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The Kähler Ricci flow on Fano manifolds (I)

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Abstract. We study the evolution of pluri-anticanonical line bundles $K_M^{-\nu}$ along the Kähler Ricci flow on a Fano manifold M. Under some special conditions, we show that the convergence of this flow is determined by the properties of the pluri-anticanonical divisors of M. For example, the Kähler Ricci flow on M converges when M is a Fano surface satisfying $c_1^2(M) = 1$ or $c_1^2(M) = 3$. Combined with the work of [CW1] and [CW2], this gives a Ricci flow proof of the Calabi conjecture on Fano surfaces with reductive automorphism groups. The original proof of this conjecture is due to Gang Tian [Tian90].

Contents

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1. Introduction

In this paper, we introduce new criteria for the convergence of the Kähler Ricci flow on general Fano manifolds. Moreover, we verify these criteria for the Kähler Ricci flow on the Fano surfaces $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$. Consequently, we give a proof of the convergence of the Kähler Ricci flow on such Fano surfaces. The existence of KE (Kähler Einstein) metrics on these Fano surfaces follows as a corollary.

In Kähler geometry, a dominating problem is to prove the celebrated Calabi conjecture ([Ca]). It states that if $c_1(M)$ (the first Chern class of a Kähler manifold M) is positive, null or negative, then M admits a KE metric. In 1976, the null case Calabi conjecture was proved by S. T. Yau. Around the same time, the negative case was proved independently by T. Aubin and S. T. Yau. However, the case of positive first Chern class is much more complicated. In [Ma], Matsushima showed that the reductivity of Aut(M) is a necessary condition for the existence of KE metric. In [Fu], A. Futaki introduced an algebraic invariant which vanishes if M admits a KE metric. Around 1988, G. Tian [Tian90] proved the Calabi conjecture for Fano surfaces with reductive automorphism groups. Prior to Tian's work, there is a series of important works [Tian87], [TY], [Siu] where existence of KE metrics on some special complex surfaces were derived.

Let (M, J) be a Fano manifold, $\{\omega_t, 0 \leq t < \infty\}$ be a one-parameter family of Kähler metric forms in $2\pi c_1(M)$, and $g(t) = g_t$ be the metrics compatible with ω_t and J. Suppose $g(t)$ evolves along the Kähler Ricci flow. Let K_M^{-1} be the anticanonical line bundle, and $h_t = \omega_t^n$ be the evolving metrics of K_M^{-1}. Let ν be a large integer such that $K_M^{-\nu}$ is very ample. In this paper, we study the evolution of the line bundles $(K_M^{-\nu}, h_t^\nu)$. Let $N_\nu = \dim H^0(K_M^{-\nu}) - 1$, and $\{S_{\nu,\beta}^t\}_{\beta=0}^{N_\nu}$ be orthonormal holomorphic sections in $H^0(K_M^{-\nu})$ with respect to the metrics ω_t and h_t^ν, i.e.,

$$\int_M \langle S_{\nu,\alpha}^t, S_{\nu,\beta}^t \rangle_{h_t^\nu} \omega_t^n = \delta_{\alpha\beta}, \qquad 0 \leq \alpha, \beta \leq N_\nu.$$

It is easy to observe that

$$F_\nu(x,t) = \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} |S_{\nu,\beta}^t|_{h_t^\nu}^2(x)$$

is a well defined function on $M \times [0, \infty)$ (independent of the choice of orthonormal basis). The Kähler Ricci flow $\{(M, \omega_t), 0 \leq t < \infty\}$ is said to be *tamed by* ν if $K_M^{-\nu}$ is very ample and $F_\nu(x,t)$ is a uniformly bounded function on $M \times [0, \infty)$. The flow is a *tamed Kähler Ricci flow* if it is tamed by some integer ν.

For a tamed flow, we show that one can give criteria for the convergence of the flow by using some invariants defined in complex algebraic geometry.

Definition 1. Suppose L is a line bundle over M with Hermitian metric h, S is a holomorphic section of L, $x \in M$. Define

$$\alpha_x(S) = \sup\{\alpha \mid \|S\|_h^{-2\alpha} \text{ is locally integrable around } x\}.$$

It is called the *local α-invariant* of the section S at point x.

See [Tian90] and [Tian91] for more details about this definition. Note that $\alpha_x(S)$ is also called the singularity exponent ([DK]), log canonical threshold ([Chl]), etc. It is determined only by the singularity type of $Z(S)$. Therefore, if $S \in H^0(K_M^{-\nu})$, then $\alpha_x(S)$ can only achieve finitely many possible values.

Definition 2. Let G be a compact subgroup of $\mathrm{Aut}(M)$, $\omega_0 \in 2\pi c_1(M)$ be a G-invariant metric, and $\mathscr{P}_{G,\nu,k}(M, \omega_0)$ be the collection of all G-invariant functions of the form $\nu^{-1} \log(\sum_{\beta=0}^{k-1} \|\tilde{S}_{\nu,\beta}\|^2_{h_0^\nu})$, where $\{\tilde{S}_{\nu,\beta}\}_{\beta=0}^{k-1} \subset H^0(K_M^{-\nu})$ $(1 \leq k \leq N_\nu + 1)$ satisfies

$$\int_M \langle \tilde{S}_{\nu,\alpha}, \tilde{S}_{\nu,\beta} \rangle_{h^\nu} \omega_0^n = \delta_{\alpha\beta}, \qquad 0 \leq \alpha, \beta \leq k-1 \leq N_\nu; \qquad h_0 = \omega_0^n.$$

The number $\sup\{\alpha \mid \sup_{\varphi \in \mathscr{P}_{G,\nu,k}} \int_M e^{-\alpha\varphi}\omega_0^n < \infty\}$ depends only on the class $[\omega_0] = 2\pi c_1(M)$. We denote this number by $\alpha_{G,\nu,k}(M)$. If $G = \{\mathrm{id}\}$, we denote $\alpha_{\{\mathrm{id}\},\nu,k}(M)$ by $\alpha_{\nu,k}(M)$.

It turns out that the values of $\alpha_{\nu,1}$ and $\alpha_{\nu,2}$ play important roles in the convergence of the tamed Kähler Ricci flow. Denote $\omega_t = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi(t)$ where $\varphi(t)$ is the Kähler potential. We have the following theorems.

Theorem 1. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν. If $\alpha_{\nu,1} > n/(n+1)$, then φ is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.*

Theorem 2. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν. If*

$$\alpha_{\nu,2} > \frac{n}{n+1} \quad \text{and} \quad \alpha_{\nu,1} > \frac{1}{2 - \frac{n-1}{(n+1)\alpha_{\nu,2}}},$$

then φ is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.

In fact, if a Kähler Ricci flow is tamed by some large integer ν, an easy argument (cf. Section 2.3) shows that the following strong partial C^0-estimate holds:

$$\left| \varphi(t) - \sup_M \varphi(t) - \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} |\lambda_\beta(t) \tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} \right| < C. \tag{1}$$

Here $\{\lambda_k(t)\}_{k=0}^{N_\nu}$ are $N_\nu + 1$ positive functions of time which satisfy

$$0 < \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_{N_\nu}(t) = 1.$$

$\{\tilde{S}^t_{\nu,\beta}\}_{\beta=0}^{N_\nu}$ is an orthonormal basis of $H^0(K_M^{-\nu})$ under the fixed metric ω_0. Intuitively, inequality (1) means that we can control $\mathrm{Osc}_M\, \varphi(t)$ by $\nu^{-1} \log \sum_{\beta=0}^{N_\nu} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}$ which only blows up along intersections of pluri-anticanonical divisors. Therefore, the estimate of $\varphi(t)$ is more or less determined by the properties of the pluri-anticanonical divisors.

In view of these theorems, it is important to check the following two conditions:

- The Kähler Ricci flow is tamed by a large integer ν;
- $\alpha_{\nu,k}$ $(k = 1, 2)$ are large enough.

$\alpha_{\nu,k}$ can be calculated by algebraic geometry methods. The first condition is weak. We believe that it holds for every Kähler Ricci flow on a Fano manifold although we cannot prove this right now. However, under some extra conditions, we can check the first condition by applying the following theorem.

Theorem 3. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow satisfying the following conditions:*

- *Volume ratio bounded from above, i.e., there exists a constant K such that*
$$\mathrm{Vol}_{g(t)}(B_{g(t)}(x,r)) \leq K r^{2n}$$
for every geodesic ball $B_{g(t)}(x,r)$ satisfying $r \leq 1$.
- *Weak compactness, i.e., for every sequence $t_i \to \infty$, by passing to a subsequence if necessary, we have*
$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g}),$$
where $(\hat{M}, \hat{g})$ is a Q-Fano metric-normal variety, " $\xrightarrow{C^\infty}$ " means Cheeger–Gromov convergence, i.e., $(M, g(t_i))$ converges to $(\hat{M}, \hat{g})$ in the Gromov–Hausdorff topology, and the convergence is in the smooth topology away from the singularities.

Then this flow is tamed.

In the case of Fano surfaces, Tian proved a similar theorem in order to solve the Kähler Einstein equation under a continuous path of complex structures. There are more estimates available in his approach since every metric there is a KE metric. In particular, Ricci curvature is constant. In our approach, more technical difficulties appear since Ricci curvature may be unbounded. The concept of Q-Fano variety was first defined in [DT]; it is a natural generalization of Fano manifold. A Q-Fano variety is an algebraic variety with a very ample line bundle whose restriction to the smooth part is the pluri-anticanonical line bundle. Let $\mathcal{P}$ be the singular part of the Q-Fano variety $\hat{M}$. We say $(\hat{M}, \hat{g})$ is *metric-normal* if $\hat{M}$ is a normal variety and the Minkowski codimension of $\mathcal{P}$ is not less than 4 with respect to the metric $\hat{g}$. In the proof of this theorem, Hörmander's L^2-estimate of the $\bar{\partial}$-operator, Perelman's fundamental estimates and the uniform control of Sobolev constants (cf. [Ye], [Zhq]) play crucial roles. Actually, Sobolev constants control and Perelman's estimates guarantee the uniform bound of $\left\| |S|_{h_t^\nu} \right\|_{C^0(M)}$ and $\left\| |\nabla S|_{h_t^\nu} \right\|_{C^0(M)}$ whenever S is a unit norm holomorphic section in $H^0(K_M^{-\nu})$. Hörmander's L^2-estimate of the $\bar{\partial}$-operator implies that the plurigenera are sequentially continuous. Therefore, for every fixed ν, we have

$$\lim_{i\to\infty} \inf_{x\in M} e^{\nu F_\nu(x,t_i)} = \lim_{i\to\infty} \inf_{x\in M} \sum_{\beta=0}^{N_\nu} |S_{\nu,\beta}^{t_i}|_{h_{t_i}^\nu}^2(x) = \inf_{x\in\hat{M}} \sum_{\beta=0}^{N_\nu} |\hat{S}_{\nu,\beta}|_{\hat{h}^\nu}^2(x).$$

This equation relates the tameness condition to the property of every limit space. If every limit space is a Q-Fano metric-normal variety, we know

$$\inf_{x\in\hat{M}} \sum_{\beta=0}^{N_\nu} |\hat{S}_{\nu,\beta}|_{h_{t_i}^\nu}^2(x) > 0$$

for some ν depending on $\hat{M}$. Then a contradiction argument shows that $e^{\nu F_\nu}$ must be uniformly bounded from below for some large ν. In other words, F_ν is uniformly bounded (the upper bound of F_ν is implied by the upper bound of $\| |S|_{h_t^\nu} \|_{C^0(M)}$) and the flow is tamed.

As applications of Theorems 1–3, we can show the convergence of the Kähler Ricci flow on a Fano surface M when $c_1^2(M) \leq 4$.

In [CW3], we proved the weak compactness of every 2-dimensional Kähler Ricci flow.

Lemma 1 ([CW3]). *Suppose $\{(M, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution on a Fano surface. Then for every sequence $t_i \to \infty$, we have the Cheeger–Gromov convergence*

$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g})$$

where $(\hat{M}, \hat{g})$ is a KRS orbifold with finite singularities. In particular, $\hat{M}$ is a Q-Fano metric-normal variety.

Moreover, the volume ratio upper bound is proved in the process of proving weak compactness. Therefore, Theorem 3 and Lemma 1 imply that every 2-dimensional Kähler Ricci flow is tamed. We remark that, in an unpublished work (cf. [Se1]), Tian has pointed out earlier the sequential convergence of the 2-dimensional Kähler Ricci flow to Kähler Ricci soliton orbifolds under the Gromov–Hausdorff topology. Under the extra condition that Ricci curvature is uniformly bounded along the flow, Lemma 1 was proved by Natasa Sesum [Se1]. However, for our purpose of using Theorem 3, these convergence theorems are not sufficient (we need Cheeger–Gromov convergence without the Ricci curvature bound). In the course of proving this lemma, the fundamental works of G. Perelman (cf. [Pe]) on the Ricci flow (no-local-collapsing theorem, pseudo-locality theorem and contradiction arguments for canonical neighborhood theorem) play crucial roles. Under some natural geometric constraints, we prove an inverse pseudo-locality theorem and obtain estimates of volume ratios of geodesic balls. We found that the arguments of volume ratios in [TV1] and [TV2] are very enlightening.

In order to show the convergence of a 2-dimensional Kähler Ricci flow, we now only need to see that $\alpha_{\nu,k}$ $(k = 1, 2)$ are large enough to satisfy the requirements of Theorem 1 or Theorem 2. If $c_1^2(M) \leq 4$, one can show that either Theorem 1 or Theorem 2 applies. However, the convergence of the Kähler Ricci flow on Fano surfaces M was proved in [CW2] when $c_1^2(M) = 2$ or $c_1^2(M) = 4$. The only remaining cases are $c_1^2(M) = 1$ and $c_1^2(M) = 3$. So we concentrate on these two cases and prove the following lemma.

Lemma 2. *Suppose M is a Fano surface ν is any positive integer.*

- *If $c_1^2(M) = 1$, then $\alpha_{\nu,1} \geq 5/6$.*
- *If $c_1^2(M) = 3$, then $\alpha_{\nu,1} \geq 2/3$ and $\alpha_{\nu,2} > 2/3$.*

Actually, the value of $\alpha_{\nu,1}$ was calculated by Ivan Cheltsov (cf. [Chl]) for every Fano surface. The value of $\alpha_{\nu,2}$ was also calculated for every cubic surface ($c_1^2(M) = 3$) by

Yalong Shi (cf. [Shi]). For the convenience of the readers, we give an elementary proof at the end of this paper.

Therefore, Theorem 1 or Theorem 2 applies to show the existence of KE metrics on M whenever $c_1^2(M) = 1$ or 3 respectively. Combining this result with the results we proved in [CW1] and [CW2], we can give an alternative proof of the celebrated theorem of Tian.

Theorem ([Tian90]). *A Fano surface M admits a KE metric if and only if* $\mathrm{Aut}(M)$ *is reductive.*

This solved a famous problem of Calabi for Fano surfaces [Ca]. This work of Tian clearly involves a deep understanding of many aspects of Kähler geometry as well as its intimate connection to algebraic geometry. It is one of the few highlights in Kähler geometry which deserve new proofs by the Ricci flow. On the other hand, the Kähler Ricci flow is a natural way to understand the Calabi conjecture in the Fano setting. Following Yau's estimate ([Yau78]), H. D. Cao [Cao85] proved that the Kähler Ricci flow with smooth initial metric always exists globally. On a KE manifold, the first named author and Tian showed that the Kähler Ricci flow converges exponentially fast toward the KE metric if the initial metric has positive bisectional curvature (cf. [CT1], [CT2]). Using his famous μ-functional, Perelman proved that scalar curvature, diameter and normalized Ricci potentials are all uniformly bounded along the Kähler Ricci flow (cf. [SeT]). These fundamental estimates of G. Perelman opened the door for a more qualitative analysis of singularities formed in the Kähler Ricci flow. As a corollary of his estimates, G. Perelman announced that the Kähler Ricci flow always converges to the KE metric on every KE manifold. The first written proof of this statement appeared in [TZ] where Tian and Zhu also generalized it to Kähler manifolds with KRS (Kähler Ricci soliton) metrics. In our humble view, the estimates of G. Perelman make the flow approach a plausible one in terms of understanding the Calabi conjecture in the Fano setting. We hope that this modest progress in the Kähler Ricci flow will attract more attention to the renown Hamilton–Tian conjecture that any Kähler Ricci flow converges to some KRS with mild singularities in the Cheeger–Gromov topology.

Application of the strong partial C^0-estimate is one of the crucial components of this paper. It sets up the framework of our proof for the convergence of 2-dimensional Kähler Ricci flow. This estimate originates from the strong partial C^0-estimate along continuous paths in Tian's original proof (cf. [Tian90]). He also conjectured that the strong partial C^0-estimate holds along continuous paths in higher dimensional Kähler manifolds (cf. [Tian91]).

The organization of this paper is as follows. In Section 2, along each tamed flow, we reduce the C^0-estimate of the potential function φ to the calculation of local α-invariants of sections $S \in H^0(K_M^{-\nu})$. In Section 3, we study the fundamental properties of pluri-anticanonical holomorphic sections along the Kähler Ricci flow. Here we discuss applications of Hörmander's L^2-estimate of the $\bar{\partial}$-operator and we deduce uniform bounds of $|S|_{h_t^\nu}$ and $|\nabla S|_{h_t^\nu}$. Using these estimates, we obtain a theorem giving some tameness criteria. In Section 4, we calculate $\alpha_{\nu,k}(M)$ $(k = 1, 2)$ when $c_1^2(M) = 1$ or 3 and show that their values are large enough to obtain the C^0-estimate of the evolving potential function $\varphi(t)$.

2. Estimates along the Kähler Ricci flow

2.1. Fundamentals of Kähler geometry

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z_1, \ldots, z_n$, ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} > 0,$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive $(1,1)$-form. Given a Kähler metric ω, the curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}, \qquad \forall i, j, k, l = 1, \ldots, n.$$

The Ricci curvature form is

$$\mathrm{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}}(\omega) dz^i \wedge dz^{\bar{j}} = -\sqrt{-1}\, \partial\bar{\partial} \log \det(g_{k\bar{l}}).$$

It is a real and closed $(1,1)$-form. Note that $[\mathrm{Ric}] = 2\pi c_1(M)$.

Now we assume that M is Fano, i.e., $c_1(M) > 0$. Fix an initial metric $\omega = \omega_0$ in the class $2\pi c_1(M)$. The normalized Ricci flow equation (cf. [Cao85]) is

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \qquad \forall i, j = 1, \ldots, n. \tag{2}$$

It is easy to check that this flow preserves the class $2\pi c_1(M)$. Therefore, every metric form ω_t can be written as $\omega_{\varphi(t)} = \omega + \sqrt{-1}\, \partial\bar{\partial}\varphi(t)$ where $\varphi(t)$ is the Kähler potential. On the level of Kähler potentials, the flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_{\varphi}^n}{\omega^n} + \varphi + u_{\omega}, \tag{3}$$

where u_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = -\sqrt{-1}\, \partial\bar{\partial} u_{\omega}, \quad \text{and} \quad \int_X (e^{-u_\omega} - 1)\omega^n = 0.$$

The flow equation (2) or (3) is referred to as the *Kähler Ricci flow* on M. It was proved by Cao [Cao85], who followed Yau's celebrated work [Yau78], that the Kähler Ricci flow exists globally for any smooth initial Kähler metric in $2\pi c_1(M)$.

For the given $\omega \in 2\pi c_1(M)$, define $\mathscr{P}(M, \omega) = \{\varphi \mid \omega + \sqrt{-1}\, \partial\bar{\partial}\varphi > 0\}$. It is shown in [Tian87] that there is a small constant $\delta > 0$ such that

$$\sup_{\varphi \in \mathscr{P}(M,\omega)} \frac{1}{V} \int_M e^{-\delta(\varphi - \sup_M \varphi)} \omega^n < \infty.$$

The supremum of such δ's depends only on the class $[\omega] = 2\pi c_1(M)$. It is called the *α-invariant* of M and denoted by $\alpha(M)$. If G is a compact subgroup of $\mathrm{Aut}(M)$ and ω is a G-invariant metric form, we define

$$\mathscr{P}_G(M, \omega) = \{\varphi \mid \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi > 0,\ \varphi \text{ is invariant under } G\}.$$

Similarly, we can define $\alpha_G(M)$. Actually, $\alpha_G(M)$ is an algebraic invariant. It is called the global log canonical threshold $\mathrm{lct}(X, G)$ by algebraic geometers. See [Chl] for more details.

2.2. *Known estimates along general Kähler Ricci flow*

There are a lot of estimates along the Kähler Ricci flow in the literature. We list some of them which are important to our arguments.

Proposition 2.1 (Perelman, cf. [SeT]). *Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is a Kähler Ricci flow solution. There are two positive constants $\mathcal{B}, \kappa$ depending only on this flow such that the following two estimates hold.*

1. *Suppose $-u_t$ is the normalized Ricci potential of ω_t, i.e.,*

$$\mathrm{Ric}(\omega_t) - \omega_t = -\sqrt{-1}\,\partial\bar{\partial}u_t, \qquad \frac{1}{V}\int_M e^{-u_t}\omega_t^n = 1.$$

   Then

$$\|R(\omega_t)\|_{C^0} + \mathrm{diam}_{g(t)} M + \|u_t\|_{C^0} + \|\nabla u_t\|_{C^0} < \mathcal{B}$$

   uniformly for every $t \in [0, \infty)$.
2. *Under the metric $g(t)$, $\mathrm{Vol}(B(x, r))/r^{2n} > \kappa$ for every $r \in (0, 1)$, $(x, t) \in M \times [0, \infty)$.*

After this fundamental work of G. Perelman, many interesting papers appeared. We include a few references here for the convenience of the readers: [CH], [CST], [CW1], [CW2], [Hei], [PSS], [PSSW1], [PSSW2], [Ru], [RZZ], [Se1], [Se2], [TZs], etc. In this subsection, we cite a few results which are directly related to our work here.

Proposition 2.2 ([Zhq], [Ye]). *There is a uniform Sobolev constant C_S along the Kähler Ricci flow solution $\{(M^n, g(t)), 0 \le t < \infty\}$. In other words, for every $f \in C^\infty(M)$, we have*

$$\left(\int_M |f|^{\frac{2n}{n-1}}\omega_t^n\right)^{\frac{n-1}{n}} \le C_S\left\{\int_M |\nabla f|^2\omega_t^n + \frac{1}{V^{1/n}}\int_M |f|^2\omega_t^n\right\}.$$

As an easy application of a normalization technique initiated in [CT1], one can prove the following property.

Proposition 2.3 (cf. [PSS]). *By properly choosing the initial condition, we have*

$$\|\dot{\varphi}\|_{C^0} + \|\nabla\dot{\varphi}\|_{C^0} < C$$

for some constant C independent of time t.

Based on these estimates, the authors proved the following properties.

Proposition 2.4 (cf. [Ru], [CW2]). *There is a constant C such that*

$$\frac{1}{V}\int_M (-\varphi)\omega_\varphi^n \leq n \sup_M \varphi - \sum_{i=0}^{n-1} \frac{i}{V}\int_M \sqrt{-1}\,\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i} + C. \tag{4}$$

In particular,

$$\frac{1}{V}\int_M (-\varphi)\omega_\varphi^n \leq n \sup_M \varphi + C. \tag{5}$$

Proposition 2.5 (cf. [Ru], [CW2]). *For every δ less than the α-invariant of M, there is a uniform constant C such that*

$$\sup_M \varphi < \frac{1-\delta}{\delta}\int_M (-\varphi)\omega_\varphi^n + C \tag{6}$$

along the flow.

Lemma 2.1 (cf. [Ru], [CW2]). *Along the Kähler Ricci flow $\{(M^n, g(t)), 0 \leq t < \infty\}$ on the Fano manifold M, the following conditions are equivalent:*

- *φ is uniformly bounded.*
- *$\sup_M \varphi$ is uniformly bounded from above.*
- *$\inf_M \varphi$ is uniformly bounded from below.*
- *$\int_M \varphi\omega^n$ is uniformly bounded from above.*
- *$\int_M (-\varphi)\omega_\varphi^n$ is uniformly bounded from above.*
- *$I_\omega(\varphi)$ is uniformly bounded.*
- *$\mathrm{Osc}_M\, \varphi$ is uniformly bounded.*

Here $I_\omega(\varphi) = \frac{1}{\mathrm{Vol}(M)}\int_M \varphi(\omega^n - \omega_\varphi^n)$.

As a simple corollary, we have

Theorem 2.1 (cf. [Ru], [CW2]). *If $\alpha_G(M) > n/(n+1)$, then φ is uniformly bounded along the Kähler Ricci flow initiating from some G-invariant metric $\omega \in 2\pi c_1(M)$.*

2.3. Estimates along the tamed Kähler Ricci flow

In this section, we only study the tamed Kähler Ricci flow.

Definition 2.1. For every positive integer ν, we can define a function F_ν on $M \times [0, \infty)$ as follows:

$$F_\nu(x,t) \triangleq \frac{1}{\nu}\log\sum_{\beta=0}^{N_\nu} |S_{\nu,\beta}^t|_{h_t^\nu}^2(x)$$

where $\{S^t_{\nu,\beta}\}_{\beta=0}^{N_\nu}$ is an orthonormal basis of $H^0(K_M^{-\nu})$ under the metrics $g_t = g(t)$ and $h_t^\nu = (\det g_t)^\nu$, i.e.,

$$\int_M \langle S^t_{\nu,\alpha}, S^t_{\nu,\beta}\rangle_{h_t^\nu} \omega_t^n = \delta_{\alpha\beta}, \quad N_\nu = \dim H^0(K_M^{-\nu}) - 1.$$

Note that this definition is independent of the choice of the orthonormal basis of $H^0(K_M^{-\nu})$.

Definition 2.2. $\{(M^n, g(t)), 0 \le t < \infty\}$ is called a *flow tamed by* ν if:

- $K_M^{-\nu}$ is very ample.
- $|F_\nu|_{C^0(M\times[0,\infty))} < \infty$.

A flow $\{(M^n, g(t)), 0 \le t < \infty\}$ is called a *tamed flow* if it is tamed by some large integer ν.

Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is a Kähler Ricci flow tamed by ν. For simplicity of notation, denote N_ν by N. Under the metrics g_t and h_t^ν, we choose $\{S^t_{\nu,\beta}\}_{\beta=0}^N$ as an orthonormal basis of $H^0(K_M^{-\nu})$. At the same time, let $\{\tilde{S}^t_{\nu,\beta}\}_{\beta=0}^N$ be an orthonormal basis of $H^0(K_M^{-\nu})$ under the metrics g_0 and h_0^ν. Then we have two embeddings

$$\Phi^t : M \to \mathbb{CP}^N, \quad x \mapsto [S^t_{\nu,0}(x) : \cdots : S^t_{\nu,N}(x)];$$
$$\Psi^t : M \to \mathbb{CP}^N, \quad x \mapsto [\tilde{S}^t_{\nu,0}(x) : \cdots : \tilde{S}^t_{\nu,N}(x)].$$

By rotating the basis if necessary, we can assume $\Phi^t = \sigma(t)\circ\Psi^t$ where

$$\sigma(t) = a(t)\,\mathrm{diag}\{\lambda_0(t), \ldots, \lambda_N(t)\}, \ \ a(t) > 0, \ \ 0 < \lambda_0(t) < \lambda_1(t) < \cdots < \lambda_N(t) = 1.$$

This indicates that the Kähler Ricci flow equation can be rewritten as

$$\begin{aligned}
\dot{\varphi} &= \log\frac{\omega_\varphi^n}{\omega^n} + \varphi + u_\omega = \frac{1}{\nu}\log\frac{\sum_{\beta=0}^N |S^t_{\nu,\beta}|^2_{h_t^\nu}}{\sum_{\beta=0}^N |S^t_{\nu,\beta}|^2_{h_0^\nu}} + \varphi + u_\omega \\
&= \frac{1}{\nu}\log\sum_{\beta=0}^N |S^t_{\nu,\beta}|^2_{h_t^\nu} - \frac{1}{\nu}\log\sum_{\beta=0}^N |a(t)\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} + \varphi + u_\omega \\
&= F_\nu(x,t) - \frac{1}{\nu}\log\sum_{\beta=0}^N |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} + \varphi + u_\omega - \frac{2}{\nu}\log a(t).
\end{aligned}$$

In other words,

$$\varphi - \frac{2}{\nu}\log a(t) = \dot{\varphi} - u_\omega - F_\nu(x,t) + \frac{1}{\nu}\log\sum_{\beta=0}^N |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}.$$

Since $F_\nu(x,t)$, $\dot{\varphi}$ and u_ω are all uniformly bounded, we obtain

$$\varphi - \frac{2}{\nu}\log a(t) \sim \frac{1}{\nu}\log\sum_{\beta=0}^N |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}.$$

Here we use $\sim$ to denote that the difference of the two sides is controlled by a constant. It follows that

$$\varphi - \sup_M \varphi \sim \frac{1}{\nu} \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} - \frac{1}{\nu} \sup_M \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}.$$

It is obvious that

$$\sup_M \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} \leq \sup_M \log \sum_{\beta=0}^{N} |\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} < C.$$

On the other hand, we have

$$\begin{aligned}\sup_M \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} &\geq \sup_M \log |\lambda_N(t)\tilde{S}^t_{N,\beta}|^2_{h_0^\nu} = \sup_M \log |\tilde{S}^t_{\nu,N}|^2_{h_0^\nu} \\ &= \log \sup_M |\tilde{S}^t_{\nu,N}|^2_{h_0^\nu} \geq \log \frac{1}{V} \int_M |\tilde{S}^t_{\nu,N}|^2_{h_0^\nu} \omega^n = -\log V.\end{aligned}$$

Therefore, $\nu^{-1} \sup_M \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}$ is uniformly bounded, which yields

$$\varphi - \sup_M \varphi \sim \frac{1}{\nu} \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}. \tag{7}$$

So we have proved the following property.

Proposition 2.6. *If $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν, then there is a constant C (depending on this flow and ν) such that*

$$\left| \varphi - \sup_M \varphi - \frac{1}{\nu} \log \sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu} \right| < C \tag{8}$$

uniformly along this flow.

Inequality (8) is called *the strong partial C^0-estimate* by Tian. Using this estimate, the control of $\|\varphi\|_{C^0(M)}$ along the Kähler Ricci flow is reduced to the control of values of local α-invariants (see Definitions 1 and 2) of holomorphic sections $S \in H^0(K_M^{-\nu})$.

Theorem 2.2. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν. If $\alpha_{\nu,1} > n/(n+1)$, then φ is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.*

Proof. Suppose not. Then there is a sequence of times t_i such that $\lim_{i\to\infty}|\varphi_{t_i}|_{C^0(M)} = \infty$.

Choose $S^t_{\nu,\beta} = a(t)\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}$, $0 \le \beta \le N$, as before. Since both $|\tilde{S}^t_{\nu,\beta}|_{h_0^\nu}$ and $\lambda_\beta(t)$ are uniformly bounded, we can assume

$$\lim_{i\to\infty}\lambda_\beta(t_i) = \bar{\lambda}_\beta, \qquad \lim_{i\to\infty}\tilde{S}^{t_i}_{\nu,\beta} = \bar{S}_{\nu,\beta}, \qquad \beta = 0, 1, \ldots, N.$$

Notice that $\bar{\lambda}_N = 1$.

Define

$$I(\alpha, t) = \int_M \Big(\sum_{\beta=0}^{N} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}\Big)^{-\alpha/\nu}\omega^n.$$

Clearly, $I(\alpha, t_i) \le \int_M |\tilde{S}^{t_i}_{\nu,N}|^{-2\alpha/\nu}_{h_0^\nu}\omega^n$. As $\bar{S}_{\nu,N} \in H^0(K_M^{-\nu})$ and $\alpha_{\nu,1} > n/(n+1)$, we can find a number $\alpha \in (n/(n+1), \alpha_{\nu,1})$ such that $\int_M |\bar{S}_{\nu,N}|^{-2\alpha/\nu}_{h_0^\nu}\omega^n < C$. By the semi-continuity of the singularity exponent (cf. [Tian90], [DK], [PS]), we have

$$\limsup_{i\to\infty} I(\alpha, t_i) \le \lim_{i\to\infty}\int_M |\tilde{S}^{t_i}_{\nu,N}|^{-2\alpha/\nu}_{h_0^\nu}\omega^n = \int_M |\bar{S}_{\nu,N}|^{-2\alpha/\nu}_{h_0^\nu}\omega^n < C.$$

Along a tamed flow, the inequality

$$\left|\varphi - \sup_M\varphi - \frac{1}{\nu}\log\sum_{\beta=0}^{N}|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu}\right| < C$$

holds. It follows that $\int_M e^{\alpha(\varphi_{t_i} - \sup_M\varphi_{t_i})}\omega^n < C$. Recalling that $\dot{\varphi} = \log(\omega^n_\varphi/\omega^n) + \varphi + u_\omega$, we have

$$\frac{1}{V}\int_M e^{-\alpha(\varphi_{t_i} - \sup_M\varphi_{t_i})}\cdot e^{\varphi_{t_i} + u_\omega - \dot{\varphi}}\omega^n_{t_i} < C.$$

Note that both $\dot{\varphi}$ and u_ω are uniformly bounded. It follows from the convexity of the exponential map that $\alpha\sup_M\varphi_{t_i} + (1-\alpha)\frac{1}{V}\int_M\varphi_{t_i}\omega^n_{t_i} < C$. In other words,

$$\sup_M\varphi_{t_i} < \frac{1-\alpha}{\alpha}\frac{1}{V}\int_M(-\varphi_{t_i})\omega^n_{t_i} + C. \tag{9}$$

Combining this with inequality (5), we have

$$\sup_M\varphi_{t_i} < n\frac{1-\alpha}{\alpha}\sup_M\varphi_{t_i} + C.$$

Since $\alpha \in (n/(n+1), 1)$, it follows that $\sup_M\varphi_{t_i}$ is uniformly bounded from above. Consequently, φ_{t_i} is uniformly bounded. This contradicts our assumption on φ_{t_i}. □

By a more careful analysis, we can improve this theorem a little bit.

Proposition 2.7. *Let* $X_t \triangleq \nu^{-1}\log(\sum_{\beta=0}^{N}|\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h_0^\nu})$. *There is a constant* C *such that*

$$\left|\frac{1}{V}\int_M\{\sqrt{-1}\,\partial\varphi_{t_i}\wedge\bar{\partial}\varphi_{t_i}-\sqrt{-1}\,\partial X_{t_i}\wedge\bar{\partial}X_{t_i}\}\wedge\omega^{n-1}\right|<C. \tag{10}$$

Proof. Since every $\tilde{S}^{t_i}_{\nu,\beta}$ is a holomorphic section, direct calculation shows that

$$\Delta X_{t_i}\geq -R$$

where Δ, R are the Laplacian operator and the scalar curvature under the metric ω. As $\Delta\varphi+n>0$, we can choose a constant C_0 such that $\Delta\varphi+\Delta X_{t_i}+C_0>0$. For the simplicity of notation, we omit the subscript t_i in the following arguments. So

$$\left|\varphi-\sup_M\varphi-X\right|<C, \qquad \Delta X+\Delta\varphi+C_0>0.$$

Having these conditions at hand, direct computation gives

$$\begin{aligned}
&\frac{1}{V}\int_M\{\sqrt{-1}\,\partial\varphi\wedge\bar{\partial}\varphi-\sqrt{-1}\,\partial X\wedge\bar{\partial}X\}\wedge\omega^{n-1}=\frac{1}{V}\int_M(X-\varphi)(\Delta X+\Delta\varphi)\omega^n\\
&=\frac{1}{V}\int_M\Big(X-\varphi+\sup_M\varphi+C\Big)(\Delta X+\Delta\varphi)\omega^n\\
&=\frac{1}{V}\int_M\Big(X-\varphi+\sup_M\varphi+C\Big)(\Delta X+\Delta\varphi+C_0)\omega^n-\frac{C_0}{V}\int_M\Big(X-\varphi+\sup_M\varphi+C\Big)\omega^n\\
&\geq-\frac{C_0}{V}\int_M\Big(X-\varphi+\sup_M\varphi+C\Big)\omega^n\geq-2C_0C.
\end{aligned}$$

On the other hand, a similar calculation shows

$$\begin{aligned}
\frac{1}{V}\int_M\{\sqrt{-1}\,\partial\varphi\wedge\bar{\partial}\varphi-\sqrt{-1}\,\partial X\wedge\bar{\partial}X\}\wedge\omega^{n-1}&\leq-\frac{C_0}{V}\int_M\Big(X-\varphi+\sup_M\varphi-C\Big)\omega^n\\
&\leq 2C_0C.
\end{aligned}$$

Consequently,

$$\left|\frac{1}{V}\int_M\{\sqrt{-1}\,\partial\varphi\wedge\bar{\partial}\varphi-\sqrt{-1}\,\partial X\wedge\bar{\partial}X\}\wedge\omega^{n-1}\right|\leq 2C_0C. \qquad \square$$

It follows from this proposition that inequality (4) implies

$$\frac{1}{V}\int_M(-\varphi)\omega_\varphi^n\leq n\sup_M\varphi-\frac{n-1}{V}\int_M\sqrt{-1}\,\partial X\wedge\bar{\partial}X\wedge\omega^{n-1}+C. \tag{11}$$

Similar to the theorems in [Tian90], we can prove the following theorem.

Theorem 2.3. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow tamed by ν, and M is a Fano manifold satisfying*

$$\alpha_{\nu,2} > \frac{n}{n+1} \quad \text{and} \quad \alpha_{\nu,1} > \frac{1}{2 - \frac{n-1}{(n+1)\alpha_{\nu,2}}}.$$

Then along this flow, φ is uniformly bounded. In particular, this flow converges to a KE metric exponentially fast.

Proof. Suppose not. We have a sequence of times t_i such that $\lim_{i\to\infty} |\varphi_{t_i}|_{C^0(M)} = \infty$. As before, we have

$$\lim_{i\to\infty} \tilde{S}_{\nu,\beta}^{t_i} = \bar{S}_{\nu,\beta}, \quad \beta = 0, 1, \ldots, N; \quad \lim_{i\to\infty} \lambda_\beta(t_i) = \bar{\lambda}_\beta; \quad \bar{\lambda}_N = 1.$$

Claim 1. $\bar{\lambda}_{N-1} = 0$.

Otherwise, $\bar{\lambda}_{N-1} > 0$. Fixing some $\alpha \in (n/(n+1), \alpha_{\nu,2})$, we calculate

$$\begin{aligned} I(\alpha, t_i) &= \int_M \Big(\sum_{\beta=0}^{N} |\lambda_\beta(t) \tilde{S}_{\nu,\beta}^{t_i}|_{h_0^\nu}^2 \Big)^{-\alpha/\nu} \omega^n \leq \int_M (|\lambda_{N-1}(t_i) \tilde{S}_{\nu,N-1}^{t_i}|_{h_0^\nu}^2 + |\tilde{S}_{\nu,N}^{t_i}|_{h_0^\nu}^2)^{-\alpha/\nu} \omega^n \\ &\leq (\lambda_{N-1}(t_i))^{-2\alpha} \int_M (|\tilde{S}_{\nu,N-1}^{t_i}|_{h_0^\nu}^2 + |\tilde{S}_{\nu,N}^{t_i}|_{h_0^\nu}^2)^{-\alpha/\nu} \omega^n. \end{aligned}$$

For simplicity of notation, we consider h_0^ν as the default metric on the line bundle $K_M^{-\nu}$ without writing it out explicitly. Then the semicontinuity property implies

$$\lim_{i\to\infty} \int_M (|\tilde{S}_{\nu,N-1}^{t_i}|^2 + |\tilde{S}_{\nu,N}^{t_i}|^2)^{-\alpha/\nu} \omega^n = \int_M (|\bar{S}_{\nu,N-1}|^2 + |\bar{S}_{\nu,N}|^2)^{-\alpha/\nu} \omega^n < \infty.$$

It follows that

$$I(\alpha, t_i) < 2(\bar{\lambda}_{N-1})^{-2\alpha/\nu} \int_M (|\bar{S}_{\nu,N-1}|^2 + |\bar{S}_{\nu,N}|^2)^{-\alpha/\nu} \omega^n < C_\alpha.$$

Recalling the definition of $I(\alpha, t_i)$, inequality (8) implies $\int_M e^{-\alpha(\varphi(t_i) - \sup_M \varphi(t_i))} \omega^n < C$. Since $\alpha > n/(n+1)$, as in the previous theorem we obtain $|\varphi_{t_i}|_{C^0(M)} < C$. This contradicts the initial assumption on φ_{t_i}. Therefore $\bar{\lambda}_{N-1} = 0$ as claimed.

Claim 2. *For every small constant ϵ, there is a constant C such that*

$$\begin{aligned} (1-\epsilon)\alpha_{\nu,2} \sup_M \varphi_{t_i} + (1 - (1-\epsilon)\alpha_{\nu,2}) \frac{1}{V} \int_M \varphi_{t_i} \omega_{t_i}^n \\ \leq -2(1-\epsilon) \frac{\alpha_{\nu,2}}{\nu} \log \lambda_{N-1}(t_i) + C. \end{aligned} \tag{12}$$

Fixing ϵ small, we have

$$\begin{aligned} I((1-\epsilon)\alpha_{\nu,2}, t_i) &= \int_M \Big(\sum_{\beta=0}^{N} |\lambda_\beta \tilde{S}^{t_i}_{\nu,\beta}|^2\Big)^{-(1-\epsilon)\alpha_{\nu,2}/\nu} \omega^n \\ &\leq \int_M \{\lambda_{N-1}(t_i)^2 (|\tilde{S}^{t_i}_{\nu,N-1}|^2 + |\tilde{S}^{t_i}_{\nu,N}|^2)\}^{-(1-\epsilon)\alpha_{\nu,2}/\nu} \omega^n \\ &< C\lambda_{N-1}(t_i)^{-2(1-\epsilon)\alpha_{\nu,2}/\nu}. \end{aligned}$$

The tameness condition implies that

$$\int_M e^{-(1-\epsilon)\alpha_{\nu,2}(\varphi_{t_i} - \sup_M \varphi_{t_i})} \omega^n < C\lambda_{N-1}(t_i)^{-2(1-\epsilon)\alpha_{\nu,2}/\nu}.$$

Plugging the equation $\dot{\varphi} = \log(\omega_\varphi^n/\omega^n) + \varphi + u_\omega$ into the previous inequality implies

$$\begin{aligned} C\lambda_{N-1}(t_i)^{-2(1-\epsilon)\alpha_{\nu,2}/\nu} &> \int_M e^{-(1-\epsilon)\alpha_{\nu,2}(\varphi_{t_i} - \sup_M \varphi_{t_i})} \cdot e^{\varphi_{t_i} + u_\omega - \dot{\varphi}_{t_i}} \omega_{t_i}^n \\ &= \int_M e^{(1-\epsilon)\alpha_{\nu,2} \sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\varphi_{t_i}} \cdot e^{u_\omega - \dot{\varphi}_{t_i}} \omega_{t_i}^n \\ &\geq e^{-\|u_\omega\|_{C^0(M)} - \|\dot{\varphi}_{t_i}\|_{C^0(M)}} \int_M e^{(1-\epsilon)\alpha_{\nu,2} \sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2})\varphi_{t_i}} \omega_{t_i}^n \\ &\geq e^{-\|u_\omega\|_{C^0(M)} - \|\dot{\varphi}_{t_i}\|_{C^0(M)}} \cdot V \cdot e^{(1-\epsilon)\alpha_{\nu,2} \sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2}) \frac{1}{V} \int_M \varphi_{t_i} \omega_{t_i}^n}. \end{aligned}$$

Taking logarithms on both sides, we obtain (12). This finishes the proof of Claim 2.

Claim 3. *For every small number $\epsilon > 0$, there is a constant C_ϵ such that*

$$\frac{1}{V}\int_M \sqrt{-1}\,\partial X_{t_i} \wedge \bar{\partial} X_{t_i} \wedge \omega^{n-1} \geq -\frac{1-\epsilon}{\nu} \log \lambda_{N-1}(t_i) - C_\epsilon. \tag{13}$$

The proof is the same as the corresponding proof in [Tian91], so we omit it.

Plugging (13) into (11), together with (12), we obtain

$$\begin{cases} \dfrac{1}{V}\displaystyle\int_M (-\varphi_{t_i})\omega_{t_i}^n \leq n \sup_M \varphi_{t_i} + \dfrac{(n-1)(1-\epsilon)}{\nu} \log \lambda_{N-1}(t_i) + C, \\ (1-\epsilon)\alpha_{\nu,2} \sup_M \varphi_{t_i} + (1-(1-\epsilon)\alpha_{\nu,2}) \dfrac{1}{V}\displaystyle\int_M \varphi_{t_i}\omega_{t_i}^n \leq -\dfrac{2(1-\epsilon)\alpha_{\nu,2}}{\nu} \log \lambda_{N-1}(t_i) + C. \end{cases}$$

Eliminating $\log \lambda_{N-1}(t_i)$, we have

$$\frac{1}{V}\int_M (-\varphi_{t_i})\omega_{t_i}^n \leq \frac{(n+1)+(n-1)\epsilon}{((n+1)-(n-1)\epsilon)\alpha_{\nu,2} - (n-1)} \alpha_{\nu,2} \sup_M \varphi_{t_i} + C.$$

From inequality (9), we have $\sup_M \varphi_{t_i} \leq \frac{1-(1-\epsilon)\alpha_{\nu,1}}{(1-\epsilon)\alpha_{\nu,1}} \frac{1}{V} \int_M (-\varphi_{t_i})\omega_{t_i}^n + C$. It follows that

$$\left\{1 - \frac{(n+1)+(n-1)\epsilon}{((n+1)-(n-1)\epsilon)\alpha_{\nu,2}-(n-1)} \cdot \alpha_{\nu,2} \cdot \frac{1-(1-\epsilon)\alpha_{\nu,1}}{(1-\epsilon)\alpha_{\nu,1}}\right\} \frac{1}{V} \int_M (-\varphi_{t_i})\omega_{t_i}^n \leq C$$

for every small constant ϵ and some large constant C depending on ϵ. Since

$$\alpha_{\nu,1} > \frac{1}{2 - \frac{n-1}{(n+1)\alpha_{\nu,2}}} = \frac{A}{A+1} \quad \text{where} \quad A = \frac{(n+1)\alpha_{\nu,2}}{(n+1)\alpha_{\nu,2}-(n-1)},$$

we can choose ϵ small enough such that

$$1 - \frac{(n+1)+(n-1)\epsilon}{((n+1)-(n-1)\epsilon)\alpha_{\nu,2}-(n-1)} \cdot \alpha_{\nu,2} \cdot \frac{1-(1-\epsilon)\alpha_{\nu,1}}{(1-\epsilon)\alpha_{\nu,1}} > 0.$$

This implies that $\frac{1}{V}\int_M (-\varphi_{t_i})\omega_{t_i}^n$ is uniformly bounded. Therefore, $|\varphi_{t_i}|_{C^0(M)}$ is uniformly bounded, a contradiction. □

Remark 2.1. The methods applied in Theorems 2.2 and 2.3 originate from [Tian91].

3. Pluri-anticanonical line bundles $K_M^{-\nu}$ and the tameness condition

In this section, we study the basic properties of normalized holomorphic sections $S \in H^0(K_M^{-\nu})$ under the evolving metrics ω_t and h_t^ν.

3.1. Uniform bounds of holomorphic sections of $K_M^{-\nu}$

Let S be a normalized holomorphic section of $H^0(K_M^{-\nu})$, i.e., $\int_M |S|^2_{h_t^\nu}\omega_t^n = 1$. In this section, we will show both $\left\| |S|_{h_t^\nu} \right\|_{C^0}$ and $\left\| |\nabla S|_{h_t^\nu} \right\|_{C^0}$ are uniformly bounded.

Lemma 3.1. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution. Then there is a constant A_0 depending only on this flow such that $|S|_{h_t^\nu} < A_0 \nu^{n/2}$ whenever $S \in H^0(K_M^{-\nu})$ satisfies $\int_M |S|^2_{h_t^\nu}\omega_t^n = 1$.*

Proof. Fix a time t and do all the calculations under the metrics g_t and h_t^ν. Recall we have a uniform Sobolev constant, so we can do analysis uniformly independent of time t.

Claim. *S satisfies the equation*

$$\Delta |S|^2 = |\nabla S|^2 - \nu R |S|^2. \tag{14}$$

This calculation can be done locally. Fix a point $x \in M$. Let U be a neighborhood of x with coordinates $\{z^1, \ldots, z^n\}$. Then $K_M^{-\nu}$ has a natural trivialization on the domain U and we can write $S = f\big(\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}\big)^\nu$ for some holomorphic function f locally. For convenience, we denote $h = \det g_{k\bar{l}}$. Therefore, direct calculation shows

$$\begin{aligned}\Delta |S|^2 &= g^{i\bar{j}}\{f\bar{f}h^\nu\}_{i\bar{j}} = g^{i\bar{j}}\{f_i\bar{f}h^\nu + \nu f\bar{f}h^{\nu-1}h_i\}_{\bar{j}}\\ &= g^{i\bar{j}}\{f_i\bar{f}_{\bar{j}}h^\nu + \nu\bar{f}h^{\nu-1}f_ih_{\bar{j}} + \nu f h^{\nu-1}\bar{f}_{\bar{j}}h_i + \nu(\nu-1)f\bar{f}h^{\nu-2}h_ih_{\bar{j}} + \nu f\bar{f}h^{\nu-1}h_{i\bar{j}}\}.\end{aligned}$$

If we choose normal coordinates at the point x, then we have $h = 1$, $h_i = h_{\bar{j}} = 0$, $h_{i\bar{j}} = -R_{i\bar{j}}$. Plugging these into the previous equality we have

$$\Delta |S|^2 = g^{i\bar{j}}\{f_i\bar{f}_{\bar{j}} - \nu f\bar{f}R_{i\bar{j}}\} = |\nabla S|^2 - \nu R|S|^2,$$

so equation (14) is proved.

From (14), we have

$$\int_M |\nabla S|^2\, d\mu = \int_M \nu R|S|^2\, d\mu \leq \nu\mathcal{B}$$

where $d\mu = \omega_t^n$. Noting that volume is fixed along the Kähler Ricci flow solution, we can omit the volume term in the Sobolev inequality by adjusting C_S. Therefore, the Sobolev inequality implies

$$\begin{aligned}\left\{\int_M |S|^{\frac{2n}{n-1}}\, d\mu\right\}^{\frac{n-1}{n}} &\leq C_S\left\{\int_M |S|^2\, d\mu + \int_M \big|\nabla|S|\big|^2\, d\mu\right\}\\ &\leq C_S\left\{\int_M |S|^2\, d\mu + \int_M |\nabla S|^2\, d\mu\right\} \leq C_S\{1 + \nu\mathcal{B}\} < C\nu.\end{aligned}$$

Here we use the property that $\bar{\nabla}S = 0$.

Note that $\Delta|S|^2 \geq -\nu R|S|^2$. Let $u = |S|^2$. We have

$$\Delta u \geq -\nu\mathcal{B}u, \qquad \|u\|_{L^{n/(n-1)}} < C\nu^{1/2}.$$

Multiplying this inequality by $u^{\beta-1}$ $(\beta > 1)$ and integrating by parts implies

$$\int_M |\nabla u^{\beta/2}|^2\, d\mu \leq \frac{\beta^2}{4(\beta-1)} \cdot (\mathcal{B}\nu) \cdot \int_M u^\beta\, d\mu.$$

Combining this with the Sobolev inequality yields

$$\left\{\int_M u^{\frac{n\beta}{n-1}}\, d\mu\right\}^{\frac{n-1}{n}} \leq C_S\left(1 + \frac{\beta^2\mathcal{B}\nu}{4(\beta-1)}\right)\int_M u^\beta\, d\mu \leq C\nu\beta\int_M u^\beta\, d\mu.$$

It follows that $\|u\|_{L^{n\beta/(n-1)}} \leq (C\nu)^{1/\beta}\beta^{1/\beta}\|u\|_{L^\beta}$. Letting $\beta = \big(\frac{n}{n-1}\big)^k$, we have

$$\|u\|_{L^\infty} \leq (C\nu)^{\sum_{k=1}^\infty(\frac{n-1}{n})^k} \cdot \left(\frac{n}{n-1}\right)^{\sum_{k=1}^\infty k(\frac{n-1}{n})^k} \cdot \|u\|_{L^{n/(n-1)}} \leq C\nu^{\sum_{k=0}^\infty(\frac{n-1}{n})^k} = C\nu^n.$$

In other words, $\|S\|_{L^\infty} \leq C\nu^{n/2}$. Letting A_0 be the last C finishes the proof. □

Corollary 3.1. *Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is a Kähler Ricci flow solution. Then*

$$F_\nu(x,t) \le \frac{2\log A_0 + n\log\nu}{\nu} < B_0, \quad \forall x \in M,\ t \in [0,\infty),\ \nu \ge 1. \tag{15}$$

Here B_0 is a constant depending only on A_0.

Proof. According to the definition of F_ν, we only need to show

$$\sum_{\beta=0}^{N_\nu} |S^t_{\nu,\beta}|^2_{h^\nu_t}(x) \le A_0^2 \nu^n$$

for every orthonormal holomorphic section basis $\{S^t_{\nu,\beta}\}_{\beta=0}^{N_\nu}$. However, by rotating the basis at the point x, we can always find a basis such that

$$|S^t_{\nu,\beta}|^2_{h^\nu_t}(x) = 0, \quad 1 \le \beta \le N_\nu.$$

Therefore, by Lemma 3.1, we have

$$\sum_{\beta=0}^{N_\nu} |S^t_{\nu,\beta}|^2_{h^\nu_t}(x) = |S^t_{\nu,0}|^2_{h^\nu_t}(x) \le A_0^2\nu^n. \qquad \square$$

Lemma 3.2. *Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is a Kähler Ricci flow solution. Then there is a constant A_1 depending only on this flow and ν such that $|\nabla S|_{h^\nu_t} < A_1$ whenever $S \in H^0(K_M^{-\nu})$ with $\int_M |S|^2_{h^\nu_t}\omega_t^n = 1$.*

Proof. Fix a time t and then do all the computations with respect to $g(t)$ and h^ν_t. As in the previous lemma, we can do uniform analysis owing to the existence of a uniform Sobolev constant.

Claim 1. $|\nabla S|^2$ *satisfies the equation*

$$\begin{aligned}\Delta|\nabla S|^2 = |\nabla\nabla S|^2 + \nu^2|\mathrm{Ric}|^2|S|^2 - \nu R|\nabla S|^2 \\ - (2\nu-1)R_{i\bar{k}}S_k\bar{S}_{\bar{i}} - \nu\{SR_i\bar{S}_{\bar{i}} + \bar{S}R_{\bar{i}}S_i\}.\end{aligned} \tag{16}$$

Locally, we can rewrite

$$S = f\left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^\nu, \quad \nabla S = \{f_i + \nu f(\log h)_i\}dz^i \otimes \left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^\nu,$$

where $h = \det g_{k\bar{l}}$. It follows that

$$|\nabla S|^2 = g^{i\bar{j}}h^\nu(f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu\bar{f}(\log h)_{\bar{j}}).$$

Choosing normal coordinates at x, we have $g_{i\bar{j}} = \delta_{i\bar{j}}$, $h = 1$, $h_i = h_{\bar{i}} = (\log h)_i = (\log h)_{\bar{i}} = 0$, $h_{i\bar{j}} = (\log h)_{i\bar{j}} = -R_{i\bar{j}}$, $(\log h)_{ij} = (\log h)_{\bar{i}\bar{j}} = 0$. Computation shows

$$\begin{aligned}
\Delta|\nabla S|^2 &= g^{k\bar{l}}\{g^{i\bar{j}}h^{\nu}(f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu\bar{f}(\log h)_{\bar{j}})\}_{k\bar{l}} \\
&= g^{k\bar{l}}\bigg\{-g^{i\bar{p}}g^{q\bar{j}}\frac{\partial g_{p\bar{q}}}{\partial z^k}h^{\nu}(f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu\bar{f}(\log h)_{\bar{j}}) \\
&\quad + \nu g^{i\bar{j}}h^{\nu-1}h_k(f_i + \nu f(\log h)_i)(\bar{f}_{\bar{j}} + \nu\bar{f}(\log h)_{\bar{j}}) \\
&\quad + g^{i\bar{j}}h^{\nu}(f_{ik} + \nu f(\log h)_{ik} + \nu f_k(\log h)_i)(\bar{f}_{\bar{j}} + \nu\bar{f}(\log h)_{\bar{j}}) \\
&\quad + g^{i\bar{j}}h^{\nu}(f_i + \nu f(\log h)_i)\nu\bar{f}(\log h)_{\bar{j}k}\bigg\}_{\bar{l}} \\
&= R_{\bar{j}\bar{i}k\bar{k}}f_i\bar{f}_{\bar{j}} + \nu h_{k\bar{k}}f_i\bar{f}_{\bar{i}} + \nu f(\log h)_{k\bar{k}i}\bar{f}_{\bar{i}} + \nu f_k(\log h)_{i\bar{k}}\bar{f}_{\bar{i}} + f_{ik}\bar{f}_{\bar{i}\bar{k}} \\
&\quad + \nu^2 f\bar{f}(\log h)_{i\bar{k}}(\log h)_{\bar{i}k} + \nu f_i\bar{f}(\log h)_{k\bar{k}\bar{i}} + \nu f_i\bar{f}_{\bar{k}}(\log h)_{k\bar{i}} \\
&= R_{j\bar{i}}f_i\bar{f}_{\bar{j}} - \nu R|\nabla f|^2 - \nu f R_i\bar{f}_{\bar{i}} - \nu R_{i\bar{k}}f_k\bar{f}_{\bar{i}} \\
&\quad + |\nabla\nabla f|^2 + \nu^2|f|^2|\mathrm{Ric}|^2 - \nu\bar{f}f_i R_{\bar{i}} - \nu R_{i\bar{k}}f_k\bar{f}_{\bar{i}} \\
&= |\nabla\nabla S|^2 + \nu^2|S|^2|\mathrm{Ric}|^2 - \nu R|\nabla S|^2 + (1-2\nu)R_{j\bar{i}}S_i\bar{S}_{\bar{j}} - \nu(SR_i\bar{S}_{\bar{i}} + \bar{S}R_{\bar{i}}S_i).
\end{aligned}$$

Claim 2. *S satisfies the equation*

$$S_{,i\bar{j}} = -\nu S R_{i\bar{j}}. \tag{17}$$

Locally, we can rewrite

$$\begin{aligned}
S &= f\left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^{\nu}, \\
\nabla S &= \{f_i + \nu f(\log h)_i\}dz^i \otimes \left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^{\nu}, \\
\bar{\nabla} S &= f_{\bar{i}}\overline{dz^i} \otimes \left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^{\nu} = 0.
\end{aligned}$$

Recall that $\Gamma_{ij}^k = g^{k\bar{l}}\frac{\partial g_{i\bar{l}}}{\partial z^j}$ vanishes at x. So $(\log h)_i$, $(\log h)_{ij}$ vanish at x. Note that f is holomorphic, and our connection is compatible with both the metric and the complex structure. So $\bar{\nabla}\nabla S$ has only one term

$$\begin{aligned}
\bar{\nabla}\nabla S &= \{\nu f(\log h)_{i\bar{j}}\}\overline{dz^j} \otimes dz^i \otimes \left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^{\nu} \\
&= -\nu f R_{i\bar{j}}\overline{dz^j} \otimes dz^i \otimes \left(\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}\right)^{\nu}.
\end{aligned}$$

It follows that $S_{,i\bar{j}} = -\nu S R_{i\bar{j}}$.

Claim 3. *There is a constant C such that $\|\nabla S\|_{L^{2n/(n-1)}} < C$ uniformly.*

Integrating both sides of (16) we have

$$\int_M |\nabla\nabla S|^2\,d\mu \le \int_M \nu R|\nabla S|^2\,d\mu + (2\nu-1)\int_M R_{i\bar{k}}S_k\bar{S}_{\bar{i}}\,d\mu + \nu\int_M \{SR_i\bar{S}_{\bar{i}} + \bar{S}R_{\bar{i}}S_i\}\,d\mu$$

where $d\mu = \omega_t^n$. Recall that $R_{i\bar{k}} = g_{i\bar{k}} - \dot{\varphi}_{i\bar{k}}$. It follows that

$$\begin{aligned}\int_M |\nabla\nabla S|^2\,d\mu &\le \int_M \nu R|\nabla S|^2\,d\mu + (2\nu-1)\int_M |\nabla S|^2\,d\mu - (2\nu-1)\int_M \dot{\varphi}_{i\bar{k}}S_k\bar{S}_{\bar{i}}\,d\mu\\ &\quad + 2\int_M \{-\nu R|\nabla S|^2 + \nu^2R^2|S|^2\}\,d\mu\\ &= 2\nu^2\int_M R^2|S|^2\,d\mu - \nu\int_M R|\nabla S|^2\,d\mu\\ &\quad + (2\nu-1)\int_M |\nabla S|^2\,d\mu + (2\nu-1)\int_M \dot{\varphi}_i\{S_{,k\bar{k}}\bar{S}_{\bar{i}} + S_k\bar{S}_{,\bar{i}\bar{k}}\}\,d\mu.\end{aligned}$$

Note that we used the property $S_{,l\bar{k}} = -\nu SR_{l\bar{k}}$. It follows that

$$\begin{aligned}\int_M |\nabla\nabla S|^2\,d\mu &\le 2\nu^2\int_M R^2|S|^2\,d\mu - \nu\int_M R|\nabla S|^2\,d\mu + (2\nu-1)\int_M |\nabla S|^2\,d\mu\\ &\quad - (2\nu-1)\nu\int_M SR\dot{\varphi}_i\bar{S}_{\bar{i}}\,d\mu + (2\nu-1)\int_M \bar{S}_{,\bar{i}\bar{k}}\dot{\varphi}_iS_k\,d\mu\\ &\le C\left\{1 + \int_M |\nabla S|\,d\mu + \int_M |\nabla\nabla S|\,|\nabla S|\,d\mu\right\}\\ &\quad \text{(as } R, \dot{\varphi}, |\nabla\dot{\varphi}|, |S|, \textstyle\int_M |S|^2\,d\mu \text{ and } \int_M |\nabla S|^2\,d\mu \text{ are all bounded)}\\ &\le C\left\{1 + V + \int_M |\nabla S|^2\,d\mu + \frac{1}{2C}\int_M |\nabla\nabla S|^2\,d\mu + 2C\int_M |\nabla S|^2\,d\mu\right\}\\ &\quad \text{(by the Hölder inequality and } xy \le x^2 + y^2\text{)}\\ &= \frac{1}{2}\int_M |\nabla\nabla S|^2\,d\mu + C\{1 + V + (2C+1)\mathcal{B}\nu\}.\end{aligned}$$

With a different C, we have proved that $\int_M |\nabla\nabla S|^2\,d\mu < C$ uniformly. On the other hand, we know

$$\int_M |\bar{\nabla}\nabla S|^2\,d\mu = \int_M \nu^2|S|^2|\mathrm{Ric}|^2\,d\mu < C\int_M |\mathrm{Ric}|^2\,d\mu < C.$$

Therefore the Sobolev inequality tells us that

$$\begin{aligned}\left(\int_M |\nabla S|^{\frac{2n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}} &\le C_S\left\{\int_M |\nabla S|^2\,d\mu + \int_M \big|\nabla|\nabla S|\big|^2\,d\mu\right\}\\ &\le C\left\{\int_M |\nabla S|^2\,d\mu + \int_M |\nabla\nabla S|^2\,d\mu + \int_M |\bar{\nabla}\nabla S|^2\,d\mu\right\}.\end{aligned}$$

This means $\|\nabla S\|_{L^{2n/(n-1)}}$ is uniformly bounded along the Kähler Ricci flow, proving Claim 3.

Fix $\beta > 1$. Multiplying by $-|\nabla S|^{2(\beta-1)}$ both sides of (16) and integrating yields

$$\begin{aligned}\frac{4(\beta-1)}{\beta^2}&\int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu\\ &= -\int_M (\nu^2|\mathrm{Ric}|^2|S|^2 + |\nabla\nabla S|^2)|\nabla S|^{2(\beta-1)}\, d\mu + \int_M \nu R|\nabla S|^{2\beta}\, d\mu\\ &\quad + \underbrace{\int_M (2\nu-1)R_{i\bar{k}}S_k\bar{S}_{\bar{i}}|\nabla S|^{2(\beta-1)}\, d\mu}_{I} + \underbrace{\nu\int_M \{SR_i\bar{S}_{\bar{i}} + \bar{S}R_{\bar{i}}S_i\}|\nabla S|^{2(\beta-1)}\, d\mu}_{II}.\end{aligned}$$

Plugging $R_{i\bar{k}} = g_{i\bar{k}} - \dot{\varphi}_{i\bar{k}}$ into I yields

$$I = (2\nu-1)\int_M |\nabla S|^{2\beta}\, d\mu - (2\nu-1)\int_M \dot{\varphi}_{i\bar{k}}S_k\bar{S}_{\bar{i}}|\nabla S|^{2(\beta-1)}\, d\mu.$$

Since $S_{,l\bar{k}} = -\nu S R_{l\bar{k}}$, by the uniform bounds of $\dot{\varphi}$, R and $|S|$, we have

$$\begin{aligned}\frac{I}{2\nu-1} &= \int_M |\nabla S|^{2\beta}\, d\mu - \int_M \dot{\varphi}_i(\nu RS\bar{S}_{\bar{i}} - S_k\bar{S}_{,\bar{i}\bar{k}})|\nabla S|^{2(\beta-1)}\, d\mu\\ &\quad + (\beta-1)\int_M \dot{\varphi}_i S_k\bar{S}_{\bar{i}}|\nabla S|^{2(\beta-2)}(-\nu S R_{l\bar{k}}\bar{S}_{\bar{l}} + S_l\bar{S}_{,\bar{l}\bar{k}})\, d\mu\\ &\le \int_M |\nabla S|^{2\beta}\, d\mu + C\nu\int_M |\nabla S|^{2\beta-1}\, d\mu + C\int_M |\nabla\nabla S|\,|\nabla S|^{2\beta-1}\, d\mu\\ &\quad + C(\beta-1)\nu\int_M |\mathrm{Ric}|\,|S|\,|\nabla S|^{2\beta-1}\, d\mu + C(\beta-1)\int_M |\nabla\nabla S|\,|\nabla S|^{2\beta-1}.\end{aligned}$$

Therefore, for some constant C (which may depend on ν), we have

$$I \le C\int_M (|\nabla S|^{2\beta-1} + |\nabla S|^{2\beta})\, d\mu + \beta C\int_M (\nu|\mathrm{Ric}|\,|S| + |\nabla\nabla S|)|\nabla S|^{2\beta-1}\, d\mu.$$

Direct calculation shows

$$\begin{aligned}II &= \nu\int_M \{-R|\nabla S|^{2\beta} + \nu|S|^2R^2|\nabla S|^{2(\beta-1)}\}\, d\mu\\ &\quad - \nu\int_M RS\bar{S}_{\bar{i}}(\beta-1)|\nabla S|^{2(\beta-2)}(-\nu\bar{S}R_{i\bar{l}}S_l + \bar{S}_{\bar{l}}S_{,li})\, d\mu\\ &\quad + \nu\int_M \{-R|\nabla S|^{2\beta} + \nu|S|^2R^2|\nabla S|^{2(\beta-1)}\}\, d\mu\\ &\quad - \nu\int_M R\bar{S}S_i(\beta-1)|\nabla S|^{2(\beta-2)}(-\nu S R_{l\bar{i}}\bar{S}_{\bar{l}} + S_l\bar{S}_{,\bar{l}\bar{i}})\, d\mu\end{aligned}$$

$$
\begin{aligned}
&= -2\nu \int_M R|\nabla S|^{2\beta}\, d\mu + 2\nu^2 \int_M |S|^2 R^2 |\nabla S|^{2(\beta-1)}\, d\mu \\
&\quad + 2(\beta-1)\nu^2 \int_M R|S|^2 R_{i\bar{l}} S_l \bar{S}_{\bar{i}} |\nabla S|^{2(\beta-2)}\, d\mu \\
&\quad - (\beta-1)\nu \int_M \{S_{,li} \bar{S}_{\bar{i}} \bar{S}_{\bar{l}} S + \bar{S}_{,\bar{l}\bar{i}} S_l S_i \bar{S}\} R |\nabla S|^{2(\beta-2)}\, d\mu \\
&\le C \int_M \{|\nabla S|^{2\beta} + |\nabla S|^{2(\beta-1)}\}\, d\mu + \beta C \int_M (\nu|\mathrm{Ric}|\,|S| + |\nabla\nabla S|)|\nabla S|^{2(\beta-1)}\, d\mu.
\end{aligned}
$$

Combining this estimate with the estimate of I we have

$$
\begin{aligned}
\frac{4(\beta-1)}{\beta^2} \int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu \le &- \int_M (\nu^2 |\mathrm{Ric}|^2 |S|^2 + |\nabla\nabla S|^2)|\nabla S|^{2(\beta-1)}\, d\mu \\
&+ C \int_M \{|\nabla S|^{2\beta} + |\nabla S|^{2(\beta-1)}\}\, d\mu \\
&+ \beta C \int_M (\nu|\mathrm{Ric}|\,|S| + |\nabla\nabla S|)\{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta-1}\}\, d\mu. \qquad (18)
\end{aligned}
$$

Since $\beta C\nu|\mathrm{Ric}|\,|S|\,|\nabla S|^{2(\beta-1)} = (\nu|\mathrm{Ric}|\,|S|\,|\nabla S|^{\beta-1}) \cdot (\beta C|\nabla S|^{\beta-1})$, we see

$$
\begin{aligned}
\int_M \beta C\nu|\mathrm{Ric}|\,|S|\,|\nabla S|^{2(\beta-1)}\, d\mu \le &\int_M \frac{1}{2}\nu^2 |\mathrm{Ric}|^2 |S|^2 |\nabla S|^{2(\beta-1)}\, d\mu \\
&+ \int_M \frac{1}{2}(\beta C)^2 |\nabla S|^{2(\beta-1)}\, d\mu.
\end{aligned}
$$

Similar deduction yields

$$
\begin{aligned}
&\beta C \int_M (\nu|\mathrm{Ric}|\,|S| + |\nabla\nabla S|)\{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta-1}\}\, d\mu \\
&\quad \le \int_M (\nu^2|\mathrm{Ric}|^2|S|^2 + |\nabla\nabla S|^2)|\nabla S|^{2(\beta-1)}\, d\mu + \beta^2 C^2 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\}\, d\mu.
\end{aligned}
$$

By adjusting the constant C, it follows from (18) that

$$
\frac{4(\beta-1)}{\beta^2} \int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu \le \beta^2 C^2 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\}\, d\mu.
$$

If $\beta \ge n/(n-1)$, we have

$$
\int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu \le (C\beta)^3 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\}\, d\mu.
$$

The Sobolev inequality tells us that

$$
\begin{aligned}
\left(\int_M |\nabla S|^{\beta\cdot\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} &\le C_S \left\{\int_M |\nabla S|^{2\beta}\, d\mu + \int_M \left|\nabla|\nabla S|^{\beta}\right|^2 d\mu\right\} \\
&\le (2C\beta)^3 \int_M \{|\nabla S|^{2(\beta-1)} + |\nabla S|^{2\beta}\}\, d\mu. \qquad (19)
\end{aligned}
$$

From this inequality and the fact $\|\nabla S\|_{L^{2n/(n-1)}}$ is uniformly bounded, the standard Moser iteration technique yields $\|\nabla S\|_{L^\infty} < A_1$ for some uniform constant A_1. □

Corollary 3.2. *Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is a Kähler Ricci flow solution. Then there is a constant A_2 depending only on this flow and ν such that $|\nabla F_\nu| < A_2$.*

3.2. Convergence of holomorphic sections of $K_M^{-\nu}$

In this subsection we use the L^2-estimate for the $\bar\partial$-operator to study the convergence of pluri-anticanonical bundles. This section is very similar to Section 5 of Tian's paper [Tian90]. For the readers' and our convenience, we write down the arguments in detail.

First let us recall an important $\bar\partial$-lemma without proof.

Proposition 3.1 (cf. [Tian90, Proposition 5.1]). *Suppose (M^n, g, J) is a complete Kähler manifold, ω is a metric form compatible with g and J, L is a line bundle on M with the Hermitian metric h, and ψ is a smooth function on M. If*

$$\mathrm{Ric}(h) + \mathrm{Ric}(g) + \sqrt{-1}\,\partial\bar\partial\psi \ge c_0\omega$$

for some uniform positive number c_0 at every point. Then for every smooth L-valued $(0,1)$-form v on M with $\bar\partial v = 0$ and $\int_M |v|^2\, d\mu_g$ finite, there exists a smooth L-valued function u on M such that $\bar\partial u = v$ and

$$\int_M |u|^2 e^{-\psi}\, d\mu_g \le \frac{1}{c_0}\int_M |v|^2 e^{-\psi}\, d\mu_g$$

where $|\cdot|$ is the norm induced by h and g.

In our application, we fix M to be a Fano manifold, and $L = K_M^{-\nu}$ for some integer ν.

This proposition ensures that the plurigenera are continuous functions in a proper moduli space of complex varieties under the Cheeger–Gromov topology.

Theorem 3.1. *Suppose (M_i, g_i, J_i) is a sequence of Fano manifolds with the following properties:*

(a) *There is an a priori constant $\mathcal{B}$ such that*

$$C_S(M_i, g_i) + \|R_{g_i}\|_{C^0(M_i)} + \|u_i\|_{C^0(M_i)} < \mathcal{B}.$$

Here $C_S(M_i, g_i)$ is the Sobolev constant of (M_i, g_i), R_{g_i} is the scalar curvature, and $-u_i$ is the normalized Ricci potential, i.e.

$$\mathrm{Ric}_{g_i} - \omega_{g_i} = -\sqrt{-1}\,\partial\bar\partial u_i, \qquad \frac{1}{V_{g_i}}\int_{M_i} e^{-u_i}\, d\mu_{g_i} = 1.$$

(b) *There is a constant K such that $K^{-1}r^{2n} \le \mathrm{Vol}(B(x,r)) \le Kr^{2n}$ for every geodesic ball $B(x,r) \subset M_i$ satisfying $r \le 1$.*

(c) *$(M_i, g_i, J_i) \xrightarrow{C^\infty} (\hat M, \hat g, \hat J)$ where $(\hat M, \hat g, \hat J)$ is a Q-Fano metric-normal variety.*

Then for every positive integer ν *such that* $K_{\hat{M}}^{-\nu}$ *is a well defined line bundle, we have:*

1. *If* $S_i \in H^0(K_{M_i}^{-\nu})$ *and* $\int_{M_i} |S_i|^2 \, d\mu_{g_i} = 1$*, then by taking a subsequence if necessary, there is* $\hat{S} \in H^0(K_{\hat{M}}^{-\nu})$ *such that*

$$S_i \xrightarrow{C^\infty} \hat{S}, \quad \int_{\hat{M}} |\hat{S}|^2 \, d\mu_{\hat{g}} = 1.$$

2. *If* $\hat{S} \in H^0(K_{\hat{M}}^{-\nu})$ *and* $\int_{\hat{M}} |\hat{S}|^2 \, d\mu_{\hat{g}} = 1$*, then there is a subsequence of holomorphic sections* $S_i \in H^0(K_{M_i}^{-\nu})$ *with* $\int_{M_i} |S_i|^2 \, d\mu_{g_i} = 1$ *such that* $S_i \xrightarrow{C^\infty} \hat{S}$*.*

Proof. For simplicity, we assume $K_{\hat{M}}^{-1}$ is a well defined line bundle over $\hat{M}$.

Let $\mathcal{P}$ be the singular set of $\hat{M}$. Since $\hat{M}$ is a Q-Fano metric-normal variety, by definition of Minkowski dimension, there is a constant $\mathcal{V}$ such that $\mathrm{Vol}(B(\mathcal{P}, r)) \leq \mathcal{V} r^4$ whenever r is small. Now we prove parts 1 and 2.

Part 1. According to the proof of Lemma 3.1, we see there is an a priori bound A_0 such that $|S_i| < A_0$.

Fix a small number δ and define $U_\delta = \hat{M} \setminus B(\mathcal{P}, \delta)$. By the definition of Cheeger–Gromov convergence, there exists a sequence of diffeomorphisms $\phi_i : U_\delta \to \phi_i(U_\delta) \subset M_i$ with the following properties:

(1) $\phi_i^* g_i \xrightarrow{C^\infty} \hat{g}$ uniformly on U_δ;

(2) $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_* \xrightarrow{C^\infty} \hat{J}$ uniformly on U_δ.

Clearly, $\phi_i^* S_i$ is a section of $(T^{(1,0)}\hat{M} \oplus T^{(0,1)}\hat{M})|_{U_\delta}$ where $T^{(1,0)}\hat{M}$ and $T^{(0,1)}\hat{M}$ are defined according to the complex structure $\hat{J}$. Note that $|\phi_i^* S_i|_{C^0(U_\delta)} < A_0$ and $\phi_i^* S_i$ is holomorphic under the complex structure $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_*$. By Cauchy's integration formula, all covariant derivatives of $\phi_i^* S_i$ with respect to $\phi_i^* g_i$ are uniformly bounded in the domain $U_{2\delta}$. Therefore, there exists a section $\hat{S}_{2\delta} \in (T^{(1,0)}\hat{M} \oplus T^{(0,1)}\hat{M})|_{U_{2\delta}}$ such that $\phi_i^* S_i \xrightarrow{C^\infty} \hat{S}_{2\delta}$ on $U_{2\delta}$. This section $\hat{S}_{2\delta}$ is automatically holomorphic with respect to $\hat{J}$ since $(\phi_i^{-1})_* \circ J_i \circ (\phi_i)_* \xrightarrow{C^\infty} \hat{J}$ on $U_{2\delta} \subset U_\delta$.

As $(M_i, g_i, J_i) \xrightarrow{C^\infty} (\hat{M}, \hat{g}, \hat{J})$, we have $\lim_{i \to \infty} V_{g_i}(M_i \setminus \phi_i(U_{2\delta})) < 2\mathcal{V}(2\delta)^4 = 32\mathcal{V}\delta^4$. It follows that

$$\begin{aligned} 1 \geq \int_{U_{2\delta}} |\phi_i^* S_i|^2 \, d\mu_{\phi_i^* g_i} &= \int_{\phi_i(U_{2\delta})} |S_i|^2 \, d\mu_{g_i} = \int_{M_i} |S_i|^2 \, d\mu_{g_i} - \int_{M_i \setminus \phi_i(U_{2\delta})} |S_i|^2 \, d\mu_{g_i} \\ &> 1 - 32 A_0^2 \mathcal{V} \delta^4. \end{aligned}$$

Therefore, for each δ, there is a limit holomorphic section $\hat{S}_{2\delta} \in H^0(K_{U_{2\delta}}^{-1})$ satisfying

$$|\hat{S}_{2\delta}|_{C^0(U_{2\delta})} \leq A_0, \quad 1 \geq \int_{U_{2\delta}} |\hat{S}_{2\delta}|^2 \, d\mu_{\hat{g}} \geq 1 - 32 A_0^2 \mathcal{V} \delta^4.$$

Letting $\delta = \delta_k = 2^{-k} \to 0$ and then taking a diagonal sequence, we obtain a subsequence of sections $\phi_{i_k}^* S_{i_k}$ satisfying

$$\phi_{i_k}^* S_{i_k}|_K \xrightarrow{C^\infty} \hat{S}|_K, \quad \forall \text{ compact set } K \subset \hat{M} \setminus \mathcal{P}.$$

This exactly means that $\phi_{i_k}^* S_{i_k} \xrightarrow{C^\infty} \hat{S}$ on $\hat{M} \setminus \mathcal{P}$. Moreover, we have $|\hat{S}|_{C^0(\hat{M}\setminus\mathcal{P})} \leq A_0$. Since $\hat{M}$ is a Q-Fano metric-normal variety, $\hat{S}$ can be naturally extended to a holomorphic section of $H^0(K_{\hat{M}}^{-1})$. Clearly, we have

$$\int_{\hat{M}} |\hat{S}|^2 \, d\mu = \int_{\hat{M}\setminus\mathcal{P}} |\hat{S}|^2 \, d\mu = 1.$$

Part 2. Fix two small positive numbers r, δ satisfying $r \gg 2\delta$. Define η_δ to be a cut-off function taking value 1 on $U_{2\delta}$, and 0 on $B(\mathcal{P}, \delta)$, with $|\nabla \eta_\delta|_{\hat{g}} < 2/\delta$. Naturally, $(\phi_i)_*(\eta_\delta \hat{S})$ can be viewed as a smooth section of the bundle $\Lambda^n(T^{(1,0)}M_i \oplus T^{(0,1)}M_i)$ by extension. Let π_i be the projection from $\Lambda^n(T^{(1,0)}M_i \oplus T^{(0,1)}M_i)$ to $\Lambda^n T^{(1,0)}M_i$ and denote $V_{\delta,i} = \pi_i((\phi_i)_*(\eta_\delta \hat{S}))$. The smooth convergence of complex structures implies that $V_{\delta,i}$ is an almost holomorphic section of $\Lambda^n T^{(1,0)}M_i$. We have

$$\lim_{i\to\infty} \sup_{\phi_i(U_{2\delta})} |\bar{\partial} V_{\delta,i}| = \lim_{i\to\infty} \sup_{\phi_i(U_{2\delta})} |\bar{\partial}(\pi_i((\phi_i)_*(\eta_\delta \hat{S})))| = 0, \tag{20}$$

where $\bar{\partial}$ is calculated under the complex structure J_i.

Notice that $V(B(\mathcal{P}, \delta)) \leq \mathcal{V}\delta^4$ for δ small. Defining $\mathcal{A} = |\hat{S}|_{C^0(\hat{M})}$, we have

$$\begin{aligned} 1 &\geq \lim_{i\to\infty} \int_{M_i} |V_{\delta,i}|^2 \, d\mu_{g_i} = \lim_{i\to\infty} \int_{M_i} |\pi_i((\phi_i)_*(\eta_\delta \hat{S}))|^2 \, d\mu_{g_i} \geq 1 - 2\mathcal{A}^2\mathcal{V}(2\delta)^4 \\ &= 1 - 32\mathcal{A}^2\mathcal{V}\delta^4. \end{aligned}$$

Recall that $V_{\delta,i}$ vanishes on $B(\mathcal{P}, \delta)$, so we have

$$\int_{M_i} |\bar{\partial} V_{\delta,i}|^2 \, d\mu_{g_i} = \int_{\phi_i(U_{2\delta})} |\bar{\partial} V_{\delta,i}|^2 \, d\mu_{g_i} + \int_{\phi_i(U_\delta \setminus U_{2\delta})} |\bar{\partial} V_{\delta,i}|^2 \, d\mu_{g_i}.$$

By (20) and the inequalities $|\nabla \eta_\delta|_{\hat{g}} < 2/\delta$ and $\mathrm{Vol}(\phi_i(U_\delta \setminus U_{2\delta})) \leq 2\mathcal{V}(2\delta)^4$, we obtain

$$\int_{M_i} |\bar{\partial} V_{\delta,i}|^2 \, d\mu_{g_i} \leq 1000\mathcal{A}^2\mathcal{V}\delta^2$$

for large i.

Let h_i be the Hermitian metric on $K_{M_i}^{-1}$ induced by g_i. Clearly, we have

$$\mathrm{Ric}(h_i) + \mathrm{Ric}(g_i) + \sqrt{-1}\,\partial\bar{\partial}(-2u_i) = 2\big(\mathrm{Ric}(g_i) - \sqrt{-1}\,\partial\bar{\partial} u_i\big) = 2\omega_{g_i}.$$

So we can to apply Proposition 3.1 to obtain a smooth section $W_{\delta,i}$ of $K_{M_i}^{-1}$ such that

$$\begin{cases} \bar\partial W_{\delta,i} = \bar\partial V_{\delta,i}, \\ \displaystyle\int_{M_i} |W_{\delta,i}|^2 e^{2u_i}\, d\mu_{g_i} \le \frac{1}{2}\int_{M_i} |\bar\partial V_{\delta,i}|^2 e^{2u_i}\, d\mu_{g_i} \le \frac{e^{2\mathcal{B}}}{2}\int_{M_i} |\bar\partial V_{\delta,i}|^2\, d\mu_{g_i} \\ \qquad < 500\mathcal{A}^2\mathcal{V}e^{2\mathcal{B}}\delta^2. \end{cases} \tag{21}$$

The triangle inequality implies

$$\begin{aligned} 1+\sqrt{500\mathcal{A}^2\mathcal{V}e^{2\mathcal{B}}\delta^2} &> \left(\int_{M_i} |V_{\delta,i}-W_{\delta,i}|^2\, d\mu_{g_i}\right)^{1/2} \\ &> \sqrt{1-32\mathcal{A}^2\mathcal{V}\delta^4} - \sqrt{500\mathcal{A}^2\mathcal{V}e^{2\mathcal{B}}\delta^2}. \end{aligned} \tag{22}$$

Therefore

$$S_{\delta,i} \triangleq \frac{V_{\delta,i}-W_{\delta,i}}{(\int_{M_i} |V_{\delta,i}-W_{\delta,i}|^2\, d\mu_{g_i})^{1/2}}$$

is a well defined holomorphic section of $K_{M_i}^{-1}$.

Direct computation shows that $W_{\delta,i}$ satisfies the inequality

$$\begin{aligned} \Delta(|W_{\delta,i}|^2) &= |\nabla W_{\delta,i}|^2 + |\bar\nabla W_{\delta,i}|^2 - R|W_{\delta,i}|^2 + 2\,\mathrm{Re}\,\langle W_{\delta,i}, \bar\partial^*\bar\partial W_{\delta,i}\rangle \\ &= |\nabla W_{\delta,i}|^2 + |\bar\nabla W_{\delta,i}|^2 - R|W_{\delta,i}|^2 + 2\,\mathrm{Re}\,\langle W_{\delta,i}, \bar\partial^*\bar\partial V_{\delta,i}\rangle \\ &\ge |\nabla W_{\delta,i}|^2 + |\bar\nabla W_{\delta,i}|^2 - (R+1)|W_{\delta,i}|^2 - |\bar\partial^*\bar\partial V_{\delta,i}|^2 \\ &\ge |\nabla W_{\delta,i}|^2 + |\bar\nabla W_{\delta,i}|^2 - 2\mathcal{B}\left\{|W_{\delta,i}|^2 + \frac{1}{2\mathcal{B}}|\bar\partial^*\bar\partial V_{\delta,i}|^2\right\}. \end{aligned} \tag{23}$$

All geometric quantities are computed under the metric g_i and complex structure J_i. Letting $f = |W_{\delta,i}|^2 + \frac{1}{2\mathcal{B}}\sup_{\phi_i(U_{\frac{r}{2}})} |\bar\partial^*\bar\partial V_{\delta,i}|^2$, we have $\Delta f \ge -2\mathcal{B} f$ on $\phi_i(U_{r/2})$. Applying local Moser iteration on $\phi_i(U_{r/2})$, we obtain

$$\begin{aligned} \|f\|_{C^0(\phi_i(U_r))} &\le C'(r,\mathcal{B},\mathcal{A})\|f\|_{L^{n/(n-1)}(\phi_i(U_{r/2}))} \\ &= C'(r,\mathcal{B},\mathcal{A})\left\{\left\||W_{\delta,i}|^2\right\|_{L^{n/(n-1)}(\phi_i(U_{r/2}))} + \frac{1}{2\mathcal{B}}\sup_{\phi_i(U_{r/2})} |\bar\partial^*\bar\partial V_{\delta,i}|^2\right\}. \end{aligned}$$

Since $\sup_{\phi_i(U_{r/2})} |\bar\partial^*\bar\partial V_{\delta,i}|^2$ tends to 0 uniformly, we have

$$\left\||W_{\delta,i}|^2\right\|_{C^0(\phi_i(U_r))} \le C''(r,\mathcal{B},\mathcal{A})\left\||W_{\delta,i}|^2\right\|_{L^{n/(n-1)}(\phi_i(U_{r/2}))}. \tag{24}$$

On the other hand, inequality (23) can be written as

$$|\nabla W_{\delta,i}|^2 + |\bar\nabla W_{\delta,i}|^2 \le \Delta(|W_{\delta,i}|^2) + 2\mathcal{B}|W_{\delta,i}|^2 + |\bar\partial^*\bar\partial V_{\delta,i}|^2.$$

Combining this with the Sobolev inequality, we can apply a cutoff function on $\phi_i(U_{r/4} \setminus U_{r/2})$ to obtain

$$\left\| |W_{\delta,i}|^2 \right\|_{L^{n/(n-1)}(\phi_i(U_{r/2}))} \leq C'''(r, \mathcal{B}, \hat{M}) \left\{ \left\| |W_{\delta,i}|^2 \right\|_{L^1(\phi_i(U_{r/4}))} + \sup_{\phi_i(U_{r/4})} |\bar{\partial}^* \bar{\partial} V_{\delta,i}|^2 \right\}.$$

Together with (24), the fact that $\sup_{\phi_i(U_{r/4})} |\bar{\partial}^* \bar{\partial} V_{\delta,i}|^2 \to 0$ implies that

$$\begin{aligned} \left\| |W_{\delta,i}|^2 \right\|_{C^0(\phi_i(U_r))} &\leq C''''(r, \mathcal{B}, \mathcal{A}, \hat{M}) \left\| |W_{\delta,i}|^2 \right\|_{L^1(\phi_i(U_{r/4}))} \\ &\leq C''''(r, \mathcal{B}, \mathcal{A}, \hat{M}) \left\| |W_{\delta,i}|^2 \right\|_{L^1(M_i)} \leq C(r, \mathcal{B}, \mathcal{A}, \mathcal{V}, \hat{M}) \delta^2. \end{aligned}$$

The last inequality follows from (21) and the fact that $|u_i| < \mathcal{B}$.

Fixing r, δ and letting $i \to \infty$, we have

$$\lim_{i\to\infty} \phi_i^*(S_{\delta,i}) = \frac{\hat{S} + \hat{W}_r}{\lim_{i\to\infty}(\int_{M_i} |V_{\delta,i} - W_{\delta,i}|^2 \, d\mu_{g_i})^{1/2}} \tag{25}$$

on the domain U_r, where $\hat{W}_r$ is a holomorphic section of $H^0(K_{U_r}^{-1})$ with $\|\hat{W}_r\|_{C^0(U_r)} \leq C\delta$. It follows from (22) and (25) that $\lim_{\delta\to 0} \lim_{i\to\infty} \phi_i^*(S_{\delta,i}) = \hat{S}$ on d U_r. Letting $\delta_k = 2^{-k}$ and taking a diagonal sequence, we obtain $\lim_{k\to\infty} \phi_{i_k}^*(S_{2^{-k},i_k}) = \hat{S}$ on U_r. Then letting $r = 2^{-l}$ and taking a diagonal sequence once again, we obtain a sequence of holomorphic sections $S_l \triangleq S_{2^{-k_l},i_{k_l}}$ such that

$$\lim_{l\to\infty} \phi_l^*(S_l) = \hat{S} \quad \text{on } \hat{M} \setminus \mathcal{P}.$$

Since every S_l is a holomorphic section (with respect to $(\phi_l^{-1})_* \circ J_l \circ (\phi_l)_*$), Cauchy's integration formula implies that this convergence is actually in the C^∞ topology. □

3.3. *Criteria for tameness*

In this section, we show when the Kähler Ricci flow is tamed.

Theorem 3.2. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow satisfying the following conditions:*

- *Volume ratio bounded from above, i.e., there exists a constant K such that*

$$\mathrm{Vol}_{g(t)}(B_{g(t)}(x, r)) \leq K r^{2n}$$

  for every geodesic ball $B_{g(t)}(x, r)$ satisfying $r \leq 1$.
- *Weak compactness, i.e., for every sequence $t_i \to \infty$, after passing to subsequence, we have*

$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g}),$$

  where $(\hat{M}, \hat{g})$ is a Q-Fano metric-normal variety.

Then this flow is tamed.

Proof. If this result is false, then for every $p_i = i!$, F_{p_i} is an unbounded function on $M \times [0, \infty)$. By Corollary 3.1, F_{p_i} has no lower bound. Therefore, there exists a point (x_i, t_i) such that

$$F_{p_i}(x_i, t_i) < -p_i. \tag{26}$$

By weak compactness, we can assume that

$$(M, g(t_i)) \xrightarrow{C^\infty} (\hat{M}, \hat{g}).$$

Moreover, as $\hat{M}$ is a Q-Fano variety, we can assume $e^{\nu F_\nu(y)} = \sum_{\alpha=0}^{N_\nu} |S_{\nu,\alpha}(y)|^2_{\hat{\omega}^\nu} > c_0$ on $\hat{M}$. Applying Theorem 3.1 and Corollary 3.2, we have

$$\lim_{i\to\infty} e^{\nu F_\nu(x_i,t_i)} > \frac{1}{2}c_0.$$

It follows that there are holomorphic sections $S_\nu^{(t_i)} \in H^0(K_M^{-\nu})$ satisfying

$$\int_M |S_\nu^{(t_i)}|^2_{h_{t_i}^\nu} \omega_{t_i}^n = 1, \qquad |S_\nu^{(t_i)}|^2_{h_{t_i}^\nu}(x_i) = e^{\nu F_\nu(x_i,t_i)} > \frac{1}{2}c_0.$$

According to Lemma 3.1, there is a constant C depending only on this flow such that

$$|S_\nu^{(t_i)}|_{h_{t_i}^\nu} < C\nu^{n/2}.$$

So we have

$$A \triangleq \int_M |(S_\nu^{(t_i)})^k|^2_{h_{t_i}^{k\nu}} \omega_{t_i}^n < VC^{2k}\nu^{nk}.$$

Therefore, $A^{-1/2}(S_\nu^{(t_i)})^k$ are unit sections of $H^0(K_M^{-k\nu})$. It follows that

$$\begin{aligned} e^{k\nu F_{k\nu}(x_i,t_i)} &\geq |A^{-1/2}(S_\nu^{(t_i)})^k|^2_{h_{t_i}^{k\nu}}(x_i) \geq V^{-1}C^{-2k}\nu^{-nk}|(S_\nu^{(t_i)})^k|^2_{h_{t_i}^{k\nu}}(x_i) \\ &\geq V^{-1}C^{-2k}\nu^{-nk}(c_0/2)^k. \end{aligned}$$

This implies that

$$k\nu \cdot F_{k\nu}(x_i, t_i) \geq -2k \log C - nk \log \nu + k \log(c_0/2) - \log V$$

for large i (depending on ν) and every k. Let $k = p_i/\nu = i!/\nu$. By (26), we have

$$-k^2\nu^2 = -p_i^2 > p_i F_{p_i}(x_i, t_i) = k\nu \cdot F_{k\nu}(x_i, t_i) \geq -2k \log C - nk \log \nu + k \log(c_0/2).$$

However, this is impossible for large k. □

In Theorem 4.4 of [CW3], we have proved the weak compactness of the Kähler Ricci flow on Fano surfaces, i.e., every sequence of evolving metrics of a Kähler Ricci flow solution on a Fano surface sub-converges to a KRS orbifold in the Cheeger–Gromov topology. Moreover, the volume ratio upper bound was proved as a lemma to prove the weak compactness. As an application of this property, we obtain

Corollary 3.3. *If* $\{(M^2, g(t)), 0 \leq t < \infty\}$ *is a Kähler Ricci flow on a Fano surface* M^2*, then it is tamed.*

Proof. According to Theorem 4.4 of [CW3], every weak limit $(\hat{M}, \hat{g})$ is a KRS orbifold with finite singularities. It has positive first Chern class and it can be embedded into projective space by its pluri-anticanonical line bundle sections (cf. [Baily]). Clearly, finite singularities have Minkowski codimension 4. Therefore, every $(\hat{M}, \hat{g})$ is a Q-Fano metric-normal variety, so Theorem 3.2 applies. □

In [RZZ], Weidong Ruan, Yuguang Zhang and Zhenlei Zhang proved that the Riemannian curvature is uniformly bounded along the Kähler Ricci flow on M^n $(n \geq 3)$ if $\int_M |\mathrm{Rm}|^n \, d\mu$ is uniformly bounded. Under the latter condition, every sequential limit is a smooth Kähler Ricci soliton manifold, therefore Theorem 3.2 applies and we have

Corollary 3.4. *Suppose* $\{(M^n, g(t)), 0 \leq t < \infty\}$ *is a Kähler Ricci flow along a Fano manifold* M^n *and* $n \geq 3$*. If*

$$\sup_{0 \leq t < \infty} \int_M |\mathrm{Rm}|^n_{g(t)} \, d\mu_{g(t)} < \infty,$$

then the flow is tamed.

4. The Kähler Ricci flow on Fano surfaces

In this section, we give an application of the theorems we developed.

4.1. Convergence of 2-dimensional Kähler Ricci flow

Since the convergence of 2-dimensional Kähler Ricci flow was studied in [CW1] and [CW2] for all cases except $c_1^2(M) = 1$ or 3, we concentrate on these two remaining cases in this section.

Lemma 4.1. *Suppose* M *is a Fano surface.*

- *If* $c_1^2(M) = 1$*, then* $\alpha_x(S) \geq 5/(6\nu)$ *for every* $S \in H^0(K_M^{-\nu})$ *and* $x \in M$*.*
- *If* $c_1^2(M) = 3$*, then* $\alpha_x(S) \geq 2/(3\nu)$ *for every* $S \in H^0(K_M^{-\nu})$ *and* $x \in M$*. Moreover, if* $\alpha_x(S_1) = \alpha_x(S_2) = 2/(3\nu)$*, then* $S_1 = \lambda S_2$ *for some constant* λ*.*

As a corollary, we have the following lemma.

Lemma 4.2. *Suppose* M *is a Fano surface.*

- *If* $c_1^2(M) = 1$*, then* $\alpha_{\nu,1} \geq 5/6$*.*
- *If* $c_1^2(M) = 3$*, then* $\alpha_{\nu,1} = 2/3$ *and* $\alpha_{\nu,2} > 2/3$*.*

Because of Lemma 4.2 and Corollary 3.3, we can apply Theorem 2.2 or Theorem 2.3 respectively to obtain the following theorem.

Theorem 4.1. *If M is a Fano surface with $c_1^2(M) = 1$ or 3, then the Kähler Ricci flow on M converges to a KE metric exponentially fast.*

Combining this with the result in [CW1] and [CW2], we have proved the following result by the Ricci flow method.

Theorem 4.2. *Every Fano surface M has a KRS metric. This metric is a KE metric if and only if $\mathrm{Aut}(M)$ is reductive.*

In particular, we have proved the Calabi conjecture on Fano surfaces by the flow method. This conjecture was first proved by Tian [Tian90] via the continuity method.

Remark 4.1. In [Chl], Cheltsov proved the following fact. Unless M is a cubic surface with bad symmetry and with Eckardt point (a point lying on three exceptional lines), there exists a finite group G such that $\alpha_G(M) > 2/3$ for every M satisfying $c_1^2(M) \leq 5$. Using this fact, we obtain the convergence of the Kähler Ricci flow on M directly if $M \sim \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$. We thank Tian and Cheltsov for pointing this out to us. However, for consistency, we still give an independent proof for the convergence of the Kähler Ricci flow on $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ without applying this fact.

4.2. Calculation of local α-invariants

In this subsection, we give an elementary proof of Lemma 4.1.

4.2.1. Local α-invariants of holomorphic sections of K_M^{-1}

Proposition 4.1. *Let $S \in H^0(\mathbb{CP}^2, 3H)$, $Z(S)$ be the divisor generated by S, and $x \in Z(S)$. Then $\alpha_x(S)$ is totally determined by the singularity type of x. It is classified as in Table 1 below.*

Proof. Direct computation. □

Proposition 4.2. *Suppose M is a Fano surface with $M \sim \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$, and $S \in H^0(K_M^{-1})$. Then $\alpha_x(S) \geq 2/3$ for every $x \in Z(S)$. Moreover, if $S_1, S_2 \in H^0(K_M^{-1})$ and $\alpha_x(S_1) = \alpha_x(S_2) = 2/3$, then there exists a nonzero constant λ such that $S_1 = \lambda S_2$.*

Proof. Let M be the blowup of $\mathbb{CP}^2$ at points $p_1, \ldots, p_6$ in generic position. Let $\pi : M \to \mathbb{CP}^2$ be the blowdown map. If $S \in H^0(K_M^{-1})$, then $\pi_*(Z(S))$ must be a cubic curve γ (maybe reducible) in $\mathbb{CP}^2$ and it must pass through every point p_i. It cannot contain any triple line. Indeed, assume it contains a triple line connecting p_1 and p_2. Then $Z(S) = 3H - aE_1 - bE_2$ for some $a, b \in \mathbb{Z}^+$. On the other hand, we know $Z(S) = 3H - \sum_{i=1}^6 E_i$, a contradiction.

Since no three p_i's are in the same line, a similar argument shows that there is no double line in $\pi_*(Z(S))$.

Table 1. Local α invariants of holomorphic anticanonical sections on projective plane

$\alpha_x(S)$	Singularity type of x	S's typical local equation
1	smooth	z
	transversal intersection of two smooth curves	zw
	ordinary double point	$z^2 - w^2(w+1)$
5/6	cusp	$z^3 - w^2$
3/4	tangential intersection of a line and a conic curve	$z(z+w^2)$
2/3	intersection of three different lines	$zw(z+w)$
1/2	a point on a double line	z^2
1/3	a point on a triple line	z^3

So Table 1 implies $\alpha_x(S) = \alpha_{\pi(x)}(\pi_*(S)) \geq 2/3$ whenever $\pi(x) \in \mathbb{CP}^2 \setminus \{p_1, \ldots, p_6\}$. Therefore we only need to consider singular points $x \in \pi^{-1}(\{p_1, \ldots, p_k\})$. Without loss of generality, we assume $x \in \pi^{-1}(p_1)$ and x is a singular point of $Z(S)$. We analyze this situation according to the singularity type of $\pi_*(x)$. Actually, x is a singular point of $Z(S)$ only if $\pi_*(x)$ is a singular point of $\pi_*(Z(S))$. By Table 1, we have the following classification.

1. $\pi_*(x) = p_1$ is an intersection point of three different lines. This case cannot happen: if such three lines exist, one of them must pass through three blowup points, impossible.
2. $\pi_*(x)$ is an intersection point of two different lines. In this case, x must be a transversal intersection point of a curve and the exceptional divisor E_1. Therefore, $\alpha_x(S) = 1$.
3. $\pi_*(x)$ is a cusp point. In this case, x must be a tangential intersection point of a smooth curve and the exceptional divisor E_1. Moreover, the tangential order is just 1. So $\alpha_x(S) = 3/4$.
4. $\pi_*(x)$ is a tangential intersection point of a line and a conic curve. In this case, x is a transversal intersection point of three curves $\gamma_1, \gamma_2, \gamma_3$ with $[\gamma_1] = E_1, [\gamma_2] \sim 2H - \sum_{l=1}^{6} E_l + E_j, [\gamma_3] \sim H - E_1 - E_j$ for some $j \in \{2, \ldots, 6\}$. Thus x is an intersection point of three exceptional lines. Clearly, $\alpha_x(S) = 2/3$.

Therefore, $\alpha_x(S) \geq 2/3$ for any $x \in \pi^{-1}\{p_1, \ldots, p_6\}$. Moreover, $\alpha_x(S) = 2/3$ only if x is a transversal intersection point of three exceptional lines.

It is well known that M is a cubic surface, and there are in total 27 exceptional lines on M. Each point can lie on at most three exceptional lines. Therefore, if $\alpha_x(S_1) = \alpha_x(S_2) = 2/3$, then $Z(S_1) = Z(S_2)$ as the union of three exceptional lines passing through x. So there is a nonzero constant λ such that $S_1 = \lambda S_2$. □

Proposition 4.3. *Suppose M is a Fano surface with $M \sim \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$, and $S \in H^0(K_M^{-1})$. Then $\alpha_x(S) \geq 5/6$ for every $x \in Z(S)$.*

Proof. In the notation of the proof of Proposition 4.2, we see $\pi_*(Z(S))$ is a cubic curve. Suppose $\pi_*(Z(S))$ is reducible; then $Z(S) = \gamma_1 + \gamma_2$ with γ_1 a line and γ_2 a conic curve.

So $Z(S)$ can pass through at most $2+5=7$ blowup points. On the other hand, it must pass through all of them, a contradiction. Therefore, $\pi_*(Z(S))$ is irreducible.

If $\pi(x) \in \mathbb{CP}^2 \setminus \{p_1, \dots, p_8\}$, we have $\alpha_x(S) = \alpha_{\pi(x)}(\pi_*(S)) \geq 5/6 > 2/3$ by Table 1. Suppose $\pi(x) \in \{p_1, \dots, p_8\}$. These eight points are in generic position. No cubic curve passes through seven of them with one point doubled. Since $\pi_*(Z(S))$ is a cubic curve passing through all of these eight points, it must pass through every point smoothly. It follows that x is a smooth point of $Z(S)$. So $\alpha_x(S) = 1$. □

4.2.2. Local α-invariants of holomorphic sections of $K_M^{-\nu}$

Proposition 4.4. *If f, g are holomorphic functions (or holomorphic sections of a line bundle) defined in a neighborhood of x, then $\alpha_x(fg) \geq \frac{\alpha_x(f)\alpha_x(g)}{\alpha_x(f)+\alpha_x(g)}$, i.e.,*

$$\frac{1}{\alpha_x(fg)} \leq \frac{1}{\alpha_x(f)} + \frac{1}{\alpha_x(g)}. \tag{27}$$

Proof. Without loss of generality, we can assume $\alpha_x(f), \alpha_x(g) < \infty$. For simplicity of notation, let $a = \alpha_x(f)$, $b = \alpha_x(g)$, $c = \frac{ab}{a+b}$. We only need to prove $\alpha_x(fg) \geq c$.

Fix a small number $\epsilon > 0$. Since $c/a + c/b = 1$, the Hölder inequality implies

$$\int_U (fg)^{-2c(1-\epsilon)}\, d\mu = \left(\int_U f^{-2a(1-\epsilon)}\, d\mu\right)^{c/a} \left(\int_U g^{-2b(1-\epsilon)}\right)^{c/b} < \infty,$$

where U is some neighborhood of x. Therefore, $\alpha_x(fg) \geq c(1-\epsilon)$. As ϵ can be arbitrarily small, we have $\alpha_x(fg) \geq c$. □

As an application of Proposition A.1.1 of [Tian90], we state the following property without proof.

Proposition 4.5. *Suppose f is a holomorphic function vanishing at x with order k. In a small neighborhood, we can express f as*

$$f = a_{ij} z_1^i z_2^j + \cdots .$$

Without loss of generality, we can assume that there is a pair (i, j) such that $i \geq j$, $i + j = k$ and $a_{ij} \neq 0$. Then $\alpha_x(f) \geq 1/i$.

Lemma 4.3. *Suppose M is a cubic surface, and let $S \in H^0(K_M^{-m})$, $x \in M$. If $\alpha_x(S) \leq 2/(3m)$, then $\alpha_x(S) = 2/(3m)$ and $S = (S')^m$ where $S' \in H^0(K_M^{-1})$ and $Z(S')$ is the union of three lines passing through x.*

Proof. We will prove this statement by induction. Suppose we have already proved it for all $k \leq m-1$; now we show it is true for $k = m$.

Claim 1. *If S splits off an anticanonical holomorphic section S', then $Z(S')$ must be the union of three lines passing through x. Moreover, $S = (S')^m$.*

Suppose $S = S' S_{m-1}$ where $S' \in H^0(K_M^{-1})$ and $S_{m-1} \in H^0(K_M^{-(m-1)})$. Since $\alpha_x(S') \geq 2/3$ and $\alpha_x(S_{m-1}) \geq 2/(3(m-1))$, by induction assumption, inequality (27) implies

$$\frac{3m}{2} \leq \frac{1}{\alpha_x(S)} \leq \frac{1}{\alpha_x(S')} + \frac{1}{\alpha_x(S_{m-1})} \leq \frac{3}{2} + \frac{3(m-1)}{2} = \frac{3m}{2}.$$

This forces that

$$\alpha_x(S) = \frac{2}{3m}, \quad \alpha_x(S') = \frac{2}{3}, \quad \alpha_x(S_{m-1}) = \frac{2}{3(m-1)}.$$

Therefore the induction hypothesis tells us that $Z(S')$ is the union of three lines passing through x, $S_{m-1} = (S'')^{m-1}$ and $Z(S'')$ is the union of three lines passing through x. As there are at most three lines passing through x on a cubic surface, we see $Z(S') = Z(S'')$. By changing coefficients if necessary, we have $S_{m-1} = (S')^{m-1}$. It follows that $S = (S')^m$ and we have finished the proof.

Claim 2. *There must be a line passing through x.*

Otherwise, there is a pencil of anticanonical divisors passing through x. In this pencil, a generic divisor is irreducible and it vanishes at x to order 2. Choose such a divisor and denote it by $Z(S')$. Locally, we can represent S by a holomorphic function f. Since $\alpha_x(f) = \alpha_x(S) \leq 2/(3m)$, we see $\mathrm{mult}_x(f) \geq \lceil 3m/(2) \rceil$. If m is odd, then $Z(S') \nsubseteq Z(S)$ implies

$$3m = K_M^{-m} \cdot K_M^{-1} \geq 2\mathrm{mult}_x(f) \geq 3m + 1,$$

impossible. Since $Z(S')$ is irreducible, we have $Z(S') \subset Z(S)$. Therefore, $S = S' S_{m-1}$. According to Claim 1, $Z(S')$ is the union of three lines and therefore $Z(S')$ is reducible. This contradicts the assumption on $Z(S')$.

So m must be an even number and $\mathrm{mult}_x(f) = 3m/2$ exactly. Now f can be written as

$$\sum_{i,j \geq 0} a_{ij} z_1^i z_2^j, \quad a_{ij} = 0 \quad \text{whenever} \quad i + j < 3m/2.$$

Using the fact that $\alpha_x(f) \leq 2/(3m)$, Proposition 4.5 implies $a_{ij} = 0$ whenever $i < 3m/2$, and $a_{3m/2,0} \neq 0$. Therefore locally f can be written as $a_{3m/2,0} z_1^{3m/2} \cdot h$ for some nonzero holomorphic function h. This means that $Z(S)$ contains a curve γ with multiplicity $3m/2$. Project $Z(S)$ to $\mathbb{CP}^2$ by the map π. The image curve $\pi(\gamma)$ must be of degree one in $\mathbb{CP}^2$, since otherwise $\pi(Z(S)) = \pi(\gamma)$ does not pass through other blowup points. Since $\pi(\gamma)$ is of degree one, it can only pass through two of the blowup points. Moreover, it vanishes at each of these two points to order $3m/2 > m$, impossible.

Claim 3. *The number of lines passing through x is greater than* 1.

Otherwise, there is exactly one line L_1 passing through x. So there is an irreducible degree 2 curve D passing through x such that $L_1 + D = Z(S')$ for some $S' \in H^0(K_M^{-1})$. Locally, we can write S as $l_1 h$ where l_1 is the defining function for L_1. As $\alpha_x(l_1) = 1$, the Hölder inequality implies that $\alpha_x(h) \le 2/(3m-2)$. Consequently, $\mathrm{mult}_x(h) \ge \lceil 3m/2 \rceil - 1$. If $2L_1 \not\subseteq Z(S)$, we have

$$m+1 = (K_M^{-m} - L_1)\cdot L_1 \ge \{h=0\}\cdot L_1 \ge \lceil 3m/2 \rceil - 1, \quad \Leftrightarrow m \le 4.$$

If $m > 4$, this inequality is wrong so we have $2L_1 \subset Z(S)$. Actually, using this argument and induction, we can show that $\lceil m/4 \rceil L_1 \subset Z(S)$.

For simplicity of notation, let $p = \lceil m/4 \rceil$. Locally, S can be written as $l_1^p h$. Clearly, $\alpha_x(h) \le 2/(3m-2p)$ and $\mathrm{mult}_x(h) \ge \lceil 3m/2 \rceil - p$. Let f_q and h_{q-2} be the lowest degree terms of f and h respectively. Then we may assume that $h_{q-2} = z_1^{j_1} z_2^{j_2} + \cdots$ and any term $z_1^i z_2^j$ in h_{q-2} satisfies $i \ge j_1$. Now we have two cases to consider.

Case 1: L_1 is tangent to $\{z_1 = 0\}$. If $(p+1)L_1 \not\subseteq Z(S)$, then

$$m+p = (K_M^{-m} - pL_1)\cdot L_1 \ge \{h=0\}\cdot L_1 \ge (\lceil 3m/2 \rceil - p)\cdot 2, \quad \Leftrightarrow p \ge \frac{m + 2\lceil m/2 \rceil}{3}.$$

Here we use the fact that $j_1 \ge \lceil 3m/2 \rceil - p$ since $\alpha_x(h) \le 2/(3m-2p)$. This contradicts our definition $p = \lceil m/4 \rceil$. Therefore, $(p+1)L_1 \subset Z(S)$.

Case 2. L_1 is not tangent to $\{z_1 = 0\}$. In this case, $f_q = \lambda z_1^{a_1} z_2^{a_2+p} + \cdots$ for some $\lambda \neq 0$. Moreover, every $z_1^i z_2^j$ in f_q satisfies $i \ge j_1$. Therefore, the fact that $\alpha_x(S) \le 2/(3m)$ and Proposition 4.5 imply $a_1 \ge \lceil 3m/2 \rceil$. It follows that $q \ge p + \lceil 3m/2 \rceil$. Under these conditions, if $(p+1)L_1 \not\subseteq Z(S)$, we have

$$m+p = (K_M^{-m} - pL_1)\cdot L_1 \ge \lceil 3m/2 \rceil, \quad \Leftrightarrow p \ge \lceil m/2 \rceil.$$

If $m \ge 3$ this is impossible, so $(p+1)L_1 \subset Z(S)$. If $m \le 2$, as $\lceil m/2 \rceil = \lceil m/4 \rceil = 1$, we already know $\lceil m/2 \rceil L_1 \subset Z(S)$.

Therefore by repeatedly rewriting S in local charts and considering Cases 1 and 2, we can prove $\lceil m/2 \rceil L_1 \subset Z(S)$. Furthermore, we have proved the following property.

- *Suppose S can be written as $l_1^n h'$ locally with $n = \lceil m/2 \rceil$. Then either $(n+1)L_1 \subset Z(S)$ or L_1 is not tangent to* $\{z_1 = 0\}$.

From this property we can show $D \subset Z(S)$. In fact, if $(n+1)L_1 \subset Z(S)$ and $D \not\subseteq Z(S)$, we have

$$\begin{aligned} 2m = K_M^{-m}\cdot D &\ge (n+1)L_1 \cdot D + \mathrm{mult}_x(h'') \\ &\ge 2(n+1) + (\lceil 3m/2 \rceil - (n+1)), \quad \Leftrightarrow m \ge 2n+1, \end{aligned}$$

where h'' is a function such that locally S is represented by $l_1^{n+1} h''$. This inequality is impossible as $n = \lceil m/2 \rceil$. If L_1 is not tangent to $\{z_1 = 0\}$, we know $\mathrm{mult}_x(h') \ge n + \lceil 3m/2 \rceil = m + 2n$. Therefore, $D \not\subseteq Z(S)$ implies that

$$2m = K_M^{-m}\cdot D \ge nL_1 \cdot D + \mathrm{mult}_x(h') \ge m + 4n, \quad \Leftrightarrow m \ge 4\lceil m/2 \rceil,$$

impossible. Therefore, no matter which case happens, we have $D \subset Z(S)$. So $D + L_1 \subset Z(S)$. It follows that S splits off an $S' \in H^0(K_M^{-1})$ with $Z(S') = L_1 + D$, contrary to Claim 1.

Claim 4. *The number of lines passing through x is greater than* 2.

Otherwise, there are only two lines L_1 and L_2 passing through x. There is a unique line L_3 not passing through x such that $L_1 + L_2 + L_3 \in K_M^{-1}$. We first prove the following property:

$$k(L_1 + L_2) \subset Z(S) \quad \text{for all } 0 \leq k \leq n = \lceil m/2 \rceil.$$

By induction, we can assume $(k-1)(L_1 + L_2) \in Z(S)$. Then S can be represented by a holomorphic function $f = l_1^{k-1} l_2^{k-1} h$ locally. Noting that $\alpha_x(l_1^{k-1} l_2^{k-1}) = 1/(k-1)$, the Hölder inequality implies

$$\alpha_x(h) \leq \frac{\frac{2}{3m}}{1 - \frac{2(k-1)}{3m}} = \frac{2}{3m - 2(k-1)}.$$

It follows that

$$\mathrm{mult}_x(h) \geq \lceil 3m/2 \rceil + 1 - k = m + n + 1 - k.$$

If $kL_1 \nsubseteq Z(S)$, we have

$$\begin{aligned} m &= (K_M^{-m} - (k-1)(L_1 + L_2)) \cdot L_1 \geq \{h = 0\} \cdot \{l_1 = 0\} \\ &\geq \mathrm{mult}_x(h) \geq m + n + 1 - k, \quad \Leftrightarrow k \geq n + 1. \end{aligned}$$

This contradicts the assumption on k. Therefore, $kL_1 \subset Z(S)$. Similarly, $kL_2 \subset Z(S)$. So $k(L_1 + L_2) \subset Z(S)$.

Now locally S can be written as $l_1^n l_2^n h$. We have $\alpha_x(h) \leq 2/(3m - 2n)$ and $\mathrm{mult}_x(h) \geq \lceil 3m/2 - n \rceil = m$. Assume $\mathrm{mult}_x(h) = m$. In local coordinates, $h = \sum_{i,j \geq 0} a_{ij} z_1^i z_2^j$. Since $\alpha_x(h) \leq 2/(3m - 2n)$, Proposition 4.5 implies $a_{ij} = 0$ whenever $i < \lceil 3m/2 - n \rceil = m$. Since $\mathrm{mult}_x(h) = m$, we see that the lowest homogeneous term of f is of the form $l_1^n l_2^n z_1^m$. The condition $\alpha_x(S) \leq 2/(3m) < 1/m$ implies that either L_1 or L_2, say L_1, is tangent to $\{z_1 = 0\}$ at x. If $(n+1)L_1 \nsubseteq Z(S)$, we have

$$\begin{aligned} m + n &= (K_M^{-m} - nL_1) \cdot L_1 \geq \{l_2^n h = 0\} \cdot L_1 \\ &\geq n + \left\{ \sum_{i,j \geq 0} a_{ij} z_1^i z_2^j = 0 \right\} \cdot L_1 \geq n + \inf\{2i + j \mid a_{ij} \neq 0\} \geq n + 2m, \end{aligned}$$

impossible. It follows that $(n+1)L_1 + nL_2 \subset Z(S)$.

Now we consider L_3. If $L_3 \nsubseteq Z(S)$, we have

$$m = L_3 \cdot K_M^{-m} \geq ((n+1)L_1 + nL_2) = 2n + 1 = 2\lceil m/2 \rceil + 1.$$

This absurd inequality implies $L_3 \subset Z(S)$. Let $S' \in K_M^{-1}$ be such that $Z(S') = L_1 + L_2 + L_3$. We have $S = S' S_{m-1}$. However, $Z(S')$ is not the union of three lines passing through x. This contradicts Claim 1.

Thus, there exist three lines $L_1, L_2, L_3 \subset Z(S)$ passing through x. Since M is a cubic surface, there exists an $S' \in H^0(K_M^{-1})$ such that $Z(S') = L_1 + L_2 + L_3$. As we argued in Claim 4, $L_1, L_2, L_3 \subset Z(S)$. Therefore $L_1 + L_2 + L_3 \subset Z(S)$ and S splits off an anticanonical holomorphic section S'. By Claim 1, we have $S = (S')^m$. □

Similarly, we can prove the following property by induction.

Lemma 4.4. *Suppose M is a Fano surface with $M \sim \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$, and let $S \in H^0(K_M^{-m})$ and $x \in M$. Then $\alpha_x(S) \geq 5/(6m)$ for every $x \in M$.*

Proof. Suppose we have proved this statement for all $k \leq m-1$.

Suppose this statement is wrong for $k = m$. Then there is a holomorphic section $S \in H^0(K_M^{-m})$ and a point $x \in M$ such that $\alpha_x(S) < 5/(6m)$. Let f be a local holomorphic function representing S. Clearly, $\mathrm{mult}_x(f) > 6m/5$. Choose $S' \in H^0(K_M^{-1})$ such that $x \in Z(S')$. Since S' is irreducible, if $Z(S') \nsubseteq Z(S)$, we have

$$m = Z(S) \cdot Z(S') > 6m/5,$$

impossible. Therefore, $Z(S') \subset Z(S)$. It follows that $S = S' S_{m-1}$ for some $S_{m-1} \in H^0(K_M^{-1})$. So Proposition 4.5 implies

$$\alpha_x(S) \geq \frac{\frac{5}{6} \cdot \frac{5}{6(m-1)}}{\frac{5}{6} + \frac{5}{6(m-1)}} = \frac{5}{6m},$$

contrary to the assumption on $\alpha_x(S)$. □

Lemma 4.1 is a combination of Lemmas 4.3 and 4.4.

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