Doc. Math. (Online first)
DOI 10.4171/DM/1060

Divided differences and multivariate holomorphic calculus

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Abstract. We review the multivariate holomorphic functional calculus for tuples in a *commutative* Banach algebra and establish a simple "naïve" extension to commuting tuples in a general Banach algebra. The approach is naïve in the sense that the naïvely defined joint spectrum maybe too big. The advantage of the approach is that the functional calculus then is given by a simple concrete formula from which all its continuity properties can easily be derived.

We apply this framework to multivariate functions arising as divided differences of a univariate function. This provides a rich set of examples to which our naïve calculus applies. Foremost, we offer a natural and straightforward proof of the Connes–Moscovici Rearrangement Lemma in the context of the multivariate holomorphic functional calculus. Secondly, we show that the Daletski–Krein type noncommutative Taylor expansion is a natural consequence of our calculus. Also Magnus' Theorem which gives a nonlinear differential equation for the log of the solutions to a linear matrix ODE follows naturally and easily from our calculus. Finally, we collect various combinatorial related formulas.

1. Introduction

This paper is a follow-up of [20], which together with [8, 21] serves as the primary motivation for the present work. [20] shows that some of the rather involved combinatorial formulas arising in the Spectral Geometry of noncommutative tori [8, 21] find their natural explanation in terms of the classical calculus of divided differences. Furthermore, a more conceptual proof of the Connes–Moscovici Rearrangement Lemma [8, Section 6.2] in the formalism of divided differences and a multivariate smooth functional calculus was given.

An important pattern are expressions of the form

$$F(a^{(0)},\dots,a^{(n)})(b_1\cdot\ldots\cdot b_n),$$

where F is a multivariate function applied to algebra elements $a^{(0)},\dots,a^{(n)}$; in (a topological tensor product) $\mathcal{A}^{\otimes n+1}$ of a Banach algebra *paired* with the algebra elements $b_1,\dots,b_n$. The pairing is most naturally defined if $\mathcal{A}^{\otimes n+1}$ is equipped with the projective tensor product topology (denoted by $\mathcal{A}_\pi^{\otimes n+1}$). The crucial property of the latter which is

Mathematics Subject Classification 2020: 47A60 (primary); 46L87, 58B34, 65D05 (secondary).
Keywords: divided difference, Banach algebra, Rearrangement Lemma, holomorphic functional calculus.

used extensively is that in this case the multiplication map

$$\mu_{n+1}: \mathcal{A}_\pi^{\otimes n+1} \to \mathcal{A}, \quad a_0 \otimes \cdots \otimes a_n \mapsto a_0 \cdot \ldots \cdot a_n \tag{1.1}$$

is continuous. In [26], Quigg characterizes the von Neumann algebras for which μ_1 is continuous relative to the spatial C^*-norm (see [26, Theorem 4.6]). In particular, Quigg proved that μ_1 is not continuous when $\mathcal{A}$ is the algebra of bounded linear operators on a separable infinite-dimensional Hilbert space (see [26, Lemma 4.4]).

Lacking a continuous functional calculus in the Banach algebra $\mathcal{A}_\pi^{\otimes n+1}$, in [20] we circumvented the issues by establishing a functional calculus, if $\mathcal{A}$ is C^*, for smooth multivariate functions; it is a $*$-homomorphism from the involutive Fréchet algebra $C^\infty(U^{n+1})$ to the involutive Banach algebra $\mathcal{A}_\pi^{\otimes n+1}$ [20, Theorem 3.2]. This approach exploited the fact that for an open subset $U \subset \mathbb{R}^n$ the algebra $C^\infty(U)$ with its natural Fréchet topology is nuclear and hence $C^\infty(U^{n+1})$ is naturally isomorphic to the projective as well as the injective tensor product of $n+1$ factors of $C^\infty(U)$.

While this approach worked, for several reasons it seems a bit unnatural. Firstly, it is a little odd that relatively deep (though now classical) results about nuclear spaces are needed. Secondly, it requires $\mathcal{A}$ to be a C^*-algebra, and the functional calculus is a "fake" version of the continuous functional calculus. On the other hand the projective tensor product $\mathcal{A}_\pi^{\otimes n+1}$ is defined for any Banach algebra and therefore using just the multivariate *holomorphic* functional calculus seems most natural. While holomorphicity is certainly narrower than smoothness, for the Rearrangement Lemma holomorphic functions (even rational ones) suffice and we have the benefit that the theory can be developed naturally in the context of Banach algebras.

This is the starting point of the current paper. We take it also as an excuse to review the quite intricate history of the multivariate holomorphic functional calculus which is by far not just an afterthought to the one variable case and which is normally not taught in standard functional analysis courses. While a uniform theory of a multivariate holomorphic functional calculus remains elusive, various approaches have been proposed [2,13,29,33]. In general, it is only well defined in a commutative Banach algebra. We show in Section 2.3, however, that there is a naïve version of the multivariate holomorphic functional calculus which has the benefit of being unambiguously defined for commuting tuples in a noncommutative Banach algebra and which is given by a simple concrete formula from which all its continuity properties can easily be derived (Theorem 4).

We also mention several more recent approaches to the multivariate holomorphic functional calculus. McIntosh and Pryde [23] studied commuting operators on a Banach space embedded in a Banach module over a Clifford algebra. Brian Jefferies explored related ideas in greater depth in his monograph *Spectral Properties of Noncommuting Operators* [19], focusing on n-tuples of noncommuting operators. Finally, the monograph by Colombo, Sabadini, and Struppa [7] proposes a novel approach to constructing a general functional calculus for not necessarily commuting n-tuples of operators, as well as a functional calculus for quaternionic operators. We mention these works here for completeness; a detailed comparison is beyond the scope of our relatively elementary presentation.

Divided differences of a holomorphic function naturally give rise to multivariate holomorphic functions. For such functions, our naïve multivariate functional calculus works seamlessly for commuting tuples in any Banach algebra. As applications we show a noncommutative Newton interpolation formula, noncommutative Taylor formulas in the tradition of Krein–Daletskii [9, 10], cf. also the "asymptotic analysis" versions of Paycha [25] (see also [17, Theorem 10]), and a short proof of Magnus' Theorem (Section 4.3). Finally, we will give a natural and straightforward proof of the Connes–Moscovici Rearrangement Lemma in the context of the multivariate holomorphic functional calculus.

The paper is organized as follows. Section 2 reviews the mulivariate holomorphic functional calculus for a commuting tuple in a commutative Banach algebra and our "naïve" extension for commuting tuples in a general Banach algebra. Section 3 studies divided differences of a holomorphic functions with the methods of the naïve holomorphic functional calculus framework. In Section 4 we present the various applications mentioned before. Finally, in Appendix A, we collect various combinatorial formulas about divided differences which are needed in the applications section.

2. Multivariate holomorphic functional calculus

2.1. Notation

We fix some notation which will be used frequently. $\mathbb{N}$ denotes the set of natural numbers including 0. Multiindices will usually be denoted by Greek letters. Sums of the form $\sum_\alpha$ will be over $\alpha \in \mathbb{N}^n$ with further restrictions indicated below the $\sum$ sign.

During this section script letters $\mathcal{A}, \mathcal{B}, \ldots$ denote *unital* Banach algebras. $\mathcal{A}_\pi^{\otimes n}$ denotes the projective tensor product completion of $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (n factors), cf., e.g., [14]. That is $\mathcal{A}_\pi^{\otimes n}$ is the completion of $\mathcal{A}^{\otimes n}$ with respect to the norm

$$\|x\|_\pi = \inf \sum_i \|a_1^{(i)}\| \cdot \ldots \cdot \|a_n^{(i)}\|,$$

where the infimum is taken over all representations of $x \in \mathcal{A}^{\otimes n}$ as a finite sum $\sum_i a_1^{(i)} \otimes \cdots \otimes a_n^{(i)}$. $\mathcal{A}_\pi^{\otimes n}$ is a Banach algebra.

Given a subset $U \subset \mathbb{C}^n$ we denote by $\mathcal{O}(U)$ the Fréchet algebra of holomorphic functions on U with the topology of uniform convergence on compact subsets.

2.2. Review of the holomorphic functional calculus

The holomorphic functional calculus in one variable can already be found in Gelfand's fundamental paper on normed rings [15, p. 20]: given an element a in a Banach algebra $\mathcal{A}$ there is a unique continuous homomorphism θ_a from the Fréchet algebra $\mathcal{O}(U)$ into $\mathcal{A}$ sending the function $\mathrm{id}_U\colon z \mapsto z$ to a.

The generalization to $f(a_1, \ldots, a_n)$ for several commuting Banach algebra elements and a holomorphic function f of n variables is surprisingly complex and from Gelfand's paper it took almost 30 years until the theory reached a mature state. More precisely, let $a = (a_1, \dots, a_n)$ be n commuting elements in $\mathcal{A}$ and let $U \subset \mathbb{C}^n$ be an open neighborhood of the *joint spectrum* $\operatorname{spec}_{\mathcal{A}}(a_1, \dots, a_n)$ (to be discussed below) one is aiming for a continuous *algebra homomorphism* $\theta\colon \mathcal{O}(U) \to \mathbb{C}$ sending the coordinate function z_j to a_j.

This aim is obstructed by various difficulties. Firstly, in a general Banach algebra there is no general accepted notion of joint spectrum for a commuting tuple $a = (a_1, \dots, a_n)$. One therefore has to be more modest and embed $a_1, \dots, a_n$ into a commutative Banach algebra where the joint spectrum can be defined using ideal theory; this will be explained below. For commuting operators on a Banach space there is a notion of joint spectrum which is independent of the choice of an ambient commutative Banach algebra [30] as well as a holomorphic functional calculus based on an algebraic approach to the Cauchy–Weil integral [29]. This celebrated approach, however, comes with the price of being dauntingly complex.

We will be more modest here and briefly review the calculus if $a_1, \dots, a_n$ are elements of a *commutative* Banach algebra $\mathcal{A}$. Historically, there exist basically two approaches. One using polynomial approximation and the Oka–Weil approximation theorem (see [1, 2, 18, 28, 31], for a modern textbook treatment see [3, Part III]), and the second one being closer to Gelfand's Cauchy integral by using a direct integral description of the functional calculus (see [4, 6, 13, 27]). Both approaches have in common, that the uniqueness statement is more involved.[1] We will use the latter approach, following mostly the excellent monograph [13, Chapter III].

So therefore from now on we assume that the Banach algebra $\mathcal{A}$ is commutative and unital. Let $M_{\mathcal{A}}$ be the maximal ideal space of $\mathcal{A}$, realized as the set of continuous unital algebra homomorphisms $\phi\colon \mathcal{A} \to \mathbb{C}$. This is a compact Hausdorff space equipped with the weak-*-topology as a subset of the unit ball in the dual $\mathcal{A}'$; $\phi \leftrightarrow \ker \phi$ is the identification between ϕ and the maximal ideal $\ker \phi \subset \mathcal{A}$. Given $a := (a_1, \dots, a_n) \in \mathcal{A}^n$, the *joint spectrum* of a is denoted by $\operatorname{spec}_{\mathcal{A}}(a) = \operatorname{spec}_{\mathcal{A}}(a_1, \dots, a_n)$ and defined as the set

$$\operatorname{spec}_{\mathcal{A}}(a) := \big\{ \big(\phi(a_1), \dots, \phi(a_n)\big) \in \mathbb{C}^n \mid \phi \in M_{\mathcal{A}} \big\}. \tag{2.1}$$

The joint spectrum is a nonempty compact subset of $\mathbb{C}^n$, and if $\mathcal{A}$ is finitely generated then $\operatorname{spec}_{\mathcal{A}}(a)$ is *polynomially convex*, cf. [13, Section III.1]. Moreover, $z = (z_1, \dots, z_n) \notin \operatorname{spec}_{\mathcal{A}}(a_1, \dots, a_n)$ is equivalent to the existence of $g_1, \dots, g_n \in \mathcal{A}$ such that

$$\sum_{j=1}^{n} (z_j - a_j) \cdot g_j = 1_{\mathcal{A}}. \tag{2.2}$$

[1] See also Zame [33] whose uniqueness statement is stronger than what we quote below. On the other hand even Zame's statement is not a plain extension of Gelfand's clear and straightforward one variable statement as he needs to *assume* a version of the spectral mapping theorem. The latter is a consequence in 1D. However, the statement of Zame is for one fixed tuple $a = (a_1, \dots, a_n)$, see [33, Theorem 3.5], while the uniqueness statement we quote talks about *all* tuples at once.

It is worth noting that the joint spectrum depends on the algebra, as demonstrated by the following example.

Example 1. Let $\Omega_r = \{z \in \mathbb{C} \mid 1 \le |z| \le r\}$, where $1 \le r$. For $r > 1$, let $\mathcal{A}_r$ be the algebra of continuous functions on Ω_r that are holomorphic in the interior of Ω_r; for $r = 1$ let $\mathcal{A}_1 = C(S^1)$ be the algebra of continuous functions on the circle of radius one. Using the maximum principle one finds that for $f \in \mathcal{A}_r$ the norm is given by

$$\|f\|_{\mathcal{A}_r} = \max\Bigl(\max_{|z|=1} \bigl|f(z)\bigr|, \max_{|z|=r} \bigl|f(z)\bigr|\Bigr). \tag{2.3}$$

Denote by $a_r \in \mathcal{A}_r$ the function $z \mapsto z$ viewed as an element of $\mathcal{A}_r$. Then one checks that $\operatorname{spec}_{\mathcal{A}_r}(a_r) = \Omega_r$, for all r. Moreover, if $r < r'$ then $\mathcal{A}_{r'} \subset \mathcal{A}_r$ is naturally a subalgebra and under this inclusion $a_{r'}$ is identified with a_r. Furthermore, $\operatorname{spec}_{\mathcal{A}_r}(a_r) \subsetneqq \operatorname{spec}_{\mathcal{A}_{r'}}(a_{r'})$.

When $r = 1$ then $\mathcal{A}_1$ is a C^*-algebra and therefore the spectrum is stable, i.e., for any C^*-algebra $\mathcal{B} \supset \mathcal{A}_1$, $\operatorname{spec}_{\mathcal{B}}(a_1) = \operatorname{spec}_{\mathcal{A}_1}(a_1)$.

On the other hand, consider the Banach subalgebra $\mathcal{C}_r \subset \mathcal{A}_r$ generated by a_r. That is any $f \in \mathcal{C}_r$ is, as a function on Ω_r, the uniform limit of a sequence of polynomials $p_n(a_r)$ in a_r. Hence, by the maximum principle, the polynomials $p_n(a_r)$ converge uniformly on the full ball $B_r := \bigl\{z \in \mathbb{C} \mid |z| \le r\bigr\}$ and hence for $f \in \mathcal{C}_r$ we have, cf. (2.3)

$$\|f\|_{\mathcal{C}_r} = \max_{|z|=r} \bigl|f(z)\bigr| = \max_{|z|\le r} \bigl|f(z)\bigr|, \tag{2.4}$$

and thus we have, again by the maximum principle, natural isometric inclusions

$$\mathcal{C}_r \hookrightarrow C(B_r), \quad \mathcal{C}_r \hookrightarrow \mathcal{A}_r \hookrightarrow C(\Omega_r) \tag{2.5}$$

and the continuous inclusion $\mathcal{C}_{r'} \subset \mathcal{C}_r$, $r < r'$.

The spectrum $\operatorname{spec}_{\mathcal{C}_r}(a_r)$ is equal (!) to $B_r \supsetneqq \operatorname{spec}_{\mathcal{A}_r}(a_r) = \Omega_r$.

The holomorphic functional calculus, as stated in [13, Theorem III.4.1], is summarized in the next theorem.

Theorem 2. *Let $\mathcal{A}$ be a unital* commutative *Banach algebra and let* $\operatorname{spec}_{\mathcal{A}}(a)$ *be the joint spectrum of $a := (a_1, \dots, a_n) \in \mathcal{A}^n$, for some $n \in \mathbb{N}^*$. For an open set $U \subset \mathbb{C}^n$ let $\mathcal{O}(U)$ be the Fréchet algebra of holomorphic functions on U equipped with the topology of compact (= locally uniform) convergence.*

Then there is a unique family *of continuous linear maps*[2]

$$\Theta_a := \Theta_a^U : \mathcal{O}(U) \to \mathcal{A}, \quad a \in \bigcup_{n=1}^{\infty} \mathcal{A}^n,\ U \supset \operatorname{spec}_{\mathcal{A}}(a),\ U \text{ open} \tag{2.6}$$

such that:

[2]The uniqueness statement here is a little involved: the family is parametrized by the $a \in \mathcal{A}^n$, $U \supset \operatorname{spec}_{\mathcal{A}}(a)$ open, and uniqueness is guaranteed only for the whole family. This was later improved by Zame [33, Theorem 3.5] whose uniqueness statement refers only to one tuple a. See, however, footnote [1].

(1) *For a polynomial,* $F \in \mathbb{C}[z_1, \ldots, z_n]$*, of the form*

$$F(z) = F(z_1, \ldots, z_n) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \cdot z_1^{\alpha_1} \cdots z_n^{\alpha_n} =: \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \cdot z^\alpha,$$

we have

$$\Theta_a(F) =: F(a) = F(a_1, \ldots, a_n) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \cdot a_1^{\alpha_1} \cdots a_n^{\alpha_n} =: \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \cdot a^\alpha.$$

(2) *For* $F \in \mathcal{O}(U)$*, if* $a_{n+1}, \ldots, a_m \in \mathcal{A}$ *and if* $\widetilde{F}$ *is the trivial extension of* F *to* $U \times \mathbb{C}^{m-n}$ *defined by* $\widetilde{F}(z_1, \ldots, z_m) = F(z_1, \ldots, z_n)$ *then, denoting* $\tilde{a} := (a_1, \ldots, a_m)$*,*

$$\Theta_{\tilde{a}}(\widetilde{F}) = \widetilde{F}(\tilde{a}) = F(a) = \Theta_a(F).$$

(3) *If* $\{F_k\}_{k \in \mathbb{N}} \subset \mathcal{O}(U)$ *is a sequence which converges locally uniformly to* $F \in \mathcal{O}(U)$ *then* $\Theta_a(F_k) \to \Theta_a(F)$*.*

Remark 3. (1) It then *follows* that $\Theta_\bullet$ is natural in all respects conceivable [13, after Theorem III.4.1], in particular, Θ_a is a homomorphism of algebras and the Gelfand transform $\widehat{F(a_1, \ldots, a_n)}$ of $F(a_1, \ldots, a_n)$ equals $F(\hat{a}_1, \ldots, \hat{a}_n)$, i.e., the Gelfand transforms of $a_1, \ldots, a_n$ inserted into F.

(2) We emphasize furthermore, that if $U \subset V$ and $F \in \mathcal{O}(V)$ then $\Theta_a^V(F) = \Theta_a^U(F_{|U})$. Also by construction it follows that if $\mathcal{A} \subset \mathcal{B}$ is contained in a larger commutative Banach algebra then for $a \in \mathcal{A}^n$ the value of $\Theta_a(F)$ will be the same regardless of whether it is taken w.r.t. to $\mathcal{A}$ or w.r.t. $\mathcal{B}$.

Idea of proof. The main technical step in the proof is to show that if $w \in C_c^\infty(U; \mathcal{A})$ is such that $w \equiv 1$ in a neighborhood of $\operatorname{spec}_{\mathcal{A}}(a)$ then there are smooth $\mathcal{A}$-valued functions $u_j \in C^\infty(\mathbb{C}^n; \mathcal{A})$, $j = 1, \ldots, n$, such that

$$\sum_{j=1}^n u_j(z)(z - a_j) = 1 - w(z).$$

It then follows that the differential form $du_1 \wedge dz_1 \wedge \cdots \wedge du_n \wedge dz_n$ is supported on U. For $F \in \mathcal{O}(U)$ one then puts

$$\Theta_a(F) := F(a) = F(a_1, \ldots, a_n) = n! \int_U F du_1 đz_1 \cdots du_n đz_n, \tag{2.7}$$

where $đz_j = \frac{1}{2\pi i} dz_j$. The integral is indeed independent of the choices (u, w) made.[3] Uniqueness then follows by invoking the Oka–Weil approximation theorem [13, Theorem III.5.1] and the Arens–Calderón lemma [13, Theorem III.5.2].

We add that the claims of Remark 3 (2) about the dependence on U resp. on the algebra $\mathcal{A}$ are an immediate consequence of the formula (2.7) and its independence on the choices made. ■

[3] It is worth nothing that for $\mathcal{A} = \mathbb{C}^n$, $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ equation (2.7) is the Cauchy–Weil formula for the holomorphic function F [32].

2.3. Naïve holomorphic functional calculus

We will need a holomorphic functional calculus for several commuting elements in a *noncommutative* Banach algebra $\mathcal{A}$. The most naïve approach that comes to mind would be the following: given commuting elements $a_1, \ldots, a_n \in \mathcal{A}$ of a not necessarily commutative Banach algebra $\mathcal{A}$. Consider the Banach algebra $\mathcal{B} \subset \mathcal{A}$ generated by $a_1, \ldots, a_n$ and use the functional calculus in $\mathcal{B}$. However, this has several disadvantages: firstly, as the Example 1 shows, the joint spectrum of $(a_1, \ldots, a_n)$ in $\mathcal{B}$ might be larger than the joint spectrum in $\mathcal{A}$. Secondly, the Banach subalgebra $\mathcal{B}$ depends on the tuple. Apart from that, the formula for the Θ_a in Theorem 2 above is not very explicit, not to speak of the difficulties of the Taylor approach to the functional calculus, see Section 2.2 above.

We propose therefore an elementary naïve approach very close to the one outlined before but with the advantage that firstly the joint spectrum does not become too large, secondly the calculus does not depend on the choice of a specific commutative subalgebra and thirdly, and maybe most importantly, that there is a concrete contour integral formula similarly to the one in the one variable case.

Therefore, let $\mathcal{A}$ be a unital Banach algebra (not necessarily commutative) and let $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$ be commuting elements. We put

$$\operatorname{spec}_{\mathrm{na}}(a) := \prod_{j=1}^{n} \operatorname{spec}_{\mathcal{A}}(a_j); \tag{2.8}$$

the subscript "na" indicates that this is a "naïve" definition of joint spectrum. The previous notion of joint spectrum leads to a potentially smaller subset of $\operatorname{spec}_{\mathrm{na}}(a)$.

However, with this notion of joint spectrum we have the following result.

Theorem 4 (Naïve holomorphic functional calculus). *Let $\mathcal{A}$ be a unital (noncommutative) Banach algebra and let $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$ be an n-tuple of commuting elements. Furthermore, let $U_j \supset \operatorname{spec}_{\mathcal{A}}(a_j)$, $1 \le j \le n$, be an open neighborhood and put $U = \prod_{j=1}^n U_j$.*

If $\mathcal{B} \subset \mathcal{A}$ is any commutative unital Banach algebra containing $a_1, \ldots, a_n$ and *their resolvents $(\lambda - a_j)^{-1}$, $\lambda \in \mathbb{C} \setminus \operatorname{spec}_{\mathcal{A}}(a_j)$, then in the commutative Banach algebra $\mathcal{B}$ we have*

$$\operatorname{spec}_{\mathcal{B}}(a) \subset \operatorname{spec}_{\mathrm{na}}(a).$$

Furthermore, the map $\Theta_a^{\mathcal{B}}$ of Theorem 2[4] *is given by*

$$\Theta_a^{\mathcal{B}}(f) = f(a) = f(a_1, \ldots, a_n) := \int_{\gamma_1} \cdots \int_{\gamma_n} f(z) \cdot \prod_{j=1}^{n} (z_j - a_j)^{-1} đz, \tag{2.9}$$

where $đz = đz_1 \cdots đz_n$, $đz_j = \frac{1}{2\pi i} dz_j$, and γ_j, $1 \le j \le n$, are integration cycles[5] *in $U_j \setminus \operatorname{spec}_{\mathcal{A}}(a_j)$ which encircle $\operatorname{spec}_{\mathcal{A}}(a_j)$ once, resp.*

[4] The superscript $\mathcal{B}$ in $\Theta_a^{\mathcal{B}}$ indicates that a priori the functional calculus of Theorem 2 depends on the ambient algebra $\mathcal{B}$.

[5] An integration cycle is a closed chain of rectifiable paths.

Additionally, denoting by $\mathcal{A}^n_U$ *the set of those* $a \in \mathcal{A}^n$ *with* $\operatorname{spec}_{\mathrm{na}}(a) \subset U$ *the map* $\mathcal{N}: \mathcal{A}^n_U \times \mathcal{O}(U) \to \mathcal{A}$ *defined by the right-hand side of* (2.9)

$$(a, f) \mapsto \mathcal{N}_a(f) := \int_{\gamma_1} \cdots \int_{\gamma_n} f(z) \cdot \prod_{j=1}^{n} (z_j - a_j)^{-1} đz = f(a), \tag{2.10}$$

is continuous, where $\mathcal{A}^n_U$ *is an open set of* $\mathcal{A}^n$ *and* $\mathcal{O}(U)$ *is the Fréchet algebra of holomorphic functions on* U *(as in Theorem* 2*).*

Remark 5. (1) One might ask, why not, in light of the discussion at the beginning of this Section 2.3, just take $\mathcal{B}$ to be the Banach algebra $\widetilde{\mathcal{B}}$ *generated* by $a_1, \ldots, a_n$ *and* their resolvents. The point we make here, however, is that the result will be the same in *any* commutative Banach algebra $\mathcal{B}$ with $\widetilde{\mathcal{B}} \subset \mathcal{B} \subset \mathcal{A}$.

(2) Our continuity statement (2.10) goes beyond the continuity statement of Theorem 2 which is for a fixed tuple a only. Also notice, that for $a, a' \in \mathcal{A}^n_U$ (a and a' might *not* commute) the corresponding algebras $\mathcal{B}$, $\mathcal{B}'$ might differ and hence the map $\mathcal{N}$ cannot easily be expressed in terms of the map $\Theta^{\bullet}_{\bullet}$ of Theorem 2. One should view the right-hand side of (2.10) as the *definition* of the map $\mathcal{N}$. Equation (2.9) then says that for given a and its corresponding algebra $\mathcal{B}$ one has $\Theta^{\mathcal{B}}_a(f) = \mathcal{N}_a(f)$.

(3) We emphasize that the algebra $\mathcal{A}$ is not necessarily commutative and that the map $\mathcal{N}$ does not depend on any choices. Furthermore, for fixed a it enjoys all the properties of the holomorphic functional calculus of Theorem 2 and it is given by a simple contour integral. The obvious disadvantage is that the naïve joint spectrum is not as small as possible and that the naïve calculus is restricted to functions being holomorphic in a neighborhood of the naïve joint spectrum.

From now on we will wherever unambiguously possible write the more suggestive $f(a)$ instead of $\mathcal{N}_a(f)$ or $\Theta^{\mathcal{B}}_a(f)$.

Proof. First fix $a \in \mathcal{A}^n$ such that $a = (a_1, \ldots, a_n)$ is an n-tuple of commuting elements.

(1) Then for $b \in \{a_1, \ldots, a_n\}$ and $\lambda \notin \operatorname{spec}_{\mathcal{A}}(b)$ one has by construction of the algebra $\mathcal{B}$ that $\lambda \notin \operatorname{spec}_{\mathcal{B}}(b)$. Let $f(z) := \lambda - z$ and $g(z) := (\lambda - z)^{-1}$. Then Theorem 2 (1) implies $\Theta^{\mathcal{B}}_b(f) = \lambda - b$ and since $f \cdot g = 1$ and since $\Theta^{\mathcal{B}}_b$ is a homomorphism of algebras, it also follows that $\Theta^{\mathcal{B}}_b(g) = (\lambda - b)^{-1}$.

(2) Next we show that the joint spectrum of a in $\mathcal{B}$, $\operatorname{spec}_{\mathcal{B}}(a)$, is contained in $\operatorname{spec}_{\mathrm{na}}(a)$. Namely, let $\lambda = (\lambda_1, \ldots, \lambda_n) \notin \operatorname{spec}_{\mathrm{na}}(a)$. Then for at least one $j \in \{1, \ldots, n\}$ we have $\lambda_j \notin \operatorname{spec}_{\mathcal{B}}(a_j)$. Then by (1) of this proof $g_j := (\lambda_j - a_j)^{-1} \in \mathcal{B}$ and hence $(\lambda_j - a_j) \cdot g_j = 1_{\mathcal{B}}$, showing that $\lambda \notin \operatorname{spec}_{\mathcal{B}}(a)$, cf. equation (2.2).

(3) Still for fixed a now choose U_j and the integration cycles γ_j as in the statement of the theorem. The right-hand side of (2.9) is a limit of Riemann sums of the form

$$\sum f\big(\gamma_1(t_{k_1}), \ldots, \gamma_n(t_{k_n})\big) \prod_{j=1}^{n} (2\pi i)^{-1} \cdot \big(\gamma_j(t_{k_j}) - a_j\big)^{-1} \big(\gamma_j(t_{k_j}) - \gamma_j(t_{k_j - 1})\big).$$

It follows from Theorem 2 and (1) of this proof that this Riemann sum is $\Theta_a^{\mathcal{B}}$ applied to the holomorphic (rational) function

$$z \mapsto \sum f\big(\gamma_1(t_{k_1}),\dots,\gamma_n(t_{k_n})\big) \prod_{j=1}^{n} (2\pi i)^{-1} \cdot \big(\gamma_j(t_{k_j}) - z_j\big)^{-1} \big(\gamma_j(t_{k_j}) - \gamma_j(t_{k_j - 1})\big).$$

In a neighborhood of $\operatorname{spec}_{\mathcal{B}}(a)$ the approximating Riemann sums therefore converge uniformly to the holomorphic function f and by the continuity statement of Theorem 2 the equality (2.9) follows.

(4) It remains to prove the continuity of the map $\mathcal{N}$. We first note that clearly the set $\mathcal{A}_U^n$ is an open subset of $\mathcal{A}^n$. So given $a \in \mathcal{A}_U^n$ choose U_j and γ_j as above. Furthermore, choose $V_j \supset \operatorname{spec}(a_j)$ open with $\overline{V_j}$ contained in the interior[6] of γ_j and put $V := \prod_j V_j$. Then $\mathcal{A}_V^n$ is an open neighborhood of a and for $b \in \mathcal{A}_V^n$ and $g, f \in \mathcal{O}(U)$ we have the estimates

$$\begin{aligned}
\big\| f(b) - g(a) \big\| &\le \big\| f(b) - f(a) \big\| + \big\| f(a) - g(a) \big\|, \\
\big\| f(a) - g(a) \big\| &\le \| f - g \|_{\infty,\gamma} \cdot \int_{\gamma_1 \times \dots \times \gamma_n} \Big\| \prod_j (z - a_j)^{-1} \Big\| \, |\text{đ} z|, \\
\big\| f(b) - f(a) \big\| &\le \| f \|_{\infty,\gamma} \cdot \int_{\gamma_1 \times \dots \times \gamma_n} \Big\| \prod_j (z - a_j)^{-1} - \prod_j (z - b_j)^{-1} \Big\| \, |\text{đ} z|,
\end{aligned}$$

where $\| \cdot \|_{\infty,\gamma}$ stands for the sup norm on the image of the curves $\gamma_1, \dots, \gamma_n$. These estimates imply the continuity of the map in equation (2.10). ∎

We record here the (expected) behavior of the naïve calculus on rational functions and on tensor products.

Proposition 6. *Under the hypothesis of Theorem 4, let γ_j, U_j be as above, $j = 1, \dots, n$. If $f_j \in \mathcal{O}(U_j)$, $1 \le j \le n$, then*

$$(f_1 \otimes \dots \otimes f_n)(a) = f_1(a_1) \cdot \ldots \cdot f_n(a_n).$$

In particular, if $p(z) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha z^\alpha$ is a polynomial (or an entire function) then

$$p(a) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \cdot a_1^{\alpha_1} \cdot \ldots \cdot a_n^{\alpha_n} = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \cdot a^\alpha.$$

Moreover, for the resolvent $R_z(\lambda) := \prod_{j=1}^n (z_j - \lambda_j)^{-1}$ we have $R_z(a) = \prod_{j=1}^n (z_j - a_j)^{-1}$.

Proof. This is immediate from equation (2.9). ∎

[6] Of course, "interior" is meant here in the sense of integration cycles, i.e., the region encircled by the γ_j.

3. Divided differences as multivariate holomorphic functional calculus

We now establish the connection between divided differences and multivariate holomorphic functional calculus. To facilitate this connection, a concise overview of divided differences, including pertinent results necessary for our discussion, is provided in Appendix A.

During this section $\mathcal{A}$ denotes a unital Banach algebra.

3.1. Divided differences of commuting elements

Let $U \subset \mathbb{C}$ be an open set and $f \in \mathcal{O}(U)$. Then the n-th divided difference

$$\mathrm{DD}^n f(z) := [z_0, \dots, z_n]f$$

is a holomorphic function of $z = (z_0, \dots, z_n) \in U^{n+1}$.

Now let $a = (a_0, \dots, a_n) \in \mathcal{A}^{n+1}$ be commuting elements and let $U \supset \bigcup_{j=0}^n \operatorname{spec} a_j$ be an open neighborhood of the union of the spectra of $a_0, \dots, a_n$. Furthermore, if $f \in \mathcal{O}(U)$ then the n-th divided difference $\mathrm{DD}^n f$ is holomorphic in U^{n+1} which is an open neighborhood of $\operatorname{spec}_{\mathrm{na}}(a)$. Thus

$$\mathrm{DD}^n f(a) = [a_0, \dots, a_n]f \in \mathcal{A}$$

is well defined by the naïve holomorphic functional calculus established in Section 2.3.

Choosing an integration cycle Γ in U which encircles $\bigcup_{j=1}^n \operatorname{spec}(a_j)$ exactly once, we have in the interior, $\operatorname{int}\Gamma$, of Γ the integral representation

$$f(\cdot) = \int_\Gamma f(\zeta)(\zeta - \cdot)^{-1} \, đ\zeta,$$

where the integral converges in the Fréchet space $\mathcal{O}(\operatorname{int}\Gamma)$. Thus by the continuity statement of Theorem 4

$$\mathrm{DD}^n f = \int_\Gamma f(\zeta) \cdot \mathrm{DD}^n\big((\zeta - \cdot)^{-1}\big) \, đ\zeta. \tag{3.1}$$

In view of the continuity properties of the naïve holomorphic functional calculus of Theorem 4 and equation (A.12) we therefore find

$$\mathrm{DD}^n f(a) = [a_0, \dots, a_n]f = \int_\Gamma f(\zeta) \cdot \prod_{j=0}^n (\zeta - a_j)^{-1} \, đ\zeta. \tag{3.2}$$

Similarly, the Genocchi–Hermite formula, cf. equation (A.7),

$$[a_0, \dots, a_n]f = \int_{\Delta_n} f^{(n)}\Big(\sum_j s_j a_j\Big) \, ds \tag{3.3}$$

holds for $a_0, \dots, a_n \in \mathcal{A}$ whenever f is holomorphic in a neighborhood of

$$\overline{\bigcup_{s \in \Delta_n} \operatorname{spec}\Big(\sum_j s_j a_j\Big)},$$

where Δ_n denotes the n-standard simplex. Also, if f is holomorphic in the ball $\{z \in \mathbb{C} \mid |z| < r\}$ and if $a_0, \ldots, a_n$ are commuting elements in $\mathcal{A}$ of norm less than r then by Proposition 3 and Proposition 6

$$[a_0, \ldots, a_n]f = \sum_{\alpha \in \mathbb{N}^{n+1}} \frac{f^{(n+|\alpha|)}(0)}{(|\alpha| + n)!} a_0^{\alpha_0} \cdot \ldots \cdot a_n^{\alpha_n}$$

with absolute convergence in norm.

3.2. Divided differences of a general tuple

We now consider arbitrary not necessarily commuting elements $a_0, a_1, \ldots \in \mathcal{A}$. Recall from Section 2.1 the projective tensor product $\mathcal{A}_\pi^{\otimes n}$.

The multiplication map

$$\mu_n : \mathcal{A}^{\otimes n} \to \mathcal{A}, \quad a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdot \ldots \cdot a_n \tag{3.4}$$

extends by continuity to a bounded linear map from the projective tensor product $\mathcal{A}_\pi^{\otimes n}$ into $\mathcal{A}$.

For $a \in \mathcal{A}_\pi^{\otimes n+1}$ and $b_1, \ldots, b_n \in \mathcal{A}$ we write

$$a(b_1 \cdot \ldots \cdot b_n) := \mu_{n+1}\big(a \cdot (b_1 \otimes \cdots \otimes b_n \otimes 1_{\mathcal{A}})\big) \in \mathcal{A}. \tag{3.5}$$

Note that

$$(a_0 \otimes \cdots \otimes a_n)(b_1 \cdot \ldots \cdot b_n) = a_0 \cdot b_1 \cdot a_1 \cdot \ldots \cdot b_n \cdot a_n. \tag{3.6}$$

Equation (3.5) defines a continuous bilinear map $\mathcal{A}_\pi^{\otimes n+1} \times \mathcal{A}_\pi^{\otimes n} \to \mathcal{A}$. For $a \in \mathcal{A}$ we introduce, cf. [20, Section 3.2]

$$\begin{aligned} a^{(j)} &:= 1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}} \otimes a \otimes 1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}}, \quad 0 \le j \le n \ (a \text{ is in the } j\text{-th slot}), \\ \nabla_a^{(j)} &:= a^{(j-1)} - a^{(j)}, \quad 1 \le j \le n. \end{aligned} \tag{3.7}$$

Note that $a^{(j)}, \nabla_a^{(j)}$ also depend on n, we suppress this from the notation. Note that we use a different sign convention as [20]. Our ∇_a here corresponds to $-\nabla_a$ in [20]. For future reference we note

$$\left.\begin{aligned} \nabla_a^{(j)} + \cdots + \nabla_a^{(n)} + a^{(n)} &= a^{(j-1)}, \\ a^{(0)} - \nabla_a^{(1)} - \cdots - \nabla_a^{(j)} &= a^{(j)}, \end{aligned}\right. \quad 1 \le j \le n. \tag{3.8}$$

Example 1. We give an application. If $n = 1$ then $\nabla_a := \nabla_a^{(1)}$ and the pairing with ∇_a gives the adjoint action of a. Namely,

$$\nabla_a(b) := (a \otimes 1_{\mathcal{A}} - 1_{\mathcal{A}} \otimes a)(b) = a \cdot b \cdot 1_{\mathcal{A}} - 1_{\mathcal{A}} \cdot b \cdot a = \mathrm{ad}_a(b), \tag{3.9}$$

and inductively

$$\nabla_a^n(b) = \mathrm{ad}_a^n(b). \tag{3.10}$$

Note that $\nabla_a^n(b)$ on the left stands for the pairing (3.5) between $(\nabla_a^{(1)})^n \in \mathcal{A}_\pi^{\otimes 2}$ and $b \in \mathcal{A}_\pi^{\otimes 1}$.

The pairing (3.5) together with (3.6) is of some use for giving proofs of combinatorial identities. E.g.

$$\begin{aligned} a^m b &= \big(a^{(0)}\big)^m(b) = \big(\nabla_a + a^{(1)}\big)^m(b) \\ &= \sum_{j=0}^{m} \binom{m}{j} \big(\nabla_a^j \big(a^{(1)}\big)^{m-j}\big)(b) = \sum_{j=0}^{m} \binom{m}{j} \cdot \operatorname{ad}_a^j(b) \cdot a^{m-j}, \end{aligned}$$

since ∇_a and $a^{(1)}$ commute.

This can be extended to the multinomial case. Namely, for arbitrary $n \in \mathbb{N}$

$$\begin{aligned} a^m \cdot b_1 \cdot \ldots \cdot b_n &= \big(\nabla_a^{(1)} + \cdots + \nabla_a^{(n)} + a^{(n)}\big)^m (b_1 \cdot \ldots \cdot b_n) \\ &= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le m}} \frac{m!}{\alpha!(m-|\alpha|)!} \operatorname{ad}_a^{\alpha_1}(b_1) \cdot \ldots \cdot \operatorname{ad}_a^{\alpha_n}(b_n) \cdot a^{m-|\alpha|}. \quad \blacksquare \end{aligned}$$

3.2.1. Divided differences of a general tuple. After these preparations we are now able to define the divided difference of a general tuple as an element of the projective tensor product.

For arbitrary $a_0, \ldots, a_n \in \mathcal{A}$ the elements $a_i^{(i)} \in \mathcal{A}_\pi^{\otimes n+1}$, $0 \le i \le n$, commute. By slight abuse of notation we put

$$\begin{aligned} [a_0, \ldots, a_n]_\pi f &:= \big[a_0^{(0)}, \ldots, a_n^{(n)}\big] f \in \mathcal{A}_\pi^{\otimes n+1} \\ &= \int_\Gamma f(\zeta) \cdot (\zeta - a_0)^{-1} \otimes \ldots \otimes (\zeta - a_n)^{-1} \, đ\zeta. \end{aligned} \tag{3.11}$$

Thus, for $a_0, \ldots, a_n \in \mathcal{A}, b_1, \ldots, b_n \in \mathcal{A}$ and appropriate $f \in \mathcal{O}(U)$ (see the beginning of this section) one obtains, in view of equation (3.2),

$$\begin{aligned} &[a_0, \ldots, a_n]_\pi f(b_1 \cdot \ldots \cdot b_n) \\ &\quad = \int_\Gamma f(\zeta) \cdot (\zeta - a_0)^{-1} \cdot b_1 \cdot (\zeta - a_1)^{-1} \cdot \ldots \cdot b_n \cdot (\zeta - a_n)^{-1} \, đ\zeta, \end{aligned} \tag{3.12}$$

where this integral is well defined by Theorem 4.

4. Applications

We now direct our focus towards demonstrating the practical implications of the constructions developed earlier by presenting a variety of applications. As in the previous section, upper case script letters $\mathcal{A}, \mathcal{B}, \ldots$ denote *unital* Banach algebras.

4.1. The noncommutative Newton interpolation formula

Proposition 1. *Let $\mathcal{A}$ be a unital Banach algebra and let $f \in \mathcal{O}(U)$ be a holomorphic function in the open set $U \subset \mathbb{C}$. Then for $a_0, \dots, a_n \in \mathcal{A}$ with* $\operatorname{spec}(a_j) \subset U$, $0 \le j \le n$, *we have*

$$f(a_n) = f(a_0) + \sum_{j=1}^{n} \big([a_0, \dots, a_j]_\pi f\big)\big((a_n - a_0) \cdot \ldots \cdot (a_n - a_{j-1})\big).$$

This is a noncommutative analogue of Newton's interpolation formula. We emphasize that the $a_0, \dots, a_n$ do not necessarily commute. Therefore, the order of the entries on the right matters.

For the proof we first establish a recursion formula for noncommutative divided differences.

Lemma 2. *Under the assumptions of Proposition 1 we have*

$$\begin{aligned}&[a_0, \dots, a_{n-1}, a_{n+1}]_\pi f(b_1 \cdot \ldots \cdot b_n) - [a_0, \dots, a_n]_\pi f(b_1 \cdot \ldots \cdot b_n)\\ &\quad = [a_0, \dots, a_n, a_{n+1}]_\pi f\big(b_1 \cdot \ldots \cdot b_n \cdot (a_{n+1} - a_n)\big).\end{aligned}$$

Proof. This is a consequence of the resolvent equation. With regard to (3.1), (3.11), (3.12), and (A.12) we note that

$$\begin{aligned}&[a_0, \dots, a_{n-1}, a_{n+1}]_\pi (\zeta - \cdot)^{-1}(b_1 \cdot \ldots \cdot b_n) - [a_0, \dots, a_n]_\pi (\zeta - \cdot)^{-1}(b_1 \cdot \ldots \cdot b_n)\\ &\quad = (\zeta - a_0)^{-1} \cdot b_1 \cdot \ldots \cdot (\zeta - a_{n-1})^{-1} \cdot b_n \cdot \big\{(\zeta - a_{n+1})^{-1} - (\zeta - a_n)^{-1}\big\}.\end{aligned}$$

By the resolvent equation the expression in curly braces equals

$$(\zeta - a_n)^{-1} \cdot (a_{n+1} - a_n) \cdot (\zeta - a_{n+1})^{-1}.$$

Integration over the contour in equation (3.12) then proves the claim. ■

Proof of Proposition 1. We proceed by induction on n. For $n = 0$ the claim is obvious. So assume it holds for n. To prove it for $n + 1$ we apply it to $a_0, \dots, a_{n-1}, a_{n+1}$ and find

$$\begin{aligned}f(a_{n+1}) &= f(a_0) + \sum_{j=1}^{n-1} \big([a_0, \dots, a_j]_\pi f\big)\big((a_{n+1} - a_0) \cdot \ldots \cdot (a_{n+1} - a_{j-1})\big)\\ &\quad + \big([a_0, \dots, a_{n-1}, a_{n+1}]_\pi f\big)\big((a_{n+1} - a_0) \cdot \ldots \cdot (a_{n+1} - a_{n-1})\big).\end{aligned}$$

By the previous lemma the last summand equals

$$\begin{aligned}&\big([a_0, \dots, a_n]_\pi f\big)\big((a_{n+1} - a_0) \cdot \ldots \cdot (a_{n+1} - a_{n-1})\big)\\ &\qquad + \big([a_0, \dots, a_{n+1}]_\pi f\big)\big((a_{n+1} - a_0) \cdot \ldots \cdot (a_{n+1} - a_n)\big)\end{aligned}$$

which are exactly the two missing summands to complete the claimed formula for $n + 1$. ■

4.2. Noncommutative Taylor formulas

Proposition 3. *Let $\mathcal{A}$ be a unital Banach algebra and let $f \in \mathcal{O}(U)$ be a holomorphic function in the open set $U \subset \mathbb{C}$. Denote by $F\colon \mathcal{A}_U \to \mathcal{A}$, $a \mapsto f(a)$ the map induced by f via the naïve holomorphic functional calculus Theorem 4. Then F is holomorphic. The Taylor formula for F about $a \in \mathcal{A}_U$ reads*

$$\begin{aligned} f(a+b) = &\sum_{j=0}^{N} \big[a^{(0)},\dots,a^{(j)}\big] f\big(b\cdot\dots\cdot b\big) \\ &+ \big[a^{(0)},\dots,a^{(N)},(a+b)^{(N+1)}\big] f\big(b\cdot\dots\cdot b\big). \end{aligned} \tag{4.1}$$

The remainder $R_N(a,b) := [a^{(0)},\dots,a^{(N)},(a+b)^{(N+1)}] f(b\cdot\dots\cdot b)$ converges to 0 as $N\to\infty$ for b small enough. Hence we have the series expansion

$$f(a+b) = \sum_{j=0}^{\infty} \big[a^{(0)},\dots,a^{(j)}\big] f\big(b\cdot\dots\cdot b\big) \tag{4.2}$$

in a neighborhood of $a \in \mathcal{A}_U$.

Therefore, the n-the derivative of F at $a \in \mathcal{A}_U$ is given by

$$D^n F(a)[b_1,\dots,b_n] = \sum_{\sigma\in S_n} \big[a^{(0)},\dots,a^{(n)}\big] f\big(b_{\sigma_1}\cdot\dots\cdot b_{\sigma_n}\big). \tag{4.3}$$

Here the sum is taken over the set S_n of all n-permutations.

Example 4. We note that this indeed contains the ordinary Taylor formula as a special case. Namely, if $\mathcal{A} = \mathbb{C}$, then $\mathcal{A}^{\otimes n} = \mathbb{C}$ and hence for $a,b\in\mathbb{C}$

$$\big[a^{(0)},\dots,a^{(n)}\big] f(b\cdot\dots\cdot b) = \frac{1}{n!} f^{(n)}(a)\cdot b^n.$$

Proof. Equation (4.1) is a consequence of the noncommutative Newton interpolation formula Proposition 1 applied to $a_0 = \dots = a_N = a$, $a_{N+1} = a+b$. To estimate the remainder $R_N(a,b)$ we write using equation (3.12) and a suitable contour Γ

$$R_N(a,b) = \int_\Gamma f(\zeta)\cdot\big((\zeta-a)^{-1}b\big)^{N+1}\cdot(\zeta-a-b)^{-1}\, đ\zeta,$$

thus for $\|b\|$ small enough

$$\big\| R_N(a,b)\big\| \le C_1\big(C_2\|b\|\big)^{N+1} \to 0,$$

if $C_2\|b\| < 1$. This proves equation (4.2).

To prove equation (4.3) we note that the n-th derivative on the left-hand side of (4.1) is given by

$$D^n F(a)[b_1,\dots,b_n] = \partial_{s_1}\cdot\dots\cdot\partial_{s_n}\big|_{s_1=\dots=s_n=0} f\Big(a+\sum_j s_j b_j\Big).$$

With regard to (4.1), using induction, this indeed equals the right-hand side of (4.3). ∎

In view of equations (3.8) and (A.15) we may rewrite the general term in (4.1) resp. (4.2) as follows

$$\begin{aligned}\big[a^{(0)},\dots,a^{(n)}\big]f &= \big[a^{(0)},a^{(0)}-\nabla_a^{(1)},\dots,a^{(0)}-\nabla_a^{(1)}-\dots-\nabla_a^{(n)}\big]_\pi f\\ &= \sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^{|\alpha|}f^{(n+|\alpha|)}(a^{(0)})}{\alpha?!}\nabla_a^\alpha,\end{aligned}\tag{4.4}$$

with $\nabla_a^\alpha := (\nabla_a^{(1)})^{\alpha_1}\cdot\ldots\cdot(\nabla_a^{(n)})^{\alpha_n}$, thus using equation (3.10)

$$\begin{aligned}&\big[a^{(0)},\dots,a^{(n)}\big]f\big(b_1\cdot\ldots\cdot b_n\big)\\ &\qquad= \sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^{|\alpha|}\cdot f^{(n+|\alpha|)}(a)\cdot \mathrm{ad}_a^{\alpha_1}(b_1)\cdot\ldots\cdot\mathrm{ad}_a^{\alpha_n}(b_n)}{\alpha?!}.\end{aligned}\tag{4.5}$$

For the notation $\alpha?!$ see (A.2). Analogously,

$$\begin{aligned}[a^{(0)},\dots,a^{(n)}]f &= [a^{(n)}+\nabla_a^{(1)}+\dots+\nabla_a^{(n)},a^{(n)}+\nabla_a^{(2)}+\dots+\nabla_a^{(n)},\dots,a^{(n)}]f\\ &= \sum_{\alpha\in\mathbb{N}^n}\frac{f^{(n+|\alpha|)}(a^{(n)})}{\alpha!?}\nabla_a^\alpha,\end{aligned}$$

thus

$$\begin{aligned}&[a^{(0)},\dots,a^{(n)}]f\big(b_1\cdot\ldots\cdot b_n\big)\\ &\qquad= \sum_{\alpha\in\mathbb{N}^n}\frac{\mathrm{ad}_a^{\alpha_1}(b_1)\cdot\ldots\cdot\mathrm{ad}_a^{\alpha_n}(b_n)\cdot f^{(n+|\alpha|)}(a)}{\alpha!\cdot(\alpha_1+1)\cdot(\alpha_1+\alpha_2+2)\cdot\ldots\cdot\big(|\alpha|+n\big)}.\end{aligned}\tag{4.6}$$

Summing up, for f as in Proposition 3 we obtain the following.

Proposition 5. *Under the conditions of Proposition 3, the noncommutative Taylor expansions are given by*

$$f(a+b) = \sum_{n=0}^{\infty}\sum_{\alpha\in\mathbb{N}^n}\frac{(-1)^{|\alpha|}\cdot f^{(n+|\alpha|)}(a)\cdot\mathrm{ad}_a^{\alpha_1}(b)\cdot\ldots\cdot\mathrm{ad}_a^{\alpha_n}(b)}{\alpha!\cdot(\alpha_n+1)\cdot(\alpha_n+\alpha_{n-1}+2)\cdot\ldots\cdot\big(|\alpha|+n\big)}\tag{4.7}$$

$$= \sum_{n=0}^{\infty}\sum_{\alpha\in\mathbb{N}^n}\frac{\mathrm{ad}_a^{\alpha_1}(b)\cdot\ldots\cdot\mathrm{ad}_a^{\alpha_n}(b)\cdot f^{(n+|\alpha|)}(a)}{\alpha!\cdot(\alpha_1+1)\cdot(\alpha_1+\alpha_2+2)\cdot\ldots\cdot\big(|\alpha|+n\big)}.\tag{4.8}$$

In the context of formal power series these formulas were proved with a slightly different method in [25, Section 1]. Note that in contrast to the abstract of loc. cit. $f^{(n+|\alpha|)}$ is on the right in equation (4.8). This is consistent, however, with [25, Theorem 1].

4.3. Exponential and Magnus' Theorem

4.3.1. The exponential. We apply our previous considerations to the exponential function. The power series expansion of the divided differences (Proposition 3 and Corollary 4) of the exponential function then read

$$
\begin{aligned}
[x_0, \ldots, x_n] \exp &= \sum_{\alpha \in \mathbb{N}^{n+1}} \frac{x^\alpha}{(|\alpha| + n)!}, \\
[a, a + x_1, \ldots, a + x_n] \exp &= e^a \cdot [0, x_1, \ldots, x_n] \exp \\
&= \sum_{\alpha \in \mathbb{N}^n} \frac{e^a}{(|\alpha| + n)!} x^\alpha, \\
[a, a + x_1, a + x_1 + x_2, \ldots, a + x_1 + \cdots + x_n] \exp &= \sum_{\alpha \in \mathbb{N}^n} \frac{e^a}{\alpha?!} x^\alpha.
\end{aligned}
$$

These expansions converge for all complex numbers $x_0, \ldots, x_n, a \in \mathbb{C}$.

For a unital Banach algebra $\mathcal{A}$ and $a, b_1, \ldots, b_n \in \mathcal{A}$, Proposition 3 specializes to the well-known Dyson expansion for the exponential function (cf. [12]). Namely, involving the Genocchi–Hermite formula (3.3), (A.7) and Proposition 3 we find

$$
\begin{aligned}
e^{a+b} &= e^a + \sum_{n=1}^{\infty} \big[a^{(0)}, \ldots, a^{(n)}\big] \exp(b \cdot \ldots \cdot b) \\
&= e^a + \sum_{n=1}^{\infty} \int_{\Delta_n} \exp\Big(\sum_{j=0}^{n} s_j a^{(j)}\Big)(b \cdot \ldots \cdot b) ds \\
&= e^a + \sum_{n=1}^{\infty} \int_{\Delta_n} e^{s_0 a} \otimes \cdots \otimes e^{s_n a}(b \cdot \ldots \cdot b) ds \\
&= e^a + \sum_{n=1}^{\infty} \int_{\Delta_n} e^{s_0 a} \cdot b \cdot e^{s_1 a} \cdot \ldots \cdot b \cdot e^{s_n a} ds,
\end{aligned}
$$

respectively, the finite Taylor formula (4.1) gives analogously,

$$
\begin{aligned}
e^{a+b} = e^a &+ \sum_{n=1}^{N} \int_{\Delta_n} e^{s_0 a} \cdot b \cdot e^{s_1 a} \cdot \ldots \cdot b \cdot e^{s_n a} ds \\
&+ \int_{\Delta_{N+1}} e^{s_0 a} \cdot b \cdot \ldots \cdot e^{s_N a} \cdot b \cdot e^{s_{N+1}(a+b)} ds.
\end{aligned}
$$

Furthermore, the noncommutative Taylor expansions, Proposition 5, give for the $\mathcal{A}$-valued divided difference (equations (4.5) and (4.6))

$$
\begin{aligned}
[a^{(0)}, \ldots, a^{(n)}] \exp\big(b_1 \cdot \ldots \cdot b_n\big) &= \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|} \cdot e^a \cdot \mathrm{ad}_a^{\alpha_1}(b_1) \cdot \ldots \cdot \mathrm{ad}_a^{\alpha_n}(b_n)}{\alpha?!} \\
&= \sum_{\alpha \in \mathbb{N}^n} \frac{\mathrm{ad}_a^{\alpha_1}(b_1) \cdot \ldots \cdot \mathrm{ad}_a^{\alpha_n}(b_n) \cdot e^a}{\alpha!?},
\end{aligned}
\tag{4.9}
$$

where the right-hand side is absolutely convergent. In the context of formal power series this formula was proved in [25, Theorem 2].

In sum, the noncommutative Newton and Taylor expansions (Proposition 1, equations (4.7) and (4.8)) now lead to the following Peano–Baker and Cambell–Baker–Hausdorff type formulas (cf. [25, Proposition 2]).

$$\begin{aligned} e^{a+b} &= e^a + \sum_{j=1}^{N} \int_{\Delta_j} e^{s_0 a} \cdot b \cdot \ldots \cdot b \cdot e^{s_j a} ds \\ &\quad + \int_{\Delta_{N+1}} e^{s_0 a} \cdot b \cdot \ldots \cdot b \cdot e^{s_N a} \cdot b \cdot e^{s_{N+1}(a+b)} ds \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|} \cdot e^a \cdot \operatorname{ad}_a^{\alpha}(b)}{\alpha!?} = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \frac{\operatorname{ad}_a^{\alpha}(b) \cdot e^a}{\alpha?!}, \end{aligned}$$

where we have used the abbreviation $\operatorname{ad}_a^{\alpha}(b) := \operatorname{ad}_a^{\alpha_1}(b) \cdot \ldots \cdot \operatorname{ad}_a^{\alpha_n}(b)$.

4.3.2. Magnus' Theorem. The calculus developed above leads to a simple proof of Magnus' Theorem [22]. Let $A\colon I \to \mathcal{A}$ be a smooth function from the interval $I \subset \mathbb{R}$ into $\mathcal{A}$; for convenience assume $0 \in I$ and let $Y\colon I \to \mathcal{A}$ be the solution of the initial value problem

$$Y'(t) = A(t) \cdot Y(t), \quad Y(0) = 1_{\mathcal{A}}. \tag{4.10}$$

Let $\Omega(t) := \log Y(t)$ which is defined at least for small t, $\Omega(0) = 0$.

From equation (4.3) we infer for a holomorphic function f that

$$\frac{d}{dt} f\big(Y(t)\big) = \big[Y(t)^{(0)}, Y(t)^{(1)}\big] f\big(Y'(t)\big),$$

hence (we omit the t argument for brevity)

$$\begin{aligned} \Omega' &= \big[Y^{(0)}, Y^{(1)}\big] \log(Y') = \frac{\log Y^{(0)} - \log Y^{(1)}}{Y^{(0)} - Y^{(1)}} \big(Y^{(1)} A\big) \\ &= \frac{\Omega^{(0)} - \Omega^{(1)}}{e^{\Omega^{(0)}} - e^{\Omega^{(1)}}} e^{\Omega^{(1)}} \big(A\big) = \frac{\operatorname{ad}_{\Omega}}{e^{\operatorname{ad}_{\Omega}} - \operatorname{Id}} (A). \end{aligned}$$

Thus $\Omega(t)$ is the solution of the nonlinear initial value problem

$$\Omega' = \frac{\operatorname{ad}_{\Omega}}{e^{\operatorname{ad}_{\Omega}} - \operatorname{Id}} (A) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_{\Omega}^n (A), \quad \Omega(0) = 0. \tag{4.11}$$

Recall from equation (3.9) that $\operatorname{ad}_{\Omega}(A) = \nabla_{\Omega}(A) = (\Omega^{(0)} - \Omega^{(1)})(A)$. Equation (4.11) is due to Magnus [22], for a more recent exposition see [5, Section 2].

4.4. The holomorphic Rearrangement Lemma

We present a version of Connes' Rearrangement Lemma [8, Lemma 6.2], [20] in the framework of the naïve multivariate holomorphic functional calculus.

Before stating the Theorem we introduce some notation. From [20] recall: for $a \in \mathcal{A}$ put $A := e^a$ and, cf. equations (3.7), (3.8), $\Delta_a^{(j)} := \exp(-\nabla_a^{(j)})$, $1 \le j \le n$. Then

$$A^{(j)} = A^{(0)} \Delta_a^{(1)} \cdot \ldots \cdot \Delta_a^{(j)}, \quad j \ge 1. \tag{4.12}$$

Furthermore, for some $0 < \delta < \pi$ denote by S_δ the strip $S_\delta := \{z \in \mathbb{C} \mid |\Im z| < \delta\}$ and by Λ_δ the sector $\Lambda_\delta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \delta\}$. Clearly, $\exp(S_\delta) \subset \Lambda_\delta$. Hence by the Spectral Mapping Theorem if $a \in \mathcal{A}$ with $\operatorname{spec}(a) \subset S_\delta$ we have $\operatorname{spec}(A = e^a) \subset \Lambda_\delta$ and $\operatorname{spec}(A^{-1}) \subset \Lambda_\delta$.

We add in passing that it follows from Gelfand Theory that for commuting elements x, y in a Banach algebra one has $\operatorname{spec}(x \cdot y) \subset \operatorname{spec}(x) \cdot \operatorname{spec}(y)$. This implies in particular that if $\operatorname{spec}(A) \subset \Lambda_\delta$ then

$$\operatorname{spec}\big(\Delta_a^{(1)} \cdot \ldots \cdot \Delta_a^{(j)}\big) \subset \Lambda_{2\delta}. \tag{4.13}$$

To see this note that $\Delta_a^{(1)} \cdot \ldots \cdot \Delta_a^{(j)} = A^{-1} \otimes 1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}} \otimes A \otimes \cdots$ which is a product of the two commuting elements $A^{-1} \otimes 1_{\mathcal{A}} \otimes \cdots$ and $1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}} \otimes A \otimes 1_{\mathcal{A}} \otimes \cdots$ both of which have spectrum contained in Λ_δ.

After these preparations we have the following.

Theorem 6 (Holomorphic Rearrangement Lemma). *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$, $A := e^a$ with $\operatorname{spec}(a) \subset S_\delta$ for some $0 < \delta < \pi/2$. Furthermore, let $f_0, \ldots, f_p$ be holomorphic functions defined in the sector $\Lambda_{2\delta}$ satisfying estimates*

$$\begin{aligned} |f_j(s)| &\le C \cdot |s|^{-\alpha_j}, \quad s \in \Lambda_{2\delta}, |s| \gg 0, \\ |f_j(s)| &\le C \cdot |s|^{-\beta_j}, \quad s \in \Lambda_{2\delta}, |s| \ll 1 \end{aligned} \tag{4.14}$$

with $\sum_j \alpha_j > 1$, $\sum_j \beta_j < 1$.

Then the functions

$$\begin{aligned} F(s_0, \ldots, s_p) &:= \int_0^\infty f_0(u \cdot s_0) \cdot \ldots \cdot f_p(u \cdot s_p)\, du, \\ G(\lambda_1, \ldots, \lambda_p) &:= \int_0^\infty f_0(u) \cdot f_1(u \cdot \lambda_1) \cdot \ldots \cdot f_p(u \cdot \lambda_p)\, du \end{aligned} \tag{4.15}$$

are holomorphic in $\Lambda_{2\delta}^{p+1}$ resp. $\Lambda_{2\delta}^{p}$. Moreover, for $s = (s_0, \ldots, s_p) \in \Lambda_\delta^{p+1}$ one has

$$F(s) = s_0^{-1} \cdot G(s_0^{-1} \cdot s_1, \ldots, s_0^{-1} \cdot s_p) \tag{4.16}$$

and for $b_1, \ldots, b_p \in \mathcal{A}$ the following holds

$$\begin{aligned} &\int_0^\infty f_0(u \cdot A) \cdot b_1 \cdot f_1(u \cdot A) \cdot \ldots \cdot b_p \cdot f_p(u \cdot A)\, du \\ &\quad = F(A^{(0)}, \ldots, A^{(p)})(b_1 \cdot \ldots \cdot b_p) \\ &\quad = A^{-1} \cdot G(\Delta_a^{(1)}, \Delta_a^{(1)} \cdot \Delta_a^{(2)}, \ldots, \Delta_a^{(1)} \cdot \ldots \cdot \Delta_a^{(p)})(b_1 \cdot \ldots \cdot b_p), \\ &\quad = A^{-1} \int_0^\infty f_0(u) \cdot f_1(u \cdot \Delta_a^{(1)}) \cdot f_2(u \cdot \Delta_a^{(1)} \Delta_a^{(2)}) \\ &\qquad \cdot \ldots \cdot f_p(u \cdot \Delta_a^{(1)} \cdot \ldots \cdot \Delta_a^{(p)})\, du (b_1 \cdot \ldots \cdot b_p). \end{aligned} \tag{4.17}$$

Remark 7. This Theorem should be compared to [20, Corollary 3.5]. The present result is not so much interesting because of its formulation (which is only very mildly more general than [8, Lemma 6.2]) but rather because of its almost trivial proof. The essence of the Rearrangement Lemma is the trivial substitution $\int_0^\infty \phi(r\cdot u)\,du = r^{-1}\cdot\int_0^\infty \phi(u)\,du$.

Proof. The theorem is more or less self-evident. The estimates (4.14) guarantee the existence of the integrals (4.15). For real $s_0 > 0$ equation (4.16) follows by changing variables $\tilde{u} = u\cdot s_0$ in the first integral in equation (4.15). For general s_0 it then follows by the uniqueness of analytic continuation. The domains of definition of the functions F and G are such that we may apply the naïve holomorphic functional calculus to the commuting elements $A^{(0)},\dots,A^{(p)}$ resp. $\Delta_a^{(1)}, \Delta_a^{(1)}\cdot\Delta_a^{(2)},\dots,\Delta_a^{(1)}\cdot\ldots\cdot\Delta_a^{(p)}$. Then formula (4.17) is now clear. ∎

A. Some combinatorial formulas

A.1. Notation

We fix some notation which will be used frequently. $\mathbb{N}$ denotes the set of natural numbers including 0. For a real number $x > -1$ we write $x! = \Gamma(x+1)$. If $x,\alpha\in\mathbb{R}^n$ then we write, whenever each term is defined, $\alpha! := \prod_j \alpha_j! = \prod_j \Gamma(\alpha_j+1)$, $x^\alpha := \prod_j x_j^{\alpha_j}$, $|\alpha| = \sum_j \alpha_j$. $|\cdot|$ should not be confused with the sum norm; the latter will play no role in this paper.

Furthermore, we introduce the abbreviations

$$\alpha!? := \alpha!\cdot(\alpha_1+1)\cdot(\alpha_1+\alpha_2+2)\cdot\ldots\cdot\big(|\alpha|+n\big) = \alpha!\cdot\prod_{j=1}^n\Big(j+\sum_{l=1}^j\alpha_l\Big), \tag{A.1}$$

$$\alpha?! := \alpha!\cdot(\alpha_n+1)\cdot(\alpha_n+\alpha_{n-1}+2)\cdot\ldots\cdot\big(|\alpha|+n\big) = \alpha!\cdot\prod_{j=1}^n\Big(j+\sum_{l=0}^{j-1}\alpha_{n-l}\Big). \tag{A.2}$$

Multiindices will usually be denoted by greek letters. Sums of the form $\sum_\alpha$ will be over $\alpha\in\mathbb{N}^n$ with further restrictions indicated below the $\sum$ sign. We use the multiindex notation for multinomial coefficients. That is

$$\binom{\alpha}{\beta} = \prod_j\binom{\alpha_j}{\beta_j} = \prod_j\frac{\alpha_j!}{\beta_j!(\alpha_j-\beta_j)!}, \tag{A.3}$$

with the usual restrictions on the parameters. In particular, we put $\binom{\alpha}{\beta}$ to 0 if $\alpha_j < \beta_j$ or $\beta_j < 0$ for at least one index j.

We denote by

$$\Delta_n = \big\{t\in\mathbb{R}^n \mid 0\le t_n\le\cdots\le t_1\le 1\big\} \tag{A.4}$$

the standard simplex. Sometimes, the functions

$$s_0 = (1-t_1),\quad s_1 = (t_1-t_2),\dots,s_n = t_n,$$
$$t_n = s_n,\quad t_{n-1} = s_{n-1}+s_n,\dots,t_1 = s_1+\cdots+s_n$$

on Δ_n will be more convenient. Note that $s_j \geq 0$ and $\sum_{j=0}^n s_j = 1$. Integration over Δ_n will always be with respect to the measure $dt := dt_1 \dots dt_n = ds_1 \dots ds_n =: ds$, which differs from the surface measure by a multiplicative constant.

A.2. Divided differences

We recall the defining formulas for the divided differences of a smooth resp. holomorphic function. Cf., e.g., [11, 24]. See also [20, Appendix A].

A.2.1. Let f be a holomorphic function on an open set $U \subset \mathbb{C}$ and consider $x_0, x_1, \dots, x_n$ a priori distinct points in U. Then one defines recursively the *divided differences*

$$\begin{aligned} [x_0]f &:= f(x_0), \\ [x_0, \dots, x_n]f &:= \frac{1}{x_0 - x_n}\big([x_0, \dots, x_{n-1}]f - [x_1, \dots, x_n]f\big). \end{aligned} \tag{A.5}$$

The first few divided differences are therefore

$$\begin{aligned} [x_0, x_1]f &= \frac{f(x_0)}{(x_0 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)}, \\ [x_0, x_1, x_2]f &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}, \end{aligned}$$

and by induction one shows the explicit formula

$$[x_0, \dots, x_n]f = \sum_{k=0}^n f(x_k) \cdot \prod_{j=0, j\neq k}^n (x_k - x_j)^{-1}, \tag{A.6}$$

resp. the Genocchi–Hermite integral formula [24, Section 1.6], [11, Section 9][7]

$$\begin{aligned} &[x_0, \dots, x_n]f \\ &= \int_{\sum_{j=0}^n s_j = 1, s_j > 0} f^{(n)}\Big(\sum_{j=0}^n s_j x_j\Big) ds_1 \dots ds_n \\ &= \int_{0 \leq t_n \leq \dots \leq t_1 \leq 1} f^{(n)}\big((1 - t_1)x_0 + \dots + (t_{n-1} - t_n)x_{n-1} + t_n x_n\big) dt_1 \dots dt_n, \end{aligned} \tag{A.7}$$

when the closed convex hull of $x_0, \dots, x_n$ is a subset of U. If γ is a closed curve in the domain of f encircling the points $x_0, \dots, x_n$ exactly once then by the Residue Theorem and equation (A.6) we have [24, Section 1.7]

$$[x_0, \dots, x_n]f = \frac{1}{2\pi i} \oint_\gamma f(\zeta) \cdot \prod_{j=0}^n (\zeta - x_j)^{-1} d\zeta. \tag{A.8}$$

[7] According to the historical remarks in [11, Section 9] the formula is due to Genocchi who communicated it to Hermite in a letter.

A.3. Basic computations

The following computations are completely elementary and put here for the convenience of the reader. We first calculate the integrals of the power functions t^α, s^α (multiindex notation!) over the standard simplex. Then we calculate the divided differences of the power function $x \mapsto x^N$, cf. [24].

Lemma 1. *Let* $\alpha = (\alpha_0, \dots, \alpha_n) =: (\alpha_0, \alpha') \in \mathbb{R}^{n+1}$. *Then*

$$\int_{\Delta_n} s^\alpha ds = \frac{\alpha!}{\big(|\alpha| + n\big)!}, \quad \textit{for } \alpha_0, \dots, \alpha_n > -1, \tag{A.9}$$

and

$$\int_{\Delta_n} t_1^{\alpha_1} \cdot \ldots \cdot t_n^{\alpha_n} dt = \frac{\big(|\alpha| + n + 1\big)\alpha!}{\alpha?!} = \frac{\alpha'!}{\alpha'?!}, \tag{A.10}$$

for $\alpha_n + 1 > 0$, $\alpha_n + \alpha_{n-1} + 2 > 0, \dots, |\alpha| + n + 1 > 0$.

Proof. (1) For $n = 1$ this is the relation between Euler's beta function and the Gamma function. Namely,

$$\int_{\Delta_1} (1-t)^{\alpha_0} t^{\alpha_1} dt = \int_0^1 (1-t)^{\alpha_0} t^{\alpha_1} dt = \frac{\Gamma(\alpha_0 + 1)\Gamma(\alpha_1 + 1)}{\Gamma\big(|\alpha| + 2\big)} = \frac{\alpha!}{\big(|\alpha| + 1\big)!}.$$

Proceeding by induction we consider $n + 1$ and write the variables in the following form $s =: (s', s_{n+1}) =: (s', t)$, $\alpha = (\alpha', \alpha_{n+1})$. Then changing integration variables $s' = (1 - s_{n+1})\tilde{s}$ yields

$$\begin{aligned}
\int_{\Delta_{n+1}} s^\alpha ds &= \int_0^1 (1-t)^{|\alpha'|+n} \cdot t^{\alpha_{n+1}}\, dt \cdot \int_{\Delta_n} (\tilde{s})^{\alpha'}\, d\tilde{s} \\
&= \frac{\alpha'!}{\big(|\alpha'| + n\big)!} \cdot \int_0^1 (1-t)^{|\alpha'|+n} \cdot t^{\alpha_{n+1}}\, dt \\
&= \frac{\big(|\alpha'| + n\big)! \cdot \alpha_{n+1}! \cdot \alpha'!}{(|\alpha| + n + 1)! \cdot \big(|\alpha'| + n\big)!} = \frac{\alpha!}{\big(|\alpha| + n + 1\big)!}.
\end{aligned}$$

(2) By direct calculation, we obtain that

$$\begin{aligned}
\int_{\Delta_n} t_1^{\alpha_1} \cdot \ldots \cdot t_n^{\alpha_n}\, dt &= \int_{\Delta_{n-1}} \int_0^{t_{n-1}} t_n^{\alpha_n}\, dt_n\, t_1^{\alpha_1} \cdot \ldots \cdot t_{n-1}^{\alpha_{n-1}}\, dt' \\
&= \frac{1}{\alpha_n + 1} \int_{\Delta_{n-1}} t_1^{\alpha_1} \cdot \ldots \cdot t_{n-2}^{\alpha_{n-2}} t_{n-1}^{\alpha_n + \alpha_{n-1} + 1}\, dt' \\
&= \frac{1}{(\alpha_n + 1) \cdot (\alpha_n + \alpha_{n-1} + 2) \cdot \ldots \cdot (\alpha_1 + \cdots + \alpha_n + n)},
\end{aligned}$$

and the claim follows. ∎

Proposition 2. *Let $N \in \mathbb{Z}$ be an integer. Then for $z_0, \dots, z_n \in \mathbb{C} \setminus \{0\}$*

$$[z_0, \dots, z_n]\mathrm{id}^N = \begin{cases} \sum_{|\alpha|=N-n} z^\alpha, & N \geq n, \\ 0, & 0 \leq N < n, \\ \frac{(-1)^n}{z_0 \cdot \ldots \cdot z_n} \sum_{|\alpha|=|N|-1} z^{-\alpha}, & N < 0. \end{cases} \tag{A.11}$$

In the sums the letter α stands for multiindices in $\mathbb{N}^{n+1}$.

The case $N < 0$ can alternatively be written as $(-1)^n \sum_{\alpha_j \leq -1, |\alpha|=N-n} z^\alpha$. Furthermore,

$$[z_0, \dots, z_n](\lambda - \mathrm{id})^{-1} = \prod_{j=0}^{n} (\lambda - z_j)^{-1}. \tag{A.12}$$

Proof. We use the contour integral formula (A.8). Let R be large enough such that

$$|z_0|, \dots, |z_n| < R$$

and consider the integral

$$\begin{aligned} I(R) &:= \oint_{|\zeta|=R} \zeta^N \cdot \prod_{j=0}^{n} (\zeta - z_j)^{-1} \, đ\zeta, \quad đ\zeta := \frac{1}{2\pi i} d\zeta \\ &= \mathrm{Res}_{\zeta=0}\, \zeta^N \cdot \prod_{j=0}^{n} (\zeta - z_j)^{-1} + [z_0, \dots, z_n]\mathrm{id}^N. \end{aligned}$$

If $N \geq 0$ the residue vanishes. Moreover, since $I(R)$ is independent of $R \to \infty$ we find $I(R) = 0$ for $N < n$ since then $I(R) \to 0$ as $R \to \infty$. For $N \geq n$, expanding the integrand yields

$$I(R) = \sum_{\alpha \in \mathbb{N}^{n+1}} \oint_{|\zeta|=R} \zeta^{N-n-|\alpha|-1} \, đ\zeta \cdot z^\alpha = \sum_{|\alpha|=N-n} z^\alpha.$$

If $N < 0$ then since $I(R) = 0$

$$\begin{aligned} [z_0, \dots, z_n]\mathrm{id}^N &= -\mathrm{Res}_{\zeta=0} (-1)^{n+1} \sum_{\alpha \geq 0} \zeta^{|\alpha|+N} \cdot z^{-\alpha} \cdot (z_0 \cdot \ldots \cdot z_n)^{-1} \\ &= \frac{(-1)^n}{z_0 \cdot \ldots \cdot z_n} \sum_{|\alpha|=|N|-1} z^{-\alpha}. \end{aligned}$$

From the formulas (A.5) or (A.8) for the divided differences one immediately infers that

$$[z_0, \dots, z_n] f(\lambda - \cdot) = (-1)^n [\lambda - z_0, \dots, \lambda - z_n] f.$$

We apply this to the function $x \mapsto x^{-1}$; together with equation (A.11) we then obtain equation (A.12). ∎

A.4. Power series expansion of divided differences

Proposition 3. *Let f be a smooth function in a neighborhood of $0 \in \mathbb{R}$. Then the Taylor series of* $\mathrm{DD}^n f(x_0, \dots, x_n) := [x_0, \dots, x_n]f$ *about* 0 *is given by*

$$\sum_{\alpha \in \mathbb{N}^{n+1}} \frac{f^{(n+|\alpha|)}(0)}{(|\alpha| + n)!} x^\alpha. \tag{A.13}$$

The radius of convergence of the series (A.13) *for $DD^n f$ is at least as large as the radius of convergence of the Taylor series of f. If f is holomorphic then so is* $\mathrm{DD}^n f$.

Proof. We apply Lemma 1 and the Genocchi–Hermite integral formula (A.7). Then

$$\begin{aligned}
[x_0, \dots, x_n]f &= \int_{\Delta_n} f^{(n)}\Big(\sum_j s_j x_j\Big)\, ds \\
&\sim \sum_{l=0}^{\infty} \frac{f^{(n+l)}(0)}{l!} \int_{\Delta_n} \Big(\sum_j s_j x_j\Big)^l ds \\
&= \sum_{\alpha \in \mathbb{N}^{n+1}} \frac{f^{(n+|\alpha|)}(0)}{|\alpha|!} \frac{|\alpha|!}{\alpha!} \int_{\Delta_n} s^\alpha ds \cdot x^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{n+1}} \frac{f^{(n+|\alpha|)}(0)}{(|\alpha| + n)!} x^\alpha.
\end{aligned}$$

Here the Multinomial Theorem was used. The remaining claims are straightforward to check. ∎

Corollary 4. *Let f be a smooth function in a neighborhood of $a \in \mathbb{R}$. Then the Taylor series of $F_1(x_1, \dots, x_n) := [a, a + x_1, \dots, a + x_n]f$ about* 0 *is given by*

$$\sum_{\alpha \in \mathbb{N}^n} \frac{f^{(n+|\alpha|)}(a)}{(|\alpha| + n)!} x^\alpha. \tag{A.14}$$

The Taylor series of $F_2(x) := [a, a + x_1, a + x_1 + x_2, \dots, a + |x|]f$ about 0 *is given by*

$$\sum_{\alpha \in \mathbb{N}^n} \frac{f^{(n+|\alpha|)}(a)}{\alpha?!} x^\alpha. \tag{A.15}$$

The radii of convergence of the series (A.14), (A.15) *are as least as large as the radius of convergence of the Taylor series of f. If f is holomorphic then so are F_1 and F_2.*

Proof. The first claim follows from Proposition 2 since

$$[a, a + x_1, \dots, a + x_n]f = [0, x_1, \dots, x_n]f(a + \cdot) = \sum_{\alpha \in \mathbb{N}^n} \frac{f^{(n+|\alpha|)}(a)}{(|\alpha| + n)!} x^\alpha.$$

For the proof of equation (A.15) we proceed as in the proof of Proposition 3 but use the second equation (A.10) in Lemma 1:

$$
\begin{aligned}
&[a, a+x_1, a+x_1+x_2, \ldots, a+x_1+\cdots+x_n]f \\
&\quad = \int_{\Delta_n} f^{(n)}(a+(s_1+\cdots+s_n)x_1+(s_2+\cdots+s_n)x_2+\cdots+s_n x_n)ds \\
&\quad = \int_{0\le t_n\le\cdots\le t_1\le 1} f^{(n)}\Big(a+\sum_j t_j x_j\Big)dt \sim \sum_{l=0}^{\infty} \frac{f^{(n+l)}(a)}{l!}\int_{\dots}\Big(\sum_j t_j x_j\Big)^l dt \\
&\quad = \sum_{\alpha\in\mathbb{N}^n} \frac{f^{(n+|\alpha|)}(a)}{\alpha!(\alpha_n+1)\cdot\ldots\cdot\big(|\alpha|+n\big)} x^\alpha = \sum_{\alpha\in\mathbb{N}^n} \frac{f^{(n+|\alpha|)}(a)}{\alpha?!} x^\alpha. \qquad \blacksquare
\end{aligned}
$$

A.4.1. A combinatorial application. We give a combinatorial application of equations (A.13), (A.14), and (A.11).

Apply equation (A.11) to $[a, a+z_1, \ldots, a+z_n]$. Then for $N \ge n$

$$
\begin{aligned}
[a, a+z_1, \ldots, a+z_n]\mathrm{id}^N &= \sum_{|\alpha|\le N-n} a^{N-n-|\alpha|} \prod_{j=1}^{n} (a+z_j)^{\alpha_j} \\
&= \sum_{0\le\beta\le\alpha,\, |\alpha|\le N-n} a^{N-n-|\beta|} z^\beta \binom{\alpha}{\beta} \\
&= \sum_{|\beta|\le N-n} a^{N-n-|\beta|} z^\beta \Big(\sum_{\alpha\ge\beta,\, |\alpha|\le N-n} \binom{\alpha}{\beta}\Big).
\end{aligned}
$$

The summation is taken over multiindices $\alpha, \beta \in \mathbb{N}^n$ with the given restrictions.

On the other hand by Corollary 4

$$
\begin{aligned}
[a, a+z_1, \ldots, a+z_n]\mathrm{id}^N &= \sum_{\beta\in\mathbb{N}^n} \frac{\partial_a^{n+|\beta|} a^N}{(n+|\beta|)!} z^\beta \\
&= \sum_{|\beta|\le N-n} a^{N-n-|\beta|} z^\beta \binom{N}{n+|\beta|}.
\end{aligned}
$$

Equating coefficients we obtain the following multinomial identities: Given $\beta \in \mathbb{N}^n$ then

$$
\sum_{0\le\beta\le\alpha, |\alpha|\le m} \binom{\alpha}{\beta} = \binom{m+n}{|\beta|+n}, \tag{A.16}
$$

$$
\sum_{0\le\beta\le\alpha, |\alpha|=m} \binom{\alpha}{\beta} = \binom{m+n-1}{|\beta|+n-1}. \tag{A.17}
$$

Note that the second equation is an immediate consequence of the first one since $\binom{m+n}{|\beta|+n} - \binom{m-1+n}{|\beta|+n} = \binom{m+n-1}{|\beta|+n-1}$.

Equation (A.17) should be compared to [16, Table 169]. In particular, for $n = 2$ it reduces to [16, Equation (5.26)].

Acknowledgments. This project was begun after [20] was finished and has long been stalled. Over the years we have benefited from numerous discussions with our mathematical friends. It is impossible to mention all of them. Explicitly, we would like to mention our long term collaborator Boris Vertman whose mathematical enthusiasm is particularly inspiring in times of frustration. The second named author would also like to thank Elmar Schrohe, Hermann Schulz-Baldes, and Achim Klenke for invitations to give talks about the project. We would like to thank the anonymous referee for valuable suggestions; in particular, we owe them the important reference [26].

Funding. This work was partially supported by FAPESP (2022/16455-6), PROBAL CAPES/DAAD (8881.700909/2022-01) and we gratefully acknowledge the financial support of the Hausdorff Center for Mathematics, Bonn.

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Communicated by Christian Bär

Received 8 November 2024; revised 22 August 2025.

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