rho}(x).$$

The sum over t of this trace is, by the Lefschetz trace formula,

$$\mathrm{Trace}(\mathsf{Frob}_L | H^2_c) - \mathrm{Trace}(\mathsf{Frob}_L | H^1_c),$$

with H_c^2 pure of weight 2, and H_c^1 mixed of weight ≤ 1. Thus, the dimension, namely, 1, of the relevant H_c^2 is the limsup, as L/k grows, of

$$\frac{1}{\#L}\sum_{t\in L^\times}\frac{1}{\#L}\sum_{x,y\in L^\times}\psi_L\big(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a)\big)\chi(x)\overline{\rho}(x).$$

So far as the limsup is concerned, we may replace this sum over $t \in L^\times$ by the sum over all $t \in L$: indeed the $t = 0$ summand is

$$\frac{1}{(\#L)^2}\Big(-\sum_{x\in L^\times}\psi_L\big(f(1/x)+g(x)\big)\chi(x)\Big)\Big(-\sum_{y\in L^\times}\psi_L\big(-f(1/y)-g(y)\big)\overline{\rho}(y)\Big).$$

Each of the factors is $O(\sqrt{\#L})$, so this $t = 0$ term is $O(1/\#L)$, and hence does not affect the limsup.

Thus, the dimension, 1, of the H_c^2 is the limsup of

$$\begin{aligned}
&\frac{1}{\#L}\sum_{t\in L}\frac{1}{\#L}\sum_{x,y\in L^\times}\psi_L\big(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a)\big)\chi(x)\overline{\rho}(y)\\
&=\frac{1}{\#L}\sum_{x,y\in L^\times,\,x^a=y^a}\psi_L\big(f(1/x)-f(1/y)+g(x)-g(y)+t(x^a-y^a)\big)\chi(x)\overline{\rho}(y)\\
&=\sum_{\zeta\in\mu_a(\overline{\mathbb{F}_p})}\frac{1}{\#L}\sum_{x\in L^\times}\psi_L\big(f(1/x)-f(1/\zeta x)+g(x)-g(\zeta x)\big)\chi(x)\overline{\rho}(\zeta x).
\end{aligned}$$

The $\zeta = 1$ summand is

$$\frac{1}{\#L}\sum_{x\in L^\times}\chi(x)\overline{\rho}(\zeta x)=\frac{\overline{\rho}(\zeta)}{\#L}\sum_{x\in L^\times}(\chi/\rho)(x)=0,$$

simply because χ/ρ is nontrivial. In each of the remaining summands, the Laurent polynomial $f(1/x) - f(1/\zeta x) + g(x) - g(\zeta x)$ inside the ψ is itself nonzero (because $\gcd(a, \gcd_{\deg}(f)) = 1$ and $\gcd(a, \gcd_{\deg}(g)) = 1$) and Artin–Schreier reduced, so each of these summands is $O(1/\sqrt{\#L})$. Thus, the limsup vanishes, the desired contradiction. ■

LEMMA 2.5. *Let $X/\mathbb{F}_q$ be smooth and geometrically connected of dimension $d \geq 1$, $\ell \neq p$, $K/\mathbb{Q}$ a finite extension, and L/K a finite Galois extension. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are nonzero lisse $\overline{\mathbb{Q}_\ell}$-sheaves on X. Suppose that both $\mathcal{F}$ and $\mathcal{G}$ are geometrically irreducible and have all their Frobenius traces in L. Suppose further that for every $\sigma \in \mathrm{Gal}(L/K)$, there exist lisse sheaves $\mathcal{F}^\sigma$ and $\mathcal{G}^\sigma$ on X whose trace functions are the σ-conjugates of those of $\mathcal{F}$ and of $\mathcal{G}$. If $\mathcal{F}$ and $\mathcal{G}$ are geometrically isomorphic, then $\mathcal{F}^\sigma$ and $\mathcal{G}^\sigma$ are geometrically isomorphic.*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are geometrically isomorphic, then because each is geometrically irreducible, there exists an $\alpha^{\deg}$ twist such that we have an arithmetic isomorphism

$$\mathcal{F} \cong \mathcal{G} \otimes \alpha^{\deg}.$$

This implies that for every finite extension $k/\mathbb{F}_q$, and every $t \in X(k)$, we have an equality

$$\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{F}) = \alpha^{\deg(k/\mathbb{F}_q)}\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{G}).$$

Because $\mathcal{F}$ is not the zero sheaf, for some $k_0/\mathbb{F}_q$ and some $t \in X(k_0)$, $\mathrm{Trace}(\mathsf{Frob}_{t,k_0}|\mathcal{F})$ is nonzero. Then $\mathrm{Trace}(\mathsf{Frob}_{t,k_0}|\mathcal{G})$ must also be nonzero, and we recover $\alpha^{\deg(k_0/\mathbb{F}_q)}$ as the ratio of nonzero traces of $\mathcal{F}$ and of $\mathcal{G}$. As these traces lie in L, it follows that $\alpha^{\deg(k_0/\mathbb{F}_q)}$ lies in L. Extending scalars from $\mathbb{F}_q$ to k_0, we reduce to the case when $\alpha \in L^\times$. Then we simply apply σ to the above equality of traces to obtain

$$\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{F}^\sigma) = \sigma(\alpha)^{\deg(k/\mathbb{F}_q)}\mathrm{Trace}(\mathsf{Frob}_{t,k}|\mathcal{G}^\sigma);$$

i.e., we have an arithmetic isomorphism

$$\mathcal{F}^\sigma \cong \mathcal{G}^\sigma \otimes \sigma(\alpha)^{\deg},$$

and hence the desired geometric isomorphism. ∎

Proposition 2.6. *Given $A \geq 1$, $B \geq 1$, $a > B$, $p \nmid ABa$, let $\mathcal{F}$ and $\mathcal{G}$ be local systems on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, whose local monodromies are of the form*

$$\mathcal{F}_{I(0)} \cong (A+B)\mathbb{1} \oplus W_{\mathcal{F}}, \quad \mathcal{G}_{I(0)} \cong (A+B)\mathbb{1} \oplus W_{\mathcal{G}},$$

with $W_{\mathcal{F}}$, $W_{\mathcal{G}}$ both irreducible of rank $a-B$ and totally wild, and

$$\mathcal{F}_{I(\infty)} \cong V_{\mathcal{F}}, \quad \mathcal{G}_{I(\infty)} \cong V_{\mathcal{G}},$$

with $V_{\mathcal{F}}$, $V_{\mathcal{G}}$ both of rank $A+a$ with all slopes < 1. Let $\mathcal{L}$ be a rank one local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, such that $\mathcal{F} \cong \mathcal{G} \otimes \mathcal{L}$. Then $\mathcal{L} \cong \overline{\mathbb{Q}_\ell}$.

Proof. We first show that $\mathcal{L}$ is tame at ∞. Indeed, if it were not, then $V_{\mathcal{G}} \otimes \mathcal{L}$ has all slopes equal to $\mathsf{Swan}_\infty(\mathcal{L}) \geq 1$, while $V_{\mathcal{F}}$ has all slopes < 1. Once we have this, it suffices to show that $\mathcal{L}_{I(0)}$ is trivial. Suppose first that $a - B > 1$. Then

$$\mathcal{G}_{I(0)} \otimes \mathcal{L}_{I(0)} \cong (A+B)\mathcal{L}_{I(0)} \oplus (\text{irreducible of rank} > 1).$$

So the one-dimensional constituents are each $\mathcal{L}_{I(0)}$. But the one-dimensional constituents of $\mathcal{F}_{I(0)}$ are each $\mathbb{1}$. Suppose next that $a - B = 1$. Then in both $\mathcal{F}_{I(0)}$ and $\mathcal{G}_{I(0)}$, the trivial constituents are in the majority. But after tensoring with $\mathcal{L}_{I(0)}$, the $\mathcal{L}_{I(0)}$ constituents are in the majority. Hence, $\mathcal{L}_{I(0)}$ is trivial. ∎

Corollary 2.7. *For $\mathcal{F}$ as in Proposition 2.6 above, suppose $\mathcal{L}$ is a rank one local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, such that $\mathcal{F}^\vee \cong \mathcal{F} \otimes \mathcal{L}$. Then $\mathcal{L} \cong \overline{\mathbb{Q}_\ell}$.*

Proof. Indeed, both $\mathcal{F}$ and $\mathcal{F}^\vee$ have the shapes of local monodromies of the Proposition. ■

Proposition 2.8. *Suppose $p \nmid ABa$, f, g are both Artin–Schreier reduced, and*

$$\gcd\big(a, \gcd_{\deg}(f)\big) = 1, \quad \gcd\big(a, \gcd_{\deg}(g)\big) = 1.$$

Then the $I(0)$-representation of $\mathcal{G}(f, g, a, \chi)$ is tensor indecomposable under each of the following conditions.

(a) *The rank $A + a \neq 4$.*

(b) *$A + a = 4$ and $p = 2$.*

(c) *$A + a = 4$, $p \neq 2$, and $(A, B, a) \neq (1, 1, 3)$.*

Proof. Indeed, the $I(0)$-representation is the direct sum $T \oplus W$ of a nonzero tame part and an irreducible wild part. In rank $\neq 4$, the result follows from [6, Corollary 10.4]. In the case of rank 4, the tame part has rank $A + B \geq 2$, so in characteristic $p = 2$ we may again apply [6, Corollary 10.4]. To apply [6, Corollary 10.4] with p odd and rank 4, we must avoid the case $A + B = 2$, i.e., the case $A = 1 = B$ and $a = 3$. ■

Proposition 2.9. *Suppose $p \nmid ABa$, f, g are both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that $\mathcal{G}(f, g, a, \chi)$ is tensor indecomposable for $I(0)$. Let us denote*

$$D := A + a, \quad t := A + B, \quad w := a - B,$$

the rank, the dimension of the tame part T, and the dimension of the wild part W of the $I(0)$-representation $V = T \oplus W$. Then $\mathcal{G}(f, g, a, \chi)$ is not tensor induced over $I(0)$ under each of the following conditions.

(a) *D is not a power.*

(b) *$w = 1$.*

(c) *$t - w > \sqrt{D}$.*

(d) *$p \nmid w$ and $w < D - \sqrt{D}$.*

Proof. Case (a) is trivial.

To treat case (b), suppose $w = 1$, and V is tensor induced: $V = U_1 \otimes \cdots \otimes U_n$ with $n \geq 2$, $\dim(U_i) = d \geq 2$, and $I(0)$ acts through $\mathrm{GL}_d(\mathbb{C}) \wr \mathsf{S}_n$. As $p \nmid aB$ and

$a - B = w = 1$, $p > 2$. Since W has dimension $w = 1$, some element $\gamma \in P(0)$ must act on W as a scalar $\zeta \neq 1$, an Nth root of unity with $N > 1$ a p-power. By [8, Lemma 3.2 (ii) with $a = 1$], γ is tensor indecomposable, so it must induce an n-cycle while permuting the n tensor factors of V. By the formula for tensor induction [1],

$$\left|\mathrm{Trace}(\gamma|_V)\right| \leq d \leq D/2 \leq D - 2$$

since $D = d^n \geq 4$. On the other hand,

$$\left|\mathrm{Trace}(\gamma|_V)\right| = |D - 1 + \zeta| \geq D - 2,$$

with equality only when $\zeta = -1$, which is impossible since $p > 2$.

To treat case (c), use the $I(0)$-tensor indecomposability of V to apply [8, part (ii) of Lemma 3.5]. It shows the existence of an element $h \in I(0)$ with $|\mathrm{Trace}(h|_V)| \leq \sqrt{D}$ if V is tensor induced. But $V = t\mathbb{1} + W$, and any $h \in I(0)$, being of finite order, has $|\mathrm{Trace}(h|_W)| \leq w$. Hence, $\sqrt{D} \geq |\mathrm{Trace}(h|_V)| \geq t - w$, a contradiction.

For case (d), we may assume $w > 1$ by (b), and then use the fact that an element $\gamma \in I(0)$ which is generator of $I(0)/P(0)$ has spectrum on W consisting of all the wth roots of some root of unity ρ (because when $p \nmid w$, W is the Kummer induction $[w]_\star \mathcal{L}$ of some rank one $\mathcal{L}$, and γ acts by cyclically permuting the w factors of the induction: because γ has finite order on V, ρ is itself a root of unity). Then we apply [8, Lemma 3.2 (i)] (with its $a = w$) to see that γ is tensor indecomposable in the $I(0)$-representation if $w < D - \sqrt{D}$. Then we repeat the argument of case (b): if V is n-tensor induced, then

$$\left|\mathrm{Trace}(\gamma|_V)\right| \leq d = D^{1/n} \leq \sqrt{D}.$$

But $\mathrm{Trace}(\gamma|_W) = 0$ since $w > 1$, and hence $\sqrt{D} \geq |\mathrm{Trace}(\gamma|_V)| = t = D - w$, i.e., $w \geq D - \sqrt{D}$, a contradiction. ■

THEOREM 2.10. *Suppose that* $p \nmid ABa(A + a)(a - B)$, f *and* g *are both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. *Then* $\mathcal{G}(f, g, a, \chi)$ *is primitive on* $\mathbb{G}_m/\overline{\mathbb{F}_p}$, *under each of the following conditions.*

(a) $w := a - B$ *is not of the form* $p^s - 1$ *for any* $s \geq 1$.

(b) $w = p^s - 1$ *and* $A \neq 1$.

(c) $w = p^s - 1$, $A = 1$, *and* $\chi \neq \chi_2$ *(for* χ_2 *the quadratic character).*

(d) $w = p^s - 1$, $A = 1$, $\chi = \chi_2$, ABa *is odd, each of* f, g *is an odd polynomial, and* $B < 2p$.

(e) $w = p^s - 1$, $A = 1$, $\chi = \chi_2$, *and* $\mathcal{G}(f, g, a, \chi)$ *has infinite* G_{geom}.

(f) $w = p^s - 1$, $A = 1$, $\chi = \chi_2$, $p \geq 5$, *each of* $f, g \in \mathbb{F}_p[x]$, *with* $g(x) = \sum_{i=0}^{B} a_i x^i$, *and either*

$$B \equiv \frac{p-1}{2} \mod (p-1)$$

or

$$\sum_{i : i \equiv \frac{p-1}{2} \mod (p-1)} a_i \neq 0.$$

PROOF. We argue by contradiction. Suppose $\mathcal{G}(f, g, a, \chi) = \pi_\star \mathcal{H}$ for some finite étale $\pi : U \to \mathbb{G}_m$ of degree $d > 1$ and some local system $\mathcal{H}$ on U. Then $d \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(\mathcal{G}(f, g, a, \chi)) = A + a$ is prime to p. Also U is geometrically connected; otherwise, $\pi_\star \mathcal{H}$ is not irreducible. Denote by X the complete nonsingular model of U, and denote by $\pi : X \to \mathbb{P}^1$ the finite flat map on the complete curves. Let

$$C := \pi^{-1}(0), \quad E = \pi^{-1}(\infty),$$

of cardinalities c, e, respectively.

For each point $x \in E$, denote by

$$\pi_x : \mathrm{Spec}\big((K_{X,x})^\wedge\big) \to \mathrm{Spec}\big((K_{\mathbb{P}^1,\infty})^\wedge\big)$$

the induced map of the spec's of completed function fields. Then for $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$, we have

$$\mathcal{G}|_{I(\infty)} = \bigoplus_{x \in E} \pi_{x\star}(\mathcal{H}_{I(x)}).$$

But $\mathcal{G}|_{I(\infty)}$ is irreducible; hence, there is precisely one point in E, call it ∞up, and

$$\mathcal{G}|_{I(\infty)} = \pi_{\infty up\star}(\mathcal{H}_{I(\infty up)}),$$

with $\mathcal{H}_{I(\infty up)}$ irreducible (because its direct image is irreducible). Because ∞up is the unique point lying over ∞, the degree of $\pi_{\infty up}$ is precisely $d := \deg(\pi)$, which is a divisor of $A + a$. Looking at degrees, we thus have

$$d \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(\mathcal{G}).$$

Therefore, $\deg(\pi_{\infty up}) = d$ is prime to p; hence, $\pi_{\infty up}$ is tame. By [4, Lemma 1.6.4.1], it follows that

$$\mathsf{Swan}_{\infty up}(\mathcal{H}) = \mathsf{Swan}_{\infty}(\mathcal{G}).$$

Similarly, we have

$$\mathcal{G}|_{I(0)} = \bigoplus_{x \in C} \pi_{x\star}(\mathcal{H}_{I(x)}),$$

while

$$\mathcal{G}|_{I(0)} = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^{A+B},$$

with $W_{B,a-B}$ irreducible of rank $w := a - B$ with all slopes $B/(a-B)$. There is precisely one point $x_0 \in C$ whose $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$ contains $W_{B,a-B}$ as a summand. More precisely, we have

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n, \quad \text{for some } n \geq 0.$$

We first consider the case $n = 0$. Then $\mathcal{H}_{I(x_0)}$ is irreducible. Moreover, it cannot be tame; i.e., it cannot be a Kummer sheaf $\mathcal{L}_\chi$: if it were, then by Frobenius reciprocity its direct image contains all $\mathcal{L}_\rho$ with $\rho^{\deg(\pi_{x_0})} = \chi$, whereas its direct image is totally wild. Looking at degrees, we have

$$\deg(\pi_{x_0}) \times \operatorname{rank}(\mathcal{H}) = \operatorname{rank}(W_{B,a-B}) = a - B.$$

As $p \nmid (a-B)$, we see that π_{x_0} has degree prime to p. Again by [4, Lemma 1.6.4.1], it follows that

$$\mathsf{Swan}_{x_0}(\mathcal{H}) = \mathsf{Swan}_0(\mathcal{G}) = B.$$

In this $n = 0$ case, we now argue as follows. On the one hand, for $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$, for the Euler–Poincaré characteristic, we have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}) = -B - A.$$

But

$$\begin{aligned}\mathsf{EP}(U, \mathcal{H}) &= \mathsf{EP}(U)\operatorname{rank}(\mathcal{H}) - \sum_{x \in C} \mathsf{Swan}_x(\mathcal{H}) - \mathsf{Swan}_{\infty up}(\mathcal{H}) \\ &= \mathsf{EP}(U)\operatorname{rank}(\mathcal{H}) - B - \sum_{x \in C, x \neq x_0} \mathsf{Swan}_x(\mathcal{H}) - A.\end{aligned}$$

Subtracting these two expressions for $\mathsf{EP}(U, \mathcal{H})$, we find that

$$\mathsf{EP}(U)\operatorname{rank}(\mathcal{H}) = \sum_{x \in C,\, x \neq x_0} \mathsf{Swan}_x(\mathcal{H}).$$

In particular, $\mathsf{EP}(U)\operatorname{rank}(\mathcal{H}) \geq 0$, and hence $\mathsf{EP}(U) \geq 0$. As U is the complement of at least two points (one in D and at least one in C) in a complete nonsingular curve, call it X, we have $\mathsf{EP}(U) = 2 - 2g_X - 1 - \#C \geq 0$. On the other hand, $2 - 2g_X - 1 - \#C \leq 0$, with equality only if $g_X = 0$ and $\#C = 1$. Because $\#C = 1$, $\deg(\pi_{x_0})$ must be $d = \deg(\pi)$. Then the entire $I(0)$-representation is wild, a contradiction, since the $I(0)$-representation has an $A + B \geq 2$ dimensional trivial part. (Alternatively, as the referee pointed out, one could use the fact that for $x \in C \smallsetminus \{x_0\}$, $\pi_{x\star}(\mathcal{H}_{I(x)})$ is tame, hence by [4, Lemma 1.6.4.1], $\mathcal{H}$ is tame at each such x, and so $\mathsf{EP}(U) = 0$ by the formula displayed above.)

Suppose next that $n \geq 1$. Then $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$ contains $\overline{\mathbb{Q}_\ell}$, which we write as $\mathcal{L}_{\mathbb{1}}$. Then by Frobenius reciprocity, $\mathcal{H}_{I(x_0)}$ contains $\mathcal{L}_{\mathbb{1}}$. Then $\deg(\pi_{x_0})$ cannot be divisible by any prime to p integer $r > 1$, for otherwise $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$ contains $\pi_{x_0\star}(\mathcal{L}_{\mathbb{1}})$, which contains all $\mathcal{L}_\rho$ with $\rho^r = \mathbb{1}$. This is impossible because the entire tame part of the $I(0)$-representation of $\mathcal{G}$ is copies of $\mathcal{L}_{\mathbb{1}}$. Thus, $\deg(\pi_{x_0}) = p^s$ for some $s \geq 0$. If $s = 0$, i.e., if $\deg(\pi_{x_0}) = 1$ is prime to p, then $\mathsf{Swan}_{x_0}(\mathcal{H}) = \mathsf{Swan}_0(\mathcal{G}) = B$, and we conclude as in the $n = 0$ case above.

Suppose next that $n \geq 1$ and $\deg(\pi_{x_0}) = p^s$ with $s \geq 1$. Then by Frobenius reciprocity, $\pi_{x_0\star}(\mathcal{L}_{\mathbb{1}})$ contains $\mathcal{L}_{\mathbb{1}}$ just once and contains no $\mathcal{L}_\rho$ for any nontrivial ρ (because it only contains $\mathcal{L}_\rho$ if $\rho^{\deg(\pi_{x_0})} = \mathbb{1}$). Therefore,

$$\pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

If $\mathcal{H}_{I(x_0)}$ were not simply $\mathcal{L}_{\mathbb{1}}$, any other irreducible constituent would either be tame (in which case its direct image would also have a wild part of rank $p^s - 1$), or would be wild, in which case its direct image would be totally wild.

Thus, in this $n \geq 1$ case, we have $n = 1$, $\mathrm{rank}(\mathcal{H}) = 1$, $\mathcal{H}_{I(x_0)} = \mathcal{L}_{\mathbb{1}}$, and

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = \pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

So the wild part of the $I(0)$-representation of $\mathcal{G}$ has dimension $w = p^s - 1$.

We now continue with the analysis of the case when $w = p^s - 1$. Looking at what remains of the $I(0)$-representation, we find

$$\mathcal{L}_{\mathbb{1}}^{A+B-1} = \bigoplus_{x \in C, x \neq x_0} \pi_{x\star}(\mathcal{H}_{I(x)}).$$

Each individual direct image $\pi_{x\star}(\mathcal{H}_{I(x)})$ is then a sum of $\mathcal{L}_{\mathbb{1}}$. Being tame, it follows that $\mathcal{H}$ is tame at each $x \neq x_0$ in C. Then $\mathcal{H}$ must be $I(x)$-trivial at each such x; otherwise, its direct image contains various $\mathcal{L}_\rho$ with nontrivial ρ. Then each π_x for $x \neq x_0$ must have degree 1: it cannot have degree divisible by a prime to p integer $r > 1$ because that introduces nontrivial tame pieces in the direct image, and it cannot have degree a strictly positive power of p because that introduces nonzero wild parts in the direct image. Thus, at each $x \neq x_0$ in C, the degree of π_x is 1. From the above displayed equation

$$\mathcal{L}_{\mathbb{1}}^{A+B-1} = \bigoplus_{x \in C, x \neq x_0} \pi_{x\star}(\mathcal{H}_{I(x)}),$$

we then see that

$$\#C = A + B,$$

and that $\mathcal{H}$ is lisse of rank one outside of the single point ∞up. Thus,

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}) = -B - A.$$

At the same time, remembering that $\mathcal{H}$ has rank one, we have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(U) - \mathsf{Swan}_{\infty up}(\mathcal{H}) = \mathsf{EP}(U) - A.$$

Thus, $\mathsf{EP}(U) - A = -B - A$, and hence

$$\mathsf{EP}(U) = -B.$$

In terms of the complete nonsingular model X of U, this gives

$$-B = \mathsf{EP}(U) = 2 - 2g_X - \#D - \#C = 2 - 2g_X - 1 - (A + B);$$

hence, $2 - 2g_X - 1 - A = 0$, i.e., $-2g_X = A - 1$. This can only hold if $g_X = 0$ and $A = 1$. Putting ∞up at ∞, $\mathcal{H}$ is lisse of rank one on $\mathbb{A}^1$, with $\mathsf{Swan}_\infty(\mathcal{H}) = 1$. Thus, $\mathcal{H}$ is $\mathcal{L}_{\psi(\alpha x)}$ for some $\alpha \neq 0$ in $\overline{\mathbb{F}_p}$. Putting x_0 at 0, the morphism π is a polynomial $H(x) \in \overline{\mathbb{F}_p}[x]$ which has degree $A + a = 1 + a = 1 + B + w = p^s + B$, which has 0 as a root of multiplicity p^s, and which has B simple zeros, each of which is nonzero. Thus, we obtain a geometric isomorphism

$$\mathcal{G}(f, g, a, \chi) \cong \big[H(x)\big]_\star \mathcal{L}_{\psi(\alpha x)}.$$

Over a large enough finite extension $k/\mathbb{F}_p$ (namely, one which contains α and the coefficients of each of the polynomials f, g, H, and with $\chi^{\#k-1} = \mathbb{1}$) both of $\mathcal{G}(f, g, a, \chi)$ and $[H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ are geometrically irreducible and geometrically isomorphic local systems on $\mathbb{G}_m/k$. Therefore, there exists some $\gamma \in \overline{\mathbb{Q}_\ell}^\times$ for which we have an *arithmetic* isomorphism

$$\mathcal{G}(f, g, a, \chi) \otimes \gamma^{\deg} \cong \big[H(x)\big]_\star \mathcal{L}_{\psi(\alpha x)}. \tag{2.10.1}$$

Recall that $\mathcal{G}(f, g, a, \chi)$ is, arithmetically, the Fourier transform

$$\mathcal{G}(f, g, a, \chi) := \mathrm{FT}_\psi\big([a]_\star\big(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x)\big)\big),$$

and hence

$$\mathcal{G}(f, g, a, \chi) \otimes \gamma^{\deg} := \mathrm{FT}_\psi\big([a]_\star\big(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x) \otimes \gamma^{\deg}\big)\big).$$

Applying the inverse Fourier transform $\mathrm{FT}_{\bar{\psi}}$ to equation (2.10.1), we get an arithmetic isomorphism

$$[a]_\star\big(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x) \otimes \gamma^{\deg}\big) \cong \mathrm{FT}_{\bar{\psi}}\big(\big[H(x)\big]_\star \mathcal{L}_{\psi(\alpha x)}\big).$$

We next prove that this cannot happen if χ has order ≥ 3. The key point is that

$$\mathrm{Gal}\big(\mathbb{Q}(\chi, \zeta_p)/\mathbb{Q}(\zeta_p)\big) = \mathrm{Gal}\big(\mathbb{Q}(\chi)/\mathbb{Q}\big).$$

So we may choose

$$\sigma \in \mathrm{Gal}\big(\mathbb{Q}(\chi, \zeta_p)/\mathbb{Q}(\zeta_p)\big)$$

so that the σ-conjugate system to $\mathcal{G}(f, g, a, \chi)$ is

$$\mathcal{G}(f, g, a, \chi)^\sigma = \mathcal{G}(f, g, a, \chi^\sigma),$$

while

$$\big([H(x)]_\star \mathcal{L}_{\psi(\alpha x)}\big)^\sigma = [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}.$$

Applying Lemma 2.5, we find that $[H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ is isomorphic to both $\mathcal{G}(f, g, a, \chi)$ and to $\mathcal{G}(f, g, a, \chi^\sigma)$. But if χ has order ≥ 3, there exists σ for which χ^σ is any character of the same order as χ, and in particular there exists σ for which $\chi^\sigma \neq \chi$. For such a σ, $\mathcal{G}(f, g, a, \chi)$ and $\mathcal{G}(f, g, a, \chi^\sigma)$ are not geometrically isomorphic, by Lemma 2.4.

For $\chi = \mathbb{1}$, the "traces nowhere vanishing" argument of [8, the proof of Proposition 4.4] shows that $\mathcal{G}(f, g, a, \mathbb{1})$ is always primitive.

We now deal with the case where $A = 1$, $w = p^s - 1$, $\chi = \chi_2$ has order 2, and $\mathcal{G}(f, g, a, \chi_2) \cong [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ with H a polynomial of degree $B + p^s$, with 0 as a root of multiplicity p^s and with B simple roots, each nonzero. For an $I(0)$-representation V, $[H(x)]_\star V$ as $I(0)$-representation is the induction through H viewed as lying in $\overline{\mathbb{F}_p}[[x]]$, call it H^{fml}. We apply this to $V := \mathcal{L}_{\psi(\alpha x)}$, which is trivial as $I(0)$-representation. We recall from [5, Corollary 6.4.5 (2)] that we may compute

$$\mathsf{Swan}_0\big([H^{fml}(x)]_\star \mathcal{L}_{\psi(\alpha x)}\big) = \mathsf{Swan}_0\big([H^{fml}(x)]_\star \mathcal{L}_{\mathbb{1}}\big)$$

as follows. Expand $H^{fml}(x)$:

$$H^{fml}(x) = x^{p^s}\Big(\sum_{m \geq 0} \alpha_m x^m\Big).$$

Then $\mathsf{Swan}_0([H^{fml}(x)]_\star \mathcal{L}_{\mathbb{1}})$ is the least prime to p integer m with $\alpha_m \neq 0$.

On the other hand, the $I(0)$-representation of $[H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ is the direct sum of $[H^{fml}(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ with B copies of $\mathcal{L}_{\mathbb{1}}$, so

$$\mathsf{Swan}_0\big([H(x)]_\star \mathcal{L}_{\psi(\alpha x)}\big) = \mathsf{Swan}_0\big([H^{fml}(x)]_\star \mathcal{L}_{\psi(\alpha x)}\big) \ \big(= \mathsf{Swan}_0\big([H^{fml}(x)]_\star \mathcal{L}_{\mathbb{1}}\big)\big).$$

But H is a polynomial of degree $B + p^s$. Thus, in $\overline{\mathbb{F}_p}[[x]]$, $H = H^{fml}$, and its expansion is

$$H(x) = x^{p^s}\Big(\sum_{m=0}^{B} a_m x^m\Big),$$

with a_0, a_B both nonzero. Moreover, if $1 \leq m \leq B - 1$ is nonzero mod p, then $a_m = 0$ (otherwise, Swan_0 would be this lower m).

Suppose now that ABa is odd and that both f, g are odd polynomials. Then $\mathcal{G}(f, g, a, \chi_2)$ is self-dual (in fact orthogonally self-dual). So if $\mathcal{G}(f, g, a, \chi_2) \cong [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$, then

$$\mathcal{H} := \big[H(x)\big]_\star \mathcal{L}_{\psi(\alpha x)}$$

is self-dual. As $\mathcal{H}$ is pure of weight zero and geometrically irreducible, its autoduality is equivalent to having $\dim(H_c^2(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{H}^{\otimes 2})) = 1$. This dimension is the limsup over larger and larger extension L of any chosen finite extension $k/\mathbb{F}_p$ which contains the coefficients of f, g, H, of the complex absolute value of

$$\begin{aligned}(1/\#L)\sum_{t\in L^\times}\big(\mathrm{Trace}(\mathsf{Frob}_{t,L}|\mathcal{H})\big)^2 &= (1/\#L)\sum_{t\in L^\times}\ \sum_{x,y\in L:H(x)=t=H(y)} \psi_L(\alpha x)\psi_L(\alpha y)\\ &= (1/\#L)\sum_{x,y\in L:H(x)=H(y)\neq 0} \psi_L\big(\alpha(x+y)\big).\end{aligned}$$

The "missing" term with $t = 0$ is $(1/\#L)(\sum_{x\in L:H(x)=0}\psi_L(\alpha x))^2$, which is at most $\deg(H)^2/\#L$, so does not affect the limsup. So the dimension of this H_c^2 is the limsup of

$$(1/\#L)\sum_{x,y\in L:H(x)=H(y)} \psi_L\big(\alpha(x+y)\big).$$

The affine curve $H(x) = H(y)$ is smooth outside the point $(0, 0)$. Indeed, its singularities are the points on the curve where $dH(x)/dx = 0 = dH(y)/dy$. From the explicit form of H above, we see that

$$dH(x)/dx = a_B(p^s + B)x^{p^s+B-1}, \quad dH(y)/dy = a_B(p^s + B)y^{p^s+B-1}.$$

The polynomial $H(x) - H(y)$ has the factorization

$$H(x) - H(y) = (x - y)\Delta_H, \quad \text{with } \Delta_H := \big(H(x) - H(y)\big)/(x - y).$$

The polynomial Δ_H is not divisible by $x - y$; indeed its leading term is

$$a_B \prod_{\zeta\in\mu_{p^s+B},\,\zeta\neq 1} (x - \zeta y).$$

The intersection of the two loci $x - y = 0$ and $\Delta_H = 0$ is the single point $(0, 0)$. Thus, the curve $\Delta_H = 0$ is lisse outside the point $(0, 0)$ (because this open set of $\Delta_H = 0$ is the complement of $x = y$ in $H(x) = H(y)$).

The sum of $\psi_L(\alpha(x + y))$ over the locus $x = y$ vanishes. So our limsup is the limsup of

$$(1/\#L)\sum_{x,y\in L:\Delta_H=0} \psi_L\big(\alpha(x+y)\big).$$

We will show that this limsup is in fact 0 provided that $B < 2p$. Suppose first that $B < p$. Then in the expansion of H, there can be no middle terms: we must have

$$H(x) = x^{p^s}(a_0 + a_B x^B).$$

Then

$$\Delta_H = a_0(x-y)^{p^s-1} + a_B \prod_{1\neq\zeta\in\mu_{p^s+B}} (x-\zeta y).$$

This is the finite part of the projective curve of equation

$$a_0 Z^B (X-Y)^{p^s-1} + a_B \prod_{1\neq\zeta\in\mu_{p^s+B}} (X-\zeta Y) = 0,$$

which has $p^s + B - 1$ points at ∞. If we invert $X - Y$, then in coordinates

$$z := Z/(X-Y), \quad x := X/(X-Y), \quad \text{and thus } Y/(X-Y) = x-1,$$

this curve becomes

$$a_0 z^B + a_B \prod_{1\neq\zeta\in\mu_{p^s+B}} \big(x - \zeta(x-1)\big) = 0.$$

This affine curve, call it $\mathcal{C}$, is defined by this polynomial, which is an Eisenstein polynomial in z for any of the factors $(x-\zeta(x-1))$ with $1 \neq \zeta \in \mu_{p^s+B}$. In particular, it is Eisenstein for the factor $2x-1$ (present because $p^s + B$ is even, as B was odd). Thus, $\mathcal{C}$ is geometrically irreducible; hence, $\Delta_H = 0$ is geometrically irreducible. The points on $\mathcal{C}$ with $z = 0$ are the points at ∞ on $\Delta_H = 0$, and z has a simple pole on $\Delta_H = 0$ at each of its zeros in $\mathcal{C}$. In particular, $2x-1$ has a pole of order B at the zero of z over $2x-1 = 0$. Over $\Delta_H = 0$, we are summing $\mathcal{L}_{\psi(\alpha((X+Y)/(X-Y)))} = \mathcal{L}_{\psi(\alpha(2x-1))}$, which has Swan $= B$ at the zero of z in $\mathcal{C}$ over $2x-1$. In particular, $\mathcal{L}_{\psi(\alpha((X+Y)/(X-Y)))}$ is not geometrically constant. Hence, this sum is $O(1/\sqrt{\#L})$, and the limsup is 0.

Suppose now that $2p > B > p$. Then in the expansion of H, there can be a middle term:

$$H(x) = x^{p^s}(a_0 + a_p x^p + a_B x^B) = a_0 x^{p^s} + a_p x^{p+p^s} + a_B x^{p^s+B}.$$

In this case, we write

$$p + p^s := pN,$$

and

$$\Delta_H = a_0(x-y)^{p^s-1} + a_p(x^N - y^N)^{p-1}\big((x^N - y^N)/(x-y)\big) + a_B \prod_{1\neq\zeta\in\mu_{p^s+B}} (x-\zeta y).$$

Because $N = 1 + p^{s-1}$ is even, the factor $(x^N - y^N)/(x - y)$ is divisible by $x + y$. Now we repeat the above argument. The curve $\Delta_H = 0$ is the finite part of the projective curve of equation

$$a_0 Z^B (X - Y)^{p^s - 1} + a_p Z^{B-p}(X^N - Y^N)^{p-1}\big((X^N - Y^N)/(X - Y)\big) + a_B \prod_{1 \neq \zeta \in \mu_{p^s+B}} (X - \zeta Y) = 0,$$

which has $p^s + B - 1$ points at ∞. If we invert $X - Y$, then in coordinates

$$z := Z/(X - Y), \quad x := X/(X - Y), \quad \text{and thus } Y/(X - Y) = x - 1,$$

we obtain the affine curve $\mathscr{C}$ of equation

$$a_0 z^B + a_p z^{B-p}\big(x^N - (x-1)^N\big)^{p-1}\big(x^N - (x-1)^N\big) + a_B \prod_{1 \neq \zeta \in \mu_{p^s+B}} \big(x - \zeta(x-1)\big) = 0.$$

The curve $\mathscr{C}$ is defined by this polynomial, which is (again) an Eisenstein polynomial in z for the factor $(2x - 1)$. From this point on, we repeat verbatim the proof in the case $B < p$ above.

In this $w = p^s - 1$, $A = 1$ case, if $\mathscr{G}(f, g, a, \chi_2)$ is induced, then it is induced from a rank one local system $\mathscr{L}_{\psi(\alpha x)}$, which has finite G_{geom}, and hence $\mathscr{G}(f, g, a, \chi)$ itself has finite G_{geom}.

In the $w = p^s - 1$, $A = 1$ case with $f, g \in \mathbb{F}_p[x]$, $p \geq 5$ and $g(x) = \sum_i a_i x^i$, we will use the hypothesis that either $B \equiv \frac{p-1}{2} \bmod (p-1)$ or

$$\sum_{i : i \equiv \frac{p-1}{2} \bmod (p-1)} a_i \neq 0$$

to show that G_{geom} is infinite. For this, it suffices to exhibit a point $t \in \mathbb{F}_p^\times$ where

$$\text{Trace}\big(\mathsf{Frob}_{t,\mathbb{F}_p} | \mathscr{G}(f, g, a, \chi_2)\big)$$

is not divisible by $\mathsf{Gauss}(\psi, \chi_2)$ as an algebraic integer. (Recall that $\mathscr{G}(f, g, a, \chi_2)$ has Frobenius traces in $\mathbb{Z}[\zeta_p]$ and is pure of weight one. Pass to $\mathscr{G}_0(f, g, a, \chi_2) := \mathscr{G}(f, g, a, \chi_2) \otimes (\mathsf{Gauss}(\psi, \chi_2)^{-\deg})$, which is pure of weight zero with traces in $\mathbb{Z}[\zeta_p][1/p]$. This twist $\mathscr{G}_0$ has arithmetic determinant of finite order. Indeed, any Frobenius determinant on $\mathscr{G}_0$ at a point $t \in \mathbb{F}_q^\times$ is an element of $\mathbb{Z}[\zeta_p][1/p]$ which is a unit for all places $\lambda \nmid p$ of $\mathbb{Q}(\zeta_p)$ (because $\mathscr{G}_0$ is part of a compatible system) and has complex absolute value 1 after all complex embeddings. As $\mathbb{Q}(\zeta_p)$ has a unique place over p, it follows (product formula) that this determinant has absolute value 1 everywhere, so is a root of unity in $\mathbb{Q}(\zeta_p)$, so it has order dividing $2p$. Thus, the arithmetic determinant of $\mathcal{G}_0$ has finite order. Then the finiteness of G_{geom} is equivalent to $\mathcal{G}_0$ having all Frobenius traces algebraic integers.)

To show that $\mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_p}|\mathcal{G}(f,g,a,\chi_2))$ is not divisible by $\mathsf{Gauss}(\psi,\chi_2)$ in $\mathbb{Z}[\zeta_p]$, we use a p-adic calculation. Define

$$\pi := \zeta_p - 1.$$

Then $\mathrm{ord}_p(\pi) = 1/(p-1)$, while $\mathrm{ord}_p(\mathsf{Gauss}(\psi,\chi_2)) = 1/2$. For $p \geq 5$, we have $1/2 > 1/(p-1)$. So we need only to find a Frobenius trace $\mathrm{Trace}(\mathsf{Frob}_{t,\mathbb{F}_p}|\mathcal{G}(f,g,a,\chi_2))$ which is divisible by π but not by π^2. This amounts to computing this Frobenius trace modulo π^2. For any $x \in \mathbb{F}_p$,

$$\psi(x) = \zeta_p^x = (1+\pi)^x \equiv 1 + \pi x \pmod{\pi^2},$$

and for any $x \in \mathbb{F}_p^\times$,

$$\chi_2(x) \equiv x^{(p-1)/2} \pmod{p}.$$

So for any Laurent polynomial $L(x) = \sum_i a_i x^i \in \mathbb{F}_p[x, 1/x]$, and any $x \in \mathbb{F}_p^\times$, we have

$$\chi_2(x)\psi\big(L(x)\big) \equiv x^{(p-1)/2}(1+\pi)^{L(x)} \equiv x^{(p-1)/2}\big(1+\pi L(x)\big) \pmod{\pi^2}.$$

Expanding out $L(x) = \sum_i a_i x^i$,

$$\begin{aligned}\sum_{x\in\mathbb{F}_p^\times} \chi_2(x)\psi\big(L(x)\big) &\equiv \sum_{x\in\mathbb{F}_p^\times} x^{(p-1)/2}\Big(1 + \pi \sum_i a_i x^i\Big) \pmod{\pi^2}\\ &\equiv \sum_{x\in\mathbb{F}_p^\times} \chi_2(x) + \pi \sum_i a_i \sum_{x\in\mathbb{F}_p^\times} x^{a_i+(p-1)/2} \pmod{\pi^2}.\end{aligned}$$

The sum $\sum_{x\in\mathbb{F}_p^\times} \chi_2(x)$ vanishes. The sum $\sum_{x\in\mathbb{F}_p^\times} x^{a_i+(p-1)/2}$ vanishes modulo p unless the exponent $a_i + (p-1)/2$ is a multiple of $p-1$, in which case this sum is $-1 \pmod{p}$. Thus,

$$-\sum_{x\in\mathbb{F}_p^\times} \chi_2(x)\psi\big(L(x)\big) \equiv \pi \sum_{i : i \equiv \frac{p-1}{2} \bmod (p-1)} a_i \pmod{\pi^2}.$$

In $\mathcal{G}(f,g,a,\chi_2)$, the relevant Laurent polynomial is $f(1/x) + g(x) + tx^a$. Here $f(1/x) = a_{-1}/x$, $g(x) = \sum_{i=0}^B a_i x^i$. Because $p \geq 5$, the $1/x$ term contributes 0. If B is not $(p-1)/2 \pmod{p-1}$, the tx^a term contributes 0, simply because $a = B + p^s - 1$ is not $(p-1)/2 \pmod{p-1}$, no matter what the value of $t \in \mathbb{F}_p^\times$. So for such B, we are done; if $\sum_{i \le B : i \equiv \frac{p-1}{2}} a_i \neq 0$, then we may choose any $t \in \mathbb{F}_p^\times$ at which to take the trace.

If, on the other hand, B is $(p-1)/2 \pmod{p-1}$, then the exponent a in tx^a, which is $a = B + p^s - 1$, is $(p-1)/2 \pmod{p-1}$. So the tx^a term contributes t, and

$$-\sum_{x \in \mathbb{F}_p^\times} \chi_2(x)\psi\big(L(x)\big) \equiv \pi\Big(t + \sum_{i \le B : i \equiv \frac{p-1}{2} \bmod (p-1)} a_i\Big) \pmod{\pi^2}.$$

We may always choose $t \in \mathbb{F}_p^\times$ so that the innermost sum is nonzero mod p. ∎

Here is an extension of the previous Theorem 2.10 to the special case $B = 1$, where we drop the hypothesis that $p \nmid (a - B)$.

THEOREM 2.11. *Suppose that* $p \nmid ABa(A + a)$, f *and* g *are both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that* $B = 1$. *Then* $\mathcal{G}(f, g, a, \chi)$ *is primitive on* $\mathbb{G}_m/\overline{\mathbb{F}_p}$.

PROOF. Repeat verbatim the first four paragraphs of the proof of Theorem 2.10, down to the point

$$\deg(\pi_{x_0}) \times \mathrm{rank}(\mathcal{H}) = \mathrm{rank}(W_{B,a-B}) = a - B$$

in the discussion of the $n = 0$ case. Because $B = 1$, $W_{B,a-B}$ has $\mathsf{Swan}_0(W_{B,a-B}) = B = 1$. In this $n = 0$ case, $\mathcal{H}$ is totally wild, and

$$\pi_{x_0\star}\mathcal{H} \cong W_{B,a-B}.$$

By [4, Lemma 1.6.4.1], we have

$$\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{H}) = \mathsf{Swan}_{x_0}(\mathcal{H}) + \mathrm{rank}(\mathcal{H})\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}).$$

In this equality, the left-hand side is $\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{H}) = \mathsf{Swan}_0(W_{B,a-B}) = 1$, while on the right, $\mathsf{Swan}_{x_0}(\mathcal{H}) \ge 1$ and $\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}) \ge 0$. Therefore, we must have

$$\mathsf{Swan}_{x_0}(\mathcal{H}) = 1 \quad \text{and} \quad \mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}) = 0.$$

Because $\mathsf{Swan}_0(\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}) = 0$, we must have $\deg(\pi_{x_0}) := D$ prime to p. Indeed, if $\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}$ is tame, then $\pi_{x_0\star}\mathcal{L}_{\mathbb{1}}$ has rank D and, being tame, is given by

$$\pi_{x_0\star}\mathcal{L}_{\mathbb{1}} = \bigoplus_{\chi \text{ with } \chi^D = \mathbb{1}} \mathcal{L}_\chi.$$

Thus, there are precisely D characters of order dividing D; hence, D is prime to p.

Once we have $\deg(\pi_{x_0}) := D$ prime to p, repeat the rest of the $n = 0$ case EP argument to get a contradiction.

Suppose now that $n \geq 1$. Exactly as in the proof of Theorem 2.10, we see that $\deg(\pi_{x_0}) = p^s$ for some $s \geq 0$. If $s = 0$, i.e., if $\deg(\pi_{x_0}) = 1$ is prime to p, then $\mathsf{Swan}_{x_0}(\mathcal{H}) = \mathsf{Swan}_0(\mathcal{G}) = B$, and we conclude as in the $n = 0$ case above.

We further see that when $s \geq 1$, we have $n = 1$, $\mathrm{rank}(\mathcal{H}) = 1$, $\mathcal{H}_{I(x_0)} = \mathcal{L}_{\mathbb{1}}$, and

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = \pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

But $\deg(\pi_{x_0})$ divides the rank of $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$, which is $1 + (a - B) = a$ (because $B = 1$). But $p \nmid a$, so $\deg(\pi_{x_0})$ cannot be p^s with $s \geq 1$. ∎

Here is an extension of Theorem 2.10 to the special case $A = 1$, where we (partially) drop the hypothesis that $p \nmid (A + a)(a - B)$.

THEOREM 2.12. *Suppose that $p \nmid ABa$, the polynomials f and g are both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that $A = 1$ and that $A + a = n_0 p^e$ with $e \geq 0$ and $1 \leq n_0 < p$. Then $\mathcal{G}(f, g, a, \chi)$ is primitive on $\mathbb{G}_m/\overline{\mathbb{F}_p}$.*

PROOF. Because $A = 1$, the $I(\infty)$-representation of $\mathcal{G}(f, g, a, \chi)$ is totally wild of rank $A + a = 1 + a$, with all slopes $A/(A + a) = 1/(a + 1)$. By Pink's argument [2, Lemma 11], if this $I(\infty)$-representation is induced, it is Kummer induced of some prime to p degree $D > 1$. As this D divides the rank $1 + a$, we see that $D | n_0$, and hence $D < p$. Thus, $\mathcal{G}(f, g, a, \chi) = \pi_\star\mathcal{H}$ for some lisse $\mathcal{H}$ on a finite étale connected

$$\pi : U \to \mathbb{G}_m$$

of degree D.

On the complete nonsingular model X of U, there is a unique point x_∞ lying over ∞ simply because $\mathcal{G}_{I(\infty)}$ is irreducible.

Now consider the unique point $x_0 \in X$ over 0 for which $\pi_{x_0\star}\mathcal{H}$ contains $W_{B,a-B}$ as $I(0)$-representation. The degree d_0 of π_{x_0} is $\leq D$, hence is $< p$, hence is prime to p. Thus, we have

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n, \quad \text{for some } n \geq 0.$$

Because $d_0 := \deg(\pi_{x_0})$ is prime to p, we have

$$\mathsf{Swan}_{x_0}(\mathcal{H}_{I(x_0)}) = \mathsf{Swan}_0\big(W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n\big) = B.$$

At any other point x_i lying over 0, the degree $d_i := \deg(\pi_{x_i})$ is again $\leq D < p$, hence is prime to p. At each such point, $\pi_{x_i}(\mathcal{H}_{I(x_0)})$ is a trivial $I(0)$-representation.

This first implies that $\mathcal{H}_{I(x_i)}$ is tame, and then that both $d_i = 1$ (otherwise, the Kummer direct image of any $\mathcal{L}_\chi$ by $[d_i]$ will not be entirely trivial) and that $\mathcal{H}_{I(x_i)}$ is just the direct sum $\overline{\mathbb{Q}_\ell}^{\mathrm{rank}(\mathcal{H})}$.

Now we give the EP argument. On the one hand, we have

$$\mathsf{EP}(U, \mathcal{H}) = \mathsf{EP}(\mathbb{G}_m, \mathcal{G}) = -\mathsf{Swan}_0(\mathcal{G}) - \mathsf{Swan}_\infty(\mathcal{G}) = -B - A,$$

while we also have

$$\begin{aligned}\mathsf{EP}(U, \mathcal{H}) &= \mathsf{EP}(U)\,\mathrm{rank}(\mathcal{H}) - \sum_{x_i \text{ over } \mathbf{0}} \mathsf{Swan}_{x_i}(\mathcal{H}) - \mathsf{Swan}_{x_\infty}(\mathcal{H}) \\ &= \mathsf{EP}(U)\,\mathrm{rank}(\mathcal{H}) - B - A.\end{aligned}$$

Comparing the two expressions for $\mathsf{EP}(U, \mathcal{H})$, we find $\mathsf{EP}(U)\,\mathrm{rank}(\mathcal{H}) = 0$, and hence $\mathsf{EP}(U) = 0$. But

$$\mathsf{EP}(U) = 2 - 2g_X - \#\{x_i \text{ over } 0\} - 1;$$

hence, $2g_X = 1 - \#\{x_i \text{ over } 0\}$. Hence, $g_X = 0$ and there is precisely one point over 0, as well as precisely one point over ∞. Thus, in suitable coordinates, U is $\mathbb{G}_m$, $x_\infty = \infty$, $x_0 = 0$, and π is the D'th power map. At $x_0 = 0$, $\mathcal{H}$ cannot be totally wild (otherwise, $[D]_\star(\mathcal{H})$ would be totally wild at 0), so it must contain some $\mathcal{L}_\chi$. Then, $[D]_\star(\mathcal{H})$ contains $[D]_\star(\mathcal{L}_\chi)$, which cannot be $I(0)$-trivial unless $D = 1$ (and $\chi = \mathbb{1}$). Thus, $D = 1$, contradiction. ■

Recall from [10, Chapter 1] the notion of (**S**+) for the G_{geom} of a local system in its given representation: that it is irreducible, primitive, tensor indecomposable, and not tensor induced.

Corollary 2.13. *Suppose that $p \nmid ABa$, the polynomials f and g are both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose that $A = 1$ and that $A + a = n_0 p^e$ with $0 \le e \le 1$ and $1 \le n_0 < p$. If $e = 0$, suppose further that $A + a$ is not a power. Then $\mathcal{G}(f, g, a, \chi)$ satisfies* (**S**+).

Proof. Indeed, the $I(\infty)$-representation is tensor indecomposable, cf. Lemma 3.4 later on. Furthermore, if $e = 1$, then $D = A + a$ cannot be a power. Thus, in all cases, $\mathcal{G}(f, g, a, \chi)$ cannot be tensor induced. ■

3. Elements with special spectra and tensor induction

Let $V = \mathbb{C}^d$. We will say an element $g \in \mathrm{GL}(V)$ has *quasi-simple spectrum*, and write that g is a qsp-*element*, if g is diagonalizable, and has at most one repeated eigenvalue but at least two distinct eigenvalues.

PROPOSITION 3.1. *Let $V = V_1 \otimes \cdots \otimes V_n$ be a tensor product of $n \geq 2$ $\mathbb{C}$-vector spaces each of dimension $d \geq 2$. Suppose $g \in (\mathrm{GL}(V_1) \otimes \cdots \otimes \mathrm{GL}(V_n)) \rtimes \mathsf{S}_n$ induces a nontrivial permutation π on the set of n tensor factors V_i and that g has simple or quasi-simple spectrum, and finite order on V. Then the following statements hold.*

(i) *Suppose $d \geq 3$. Then π is either an n-cycle or a 2-cycle.*

(ii) *If $d = 2$, then π is either an n-cycle, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle.*

PROOF. Write $\pi = \sigma_1\sigma_2 \ldots \sigma_l$ as a product of disjoint cycles of non-increasing lengths

$$k_1 \geq k_2 \geq \cdots \geq k_l \geq 1. \tag{3.1.1}$$

If $l = 1$, then π is an n-cycle, and we are done. Hence, we will assume $l \geq 2$, and so $\dim(V) = d^n \geq 4$.

First we note that if $g = X \otimes Y$ is tensor decomposable, then both X and Y have simple spectra. Indeed, if X say of size $s \times s$ with $s > 1$ has only a single eigenvalue, then each of the eigenvalues of g repeats $\geq s$ times, contrary to the assumption that g has a simple eigenvalue. Hence, X has at least two distinct eigenvalues $\alpha_1 \neq \alpha_2$. Now if Y admits a multiple eigenvalue $\beta_1 = \beta_2$, then $\alpha_1\beta_1$ and $\alpha_2\beta_1$ are two distinct multiple eigenvalues of g, again a contradiction. Hence, Y has a simple spectrum, and similarly does X.

Suitably conjugating g in $\mathrm{GL}(V)$, we may assume that

$$\pi = (1, 2, \ldots, k_1)(k_1 + 1, k_1 + 2, \ldots, k_1 + k_2) \ldots \left(\sum_{i=1}^{l-1} k_1 + 1, \sum_{i=1}^{l-1} k_i + 2, \ldots, n\right).$$

Now we can write $g = X \otimes Y$, where

$$X \in \mathrm{GL}(V_1 \otimes V_2 \otimes \cdots \otimes V_{k_1+\cdots+k_{l-1}})$$

permutes the $n - k_l$ tensor factors $V_1, \ldots, V_{n-k_l}$, inducing the permutation

$$(1, 2, \ldots, k_1)(k_1 + 1, k_1 + 2, \ldots, k_1 + k_2) \ldots \left(\sum_{i=1}^{l-2} k_1 + 1, \sum_{i=1}^{l-2} k_i + 2, \ldots, n - k_l\right),$$

and

$$Y \in \mathrm{GL}(V_{n-k_l+1} \otimes V_{n-k_l+2} \otimes \cdots \otimes V_n)$$

inducing the k_l tensor factors cyclically.

By the previous remark, both X and Y have simple spectra, and we may rescale X and Y so that both have finite order. Also, since π is nontrivial, we have $k_1 \geq 2$ and

$k_l \le k_1$ by (3.1.1). Assume first that $d \ge 3$. Then, applying [10, Proposition 5.2.3], we see that $k_1 = 2, k_2 = \cdots = k_{l-1} = 1$. Now if $k_l = 1$, then we arrive at (i). If $k_l = 2$, then we must have $l = 2$ by (3.1.1). In this case, the proof of [10, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V, a contradiction.

Assume now that $d = 2$. Again applying [10, Proposition 5.2.3], we have that either

(a) $k_1 = 2$ and $k_2 = \cdots = k_{l-1} = 1$, or

(b) $k_1 = 3$ and $k_2 = \cdots = k_{l-1} = 1$, or

(c) $k_1 = 3, k_2 = 2$, and $k_3 = \cdots = k_{l-1} = 1$.

In the case of (a), we cannot have $(l, k_l) = (2, 2)$ again by [10, Lemma 5.2.2]. So $k_l = 1$, and we arrive at (ii).

Suppose we are in the case of (b). If $k_l = 1$, then we arrive at (ii). If $k_l = 2$, then $l = 2$ by (3.1.1), and (ii) holds again. If $k_l = 3$, then $l = 2$ by (3.1.1), and the proof of [10, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V (namely, $\gamma\delta$ and $\gamma\delta\zeta_3$ in its notation), a contradiction.

Finally, assume we are in the case of (c). If $k_l = 1$, then we arrive at (ii). If $k_l = 2$, then $l = 3$ by (3.1.1), and the proof of [10, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V, again a contradiction. ■

We rule out the case of n-cycle of Proposition 3.1 in a more special situation.

Proposition 3.2. *Let $r \ge 2$ be a prime and let $V = V_1 \otimes \cdots \otimes V_r$ be a tensor product of r $\mathbb{C}$-vector spaces each of dimension $d \ge 2$. Suppose $g \in (\mathrm{GL}(V_1) \otimes \cdots \otimes \mathrm{GL}(V_r)) \rtimes \mathsf{S}_r$ induces an r-cycle on the set of r tensor factors V_i. Assume in addition that g is conjugate to*

$$\mathrm{diag}(\underbrace{1, \ldots, 1}_{t\ \text{times}}, \alpha, \alpha\zeta, \alpha\zeta^2, \ldots, \alpha\zeta^{w-1}),$$

where $t \ge 2$, $w \ge 1$, $\alpha \in \mathbb{C}^\times$ is a root of unity, $\zeta = \exp(2\pi i/w)$, and $(\alpha, w) \ne (1, 1)$. Then $d = r = 2$, and either $(t, w, \alpha) = (3, 1, -1)$ or $t = w = 2$ and $\alpha = \pm 1$.

Proof. (a) The assumptions imply that g is a qsp-element of finite order, with 1 being the only multiple eigenvalue. Again conjugating g suitably in $\mathrm{GL}(V)$, we may assume that

$$g : V_1 \mapsto V_2 \mapsto \cdots \mapsto V_r \mapsto V_1.$$

In particular, g^r induces a semisimple element of $\mathrm{GL}(V_1)$, and thus we can find a basis $(e^1_1, \ldots, e^1_d)$ of V_1 in which g^r acts as $\mathrm{diag}(x_1, x_2, \ldots, x_d)$ for some roots of unity $x_i \in \mathbb{C}^\times$. Defining $e^i_j = g^{i-1}(e^1_j)$ for $2 \le i \le r$ and $1 \le j \le d$, we see that $(e^i_1, \ldots, e^i_d)$ is a basis of V_i. Now arguing as in the proof of [10, Proposition 5.2.1], we see that the spectrum of g can be written (counting multiplicities) as

$$\text{Spec}(g) = \{\underbrace{1, \ldots, 1}_{t \text{ times}}\} \sqcup Z = X \sqcup Y, \tag{3.2.1}$$

where $X = \{x_1, x_2, \ldots, x_d\}$, and Y consists of $(d^r - d)/r$ r-tuples, each being all the rth roots of some $x_{i_1}x_{i_2}\ldots x_{i_r}$ with $1 \leq i_1, i_2, \ldots, i_r \leq d$ being not all the same, and

$$Z := \{\alpha, \alpha\zeta, \alpha\zeta^2, \ldots, \alpha\zeta^{w-1}\}.$$

In particular, Y is stable under the multiplication by the subgroup μ_r of $\mathbb{C}^\times$.

Suppose $d = r = 2$. Then (3.2.1) shows that

$$x_1 + x_2 = \text{Trace}(g) = t + \alpha \sum_{i=1}^{w} \zeta^i.$$

If $w = 2$, then we get $t = 2$ and then $2 = x_1 + x_2$, which implies $x_1 = x_2 = 1$ for the roots of unity x_1, x_2. In this case, $X = \{1, 1\}$ and $Y = \{1, -1\}$, and so $\alpha = \pm 1$. If $w = 1$, then we get $t = 3$ and then $3 = x_1 + x_2 - \alpha$ which implies $x_1 = x_2 = 1 = -\alpha$ for the roots of unity x_1, x_2, α.

(b) We will now assume $(d, r) \neq (2, 2)$, so that

$$(d^r - d)/r \geq 2. \tag{3.2.2}$$

In particular, Y contains at least two μ_r-cosets of $\beta, \gamma \in \mathbb{C}^\times$ (counting multiplicities).

First we show that

$$r | w. \tag{3.2.3}$$

Suppose that $r \nmid w$. Then $|\{\beta, \beta\zeta_r\} \cap Z| \leq 1$, and similarly $|\{\gamma, \gamma\zeta_r\} \cap Z| \leq 1$. It follows from (3.2.1) that at least one of $\beta, \beta\zeta_r$ is 1, which means that the μ_r-coset of β is μ_r. Similarly, the μ_r-coset of γ is μ_r. Thus, Y contains ζ_r twice, and hence $\zeta_r \neq 1$ is a multiple eigenvalue of g, a contradiction.

In particular, the elements in Z sum up to zero, and so

$$\text{Trace}(g) = t. \tag{3.2.4}$$

Next we show that the multi-set Y contains 1 at most once. Indeed, if Y contains 1 at least twice, then since Y is μ_r-stable, Y contains ζ_r at least twice, again a contradiction.

(c) Suppose that X contains 1 at least twice. Then, without loss we may assume $x_1 = x_2 = 1$. Now if $x_j \neq 1$ for some $j > 2$, then Y contains δ at least twice for

$$\delta^r = x_1^{r-1}x_j = x_2^{r-1}x_j \neq 1,$$

and thus $\delta \neq 1$ is a multiple eigenvalue of g, a contradiction. It follows that

$$x_1 = x_2 = \cdots = x_d = 1,$$

which means that g^r acts trivially on V_1 and hence $g^r = \mathrm{id}_V$. The formula for tensor induction [1] and (3.2.4) then show that

$$t = \mathrm{Trace}(g) = d.$$

Note that, in this case, Y consists of $(d^r - d)/r$ copies of μ_r and X contains 1 exactly d times. So the multiplicity of 1 as an eigenvalue of g is

$$d + (d^r - d)/r.$$

But this multiplicity is at most $t + 1 = d + 1$, so we arrive at $(d^r - d)/r \leq 1$, contrary to (3.2.2).

We have therefore shown that X contains 1 at most once. But in (c) we showed that Y also contains 1 at most once. On the other hand, the multiplicity of 1 in $\mathrm{Spec}(g)$ is at least $t \geq 2$. So we conclude that $t = 2$, and each of X and Y contains 1 exactly once. In such a case, the μ_r-invariance of Y implies that $\zeta_r \in Y$. Since $\zeta_r \neq 1$, ζ_r belongs to the set Z which is μ_w-invariant. By (3.2.3), Z is also μ_r-invariant, and hence $1 = (\zeta_r)(\zeta_r)^{-1}$ belongs to Z. But then the multiplicity of 1 in $\mathrm{Spec}(g)$ becomes 3, a contradiction. ∎

Next we will prove an auxiliary result on finite permutation groups.

Lemma 3.3. *Let p be a prime, and let $J = P \rtimes C$ be a transitive subgroup of S_n with $n > 1$ such that P is a transitive normal p-subgroup and $C = \langle \gamma \rangle$ is a cyclic p'-group. Suppose that every element in the coset γP is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Then one of the following statements holds.*

(i) $p = n = |J| = |P| = 2$.

(ii) $p = n = |J| = |P| = 3$.

(iii) $p = n = 3$, $P = C_3$, *and* $J = \mathsf{S}_3$. *Furthermore, γ is a 2-cycle.*

(iv) $n = 4$, $p = 2$, $P = C_2^2$, *and* $J = \mathsf{A}_4$. *Furthermore, γ is a 3-cycle.*

Proof. Let ρ denote the corresponding permutation character of J. Then the transitivity of J means that

$$\sum_{x \in J} \rho(x) = |J|. \tag{3.3.1}$$

Also, since P is transitive, we have

$$\sum_{x\in P} \rho(x) = |P|, \quad \text{and} \quad n = p^c \leq |P| \text{ for some } c \in \mathbb{Z}_{\geq 1}. \tag{3.3.2}$$

(a) First consider the case $J = P$. Then either $p = 2$ and every nontrivial element x in P is a 2-cycle, in which case $\rho(x) = n - 2$, or $p = 3$ and every nontrivial element x in P is a 3-cycle, in which case $\rho(x) = n - 3$. Using (3.3.2), in the former case we have

$$|P| = \sum_{x\in P} \rho(x) = n + (|P| - 1)(n - 2) = 2 + |P|(n - 2),$$

i.e., $|P|(n - 3) = -2$. As $|P| \geq 2$, we must have that $n = 2$ and hence $|P| = 2$, as stated in (i). In the latter case we have

$$|P| = \sum_{x\in P} \rho(x) = n + (|P| - 1)(n - 3) = 3 + |P|(n - 3),$$

i.e., $|P|(n - 4) = -3$. As $|P| \geq 3$, we conclude that $n = 3$ and hence $|P| = 3$, as stated in (ii).

(b) From now on we will assume that $J > P$, i.e., $\gamma \notin P$. By assumption, $\rho(x) \geq n - 5$ for all $x \in \gamma P$; furthermore, $x^6 = 1$, so $J/P \hookrightarrow C_6$. It follows from (3.3.1) and (3.3.2) that

$$6|P| \geq |J| = \sum_{x\in J} \rho(x) = \sum_{x\in P} \rho(x) + \sum_{x\in \gamma P} \rho(x) \geq (n - 4)|P|, \tag{3.3.3}$$

whence $p^c = n \leq 10$.

Assume in addition that $p \geq 5$. Then in fact we have $n = p \in \{5, 7\}$ and hence $P \cong C_p$. Now

$$\mathbf{N}_{\mathsf{S}_p}(P) = P \rtimes \langle\sigma\rangle,$$

where σ is a $(p - 1)$-cycle; in particular, any $1 \neq \sigma^i$ has a unique fixed point. As $P \lhd J$ and $\gamma \notin P$, we have $1 \neq \sigma^j \in \gamma P$ for some $j \in \mathbb{Z}$. Thus, σ^i is either a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle, none of which can have exactly one fixed point.

Now we consider the case $p = 3$. As $\gamma \neq 1$ is a $3'$-element, it must be a 2-cycle, and thus $\gamma^2 = 1$. It follows that $J/P = C_2$, so instead of (3.3.3) we now have

$$2|P| = |J| = \sum_{x\in J} \rho(x) = \sum_{x\in P} \rho(x) + \sum_{x\in \gamma P} \rho(x) \geq (n - 4)|P|.$$

Thus, $3^c = n \leq 6$. It follows that $n = 3$, $P = C_3$ and $J = \mathsf{S}_3$ as $J > P$, and we arrive at (iii).

Finally, let $p = 2$. As $\gamma \neq 1$ is a $2'$-element, it must be a 3-cycle, and thus $\gamma^3 = 1$. It follows that $J/P = C_3$. Furthermore, any element $x \in J$ belongs to γP if and only if $x^{-1} \in \gamma^{-1}P$, and so we also have $\rho(y) \geq n - 5$ for all $y \in \gamma^{-1}P$. So instead of (3.3.3) we now have

$$3|P| = |J| = \sum_{x \in J} \rho(x) = \sum_{x \in P} \rho(x) + \sum_{x \in \gamma P} \rho(x) + \sum_{x \in \gamma^{-1} P} \rho(x) \geq (2n - 9)|P|.$$

Thus, $2^c = n \leq 6$. The case $n = 2$ is impossible since $J > P \geq C_2$. So $n = 4$. Since the subgroup P of a Sylow 2-subgroup of S_4, which is dihedral of order 8, is normalized by the 3-cycle γ, we conclude that $P \cong C_2^2$ and so $J = \mathsf{A}_4$, as stated in (iv). ■

Now we establish some basic lemmas about tensor indecomposability and lack of tensor induction for $I(\infty)$ of ℓ-adic local systems. Recall that a representation ρ of a group G on a vector space V is said to be tensor decomposable if there exists a vector space isomorphism $V \cong V_1 \otimes V_2$ with both $\dim(V_i) \geq 2$ such that the image $\rho(g) \in \mathrm{GL}(V)$ of each element $g \in G$ can be written as $A_1 \otimes A_2$ with each $A_i \in \mathrm{GL}(V_i)$. This notion of tensor decomposability of a representation ρ does not in general imply that ρ is a tensor product of representations $\rho_1 \otimes \rho_2$.

Lemma 3.4. *Let $\mathcal{F}$ be an irreducible $I(\infty)$-representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and* $\gcd(N, D) = 1$. *Suppose further that $p^2 \nmid D$. Then $\mathcal{F}$ is tensor indecomposable.*

Proof. By (the $I(\infty)$-version of) [9, Proposition 2.2], if $\mathcal{F}$ is tensor decomposable, it is also linearly tensor decomposable, and hence we can write it as $\mathcal{A} \otimes \mathcal{B}$ where both $\mathcal{A}, \mathcal{B}$ are $I(\infty)$-representations of dimensions ≥ 2. Because $p^2 \nmid D$, at least one of $\mathcal{A}, \mathcal{B}$ has dimension prime to p, say $\mathcal{A}$ has dimension prime to p. By the argument proving [9, Proposition 2.2 (ii)], we may do so in such a way that $\mathcal{A}$ has $G_{\mathrm{geom}} \leq \mathrm{SL}_{\dim(\mathcal{A})}$, and then infer that both $\mathcal{A}, \mathcal{B}$ have all slopes $\leq N/D$. Each of $\mathcal{A}, \mathcal{B}$ is irreducible (otherwise, their tensor product is reducible). Let λ be the unique slope of $\mathcal{A}$ (unique because $\mathcal{A}$ is $I(\infty)$-irreducible). Then for $d := \dim(\mathcal{A})$, $d\lambda \in \mathbb{Z}$. This integrality shows that $\lambda < N/D$; indeed, if $\lambda = N/D$, then $dN/D \in \mathbb{Z}$ with $d < D$, impossible because $\gcd(N, D) = 1$. Similarly, $\mathcal{B}$ has unique slope $\mu < N/D$, and hence $\mathcal{A} \otimes \mathcal{B}$ has slopes $\leq \sup(\lambda, \mu) < N/D$, contradiction. ■

Lemma 3.5. *Let $\mathcal{F}$ be an $I(\infty)$-representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and* $\gcd(N, D) = 1$. *Then $\mathcal{F}$ is $I(\infty)$-irreducible.*

Proof. Indeed, any nonzero irreducible subrepresentation V has all slopes N/D, and the product $\dim(V) \times N/D \in \mathbb{Z}$, impossible if $\dim(V) < D$. ■

Combining Lemmas 3.4 and 3.5, we get the following corollary.

Corollary 3.6. *Suppose* $\gcd(a, A) = 1$ *and* $p^2 \nmid D := A + a$. *Then* $\mathcal{G}(f, g, a, \chi)$ *is both* $I(\infty)$*-irreducible and* $I(\infty)$*-tensor indecomposable.*

Proof. Here the $I(\infty)$-slopes are $A/(A + a)$ with $\gcd(A, a + A) = 1$. ■

Lemma 3.7. *Let* $\mathcal{F}$ *be an irreducible* $I(\infty)$*-representation of rank* $D \geq 2$ *all of whose slopes are* N/D *with* $N \geq 1$ *and* $\gcd(N, D) = 1$. *Suppose further that* $p^2 \nmid D$. *Suppose that* $D = d^n$ *with* $n \geq 2$, $n < p$, *and* $\gcd(n, D) = 1$. *Then* $\mathcal{F}$ *is not* n*-tensor induced.*

Proof. If $\mathcal{F}$ were n-tensor induced, the map $I(\infty) \mapsto \mathsf{S}_n$, giving the action on the tensor factors, is trivial on $P(\infty)$ simply because $P(\infty)$ is a pro-p group while S_n for $n < p$ has order prime to p. So the image of $I(\infty)$ is a cyclic subgroup of S_n, generated by the image π of a chosen element $\gamma \in I(\infty)$ which generates $I(\infty)/P(\infty)$. We first claim that π is an n-cycle. If not, write it as a product of disjoint cycles to see that γ preserves a tensor decomposition, and (hence) that every power of γ, times any element of $P(\infty)$, preserves this same tensor decomposition. Thus, the entire group $I(\infty)$ preserves this tensor decomposition. By Lemma 3.4, this contradicts the tensor indecomposability of $\mathcal{F}$. Once γ induces an n-cycle, γ (and then the entire group $I(\infty)$) preserves the tensor decomposition of the Kummer pullback $[n]^\star\mathcal{F}$. But this Kummer pullback $[n]^\star\mathcal{F}$ has rank D and all slopes nN/D, so it is irreducible when $\gcd(n, D) = 1$ (because then $\gcd(nN, D) = 1$) and hence is tensor indecomposable, the desired contradiction. ■

Lemma 3.8. *Suppose* $\mathcal{F}$ *is an* $I(\infty)$*-representation of the form* $T \oplus W$, *with* T *tame of rank* $t \geq 1$ *and with* W *irreducible of rank* $w \geq 1$ *with all slopes* m/w *with* $m \geq 1$ *and* $\gcd(m, w) = 1$. *Suppose further that* $t + w \neq 4$. *Suppose that* $D := t + w$, *the rank of* $\mathcal{F}$, *is a power* $D = d^n$ *with* $n \geq 2$, $n < p$, *and* $\gcd(n, w) = 1$. *Then* $\mathcal{F}$ *is not* n*-tensor induced.*

Proof. By [6, Corollary 10.4], $\mathcal{F}$ is tensor indecomposable. If it were n-tensor induced, then precisely as in the proof of Lemma 3.7, the image of γ must be an n-cycle. Then $[n]^\star\mathcal{F}$ is tensor decomposed. But $[n]^\star\mathcal{F} = [n]^\star T \oplus [n]^\star W$. Here $[n]^\star T$ is tame of the same rank t, and $[n]^\star W$ has rank w and all slopes nm/w. Then $[n]^\star W$ is irreducible by Lemma 3.5, and by [6, Corollary 10.4], $[n]^\star\mathcal{F}$ is tensor indecomposable, the desired contradiction. ■

Lemma 3.9 (Compare to [9, Lemma 3.2]). *Suppose* $A, a \geq 1$, *and* $\mathcal{F}$ *an* $I(\infty)$*-representation of rank* $D := A + a$ *all of whose slopes are* $A/(A + a)$. *Suppose that* $\mathcal{F}$ *is tensor indecomposable over* $I(\infty)$. *Suppose that* $\mathcal{F}$ *is* n*-tensor induced for* *some $n \geq 2$. Consider the map $\phi : I(\infty) \to \mathsf{S}_n$ giving the action on the tensor factors. If $(n-2)A < a$, then ϕ is trivial on $P(\infty)$, and the image of ϕ is the cyclic group generated by an n-cycle. Moreover, n is prime to p.*

Proof. To show that ϕ is trivial on $P(\infty)$, view $\mathsf{S}_n \leq \mathrm{O}_{n-1}$ by the deleted permutation representation. It suffices to show that $\phi : I(\infty) \to \mathrm{O}_{n-1}$ has $\mathsf{Swan}_\infty < 1$. Note that

$$\mathsf{Swan}_\infty \leq (n-1)(\text{the largest slope of } \mathcal{F}) = (n-1)A/(A+a).$$

Thus, $\mathsf{Swan}_\infty < 1$ is the condition

$$(n-1)A < A + a, \quad \text{i.e.,} \quad (n-2)A < a.$$

Let $\gamma \in I(\infty)$ be a generator of $I(\infty)/P(\infty)$. Then the image of ϕ is the cyclic subgroup of S_n generated by $\phi(\gamma)$. If $\phi(\gamma)$ were not an n-cycle, it (and every power of it, and hence every element of $I(\infty)$) would preserve some given tensor decomposition of $\mathcal{F}$, contradicting the tensor indecomposability of $\mathcal{F}$ over $I(\infty)$. Because $I(\infty)/P(\infty)$ has (pro) order prime to p, its image under ϕ has order prime to p. ∎

Remark 3.10. In the above Lemma 3.9, the condition $(n-2)A < a$ is always satisfied for $n = 2$. For the extreme case $A = 1$, the condition is $n < a + 2$, which is satisfied for *all* $n \geq 2$. (Indeed, the rank $D = a + 1$ must be an nth power, say $a + 1 = d^n$. Now if $n \geq a + 2 > a + 1$, then $n > d^n$, which is false for all $n \geq 1$ and all $d \geq 2$: worst case being $d = 2$, for which $2^n \geq n + 1$.) If in this $A = 1$ case we also had $p^2 \nmid (a+1)$ and both $\gcd(n, 1+a) = 1$ and $n < p$ whenever $1 + a = d^n$, then we would rule out $\mathcal{F}$ being tensor induced, by using Lemma 3.7. (But already for $n = 2$, we have a problem when $1 + a$ is an even square.)

Theorem 3.11. *Suppose $p \nmid ABa$, f, g are both Artin–Schreier reduced, and*

$$\gcd\big(a, \gcd_{\deg}(f)\big) = 1, \quad \gcd\big(a, \gcd_{\deg}(g)\big) = 1.$$

Suppose that $\mathcal{G} = \mathcal{G}(f, g, a, \chi)$ is n-tensor induced as a representation of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$ for some $n \geq 2$. Assume in addition that $p \nmid w = a - B$ and $D = a + A > 4$. Then all the following conditions hold.

(i) *Either $n = p = 3$ or $(D, n, p) = (16, 4, 2)$.*

(ii) *$\mathcal{G}$ is tensor decomposable over $I(\infty)$.*

(iii) *If in addition $\gcd(a, A) = 1$, then $p^n | D$.*

Proof. By assumption, $G = G_{\mathrm{geom}}$ stabilizes a tensor-induced decomposition $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of the underlying representation V, with $d := \dim(V_i)$ and $D = d^n$.

Let $\pi : G \to \mathsf{S}_n$ denote the permutation representation of G while acting on the n tensor factors of V. By Proposition 2.8, $\mathcal{G}$ is tensor indecomposable over $I(0)$; hence, $\pi(I(0))$ is a transitive subgroup of S_n. Furthermore, since $D > 4$, we must have $w > 1$ by Proposition 2.9 (b).

Fix a p'-generator γ of $I(0)$ over $P(0)$, and write $\pi(I(0)) = J = P \rtimes C$, where $P = \pi(P(0))$ and $C = \langle \pi(\gamma) \rangle$. By Lemma 2.1, the condition $p \nmid w$ implies that the action of γ on V has spectrum

$$\text{diag}\big(\underbrace{1, \dots, 1}_{t \text{ times}}, \alpha, \alpha\zeta, \alpha\zeta^2, \dots, \alpha\zeta^{w-1}\big), \tag{3.11.1}$$

where $t = A + B \geq 2$, $w = a - B \geq 1$, $\alpha \in \mathbb{C}^\times$ is a root of unity, $\zeta = \exp(2\pi i/w)$. As $w > 1$, $\gamma|_V$ is a qsp-element. In fact, this also holds for any element in the coset $\gamma P(0)$ for the same reason.

Since $J \leq \mathsf{S}_n$ is transitive and $P \lhd J$, P acts on the set $\{V_1, V_2, \dots, V_n\}$ with $e \geq 1$ orbits $\Omega_1, \dots, \Omega_e$, all of length n/e and permuted cyclically by $\pi(\gamma)$. Suppose that $e > 1$. Letting U_j be the tensor product of the V_i in Ω_j for $1 \leq j \leq e$, we see that γ permutes the e tensor factors of the decomposition $V = U_1 \otimes U_2 \otimes \ldots \otimes U_e$ cyclically, say

$$U_1 \mapsto U_2 \mapsto U_3 \mapsto \cdots \mapsto U_e \mapsto U_1.$$

Choosing a prime divisor r of e, we see that γ permutes the r sets

$$\Delta_j := \big\{U_i \mid 1 \leq i \leq r,\ i \equiv j \pmod r\big\}, \quad 1 \leq j \leq r,$$

cyclically. Letting W_j be the tensor product of the U_i in Δ_j, $1 \leq j \leq r$, we now have that the element γ with spectrum (3.11.1) permutes the r tensor factors of the decomposition $V = W_1 \otimes W_2 \otimes \cdots \otimes W_r$ cyclically. But this is impossible by Proposition 3.2 and the assumption that $D > 4$.

Thus, $P \leq \mathsf{S}_n$ is a transitive subgroup; in particular, $n = p^c$ for some $c \geq 1$. By Proposition 3.2 applied to any $\gamma' \in \gamma P(0)$, $\pi(\gamma')$ cannot be an n-cycle (because $D > 4$). Applying Proposition 3.1 to γ', we see that any element in the coset $\pi(\gamma)P$ is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Hence, we can apply Lemma 3.3 to J.

In the cases of Lemmas 3.3 (i) or 3.3 (ii), $\pi(\gamma)P = P$, and so some element $\gamma_1 \in \gamma P(0)$ has $\pi(\gamma_1)$ being an n-cycle, a contradiction. Thus, we are in the case of Lemma 3.3 (iii), whence $n = p = 3$, or of Lemma 3.3 (iv), whence $(n, p) = (4, 2)$. Observe that in the latter case $D = 16$. Indeed, in this case $\pi(\gamma)$ is a 3-cycle, so we may write the action of γ as $X \otimes Y$, where $X \in \mathrm{GL}(V_1 \otimes V_2 \otimes V_3)$ permutes V_1, V_2, V_3 cyclically, and $Y \in \mathrm{GL}(V_4)$. By the proof of Proposition 3.1, X has simple spectrum. Applying Proposition 3.1 to X, we see that $d = 2$ and hence $D = 16$. Thus, we have proved (i).

In these two remaining cases, we now show that

$$a > (n-2)A. \tag{3.11.2}$$

Assume we are in the case of Lemma 3.3 (iii), so that $\pi(\gamma)$ is a 2-cycle and $D = d^3$. Since $\gamma|_V$ has finite order and flips, say, V_1 and V_2, the formula for tensor induction [1] shows that

$$\left|\mathrm{Trace}(\gamma|_V)\right| \le d^2 \le D/2.$$

Now, (3.11.1) implies that $\mathrm{Trace}(\gamma|_V) = t$, whence $t + w = D \ge 2t$, and so

$$a - B = w \ge t = A + B,$$

implying (3.11.2).

Next suppose we are in the case of Lemma 3.3 (iv), so that $\pi(\gamma)$ is a 3-cycle and $D = d^4 = 16$. Since $\gamma|_V$ has finite order and permutes, say, V_1, V_2, and V_3, cyclically, the formula for tensor induction [1] shows that

$$\left|\mathrm{Trace}(\gamma|_V)\right| \le d^2 = D/4.$$

Now, (3.11.1) implies that $\mathrm{Trace}(\gamma|_V) = t$, whence $t + w = D \ge 4t$, and so

$$a - B = w \ge 3t = 3A + 3B,$$

implying (3.11.2).

Thus, we have proved (3.11.2). Now, if $\mathcal{G}$ is tensor indecomposable over $I(\infty)$, then the equality $n = p$ contradicts Lemma 3.9. So $\mathcal{G}$ is tensor decomposable over $I(\infty)$, proving (ii).

Assume in addition that $\gcd(a, A) = 1$. Then the tensor decomposability over $I(\infty)$ of $\mathcal{G}$ implies by Lemma 3.4 that $p|D$. But $D = d^n$, so $p^n|D$, establishing (iii). ∎

Corollary 3.12. *Suppose $p \nmid ABa$, f, g are both Artin–Schreier reduced, and* $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. *Suppose in addition that $p \nmid w = a - B$ and $D = a + A > 4$. If $\mathcal{G}(f, g, a, \chi)$ is primitive (e.g., by Theorem 2.10), then it satisfies* **(S+)** *if either $p \ge 5$ or $p^2 \nmid D$.*

Remark 3.13. In cases when $p|w$, there are other ways to prove **(S+)**. We can sometimes apply Theorems 2.11 or 2.12 to prove primitivity, and we can sometimes apply Propositions 2.8 and 2.9 to prove the absence of tensor induction.

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