Ann. Inst. H. Poincaré C
Anal. Non Linéaire 43 (2026), 749–758
DOI 10.4171/AIHPC/176

Corrigendum to "Fine multibubble analysis in the higher-dimensional Brezis–Nirenberg problem"

Tobias König and Paul Laurain

Abstract. In this corrigendum, we correct an inaccuracy in the proofs of Propositions 2.5 and 2.6 in Ann. Inst. H. Poincaré C Anal. Non Linéaire 41, 1239–1287 (2024). Indeed, to obtain the refined error terms $q_{i,\varepsilon}$ and $p_{i,\varepsilon}$ we subtract only a radial term in Ann. Inst. H. Poincaré C Anal. Non Linéaire 41, 1239–1287 (2024), but in fact to ensure the desired properties of $q_{i,\varepsilon}$ and $p_{i,\varepsilon}$ one needs to subtract the whole Hessian. This change does not affect the main result of Ann. Inst. H. Poincaré C Anal. Non Linéaire 41, 1239–1287 (2024), Theorem 1.1, up to the expression of the refined local asymptotics which must include all second derivatives. Moreover, the mentioned refinement can be accommodated in a natural way into the existing proofs of Proposition 2.5 and 2.6 given in Ann. Inst. H. Poincaré C Anal. Non Linéaire 41, 1239–1287 (2024).

1. Statement of the main theorem

In Theorem 1.1, *only* (i) *has to be changed.* For multiindices $\alpha \in \mathbb{N}_0^2$, we consider functions W_α which satisfy

$$-\Delta W_\alpha - N(N+2)B^{\frac{4}{N-2}} W_\alpha = f_\alpha \text{ on } \mathbb{R}^N, \quad W_\alpha(x) = \frac{1}{\alpha!}x^\alpha + o(|x|^2), \tag{1.1}$$

with

$$f_\alpha = \begin{cases} 0 & \text{if } x^\alpha = x_j x_k \text{ for some } j \neq k, \\ -B & \text{if } x^\alpha = x_j^2 \text{ for some } j. \end{cases}$$

We construct these functions in Lemma A.3 below. The only change in our main theorem is the following:

(i) **Refined local asymptotics:** For any $i = 1, \dots, n$, denote $B_{i,\varepsilon} := B_{\mu_{i,\varepsilon}, x_{i,\varepsilon}}$ and

$$W_{i,\varepsilon}(x) := \mu_{i,\varepsilon}^2 \sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = 2} W_\alpha\Big(\frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) \partial_\alpha (u_\varepsilon - B_{i,\varepsilon})(x_{i,\varepsilon}). \tag{1.2}$$

Then for $\delta > 0$ small enough, and every $\nu \in (2,3)$,

$$|(u_\varepsilon - B_{i,\varepsilon} - W_{i,\varepsilon})(x)| \lesssim \Big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\Big)|x - x_{i,\varepsilon}|^\nu$$

for all $x \in B(x_{i,\varepsilon}, \delta)$.

Mathematics Subject Classification 2020: 35B40 (primary); 35J60 (secondary).
Keywords: non-linear elliptic PDE, asymptotic analysis.

We also make more precise the following remark.

Remarks 1.1. (a) In order to keep the statement of our theorem reasonable, in the refined local asymptotics, we just give the expansion up to the first term after the bubble. But, in fact we can go further, as shown by Proposition 2.6. More precisely, our technique, which consists in subtracting recursively a suitable solution of the inhomogeneous linearized equation, can give bounds on remainder terms $q_{i,\varepsilon}^{(l)}$ *of arbitrary order* $l \in \mathbb{N}_0$. Let us sketch the general framework. Indeed, set $q_{i,\varepsilon}^{(0)} = u_\varepsilon - B_{i,\varepsilon}$ and define recursively

$$W_{i,\varepsilon}^{(l+1)} := \sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = l+2} W_\alpha^{(l+1)}\Big(\frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\partial_\alpha(q_{i,\varepsilon}^{(l)}(x_{i,\varepsilon})),$$

where $W_\alpha^{(l+1)}$ is a solution to

$$(-\Delta - N(N+2)B^{\frac{4}{N-2}})W_\alpha^{(l+1)} = f_\alpha(x, W^{(1)}, \dots, W^{(l)}) \quad \text{on } \mathbb{R}^N$$

with

$$W_\alpha^{(l+1)}(x) = \frac{1}{\alpha!}x^\alpha + o(|x|^{l+2}) \quad \text{as } x \to 0.$$

Here, the inhomogeneities f_α are determined recursively from the equation satisfied by the remainder term. Then by carrying out the technique used in Section 2, the remainder terms

$$q_{i,\varepsilon}^{(l)} = u - B_{i,\varepsilon} - \sum_{k=1}^{l} W_{i,\varepsilon}^{(k)}$$

can be expected to satisfy the estimate

$$|q_{i,\varepsilon}(x)| \lesssim \Big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3+l-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\Big)|x - x_{i,\varepsilon}|^\nu$$

for all $x \in B(x_{i,\varepsilon}, \delta)$ and $\nu \in (l+1, l+2)$. We carry this program out rigorously for $l = 1, 2$ in this paper; see Propositions 2.5 and 2.6.

2. Modification of the proofs of Propositions 2.5 and 2.6

We modify the definitions of $q_{i,\varepsilon}$ *and* $r_{i,\varepsilon}$ *as follows.* We set

$$r_{i,\varepsilon} := u_{i,\varepsilon} - B_{i,\varepsilon}$$

and then

$$q_{i,\varepsilon} := r_{i,\varepsilon} - W_{i,\varepsilon}$$

and

$$p_{i,\varepsilon} := q_{i,\varepsilon} - \widetilde{W}_{i,\varepsilon}$$

on $\mathsf{b}_{i,\varepsilon}$. Here, $W_{i,\varepsilon}$ is the function defined in (1.2). Similarly, the term $\widetilde{W}_{i,\varepsilon}$ is defined to be

$$\widetilde{W}_{i,\varepsilon} := \mu_{i,\varepsilon}^3 \sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = 3} \widetilde{W}_\alpha \Big(\frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}} \Big) \partial_\alpha (q_{i,\varepsilon})(x_{i,\varepsilon}),$$

for functions $\widetilde{W}_\alpha$ satisfying, for every multiindex $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = 3$,

$$\begin{aligned} -\Delta \widetilde{W}_\alpha - N(N+2) B^{\frac{4}{N-2}} \widetilde{W}_\alpha &= f_\alpha \quad \text{on } \mathbb{R}^N, \\ \widetilde{W}_\alpha(x) &= \frac{1}{\alpha!} x^\alpha + o(|x|^3) \quad \text{as } x \to 0, \end{aligned} \tag{2.1}$$

with

$$f_\alpha = \begin{cases} 0 & \text{if } x^\alpha = x_j x_k x_l \text{ for some } j \neq k \neq l \neq j, \\ -B x_l & \text{if } x^\alpha = x_j^2 x_l \text{ for some } j, l. \end{cases}$$

We construct the functions W_{jkl} in Lemma A.4 below.

Notice that the correction terms $W_{i,\varepsilon}$ and $\widetilde{W}_{i,\varepsilon}$ are chosen to ensure that

$$\partial_\alpha q_{i,\varepsilon}(x_{i,\varepsilon}) = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \leq 2 \tag{2.2}$$

and

$$\partial_\alpha p_{i,\varepsilon}(x_{i,\varepsilon}) = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \leq 3. \tag{2.3}$$

2.1. The bound on $q_{i,\varepsilon}$

The statement of Proposition 2.5 *is unchanged; only the proof needs some adjustments.*

Proof of Proposition 2.5. Let $Q_{i,\varepsilon}(x) := \frac{q_{i,\varepsilon}(x)}{|x - x_{i,\varepsilon}|^\nu}$, fix a point $z_{i,\varepsilon}$ with $Q_{i,\varepsilon}(z_{i,\varepsilon}) \geq \frac{1}{2} \|Q_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}$ and let $d_{i,\varepsilon} := |x_{i,\varepsilon} - z_{i,\varepsilon}|$. (Notice that, by (2.2) and Taylor’s theorem, we have $\|Q_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} < \infty$.)

When $d_{i,\varepsilon} \gtrsim 1$, we have

$$\begin{aligned} Q_{i,\varepsilon}(x) \lesssim \frac{q_{i,\varepsilon}(z_{i,\varepsilon})}{d_{i,\varepsilon}^\nu} &\lesssim |B_{i,\varepsilon}(z_{i,\varepsilon})| + \Big| \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} W\Big(\frac{z_{i,\varepsilon} - x_{i,\varepsilon}}{\mu_{i,\varepsilon}} \Big) \Big| \\ &\lesssim \mu_\varepsilon^{\frac{N-2}{2}} + \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} \lesssim \mu_\varepsilon^{\frac{N-2}{2}}, \end{aligned}$$

where we used Lemma 2.3 and the fact that W is bounded by Lemma A.1. So it remains to treat the case $d_{i,\varepsilon} = o(1)$ in the following.

In the following, let us assume $N \geq 6$. Then $\frac{N+2}{N-2} \leq 2$. Using the definition of the W_α in (1.1) and of $W_{i,\varepsilon}$ in (1.2), we find

$$\begin{aligned} -\Delta W_{i,\varepsilon} - N(N+2) B_{i,\varepsilon}^{\frac{4}{N-2}} W_{i,\varepsilon} &= \mu_{i,\varepsilon}^{\frac{N-2}{2}} B_{i,\varepsilon} (\Delta u_\varepsilon - \Delta B_{i,\varepsilon})(x_{i,\varepsilon}) \\ &= -\mu_{i,\varepsilon}^{\frac{N-2}{2}} B_{i,\varepsilon} \varepsilon V(x_{i,\varepsilon}) u(x_{i,\varepsilon}) = -\varepsilon V(x_{i,\varepsilon}) B_{i,\varepsilon}. \end{aligned}$$

With this, the equation satisfied by $q_{i,\varepsilon}$ can be written as

$$-\Delta q_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}} q_{i,\varepsilon} = \varepsilon B_{i,\varepsilon}(V(x_{i,\varepsilon}) - V(x)) - \varepsilon V r_{i,\varepsilon} + \mathcal{O}(r_{i,\varepsilon}^{\frac{N+2}{N-2}}), \quad \text{on } \mathsf{b}_{i,\varepsilon}.$$

When $N = 4, 5$, and hence $\frac{N+2}{N-2} > 2$, the last term need to be replaced by $\mathcal{O}(r_{i,\varepsilon}^2 B_{i,\varepsilon}^{\frac{6-N}{N-2}})$.

The rest of the proof is verbatim the same. ■

2.2. The bound on $p_{i,\varepsilon}$

The statement of Proposition 2.6 *is unchanged; only the proof needs some adjustments.*

Proof of Proposition 2.6. The proof works exactly the same as that of Propositions 2.4 and 2.5. There is only one subtlety we point out, the rest is exactly the same. Let $P_{i,\varepsilon}(x) := \frac{p_{i,\varepsilon}(x)}{|x-x_{i,\varepsilon}|^\nu}$, fix a point $z_{i,\varepsilon}$ with $P_{i,\varepsilon}(z_{i,\varepsilon}) \geq \frac{1}{2}\|P_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}$ and let $d_{i,\varepsilon} := |x_{i,\varepsilon} - z_{i,\varepsilon}|$. (Notice that, by (2.3) and Taylor's theorem, we have $\|P_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} < \infty$.)

The rest of the proof is verbatim the same. ■

3. The main expansions

The statements of Propositions 3.1 *and* 3.2 *are the same; only the proofs require some adjustments.*

Proof of Proposition 3.1. *The proof is verbatim the same up to the treatment of the second term of* (3.6).

We now treat the second main term of (3.6), namely

$$N(N+2)\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} W_{i,\varepsilon} \frac{1}{\omega_{N-1}(N-2)|x-x_{i,\varepsilon}|^{N-2}}\, \mathrm{d}x.$$

Recall that $W_{i,\varepsilon}(x) = \mu_{i,\varepsilon}^2 \sum_{|\alpha|=2} c_{\alpha,\varepsilon} W_\alpha(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}})$ with $c_{\alpha,\varepsilon} = \partial_\alpha(u_\varepsilon - B_{i,\varepsilon})(x_{i,\varepsilon})$. Now if $x^\alpha = x_j x_k$ for some $j \neq k$, then $W_\alpha(x) = f(x)Y_2(x/|x|)$ for some spherical harmonic Y_2 of degree 2; see Lemma A.3. Hence its integral over the ball $\mathsf{b}_{i,\varepsilon}$ against the radial function $|x-x_{i,\varepsilon}|^{-N+2}B_{i,\varepsilon}^{\frac{4}{N-2}}$ vanishes. Thus only the terms with $x^\alpha = x_j^2$ remain. In that case, $W_\alpha(x) = f(x)Y_2(x/|x|) + W(x)$, where $W(x) = W(|x|)$ is the function from Lemma A.1. Again, the term $f(x)Y_2(x/|x|)$ integrates to zero against $|x-x_{i,\varepsilon}|^{-N+2}B_{i,\varepsilon}^{\frac{4}{N-2}}$. Hence it remains to integrate against $|x-x_{i,\varepsilon}|^{-N+2}B_{i,\varepsilon}^{\frac{4}{N-2}}$ the term

$$\begin{aligned}\mu_{i,\varepsilon}^2 \sum_j \partial_{jj}(u_\varepsilon - B_{i,\varepsilon})(x_{i,\varepsilon}) W\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) &= \mu_{i,\varepsilon}^2 W\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\Delta(u_\varepsilon - B_{i,\varepsilon})(x_{i,\varepsilon}) \\ &= \varepsilon V(x_{i,\varepsilon})\mu_{i,\varepsilon}^2 W\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big).\end{aligned}$$

We obtain, using that $N(N+2)B^{\frac{4}{N-2}}W = -\Delta W + B$,

$$\begin{aligned}
N(N+2)&\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} W_{i,\varepsilon} \frac{1}{\omega_{N-1}(N-2)|x-x_{i,\varepsilon}|^{N-2}}\,\mathrm{d}x \\
&= \varepsilon V(x_{i,\varepsilon}) N(N+2)\mu_{i,\varepsilon}^2 \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} W\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) \frac{1}{\omega_{N-1}(N-2)|x-x_{i,\varepsilon}|^{N-2}}\,\mathrm{d}x \\
&= \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon}) \int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} (-\Delta W) \frac{1}{\omega_{N-1}(N-2)|z|^{N-2}}\,\mathrm{d}z \\
&\quad + \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon}) \int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} B \frac{1}{\omega_{N-1}(N-2)|z|^{N-2}}\,\mathrm{d}z + o(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}).
\end{aligned}$$

Then the proof goes on verbatim up to the following:

For the next term, by Proposition 2.4 and straightforward estimates, we obtain

$$\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} |r_{i,\varepsilon}| H(\cdot, x_{i,\varepsilon})\,\mathrm{d}x \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+5} + \mu_\varepsilon^{\frac{N+2}{2}} = o(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}) + \mathcal{O}(\mu_\varepsilon^{\frac{N+2}{2}}).$$

And verbatim up to the end. ∎

Proof of Proposition 3.2. *The proof is verbatim the same up to the following:*

To treat the singular term, write $u_\varepsilon = B_{i,\varepsilon} + r_{i,\varepsilon} = B_{i,\varepsilon} + W_{i,\varepsilon} + q_{i,\varepsilon}$ and expand $u_\varepsilon^{\frac{N+2}{N-2}}$ in (3.8). By antisymmetry, the term involving $B_{i,\varepsilon}^{\frac{N+2}{N-2}}$ vanishes. Moreover, since $W_{i,\varepsilon}$ only contains spherical harmonics of degrees 0 and 2, the term $B_{i,\varepsilon}^{\frac{4}{N-2}} W_{i,\varepsilon}$ also vanishes when integrated against $\frac{x-x_{i,\varepsilon}}{|x-x_{i,\varepsilon}|^N}$ (which is a radial function times a spherical harmonic of degree 1).

Then the same up to the following:

We now discuss the main contribution from $q_{i,\varepsilon}$, namely the term

$$\frac{N(N+2)}{\omega_{N-1}} \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} \widetilde{W}_{i,\varepsilon} \frac{(x-x_{i,\varepsilon})_l}{|x-x_{i,\varepsilon}|^N}\,\mathrm{d}x.$$

Recall that $\widetilde{W}_{i,\varepsilon}(x) = \mu_{i,\varepsilon}^3 \sum_{|\alpha|=3} c_{\alpha,\varepsilon} \widetilde{W}_\alpha(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}})$ with $c_{\alpha,\varepsilon} = \partial_\alpha(q_{i,\varepsilon})(x_{i,\varepsilon})$. Now if $x^\alpha = x_j x_k x_l$ for some $j \neq k \neq l \neq j$, then $\widetilde{W}_\alpha(x) = f(x) Y_3(x/|x|)$ for some spherical harmonic Y_3 of degree 2; see Lemma A.4. Hence its integral over the ball $\mathsf{b}_{i,\varepsilon}$ against $|x-x_{i,\varepsilon}|^{-N} B_{i,\varepsilon}^{\frac{4}{N-2}} (x-x_{i,\varepsilon})_l$ (which is a radial function times a spherical harmonic of degree 1) vanishes. Thus only the terms with $x^\alpha = x_j^2 x_k$ remain. In that case, $\widetilde{W}_\alpha(x) = f(|x|) Y_e(x/|x|) + \widetilde{W}(x)\frac{x_k}{|x|}$, where $\widetilde{W}(x) = \widetilde{W}(|x|)$ is the function from Lemma A.2. Again, the term $f(|x|) Y_3(x/|x|)$ integrates to zero against $|x-x_{i,\varepsilon}|^{-N} B_{i,\varepsilon}^{\frac{4}{N-2}} (x-x_{i,\varepsilon})_l$. Moreover, also the term $\widetilde{W}(x)\frac{x_k}{|x|}$ integrates to zero unless $k = l$. In summary, it remains to integrate against $|x-x_{i,\varepsilon}|^{-N} B_{i,\varepsilon}^{\frac{4}{N-2}}(x-x_{i,\varepsilon})_l$ the term

$$\begin{aligned}
&\mu_{i,\varepsilon}^3 \sum_j \partial_{jjl}(q_{i,\varepsilon})(x_{i,\varepsilon}) \widetilde{W}\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\frac{(x-x_{i,\varepsilon})_l}{|x-x_{i,\varepsilon}|}\\
&\quad = \mu_{i,\varepsilon}^3 \widetilde{W}\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\frac{(x-x_{i,\varepsilon})_l}{|x-x_{i,\varepsilon}|}\partial_l \Delta(q_{i,\varepsilon})(x_{i,\varepsilon})\\
&\quad = \mu_{i,\varepsilon}^{-\frac{N}{2}+4} \widetilde{W}\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\frac{(x-x_{i,\varepsilon})_l}{|x-x_{i,\varepsilon}|}\varepsilon \partial_l V(x_{i,\varepsilon}).
\end{aligned}$$

In conclusion, we can write

$$\begin{aligned}
&\frac{N(N+2)}{\omega_{N-1}} \int_{\mathrm{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} \widetilde{W}_{i,\varepsilon} \frac{(x-x_{i,\varepsilon})_l}{|x-x_{i,\varepsilon}|^N}\,\mathrm{d}x\\
&\quad = \frac{N(N+2)}{\omega_{N-1}} \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+4} \partial_l V(x_{i,\varepsilon}) \int_{\mathrm{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} \widetilde{W}\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) \frac{(x-x_{i,\varepsilon})_l^2}{|x-x_{i,\varepsilon}|^{N+1}}\,\mathrm{d}x\\
&\quad = \frac{N(N+2)}{\omega_{N-1}} \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} \partial_l V(x_{i,\varepsilon}) \int_{B(0,R_\varepsilon)} B^{\frac{4}{N-2}}(z) \widetilde{W}(z) \frac{z_l^2}{|z|^{N+1}}\,\mathrm{d}z,
\end{aligned}$$

with $R_\varepsilon = \delta_0 \mu_{i,\varepsilon}^{-1}$. Now we use the equation $-\Delta \widetilde{W} - N(N+2) B^{\frac{4}{N-2}} \widetilde{W} = -B|z|$ for $\widetilde{W}$. This gives

$$\begin{aligned}
&\frac{N(N+2)}{\omega_{N-1}} \int_{\mathrm{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} \widetilde{W}_{i,\varepsilon} \frac{(x-x_{i,\varepsilon})_l}{|x-x_{i,\varepsilon}|^N}\,\mathrm{d}x\\
&= \frac{1}{\omega_{N-1}} \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} \partial_l V(x_{i,\varepsilon}) \Big(\int_{B(0,R_\varepsilon)} (-\Delta \widetilde{W})(z) \frac{z_l^2}{|z|^{N+1}}\,\mathrm{d}z + \int_{B(0,R_\varepsilon)} B(z) \frac{z_l^2}{|z|^N}\,\mathrm{d}z \Big).
\end{aligned}$$

The second summand cancels precisely with the term (3.9) from the left-hand side pointed out above. *And it goes verbatim the same up to the end.* ■

A. Some computations

With respect to the initial paper, Lemmas A.1 *and* A.2 *are just made more precise with* (A.4) *and* (A.5)*. Then Lemmas* A.3 *and* A.4 *are new and dedicated to the construction of the functions* W_{jk} *and* W_{jkl}.

In the following we denote

$$v(r) = \frac{1-r^2}{(1+r^2)^{N/2}}, \quad \tilde{v}(r) = \frac{r}{(1+r^2)^{N/2}}. \tag{A.1}$$

These functions are chosen such that

$$\partial_\lambda B(x) = c v(|x|), \quad \partial_{x_i} B(x) = \tilde{c} \tilde{v}(|x|) \frac{x_i}{|x|},$$

for some constants $c, \tilde{c}$. In particular, v and $\tilde{v}$ are solutions to the homogeneous ODEs

$$-v'' - \frac{N-1}{r} v' - N(N+2) B^{\frac{4}{N-2}} v = 0 \quad \text{on } (0,\infty) \tag{A.2}$$

and

$$-\tilde{v}'' - \frac{N-1}{r} \tilde{v}' + \frac{N-1}{r^2} \tilde{v} - N(N+2) B^{\frac{4}{N-2}} \tilde{v} = 0 \quad \text{on } (0,\infty), \tag{A.3}$$

respectively. In the following lemmas, we study the asymptotic properties at 0 and ∞ of solutions to inhomogeneous versions of (A.2) and (A.3). These will be crucial for handling the refined terms $q_{i,\varepsilon}$ and $p_{i,\varepsilon}$ in the main part of our argument.

Lemma A.1. *For v as in (A.1), let W be given by*

$$W(r) = v(r) \int_0^r \frac{1}{s^{N-1} v(s)^2} \left(\int_0^s B(t) t^{N-1} v(t)\, \mathrm{d}t \right) \mathrm{d}s. \tag{A.4}$$

Then W solves

$$-W'' - \frac{N-1}{r} W' - N(N+2) B^{\frac{4}{N-2}} W = -B \quad \text{on } (0,\infty).$$

Moreover,

$$\frac{W(r)}{r^2} \to \frac{1}{2N} \quad \text{as } r \to 0.$$

As $r \to \infty$,

$$W(r) = \begin{cases} \dfrac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{(N-2)\Gamma(N-1)} + o(1) & \text{if } N \geq 5, \\ \dfrac{1}{2} \ln r + o(\ln r) & \text{if } N = 4, \end{cases}$$

and

$$W'(r) = \begin{cases} o(r^{-1}) & \text{if } N \geq 5, \\ o(r^{-1} \ln r) & \text{if } N = 4. \end{cases}$$

Proof. It can be checked by a direct computation that W satisfies the claimed equation. The expression for (A.4) can be derived using the method of variation of constants, which is already carried out in the proof of [2, Lemma A.1]. The estimates at ∞ are verbatim the same, and the claimed asymptotic behavior as $r \to 0$ can be read off directly from (A.4), using that $B(r) \to 1$ and $v(r) \to 1$ as $r \to 0$. ■

A very similar argument, whose details we omit, yields the asymptotics of the ODE solution governing the correction term $\widetilde{W}_{i,\varepsilon}$.

Lemma A.2. *For $\tilde{v}$ as in (A.1), let $\widetilde{W}$ be given by*

$$\widetilde{W}(r) = \tilde{v}(r) \int_0^r \frac{1}{s^{N-1} \tilde{v}(s)^2} \left(\int_0^s B(t) t^N \tilde{v}(t)\, \mathrm{d}t \right) \mathrm{d}s. \tag{A.5}$$

Then $\widetilde{W}$ solves

$$-\widetilde{W}'' - \frac{N-1}{r}\widetilde{W}' + \frac{N-1}{r^2}\widetilde{W} - N(N+2)B^{\frac{4}{N-2}}\widetilde{W} = -B(r)r \quad \textit{on } (0,\infty).$$

Moreover,

$$\frac{\widetilde{W}(r)}{r^3} \to \frac{1}{2(N+2)} \quad \textit{as } r \to 0$$

and, when $N \geq 5$,

$$\lim_{r\to\infty} \widetilde{W}(r)r^{-1} = \lim_{r\to\infty} \widetilde{W}'(r) = \frac{a_N}{N},$$

with

$$a_N = \frac{N}{4}\frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{\Gamma(N-1)}.$$

Using the inhomogeneous solutions W and $\widetilde{W}$, we can now construct the functions W_{jk} and W_{jkl} from (1.1) and (2.1).

Lemma A.3. *For multiindices $\alpha \in \mathbb{N}_0^N$, we consider functions W_α which satisfy*

$$-\Delta W_\alpha - N(N+2)B^{\frac{4}{N-2}}W_\alpha = f_\alpha \ \textit{ on } \mathbb{R}^N, \quad W_\alpha(x) = \frac{1}{\alpha!}x^\alpha + o(|x|^2), \tag{A.6}$$

with

$$f_\alpha = \begin{cases} 0 & \textit{if } x^\alpha = x_j x_k \textit{ for some } j \neq k, \\ -B & \textit{if } x^\alpha = x_j^2 \textit{ for some } j. \end{cases}$$

Proof. If $x^\alpha = x_j x_k$ with $j \neq k$, we make the ansatz $W_\alpha = f(|x|)Y_{jk}(x/|x|)$, with $Y_{jk}(\omega) = \omega_j\omega_k$ for $\omega \in \mathbb{S}^{N-1}$. Observing that Y_{jk} is a spherical harmonic of degree 2, W_α solves the equation in (A.6) if and only if f solves the ODE

$$-f''(r) - \frac{N-1}{r}f'(r) + \frac{2N}{r^2}f(r) + N(N+2)f(r)B^{\frac{4}{N-2}}(r) = 0 \quad \text{on } (0,\infty). \tag{A.7}$$

By the proof of [1, Proposition A.1], there is a solution to (A.7) which satisfies $f(r) \sim r^2$ for $r \in (0,\infty)$. Up to replacing f by a suitable scalar multiple, we may thus assume that $\lim_{r\to 0} f(r)r^{-2} = 1$. It follows that

$$W_\alpha(x) = f(|x|)Y_{jk}(x/|x|) = f(|x|)|x|^{-2}x_jx_k = (1+o(1))x_jx_k = \frac{1}{\alpha!}x^\alpha + o(|x|^2).$$

If on the other hand $x^\alpha = x_j^2$, we set

$$W_\alpha(x) = f(|x|)Y_j(x/|x|) + W(|x|),$$

where $Y_j(\omega) = \frac{1}{2}\omega_j^2 - \frac{1}{2N}$, f is a solution to (A.7) with $\lim_{r\to 0} f(r)r^{-2} = 1$ and W is the function from Lemma A.1. Observing that Y_j is a spherical harmonic of degree 2, W_{jj} satisfies the equation in (A.6). Moreover,

$$W_{jj}(x) = f(|x|)\Big(\frac{1}{2}\frac{x_j^2}{|x|^2} - \frac{1}{2N}\Big) + W(|x|) = \frac{1}{\alpha!}x^\alpha + \Big(W(|x|) - \frac{1}{2N}f(|x|)\Big) + o(|x|^2).$$

By Lemma A.1, we have $W(|x|) - \frac{1}{2N} f(|x|) = W(|x|) - \frac{1}{2N}|x|^2 = o(|x|^2)$, and the proof is complete. ■

Lemma A.4. *For every multiindex $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = 3$, there are functions $\widetilde{W}_\alpha$ satisfying*

$$\begin{aligned} -\Delta \widetilde{W}_\alpha - N(N+2)B^{\frac{4}{N-2}} \widetilde{W}_\alpha &= f_\alpha \quad \text{on } \mathbb{R}^N, \\ \widetilde{W}_\alpha(x) &= \frac{1}{\alpha!} x^\alpha + o(|x|^3) \quad \text{as } x \to 0, \end{aligned} \tag{A.8}$$

with

$$f_\alpha = \begin{cases} 0 & \text{if } x^\alpha = x_j x_k x_l \text{ for some } j \neq k \neq l \neq j, \\ -B x_l & \text{if } x^\alpha = x_j^2 x_l \text{ for some } j, l. \end{cases}$$

The proof is similar to the previous one. However, in the case $x^\alpha = x_j^2 x_l$, the needed decomposition is a bit more subtle, so let us give full details also here.

Proof of Lemma A.4. If $x^\alpha = x_j x_k x_l$ with $j \neq k \neq l \neq j$, we make the ansatz $\widetilde{W}_\alpha = f(|x|) Y_{jkl}(x/|x|)$, with $Y_{jkl}(\omega) = \omega_j \omega_k \omega_l$ for $\omega \in \mathbb{S}^{N-1}$. Observing that Y_{jkl} is a spherical harmonic of degree 3, W_α solves the equation in (A.6) if and only if f solves the ODE

$$-f''(r) - \frac{N-1}{r} f'(r) + \frac{2N}{r^2} f(r) + N(N+2) f(r) B^{\frac{4}{N-2}}(r) = 0 \quad \text{on } (0, \infty). \tag{A.9}$$

By the proof of [1, Proposition A.1], there is a solution to (A.7) which satisfies $f(r) \sim r^3$ for $r \in (0, \infty)$. Up to replacing f by a suitable scalar multiple, we may thus assume that $\lim_{r\to 0} f(r) r^{-3} = 1$. It follows that

$$\widetilde{W}_\alpha(x) = f(|x|) Y_{jkl}(x/|x|) = f(|x|)|x|^{-3} x_j x_k = (1 + o(1)) x_j x_k x_l = \frac{1}{\alpha!} x^\alpha + o(|x|^3).$$

If on the other hand $x^\alpha = x_j^2 x_l$, first assume that $j \neq l$. Observing that $x_j^2 x_l - x_i^2 x_l$ is a homogeneous harmonic polynomial of degree 3 for every $i \notin \{j, l\}$, so is

$$\sum_{i \notin \{j,l\}} (x_j^2 x_l - x_i^2 x_l) \pm x_j^2 x_l \pm x_l^3 = (N-1) x_j^2 x_l - |x|^2 x_l + x_l^3.$$

Similarly, $x_l^3 - 3x_j^2 x_l$ for every $j \neq l$ is a homogeneous harmonic polynomial of degree 3, hence so is

$$\sum_{j \neq l} x_l^3 - 3x_j^2 x_l \pm 3x_l^3 = (N+2) x_l^3 - 3|x|^2 x_l.$$

Subtracting appropriate scalar multiples of the found expressions from each other, we obtain that

$$\widetilde{Y}_{jl}(\omega) = \frac{1}{2} \omega_j^2 \omega_l - \frac{1}{2(N+2)} \omega_l$$

is a spherical harmonic of degree 3.

We now set

$$\widetilde{W}_\alpha(x) = f(|x|)\widetilde{Y}_{jl}(x/|x|) + \widetilde{W}(|x|)\frac{x_l}{|x|},$$

f is a solution to (A.9) with $\lim_{r\to 0} f(r)r^{-3} = 1$ and $\widetilde{W}$ is the function from Lemma A.2. Then $\widetilde{W}_\alpha$ satisfies the PDE in (A.8). Moreover,

$$\begin{aligned}\widetilde{W}_\alpha(x) &= f(|x|)\Big(\frac{1}{2}\frac{x_j^2 x_l}{|x|^3} - \frac{1}{2(N+2)}\frac{x_l}{|x|}\Big) + \widetilde{W}(|x|)\frac{x_l}{|x|} \\ &= \frac{f(|x|)}{|x|^3}\Big(\frac{1}{2x_j^2 x_l} - \frac{1}{2(N+2)}x_l|x|^2\Big) + \frac{\widetilde{W}(|x|)}{|x|^3}x_l|x|^2 \\ &= \frac{1}{2}x_j^2 x_l + o(|x|^3) = \frac{1}{\alpha!}x^\alpha + o(|x|^3).\end{aligned}$$

Here we crucially used that $\frac{\widetilde{W}(r)}{r^3} \to \frac{1}{2(N+2)}$ as $r \to 0$ by Lemma A.2.

Finally, if $j = l$, that is, if $x^\alpha = x_j^3$, we use instead of $\widetilde{Y}_{jl}$ the function

$$\widetilde{Y}_j(\omega) = \frac{1}{6}\omega_j^3 - \frac{1}{2(N+2)}w_\alpha.$$

The rest of the argument is identical. ■

Acknowledgments. The authors are grateful to Shuibo Huang for valuable comments on our work [1], which also applied to [2] and led to the redaction of this corrigendum.

Funding. Tobias König is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 555837013.

References

[1] T. König and P. Laurain, Multibubble blow-up analysis for the Brezis–Nirenberg problem in three dimensions. 2025, arXiv:2208.12337v4, to appear in *Amer. J. Math.*

[2] T. König and P. Laurain, Fine multibubble analysis in the higher-dimensional Brezis–Nirenberg problem. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **41** (2024), no. 5, 1239–1287
Zbl 1547.35366 MR 4782462

Received 23 December 2025; accepted 27 January 2026.

Tobias König
Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60325 Frankfurt am Main, Germany; koenig@mathematik.uni-frankfurt.de

Paul Laurain
Laboratoire d'Analyse et de Mathématiques Appliquées, Université Gustave Eiffel, Bâtiment Copernic, 5 boulevard Descartes, 77420 Champs-sur-Marne, France; paul.laurain@univ-eiffel.fr