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Linking number obstructions to satellite homomorphisms

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Abstract. We prove that satellite operations that satisfy a certain positivity condition and have winding number other than one are not homomorphisms. The argument uses the d-invariants of branched covers. In the process, we prove a technical result relating d-invariants and the Torelli group which may be of independent interest.

1. Introduction

Two oriented knots in S^3 are called concordant if they cobound a smoothly embedded annulus in $S^3 \times I$. The relation of concordance is an equivalence relation on the set of knots, and the resulting collection of equivalence classes is denoted by $\mathcal{C}$. A knot is called slice if it is concordant to the unknot or, equivalently, if it bounds a smoothly embedded disc in B^4. There have been many applications of concordance to low-dimensional topology: for example, the existence of a knot that is non-trivial in $\mathcal{C}$ but bounds a locally flat topologically embedded disk in B^4 implies the existence of an exotic $\mathbb{R}^4$ [15]. Interestingly, $\mathcal{C}$ takes on the structure of an abelian group under connected sum [11], and there has been a large amount of work on analyzing the algebraic structure of this group [4, 5, 12, 21, 26, 27, 31, 32, 39]. The classical satellite construction, wherein a pattern knot P in a solid torus and a companion knot K are combined to give a satellite knot $P(K)$, induces well-defined functions on $\mathcal{C}$ which are much studied [6–9, 20, 24, 36, 48]. For example, [8] uses satellite maps to give evidence that the knot concordance group may have a fractal structure with respect to a certain naturally defined metric, and [7,24,36] analyze the injectivity and surjectivity of satellite maps. However, the following conjecture of Hedden suggests that satellite maps on $\mathcal{C}$ almost never interact nicely with the connected sum group structure.

Conjecture 1.1 (Hedden [18,47]). *Let P be a pattern knot in the solid torus. Let $[P]: \mathcal{C} \to \mathcal{C}$ be the map that sends the concordance class of K to the concordance class of $P(K)$. If $[P]$ is a group homomorphism, then $[P]$ is either the zero, the identity, or the reversal map.*

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Throughout the paper, we will use "P induces a homomorphism" interchangeably with "$[P]$ is a homomorphism".

Many patterns can be obstructed from inducing homomorphisms by the unexciting requirement that $P(U)$ must be a slice knot, where U denotes the unknot. However, if $P(U)$ is slice, then classical invariants, such as the Alexander polynomial and the Tristram–Levine signature function, cannot obstruct P from inducing a homomorphism on $\mathcal{C}$ [30, 35]. The first interesting non-homomorphism in the literature was therefore not identified until the work of Gompf [14], who showed that the Whitehead pattern does not induce a homomorphism on $\mathcal{C}$, despite inducing the zero map on the topological concordance group. Later, Levine [24] and Hedden [17] used the τ invariant from Heegaard Floer homology to show that the Mazur pattern and $(n, 1)$ cables $(n > 1)$ also do not induce homomorphisms. Most recently, Miller [35] used Casson–Gordon signatures to give an obstruction to a pattern inducing a homomorphism even on the topological concordance group.

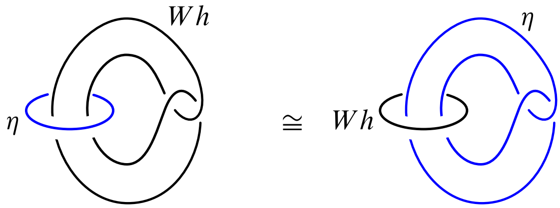
Our first main result already implies Hedden's conjecture for a large collection of patterns. For a pattern P in the solid torus and prime power q, the q-fold cyclic branched cover of S^3 along $P(U)$ is a rational homology sphere $\Sigma_q(P(U))$. When q divides the winding number of P, the curve η that bounds a meridional disk in the solid torus lifts to a q-component framed oriented link $\eta_1 \cup \cdots \cup \eta_q$ in $\Sigma_q(P(U))$. (Recall that the winding number of an oriented pattern P in $S^1 \times D^2$ is the unique $w \in \mathbb{Z}$ such that the element of $H_1(S^1 \times D^2; \mathbb{Z})$ represented by P equals w times the homology class represented by $S^1 \times \{*\}$.) For any two distinct components η_i and η_j, the linking number $\operatorname{lk}(\eta_i, \eta_j)$ is a rational number; see Section 2 for the definition.

Theorem 1.2. *Let P be a pattern in the solid torus with winding number w, and let q be a prime power dividing w. Suppose that the lifts $\eta_1, \dots, \eta_q$ of η to $\Sigma_q(P(U))$ are null-homologous in $\Sigma_q(P(U))$. If $\operatorname{lk}(\eta_i, \eta_j) \geq 0$ for all $i \neq j$ but is not identically zero, then P does not induce a homomorphism of the smooth concordance group.*

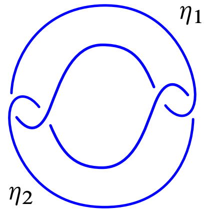
Since a pattern P induces a homomorphism exactly when the mirrored pattern $\overline{P}$ does, Theorem 1.2 can also be used to obstruct a pattern with $\operatorname{lk}(\eta_i, \eta_j) \leq 0$ for all $i \neq j$ from inducing a homomorphism. We prove Theorem 1.2 by showing that the assumptions imply that $P(-K)$ is generally not concordant to $-P(K)$; that is, P does not induce a *pseudo-homomorphism*, using the language of [35, Section 5].

Example 1.3. Let $\mathrm{Wh} \cup \eta$ denote the positive Whitehead link, which determines the Whitehead doubling pattern (see Section 2 for a discussion of the relationship between patterns and 2-component links). Then, $\Sigma_2(\mathrm{Wh}(U))$ is S^3 and $\eta_1 \cup \eta_2$ is the $-T(2,4)$ torus link, as illustrated in Figure 1. Similarly, for any $n > 1$ and q any prime power dividing n, we have that

$$\Sigma_q(C_{n,1}(U)) = S^3$$



Symmetry of the pattern for Whitehead doubles.



A diagram of the 2-fold branched cover of $Wh(K)$.

Figure 1. Whitehead doubles.

and the lifts of η form the $T(n, q)$ torus link. We therefore see that Theorem 1.2 recovers the results of [14] and [17] that Whitehead doubling and the $(n, 1)$ cable map do not induce homomorphisms on concordance.

Example 1.4. Up to mirroring and reversal, there are 19 patterns that have $P(U)$ slice and that can be presented by a 2-component link $P \cup \eta$ with no more than 8 crossings [35, Section 5]. Four of these satellite operators (L2a1, L7a5, L6a2, and L8a9, in the notation of [35]) induce pseudo-homomorphisms and two more (L7a6 and L8a8) are winding number one: Theorem 1.2 cannot be applied in either case. However, the remaining thirteen patterns (L4a1=$C_{2,1}$, L5a1=Wh, L6a1, L6a3=$C_{3,1}$, L7a4, L8a1a, L8a1b, L8a6, L8a10, L8a11, L8a12, L8a13, and L8a14=$C_{4,1}$) are all obstructed from inducing homomorphisms by Theorem 1.2 and $q \in \{2, 3, 4\}$. While these patterns were already known not to induce pseudo-homomorphisms, previous proofs relied on a variety of ad hoc techniques: it is pleasing to obstruct all thirteen with a single method.

Previous work has shown that if the curves $\eta_1, \ldots, \eta_q$ generate the first homology of $\Sigma_q(P(U))$, then P does not induce a homomorphism [35, Theorem A]. Our work was motivated in part by a desire to obstruct homomorphisms when each component $\eta_1, \ldots, \eta_q$ is null-homologous, but we are also able to strengthen Theorem 1.2 to the case when the lifts $\eta_1, \ldots, \eta_q$ are not null-homologous in $\Sigma_q(P(U))$.

Theorem 1.5. *Let P be a pattern in the solid torus with winding number w, and let q be a prime power dividing w. Let $\eta_1, \ldots, \eta_q$ denote the lifts of η to $\Sigma_q(P(U))$, and* *let n denote the order of $[\eta_1]$ in $H_1(\Sigma_q(P(U)))$. Suppose that either n is odd or w is a nonzero multiple of n. If $\operatorname{lk}(\eta_i, \eta_j) \geq 0$ for all $i \neq j$ but is not identically zero, then P does not induce a homomorphism of the smooth concordance group.*

We remark that all lifts of η to $\Sigma_q(P(U))$ represent elements of $H_1(\Sigma_q(P(U)))$ having the same order, since these lifts are related by the covering transformation of $\Sigma_q(P(U))$, which is a diffeomorphism.

Theorem 1.5 gives us a particularly nice consequence when $q = 2$.

Corollary 1.6. *Let P be a pattern in the solid torus with even winding number such that the two lifts of η to $\Sigma_2(P(U))$ have nonzero rational linking number. Then, P does not induce a homomorphism.*

Proof. The order of the first homology of the double branched cover of S^3 along a knot, and hence the order of $[\eta_1]$ in $H_1(\Sigma_2(P(U)))$, is odd. ■

Theorem 1.5 arises as an almost immediate corollary of the following somewhat peculiar result.

Theorem 1.7. *Let P be a pattern in the solid torus with winding number w and q a prime power dividing w. Denote the q lifts of η to $\Sigma_q(P(U))$ by $\eta_1, \ldots, \eta_q$. Suppose that, for $i \neq j$, the numbers $\operatorname{lk}(\eta_i, \eta_j)$ are non-negative and not identically zero, and let n be the order of $[\eta_1]$ in $H_1(\Sigma_q(P(U)))$. If R is any pattern whose winding number is a nonzero multiple of n, then the composition $P \circ R$ does not induce a pseudo-homomorphism.*

Proof of Theorem 1.5, assuming Theorem 1.7. If n is odd, let Q be the "$(n, 1)$ alternating cable" given by the n-strand braid $\prod_{i=1}^{n-1} \sigma_i^{(-1)^{i+1}}$, illustrated in the center of Figure 9 for $n = 3$. Note that $Q(U) = U$ is slice and Q is isotopic to $-Q$ in the solid torus, and hence, $Q(-K)$ is isotopic to $-Q(K)$ for all knots K [35]. So, Q does induce a pseudo-homomorphism, but Theorem 1.7 implies that $P \circ Q$ does not induce a pseudo-homomorphism. We therefore have our desired conclusion.

If w is a nonzero multiple of n, then Theorem 1.7 implies that $P \circ P$ does not induce a pseudo-homomorphism, and so, we obtain our desired conclusion. ■

We also obtain the following result, showing that our obstruction is robust under composition with arbitrary patterns of nonzero winding number.

Corollary 1.8. *Let P be a pattern satisfying the hypotheses from Theorem 1.2 and Q an arbitrary pattern with nonzero winding number. Then, $P \circ Q$ does not induce a homomorphism.*

Proof. By the assumption on P, there exists a prime power q for which the lifts of η to $\Sigma_q(P(U))$ are null-homologous. We thus have that the order of η_1 in homology is

1, and so, $n = 1$ in the notation of Theorem 1.7. Any Q with nonzero winding number is clearly a nonzero multiple of $n = 1$, and so, we can conclude that $P \circ Q$ does not induce a homomorphism by Theorem 1.7. ■

Additional evidence for the strength of Theorem 1.5 comes from recent work of Johanningsmeier, Kim, and the second author [22]. They verified Conjecture 1.1 for patterns with winding numbers that are even but not divisible by eight by showing that all such patterns are obstructed from inducing homomorphisms by Theorem 1.5.

The rough strategy to prove Theorem 1.2 is as follows. If P acts as a homomorphism, then the knot $J_k = P(T_{2,2k+1})\#P(-T_{2,2k+1})$ must be slice, and so, the q-fold cyclic branched cover of J_k would bound a rational homology ball and hence have some Heegaard Floer d-invariants vanishing. (For the relevant background on the d-invariants, see Section 3.) In particular, this would imply that the maximal d-invariant of $\Sigma_q(J_k)$ would be non-negative, as observed in Corollary 3.2. The linking hypothesis allows us to show that, for sufficiently large k, all d-invariants of $\Sigma_q(J_k)$ are highly negative, thereby obstructing P from inducing a homomorphism.

In order to carry out the d-invariant bound, we prove the following technical result which may be of interest to those working in Heegaard Floer homology and/or mapping class groups. Given a 3-manifold Y, an embedding $i\colon \Sigma_g \to Y$ of a closed genus g surface into Y, and an element ϕ of the mapping class group of Σ_g, let Y_ϕ be the manifold obtained from Y after removing $i(\Sigma_g)$ and reattaching via ϕ. See Equation (3.2) in Section 3.4 for a precise definition.

Theorem 1.9. *Let $N, g \geq 0$. There exists a constant $C_{N,g}$ depending only on N and g with the following property: for any integer homology sphere Y, any embedding $i\colon \Sigma_g \to Y$, and any element ϕ in the Torelli group of Σ_g expressible as a product of at most N separating Dehn twists and/or bounding pair maps,*

$$|d(Y) - d(Y_\phi)| \leq C_{N,g}.$$

There is a similar statement for performing such "Torelli surgeries" in rational homology spheres, but we do not state it here for ease of reading. In forthcoming work of Afton, Kuzbary, and Lidman, this relationship with the Torelli group will be explored further.

1.1. Proof strategy for Theorem 1.2

The (3-manifold, framed link) pair $(\Sigma_q(P(U)), \bigcup_{i=1}^{q} \eta_i)$ contains all the information necessary to construct $\Sigma_q(P(K))$ for any knot K, as discussed in detail at the beginning of Section 2. Our key idea is to construct a cobordism from $(\Sigma_q(P(U)), \bigcup_{i=1}^{q} \eta_i)$ to (S^3, H_q), where H_q is the split union of the (-1)-framed positive Hopf link with a $(q-2)$-component 0-framed unlink. We use three types of modifications, discussed in Section 2: null-homologous surgeries, simple negative definite cobordisms, and clasper surgeries. In Section 3, we analyze how these three operations relate the d-invariants of $\Sigma_q(P(K))$, encoded by $(\Sigma_q(P(U)), \bigcup_{i=1}^{q} \eta_i)$, to those of $\Sigma_2(C_{2,1}(K))$, encoded by (S^3, H_q).

We are able to conclude that there exists a constant $c_{P,q}$ depending only on P and q such that for any knot K one has

$$d_{\max}(\Sigma_q(P(K))) \leq d_{\max}(\Sigma_2(C_{2,1}(K))) + c_{P,q}.$$

We then use well-known formulas to observe that

$$\lim_{k\to\infty} d_{\max}(\Sigma_2(C_{2,1}(T_{2,2k+1}))) + d_{\max}(\Sigma_2(C_{2,1}(-T_{2,2k+1}))) = -\infty$$

and use this to see that, for sufficiently large k,

$$d_{\max}(\Sigma_q(P(T_{2,2k+1})\#P(-T_{2,2k+1}))) < 0.$$

We then conclude that $\Sigma_q(P(T_{2,2k+1})\#P(-T_{2,2k+1}))$ cannot bound a rational homology ball and hence that $P(T_{2,2k+1})\#P(-T_{2,2k+1})$ is not slice.

Remark 1.10. This project was an offshoot of an attempt to approach the problem of concordance homomorphisms using Chern–Simons theory as applied previously by Hedden, Kirk, and Pinzón-Caicedo (see, for example, [19, 20]). The key idea would be to build a certain negative definite cobordism from $\Sigma_q(P(T_{r,s})\#P(-T_{r,s}))$ to a 3-manifold built out of surgeries on $T_{r,s}$. Roughly, as the parameters of the torus knots grow, the minimal Chern–Simons invariants of this manifold would decrease, and once small enough, would preclude the branched cover from bounding a rational homology ball. The input of Heegaard Floer homology allowed us to use this argument, with the unboundedness of $d(\Sigma_2(P(T_{2,2k+1})\#P(-T_{2,2k+1})))$ playing the role of "small minimal Chern–Simons invariant", while simultaneously understanding the behavior of the invariant under other topological operations, such as Dehn surgery and clasper surgery (see Section 3 for concrete statements). It would be interesting to prove some of the analogous statements in Section 3 in the realm of Chern–Simons theory.

Organization

In Section 2, we describe certain cyclic branched covers of satellite knots in terms of (3-manifold, framed link) pairs and define and prove results about null-homologous surgery, twisting along an unknot, and clasper surgery. In Section 3, we constrain how each of these operations can change the d-invariants of the associated 3-manifolds and prove Theorem 1.9. Finally, in Section 4, we prove our main results, Theorem 1.2 and Theorem 1.7.

2. Topological constructions

Since we base our argument in a specific topological relationship involving cyclic branched covers of satellite knots, we begin this section with a convenient decomposition of these covers, originally due to Seifert [49]. Given a pattern P in a parametrized solid torus $S^1 \times D^2$, we can take the standard embedding of $S^1 \times D^2$ into S^3 and consider the 2-component link given by $P \cup (* \times \partial D^2)$. This process is reversible: given an ordered oriented link $P \cup \eta$ in S^3 such that η is unknotted, we can canonically identify $S^3 \smallsetminus \nu(\eta)$ with a parametrized solid torus $S^1 \times D^2$ by requiring that a meridian of η be identified with $S^1 \times \{*\}$ for some $* \in \partial D^2$. From this perspective, the winding number is precisely the linking number of P with η.

One virtue of this link description of a pattern is that it leads to a useful decomposition of the exterior of a satellite knot:

$$E_{P(K)} = (S^3 \smallsetminus \nu(\eta \cup P)) \cup E_K, \tag{2.1}$$

where ∂E_K is identified with $\partial \overline{\nu(\eta)}$ in such a way that λ_K, the 0-framed longitude of K, is identified with μ_η and similarly μ_K is identified with λ_η.

This induces a convenient decomposition of certain branched covers of S^3 along $P(K)$, as follows. Let q be a positive divisor of the winding number $w = \operatorname{lk}(P, \eta)$, and observe that the curve η lifts to q distinct curves $\eta_1, \dots, \eta_q$ in the q-fold cyclic branched cover $\Sigma_q(P(U))$. The q-fold cover of S^3 branched over $P(K)$ then decomposes as

$$\Sigma_q(P(K)) = \big(\Sigma_q(P(U)) \setminus \nu(\eta_1 \cup \cdots \cup \eta_q)\big) \cup \bigcup_{i=1}^{q} E_K^{(i)}. \tag{2.2}$$

Note that $E_K^{(i)}$ is attached along $\partial \overline{\nu(\eta_i)}$ via a gluing map that identifies $\mu_K^{(i)}$ with the ith lift of a 0-framing of η, and $\lambda_K^{(i)}$ with the ith lift of a meridian of η.

Motivated by this topological description of covers branched along satellite knots, we introduce the following notation: given a framed q-component link L in a 3-manifold Y and a knot K in S^3, we define

$$(Y, L) * E_K = (Y \smallsetminus \nu(L)) \cup \bigcup_{i=1}^{q} E_K^{(i)},$$

where E_K denotes the exterior of K and $\partial E_K^{(i)}$ is identified with $\partial\nu(L_i)$ such that the Seifert longitude λ_K is identified with μ_{L_i} and that μ_K is identified with the framing curve γ_{L_i}.[1] In this paper, $*$ will always refer to the above operation, rather than the join of two topological spaces. Moreover, the decomposition from Equation (2.2) can thus be denoted by

$$(\Sigma_q(P(U)), \eta_1 \cup \cdots \cup \eta_q) * E_K.$$

We note for later reference that if Y is an integer homology sphere and L is any link in L, the manifold $(Y, L) * E_K$ must also be an integer homology sphere, since knot exteriors are homology solid tori.

Example 2.1. Let U be an n-framed unknot in S^3. Then, for any knot K, we have that

$$(S^3, U) * E_K = E_U \cup E_K,$$

where λ_K is identified with μ_U and μ_K is identified with a curve representing $n[\mu_U]+[\lambda_U]$ in $H_1(\partial E_U)$. Therefore, the curve representing $[\mu_K]-n[\lambda_K]$ in $H_1(E_K)$ is identified with λ_U, which bounds a meridional disc in $E_U = S^1 \times D^2$, and we have that

$$(S^3, U) * E_K = S^3_{-1/n}(K).$$

Example 2.2. Let K be any knot. The Akbulut–Kirby algorithm [1, Section 2] applied to the Möbius band bounded by $C_{2,1}(K)$ gives that

$$\Sigma_2(C_{2,1}(K)) = S^3_{+1}(K \# K^r).$$

Alternately, we obtain that

$$\Sigma_2(C_{2,1}(K)) = (S^3, H) * E_K,$$

where H denotes the (-1)-framed positive Hopf link, as follows.

Recall that the exterior of $C_{2,1}(K)$ is obtained from the exterior of $C_{2,1} \cup \eta$ by gluing in E_K so that λ_K is identified with μ_η and μ_K is identified with a curve representing λ_η. In Figure 2, we see two diagrams for $C_{2,1} \cup \eta$: the usual one on the left and a diagram where $C_{2,1}$ is the standard unknot on the right. In Figure 3, we branch over the disc bounded by $C_{2,1}$ to obtain a schematic diagram for $\Sigma_2(C_{2,1}(K))$. Note that the -1 labels on the two lifts of η are not surgery framings but rather indicate that μ_K must be glued to the $(-1, 1)$-curve of each η_i. They come from the fact that the curve η on the right of Figure 2 has writhe equal to $+1$. We therefore obtain as desired that $\Sigma_2(C_{2,1}(K)) = (S^3, H) * E_K$.

[1]For example, if L_i is null-homologous and f_i is the framing number of L_i, then γ_{L_i} is the push-off of L_i to $\partial N(L_i)$ such that $\mathrm{lk}(L_i, \gamma_{L_i}) = f_i$.

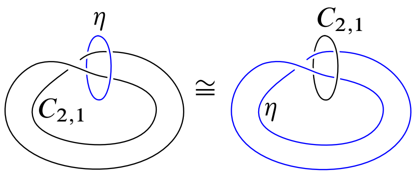


Figure 2. The $(2,1)$-cable pattern.

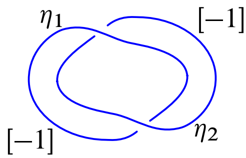


Figure 3. A diagram for the 2-fold branched cover of $C_{2,1}(K)$, where the -1 labels indicate that the copies of E_K are being glued in to identify μ_K with the $(-1,1)$ curve of each η_i.

We now remind the reader that, for disjoint oriented curves γ_1 and γ_2 in a rational homology sphere Y, there is an associated linking number $\mathrm{lk}_Y(\gamma_1,\gamma_2) \in \mathbb{Q}$ [50, Chapter 10, Section 77] that depends on the isotopy classes of the two curves.[2] We remark that when one of γ_1 or γ_2 is null-homologous, then $\mathrm{lk}_Y(\gamma_1,\gamma_2)$ is an integer. This is certainly guaranteed when Y is an integer homology sphere.

When it is clear from context what the ambient 3-manifold is, we will sometimes abbreviate $\mathrm{lk}_Y(\gamma_1,\gamma_2)$ as simply $\mathrm{lk}(\gamma_1,\gamma_2)$.

Now, returning to our consideration of $\eta_1 \cup \cdots \cup \eta_q$ in $\Sigma_q(P(U))$, we observe that orienting one of the η_i determines an orientation for all the rest by requiring that the link be covering transformation invariant as an oriented link. Therefore, for $i \neq j$, we have that $\mathrm{lk}_{\Sigma_q(P(U))}(\eta_i,\eta_j) \in \mathbb{Z}$ is well defined.

Until Section 4.2, we assume that each lift η_i is null-homologous in $\Sigma_q(P(U))$. One could equivalently assume that any one lift is null-homologous by picking η_1 arbitrarily and using a primitive covering transformation f to get $\eta_i = f^{i-1}(\eta_1)$. It follows that the longitudes of each η_i canonically correspond to elements of $\mathbb{Z}$, by requiring that the Seifert longitude corresponds to 0. Define $\mathrm{fr}(\eta_i)$ to be the integer corresponding to the ith lift of the 0-framed longitude of η.

Lemma 2.3. *Given the notation above,* $\mathrm{fr}(\eta_i) = -\sum_{j\neq i} \mathrm{lk}(\eta_i,\eta_j)$.

Proof. Denote the ith lift of the 0-framed longitude of η by λ_i. Recall that η is an unknot in S^3 and hence bounds a disc D that we may assume is transverse to P. In

[2] The reader may be more familiar with the $\mathbb{Q}/\mathbb{Z}$ torsion linking form: the image of $\mathrm{lk}_Y(\gamma_1,\gamma_2)$ in $\mathbb{Q}/\mathbb{Z}$ is an invariant of the homology classes $[\gamma_1],[\gamma_2] \in H_1(Y;\mathbb{Z})$, but with the additional information of the isotopy class the rational number is well defined.

particular, the 0-framed longitude of η is given by pushing η into the interior of D. The lift of D to $\Sigma_q(P(U))$ is a (possibly disconnected) surface F with boundary $\bigcup_{k=1}^q \eta_k$, and λ_i is given by pushing η_i into the interior of F. Modifying F by removing the boundary annulus near η_i gives a surface $\widehat{F}$ with boundary

$$\partial \widehat{F} = \lambda_i \cup \bigcup_{j \neq i} \eta_j .$$

It follows that λ_i is homologous to $-\bigcup_{j\neq i} \eta_j$ in the complement of η_i, and therefore that

$$\mathrm{fr}(\eta_i) = \mathrm{lk}(\eta_i, \lambda_i) = \mathrm{lk}\Big(\eta_i, -\bigcup_{j \neq i} \eta_j\Big) = -\sum_{j \neq i} \mathrm{lk}(\eta_i, \eta_j),$$

as desired. ∎

We will need three ways to modify a (3-manifold, framed link) pair (Y, L): null-homologous surgery, twisting along an unknot, and clasper surgery. (The reader should think of Y as $\Sigma_q(P(U))$ and L as the lift of η.) We now define these operations, analyze how they change the linking and framing information of L, and prove some key results for later use. We remark that in each case an operation from (Y, L) to (Y', L') will induce a parallel operation from $(Y, L) * E_K$ to $(Y', L') * E_K$.

2.1. Null-homologous surgery

Let L be a link in Y and γ a curve that is null-homologous in $Y \smallsetminus \nu(L)$. Observe that surgery along γ transforms (Y, L) to a new (Y', L'), with the key property that L' has the same linking numbers and framings as L. We will call this operation a *null-homologous surgery* on a 3-manifold, link pair.

Proposition 2.4. *Let Y be a rational homology sphere and L a framed null-homologous link in Y. There exists a framed link in L' in S^3 with the same linking and framing information as L, a natural number k, and a sequence of natural numbers $m_1, \dots, m_k$ with the property that there exists a finite sequence of null-homologous surgeries from (S^3, L') to $(Y \#_{i=1}^k L(m_i, 1), L)$.*

By the phrase "a sequence of null-homologous surgeries", we mean that there exists a link $J = J_1 \cup \cdots \cup J_r$ such that J is null-homologous in $Y \smallsetminus \nu(L)$, and for each i, J_{i+1} is null-homologous in the complement of the image of L after surgery on $J_1, \dots, J_i$. We will prove Proposition 2.4 by establishing the following equivalent formulation.

Lemma 2.5. *Let $L = L_1 \cup \cdots \cup L_n$ be a null-homologous framed link in a rational homology sphere Y. Then, there exist two disjoint framed links $L' = L'_1 \cup \cdots \cup L'_n$,*

$J = J_1 \cup \cdots \cup J_k$ in S^3 with $\mathrm{lk}_Y(L_i, L_j) = \mathrm{lk}_{S^3}(L'_i, L'_j)$ for all $i \neq j$ which satisfies the following properties.

(1) $\mathrm{lk}_{S^3}(J_i, J_j) = \mathrm{lk}_{S^3}(J_i, L'_j) = 0$ *for all i, j (i.e., J has pairwise linking zero and is null-homologous in the exterior of L').*

(2) *Framed surgery on J produces the connected sum of Y and lens spaces of the form $L(m, 1)$.*

(3) *The image of L' under surgery on J is contained in the Y-summand and agrees with L as a framed link.*

Lemma 2.5 immediately establishes Proposition 2.4: since (Y, L) is obtained from (S^3, L') by surgery on a link with pairwise linking zero and trivial linking with L', we see that after applying surgery on each component of J the sublink of the remaining components of J will still be null-homologous.

Proof of Lemma 2.5. After connected sum with lens spaces of the form $L(m, 1)$, Y can be expressed as surgery on a framed link in S^3 with pairwise linking zero [29, Lemma 2.4]. This link will be J. Viewing $Y \#_i L(m_i, 1)$ as surgery on J, we can view the link L as a link L'' in S^3 which is disjoint from J. Note that L'' may have linking with J.

We first claim that we may handle-slide the components of L'' over J so that they do not link J algebraically. To see this, consider the linking matrix A_J associated to the framed link J. Now, let $\overrightarrow{lk_{S^3}(L''_1, J)}$ be the integral vector with ith entry given by $lk_{S^3}(L''_1, J_i)$. The fact that L is null-homologous in Y, and hence in $Y \#_i L(m_i, 1)$, means that the vector $\overrightarrow{lk_{S^3}(L''_1, J)}$ is contained in the integral column space of A_J; that is, there exists an integral vector $\vec{\alpha}$ such that

$$\overrightarrow{lk_{S^3}(L''_1, J)} = A_J \cdot \vec{\alpha} = \sum_{i=1}^{k} \alpha_i \mathrm{col}_i(A_J).$$

Next, note that if we handle-slide L''_1 over J (where the handle-slide happens away from the other components of L''), we do not change the isotopy type of L'' as a link in $Y \#_i L(m_i, 1)$, but we do change the corresponding vector $\overrightarrow{lk_{S^3}(L''_1, J)}$. In particular, when handle-sliding over the ith component of J, we add the ith column of A_J to $\overrightarrow{lk_{S^3}(L''_1, J)}$ (see [15, Section 5.1, p. 142]). The sequence of handle-slides of L''_1 over J with multiplicities given by $-\alpha$ produce a new link with

$$\mathrm{lk}_{S^3}(L''_1, J_i) = A_J \cdot \vec{\alpha} + A_J \cdot (\vec{-\alpha}) = 0$$

for all i. We now repeat the same argument with L''_2. Since $\mathrm{lk}_{S^3}(L''_1, J_i) = 0$ for all i, handle-sliding L''_2 over J does not change the linking of L''_1 with J. Continue this process for all components of L''. The result after the sequence of handle-slides is the claimed link L'.

It remains to show that $\mathrm{lk}_Y(L_i, L_j) = \mathrm{lk}_{S^3}(L'_i, L'_j)$ for all $i \neq j$. Since L' has no linking with J, we can find a Seifert surface S_i for each component of L'_i. Notice that the algebraic intersection number of each component of J with S_i is zero. To make sure that the geometric intersection is zero as well, notice that the points in $S_i \cap J_r$ can be divided into positive and negative according to the sign of the intersection, and a positive point x_+ pairs with the first negative point x_- that appears when traversing J. Therefore, it is possible to modify S_i by replacing disk neighborhoods of a pair (x_+, x_-) with a tube $[0,1] \times S^1$ obtained as the boundary of a (small enough) tubular neighborhood of the arc in J_r that joins x_+ with x_-. After tubing along all the components of J, one obtains a surface disjoint from J. (Note that we will have some S_i's intersecting L'_j's with $i \neq j$.) Since S_i is disjoint from J, we see that $\mathrm{lk}_{S^3}(L'_i, L'_j)$ is unchanged by surgery on J. Consequently, $\mathrm{lk}_Y(L_i, L_j) = \mathrm{lk}_{S^3}(L'_i, L'_j)$. ■

Remark 2.6. The addition of lens spaces is not necessary if Y is a homology sphere. An example of a rational homology sphere that cannot be obtained by integral surgery on a link in S^3 with pairwise linking zero is the 3-manifold obtained by 0-surgery on the 2-component link $T_{2,4}$. To see this, use the integral surgery presentation to identify

$$H_1(S^3_0(T_{2,4})) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

so that the linking form is

$$\mathrm{lk}((a,b),(c,d)) = (ad + bc)/2 \in \mathbb{Q}/\mathbb{Z}.$$

Consequently, every element has vanishing self-linking. However, a three-manifold with non-trivial first homology which is obtained by surgery on a link with pairwise linking zero would have elements with non-trivial self-linking. Nevertheless, the connected sum of this rational homology sphere with $-L(2,1) = S^3_2(U)$ can be represented as surgery along a framed link with diagonal linking matrix with diagonal entries $(-2, 2, 2)$.

2.2. A simple negative-definite cobordism from twisting along an unknot

As above, let $L = L_1 \cup \cdots \cup L_n$ be a link in a 3-manifold Y. Choose $i \neq j$. Let α be an unknot in Y bounding a disc D such that the geometric intersection of D with L_k is a single positively oriented point if $k = i$, a single negatively oriented point if $k = j$, and the empty set otherwise, as illustrated in Figure 4. Then, doing -1 framed surgery along α does not change Y but transforms L to a link L' with

$$\mathrm{lk}(L'_k, L'_\ell) = \begin{cases} \mathrm{lk}(L_k, L_\ell) - 1 & \text{if } \{k,\ell\} = \{i,j\}, \\ \mathrm{lk}(L_k, L_\ell) & \text{else} \end{cases} \tag{2.3}$$

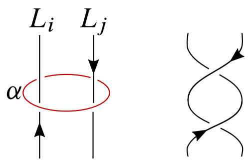


Figure 4. Positive twisting along an unknot α changes $\mathrm{lk}(L_i, L_j)$ by -1 and $\mathrm{fr}(L_i)$ and $\mathrm{fr}(L_j)$ by $+1$.

and

$$\mathrm{fr}(L'_k) = \begin{cases} \mathrm{fr}(L_k) + 1 & \text{if } k \in \{i, j\}, \\ \mathrm{fr}(L_k) & \text{else.} \end{cases}$$

We call the 4-manifold

$$W = Y \times I \bigcup_{\alpha \times \{1\}} \text{2-handle}$$

an *elementary negative-definite cobordism* from (Y, L) to (Y, L'). Note that if W is an elementary negative definite cobordism from (Y, L) to (Y', L'), then for any knot in S^3 we can glue a collection of copies of $E_K \times I$ to obtain a negative definite cobordism from $(Y, L) * E_K$ to $(Y, L') * E_K$.

Proposition 2.7. *Let L be a framed q-component link in S^3 such that*

(1) $\mathrm{lk}(L_i, L_j) \geq 0$ *for all* $i \neq j$,

(2) $\mathrm{lk}(L_{i_0}, L_{j_0}) \geq 1$ *for some* $i_0 \neq j_0$, *and*

(3) $fr(L_i) = -\sum_{j \neq i} \mathrm{lk}(L_i, L_j)$ *for all* i.

Then, there exists a sequence of elementary negative definite cobordisms from (S^3, L) to (S^3, L'), where L' is a framed link in S^3 satisfying

$$\mathrm{lk}(L'_i, L'_j) = \begin{cases} 1 & \textit{if } \{i, j\} = \{i_0, j_0\} \\ 0 & \textit{else}. \end{cases} \quad \textit{and} \quad fr(L'_i) = \begin{cases} -1 & \textit{if } i \in \{i_0, j_0\} \\ 0 & \textit{else}. \end{cases}$$

In particular, for any knot K, there is a negative definite cobordism from $(S^3, L) * E_K$ to $(S^3, L') * E_K$.

Proof of Proposition 2.7. By renumbering the components of L, we may assume for simplicity in notation that $i_0 = 1$ and $j_0 = 2$. We construct a $(\sum_{i \neq j} \mathrm{lk}(L_i, L_j) - 1)$-component link J in $S^3 \smallsetminus L$ as follows. For each $i < j$, let $\alpha_{i,j}$ be an arc in S^3 connecting L_i to L_j and otherwise disjoint from L. By transversality, we can assume that all such arcs are disjoint. We can choose a disc $D_{i,j}$ living arbitrarily close to $\alpha_{i,j}$,

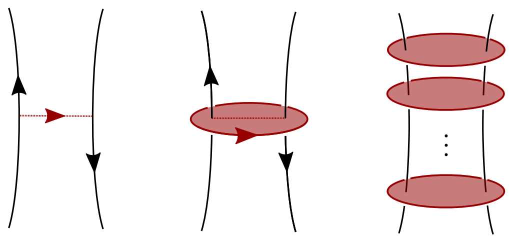


Figure 5. An arc $\alpha_{i,j}$ connecting L_i to L_j (left); a disc $D_{i,j}$ lying very close to $\alpha_{i,j}$ (center); and $\mathrm{lk}(L_i, L_j)$ parallel push-offs of $D_{i,j}$ (right).

with a boundary that is homotopic to $\alpha_{i,j}\mu_{L_i}\overline{\alpha}_{i,j}\overline{\mu}_{L_j}$ in $S^3 \smallsetminus L$, as illustrated on the left and center of Figure 5. In particular,

$$\mathrm{lk}(\partial D_{i,j}, L_k) = \begin{cases} 1, & k = i, \\ -1, & k = j, \\ 0 & \text{else.} \end{cases}$$

For $i < j$ such that $(i, j) \neq (1, 2)$ and $\mathrm{lk}(L_i, L_j) \neq 0$, let $\mathcal{D}_{i,j}$ be the disjoint union of $\mathrm{lk}(L_i, L_j)$ parallel push-offs of $D_{i,j}$, as illustrated on the right of Figure 5. Let $\mathcal{D}_{1,2}$ be the disjoint union of $\mathrm{lk}(L_1, L_2) - 1$ parallel push-offs of $D_{1,2}$ (or the empty set, if $\mathrm{lk}(L_1, L_2) = 1$). Let $\mathcal{D}$ be the union of all of these discs, and denote by $\mathcal{L}$ its boundary. By construction, $\mathcal{L}$ is an unlink in S^3 that is disjoint from L. By attaching (-1)-framed 2-handles to $S^3 \times [0, 1]$ along $\mathcal{L} \subset S^3 \times \{1\}$, we obtain a negative definite cobordism from (S^3, L) to (S^3, L') for some link L'. It remains to show that the linking and framing information of L' is as desired. But this follows immediately from equation (2.3), given our choice of the number of components of

$$\mathcal{L}_{i,j} = \partial D_{i,j}. \qquad \blacksquare$$

2.3. Clasper surgeries

Although null-homologous surgeries do not change linking information, it is not true that any two links with the same pairwise linking can be transformed into each other by a sequence of null-homologous surgeries. A simple example of this is the 3-component unlink and the Borromean rings. Nevertheless, the latter can be transformed into the former via a well-studied generalization of Dehn surgery called clasper or Borromean surgery; this is an operation commonly studied in quantum topology. This operation will allow us to transform the lift of η into a simpler and familiar link, and this will in turn allow us to prove our main theorem.

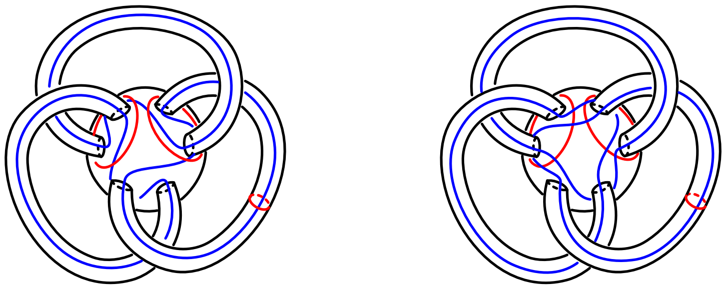


Figure 6. Pictured are two Heegaard diagrams representing a clasper surgery from S^3 to S^3. If we ignore the red curves, the handlebodies with compressing disks determined by the blue curves are exactly the genus 3 handlebodies involved in the definition of clasper surgery.

Before defining clasper surgery, we introduce some notation. Denote by H_l (H_r, respectively) the genus three handlebody represented by the blue curves on the left (right, respectively) of Figure 6.

Definition 2.8. A *clasper surgery on a 3-manifold* is the result of removing from a 3-manifold M an embedded copy of the genus three handlebody H_l on the left of Figure 6 and replacing it with the handlebody H_r on the right of the same figure, or the reverse. A *clasper surgery on a link* in a 3-manifold is defined analogously where the genus three handlebody must be disjoint from the link.

We rely on the Alexander method [10, Chapter 2, Section 3] to define the gluing ψ of the handlebody H_r to the complement of H_l. Indeed, an element of the mapping class group of the genus 3 surface S_3 is completely determined by its action on a set of ordered and oriented curves that cut the surface into a disk. Since the set of blue and red curves on each side of Figure 6 cuts S_3 into a disk, the map ψ is defined as the map that sends the set of red and blue curves on the left to the corresponding set of blue and red curves on the right[3]. Note that since the blue curves on the left and right of Figure 6 are pairwise homologous, the map ψ is an element of the Torelli group of the genus 3 surface S_3. As a consequence, clasper surgery on a 3-manifold does not change the homology. For more details on the clasper surgery construction, see [34, Section 6] or [33].

We remark that if one does clasper surgery to the unknotted (genus three) handlebody in S^3, the result is still S^3. Similarly, modifying the standard genus three splitting for $\#_3 S^2 \times S^1$ by clasper surgery yields $\mathbb{T}^3$. Since we do not need this fact, we leave the verification as an exercise for the reader.

[3]Here, we are assuming the ordering and orientation of curves are the same on the left and right of Figure 6.

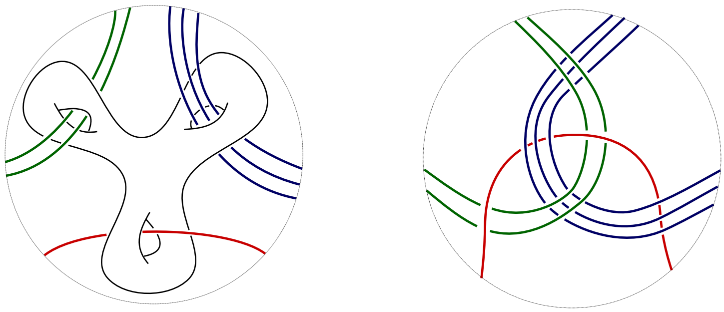


Figure 7. Clasper surgery locally transforms between the links on the left and right. Note that the depicted arcs may belong to any components of the link.

In the case of links, the result of clasper surgery is to tie the strands of the link "passing through" the one-handles of the handlebody into the Borromean rings, as illustrated in Figure 7.

Similar to the manifold case, clasper surgery on a link does not change pairwise linking numbers or framings. (However, it can change the higher-order linking invariants of Milnor [37]!) What is more interesting is that the converse is also true.

Theorem 2.9 (Proof of [34, Lemma 2], see also [38]). *Let $L = L_1 \cup \cdots \cup L_n$ and $L' = L'_1 \cup \cdots \cup L'_n$ be two framed links in S^3 with the same number of components. If the linking matrices for L and L' agree, then L and L' are related by a sequence of clasper surgeries.*

A special case of this is the following proposition.

Proposition 2.10. *Let L be a framed q-component link in S^3 with the property that*

$$\mathrm{lk}(L_i, L_j) = \begin{cases} 1 & \text{if } \{i, j\} = \{i_0, j_0\}, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad fr(L_i) = \begin{cases} -1 & \text{if } i \in \{i_0, j_0\}, \\ 0 & \text{else}. \end{cases}$$

Then, there is a clasper surgery from (S^3, L) to (S^3, H_q), where H_q is the split union of the (-1)-framed positive Hopf link with a 0-framed $(q-2)$-component unlink.

3. d-invariants

The goal of this section is to constrain the d-invariants of 3-manifolds obtained by the three topological transformations studied in Section 2: null-homologous surgeries, negative-definite cobordisms, and clasper surgeries. Along the way in Theorem 1.9, we establish a relationship between d-invariants and the Torelli group so as to study the behavior of clasper surgeries. Since the d-invariants are governed by Spin^c structures on 3- and 4-manifolds, we begin with a review of those in Section 3.1. We then give the key properties and computations for the Ozsváth–Szabó d-invariants [41] in Section 3.2. In Section 3.3, we give some basic constraints on the d-invariants of surgeries on null-homologous knots in rational homology spheres; Proposition 3.6 below. (This is surely known to experts, but we could not find it in the literature.) Finally, we prove Theorem 1.9 in Section 3.4. Those familiar with the basics of d-invariants may wish to begin with Section 3.3.

3.1. Spin^c structures in low dimensions

For Y a closed, connected, oriented 3-manifold Y, a result of Turaev [51] states that Spin^c structures on Y can be thought of as equivalence classes of non-vanishing vector fields where two vector fields are equivalent if they are homotopic in the complement of a finite union of balls. This geometric approach to Spin^c structures is by no means their only definition, but with the intention of keeping this section user-friendly, we avoid an in-depth discussion of the definition(s) and focus instead on the properties that are key to establishing our results. One of these relevant properties is the relationship between $\mathrm{Spin}^c(Y)$ and $H^2(Y;\mathbb{Z})$. Taking the first Chern class of a Spin^c structure gives a map

$$c_1 : \mathrm{Spin}^c(Y) \to H^2(Y;\mathbb{Z}).$$

(The first Chern class of a Spin^c structure $\mathfrak{s}$ is defined as the first Chern class of a complex line bundle obtained as an orthogonal complement to the vector field $\mathfrak{s}$ at each point.) The image of c_1 is precisely $2H^2(Y;\mathbb{Z})$. A Spin^c structure $\mathfrak{s}$ is said to be *torsion* if $c_1(\mathfrak{s})$ is a torsion element of $H^2(Y;\mathbb{Z})$. A Spin^c 3-manifold is a pair $(Y,\mathfrak{s})$.

Given two 3-manifolds Y_0 and Y_1 and $\mathfrak{s}_i \in \mathrm{Spin}^c(Y_i)$ for $i = 0, 1$, using the description of Spin^c structures as vector fields, it is easy to see that there is a well-defined element $\mathfrak{s}_0 \# \mathfrak{s}_1 \in \mathrm{Spin}^c(Y_0 \# Y_1)$. Further, $c_1(\mathfrak{s}\#\mathfrak{s}')$ is identified with $c_1(\mathfrak{s}) \oplus c_1(\mathfrak{s}')$ under the isomorphism $H^2(Y\#Y';\mathbb{Z}) \cong H^2(Y;\mathbb{Z}) \oplus H^2(Y';\mathbb{Z})$. Hence, $\mathfrak{s}\#\mathfrak{s}'$ is torsion if and only if both $\mathfrak{s}, \mathfrak{s}'$ are.

Next, let X be a compact, connected, orientable 4-manifold. Recall that an element of $H^2(X;\mathbb{Z})$ is characteristic if its pairing with each element $\alpha \in H_2(X;\mathbb{Z})$ agrees mod 2 with the self-intersection of α. Similar to the 3-dimensional case, first Chern classes give a map

$$c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z}).$$

In this case, the image of c_1 is in bijective correspondence with the set of *characteristic* elements of $H^2(X;\mathbb{Z})$. Moreover, given a Spin^c structure $\mathfrak{t}$ on a 4-manifold X, there is a restricted Spin^c structure on ∂X and this is natural with respect to the first

Chern class, that is,

$$c_1(\mathfrak{t}|_{\partial X}) = c_1(\mathfrak{t})|_{\partial X}.$$

Similarly, if $W : Y \to Y'$ is a cobordism and there exists a Spin^c structure $\mathfrak{t}$ on W which restricts to $\mathfrak{s}$ (respectively, $\mathfrak{s}'$) on Y (respectively, Y'), then we say that $\mathfrak{s}$ and $\mathfrak{s}'$ are cobordant.

Finally, we remark that Spin^c structures are in fact affine over the second cohomology group. This implies that, for a Spin^c structure ν on a 3- or 4-manifold M, we obtain a different Spin^c structure $\nu + \alpha$ for $\alpha \in H^2(M;\mathbb{Z})$. This affine structure is also natural with regard to restriction to the boundary and satisfies $c_1(\nu+\alpha) = c_1(\nu) + 2\alpha$.

3.2. Review of properties of d-invariants

We begin with the simplest version of d-invariants, also sometimes called correction terms, introduced by Ozsváth–Szabó in [41]. For a Spin^c structure $\mathfrak{s}$ on a rational homology sphere Y, the correction term $d(Y,\mathfrak{s})$ is a rational number, and is an invariant of Spin^c rational homology cobordism. More precisely, if W is a cobordism from Y_0 to Y_1 with the same rational homology as $[0,1] \times S^3$, then for any Spin^c structure $\mathfrak{t}$ on W one has $d(Y_0, \mathfrak{t}|_{Y_0}) = d(Y_1, \mathfrak{t}|_{Y_1})$.

Theorem 3.1 ([41, Theorems 1.2 and 9.6]). *Let $(Y,\mathfrak{s})$ and $(Y',\mathfrak{s}')$ be Spin^c rational homology spheres. Then,*

(1) $d(-Y,\mathfrak{s}) = -d(Y,\mathfrak{s})$;

(2) $d(Y \# Y', \mathfrak{s}\#\mathfrak{s}') = d(Y,\mathfrak{s}) + d(Y',\mathfrak{s}')$;

(3) *if $(Y,\mathfrak{s})$ bounds a negative-definite Spin^c 4-manifold $(W,\mathfrak{t})$, then*

$$d(Y,\mathfrak{s}) \geq \frac{c_1(\mathfrak{t})^2 + b_2(W)}{4}.$$

If $(W,\mathfrak{t}) : (Y,\mathfrak{s}) \to (Y',\mathfrak{s}')$ is a negative-definite Spin^c cobordism, then

$$d(Y',\mathfrak{s}') - d(Y,\mathfrak{s}) \geq \frac{c_1(\mathfrak{t})^2 - 3\sigma(W) - 2\chi(W)}{4}.$$

Note that the second claim in Theorem 3.1 (3) follows from the first by applying Theorem 3.1 (1) and Theorem 3.1 (2). Also, since integer homology spheres have a single Spin^c structure, we omit this from the notation when we work with such manifolds.

Corollary 3.2. *Let Y be a rational homology sphere that bounds a rational homology ball. Then,*

$$d_{\max}(Y) := \max\{d(Y,\mathfrak{s}) : \mathfrak{s} \in \mathrm{Spin}^c(Y)\} \geq 0.$$

Proof. Let $\mathfrak{t}$ be a Spin^c structure on the hypothesized rational homology ball W. By Theorem 3.1 (3), we have that

$$d_{\max}(Y) \geq d(Y, \mathfrak{t}|_Y) \geq 0. \qquad \blacksquare$$

As mentioned in the introduction, our arguments rely on the computation of the d-invariants for surgeries along knots of the form $T_{2,2k+1} \# T_{2,2k+1}$, so we record those for later use here.

Theorem 3.3 ([42, Corollary 1.5]). *Let K be an alternating knot. Then,*

$$d(S^3_1(K)) = 2\min\{0, -\lceil -\sigma(K)/4 \rceil\}.$$

Consequently,

$$d(S^3_1(T_{2,2k+1} \# T_{2,2k+1})) = \begin{cases} -2k, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

3.3. d-invariants of surgeries

The goal of this subsection is to prove Proposition 3.6, which is concerned with the d-invariants of nonzero surgeries on null-homologous knots in rational homology spheres. Our analysis will require the study of correction terms for 3-manifolds N with $b_1(N) > 0$. An issue thus arises since d-invariants are not defined for Spin^c structures in a general 3-manifold. However, if $\mathfrak{s}$ is a torsion Spin^c structure, i.e., $c_1(\mathfrak{s})$ is torsion in $H^2(Y)$, and $(Y, \mathfrak{s})$ has *standard HF^∞*, then Ozsváth–Szabó defined a correction term $d_b(Y, \mathfrak{s}) \in \mathbb{Q}$ that is again an invariant of Spin^c rational homology cobordism [41, Section 9], where b stands for “bottom”. Having standard HF^∞ means that $HF^\infty(Y, \mathfrak{s}) \cong HF^\infty(\#_{b_1(Y)} S^2 \times S^1, \mathfrak{s}_0)$, where $\mathfrak{s}_0$ is the unique torsion Spin^c structure on $\#_{b_1(Y)} S^2 \times S^1$. We remark that if Y is a rational homology sphere and $\mathfrak{s}$ is any Spin^c structure on Y, then $(Y, \mathfrak{s})$ has standard HF^∞ [43, Theorem 10.1]. Similar to Theorem 3.1, we have the following theorem.

Theorem 3.4 ([43, Theorem 1.5, Proposition 2.5, and Theorem 10.1], [41, Theorem 9.15], [25, Proposition 4.3], [45, Proposition 4.0.5]). *Let $(N, \mathfrak{s})$ and $(N', \mathfrak{s}')$ be 3-manifolds with standard HF^∞ equipped with torsion Spin^c structures. Then,*

(1) *if N is a rational homology sphere, then $d(N, \mathfrak{s}) = d_b(N, \mathfrak{s})$;*

(2) *$(-N, \mathfrak{s})$ has standard HF^∞, but $d_b(-N, \mathfrak{s})$ and $d_b(N, \mathfrak{s})$ are not necessarily related;*

(3) *$(N \# N', \mathfrak{s} \# \mathfrak{s}')$ has standard HF^∞ and*

$$d_b(N \# N', \mathfrak{s} \# \mathfrak{s}') = d_b(N, \mathfrak{s}) + d_b(N', \mathfrak{s}');$$

(4) *If* $(N, \mathfrak{s})$ *is the boundary of a negative semidefinite Spin*^{c} *4-manifold* $(W, \mathfrak{t})$ *with* $b_1(W) = 0$*, then*

$$d_b(N, \mathfrak{s}) \geq \frac{c_1(\mathfrak{t})^2 + b_2^-(W) - 2b_1(Y)}{4};$$

(5) *if* N *is a circle bundle over a genus* g *surface with Euler number* n *and* $\mathfrak{s}$ *is any torsion Spin*^{c} *structure, then* $(N, \mathfrak{s})$ *has standard* HF^∞ *if and only if* $n \neq 0$ *or* $g = 0$.

Recall that a 4-manifold is *negative semidefinite* if the intersection form is negative definite on the non-degenerate part. The primary example is any 4-manifold with vanishing intersection form. For the readers familiar with other notions of d-invariants, we remark that $d_b(-N, \mathfrak{s}) = -d_{top}(N, \mathfrak{s})$, where d_{top} is the "topmost" d-invariant; see, for example, [25].

In order to prove Proposition 3.6, we will need to guarantee that the Spin^c structures on the 3-manifolds under consideration are torsion. The following homological lemma accomplishes this.

Lemma 3.5. *Let* $(X, \mathfrak{t})$ *be a* Spin^c *4-manifold and* S_g *an orientable embedded surface of genus g and self-intersection n. Let* $N_{n,g}$ *be the boundary of a tubular neighborhood of* S_g*, i.e., a circle bundle over* S_g *with Euler number n. If either* $n \neq 0$ *or* $\langle c_1(\mathfrak{t}), [S_g]\rangle_X = 0$*, then* $\mathfrak{t}|_{N_{n,g}}$ *is torsion.*

Proof. Recall that a Spin^c structure $\mathfrak{s}$ on $N_{n.g}$ is torsion if $c_1(\mathfrak{s})$ is a torsion element of $H^2(N_{n,g}; \mathbb{Z})$, and that

$$H_2(N_{n,g}; \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g} & \text{if } n \neq 0, \\ \mathbb{Z}^{2g+1} & \text{if } n = 0 \end{cases}$$

generated entirely by the tori T_γ obtained from restricting the circle bundle to loops γ in S_g whenever $n \neq 0$, and these tori plus the homology class of S_g when $n = 0$ (see [15, Exercise 5.4.4]).

Thus, to show that $\mathfrak{t}|_{N_{n,g}}$ is torsion, it suffices to show that $c_1(\mathfrak{t}|_{N_{n,g}})$ evaluates to be trivial on the generators of $H_2(N_{n,g}; \mathbb{Z})$. By the naturality of c_1 for Spin^c structures, we have

$$\langle c_1(\mathfrak{t}|_{N_{n,g}}), [T_\gamma]\rangle_{N_{n,g}} = \langle c_1(\mathfrak{t})_{N_{n,g}}, [T_\gamma]\rangle_{N_{n,g}} = \langle c_1(\mathfrak{t}), [T_\gamma]\rangle_X.$$

However, in a neighborhood of S_g in X, one can extend the circle fibers over the disk fiber, and so, T_γ bounds a solid torus. Consequently, $\langle c_1(\mathfrak{t}), [T_\gamma]\rangle_X = 0$, and this completes the proof. ∎

To establish notation, given a null-homologous knot K in a 3-manifold Y, let

$$W_n(K) = [0,1] \times Y \cup D^2 \times D^2 \tag{3.1}$$

denote the 2-handle cobordism from Y to $Y_n(K)$. Note that

$$\mathrm{Spin}^c(Y_n(K)) = \mathrm{Spin}^c(Y) \oplus \mathbb{Z}/n.$$

Every Spin^c structure on $Y_n(K)$ extends to a Spin^c structure on $W_n(K)$. Furthermore, the projection from $\mathrm{Spin}^c(Y_n(K))$ to $\mathrm{Spin}^c(Y)$ in the splitting above can be realized by extending a Spin^c structure on $Y_n(K)$ over $W_n(K)$ and then restricting to Y. See [44] for more details.

Proposition 3.6. *Let $n \neq 0$ be an integer and $g \geq 0$. There exists a constant $C_{g,n}$ that depends on only $|n|$ and g with the following property. For any null-homologous knot K that bounds a genus g surface in a rational homology sphere Y, and any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and $\mathfrak{s}_n \in \mathrm{Spin}^c(Y_n(K))$ that are cobordant over $W_n(K)$, we have*

$$|d(Y_n(K), \mathfrak{s}_n) - d(Y, \mathfrak{s})| \leq C_{g,n}.$$

Before proving the proposition, a few remarks are in order.

Remark 3.7.
- The distinction between $|n|$ and n is purely cosmetic, since we can take the max of $C_{g,n}$ and $C_{g,-n}$. The advantage of stating it this way is simply for ease of exposition later.
- This result was proved by Frøyshov for ± 1-framed surgeries for the instanton h-invariant (see [13, Corollary 1]).
- For knots in S^3, the result follows easily from the Ni–Wu d-invariant surgery formula [40]. The arguments there can be generalized to prove Proposition 3.6 if one allows the constant to depend on the ambient manifold Y as well.
- There are analogs of Proposition 3.6 for more general cobordisms between 3-manifolds, but we do not state or prove those here for ease of exposition.

Proof of Proposition 3.6. Consider a rational homology sphere Y and $K \subset Y$ a null-homologous knot. Let $W = W_n(K)$ be as in Equation (3.1). Assume for simplicity that $n < 0$.

To get a lower bound on $d(Y_n(K), \mathfrak{s}_n) - d(Y, \mathfrak{s})$, notice that W is a negative definite manifold with oriented boundary $-Y \amalg Y_n(K)$. By Theorem 3.1 (4),

$$d(Y_n(K), \mathfrak{s}_n) - d(Y, \mathfrak{s}) \geq \frac{c_1(\mathfrak{t})^2 + 1}{4}$$

for any Spin^c structure $\mathfrak{t}$ on W which restricts to $\mathfrak{s}$ on Y and $\mathfrak{s}_n$ on $Y_n(K)$. We would like to choose $\mathfrak{t}$ giving a lower bound on $c_1(\mathfrak{t})^2$ independent of $Y, K, \mathfrak{s}, \mathfrak{s}_n$. Let $[S]$

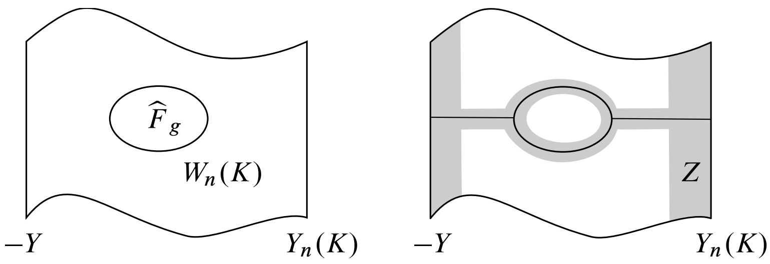


Figure 8. The surface $\widehat{F}_g$ in $W_n(K)$ (left) and a cobordism Z from $Y \sqcup -Y_n(K)$ to $Y \# N_{n,g} \# -Y_n(K)$ (right).

be a generator of $H^2(W)$, and note that $(k[S])^2 = \frac{k^2}{n}$. By naturality of the affine H^2 structure for Spin^c structures, observe that $\mathfrak{t}$ and $\mathfrak{t} + n[S]$ restrict to the same Spin^c structure on both Y and $Y_n(K)$. Therefore, by possibly adding copies of $n[S]$ to $\mathfrak{t}$, or equivalently, $2n[S]$ to $c_1(\mathfrak{t})$, we may choose $\mathfrak{t}$ such that $c_1(\mathfrak{t}) = k[S]$, where $-n \leq k \leq n$, and $\mathfrak{t}$ restricts to $\mathfrak{s}$ and $\mathfrak{s}_n$ (see also [44, Lemma 2.2]). In this case, we obtain $|c_1(\mathfrak{t})|^2 \leq |n|$. Thus, we obtain a lower bound on $d(Y_n(K), \mathfrak{s}_n) - d(Y, \mathfrak{s})$, which depends only on n.

Unfortunately, $-W$ is not negative definite, and so, we cannot simply reverse orientations to obtain a lower bound. Luckily, it will turn out that we can "excise the homology" from $-W$ so that the resulting manifold is negative semidefinite and we can obtain a similar bound. However, this will force us to consider a correction term determined by the particular representative of homology we excise. To be more precise, for F a genus g Seifert surface for K in Y, denote by $\widehat{F}_g$ the closed surface obtained by pushing $\{1\} \times F$ into the interior of $[0, 1] \times Y \subset W$ and capping it off with the core of the 2-handle. Notice that $H_2(W; \mathbb{Z})$ is generated by the homology class of $\widehat{F}_g$, so "excising" second homology from W amounts to removing a tubular neighborhood of $\widehat{F}_g$. To this end, construct a 4-manifold Z by taking the union of collar neighborhoods of Y, $Y_n(K)$ in W and tubular neighborhoods of arcs joining Y, $Y_n(K)$ to $\widehat{F}_g$ (see Figure 8 for a schematic diagram of this construction).

We will obtain a lower bound for $d(Y, \mathfrak{s}) - d(Y_n(K), \mathfrak{s}_n)$ from Theorem 3.4 (4) applied to the manifold $X = -W \setminus -Z$. In order to do so, one has to understand the boundary of X, its intersection form, and the restriction of elements of Spin$^c(X)$ to ∂X. With this in mind, denote by $N_{n,g}$ the boundary of a neighborhood of $\widehat{F}_g$, namely, a circle bundle of Euler number n over a genus g surface. We then have

$$\partial Z = -Y \sqcup (Y \# N_{n,g} \# -Y_n(K)) \sqcup Y_n(K)$$

and conclude that $\partial X = Y \# N_{n,g} \# -Y_n(K)$.

Next, using the Mayer–Vietoris sequence for the decomposition $-W = X \underset{\partial X}{\cup} -Z$, we get that the inclusion of ∂X into X induces an isomorphism on rational second homology and that $b_1(X) = 0$. In addition, since any surface in X can be pushed off itself in a collar of ∂X, the intersection form on X vanishes. It follows that X is a negative semidefinite manifold (as well as a positive semidefinite one). Finally, we study the Spin^c structures on X so that we can invoke Theorem 3.4 (4). By assumption, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and $\mathfrak{s}_n \in \mathrm{Spin}^c(Y_n(K))$ are cobordant over W; that is, there exists a Spin^c structure $\mathfrak{t}$ on W which restricts to $\mathfrak{s}_n$ and $\mathfrak{s}$ on the ends. Thus, $(Y \# N_{n,g} \# -Y_n(K), \mathfrak{s} \# \nu \# \mathfrak{s}_n)$ is the Spin^c boundary of $(X, \mathfrak{t}|_X)$, where ν is the Spin^c structure induced by restriction of $\mathfrak{t} \in \mathrm{Spin}^c(W)$ to $N_{n,g}$.

To complete the proof, notice first that a combination of Theorem 3.4 (4)–(5) gives

$$\begin{aligned} d(Y, \mathfrak{s}) + d_b(N_{n,g}, \nu) - d(Y_n(K), \mathfrak{s}_n) &\geq \frac{c_1(\mathfrak{t})^2 + b_2^-(X) - 2b_1(\partial X)}{4} \\ &= \frac{0 + 0 - 4g}{4} = -g. \end{aligned}$$

To see this, note that the intersection form of X vanishes, so $c_1(\mathfrak{t})^2 = b_2^-(X) = 0$. Since there are only finitely many torsion Spin^c structures on $N_{n,g}$, the maximum value of $|d_b(N_{n,g}, \nu)|$ over the torsion elements of $\mathrm{Spin}^c(N_{n,g})$ is an invariant of $N_{n,g}$ and hence only depends on n and g. We thus get the lower bound for $d(Y, \mathfrak{s}) - d(Y_n(K), \mathfrak{s}_n)$ sought. ■

3.4. d-invariants and the Torelli group

In this subsection, we prove Theorem 1.9 regarding the effect Torelli surgeries have on d-invariants. To begin, let Σ_g be a fixed genus g surface. If $\mathcal{M}_g$ denotes the mapping class group of Σ_g, then the Torelli group $\mathcal{T}_g$ is the subgroup of $\mathcal{M}_g$ consisting of elements $\phi : \Sigma_g \to \Sigma_g$ for which the induced map $\phi_* : H_1(\Sigma_g; \mathbb{Z}) \to H_1(\Sigma_g; \mathbb{Z})$ is the identity. Note that $\mathcal{T}_1$ is trivial, $\mathcal{T}_2$ is generated by Dehn twists along separating curves, and $\mathcal{T}_g$ is generated by bounding pair maps for $g \geq 3$, as follows from [3, 23, 46]. These last maps are defined as follows: consider a pair of non-separating curves $\alpha_\pm$ on Σ_g that separates Σ_g into two subsurfaces having $\alpha_- \sqcup \alpha_+$ as their common (unoriented) boundary. A bounding pair map consists of the composition of a positive Dehn twist along α_+ and a negative Dehn twist along α_-. For more information on the above, see [16].

Next, given a 3-manifold Y, an embedding $i : \Sigma_g \to Y$ of a closed genus g surface into Y, and an element ϕ of the mapping class group of Σ_g, we define Y_ϕ as the union

$$Y \setminus \nu(i(\Sigma_g))^\circ \bigcup_\phi [0,1] \times \Sigma_g, \tag{3.2}$$

where the gluing takes place along the boundary using the identity on the component $\{0\} \times \Sigma_g$ and ϕ on $\{1\} \times \Sigma_g$. Observe that if ϕ is an element the Torelli group, then Y_ϕ and Y have the same homology groups. The following theorem is the main result of this subsection.

Theorem 1.9. *Let $N, g \geq 0$. There exists a constant $C_{N,g}$ depending only on N and g with the following property: for any integer homology sphere Y, any embedding $i\colon \Sigma_g \to Y$, and any element ϕ in the Torelli group of Σ_g expressible as a product of at most N separating Dehn twists and/or bounding pair maps,*

$$|d(Y) - d(Y_\phi)| \leq C_{N,g}.$$

Since a clasper surgery can be accomplished by a surgery as in Theorem 1.9 with $g = 3$ and a fixed element of the Torelli group, we have the following corollary.

Corollary 3.8. *There exists a constant C such that if M is an integer homology sphere obtained by clasper surgery on another integer homology sphere N, then $|d(M) - d(N)| \leq C$.*

The proof of Theorem 1.9 is similar to that of Proposition 3.6. In particular, for a separating Dehn twist (which simply corresponds to surgery along a null-homologous knot) or bounding pair map, we consider the associated 2-handle cobordism and use this to constrain the d-invariants. To handle the case of bounding pair maps, we will need the *twisted* d-invariant due to Behrens–Golla [2], an even more general notion of d-invariant that works in the absence of standard HF^∞ (see also the work of Levine–Ruberman [25]). For a torsion Spin^c structure $\mathfrak{s}$ on a 3-manifold Y, they define an invariant $\underline{d}(Y, \mathfrak{s}) \in \mathbb{Q}$ (not to be confused with the Hendricks–Manolescu "d-lower") which satisfies the following properties similar to d and d_b.

Theorem 3.9 (Behrens–Golla [2]). *Let Y be a closed, connected, oriented 3-manifold and $\mathfrak{s}$ a torsion* Spin^c *structure on Y. Then, we have the following:*

(1) *if Y is a rational homology sphere, $\underline{d}(Y, \mathfrak{s}) = d(Y, \mathfrak{s})$;*

(2) *$\underline{d}(Y \# Y', \mathfrak{s} \# \mathfrak{s}') = \underline{d}(Y, \mathfrak{s}) + d(Y', \mathfrak{s}')$ for any* Spin^c *rational homology sphere Y';*

(3) *if $(W, \mathfrak{t})$ is a negative semidefinite Spin^c 4-manifold with boundary $(Y, \mathfrak{s})$, then*
$$\underline{d}(Y, \mathfrak{s}) \geq \frac{c_1^2(\mathfrak{t}) + b_2^-(W) - 2b_1(Y)}{4}.$$

Proof of Theorem 1.9. First, suppose that ϕ is a single positive (respectively, negative) Dehn twist along a curve α which separates Σ_g into two subsurfaces S_1, S_2, each bounded by α. Then, the image of α in Y is a null-homologous knot K of genus at most g and Y_ϕ is obtained by -1-framed (respectively, $+1$-framed) Dehn surgery along K. We obtain a bound $|d(Y) - d(Y_\phi)| \le C_{g,1}$ from Proposition 3.6, where $C_{g,1}$ is independent of the sign of the twist or particular separating curve (but does depend on g). Next, suppose that ϕ is a composition of Dehn twists along separating curves $\alpha_1, \dots, \alpha_N$. Then, we can view Y_ϕ as a sequence of ± 1-surgeries along the images of $\alpha_1, \dots, \alpha_N$ in Y [28, pp. 539–540]. Repeated application of Proposition 3.6 shows that $|d(Y) - d(Y_\phi)| \le NC_{g,1}$ for some constant that depends only on g.

To finish the proof, it therefore suffices to show that there is a constant A_g such that if ϕ is a bounding pair map, then $|d(Y) - d(Y_\phi)| \le A_g$, where again A_g is independent of Y. Let $\alpha_\pm$ be two curves in Σ_g which cobound a subsurface of Σ_g. Assume that ϕ is the composition of a positive Dehn twist along α_+ and a negative Dehn twist along α_-. Let $K_\pm$ be the (oriented) images of $\alpha_\pm$ in Y, and let F be a subsurface of S in Y which K_+ and K_- cobound. In this case, Y_ϕ is obtained from Y by doing surgery on the link $K_+ \cup K_-$. Let W be the 2-handle cobordism from Y to Y_ϕ. Unfortunately, we cannot use S to constrain the genera of either K_+ or K_- like in the separating case because $\alpha_\pm$ are assumed to be non-separating in S. Nevertheless, we can find a primitive, homologically essential, square zero, closed, and oriented surface in W obtained by the union of F and one core disk for each of the 2-handles, with orientations determined by those of $K_\pm$. Call this surface $\widehat{F}$. Note that it has the same genus as F, which is at most g.

We now argue that $\widehat{F}$ has vanishing algebraic self-intersection in W. First, observe that an F-framed push-off of K_+ is homologous to K_- in the exterior of $K_+ \cup K_-$ and vice versa. It follows that

$$lk(K_+, K_-) = -\operatorname{lk}(K_+, K_+^F),$$

where K_+^F refers to a push-off of K_+ in F. (The seemingly opposite signs in the above formula come from the fact that $K_+ \cup -K_-$ bounds.)

Cutting and regluing by the positive Dehn twist along α_+ corresponds to (-1)-surgery on α_+, where these surgery coefficients are relative to the framings induced by F, not the Seifert framing [28, pp. 539–540]. More precisely, the surgery framing curve is homologous to $-\mu_{K_+} + K_+^F$. A similar argument shows the negative Dehn twist along α_- corresponds to $(+1)$-surgery along α_-, again relative to the framing induced by F. Therefore, relative to the Seifert framings, the linking matrix for the surgery on $K_+ \cup K_-$ is given by

$$\begin{pmatrix} -\ell - 1 & \ell \\ \ell & -\ell + 1 \end{pmatrix},$$

where $\ell = lk(K_+, K_-)$. This describes a natural basis for $H_2(W)$ such that $\widehat{F}$ corresponds to $(1, 1)$, and we see that $\widehat{F}$ is square zero.

As in the proof of Proposition 3.6, we break our cobordism W into two pieces. Let Z be the union of collar neighborhoods of Y_ϕ and Y, a neighborhood of $\widehat{F}$, and neighborhoods of arcs running from Y_ϕ and Y to $\widehat{F}$; and let $X := W \smallsetminus Z$. If there existed a surface Σ in X with nonzero self-intersection, then $\Sigma, \widehat{F}$ would be linearly independent and hence form a basis for the rational second homology of W. With respect to this basis, the intersection form would be given by $\left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which would contradict the nondegeneracy of the intersection form on W, a 4-manifold with integer homology sphere boundary components. Therefore, X has vanishing intersection form and in particular is a negative semidefinite 4-manifold with boundary $-Y \# Y_\phi \# \widehat{F} \times S^1$. We would hope for an inequality of the d-invariants, as with null-homologous surgeries. The problem is that $\widehat{F} \times S^1$ does *not* have standard HF^∞ when the genus of $\widehat{F}$ is at least 1, so we must use the twisted correction terms.

We would like to invoke Theorem 3.9 (3). Note that the integer homology spheres Y and Y_ϕ have unique Spin^c structures and $\widehat{F} \times S^1$ has a unique torsion Spin^c structure since $H^2(\widehat{F} \times S^1)$ is torsion-free. Thus, to utilize Theorem 3.9 (3), we just need to show that there exists a Spin^c structure on X that restricts to be torsion on $\widehat{F} \times S^1$. By Lemma 3.5, it suffices to construct a Spin^c structure on X whose first Chern class evaluates to be trivial on $\widehat{F}$. We will construct our Spin^c structure on W and then restrict to X. (Note that Spin^c structures on a closed 3-manifold are the same as on once-punctured 3-manifolds, so this restriction will not cause any issues.)

If ℓ is odd, then W has even intersection form, and so, 0 is a characteristic element in $H^2(W;\mathbb{Z})$. Since characteristic elements are the first Chern classes of Spin^c structures, there is a Spin^c structure $\mathfrak{t}$ on W which satisfies $c_1(\mathfrak{t}) = 0$; this necessarily restricts to $c_1 = 0$ on $\widehat{F} \times S^1$, which implies that the restriction of $\mathfrak{t}$ to X is torsion. The case that ℓ is even goes as follows. A Spin^c structure on $\widehat{F} \times S^1$ is torsion if and only if c_1 pairs trivially with $\widehat{F}$. Let $a \in H^2(W) = H^2(W, \partial W)$ be the element which pairs to ± 1 on $T_\pm$, where $T_\pm$ are generators of $H_2(W)$ obtained by capping off Seifert surfaces for $K_\pm$ in Y with the cores of the corresponding 2-handle. One can quickly check that a is characteristic, and therefore $a = c_1(\mathfrak{t})$ for some Spin^c structure $\mathfrak{t}$ on W. Further, $a = c_1(\mathfrak{t})$ pairs to zero with $\widehat{F}$, since $[\widehat{F}] = T_+ + T_-$. In conclusion, independent of ℓ, there is a Spin^c structure on W (and hence X) that restricts to be torsion on $\widehat{F} \times S^1$.

By the above discussion, Theorem 3.9 (3) gives the bound

$$\underline{d}(-Y \# Y_\phi \# \widehat{F} \times S^1) \geq \frac{-2b_1(\partial X)}{4} \geq -\frac{2g+1}{2},$$

where we do not write any Spin^c structures because each summand has a unique torsion one. (There is no c_1^2 or b_2^- term because the intersection form on X vanishes.) We thus obtain a lower bound for $\underline{d}(-Y \# Y_\phi \# \widehat{F} \times S^1)$ that depends only on the genus of F. By Theorem 3.9,

$$\underline{d}(-Y\#Y_\phi\#\widehat{F}\times S^1) = -d(Y) + d(Y_\phi) + \underline{d}(\widehat{F}\times S^1).$$

Therefore, we get $d(Y) - d(Y_\phi) \leq C$, where C depends only on the genus of the subsurface F. Since the genus of F is at most g, we obtain a lower bound independent of the subsurface. By reversing orientations and repeating the process, we gain the opposite inequality $d(Y_\phi) - d(Y) \leq C$. This completes the proof. ∎

Remark 3.10. The proof shows something slightly stronger than Theorem 1.9. Namely, the constant can be refined further if one knows the genera of the subsurfaces involved in the factorization into separating Dehn twists and bounding pair maps.

4. Proofs of the main theorems

4.1. The proof of Theorem 1.2

We are now ready to prove the following result, which will quickly imply Theorem 1.2.

Proposition 4.1. *Let P be a pattern in the solid torus with winding number w. Suppose that there exists a prime power q dividing w such that if $\eta_1, \dots, \eta_q$ denote the lifts of η to $\Sigma_q(P(U))$, we have that*

(1) $\operatorname{lk}(\eta_i, \eta_j) \geq 0$ *for all $i \neq j$, but is not identically zero, and*

(2) *$\eta_1, \dots, \eta_q$ are null-homologous in $\Sigma_q(P(U))$.*

Then, there exists a constant $C_{P,q}$ depending only on P and q such that, for any knot K,

$$d_{\max}(\Sigma_q(P(K))) \leq d(\Sigma_2(C_{2,1}(K))) + C_{P,q}.$$

Proof. Step 1: converting $(\Sigma_q(P(U)), \bigcup_{i=1}^q \eta_i)$ to (S^3, L) via null-homologous surgery. Applying Proposition 2.4 to $(\Sigma_q(P(U)), \bigcup_{i=1}^q \eta_i)$ tells us that there exist (i) a framed link L in S^3 with the same linking-framing information as $\bigcup_{i=1}^q \eta_i$, (ii) natural numbers n, and (iii) a sequence $m_1, \dots, m_n$ of natural numbers such that a finite sequence of null-homologous integral surgeries along a link $\Gamma = \gamma_1 \cup \dots \cup \gamma_k$ transforms (S^3, L) into $(\Sigma_q(P(U))\#_{i=1}^n L(m_i, 1), \bigcup_{i=1}^q \eta_i)$. For convenience in notation, let $Y := \#_{i=1}^n L(m_i, 1)$ and define

$$c_0 := \max\{|d(Y, \mathfrak{s})| : \mathfrak{s} \in \operatorname{Spin}^c(Y)\}. \tag{4.1}$$

Since Γ is disjoint from L, we can consider each component γ of Γ as a knot:

$$\gamma \subset S^3 \smallsetminus \nu(L) \subset (S^3, L) * E_K.$$

Moreover, since we are considering a sequence of null-homologous surgeries in $S^3 \smallsetminus L$, each knot γ_{i+1} bounds a surface in the result of surgery on $\gamma_1, \dots, \gamma_i$ in $(S^3, L) * E_K$ whose genus is bounded above independently of K (i.e., only in terms of P and q). (See the discussion following Proposition 2.4 for what is meant by a sequence of null-homologous surgeries.) Finally, surgery on $(S^3, L) * E_K$ along $\gamma_1, \dots, \gamma_k$ using the same coefficients as before transforms $(S^3, L) * E_K$ to

$$\begin{aligned}\left(\Sigma_q(P(U))\#Y, \bigcup_{i=1}^{q} \eta_i\right) * E_K &= \left[\left(\Sigma_q(P(U)), \bigcup_{i=1}^{q} \eta_i\right) * E_K\right]\#Y \\ &= \Sigma_q(P(K))\#Y.\end{aligned}$$

Let W_K denote the 4-manifold obtained by taking $[0,1] \times ((S^3, L) * E_K)$ and attaching 2-handles along each $\{1\} \times \gamma$ for each component γ of Γ with the surgery framings, and note that W_K is the surgery 2-handle cobordism from $(S^3, L) * E_K$ to $\Sigma_q(P(K))\#Y$. Let $\mathfrak{s}_\Sigma$ denote a Spin^c structure on $\Sigma_q(P(K))$ with maximal corresponding d-invariant, i.e., such that

$$d(\Sigma_q(P(K)), \mathfrak{s}_\Sigma) = d_{\max}(\Sigma_q(P(K))). \tag{4.2}$$

Choose any Spin^c-structure $\mathfrak{s}_Y$ on Y. As noted before Proposition 3.6, there exists a Spin^c structure $\mathfrak{t}$ on W_K that restricts to $\mathfrak{s}_\Sigma\#\mathfrak{s}_Y$ on $\Sigma_q(P(K))\#Y$. Since $(S^3, L) * E_K$ is an integer homology sphere, $\mathfrak{t}$ restricts to the unique Spin^c structure on $(S^3, L) * E_K$.

By repeated application of Proposition 3.6, there exists a constant c_1 depending only on the framing of the surgeries and the genera of the knots $\gamma_1, \dots, \gamma_k$ (in particular, independent of K) such that

$$|d((S^3, L) * E_K) - d(\Sigma_q(P(K))\#Y, \mathfrak{s}_\Sigma\#\mathfrak{s}_Y)| \le c_1. \tag{4.3}$$

We can now use the additivity of d-invariants under connected sum Theorem 3.1 2, together with (4.1), (4.2), and (4.3) above, to obtain that

$$|d((S^3, L) * E_K) - d_{\max}(\Sigma_q(P(K)))| \le |d(Y, \mathfrak{s}_Y)| + c_1 \le c_0 + c_1. \tag{4.4}$$

Step 2: reducing the linking of L via negative-definite cobordisms. By assumption, $\mathrm{lk}(\eta_i, \eta_j) \ge 0$ for all $i \ne j$ and $\mathrm{lk}(\eta_i, \eta_j) > 0$ for some $i \ne j$. By Lemma 2.3, we know that $\mathrm{fr}(\eta_i) = -\sum_{j \ne i} \mathrm{lk}(\eta_i, \eta_j)$ for all i. Since (S^3, L) is obtained from $(\Sigma_q(P(U)), \bigcup_{i=1}^{q} \eta_i)$ by null-homologous surgeries, which do not change linking-framing information, the link L also satisfies the hypotheses of Proposition 2.7. Therefore, the following conclusion holds: there exists a negative definite cobordism W from (S^3, L) to (S^3, L'), where L' is a q-component link with the same linking-framing information as H_q. Moreover, this cobordism is of a particularly simple type, consisting of attaching 2-handles to $I \times S^3$ along curves which are disjoint from L. It follows that, for any knot K, by cutting out $I \times \nu(L)$ and gluing in $I \times \bigcup_{i=1}^{q} E_K$, we obtain a negative definite cobordism from $(S^3, L) * E_K$ to $(S^3, L') * E_K$. By the behavior of d-invariants under negative-definite cobordisms (Theorem 3.1 3), we therefore have that, for any knot K,

$$d((S^3, L) * E_K) \leq d((S^3, L') * E_K). \tag{4.5}$$

Step 3: converting L' to H_q via clasper moves. We now apply Proposition 2.10 to (S^3, L') to observe that there exists some length M sequence of clasper surgeries transforming (S^3, L') into (S^3, H_q), where we remind the reader that H_q is the split union of a -1-framed positive Hopf link H with a $(q-2)$-component 0-framed unlink. For any knot K, this induces a length M sequence of clasper surgeries transforming $(S^3, L') * E_K$ into $(S^3, H_q) * E_K$. By repeated application of Corollary 3.8, there exists a constant c_2 depending only on M, and in particular independent of K, such that for any K we have

$$|d((S^3, L') * E_K) - d((S^3, H_q) * E_K)| \leq c_2. \tag{4.6}$$

Step 4: finishing the proof. By sequentially using (4.4), (4.5), and (4.6), we therefore have that for any knot K

$$\begin{aligned} d_{\max}(\Sigma_q(P(K))) &\leq d((S^3, L) * E_K) + c_0 + c_1 \\ &\leq d((S^3, L') * E_K) + c_0 + c_1 \\ &\leq d((S^3, H_q) * E_K) + c_0 + c_1 + c_2. \end{aligned}$$

Since $H_q = H \cup U_{q-2}$ is a split link, we have that

$$\begin{aligned} (S^3, H_q) * E_K &\cong [(S^3, H) * E_K]\#[(S^3, U_{q-2}) * E_K] \\ &\cong [(S^3, H) * E_K]\#_{i=1}^{q-2}[(S^3, U) * E_K] \\ &\cong \Sigma_2(C_{2,1}(K))\#_{i=1}^{q-2} S^3 \\ &\cong \Sigma_2(C_{2,1}(K)), \end{aligned}$$

where the second-to-last equality follows from Example 2.2 and Example 2.1. So, $C_{P,q} := c_0 + c_1 + c_2$ is our desired constant. ■

We are now ready to prove Theorem 1.2, which states that under the hypotheses of Proposition 4.1 we have that P does not induce a homomorphism.

Proof of Theorem 1.2. Suppose for a contradiction that $P(T_{2,2k+1})\#P(-T_{2,2k+1})$ is slice for all k. This implies that the rational homology spheres

$$Z_k := \Sigma_q(P(T_{2,2k+1})\#P(-T_{2,2k+1})) = \Sigma_q(P(T_{2,2k+1}))\#\Sigma_q(P(-T_{2,2k+1}))$$

must all bound rational homology balls, and hence by Corollary 3.2 we must have that $d_{\max}(Z_k) \geq 0$ for all k. Now, let $C := C_{P,q}$ be as in Proposition 4.1, and observe that

$$\begin{aligned} d_{\max}(Z_k) &= d_{\max}(\Sigma_q(P(T_{2,2k+1}))) + d_{\max}(\Sigma_q(P(-T_{2,2k+1})) \\ &\leq d(\Sigma_2(C_{2,1}(T_{2,2k+1}))) + d(\Sigma_2(C_{2,1}(-T_{2,2k+1})) + 2C \\ &= d(S^3_1(T_{2,2k+1}\#T_{2,2k+1})) + d(S^3_1(-T_{2,2k+1}\# - T_{2,2k+1}) + 2C, \end{aligned}$$

where for the last equality we use the well-known fact that for any knot K one has $\Sigma_2(C_{2,1}(K)) \cong S^3_1(K\#K^r)$ as was explained in Example 2.2 and the fact that torus knots are reversible.

We can now apply Theorem 3.3 to obtain that

$$d_{\max}(Z_k) \leq -2k + 2C,$$

which is strictly negative for sufficiently large k, providing a contradiction. ■

4.2. The case of non-null-homologous lifts

The arguments of the previous subsection do not generalize well to the setting where the lifts of η to $\Sigma_q(P(U))$ represent non-trivial elements of first homology. We therefore prove Theorem 1.7 by working with composite patterns, relying on the elementary observation that if $P \circ R$ does not induce a homomorphism, then at least one of P and R must also not induce a homomorphism.

Definition 4.2. Given patterns $P \cup \eta^P$ and $Q \cup \eta^Q$, the composite pattern $(P \circ Q) \cup \eta^{P\circ Q}$ is defined as the image of (P, η^Q) in $S^3 = E_Q \cup E_{\eta^P}$, where the identification of ∂E_Q with ∂E_{η^P} glues λ_Q to μ_{η^P} and μ_Q to λ_{η^P}.

See Figure 9 for an example of a composite pattern.

Theorem 4.3. *Let $P \cup \eta^P$ and $R \cup \eta^R$ be patterns and n divide the winding number w_P of P. Then, the following hold.*

(1) $H_1(\Sigma_n((P \circ R)(U))) \cong H_1(\Sigma_n(P(U)))$.

(2) *Under this isomorphism, there is a correspondence of homology classes:*

$$\big[(\lambda^{\eta^{P\circ R}})_i\big] \leftrightarrow w_R\big[(\lambda^{\eta^P})_i\big]$$

for all $i = 1, \ldots, n$.

(3) $\mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta^{P\circ R}}, \lambda_j^{\eta^{P\circ R}}) = w_R^2\, \mathrm{lk}_{\Sigma_n(P(U))}(\lambda_i^{\eta^P}, \lambda_j^{\eta^P})$ *for all distinct* $1 \leq i, j \leq n$.

We emphasize that, in the statement of Theorem 4.3, we are not assuming that any of the lifts of η_P, η_R, or $\eta_{P\circ R}$ are null-homologous, and so, the linking numbers are rationally rather than integrally valued.

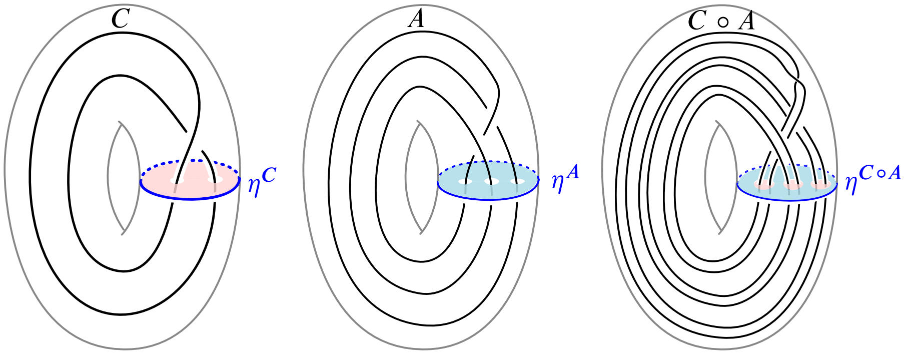


Figure 9. Left: the pattern $C \cup \eta^C$ that defines the $(2,1)$-cabling operation. Center: the pattern $A \cup \eta^A$ that defines the 3-stranded alternating cabling. Right: the composition $(C \circ A) \cup \eta^{C\circ A}$.

Proof. Recall that Equation (2.2) gives decompositions

$$\Sigma_n((P\circ R)(U)) = \left(\Sigma_n((P\circ R)(U)) \smallsetminus \bigcup_{i=1}^{n} \nu(\eta_i^{P\circ R})\right) \cup \bigcup_{i=1}^{n} E_U^{(i)}, \tag{4.7}$$

and

$$\Sigma_n(P(R(U))) = \left(\Sigma_n(P(U)) \smallsetminus \bigcup_{i=1}^{n} \nu(\eta_i^{P})\right) \cup \bigcup_{i=1}^{n} (S^3 \smallsetminus \nu(R\cup\eta^R)\cup E_U)^{(i)}, \tag{4.8}$$

where the second identification relies on the fact that n divides w_P and the fact that $(S^3 \smallsetminus \nu(R\cup\eta^R)\cup E_U) = E_{R(U)}$ as in Equation (2.1). The first claim follows after observing that E_U and $E_{R(U)}$ have the same homology. Notice that $\lambda_i^{\eta^{P\circ R}}$ is isotopic to $\lambda_i^{\eta^R}$ in $\Sigma_n((P\circ R)(U))$, since in Equation (4.7) $\lambda_i^{\eta^{P\circ R}}$ is identified with $\mu_U^{(i)}$, and in Equation (4.8) $\mu_U^{(i)}$ is identified with $\lambda_i^{\eta^R}$.

Next, let D be a meridional disk of the solid torus $S^3 \setminus \nu(\eta^R)$ that intersects R transversely, and for each point $x \in R\cap D$ denote by N_x a disk neighborhood of x in $R\cap D$. The unknotted circles ∂N_x can be divided into two sets $(R\cap D)^{\pm}$ according to whether the orientation agrees with that of μ_R or not. By the definition of winding number, there is a pairing between the elements of $(R\cap D)^-$ and all but w_R elements of $(R\cap D)^+$ via tubes contained in $S^3 \setminus \nu(\eta^R \cup R)$. If S is the surface obtained as the union of $D \setminus \bigcup_{x\in R\cap D} N_x$ and large enough tubes, then S is contained in $S^3 \smallsetminus \nu(R\cup\eta^R)$ and its boundary ∂S consists of the disjoint union of $\partial D = \lambda^{\eta^R}$ and w_R copies of negatively oriented meridians of R. Since

$$S^3 \smallsetminus \nu(R\cup\eta^R) \subset E_{R(U)} = (S^3 \smallsetminus \nu(R\cup\eta^R)\cup E_U),$$

then for each $i = 1, \dots, n$ there is a copy S^i of S contained in the branched cover $\Sigma_n(P(R(U)))$. It follows that $\lambda_i^{\eta^{P\circ R}} = \lambda_i^{\eta^R}$ is homologous to $w_R \mu_{R(U)}^{(i)}$ in $E_{R(U)}^{(i)}$ and thus in $\Sigma_n(P(R(U)))$. To establish the second claim, it is enough to observe that $\mu_{R(U)}^{(i)}$ is identified with $\lambda_i^{\eta_P}$ in $\Sigma_n(P(R(U)))$.

To establish the precise relationship between linking numbers stated as the third claim, notice that for any $j \neq i$ the surface S^i is contained in the complement of $\lambda_j^{\eta^{P\circ R}}$, and so,

$$\begin{aligned} \mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta^{P\circ R}}, \lambda_j^{\eta^{P\circ R}}) &= w_R \, \mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta_P}, \lambda_j^{\eta^{P\circ R}}) \\ &= w_R^2 \, \mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta_P}, \lambda_j^{\eta_P}). \end{aligned} \tag{4.9}$$

Here, we are using the following fact: if J_1 and J_2 are rationally null-homologous knots which are homologous in the complement of another rationally null-homologous knot K in a 3-manifold Y, then $\mathrm{lk}_Y(J_1, K) = \mathrm{lk}_Y(J_2, K) \in \mathbb{Q}$ (and not just in $\mathbb{Q}/\mathbb{Z}$) (see [50, Chapter 10, Section 77]).

Lastly, consider an unknotting sequence for $R(U)$: that is, a collection $c_1, \dots, c_M$ of curves in S^3, each of which bounds a disk that intersects $R(U)$ in exactly two points and with opposite orientations, and such that ± 1 surgery along all these curves transforms $(S^3, R(U))$ into (S^3, U). Lifting each of these curves to each of the n copies of $E_{R(U)}$ in equation (4.8) gives us an nM-component link Γ in $\Sigma_n((P \circ R)(U))$. Doing surgery to $\Sigma_n((P \circ R)(U))$ along Γ using the lifted framings results in $\Sigma_n(P(U))$. Moreover, since each component of Γ is null-homologous in some $E_{R(U)}^{(i)}$, this is a null-homologous surgery from $(\Sigma_n((P \circ R)(U)), \bigcup_{i=1}^n \mu_{R(U)}^{(i)})$ to $(\Sigma_n(P(U)), \bigcup_{i=1}^n \mu_U^{(i)})$. Therefore, since null-homologous surgeries preserve linking numbers, we have that

$$\mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\mu_{R(U)}^{(i)}, \mu_{R(U)}^{(j)}) = \mathrm{lk}_{\Sigma_n(P(U))}(\mu_U^{(i)}, \mu_U^{(j)}) \quad \text{for all } i \neq j.$$

But $\mu_{R(U)}^{(i)}$ is identified with η_P^i in $\Sigma_n(P(R(U))$ and $\mu_U^{(i)}$ is identified with η_P^i in $\Sigma_n(P(U))$, so

$$\mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta_P}, \lambda_j^{\eta_P}) = \mathrm{lk}_{\Sigma_n(P(U))}(\lambda_i^{\eta_P}, \lambda_j^{\eta_P}).$$

Therefore, by (4.9),

$$\begin{aligned} w_R^2 \, \mathrm{lk}_{\Sigma_n(P(U))}(\lambda_i^{\eta_P}, \lambda_j^{\eta_P}) &= w_R^2 \, \mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta_P}, \lambda_j^{\eta_P}) \\ &= \mathrm{lk}_{\Sigma_n((P\circ R)(U))}(\lambda_i^{\eta^{P\circ R}}, \lambda_j^{\eta^{P\circ R}}), \end{aligned}$$

which completes the proof. ∎

We are now ready to prove Theorem 1.7.

Theorem 1.7. *Let P be a pattern in the solid torus with winding number w and q a prime power dividing w. Denote the q lifts of η to $\Sigma_q(P(U))$ by $\eta_1, \ldots, \eta_q$. Suppose that, for $i \neq j$, the numbers $\mathrm{lk}(\eta_i, \eta_j)$ are non-negative and not identically zero, and let n be the order of $[\eta_1]$ in $H_1(\Sigma_q(P(U))$. If R is any pattern whose winding number is a nonzero multiple of n, then the composition $P \circ R$ does not induce a pseudo-homomorphism.*

Proof. While the lifts of η_P may not be null-homologous, if n divides w_R, then by Theorem 4.3 (2) the lifts of $\eta_{P \circ R}$ are null-homologous in $\Sigma_q(P \circ R(U))$. Moreover, it follows from Theorem 4.3 (3) that the lifted curves η_i in $\Sigma_q(P \circ R(U))$ also have non-negative linking which is not identically vanishing. Theorem 1.2 now implies that $P \circ R$ cannot be a pseudo-homomorphism. ∎

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