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Spectral Flow, Twisted Modules, and MLDE of Quasi-Lisse Vertex Algebras

by

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Abstract

We calculate the dimensions of intertwining operators among highest weight modules and their contragredient modules of $L_k(\mathfrak{sl}_2)$ at the admissible level. We derive a series of $\Gamma^0(2)$-MLDEs satisfied by normalized characters of some ordinary twisted modules of some quasi-lisse vertex algebras. Examples include simple affine vertex algebras $L_{-2+2/u}(\mathfrak{sl}_2)$ $(u = 3, 5, 7)$, $L_{-1/2}(\mathfrak{sl}_2)$, $L_{-3/2}(\mathfrak{sl}_3)$, $L_{-2}(\mathfrak{so}_8)$, and the Bershadsky–Polyakov algebra $\mathrm{BP}^{-9/4}$. We also derive the full characters of some nonvacuum modules of $L_{-2}(\mathfrak{so}_8)$ using the spectral flow automorphism and SCFT/VOA correspondence.

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§1. Introduction

Four-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) in physics have rich mathematical structures. In [BLL+15], the authors propose a correspondence between the Schur sectors of $4d$ $\mathcal{N} = 2$ SCFTs and $2d$ vertex operator algebras (VOAs). This correspondence has fueled a lot of work in the past years, including some conjectures about the chiral algebra in the context of theories of class $\mathcal{S}$ [Gai12, GMN13, BPRvR15, LP15]. For the genus zero case, the conjecture has been proved in terms of a functorial construction [Ara19]. Class $\mathcal{S}$ theories have the Coulomb branch operators with integral scaling dimension, yet there is another class of $\mathcal{N} = 2$ SCFTs called Argyres–Douglas (AD) theories

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[AD95, Xie13] which usually have fractional scaling dimensions for the Coulomb branch operators. These AD theories can be constructed by compactifying $6d$ $(2,0)$ theory on a Riemann surface with irregular singularities. The corresponding VOAs of a class of AD theories are identified with certain affine vertex algebras $L_k(\mathfrak{g})$ at admissible level k, or affine $\mathcal{W}_k(\mathfrak{g}, f)$-algebras [CS16, BN16b, BN16a, SXY17, XYY21, XY21c]. Dualities of $4d$ theories imply nontrivial isomorphism and collapsing levels of VOAs [XY21c, XY21a, LXY23], some of which were proved rigorously [AVEM24, AMP21].

One consequence of this SCFT/VOA correspondence is that the Schur index of the 4d SCFT is equal to the normalized vacuum character of the corresponding VOA. Hence, the character formula provides a valuable tool to study the spectrum of 4d SCFTs. The character here means that the trace $\mathrm{ch}_\lambda(\tau, z) = \mathrm{tr}_{L(\lambda)}\, e^{2\pi i(\tau(L_0 - c(k)/24) + \frac{1}{2} z h_{(0)})}$ over an irreducible $L_k(\mathfrak{g})$-module $L(\lambda)$. In [KW88], the authors derived character formulas for the irreducible representations of an affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}$ at a rational level k, and also investigated the modular property of these characters. In particular, the transformed character $\mathrm{ch}_\lambda(-\frac{1}{\tau}, \frac{z}{\tau})$ can be written as a linear combination of characters of admissible representations with a shifted conformal vector, while $\mathrm{ch}_\lambda(-\frac{1}{\tau}, z)$ is a linear combination of the characters of some $\mathbb{Z}_2$-twisted modules. The character formulas were used to derive the Schur index of a large class of AD theory [XY21b], while the modular properties also have applications in physics [Raz12].

Another conjecture of the SCFT/VOA correspondence is the identification between the Higgs branch of vacua of an $\mathcal{N} = 2$ SCFT and the associated variety of the corresponding VOA [BR18, SXY17], which were also used to propose lisse VOAs from $4d$ SCFTs [XY21a]. The VOA corresponding to a 4d SCFT is often of the quasi-lisse type whose associated variety has finitely many symplectic leaves. The normalized character of an ordinary representation of a quasi-lisse VOA was shown to satisfy a modular linear differential equation (MLDE), and solving the MLDE gives the explicit expression for the character of the affine Lie algebra associated with the Deligne–Cvitanovic (DC) series [AK18]. The MLDE for VOAs corresponding to several families of AD theories and $\mathcal{N} = 4$ super Yang–Mills with $\mathfrak{su}(n)$ gauge group were also discussed in [BR18]. Recently in [ZPW22] the authors constructed flavored MLDEs for the Schur index of $\mathfrak{a}_1$ class $\mathcal{S}$ theories based on a compact formula for the index they found earlier [PP22]. Both works used the Higgs branch structure to probe the singular vector of the corresponding VOA and then derived the MLDE.

The SCFT/VOA correspondence also goes beyond the Schur index and the vacuum module. One generalization is to consider the lens space index [RY13] instead of the normal index. In [FS18], the lens space index was identified with the characters of the twisted modules. Given an automorphism g of a VOA V of finite order, the basic properties of g-twisted modules for V are systematically studied by Li [Li96] using twisted local systems. In particular, he showed that $\mathbb{Z}_2$-twisted modules of the affine VOA mentioned above can be obtained from its untwisted modules via Li's Δ-operators. Later on, in the important work [DLM00], the authors showed that the trace functions of the g-twisted modules for the C_2-cofinite rational VOA satisfy certain twisted modular linear differential equations (MLDE) and possess some modular invariance properties. Li [Li23] generalized some of the results obtained by Dong, Li, and Mason to the quasi-lisse vertex operator (super)algebra case, as well as proved that characters of twisted modules of quasi-lisse vertex algebras satisfy certain twisted MLDEs. Another generalization is to consider the index in the presence of defects, which were identified with the twisted modules using spectral flow [CS16, CGS16, CGS17]. The spectral flow used in these works has a long history in the conformal field theory (see, e.g., [SS87, LMRS02, Rid09, Rid10, CR12, CR13, KRW22]).

As reviewed above, there is an increasing interest in understanding twisted modules and spectral flowed modules of VOAs from the perspective of SCFT/VOA correspondence, as it provides knowledge of lens space indices and defects of the corresponding SCFTs. Especially if one can show that characters of these modules are solutions of certain MLDEs, closed-form expression might also be within grasp as in the ordinary module case. Since conventional methods in physics usually give only power series, such closed-form expressions are valuable and may review more interesting properties. Mathematically, people introduce the twisted modules of V to study the module category of G invariants V^G, where G is the finite automorphism group of V.

We start with simple affine vertex algebras $L_k(\mathfrak{sl}_2)$ at admissible level. It is easy to see that the Dynkin labels (eigenvalues of $h_{(0)}$ on the highest weight vector) of all $\mathbb{Z}_2$-twisted modules of $L_k(\mathfrak{sl}_2)$ at admissible level in category $\mathcal{O}$ are $\{r - st - \frac{1}{2}k \mid 0 \leq r \leq p-2,\ 0 \leq s \leq q-1\}$. And all irreducible highest weight $\mathbb{Z}_2$-twisted modules of $L_k(\mathfrak{sl}_2)$ at admissible level in category $\mathcal{O}$ can be obtained by using $\sigma^{-\frac{1}{2}}$ spectral flow on the untwisted modules in category $\mathcal{O}$. In particular, all of those irreducible twisted modules are ordinary modules at boundary admissible levels. The category of ordinary $\mathbb{Z}_2$-twisted modules of $L_k(\mathfrak{sl}_2)$ at admissible level is semisimple, following from the property of Li's delta operator [Li97], and [AM95, GK11, Ara16].

Following the idea in [DLM97] we compute the dimension of intertwining operators among modules of $L_k(\mathfrak{sl}_2)$ at admissible level and their contragredient modules.

Theorem 1.1. *For admissible weights $j_i = n_i - 1 - (l_i - 1)t$ $(i = 1, 2)$ of the vertex affine algebra $L_k(\mathfrak{sl}_2)$, when $n_2 - n_1 \geq 0$, the dimension of the intertwining operators of type $\binom{L(k,-j_1+j_2)}{(L(k,j_1))^* \ L(k,j_2)}$ or of type $\binom{L(k,-j_1+j_2)}{L(k,j_2) \ (L(k,j_1))^*}$ is 1. The dimension of the intertwining operators of type $\binom{M}{(L(k,j_1))^* \ L(k,j_2)}$ or of type $\binom{M}{L(k,j_2) \ (L(k,j_1))^*}$ for any other irreducible highest weight modules or their contragredient modules is 0. For brevity, we write this as*

$$(L(k, j_1))^* \times L(k, j_2) = L(k, j_2) \times (L(k, j_1))^* = \begin{cases} L(k, -j_1 + j_2) & \text{if } n_2 - n_1 \geq 0, \\ (L(k, j_1 - j_2))^* & \text{if } n_2 - n_1 < 0. \end{cases}$$

We also show that

$$(L(k, j_1))^* \times (L(k, j_2))^* = \sum_{i=\max\{0,n_1+n_2-p\}}^{\min\{n_1-1,n_2-1\}} (L(k, j_1 + j_2 - 2i))^*.$$

Then, combining results in [DLM97] and [Li97, Theorem 2.15] (see also [AP19, Proposition 3.1]), we read off the dimension of intertwining operators between highest weight modules of $L_k(\mathfrak{sl}_2)$ (or its contragredient dual) and $\mathbb{Z}_n$-twisted modules (see Section 3.5).

We derive specific twisted MLDEs satisfied by the normalized characters of certain ordinary twisted modules over some quasi-lisse VOAs. For $L_{-2+2/u}(\mathfrak{sl}_2)$ $(u = 3, 5, 7)$, if we only consider q-series, the normalized characters of their ordinary $\mathbb{Z}_2$-twisted modules form a complete solutions of a $(u+1)/2$-order $\Gamma^0(2)$-MLDE (see Section 4 for a definition). We also construct two ordinary $\mathbb{Z}_2$-twisted modules for $L_{-1/2}(\mathfrak{sl}_2)$, and show that the characters of these twisted modules form complete solutions of a second-order $\Gamma^0(2)$-MLDE. For $L_{-3/2}(\mathfrak{sl}_3)$, we construct an ordinary $\mathbb{Z}_2$-twisted module by the spectral flow on the vacuum module along $\frac{1}{2}\bar{\Lambda}_1^\vee$ in the weight lattice, whose normalized character satisfies a second-order $\Gamma^0(2)$-MLDE. For the BP^k-algebra with $k = -9/4$, we show that the character of one of its $\mathbb{Z}_2$-twisted modules is a solution of a third-order MLDE in the full $\mathrm{SL}(2, \mathbb{Z})$ group. Finally, we study the characters of the spectral flowed modules of the affine vertex algebra $L_{-2}(\mathfrak{so}_8)$. Since it is nonadmissible, one cannot use the Kac–Wakimoto formula to write down the characters of simple modules directly. Fortunately, we can get all simple modules of $L_{-2}(\mathfrak{so}_8)$ except for $L(-\Lambda_2)$ from the spectral flow along certain directions on the vacuum module $L(-2\Lambda_0)$. In [Ara19] it was proved that the VOA corresponding to genus zero $\mathfrak{a}_1$ class $\mathcal{S}$ theory is $L_{-2}(\mathfrak{so}_8)$. By [PP22, formula (4.45)], one can compute the full character of $L(-2\Lambda_0)$ (the Schur index of the genus one $\mathfrak{a}_1$ class $\mathcal{S}$ theory) rigorously. Then we obtain the full characters of all irreducible modules of $L_{-2}(\mathfrak{so}_8)$ except for $L(-\Lambda_2)$ by using spectral flow twist. We also get one ordinary $\mathbb{Z}_2$-twisted module, whose character satisfies a second-order $\Gamma^0(2)$-MLDE. In general, for the simple affine vertex algebra $L_k(\mathfrak{g})$ with $\mathfrak{g}$ being a Lie algebra of DC series and $k = -h^\vee/6-1$, we make the following conjecture on the $\Gamma^0(2)$-MLDEs of characters of its $\mathbb{Z}_2$-twisted modules.

Conjecture 1.2. *Let V be a simple affine vertex algebras associated with the* DC *series,*

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

at level $-h^\vee/6-1$. For each V, there exists an irreducible ordinary $\mathbb{Z}_2$-twisted module M such that the normalized character $\mathrm{ch}_V[M](q)$ satisfies a second-order $\Gamma^0(2)$*-MLDE*

$$(D_q^{(2)} + a_1\Theta_{0,2} + a_2\Theta_{1,1})\,\mathrm{ch}[M](q) = 0,$$

where $a_1, a_2 \in \mathbb{Q}$.

Remark 1.3. The above conjecture is proved when $\mathfrak{g}$ is a classical Lie algebra in the DC series. The specific form of $\Gamma^0(2)$-MLDEs can be found in Sections 4 and 5.

Remark 1.4. For simple affine VOAs, $L_{-4/3}(\mathfrak{sl}_2)$ and $L_{-3/2}(\mathfrak{sl}_3)$, one has a distinguished ordinary $\mathbb{Z}_2$-twisted module $\sigma^{-\frac{1}{2}\Lambda}(M_{\mathrm{vac}})$, where M_{vac} is a vacuum module and Λ is a specific weight. We believe that this should be true for all affine VOAs $L_k(\mathfrak{g})$ at admissible level k. However, for general V, we do not have a systematic approach to get the ordinary $\mathbb{Z}_2$-twisted modules from irreducible weak V-modules in the category $\mathcal{O}$.

In this work, the usage of the spectral flow is crucial, as we use it to obtain twisted modules and new modules. In particular, we hope to prove Conjecture 1.2 and address some of the questions raised in the above Remark 1.3 in future work.

We summarize the main contents in this note. In Section 2 we recall some basic notions of affine vertex algebras at the admissible level, their (twisted) characters, and the modularity of their characters under the transformations $\tau \to -\frac{1}{\tau}$. In Section 3, we further calculate the dimensions of intertwining operators between the highest weight modules and their contragredient modules for $L_k(\mathfrak{sl}_2)$ at the admissible level. In Section 4 we study the $\mathbb{Z}_2$-twisted modules of some affine VOAs at the admissible level, then derive $\Gamma^0(2)$-MLDEs that their characters satisfy. In Section 5 we discuss the generalization of the result in the previous section to $L_{-2}(\mathfrak{so}_8)$.

§2. Character formulas at admissible level and their modularity

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank r with the standard bilinear form $\langle\cdot,\cdot\rangle$ normalized by $\langle\theta,\theta\rangle = 2$, where θ is the highest root. Let Δ, Δ_+, and Π be roots, positive roots, and simple roots of $\mathfrak{g}$. Let $\{\bar{\Lambda}_1,\dots,\bar{\Lambda}_r\}$ be the fundamental weights for $\mathfrak{g}$. Let $h^\vee$ and ρ be the dual Coxeter number and Weyl vector of $\mathfrak{g}$, respectively. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\hat{\mathfrak{g}} = \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}D$ be the affine Kac–Moody algebra associated with $\mathfrak{g}$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_-\oplus\hat{\mathfrak{h}}\oplus\hat{\mathfrak{n}}_+$ be the triangular decomposition of $\hat{\mathfrak{g}}$. Let $\widehat{\Delta}$, $\widehat{\Delta}_+$, $\widehat{\Delta}_+^{\mathrm{re}}$ be roots, positive roots, positive real roots of $\hat{\mathfrak{g}}$. Let the $\{\Lambda_0,\dots,\Lambda_r\}$ be the fundamental weights of $\hat{\mathfrak{g}}$. Let $\hat{\rho}$ be the affine Weyl vector. Given $\lambda\in\hat{\mathfrak{h}}^*$, the set of λ-integral roots is $\widehat{\Delta}^\lambda := \{\alpha\in\widehat{\Delta}^{\mathrm{re}} \mid \lambda(\alpha^\vee)\in\mathbb{Z}\}$. We define λ to be an *admissible weight* if the following two properties hold:

- $(\lambda+\rho)(\alpha^\vee)\notin\mathbb{Z}_{\le 0}$ for all $\alpha\in\widehat{\Delta}_+^{\mathrm{re}}$,
- $\mathbb{Q}\widehat{\Delta}^\lambda = \mathbb{Q}\widehat{\Delta}$.

If there exists a further isometry ϕ of $\hat{\mathfrak{h}}^*$ such that $\phi(\widehat{\Delta}^\lambda) = \widehat{\Delta}$, λ is called a *principal* admissible weight.

Theorem 2.1 ([KW88]). *Let $\hat{\mathfrak{g}}$ be a Kac–Moody Lie algebra with a symmetrizable generalized Cartan matrix, and $\lambda\in\hat{\mathfrak{h}}^*$ be an admissible weight. Then the character of $L(\lambda)$ is given by the following formula:*

$$\operatorname{ch} L(\lambda) = \frac{1}{R}\cdot\sum_{w\in\widehat{W}^\lambda}\varepsilon(w)e^{w(\lambda+\hat{\rho})},$$

where $R := e^{\hat{\rho}}\prod_{n=1}^\infty(1-q^n)^r\prod_{\alpha\in\widehat{\Delta}_+}(1-e^\alpha q^n)(1-e^{-\alpha}q^{n-1})$ is the Kac–Weyl denominator, and $\widehat{W}^\lambda := \{r_\alpha \mid \alpha\in\widehat{\Delta}^\lambda\}$.

The normalized character of an irreducible highest weight $\hat{\mathfrak{g}}$-module $L(\lambda)$ at level k evaluated at $h = 2\pi i(-\tau D + z + tK)$, is defined as

$$\operatorname{ch}_\lambda(\tau,z,t) = q^{m_\lambda}\operatorname{tr}_{L(\lambda)} e^h,$$

where $z\in\mathfrak{h}$, $\tau,t\in\mathbb{C}$, $\operatorname{Im}\tau>0$, $q = e^{2\pi i\tau}$, $m_\lambda = \frac{|\lambda+\hat{\rho}|^2}{2(k+h^\vee)} - \frac{\dim\mathfrak{g}}{24}$.

It is known that all admissible weights of $A_1^{(1)}$ are principal admissible. The level $\lambda(K) = k = \frac{t}{u}$ of an admissible weight satisfies the following condition:

$$k+2 = \frac{v}{u}\quad\text{with } v,u\in\mathbb{Z}_{\ge 1},\ (v,u)=1,\ v\ge 2.$$

All admissible weights at level $k = \frac{t}{u}$ are given by [KW88]

$$\begin{aligned} P^k &= \big\{\lambda_{k,i,j} := (k-i+j(k+2))\Lambda_0 + (i-j(k+2))\Lambda_1 \;\big|\; 0 \le i \le v-2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \le j \le u-1\big\} \\ &= \big\{t_{-\frac{j\alpha}{2}}(\widetilde{\Lambda^0} - (u-1)(k+2)\Lambda_0)\big\}, \end{aligned}$$

where $\widetilde{\Lambda^0} = (u(k+2)-2-i)\Lambda_0 + i\Lambda_1$.

One can write down the normalized character $\mathrm{ch}_{\lambda_{k,i,j}}$ [KW88]:

$$\begin{aligned} \mathrm{ch}_{\lambda_{k,i,j}}(\tau,z,t) &= \frac{A_{\lambda+\rho}(h)}{A_\rho(h)} \\ &= \frac{(\Theta_{a^+,b} - \Theta_{a^-,b})(\tau,\frac{z}{u},\frac{t}{u^2})}{(\Theta_{1,2} - \Theta_{-1,2})(\tau,z,t)} \\ &= \frac{(\Theta_{a^+,b} - \Theta_{a^-,b})(\tau,\frac{z}{u},\frac{t}{u^2})}{-ie^{-4\pi i t}\vartheta_{11}(\tau,z)}, \end{aligned}$$

where theta functions $\Theta_{k,i}$ are defined in Appendix A and A_λ is defined as

$$A_\lambda(h) := \sum_{w\in\widehat{W}^\lambda} \varepsilon(w)\Theta_{w(\lambda)}(h).$$

Here,

$$a^+ := u((i+1)-j(k+2)), \quad a^- := u(-(i-1)-j(k+2)), \quad b := u^2(k+2).$$

The modular S transformation property of ch_λ is

$$\mathrm{ch}_{\lambda_{k,i,j}}\Big(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{|z|^2}{2\tau}\Big) = \sum_{\substack{0\le j'\le u-1\\ 0\le i'\le v-2}} a^{(k)}_{(i,j),(i',j')}\,\mathrm{ch}_{\lambda_{k,i',j'}}(\tau,z,t),$$

where

$$a^{(k)}_{(i,j),(i',j')} := \sqrt{\frac{2}{u^2(k+2)}}\,e^{i\pi(j'(i+1)+j(i'+1))}e^{-i\pi jj'(k+2)} \times \sin\frac{(i+1)(i'+1)\pi}{k+2}.$$

§2.1. Twisted modular transformation

Let $N := \mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \times i\mathbb{R}$ be the Heisenberg group with multiplication

$$(\alpha,\beta,u)\cdot(\alpha',\beta',u') := (\alpha+\alpha',\beta+\beta',u+u'+\pi i(\langle\alpha,\beta'\rangle - \langle\alpha',\beta\rangle)),$$

where $\mathfrak{h}_{\mathbb{R}}$ is the set of all real linear combinations of the root vector h_α corresponding to α for all $\alpha \in \Delta$. One defines the action of modular group $\mathrm{SL}_2(\mathbb{Z})$ and

Heisenberg group N on the space $\mathfrak{h}^*$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, t) := \Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t - \frac{c|z|^2}{2(c\tau + d)}\Big),$$

and

$$(\alpha, \beta, u) \cdot h := t_\beta h + 2\pi i \alpha + (u - \pi i \langle \alpha, \beta \rangle)\delta,$$

where t_β is the translation operator defined as

$$t_\beta(h) := h + \langle h, \delta \rangle \beta - \Big(\frac{\langle \beta, \beta \rangle}{2} \langle h, \delta \rangle + \langle h, \beta \rangle\Big)\delta.$$

One can check that the action of Heisenberg group N on $\mathfrak{h}^*$ is a group action, i.e.,

$$((\alpha, \beta, u)(\alpha', \beta', u')) \cdot h = (\alpha, \beta, u) \cdot ((\alpha', \beta', u') \cdot h).$$

Lemma 2.2 ([Wak01]). *For $(\alpha, \beta, u) \in N$ and $h = (\tau, z, t) = 2\pi i(-\tau D + z + tK)$, the following holds:*

$$(\alpha, \beta, u) \cdot (\tau, z, t) = \Big(\tau, z + \alpha - \tau\beta, t + \frac{u}{2\pi i} - \frac{\langle \alpha, \beta \rangle}{2} + \frac{\tau}{2}|\beta|^2 - \langle \beta, z \rangle\Big).$$

Proof. By direct calculation, one has

$$\begin{aligned} t_\beta(\Lambda_0) &= \Lambda_0 + \beta - \frac{|\beta|^2}{2}\delta, \\ t_\beta(\delta) &= \delta, \\ t_\beta(z) &= z - \langle z, \beta \rangle \delta. \end{aligned}$$

Thus,

$$\begin{aligned} t_\beta(h) &= t_\beta(-2\pi i \Lambda_0 + 2\pi i z + 2\pi i \delta t) \\ &= -2\pi i \tau\Big(\Lambda_0 + \beta - \frac{|\beta|^2}{2}\delta\Big) + 2\pi i(z - \langle z, \beta \rangle \delta) + 2\pi i t \delta \\ &= -2\pi i \Lambda_0 \tau + 2\pi i\Big(z - \tau\beta) + 2\pi i \delta(t - \langle z, \beta \rangle + \tau\frac{|\beta|^2}{2}\Big). \end{aligned}$$

Furthermore,

$$\begin{aligned} (\alpha, \beta, u) \cdot h = {} & -2\pi i \Lambda_0 \tau + 2\pi i(z + \alpha - \tau\beta) \\ & + 2\pi i \delta\Big(t + \frac{u}{2\pi i} - \frac{\langle \alpha, \beta \rangle}{2} - \langle z, \beta \rangle + \tau\frac{|\beta|^2}{2}\Big). \end{aligned}$$

We are done. □

The actions of groups $\mathrm{SL}_2(\mathbb{Z})$ and N on $\mathfrak{h}^*$ are compatible.

Lemma 2.3. *For* $(\alpha,\beta,u) \in N$ *and* $h \in \mathfrak{h}^*$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*, the following holds:*

$$\begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \cdot (\alpha,\beta,u)\cdot h = (a\alpha + b\beta, c\alpha + d\beta, u)\cdot \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \cdot h.$$

The metaplectic group is defined as

$$\mathrm{Mp}_2(\mathbb{Z}) := \big\{(A,j) \;\big|\; A \in \mathrm{SL}_2(\mathbb{Z}),\ j \text{ is a holomorphic function in } \tau \in \mathbb{H} \text{ such that } j(\tau)^2 = c\tau + d\big\}.$$

Given a holomorphic function F on $Y = \mathbb{H}\times\mathbb{C}\times\mathbb{C}$, one has the right action of $\mathrm{Mp}_2(\mathbb{Z})$ and N on F:

$$F|_{(A,j)}(\tau,z,t) := \frac{1}{j(\tau)^r}\cdot F(A\cdot(\tau,z,t)),$$
$$F|_{(\alpha,\beta,u)}(\tau,z,t) := F((\alpha,\beta,u)\cdot(\tau,z,t)).$$

Then one defines very important functions for $\alpha,\beta \in \mathfrak{h}^*$:

$$F^{\alpha,\beta}(\tau,z,t) := F((\alpha,\beta,0)\cdot(\tau,z,t)), \tag{2.1}$$

namely

$$F^{\alpha,\beta}(\tau,z,t) = F\Big(\tau, z + \alpha - \tau\beta, t - \frac{\langle\alpha,\beta\rangle}{2} - \langle\beta,z\rangle + \frac{\tau}{2}|\beta|^2\Big).$$

Modular transformation of these functions are given as follows.

Lemma 2.4 ([Wak01]). *Under the action of* $(A,j) \in \mathrm{Mp}_2(\mathbb{Z})$*,*

- $(F|_{(A,j)})^{\alpha,\beta} = F^{a\alpha+b\beta,c\alpha+d\beta}|_{(A,j)}$,
- $F^{\alpha,\beta}|_{(A,j)} = (F|_{(A,j)})^{a'\alpha+b'\beta,c'\alpha+d'\beta}$,

where $A^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$.

§2.2. Twisted characters and twisted modules

Now we consider the normalized character of $L(\lambda)$ at level k. Then given $\alpha,\beta \in \mathfrak{h}^*$, by letting

$$F(\tau,z,t) = \mathrm{ch}_\lambda(\tau,z,t)$$

in (2.1), one has

$$\begin{aligned}\mathrm{ch}_\lambda^{\alpha,\beta}(\tau,z,t) &= q^{m_\lambda}\,\mathrm{tr}_{L(\lambda)}\, e^{-2\pi i\tau D+2\pi i(z+\alpha-\tau\beta)+2\pi i k(t-\frac{\langle\alpha,\beta\rangle}{2}-\langle z,\beta\rangle+\tau\frac{|\beta|^2}{2})}\\ &= e^{2\pi ikt}\,\mathrm{tr}_{L(\lambda)}\, e^{2\pi i((z+\alpha)+k(-\frac{\langle\alpha,\beta\rangle}{2}-\langle z,\beta\rangle))} q^{-\beta+k\frac{|\beta|^2}{2}} q^{-D-\frac{c(k)}{24}}\\ &= e^{2\pi ikt}\,\mathrm{tr}_{L(\lambda)}\, e^{2\pi i(z+\alpha-k\frac{\langle\alpha,\beta\rangle}{2}-k\langle z,\beta\rangle)} q^{-D-\beta+k\frac{|\beta|^2}{2}-\frac{c(k)}{24}},\end{aligned} \tag{2.2}$$

where $c(k)=\frac{k\cdot\dim\mathfrak{g}}{k+h^\vee}$. The twisted character (2.2) is very important. When $z=0$, multiplying (2.2) with $\eta(\tau)^k$ produces the theta function defined on vertex algebra by Miyamoto [Miy00, Definition 1]. When $\alpha=0$, (2.2) can be written as

$$\mathrm{ch}_\lambda^{0,\beta}(\tau,z,t) = (\mathbf{y}e^{-2\pi i\langle z,\beta\rangle} q^{\frac{|\beta|^2}{2}})^k\,\mathrm{tr}_{L(\Lambda)}\, e^{2\pi iz} q^{-\beta} q^{-D-\frac{c(k)}{24}} \tag{2.3}$$

where $\mathbf{y} := e^{2\pi it}$.

2.2.1. Twisted modules of a vertex operator algebra. Let (V,ω) be a $\mathbb{Q}$-graded VOA, where ω is the conformal vector. The Fourier coefficients of the vertex operator

$$Y(\omega,z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy the defining relations of the Virasoro algebra with central charge c. And we let $L_0=\omega_{(-1)}$. Let g be the automorphism of V of finite order T. We first recall the definition of a g-twisted module.

Definition 2.5. An ordinary g-twisted V-module is a $\mathbb{C}$-linear space M equipped with a linear map

$$\begin{aligned} V &\to \mathrm{End}(V)[[z^{\frac{1}{T}}, z^{-\frac{1}{T}}]],\\ v &\mapsto Y_M(v,z)=\sum_{n\in\mathbb{Q}} v_{(n)}z^{-n-1},\end{aligned}$$

satisfying

- for $v\in V$ and $w\in M$, $v_{(m)}w=0$ if m is large enough;
- $Y_M(\mathbf{1},z)=\mathrm{id}_V$;
- for $v\in V^r=\{v\in V\mid gv=e^{\frac{2\pi ir}{T}}v\}$, and $0\le r\le T-1$,

$$Y_M(v,z)=\sum_{n\in\frac{r}{T}+\mathbb{Z}} v_{(n)}z^{-n-1}.$$

- (**Jacobi identity**) For $u \in V^r$,

$$z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y_M(v,z_1)Y_M(u,z_2) - (-1)^{|v||w|}z_0^{-1}\delta\Big(\frac{z_0-z_1}{-z_0}\Big)Y_M(v,z_2)Y_M(u,z_1)$$
$$= z_2^{-1}\Big(\frac{z_1-z_0}{z_2}\Big)^{-\frac{r}{T}}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y_M(Y(u,z_0)v,z_2).$$

- (**grading**)
  - $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M \mid L_0 w = \lambda w\}$,
  - M_λ is finite-dimensional,
  - for a fixed λ, $M_{\frac{n}{T}+\lambda} = 0$ for all small enough integers n.

Remark 2.6. If the grading condition is dropped, one calls it the weak g-twisted module.

Definition 2.7. Let M^* be a restricted dual of an ordinary g-twisted module M. It has a standard g-twisted V-module structure, i.e., $Y_M'(\cdot,z)$, given by

$$\langle Y_M'(v,z)w',w\rangle = \langle w', Y_M(e^{zL_1}(-z^{-2})^{L_0}v,z^{-1})w\rangle,$$

for $v \in V$, $w' \in M^*$, and $w \in M$. We call $(M^*, Y_M'(\cdot,z))$ the contragredient dual of M or the contragredient module of M.

Now we review some basic facts about twisted modules under the vertex algebra setting. Let $v \in V_1$ be a weight-1 vector satisfying the Heisenberg λ-bracket relation

$$[v_\lambda v] = k\lambda, \quad [\omega_\lambda v] = (T+\lambda)v.$$

Suppose further that $v_{(0)}$ acts semisimply on V such that the eigenvalues of $v_{(0)}$ belong to $\frac{1}{T}\mathbb{Z}$. Li's Δ-operator is defined as [Li97]

$$\Delta(z) := z^{v_{(0)}} \exp\Big(\sum_{n=1}^{\infty}\frac{v_{(n)}}{-n}(-z)^{-n}\Big).$$

When $g(v) = v$, one can obtain a weak $ge^{2\pi i v_{(0)}}$-twisted module from a weak g-twisted V-module M by using Li's Δ-operator as follows.

Proposition 2.8 ([Li96]). *The module $(M, Y_M(\Delta(z)\cdot,z))$ is a weak $ge^{2\pi i v_{(0)}}$-twisted module.*

When $g = \mathrm{id}$, $e^{2\pi i v_{(0)}}$ is an automorphism of V of order T. Then $(M, Y_M(\Delta(z)\cdot,z))$ is a weak $e^{2\pi i v_{(0)}}$-twisted module. Let $v' \in V_1$ satisfy $v_{(0)}v' = 0$.

Define

$$\widehat{L}_0 := \operatorname{Res}_z zY_M(\Delta(z)\omega, z),$$
$$\widehat{v'}_{(0)} := \operatorname{Res}_z Y_M(\Delta(z)v', z).$$

By direct calculation, one has

$$\widehat{L}_0 = L_0 + v_{(0)} + \frac{1}{2}\langle v, v\rangle, \quad \widehat{v'}_{(0)} = v'_{(0)} + \langle v, v'\rangle.$$

§2.3. Spectral flow

Let us give a brief introduction to the spectral flow automorphisms (see [Rid09, CR12, CR13] for details). Let $\alpha \in \Delta$ denote a root of $\mathfrak{g}$ with root vector e^α and coroot $\alpha^\vee$. Each element $w \in W$ permutes the roots and induces an automorphism of $\mathfrak{g}$ via $w(e^\alpha) = e^{w(\alpha)}$. This action can be generalized to an affine Lie algebra $\hat{\mathfrak{g}}$ as follows. The corresponding root vectors of real roots are e^α_n, and the root vectors corresponding to imaginary roots are denoted by h^i_n. Then one can identify h^i with the simple coroot $\alpha_i^\vee$ of $\mathfrak{g}$. The coroot lattice acts on the roots of $\hat{\mathfrak{g}}$ by translation in the imaginary direction. Then each simple coroot $\alpha_i^\vee$ of $\mathfrak{g}$ defines an independent transformation τ_i on the root vector of an affine Lie algebra $\hat{\mathfrak{g}}$ via

$$\tau_i(e^\alpha_n) = e^\alpha_{n-\langle\alpha,\alpha_i^\vee\rangle} \ (n \in \mathbb{Z}), \quad \tau_i(h^j_n) = h^j_n \ (n \neq 0).$$

We finally obtain that the spectral flow automorphisms τ_i act on generators of the affine Lie algebra $\hat{\mathfrak{g}}$ and L_0 as

$$\tau_i(e^\alpha_n) = e^\alpha_{n-\langle\alpha,\alpha_i^\vee\rangle}, \quad \tau_i(h^j_n) = h^j_n - \delta_{n,0}\langle\alpha_i^\vee, \alpha_j^\vee\rangle K,$$
$$\tau_i(K) = K, \quad \tau_i(L_0) = L_0 - h^i_0 + \frac{2}{\langle\alpha_i, \alpha_i\rangle}K.$$

Let $\bar{\lambda} = \sum_{i=1}^r a_i\alpha_i^\vee$ be a weight vector of $\mathfrak{g}$. We let $\sigma^{\bar{\lambda}} := \prod_{i=1}^r \tau_i^{a_i}$. For the affine Lie algebra $\widehat{\mathfrak{sl}_2}$, let $\sigma = \tau_1^{\frac{1}{2}}$; the powers of σ acts as

$$\sigma^\ell(e_n) = e_{n-\ell}, \quad \sigma^\ell(h_n) = h_n - \ell\delta_{n,0}K, \quad \sigma^\ell(f_n) = f_{n+\ell},$$
$$\sigma^\ell(K) = K, \quad \sigma^\ell(L_0) = L_0 - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2 K.$$

One can use spectral flow automorphism to modify the action of $\widehat{\mathfrak{sl}_2}$ on any module M, thereby obtaining new modules $\sigma^*(M)$. Explicitly, the modified algebra action defining these new modules is given by

$$X \cdot \sigma^*|v\rangle = \sigma^*(\sigma^{-1}(X)|v\rangle) \quad (X \in \mathfrak{sl}_2).$$

For example, if $|\lambda, \Delta\rangle$ is a vector of weight λ and conformal dimension Δ, then the weight and conformal dimension of the vector $(\sigma^\ell)^*|\lambda, \Delta\rangle \in (\sigma^\ell)^*(M)$ becomes

$$h_0(\sigma^\ell)^*|\lambda, \Delta\rangle = (\lambda + \ell K)(\sigma^\ell)^*|\lambda, \Delta\rangle \tag{2.4}$$

$$L_0(\sigma^\ell)^*|\lambda, \Delta\rangle = \Big(L_0 + \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2 K\Big)(\sigma^\ell)^*|\lambda, \Delta\rangle \tag{2.5}$$

From now on, we denote the new module by $\sigma^\ell(M)$. In order to obtain the character of new module $\sigma^\ell(M)$, we need to compute the character of module M.

Definition 2.9. The normalized character of a $\hat{\mathfrak{g}}$-module is defined as

$$\mathrm{ch}[M](\mathbf{y}, \mathbf{z}, q) = \mathrm{tr}_M\, \mathbf{y}^k \mathbf{z}^{h_0} q^{L_0 - c/24},$$

where $\mathbf{y} = e^{2\pi i t}$, $\mathbf{z} = e^{2\pi i z}$, and $q = e^{2\pi i \tau}$.

If M is an irreducible highest weight module $L(\lambda)$, then $\mathrm{ch}[L(\lambda)](\mathbf{y}, \mathbf{z}, q) = \mathrm{ch}_\lambda(\tau, z, t)$. In the following context, we use $\mathrm{ch}[M](q)$ to denote $\mathrm{ch}[M](1, 1, q)$ when M is an ordinary module.

The character of new module $\sigma^\ell(M)$ can be written in terms of the character of module M as

$$\mathrm{ch}[\sigma^\ell(M)](\mathbf{y}, \mathbf{z}, q) = \mathrm{ch}[M](\mathbf{y}\mathbf{z}^\ell q^{\ell^2/4}, \mathbf{z} q^{\ell/2}, q). \tag{2.6}$$

One can check that the character of module M satisfies the relation

$$\begin{aligned}\mathrm{ch}[\sigma^{\ell+\ell'}(M)](\mathbf{y}, \mathbf{z}, q) &= \mathrm{ch}[\sigma^\ell \circ \sigma^{\ell'}(M)](\mathbf{y}, \mathbf{z}, q) \\ &= \mathrm{ch}[M](\mathbf{y}\mathbf{z}^{\ell+\ell'} q^{(\ell+\ell')^2/4}, \mathbf{z} q^{(\ell+\ell')/2}, q).\end{aligned}$$

2.3.1. Li's delta operator, spectral flow, and twisted modules. Let M be a highest weight $\hat{\mathfrak{g}}$-module or its contragredient module. Let $v \in \mathfrak{h}_{\mathbb{R}}^*$. We have

$$\begin{aligned}\Delta(z) e^\alpha_{-1}\mathbf{1} &= z^{\langle v, \alpha\rangle} e_{-1}\mathbf{1}, \\ \Delta(z) h^j_{-1}\mathbf{1} &= h^j_{-1}\mathbf{1} + \langle v, h^j\rangle k z^{-1}\mathbf{1}.\end{aligned}$$

Thus, it induces the spectral flow twist τ:

$$\begin{aligned}\tau(e^\alpha_n) &= e^\alpha_{n+\langle v, \alpha\rangle}, \\ \tau(h^j_n) &= h^j_n + \langle v, h^j\rangle k \delta_{n,0}.\end{aligned}$$

If $\langle v, \alpha\rangle \in \frac{1}{T}\mathbb{Z}$ for all roots α, according to Proposition 2.8, $(M, Y_M(\Delta(z)\cdot, z))$ is a weak $e^{2\pi i v_{(0)}}$ -twisted ($\mathbb{Z}_T$-twisted) module. But the weak $\mathbb{Z}_T$-twisted modules are not necessarily coming from spectral flow.

In the future, we shall call $(M, Y_M(\Delta(z)\cdot, z))$ the spectral flowed module or $e^{2\pi i v_{(0)}}$-twisted module. When $T = 2$, we call it the $\mathbb{Z}_2$-twisted module.

Proposition 2.10. *Let $L(\lambda)$ be a highest weight $L_k(\mathfrak{sl}_2)$-module, where $k=-2+\frac{v}{u}$ is an admissible level. Let $\lambda_{k,i,j}$ and $\Delta_{k,i,j}$ be the highest weight and conformal dimension of $L(\lambda)$. The highest weight and the conformal dimension of the $\mathbb{Z}_2$-twisted $\sigma^{-\frac{1}{2}}(L(\lambda))$ become*

$$\sigma^{-\frac{1}{2}}(\lambda_{k,i,j}) = \Big(k - \Big(i-(k+2)j-\frac{k}{2}\Big)\Big)\Lambda_0 + \Big(i-(k+2)j-\frac{k}{2}\Big)\Lambda_1$$
$$(i=0,\dots,v-2,\ j=0,\dots,u-1),$$
$$\sigma^{-\frac{1}{2}}(\Delta_{k,i,j}) = \frac{1}{16}\Big(k-4i+4(2+k)j+\frac{4(-1+(i+1-(2+k)j)^2)}{2+k}\Big)$$
$$(i=0,\dots,v-2,\ j=0,\dots,u-1).$$

Proof. The statement follows from (2.4), (2.5), and [AM95, Theorem (3.5.3)]. □

2.3.2. Half-integer spectral flow for $L_k(\mathfrak{sl}_2)$ at the boundary admissible level. For $\widehat{\mathfrak{sl}_2}$, $h^\vee=2$, the boundary admissible levels are $k=-2+\frac{2}{u}$, where $u=2n+1$ is a positive odd integer. All admissible weights are

$$\Lambda_{k,j} := t_{-\frac{j}{2}}\cdot(k\Lambda_0) = \Big(k+\frac{2j}{u}\Big)\Lambda_0 - \frac{2j}{u}\Lambda_1 \quad j=0,1,\dots,u-1,$$

where Λ_0 and Λ_1 are fundamental weights of $\widehat{\mathfrak{sl}_2}$. All characters of irreducible modules can be written in terms of the Jacobi theta function $\theta_1(\mathbf{z};q)$ (see Appendix A for the definition) as

$$\mathrm{ch}[L(\Lambda_{k,j})](\mathbf{y},\mathbf{z},q) = \mathbf{y}^k\mathbf{z}^{-\frac{2j}{u}}q^{\frac{j^2}{2u}}\frac{\theta_1(\mathbf{z}^2q^{-j};q^u)}{\theta_1(\mathbf{z}^2;q)} \quad (j=0,1,\dots,u-1).$$

We should emphasize that the normalized characters of $L_{\frac{-4n}{2n+1}}(\mathfrak{sl}_2)$ coincide with superconformal indices of the $4d$ supersymmetric gauge theories which are called (A_1,D_{2n+1}) theories. In particular, when $u=3$, level $k=-4/3$, $L_{-4/3}(\mathfrak{sl}_2)$ coincides with the $\mathfrak{a}_1$ DC series of simple Lie algebras. This example was also intensively studied in detail from different perspectives in [Ada05] and [CR13]. One can generalize the integer flow parameter ℓ to a half-integer, (2.6) stays the same. For the $L_k(\mathfrak{sl}_2)$ at boundary admissible levels $k=-2+\frac{2}{u}$, using (2.6), all characters of these twisted modules take the form

$$\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,j}))](\mathbf{y},\mathbf{z},q) = (\mathbf{y}\mathbf{z}^{-\frac{1}{2}}q^{\frac{1}{16}})^k(\mathbf{z}q^{-\frac{1}{4}})^{-\frac{2j}{u}}q^{\frac{j^2}{2u}}\frac{\theta_1(\mathbf{z}^2q^{-j-\frac{1}{2}};q^u)}{\theta_1(\mathbf{z}^2q^{-\frac{1}{2}};q)}$$
$$(j=0,1,\dots,u-1).$$

Fix a boundary admissible level k; all $\sigma^{-\frac{1}{2}}(L(\Lambda_{k,j}))$ are ordinary $\mathbb{Z}_2$-twisted modules. The normalized characters of these modules satisfy a modular linear differential equation (see Section 4).

2.3.3. Modular transformation. Applying (2.4) to (2.2), one has the following S-transformation for twisted characters:

$$\mathrm{ch}_\lambda^{\alpha,\beta}(\tau,z,t)|_S = (\mathrm{ch}_\lambda(\tau,z,t)|_S)^{\beta,-\alpha},$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Example 2.11 (Boundary admissible level). We follow the same notation as in Section 2.3.2. Let $\mathrm{Spec}(T)$ be the eigenvalues of T. First, note that $\mathrm{Spec}((\epsilon h_1)_{(0)}) \in 2\epsilon\mathbb{Z}$ ($\epsilon \in \mathbb{Q}$). The twisted character (2.3) equals

$$\begin{aligned} \mathrm{ch}_\lambda^{0,-\frac{\ell}{2}\alpha_1}(\tau,z,t) &= (\mathbf{y}e^{-2\pi i\langle\alpha_1 z,-\frac{\ell}{2}\alpha_1\rangle}q^{\frac{|-\frac{\ell}{2}\alpha_1|^2}{2}})^m \,\mathrm{tr}_{L(\lambda)}\, e^{2\pi i\alpha_1 z}q^{\frac{\ell}{2}\alpha_1}q^{L_0-\frac{c(k)}{24}} \\ &= (\mathbf{y}\mathbf{z}^\ell q^{\frac{\ell^2}{4}})^k \,\mathrm{tr}_{L(\lambda)}(\mathbf{z}q^{\frac{\ell}{2}})^{(h_1)_{(0)}}q^{L_0-\frac{c(k)}{24}}, \end{aligned} \tag{2.7}$$

where we identify $(h_1)_0$ with α_1. Equation (2.7) is the same as (2.6) by letting M be an irreducible module of $L_k(\mathfrak{sl}_2)$-module, $L(\lambda)$.

We split the discussion into two cases:

- $\mathrm{Spec}((\epsilon h_1)_{(0)}) \in \mathbb{Z}$, i.e., $\epsilon = \frac{\ell}{2}$, $\ell \in \mathbb{Z}$. Let σ be a spectral flow automorphism on $\hat{\mathfrak{g}}$ (see Section 2.3). In this case, (2.7) is the character of $\sigma^\ell(L(\lambda))$.
- $\mathrm{Spec}((\epsilon h_1)_{(0)}) \in \frac{1}{T}\mathbb{Z}$, $T \in \mathbb{Z}_{>0}$, i.e., $\epsilon = \frac{\ell}{2}$, $\ell \in \frac{1}{T}\mathbb{Z}$, $T \in \mathbb{Z}_{>0}$. Equation (2.7) now becomes the character of the $e^{2\pi i(\epsilon h_1)_{(0)}}$-twisted module $\sigma^\ell(L(\lambda))$. The S-transformation of $\mathrm{ch}_{\Lambda_{k,j}}^{0,-\frac{\ell}{2}\alpha_1}(\tau,z,t)$ is

$$\begin{aligned} \mathrm{ch}_{\Lambda_{k,j}}^{0,-\frac{\ell}{2}\alpha_1}(\tau,z,t)|_S &= (\mathrm{ch}_{\Lambda_{k,j}}(\tau,z,t)|_S)^{-\frac{\ell}{2}\alpha_1,0} \\ &= \sum_{\Lambda_{k,j'}\in P_k} a_{\Lambda_{k,j},\Lambda_{k,j'}}(\mathrm{ch}_{\Lambda_{k,j'}}^{-\frac{\ell}{2}\alpha_1,0}(\tau,z,t)), \end{aligned}$$

where

$$a_{\Lambda_{k,j},\Lambda_{k,j'}} = (-1)^{j+j'}e^{-\frac{2\pi i jj'}{u}}\frac{1}{\sqrt{u}}\sin\frac{u\pi}{2}$$

and

$$\mathrm{ch}_{\Lambda_{k,j'}}^{-\frac{\ell}{2}\alpha_1,0}(\tau,z,t) = \mathrm{ch}_{\Lambda_{k,j'}}\Big(\tau,z-\frac{\ell}{2},t\Big).$$

As one can see, the S transformation of the character of $e^{2\pi i(\epsilon h_1)_{(0)}}$-twisted module $L(\Lambda_{k,j})$ can be written as a linear combination of the characters of untwisted modules with the same S-matrix as in the untwisted case.

§3. Contragredient modules of highest weight modules

First, we fix some notation. For brevity, we denote the irreducible module $L(a\Lambda_0+b\Lambda_1)$ of $L_k(\mathfrak{sl}_2)$ by $L(k=a+b,b)$. Throughout this section we consider the admissible level $k = -2 + t$, where $t = \frac{p}{q}$ with $p, q \in \mathbb{Z}_{\geq 1}$, $(p, q) = 1$, $p \geq 2$. The set of all irreducible highest weight modules of $L_k(\mathfrak{sl}_2)$ is then given by $\{L(k, (n-1) - (l-1)t) \mid 1 \leq n \leq p-1,\ 1 \leq l \leq q\}$. We use the notation e, f, h to refer to both the standard generators of $\mathfrak{sl}_2$ and their images $e(-1)\mathbf{1}$, $f(-1)\mathbf{1}$, $h(-1)\mathbf{1}$ in $L_k(\mathfrak{sl}_2)$. This slight abuse of notation should not cause confusion, as the intended meaning will be clear from the context. For clarity of exposition, we occasionally write $v(0)$ instead of $v_{(0)}$, where v belongs to some VOA.

§3.1. Motivation

Given an admissible irreducible highest weight $L_{-4/3}(\mathfrak{sl}_2)$-module $L(-\frac{4}{3}, -\frac{4}{3})$, one can obtain the ordinary twisted modules and contragredient modules by taking half-integer and integer spectral flow respectively (Figure 1). In general, let $\{M_i\}$ be the collection of all highest weight modules for $L_k(\mathfrak{sl}_2)$ at the admissible level, one can obtain all the $\mathbb{Z}_2$-twisted modules of $L_k(\mathfrak{sl}_2)$ at the admissible level either from $\{\sigma^{-\frac{1}{2}}(M_i)\}$ or $\{\sigma^{\frac{1}{2}}(M_i^*)\}$. In this section, we shall calculate Zhu's bimodules of the contragredient modules of highest weight modules and the dimension of the intertwining operators among them.

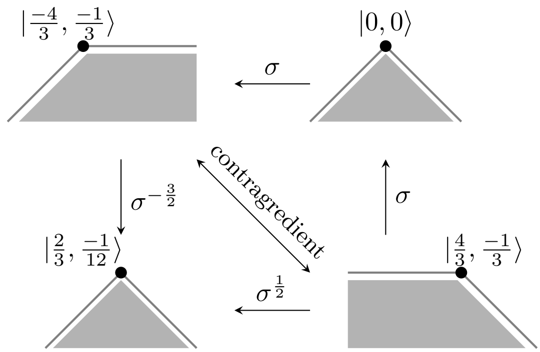

Figure 1. The relations between admissible irreducible highest weight $L_{-4/3}(\mathfrak{sl}_2)$-modules, contragredient modules, and ordinary $\mathbb{Z}_2$-twisted modules, where each state is labeled by $|\lambda, \Delta\rangle$, its $\mathfrak{sl}_2$-weight λ, and conformal dimension Δ.

§3.2. Zhu's bimodules

Assume V is $\mathbb{Z}$-graded. Denote the conformal weight of $v \in V$ by $\mathrm{wt}\, v$. Let M be a V-module. Let $O(M)$ be the linear span of elements

$$\mathrm{Res}_z\Big(Y(a, z)\frac{(z+1)^{\mathrm{wt}\, a}}{z^2}v\Big),$$

where $a \in V$ and $v \in M$. In particular, for $V = L_{-4/3}(\mathfrak{sl}_2)$, $O(M)$ is spanned by

$$\mathrm{Res}_z\Big(Y(e,z)\frac{(z+1)}{z^{m+1}}\Big)v = (e(-m-1)+e(-m))v,$$
$$\mathrm{Res}_z\Big(Y(f,z)\frac{(z+1)}{z^{m+1}}\Big)v = (f(-m-1)+f(-m))v,$$
$$\mathrm{Res}_z\Big(Y(h,z)\frac{(z+1)}{z^{m+1}}\Big)v = (h(-m-1)+h(-m))v,$$

for any positive integer m and for $v \in M$. Zhu's algebra of V is defined as $A(V) = V/O(V)$, and $A(M) = M/O(M)$ is an $A(V)$-bimodule called Zhu's bimodule [FZ92, Definitions (1.5.5), (1.5.6)].

In [DLM97, Section 3] the authors extended the definition Zhu's algebra and Zhu's bimodule to any $\mathbb{Q}$-graded vertex operator algebra V. Given the vertex operator algebra V with an automorphism g of finite order, the authors constructed the twisted Zhu algebra $A_g(V)$ [DLM98]. When $V = L_k(\mathfrak{sl}_2)$ and $g = e^{\frac{\pi i h_{(0)}}{2}}$, there is an isomorphism between $(A(V), \omega + \frac{1}{2}h_{(-2)}\mathbf{1})$ and $(A_g(V), \omega)$, where ω is the Sugawara conformal vector. This isomorphism is given by

$$z^{\frac{1}{4}h_{(0)}} \exp\Big(\sum_{n=1}^{\infty}\frac{\frac{1}{4}h_{(n)}}{-n}(-z)^{-n}\Big).$$

The algebra $(A(V), \omega + \frac{1}{2}h_{(-2)}\mathbf{1})$ was calculated in [DLM97]. Then we have

$$A_g(V) \cong \mathbb{C}[x]/\Big\langle \prod_{r=0}^{p-2}\prod_{s=0}^{q-1}\Big(x+\frac{1}{2}k-r+st\Big)\Big\rangle, \tag{3.1}$$

where $t = k+2$.

Let $M(k,j)$ be the Verma module of $\widehat{\mathfrak{sl}_2}$ at level k with highest weight $(k-j)\Lambda_0 + j\Lambda_1$. By [FZ92], one has the following isomorphism F:

$$F\colon L(j)\otimes U(\mathfrak{sl}_2) \to A(M(k,j)),$$
$$F\colon v\otimes a_1\cdots a_n \mapsto [a_n(-1)\cdots a_1(-1)v],$$

whose inverse is given by

$$F^{-1}\colon A(M(k,j)) \to L(j)\otimes U(\mathfrak{sl}_2),$$
$$F^{-1}\colon [a_1(-1-i_1)\cdots a_n(-1-i_n)v] \to (-1)^{i_1+\cdots+i_n} v\otimes a_n\cdots a_1,$$

where the tensor products are over $U(\mathfrak{sl}_2)$. Here, $a_j(-i)$ means $(a_j)_{(-i)}$.

Consider the example of $L_{-4/3}(\mathfrak{sl}_2)$. It has three irreducible modules in category $\mathcal{O}$, i.e., $L(-\frac{4}{3}, 0)$, $L(-\frac{4}{3}, -\frac{2}{3})$, $L(-\frac{4}{3}, -\frac{4}{3})$. By [MFF86], the maximal submodule of the Verma module $M(-\frac{4}{3},-\frac{2}{3})$ is generated by singular vectors v_1 and v_2,

$$v_1 = \Big(\frac{2}{9}e_{(-2)} - \frac{1}{3}e_{(-1)}h_{(-1)} + e^2_{(-1)}f_{(0)}\Big)v_{-2/3}$$
$$v_2 = \Big(-\frac{10}{9}f_{(-1)} - \frac{5}{3}h_{(-1)}f_{(0)} + e_{(-1)}f^2_{(0)}\Big)v_{-2/3},$$

and the maximal submodule of $M(-\frac{4}{3},-\frac{4}{3})$ is generated by singular vectors $e_{(-1)}v_{-2/3}$ and

$$\begin{aligned}\Big(&\frac{280}{81}f_{(-2)} + \frac{70}{27}h_{(-2)}f_{(0)} - \frac{10}{9}e_{(-2)}f^2_{(0)}\\ &+ \frac{140}{27}h_{(-1)}f_{(-1)} + \frac{35}{9}h^2_{(-1)}f_{(0)} - \frac{5}{3}h_{(-1)}e_{(-1)}f^2_{(0)}\\ &- \frac{70}{9}e_{(-1)}f_{(-1)}f_{(0)} - \frac{10}{3}e_{(-1)}h_{(-1)}f^2_{(0)} + e^2_{(-1)}f^3_{(0)}\Big)v_{-2/3},\end{aligned} \tag{3.2}$$

where $v_{-2/3}$ and $v_{-4/3}$ are highest weight vectors of $M(-\frac{4}{3},-\frac{2}{3})$ and $M(-\frac{4}{3},-\frac{4}{3})$, respectively.

We next use the singular vectors to compute $A(L(-\frac{4}{3},-\frac{2}{3}))$, $A(L(-\frac{4}{3},-\frac{4}{3}))$. For $A(L(-\frac{4}{3},-\frac{2}{3}))$, the preimages of equivalence classes of v_1 and v_2 are

$$-\frac{2}{9}v\otimes e - \frac{1}{3}v\otimes he + fv\otimes e^2, \tag{3.3}$$

$$-\frac{10}{9}v\otimes f - \frac{5}{3}fv\otimes h + f^2v\otimes e. \tag{3.4}$$

Thus

$$A\Big(L_{\mathfrak{sl}_2}\Big(-\frac{4}{3},-\frac{2}{3}\Big)\Big) = \Big(L\Big(-\frac{2}{3}\Big)\otimes U(\mathfrak{sl}_2)\Big)/I_1,$$

where I_1 is generated by (3.3) and (3.4), and the tensor products are over $U(\mathfrak{sl}_2)$. For $A(L(-\frac{4}{3},-\frac{4}{3}))$, the preimage of equivalence classes of (3.2) is

$$\begin{aligned}-\frac{280}{81}v\otimes f - \frac{70}{27}fv\otimes h + \frac{10}{9}f^2v\otimes e + \frac{140}{27}v\otimes hf + \frac{35}{9}fv\otimes h^2\\ - \frac{5}{3}f^2v\otimes eh - \frac{70}{9}fv\otimes fe - \frac{10}{3}f^2v\otimes he + f^3v\otimes e^2.\end{aligned}$$

Denote by $L(-\frac{2}{3})^*$ and $L(-\frac{4}{3})^*$ the dual of highest weight $\mathfrak{sl}_2$-modules with weights $-\frac{2}{3}$ and $-\frac{4}{3}$. We now consider the $A(L(-\frac{4}{3},0))$-module $L(-\frac{2}{3})^*\otimes A(L(-\frac{4}{3},-\frac{2}{3}))$. Let v' be the lowest weight vector of $L(-\frac{2}{3})^*$. Then $I_1\otimes v' \cong \langle -\frac{10}{9}v\otimes ev' + fv\otimes e^2v', -\frac{10}{9}fv\otimes v' + f^2v\otimes ev'\rangle$. It is isomorphic to

$$A\Big(L\Big(-\frac{4}{3},-\frac{2}{3}\Big)\Big)\otimes L\Big(-\frac{2}{3}\Big)^* \cong \mathbb{C}(v\otimes v').$$

Thus by [FZ92], we have

$$L\Big(-\frac{4}{3},-\frac{2}{3}\Big)\times\Big(L\Big(-\frac{4}{3},-\frac{2}{3}\Big)\Big)^{*}=L\Big(-\frac{4}{3},0\Big).$$

Similarly, we also have

$$\begin{aligned}
L\Big(-\frac{4}{3},-\frac{4}{3}\Big)\times\Big(L\Big(-\frac{4}{3},-\frac{4}{3}\Big)\Big)^{*}&=L\Big(-\frac{4}{3},0\Big),\\
L\Big(-\frac{4}{3},-\frac{2}{3}\Big)\times\Big(L\Big(-\frac{4}{3},-\frac{4}{3}\Big)\Big)^{*}&=\Big(L\Big(-\frac{4}{3},-\frac{2}{3}\Big)\Big)^{*},\\
L\Big(-\frac{4}{3},-\frac{4}{3}\Big)\times\Big(L\Big(-\frac{4}{3},-\frac{2}{3}\Big)\Big)^{*}&=L\Big(-\frac{4}{3},-\frac{2}{3}\Big).
\end{aligned}$$

§3.3. The contragredient modules of highest weight modules

As one can see from the above example, directly computing Zhu's bimodules depends on the explicit form of singular vectors. In practice, it is extremely tedious to convert the singular vectors given in [MFF86] into their normal forms. It was noted in [DLM97] that the dimension of intertwining operators among the modules at the admissible level remains the same after a shift of the conformal vector. By a proper shift of the conformal vector, there are nice and compact projection formulas for singular vectors ([Fuc89, DLM97]), which can help us to compute the dimensions of intertwining operators by avoiding finding an explicit form of singular vectors.

Since for $\gamma\in\mathbb{Q}_+$, one has ([MFF86])

$$\begin{aligned}
[f(0),e(0)^{\gamma}]&=-(\gamma e(0)^{\gamma-1}+\gamma(\gamma-1)e(0)^{\gamma-1}),\\
[h(0),e(0)^{\gamma}]&=2\gamma e(0)^{\gamma},
\end{aligned}$$

then by using the same argument as in [MFF86] we have the following proposition.

Proposition 3.1. *Let $j=n-1-(l-1)t$, where n and l are positive integers satisfying $1\le n\le p-1$, $1\le l\le q$. Let $(M(k,j))^{*}:=U(\widehat{\mathfrak{sl}_2})\otimes_{U(\mathfrak{sl}_2\otimes\mathbb{C}[t]\oplus\mathbb{C}K)}M(j)^{*}$, where $M(j)^{*}$ is the universal lowest weight module of $\mathfrak{sl}_2$ with lowest weight $j\bar{\Lambda}_1$. Let v' be the lowest weight vector in $M(j)^{*}$. Let $v=1\otimes v'$ in $(M(k,j))^{*}$. Set*

$$\begin{aligned}
E_1(n,l)&=e(0)^{n+(l-1)t}f(-1)^{n+(l-2)t}e(0)^{n+(l-3)t}f(-1)^{n+(l-4)t}\\
&\qquad\cdots f(-1)^{n-(l-2)t}e(0)^{n-(l-1)t},\\
E_2(n,l)&=f(-1)^{p-n+(q-l)t}e(0)^{p-n+(q-l-1)t}f(-1)^{p-n+(q-l-2)t}e(0)^{p-n+(q-l-3)t}\\
&\qquad\cdots e(0)^{p-n-(q-l+1)t}f(-1)^{p-n-(q-l)t}.
\end{aligned}$$

Then $v_{-j,1}=E_1(n,l)v$, $v_{-j,2}=E_2(n,l)v$ are singular vectors of $(M(k,j))^{*}$.

Next we consider the projection formula. Let P_1 be the projection $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$ such that $P_1(a \otimes t^n) = a$ for any $a \in \mathfrak{g}$ and $P_1(c) = 0$. Let $H_\alpha = fe - \alpha h - \alpha(\alpha+1)$.

Proposition 3.2 ([MFF86]). *The following projection formulas hold:*

$$P_1(E_1(n,l)) = \Big(\prod_{r=1}^{n}\prod_{s=1}^{l-1} H_{-r-st}\Big)e^n,$$

$$P_1(E_2(n,l)) = \Big(\prod_{r=0}^{p-n-1}\prod_{s=1}^{q-l} H_{r+st}\Big)f^{p-n}.$$

Define a subalgebra of $\widehat{\mathfrak{sl}_2}$:

$$N_- = \mathbb{C}e + t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{sl}_2.$$

Let $B_0 = (t^{-1}+1) \otimes \mathbb{C}e + (t^{-2}+t^{-1})\mathbb{C}[t^{-1}] \otimes \mathfrak{g}$. Since B_0 is an ideal of N_-, $U(N_-)B_0 = B_0U(N_-)$ is an ideal of $U(N_-)$. Set $K_0 = N_-/B_0$. Define

$$T_+ = e(0) + B_0, T_- = f(-1) + B_0, T_0 = h(-1) + B_0.$$

They obey the following $\mathfrak{sl}_2$-relationships:

$$[T_0, T_+] = -2T_+, \quad [T_0, T_-] = 2T_-, \quad [T_+, T_-] = T_0.$$

Define $G_\alpha = T_-T_+ - \alpha T_0 + \alpha(\alpha+1)$. They satisfy the following relationships:

$$G_\alpha G_\beta = G_\beta G_\alpha, \quad T_+^m G_\alpha = G_{\alpha-m}T_+^m, \quad T_-^m G_\alpha = G_{\alpha+m}T_-^m,$$
$$T_-^m T_+^m = G_0 G_1 \cdots G_{m-1}, \quad T_+^m T_-^m = G_{-1}G_{-2}\cdots G_{-m},$$

for any complex numbers α, β and for any positive integer m.

Let P be the natural quotient map from $U(N_-)$ onto $U(K_0)$. Using a similar method to that suggested in [MFF86] we obtain the following proposition.

Proposition 3.3. *The following formulas hold:*

$$P(E_1(n,l)) = \Big(\prod_{r=1}^{n}\prod_{s=1}^{l-1} G_{-r-st}\Big)T_+^n,$$

$$P(E_2(n,l)) = \Big(\prod_{r=0}^{p-n-1}\prod_{s=1}^{q-l} G_{r+st}\Big)T_-^{p-n}.$$

§3.4. Dimensions of intertwining operators

For the contragredient modules of the highest weight modules, we choose the new conformal vector $\omega_z = \omega - \frac{1}{2}zh(-2)\mathbf{1}$, where $0 < z < 1$. Let M be any weak

$L_k(\mathfrak{sl}_2)$-module. Denote the conformal weight of $v \in M$ with respect to ω_z by $\mathrm{wt}\, v$. Since for $v \in M$,

$$\mathrm{wt}\, h = 1, \quad \mathrm{wt}\, e = 1 + z, \quad \mathrm{wt}\, f = 1 - z,$$

we have

$$\begin{aligned} \mathrm{Res}_z \frac{(1+z)^{[\mathrm{wt}\, f]}}{z^m} Y(f,z)u &= f(-m)u, \\ \mathrm{Res}_z \frac{(1+z)^{[\mathrm{wt}\, e]}}{z^m} Y(e,z)u &= (e(-m) + e(1-m))u, \\ \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, h}}{z^{m+1}} Y(h,z)u &= (h(-m-1) + h(-m))u, \end{aligned}$$

for any positive integer m and for $u \in M$, where $[a]$ means the maximal integer less than a. We shall compute Zhu's bimodule for the $\mathbb{Q}$-graded VOA $(L_k(\mathfrak{sl}_2), \omega_z)$ (see [DLM97, Section 3] for general definitions).

Proposition 3.4. *Let $j = n - 1 - (l-1)t$ be an admissible weight. Then the $A(L_k(\mathfrak{sl}_2))$-bimodule $A((L(k,j))^*)$ is isomorphic to the quotient space of $\mathbb{C}[x,z]$ modulo the subspace*

$$\mathbb{C}[x,z]z^n + \mathbb{C}[x]f'_{j,0}(x,z) + \mathbb{C}[x]f'_{j,1}(x,z) + \cdots + \mathbb{C}[x]f'_{j,n-1}(x,z),$$

where $f'_{j,i}(x,z) = z^i \prod_{r=0}^{p-n-1} \prod_{s=0}^{q-l} (x + r + i - st)$. The left and right actions of $A(L_k(\mathfrak{sl}_2))$ on $A(L(k,j)^*)$ are given by

$$x * f(x,y) = \Big(x + j - 2y\frac{\partial}{\partial y}\Big) f(x,y), \quad f(x,y) * x = x f(x,y)$$

for any $f(x,y) \in \mathbb{C}[x,y]$, where $h_{(0)}v = jv$.

Proof. First, $(M(k,j))^* \cong U(N_-)$ as a vector space. We have

$$O((M(\ell,j))^*) \cong f(-1)U(N_-) + B_0 U(N_-).$$

Recall $v_{-j,1}$, $v_{-j,2}$ are two singular vectors of $(M(k,j))^*$. Then we have

$$\begin{aligned} &A\big((L(k,j))^*\big) \\ &\quad \cong (M(k,j))^* / \big(O\big((M(k,j))^*\big) + U(N_-)v_{-j,1} + U(N_-)v_{-j,2}\big) \\ &\quad \cong U(N_-)/(B_0 U(N_-) + f(-1)U(N_-) + U(N_-)E_1(n,k) + U(N_-)E_2(n,l)) \end{aligned}$$

as $A(L_k(\mathfrak{sl}_2))$-bimodules. Note that $U(N_-)/B_0 U(N_-) \cong U(K_0)$. Thus

$$A\big((L(k,j))^*\big) \cong U(K_0)/(U(K_0)P(E_1(n,l)) + U(K_0)P(E_2(n,l)) + T_- U(K_0)).$$

For any nonnegative integers a, b, d, using the above relationships, we have (see Appendix B for the detail)

$$\begin{aligned}&T_-^aT_0^bT_+^dP(E_1(n,l))\\ &=T_-^a\Big(\prod_{r=1}^{n}\prod_{s=1}^{l-1}(r+st+d)(T_0+r+st+d-1)\Big)T_0^bT_+^{n+d} \bmod T_-U(K_0).\end{aligned}\tag{3.5}$$

Noticing that $r+st+d\neq 0$ for any $1\le r\le n$, $1\le s\le l-1$, $d\in\mathbb{Z}_+$ we obtain

$$\begin{aligned}&U(K_0)P(E_1(n,l))+T_-U(K_0)\\ &\quad=T_-U(K_0)+\sum_{d=0}^{\infty}\mathbb{C}[T_0]\Big(\prod_{r=0}^{n-1}\prod_{s=1}^{l-1}(T_0+r+st+d)\Big)T_+^{n+d}.\end{aligned}$$

Similarly, let a, b, d be any nonnegative integers. If $d<p-n$, we have (see Appendix B for details)

$$\begin{aligned}&T_-^aT_0^bT_+^dP(E_2(n,l))\\ &\quad=T_-^{a+p-n-d}(T_0+2(p-n-d))^b\\ &\qquad\times\prod_{r=0}^{p-n-1}\prod_{s=1}^{q-k}\prod_{i=1}^{d}G_{r+st-p+n}G_{-i-p+n+d} \bmod T_-U(K_0).\end{aligned}\tag{3.6}$$

If $d=m+p-n$ for some $m\in\mathbb{Z}_+$, we have

$$\begin{aligned}&T_-^aT_0^bT_+^dP(E_2(n,l))\\ &\quad=T_-^a\prod_{r=1}^{p-n}\prod_{s=0}^{q-l}(st-m-r)(-T_0+st-m-r+1)T_0^bT_+^m \bmod T_-U(K_0).\end{aligned}$$

Since $-st+m+r-1\neq 0$ for any $1\le r\le p-n$, $0\le s\le q-k$, we obtain

$$\begin{aligned}&U(K_0)P(E_2(n,l))+T_+U(K_0)\\ &\quad=T_-U(K_0)+\sum_{m=0}^{\infty}\mathbb{C}[T_0]\Big(\prod_{r=0}^{p-n-1}\prod_{s=0}^{q-l}(T_0-st+m+r)\Big)T_+^m.\end{aligned}$$

Thus

$$\begin{aligned}&U(K_0)P(E_1(n,l))+U(K_0)P(E_2(n,l))+T_-U(K_0)\\ &\quad\subset T_+U(K_0)+U(K_0)T_+^n+\sum_{i=0}^{n-1}\mathbb{C}[T_0]\Big(\prod_{r=0}^{p-n-1}\prod_{s=0}^{q-k}(T_0-st+m+r)\Big)T_+^m.\end{aligned}$$

On the other hand, since $-r-st-d\neq s't'-m-r'$ for any $0\le r\le n-1$, $1\le s\le l-1$, $0\le r\le p-n-1$, $0\le s\le q-l$, $d,m\in\mathbb{Z}_+$, $\prod_{r=0}^{p-n-1}\prod_{s=0}^{q-l}(x-st+m+r)$

and $\prod_{r=0}^{p-n-1}\prod_{s=0}^{q-l}(x-st+m+r)$ are relatively prime. Then we obtain

$$\mathbb{C}[T_0]T_+^{n+i} \subset U(K_0)P(E_1(n,l)) + U(K_0)P(E_2(n,l)) + T_-U(K_0)$$

for any $i \in \mathbb{Z}_+$. This shows that

$$\begin{aligned} &U(K_0)P(E_1(n,l)) + U(K_0)P(E_2(n,l)) + T_-U(K_0) \\ &\quad \supset T_-U(K_0) + U(K_0)T_+^n + \sum_{i=0}^{n-1}\mathbb{C}[T_0]\Big(\prod_{r=0}^{p-n-1}\prod_{s=0}^{q-k}(T_0 - st + m + r)\Big)T_+^i. \end{aligned}$$

Set $x = T_0$, $y = T_-$. Then the proposition follows from a similar argument to that in [DLM97, Proposition 4.3, Lemma 4.5]. □

Now we can prove the main theorem in this section.

Theorem 3.5. *For admissible weight $j_i = n_i - 1 - (l_i - 1)t$ $(i = 1, 2)$, we have*

$$L(k,j_1) \times L(k,j_2) = \sum_{i=\max\{0,n_1+n_2-p\}}^{\min\{n_1-1,n_2-1\}} L(\ell, j_1 + j_2 - 2i), \tag{3.7}$$

$$(L(k,j_1))^* \times L(k,j_2) = L(k,j_2) \times (L(k,j_1))^*$$

$$= \begin{cases} L(k, -j_1 + j_2) & \text{if } n_2 - n_1 \geq 0, \\ (L(k, j_1 - j_2))^* & \text{if } n_2 - n_1 < 0, \end{cases} \tag{3.8}$$

$$(L(k,j_1))^* \times (L(k,j_2))^* = \sum_{i=\max\{0,n_1+n_2-p\}}^{\min\{n_1-1,n_2-1\}} (L(\ell, j_1 + j_2 - 2i))^*. \tag{3.9}$$

Proof. Equation (3.7) was proved in [DLM97]. We use a similar method to prove (3.8) and (3.9).

We prove (3.8). For any admissible weight $-j$, let $\mathbb{C}v_{-j}$ be the one-dimensional module for Lie algebra $\mathbb{C}h$ such that $hv_{-j} = -jv_{-j}$. Then $\mathbb{C}v_{-j}$ is the lowest weight space of $L(k,-j)$. By Frenkel–Zhu's theorem [FZ92, Li99], we need to calculate the $A(L_k(\mathfrak{sl}_2))$-module $A(L(k,-j_1)) \otimes_{A(L(k,0))} \mathbb{C}v_{j_2}$. Note $ev_{j_2} = 0$. We get

$$A(L(k,-j_1)) \otimes_{A(L(k,0))} \mathbb{C}v_{j_2} \cong \mathbb{C}[x,z]/J,$$

where J is the subspace of $\mathbb{C}[x,z]$ spanned by

$$\{x - j_2, z\}.$$

Thus, $\mathbb{C}[x,z]/J \cong v_{-j_1} \otimes v'_{j_2}$. And $x * (v_{-j_1} \otimes v'_{j_2}) = j_2 - j_1$, as required.

For (3.9), let $\mathbb{C}v_{-j}$ be the one-dimensional module for Lie algebra $\mathbb{C}h$ such that $hv_{-j} = -jv_{-j}$. Using Proposition 3.4 we get

$$A(L(k,j_1)) \otimes_{A(L(k,0))} \mathbb{C}v_{-j_2} \cong \mathbb{C}[x,z]/J,$$

where J is the subspace of $\mathbb{C}[x,z]$ spanned by

$$\{x+j_2,\ \mathbb{C}[x,z]z^{n_1},\ f'_{j_1,i}(-j_2,1)\mathbb{C}[x]z^i,\ i=0,1,\ldots,n_1-1\}.$$

If j_2 does not satisfy the relation $0 \le l_2 - 1 \le q - l_1$, then

$$f'_{j_1,i}(-j_2,1) = \prod_{r=0}^{p-n_1-1}\prod_{s=0}^{q-l_1}(-j_2+r+i-st) \neq 0$$

for $0 \le i \le n_1 - 1$. Thus $A(L(k,-j_1)) \otimes_{A(L(k,0))} \mathbb{C}v_{j_2} = 0$ so that all the corresponding intertwining operators are zero.

Suppose $0 \le l_2 - 1 \le q - l_1$. As before $\mathbb{C}[x]z^i = 0$ in $\mathbb{C}[x,z]/J$ if $f'_{j_1,i}(j_2,1) \neq 0$. Notice that $f'_{j_1,i}(j_2,1) = 0$ if and only if $-j_2 + r + i - st = 0$ for some $0 \le r \le p - n_1 - 1$ and $0 \le s \le q - l_1$. This implies that $r + i = n_2 - 1$. Thus $n_1 + n_2 - p \le i \le n_2 - 1$. Therefore

$$\max\{0, n_1+n_2-p\} \le i \le \min\{n_1-1, n_2-1\}.$$

If $n_1 + n_2 - p \le i \le n_2 - 1$, then $\mathbb{C}[x]z^i$ is not zero in $\mathbb{C}[x,z]/J$. Thus

$$\mathbb{C}[x,z]/J \cong \bigoplus_{\substack{\max\{0,n_1+n_2-p\}\\ \le i \le \min\{n_1-1,n_2-1\}}} \mathbb{C}y^i.$$

We get $x * z^i = (-j_2 - j_1 + 2i)z^i$, as required. □

§3.5. Intertwining operators among twisted modules

Combining Theorem 3.5 and [Li97, Theorem 2.15] (see also [AP19, Proposition 3.1]), we have the following corollary.

Corollary 3.6. *For $n > 1$,*

$$L(k,j_1) \times \sigma^{-\frac{1}{n}}(L(k,j_2)) = \sum_{i=\max\{0,n_1+n_2-p\}}^{\min\{n_1-1,n_2-1\}} \sigma^{-\frac{1}{n}}(L(k,j_1+j_2-2i)),$$

$$(L(k,j_1))^* \times \sigma^{-\frac{1}{n}}\left((L(k,j_2))^*\right) = \sum_{i=\max\{0,n_1+n_2-p\}}^{\min\{n_1-1,n_2-1\}} \sigma^{-\frac{1}{n}}\left((L(k,j_1+j_2-2i))^*\right),$$

$$(L(k,j_1))^* \times \sigma^{-\frac{1}{n}}(L(k,j_2)) = \begin{cases} \sigma^{-\frac{1}{n}}(L(k,-j_1+j_2)) & \text{if } n_2-n_1 \ge 0,\\ \sigma^{-\frac{1}{n}}\left((L(k,j_1-j_2))^*\right) & \text{if } n_2-n_1 < 0,\end{cases}$$

$$L(k,j_2) \times \sigma^{-\frac{1}{n}}\left((L(k,j_1))^*\right) = \begin{cases} \sigma^{-\frac{1}{n}}(L(k,-j_1+j_2)) & \text{if } n_2-n_1 \ge 0,\\ \sigma^{-\frac{1}{n}}\left((L(k,j_1-j_2))^*\right) & \text{if } n_2-n_1 < 0.\end{cases}$$

§4. Twisted modules from spectral flow and their MLDEs

In [Li23], we have the following result.

Theorem 4.1. *If V is a quasi-lisse vertex superalgebra and g is an automorphism of V of finite order, then the supercharacter of a simple ordinary g-twisted module satisfies a twisted modular linear differential equation (see* [Li23, Section 5] *for more details).*

In this section and the next, we shall provide examples for this theorem. We also discuss their applications in physics.

Firstly, let us review some useful facts about modular forms and modular differential operators. See any standard reference, or [KNS13], for further details. The ordinary Eisenstein series are modular forms for the full modular group Γ of weight $2k$ with $k \geq 2$. We define our Eisenstein series, following the notation of [BR18]:

$$\mathbb{E}_k(\tau) = -\frac{B_{2k}}{2k!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n},$$

where B_{2k} is the $2k$th Bernoulli number. The ring of modular forms for the full modular group Γ is freely generated by $\mathbb{E}_4(\tau)$ and $\mathbb{E}_6(\tau)$, so we have

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma,\mathbb{C}) = \mathbb{C}[\mathbb{E}_4(\tau),\mathbb{E}_6(\tau)].$$

We also make use of a class of twisted Eisenstein series that are modular forms for certain congruence subgroups of Γ,

$$\begin{aligned}\mathbb{E}_k\begin{bmatrix}\varphi\\ \vartheta\end{bmatrix}(\tau) \equiv -\frac{B_k(\lambda)}{k!} + \frac{1}{(k-1)!}\sum_{r\geq 0}{}'\,\frac{(r+\lambda)^{k-1}\vartheta^{-1}q^{r+\lambda}}{1-\vartheta^{-1}q^{r+\lambda}}\\ + \frac{(-1)^k}{(k-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{k-1}\vartheta q^{r-\lambda}}{1-\vartheta q^{r-\lambda}},\end{aligned}$$

where $\varphi = e^{2\pi i\lambda}$ with $\lambda \in [0,1)$ and now $B_k(x)$ is the kth Bernoulli polynomial. The prime in the first summation means that the $r = 0$ term should be omitted when $\varphi = \vartheta = 1$. The spaces of modular forms for $\Gamma(2)$, $\Gamma^0(2)$ all admit a simple description in terms of theta functions. For example,

$$M_{2k}(\Gamma^0(2)) = \operatorname{span}_{\mathbb{C}}\{\overline{\Theta}_{r,s}(\tau) \mid r+s=k\},$$

where the $\overline{\Theta}_{r,s}$ take the form

$$\overline{\Theta}_{r,s}(\tau) := \theta_2(\tau)^{4r}\theta_3(\tau)^{4s} + \theta_2(\tau)^{4s}\theta_3(\tau)^{4r}, \quad r \leq s,$$

where $\theta_i(\tau)$ are defined in Appendix A. We define the kth-order modular differential operators $D_q^{(k)}$ as

$$D_q^{(k)}\chi(q) := \partial_{(2k-2)} \circ \cdots \circ \partial_{(2)} \circ \partial_{(0)}\chi(q).$$

Then the twisted modular linear differential operators that are holomorphic and monic have the generic form

$$\mathcal{D}_q^{(k)} \equiv D_q^{(k)} + \sum_{r=1}^{k} f_r(q) D_q^{(k-r)}, \quad f_r(q) \in M_{2k}(\widetilde{\Gamma}, \mathbb{C}),$$

where $\widetilde{\Gamma}$ denotes some congruence subgroup of Γ. We call the corresponding twisted MLDE, $\widetilde{\Gamma}$-MLDE; when $\widetilde{\Gamma}$ is the full $\mathrm{SL}(2,\mathbb{Z})$, it is called MLDE.

§4.1. $L_k(\mathfrak{sl}_2)$ at boundary admissible levels $k = -2 + \frac{2}{u}$

In this section we give some specific $\Gamma^0(2)$-MLDEs satisfied by the characters of irreducible ordinary $\mathbb{Z}_2$-twisted modules for $L_k(\mathfrak{sl}_2)$ at boundary admissible level $k = -2 + \frac{2}{u}$. Suppose that M is an irreducible $\mathbb{Z}_2$-twisted V-module. We say M is a simple object in category $\mathcal{O}$ if the top component $M(0)$, i.e., the lowest conformal weight vector space in M, is a simple module of the $\mathbb{Z}_2$-twisted Zhu algebra. Following Proposition 2.10, and formula (3.1), we have the following theorem.

Theorem 4.2. *All irreducible $\mathbb{Z}_2$-twisted modules of $L_k(\mathfrak{sl}_2)$ at admissible level in category $\mathcal{O}$ can be obtained by using $\ell = -\frac{1}{2}$ spectral flow on the untwisted modules in category $\mathcal{O}$. In particular, for boundary admissible level, all of those irreducible twisted modules are ordinary modules. We find that the q-series characters satisfy the relation*

$$\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,j}))](q) = \mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,u-1-j}))](q).$$

Furthermore, the number of independent characters in the series q is $\frac{u+1}{2}$.

Now let us give some concrete examples for small values of u.

Example 4.3. For $L_{-4/3}(\mathfrak{sl}_2)$, the number of independent q-series characters of all $\mathbb{Z}_2$-twisted modules in category $\mathcal{O}$ is two. We denote these two characters of ordinary twisted modules as $\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,0}))]$ and $\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,1}))]$. They are annihilated by a second-order $\Gamma^0(2)$-MLDE which we display here:

$$\left(D_q^{(2)} - \frac{1}{96}\overline{\Theta}_{1,1}(\tau)\right)\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,i}))](q) = 0. \tag{4.1}$$

Since the modular form $M_{2k}(\Gamma^0(2))$ spanned by the $\overline{\Theta}_{r,s}$ can be rewritten in terms of twisted Eisenstein series, we can rewrite the above MLDE (4.1) as

$$\left(D_q^{(2)} + \frac{4}{3}\mathbb{E}_4\begin{bmatrix}-1\\1\end{bmatrix} + \frac{28}{3}\mathbb{E}_4\begin{bmatrix}1\\-1\end{bmatrix} + \frac{28}{3}\mathbb{E}_4\begin{bmatrix}-1\\-1\end{bmatrix}\right)\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,i}))](q) = 0.$$

Actually, the twisted module character $\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,1}))](1, \mathbf{z}, q)$ has two different physical interpretations. In [CGS17], the authors computed the defect Schur indices of the (A_1, A_3) Argyres–Douglas theory,

$$\begin{aligned}\mathcal{I}_{\mathbb{S}}(q,x) = (q)_\infty^2 \sum_{\substack{\ell_1,\dots,\ell_3,\\ k_1,\dots,k_3=0}}^{\infty} &\frac{(-1)^{\sum_{i=1}^3(k_i+\ell_i)} q^{\frac{1}{2}\sum_{i=1}^3(k_i+\ell_i)+\ell_2(\ell_1+\ell_3)}}{\prod_{i=1}^3 (q)_{k_i}(q)_{\ell_i}}\\ &\times (x)^{\ell_1-k_1}\left(q^{\frac{\ell_1-k_1}{2}} + q^{\frac{k_1-\ell_1}{2}} - q^{\frac{\ell_1+k_1}{2}}\right)\delta_{k_2,\ell_2}\delta_{k_1+k_3,\ell_1+\ell_3}.\end{aligned}$$

The corresponding VOA of the (A_1, A_3) AD theory is just $L_{-4/3}(\mathfrak{sl}_2)$. One can compare the above surface defect index with the $\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,1}))](1, \mathbf{z}, q)$ by expanding the expression in terms of q. We find that their corresponding coefficients for each term are equal up to q^{10}. In [FS18], the authors compute the lens space index of the (A_1, A_3) AD theory which is the supersymmetric partition function on $S^1 \times L(r, 1)$. For example, we have checked that $\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,1}))](1, \mathbf{z}, q)$ agrees with the lens space index $\mathcal{I}^{\mathrm{Mac}}_{(A_1,A_3)}(1, \mathbf{z}, q)$ up to q^{10} and an overall factor by identifying their "twisting parameter" with the spectral flow parameter. One advantage of our expression is that the operator spectrum of the 4d theory is much easier to read off, and modular properties are also apparent.

Example 4.4. For $L_{-8/5}(\mathfrak{sl}_2)$, the number of independent q-series characters of all $\mathbb{Z}_2$-twisted modules in category $\mathcal{O}$ is three. They satisfy a third-order $\Gamma^0(2)$-MLDE

$$\left[D_q^{(3)} - \left(\frac{7}{450}\overline{\Theta}_{0,2}(\tau) + \frac{31}{1800}\overline{\Theta}_{1,1}(\tau)\right)D_q^{(1)} - \frac{1}{400}\overline{\Theta}_{1,2}(\tau)\right]\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\lambda_{k,j}))](q) = 0.$$

Example 4.5. For $L_{-12/7}(\mathfrak{sl}_2)$, the number of independent q-series characters of all $\mathbb{Z}_2$-twisted modules in category $\mathcal{O}$ is four. They satisfy a fourth-order $\Gamma^0(2)$-MLDE

$$\begin{aligned}&[D_q^{(4)} - \left(\frac{1}{18}\overline{\Theta}_{0,2}(\tau) + \frac{17}{1008}\overline{\Theta}_{1,1}(\tau)\right)D_q^{(2)} + \left(\frac{50}{9261}\overline{\Theta}_{0,3}(\tau) - \frac{883}{49392}\overline{\Theta}_{1,2}(\tau)\right)D_q^{(1)}\\ &\quad + \left(\frac{9}{10976}\overline{\Theta}_{1,3}(\tau) - \frac{225}{175616}\overline{\Theta}_{2,2}(\tau)\right)]\,\mathrm{ch}[\sigma^{-\frac{1}{2}}(L(\Lambda_{k,j}))](q) = 0.\end{aligned}$$

§4.2. $L_{-1/2}(\mathfrak{sl}_2)$

For $L_{-1/2}(\mathfrak{sl}_2)$, there are four admissible highest weight modules whose highest weights are $-\frac{1}{2}\Lambda_0$, $-\frac{3}{2}\Lambda_0+\Lambda_1$, $-\frac{1}{2}\Lambda_1$, and $\Lambda_0-\frac{3}{2}\Lambda_1$. The characters of these modules are

$$\begin{aligned}
\mathrm{ch}\Big[L\Big(-\frac{\Lambda_0}{2}\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}}{2}\Big[\frac{\eta(\tau)}{\theta_4(\mathbf{z};q)}+\frac{\eta(\tau)}{\theta_3(\mathbf{z};q)}\Big],\\
\mathrm{ch}\Big[L\Big(-\frac{\Lambda_1}{2}\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}}{2}\Big[\frac{-i\eta(\tau)}{\theta_1(\mathbf{z};q)}+\frac{\eta(\tau)}{\theta_2(\mathbf{z};q)}\Big],\\
\mathrm{ch}\Big[L\Big(-\frac{3}{2}\Lambda_0+\Lambda_1\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}}{2}\Big[\frac{\eta(\tau)}{\theta_4(\mathbf{z};q)}-\frac{\eta(\tau)}{\theta_3(\mathbf{z};q)}\Big],\\
\mathrm{ch}\Big[L\Big(\Lambda_0-\frac{3}{2}\Lambda_1\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}}{2}\Big[\frac{-i\eta(\tau)}{\theta_1(\mathbf{z};q)}-\frac{\eta(\tau)}{\theta_2(\mathbf{z};q)}\Big].
\end{aligned}$$

Now, considering the action of $\ell=-\frac{1}{2}$ spectral flow on these irreducible highest weight modules, the characters become

$$\begin{aligned}
\mathrm{ch}\Big[\sigma^{-\frac{1}{2}}\Big(L\Big(-\frac{\Lambda_0}{2}\Big)\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}\mathbf{z}^{\frac{1}{4}}q^{-\frac{1}{32}}}{2}\Big[\frac{\eta(q\tau)}{\theta_4(\mathbf{z}q^{-\frac{1}{4}};q)}+\frac{\eta(\tau)}{\theta_3(\mathbf{z}q^{-\frac{1}{4}};q)}\Big],\\
\mathrm{ch}\Big[\sigma^{-\frac{1}{2}}\Big(L\Big(-\frac{\Lambda_1}{2}\Big)\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}\mathbf{z}^{\frac{1}{4}}q^{-\frac{1}{32}}}{2}\Big[\frac{\eta(\tau)}{\theta_4(\mathbf{z}q^{-\frac{1}{4}};q)}-\frac{\eta(\tau)}{\theta_3(\mathbf{z}q^{-\frac{1}{4}};q)}\Big],\\
\mathrm{ch}\Big[\sigma^{-\frac{1}{2}}\Big(L\Big(-\frac{3}{2}\Lambda_0+\Lambda_1\Big)\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}\mathbf{z}^{\frac{1}{4}}q^{-\frac{1}{32}}}{2}\Big[\frac{-i\eta(\tau)}{\theta_1(\mathbf{z}q^{-\frac{1}{4}};q)}+\frac{\eta(\tau)}{\theta_2(\mathbf{z}q^{-\frac{1}{4}};q)}\Big],\\
\mathrm{ch}\Big[\sigma^{-\frac{1}{2}}\Big(L\Big(\Lambda_0-\frac{3}{2}\Lambda_1\Big)\Big)\Big](\mathbf{y},\mathbf{z},q) &= \frac{\mathbf{y}^{-\frac{1}{2}}\mathbf{z}^{\frac{1}{4}}q^{-\frac{1}{32}}}{2}\Big[\frac{-i\eta(\tau)}{\theta_1(\mathbf{z}q^{-\frac{1}{4}};q)}-\frac{\eta(\tau)}{\theta_2(\mathbf{z}q^{-\frac{1}{4}};q)}\Big].
\end{aligned}$$

For the character $\mathrm{ch}[L(-\frac{\Lambda_0}{2})]$, it is a solution of a third-order MLDE under the full $\mathrm{SL}(2,\mathbb{Z})$ group,

$$\Big[D_q^{(3)}-\frac{235}{4}\mathbb{E}_4(\tau)D_q^{(1)}-\frac{455}{8}\mathbb{E}_6(\tau)\Big]\,\mathrm{ch}\Big[L\Big(-\frac{\Lambda_0}{2}\Big)\Big](q)=0.$$

There are two independent well-defined q-series characters,

$$\begin{aligned}
f_1(q) &= \mathrm{ch}\Big[\sigma^{-\frac{1}{2}}\Big(L\Big(-\frac{\Lambda_0}{2}\Big)\Big)\Big](q),\\
f_2(q) &= \mathrm{ch}\Big[\sigma^{-\frac{1}{2}}\Big(L\Big(-\frac{3}{2}\Lambda_0+\Lambda_1\Big)\Big)\Big](q).
\end{aligned}$$

They satisfy a second-order $\Gamma^0(2)$-MLDE,

$$\Big[D_q^{(2)}-\frac{5}{48}\overline{\Theta}_{0,1}(\tau)D_q^{(1)}+\Big(\frac{25}{9216}\overline{\Theta}_{0,2}(\tau)-\frac{41}{9216}\overline{\Theta}_{1,1}(\tau)\Big)\Big]f_i(q)=0.$$

§4.3. $L_{-3/2}(\mathfrak{sl}_3)$

For $L_{-3/2}(\mathfrak{sl}_3)$, there are four irreducible admissible highest weight modules in category $\mathcal{O}$ [Ara16, Per08]. Their characters can be written in terms of the Jacobi theta function [KW88, KW17]:

$$\mathrm{ch}\Big[L\Big(-\frac{3}{2}\Lambda_0\Big)\Big](\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,q) = \mathbf{y}^{-\frac{3}{2}}\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{-1}\frac{\theta_1(\mathbf{z}_1;q^2)\theta_1(\mathbf{z}_2;q^2)\theta_1(\mathbf{z}_1\mathbf{z}_2;q^2)}{\theta_1(\mathbf{z}_1;q)\theta_1(\mathbf{z}_2;q)\theta_1(\mathbf{z}_1\mathbf{z}_2;q)},$$
$$\mathrm{ch}\Big[L\Big(-\frac{3}{2}\Lambda_1\Big)\Big](\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,q) = -\mathbf{y}^{-\frac{3}{2}}\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{-1}\frac{\theta_1(\mathbf{z}_2;q^2)\theta_4(\mathbf{z}_1;q^2)\theta_4(\mathbf{z}_1\mathbf{z}_2;q^2)}{\theta_1(\mathbf{z}_1;q)\theta_1(\mathbf{z}_2;q)\theta_1(\mathbf{z}_1\mathbf{z}_2;q)},$$
$$\mathrm{ch}\Big[L\Big(-\frac{3}{2}\Lambda_2\Big)\Big](\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,q) = -\mathbf{y}^{-\frac{3}{2}}\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{-1}\frac{\theta_1(\mathbf{z}_1;q^2)\theta_4(\mathbf{z}_1\mathbf{z}_2;q^2)\theta_4(\mathbf{z}_2;q^2)}{\theta_1(\mathbf{z}_1;q)\theta_1(\mathbf{z}_2;q)\theta_1(\mathbf{z}_1\mathbf{z}_2;q)},$$
$$\mathrm{ch}\Big[L\Big(-\frac{\Lambda_0+\Lambda_1+\Lambda_2}{2}\Big)\Big](\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,q)$$
$$= \mathbf{y}^{-\frac{3}{2}}(\mathbf{z}_1\mathbf{z}_2)^{\frac{3}{2}}q^{\frac{3}{2}}\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{-1}\frac{\theta_1(\mathbf{z}_2^{-1}q^{-1};q^2)\theta_1(\mathbf{z}_1^{-1}q^{-1};q^2)\theta_1(\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}q^{-2};q^2)}{\theta_1(\mathbf{z}_1;q)\theta_1(\mathbf{z}_2;q)\theta_1(\mathbf{z}_1\mathbf{z}_2;q)},$$

where $z=\sum_{i=1}^{2}\mathfrak{z}_i\bar{\Lambda}_i$ and $\mathbf{z}_i=e^{2\pi i\mathfrak{z}_i}$. Now we consider the action of spectral flow twist on these irreducible highest weight modules. Firstly, we consider the spectral flow along the $\frac{1}{2}\bar{\Lambda}_1^\vee$ direction, and the character becomes

$$\mathrm{ch}\Big[\sigma^{\frac{1}{2}\bar{\Lambda}_1^\vee}\Big(L\Big(-\frac{3}{2}\Lambda_0\Big)\Big)\Big](\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,q)$$
$$= (\mathbf{y}\mathbf{z}_1^{\frac{1}{3}}\mathbf{z}_2^{\frac{1}{6}}q^{\frac{1}{12}})^{-\frac{3}{2}}\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{-1}\frac{\theta_1(\mathbf{z}_1q^{\frac{1}{2}};q^2)\theta_1(\mathbf{z}_2;q^2)\theta_1(\mathbf{z}_1\mathbf{z}_2q^{\frac{1}{2}};q^2)}{\theta_1(\mathbf{z}_1q^{\frac{1}{2}};q)\theta_1(\mathbf{z}_2;q)\theta_1(\mathbf{z}_1\mathbf{z}_2q^{\frac{1}{2}};q)}.$$

This spectral flowed module is an ordinary module. It satisfies a second-order $\Gamma^0(2)$-modular linear differential equation,

$$\Big(D_q^{(2)}-\frac{5}{576}\overline{\Theta}_{0,2}(\tau)-\frac{11}{576}\overline{\Theta}_{1,1}(\tau)\Big)\,\mathrm{ch}\Big[\sigma^{\frac{1}{2}\bar{\Lambda}_1^\vee}\Big(L\Big(-\frac{3}{2}\Lambda_0\Big)\Big)\Big](q)=0.$$

Secondly, consider the spectral flow of the character $\mathrm{ch}[L(-\frac{\Lambda_0+\Lambda_1+\Lambda_2}{2})]$ along the $\frac{1}{3}(\bar{\Lambda}_1^\vee+\bar{\Lambda}_2^\vee)$ direction,

$$\mathrm{ch}\Big[\sigma^{\frac{1}{3}\bar{\Lambda}_1^\vee+\frac{1}{3}\bar{\Lambda}_2^\vee}\Big(L\Big(-\frac{\Lambda_0+\Lambda_1+\Lambda_2}{2}\Big)\Big)\Big](y,\mathbf{z}_1,\mathbf{z}_2,q)$$
$$= (\mathbf{y}\mathbf{z}_1^{\frac{1}{3}}\mathbf{z}_2^{\frac{1}{3}}q^{\frac{1}{9}})^{-\frac{3}{2}}(\mathbf{z}_1\mathbf{z}_2q^{\frac{3}{2}})^{\frac{3}{2}}q^{\frac{3}{2}}\Big(\frac{\eta(2\tau)}{\eta(\tau)}\Big)^{-1}$$
$$\times\frac{\theta_1((\mathbf{z}_2q^{\frac{1}{3}})^{-1}q^{-1};q^2)\theta_1((\mathbf{z}_1q^{\frac{1}{3}})^{-1}q^{-1};q^2)\theta_1((\mathbf{z}_1\mathbf{z}_2q^{-\frac{2}{3}})^{-1}q^{-2};q^2)}{\theta_1(\mathbf{z}_1q^{\frac{1}{3}};q)\theta_1(\mathbf{z}_2q^{\frac{1}{3}};q)\theta_1(\mathbf{z}_1\mathbf{z}_2q^{\frac{2}{3}};q)}.$$

This spectral flowed character matches with the lens space index of (A_1,D_4) AD theory [FS18] with a suitable change of variables.

§4.4. Bershadsky–Polyakov algebra BP^k with $k = -\frac{9}{4}$

In previous examples, we consider the spectral flowed modules of affine VOA $L_k(\mathfrak{g})$. Now we consider an example of the affine W-algebra [BS95, FF90a, FF90b, KRW03], the $W^k(\mathfrak{sl}_3, f_{\min})$ which agrees with the BP^k-algebra and was defined in [Ber91], and previously studied in [AK21, FR22]. First, let us review the definition of BP^k-algebra.

Definition 4.6 ([AK21, FKR21]). Given $k \in \mathbb{C}$, $k \neq -3$, the level k universal Bershadsky–Polyakov algebra BP^k is the vertex operator algebra with vacuum $\mathbf{1}$ that is strongly and freely generated by fields $J(z)$, $G^+(z)$, $G^-(z)$, and $L(z)$ satisfying the complicated operator product expansions. The conformal weights of the generating fields $J(z)$, $G^+(z)$, $G^-(z)$, and $L(z)$ are 1, $\frac{3}{2}$, $\frac{3}{2}$, and 2 respectively, and the central charge is

$$c_{u,v}^{\mathrm{BP}} = -\frac{(2k+3)(3k+1)}{k+3}.$$

The action of the spectral flow twist σ^ℓ, $\ell \in \frac{1}{2}\mathbb{Z}$ for the vertex algebra BP^k on the modes of the generating field $J(z)$, $G^+(z)$, $G^-(z)$, and $L(z)$ is

$$\begin{aligned}
\sigma^\ell(J_n) &= J_n - \frac{2k+3}{3}\ell\delta_{n,0}\mathbf{1},\\
\sigma^\ell(G_r^+) &= G_{r-\ell}^+,\\
\sigma^\ell(G_r^-) &= G_{r+\ell}^-,\\
\sigma^\ell(L_n) &= L_n - \ell J_n + \frac{2k+3}{6}\ell^2\delta_{n,0}\mathbf{1}.
\end{aligned}$$

Let $|\lambda, \Delta\rangle \in M$ be a vector of weight λ and conformal dimension Δ in any module M. Then the state $(\sigma^\ell)^*|\lambda, \Delta\rangle \in (\sigma^\ell)^*(M)$ satisfies

$$\begin{aligned}
h_0(\sigma^\ell)^*|\lambda, \Delta\rangle &= \Big(\lambda + \ell\frac{2k+3}{3}\Big)(\sigma^\ell)^*|\lambda, \Delta\rangle,\\
L_0(\sigma^\ell)^*|\lambda, \Delta\rangle &= \Big(\Delta + \ell\lambda + \frac{2k+3}{6}\ell^2\Big)(\sigma^\ell)^*|\lambda, \Delta\rangle.
\end{aligned}$$

We consider the special case where the level $k = -\frac{9}{4}$ and $c_{u,v}^{\mathrm{BP}} = -\frac{23}{2}$, the lowest conformal weight vector is $|\lambda, \Delta\rangle = |\frac{1}{4}, -\frac{3}{8}\rangle$. After the spectral flow twist $\sigma^{\frac{1}{2}}$, the weight and conformal dimension become

$$\lambda' = \frac{1}{4} + \frac{1}{2} \times \Big(-\frac{1}{2}\Big) = 0, \quad \Delta' = -\frac{3}{8} + \frac{1}{2} \times \frac{1}{4} - \frac{1}{4} \times \frac{1}{4} = -\frac{5}{16}.$$

According to [CGS17], the character of a spectral flowed module can be written as

$$\mathrm{ch}[\sigma^{\frac{1}{2}}(L(\lambda))](q) = q^{\frac{1}{6}}(1 + 4q + 10q^2 + 24q^3 + 51q^4 + 100q^5 + \mathcal{O}(q^6)).$$

This character satisfies a third-order MLDE under the full $\mathrm{SL}(2,\mathbb{Z})$ group,

$$[D_q^{(3)} - 25\mathbb{E}_4(\tau)D_q^{(1)} - 175\mathbb{E}_6(\tau)]\,\mathrm{ch}[\sigma^{\frac{1}{2}}(L(\lambda))](q) = 0.$$

§5. $L_{-2}(\mathfrak{so}_8)$

In this section we prove a relation between simple modules and spectral flowed modules of $L_{-2}(\mathfrak{so}_8)$. We also find an ordinary module $\sigma^{\frac{1}{2}\bar{\Lambda}_2}(L(-\Lambda_2))$, whose character satisfies a second-order $\Gamma^0(2)$-MLDE.

Firstly, recall the result for the simple modules of $L_{-2}(\mathfrak{so}_8)$.

Theorem 5.1 ([Per13]). *The set $\{L(-2\Lambda_0), L(-2\Lambda_1), L(-2\Lambda_3), L(-2\Lambda_4), L(-\Lambda_2)\}$ provides a complete list of irreducible weak $L_{-2}(\mathfrak{so}_8)$-modules from the category $\mathcal{O}$. Among these, $L(-2\Lambda_0)$ is the unique irreducible ordinary module for $L_{-2}(\mathfrak{so}_8)$. And every ordinary $L_{-2}(\mathfrak{so}_8)$-module is completely reducible.*

We consider the vacuum module with a spectral flow twist. The fundamental weights $\bar{\Lambda}_i$ of the Lie algebra $\mathfrak{so}_8$ can be written in terms of the linear combination of simple roots α_i:

$$\bar{\Lambda}_1 = \alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_4, \quad \bar{\Lambda}_2 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4,$$
$$\bar{\Lambda}_3 = \frac{1}{2}\alpha_1 + \alpha_2 + \alpha_3 + \frac{1}{2}\alpha_4, \quad \bar{\Lambda}_4 = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3 + \alpha_4.$$

Since $\mathfrak{so}_8$ is simply laced, $\alpha_i = \alpha_i^\vee$ and $\bar{\Lambda}_i = \bar{\Lambda}_i^\vee$, and we shall use roots (weights) or coroots (coweights) without distinction. The highest root of finite Lie algebra $\mathfrak{so}_8$ is $\theta = \bar{\Lambda}_2$; therefore, the marks and comarks of affine Lie algebra $\widehat{\mathfrak{so}}_8$ are $(a_i) = (a_i^\vee) = (1,1,2,1,1)$, and the level of affine weight $\Lambda = \sum_{i=0}^{4} \lambda_i\Lambda_i$ is then given by

$$k = \lambda_0 + \lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4.$$

Theorem 5.2. *The $L(-2\Lambda_i)$ with $i = 1,3,4$ are the same as the spectral flowed modules $\sigma^{\bar{\Lambda}_i}(L(-2\Lambda_0))$.*

Proof. The powers of τ_i acts as

$$\tau_i^\ell(h_n^j) = h_n^j - \ell(\alpha_i,\alpha_i)\delta_{n,0}K.$$

We can compute the action of this automorphism along $\bar{\Lambda}_1$ on h_0^i:

$$\tau_1\tau_2\tau_3^{\frac{1}{2}}\tau_4^{\frac{1}{2}}(h_0^1) = h_0^1 + K, \quad \tau_1\tau_2\tau_3^{\frac{1}{2}}\tau_4^{\frac{1}{2}}(h_0^2) = h_0^2,$$
$$\tau_1\tau_2\tau_3^{\frac{1}{2}}\tau_4^{\frac{1}{2}}(h_0^3) = h_3^0, \qquad \tau_1\tau_2\tau_3^{\frac{1}{2}}\tau_4^{\frac{1}{2}}(h_0^4) = h_0^4,$$

where $\sigma^{\bar{\Lambda}_1} = \tau_1\tau_2\tau_3^{\frac{1}{2}}\tau_4^{\frac{1}{2}}$. This means the spectral flow automorphism changes the highest weight $-2\Lambda_0$ into another highest weight $-2\Lambda_1$. One can get highest weight $-2\Lambda_3$ and $-2\Lambda_4$ in the same way. According to [DLM96], the spectral flow twist along the $\bar{\Lambda}_i$ direction preserves the irreducibility of the module $L(-2\Lambda_i)$. The result follows. □

In fact, the closed form expressions of characters of all simple modules of $L_{-2}(\mathfrak{so}_8)$ are conjectured from the SCFT/VOA correspondence ([PP22, ZPW22]) as

$$
\begin{aligned}
\mathrm{ch}[L(-2\Lambda_0)] &= \mathcal{I}_{0,4},\\
\mathrm{ch}[L(-2\Lambda_1)] &= \mathcal{I}_{0,4} - 2R_1,\\
\mathrm{ch}[L(-\Lambda_2)] &= -2\mathcal{I}_{0,4} + 2R_1 + 2R_2,\\
\mathrm{ch}[L(-2\Lambda_3)] &= \mathcal{I}_{0,4} - R_1 - R_2 - R_3 - R_4,\\
\mathrm{ch}[L(-2\Lambda_4)] &= \mathcal{I}_{0,4} - R_1 - R_2 - R_3 + R_4,
\end{aligned}
\tag{5.1}
$$

where $\mathcal{I}_{0,4}$ is the Schur index of an SU(2) gauge theory with four hypermultiplets, which is equal to

$$
\begin{aligned}
&\mathrm{ch}[L(-2\Lambda_0)](q; , \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)\\
&\quad = \frac{1}{2}\frac{\eta(\tau)^2}{\theta_1(\tilde{z}_1\tilde{z}_2^2\tilde{z}_3\tilde{z}_4; q)\theta_1(\tilde{z}_1; q)\theta_1(\tilde{z}_4; q)\theta_1(\tilde{z}_3; q)}\\
&\qquad \times \sum_{\vec{\alpha}=\pm}\left(\prod_{i=1}^{4}\alpha_i\right)E_2\left[\frac{1}{(\tilde{z}_1^{\frac{1}{2}}\tilde{z}_2\tilde{z}_3^{\frac{1}{2}}\tilde{z}_4^{\frac{1}{2}})^{\alpha_1}(\tilde{z}_1^{\frac{1}{2}})^{\alpha_2}(\tilde{z}_4^{\frac{1}{2}})^{\alpha_3}(\tilde{z}_3^{\frac{1}{2}})^{\alpha_4}}\right],
\end{aligned}
\tag{5.2}
$$

and the R_j functions are

$$
R_j(\widetilde{\boldsymbol{m}}_i, \tau) = \frac{i}{2}\frac{\theta_1(2\widetilde{\boldsymbol{m}}_j; q)}{\eta(\tau)}\prod_{l\neq j}\frac{\eta(\tau)}{\theta_1(\widetilde{\boldsymbol{m}}_j + \widetilde{\boldsymbol{m}}_l; q)}\frac{\eta(\tau)}{\theta_1(\widetilde{\boldsymbol{m}}_j - \widetilde{\boldsymbol{m}}_l; q)}, \quad j = 1, \ldots, 4.
$$

Here, $\widetilde{m}_i = e^{2\pi i\widetilde{\boldsymbol{m}}_i}$ are related to the Cartan element $\tilde{z}_i = e^{2\pi i\tilde{\boldsymbol{z}}_i}$ of $\mathfrak{so}_8$ as

$$
\tilde{z}_1 = \frac{\widetilde{m}_1}{\widetilde{m}_2}, \quad \tilde{z}_2 = \frac{\widetilde{m}_2}{\widetilde{m}_3}, \quad \tilde{z}_3 = \widetilde{m}_3\widetilde{m}_4, \quad \tilde{z}_4 = \frac{\widetilde{m}_3}{\widetilde{m}_4}.
$$

Proposition 5.3. *The characters of simple modules of $L_{-2}(\mathfrak{so}_8)$ are*

$$
\begin{aligned}
&\mathrm{ch}[L(-2\Lambda_1)](q; , \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)\\
&\quad = \frac{1}{2}(y\tilde{z}_1\tilde{z}_2\tilde{z}_3^{\frac{1}{2}}\tilde{z}_4^{\frac{1}{2}}q^{\frac{1}{2}})^{-2}\frac{\eta(\tau)^2}{\theta_1(\tilde{z}_1q\tilde{z}_2^2\tilde{z}_3\tilde{z}_4; q)\theta_1(\tilde{z}_1q; q)\theta_1(\tilde{z}_4; q)\theta_1(\tilde{z}_3; q)}\\
&\qquad \times \sum_{\vec{\alpha}=\pm}\left(\prod_{i=1}^{4}\alpha_i\right)E_2\left[\frac{1}{(\tilde{z}_1^{\frac{1}{2}}q^{\frac{1}{2}}\tilde{z}_2\tilde{z}_3^{\frac{1}{2}}\tilde{z}_4^{\frac{1}{2}})^{\alpha_1}(\tilde{z}_1^{\frac{1}{2}}q^{\frac{1}{2}})^{\alpha_2}(\tilde{z}_4^{\frac{1}{2}})^{\alpha_3}(\tilde{z}_3^{\frac{1}{2}})^{\alpha_4}}\right],
\end{aligned}
$$

$$\begin{aligned}&\mathrm{ch}[L(-2\Lambda_3)](q;,\tilde z_1,\tilde z_2,\tilde z_3,\tilde z_4)\\&\quad=\frac12(y\tilde z_1^{\frac12}\tilde z_2\tilde z_3\tilde z_4^{\frac12}q^{\frac12})^{-2}\frac{\eta(\tau)^2}{\theta_1(\tilde z_1\tilde z_2^2\tilde z_3q\tilde z_4;q)\theta_1(\tilde z_1;q)\theta_1(\tilde z_4;q)\theta_1(\tilde z_3q;q)}\\&\quad\quad\times\sum_{\vec\alpha=\pm}\left(\prod_{i=1}^4\alpha_i\right)E_2\left[\frac{1}{(\tilde z_1^{\frac12}\tilde z_2\tilde z_3^{\frac12}q^{\frac12}\tilde z_4^{\frac12})^{\alpha_1}(\tilde z_1^{\frac12})^{\alpha_2}(\tilde z_4^{\frac12})^{\alpha_3}(\tilde z_3^{\frac12}q^{\frac12})^{\alpha_4}}\right],\\&\mathrm{ch}[L(-2\Lambda_4)](q;,\tilde z_1,\tilde z_2,\tilde z_3,\tilde z_4)\\&\quad=\frac12(y\tilde z_1^{\frac12}\tilde z_2\tilde z_3^{\frac12}\tilde z_4q^{\frac12})^{-2}\frac{\eta(\tau)^2}{\theta_1(\tilde z_1\tilde z_2^2\tilde z_3\tilde z_4q;q)\theta_1(\tilde z_1;q)\theta_1(\tilde z_4q;q)\theta_1(\tilde z_3;q)}\\&\quad\quad\times\sum_{\vec\alpha=\pm}\left(\prod_{i=1}^4\alpha_i\right)E_2\left[\frac{1}{(\tilde z_1^{\frac12}\tilde z_2\tilde z_3^{\frac12}\tilde z_4^{\frac12}q^{\frac12})^{\alpha_1}(\tilde z_1^{\frac12})^{\alpha_2}(\tilde z_4^{\frac12}q^{\frac12})^{\alpha_3}(\tilde z_3^{\frac12})^{\alpha_4}}\right].\end{aligned}$$

Proof. According to [Ara19], $L_{-2}(\mathfrak{so}_8)$ is the vertex algebra associated to the genus zero $\mathfrak{a}_1$ class $\mathcal{S}$ theory, whose Schur index was given by [PP22, formula (1.2)]. It is essentially the same as the formula in [Ara19, Proposition 10.5] by taking $G=\mathrm{SL}_2$ and $b=4$. The contour integral in [PP22, formula (1.2)] was computed in [PP22, Section 4.45], which implies that [PP22, formula (1.2)] equals (5.2). The result follows from Theorem 5.2. □

We can also check that the characters of vacuum modules $L(-2\Lambda_0)$ and the partition function of the curved $\beta\gamma$ system [ELS20] are equal up to q^{10} after expanding the expression in terms of the parameter q. The partition function of the curved $\beta\gamma$ system on complex Grassmannian $\mathrm{Gr}(2,4)$ is given by

$$Z_{\mathfrak{so}_8}(t,\boldsymbol{m}^{\mathfrak{sl}_4},\tau)=\frac{i\eta(\tau)\theta_1(2\sigma,\tau)}{\prod_{\omega\in\rho}\theta_1(\sigma+(\boldsymbol{m}^{\mathfrak{sl}_4},\omega),\tau)},$$

where $t=e^{2\pi i\sigma}$, $q=e^{2\pi i\tau}$, and $\boldsymbol{m}^{\mathfrak{g}}\in\mathfrak{h}_{\mathfrak{g}}$, the Cartan subalgebra of Lie algebra $\mathfrak{g}$. The product in the denominator is over the weights in the representation ρ of $\mathfrak{sl}_4$ with highest weight $\bar\Lambda_{\mathfrak{sl}_4}$. The authors of [ELS20] found that this partition function is also given by

$$Z_{\mathfrak{so}_8}(t,\boldsymbol{m}^{\mathfrak{sl}_4},\tau)=\mathrm{ch}[L(-2\Lambda_0)](\boldsymbol{m}^{\mathfrak{so}_8},\tau)-\mathrm{ch}[L(-2\Lambda_4)](\boldsymbol{m}^{\mathfrak{so}_8},\tau),$$

with the following identifications of parameters:

$$\boldsymbol{m}_i^{\mathfrak{so}_8}=\boldsymbol{m}_i^{\mathfrak{sl}_4},\ \text{ for } i=1,2,3,\quad \boldsymbol{m}_4^{\mathfrak{so}_8}=\sigma-\frac{\boldsymbol{m}_1^{\mathfrak{sl}_4}}{2}-\boldsymbol{m}_2^{\mathfrak{sl}_4}-\frac{\boldsymbol{m}_3^{\mathfrak{sl}_4}}{2}.$$

Using the expression for $\mathrm{ch}[L(-2\Lambda_0)]$, one also sees that

$$Z_{\mathfrak{so}_8}(t,\boldsymbol{m}^{\mathfrak{sl}_4},\tau)=\mathrm{ch}[L(-2\Lambda_0)](\boldsymbol{m}^{\mathfrak{so}_8},\tau)-\mathrm{ch}[\sigma^{\bar\Lambda_4}(L(-2\Lambda_0))](\boldsymbol{m}^{\mathfrak{so}_8},\tau).$$

Therefore, we have

$$\mathrm{ch}[L(-2\Lambda_4)](\boldsymbol{m}^{\mathfrak{so}_8},\tau)=\mathrm{ch}[\sigma^{\bar{\Lambda}_4}(L(-2\Lambda_0))](\boldsymbol{m}^{\mathfrak{so}_8},\tau).$$

Triality gives similar results for $i=1$ and 2.

Now let us consider the character of the spectral flowed module of $L(-\Lambda_2)$ along the $\frac{1}{2}\bar{\Lambda}_2$ direction based on the third formula in (5.1):

$$\begin{aligned}&\mathrm{ch}[\sigma^{\frac{1}{2}\bar{\Lambda}_2}(L(-\Lambda_2))](q;\tilde{z}_1,\tilde{z}_2,\tilde{z}_3,\tilde{z}_4)\\&\quad=(y\tilde{z}_1^{\frac{1}{2}}\tilde{z}_2\tilde{z}_3^{\frac{1}{2}}\tilde{z}_4^{\frac{1}{2}}q^{\frac{1}{4}})^{-2}\,\mathrm{ch}[L(-\Lambda_2)](q;\tilde{z}_1,\tilde{z}_2q^{-\frac{1}{2}},\tilde{z}_3,\tilde{z}_4).\end{aligned}$$

Moreover, $\mathrm{ch}[\sigma^{\frac{1}{2}\bar{\Lambda}_2}(L(-\Lambda_2))]$ is the character of an ordinary $\mathbb{Z}_2$-twisted module as it converges under the limit $\tilde{z}_i\to 1$. It equals the defect index $\mathcal{I}^{\mathrm{defect}}_{0,4}(k=1)$ of [ZPW22], which satisfies the second-order $\Gamma^0(2)$-MLDE

$$\left(D_q^{(2)}+\frac{1}{144}\overline{\Theta}_{0,2}(\tau)-\frac{37}{288}\overline{\Theta}_{1,1}(\tau)\right)\mathrm{ch}[\sigma^{\frac{1}{2}\bar{\Lambda}_2}(L(-\Lambda_2))](q)=0,$$

without $D_q^{(1)}$ term.

Appendix A. Theta functions

One can use classical theta functions to define *Jacobi theta functions* of degree two:

$$\begin{aligned}\theta_3(\mathbf{z};q)\equiv\vartheta_{00}(\tau,z)&=\Theta_{2,2}(\tau,z)+\Theta_{0,2}(\tau,z)\\&=\sum_{n\in\mathbb{Z}}\mathbf{z}^{2(n+\frac{1}{2})}q^{2(n+\frac{1}{2})^2}+\mathbf{z}^{2n}q^{2n^2}=\sum_{n\in\mathbb{Z}}\mathbf{z}^nq^{\frac{n^2}{2}},\\\theta_4(\mathbf{z};q)\equiv\vartheta_{01}(\tau,z)&=-\Theta_{2,2}(\tau,z)+\Theta_{0,2}(\tau,z)\\&=\sum_{n\in\mathbb{Z}}-\mathbf{z}^{2(n+\frac{1}{2})}q^{2(n+\frac{1}{2})^2}+\mathbf{z}^{2n}q^{2n^2}=\sum_{n\in\mathbb{Z}}(-1)^n\mathbf{z}^nq^{\frac{n^2}{2}},\\\theta_2(\mathbf{z};q)\equiv\vartheta_{10}(\tau,z)&=\Theta_{1,2}(\tau,z)+\Theta_{-1,2}(\tau,z)\\&=\sum_{n\in\mathbb{Z}}\mathbf{z}^{2(n+\frac{1}{4})}q^{2(n+\frac{1}{4})^2}+\mathbf{z}^{2(n-\frac{1}{4})}q^{2(n-\frac{1}{4})^2}=\sum_{n\in\mathbb{Z}}\mathbf{z}^{(n+\frac{1}{2})}q^{\frac{(n+\frac{1}{2})^2}{2}},\\\theta_1(\mathbf{z};q)\equiv-\vartheta_{11}(\tau,z)&=i\Theta_{1,2}(\tau,z)-i\Theta_{-1,2}(\tau,z)\\&=\sum_{n\in\mathbb{Z}}i\mathbf{z}^{2(n+\frac{1}{4})}q^{2(n+\frac{1}{4})^2}-i\mathbf{z}^{2(n-\frac{1}{4})}q^{2(n-\frac{1}{4})^2}\\&=\sum_{n\in\mathbb{Z}}e^{\pi i(n+\frac{1}{2})}\mathbf{z}^{(n+\frac{1}{2})}q^{\frac{(n+\frac{1}{2})^2}{2}}.\end{aligned}$$

Using $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, one can get the following infinite product identities:

$$\prod_{n=1}^{\infty}(1+\mathbf{z}q^{n-\frac{1}{2}})(1+\mathbf{z}^{-1}q^{n-\frac{1}{2}}) = q^{\frac{1}{24}}\frac{\vartheta_{00}(\tau,z)}{\eta(q)},$$

$$\prod_{n=1}^{\infty}(1-\mathbf{z}q^{n-\frac{1}{2}})(1-\mathbf{z}^{-1}q^{n-\frac{1}{2}}) = q^{\frac{1}{24}}\frac{\vartheta_{01}(\tau,z)}{\eta(q)},$$

$$\prod_{n=1}^{\infty}(1+\mathbf{z}q^{n})(1+\mathbf{z}^{-1}q^{n-1}) = q^{-\frac{1}{12}}\mathbf{z}^{-\frac{1}{2}}\frac{\vartheta_{10}(\tau,z)}{\eta(q)},$$

$$\prod_{n=1}^{\infty}(1-\mathbf{z}q^{n})(1-\mathbf{z}^{-1}q^{n-1}) = -iq^{-\frac{1}{12}}\mathbf{z}^{-\frac{1}{2}}\frac{\vartheta_{11}(\tau,z)}{\eta(q)},$$

where q is, as always, $e^{2\pi i\tau}$.

We summarize some basic facts about the affine Lie algebra of $\widehat{\mathfrak{sl}_2}$.

Cartan matrix	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$
Simple roots	$\{\alpha_0, \alpha_1\}$
$\mathfrak{h}^*$	$\mathrm{Span}_{\mathbb{C}}\{\alpha_0, \alpha_1, \Lambda_0\}$
Bilinear form on $\mathfrak{h}^*$	$\langle \alpha_0, \alpha_0 \rangle = 2$
	$\langle \alpha_1, \alpha_1 \rangle = 2$
	$\langle \alpha_1, \alpha_0 \rangle = \langle \alpha_0, \alpha_1 \rangle = -2$
	$\langle \alpha_0, \Lambda_0 \rangle = 1$
	$\langle \Lambda_0, \Lambda_0 \rangle = \langle \alpha_1, \Lambda_0 \rangle = 0$
Basic imaginary root	$\delta = \alpha_0 + \alpha_1$
$\mathfrak{h}$	$\mathrm{Span}_{\mathbb{C}}\{\alpha_0^\vee = h_1, \alpha_2^\vee = h_2, d\}$
Central element	$c = \alpha_0^\vee + \alpha_1^\vee$
Fundamental weights	$\{\Lambda_0, \Lambda_1\}$
	$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$, $\Lambda_i(d) = 0$, $i = 0, 1$
Lattice M	$\mathbb{Z}h_1$
Lattice M^*	$\frac{1}{2}\mathbb{Z}\alpha_1$
Integral forms P	$\{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z},\ i = 0, 1\}$
Positive integral forms P_+	$\{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0},\ i = 0, 1\}$
Weyl group	$t(M) \rtimes \overline{W}$, $t(M) = \{t_m \mid m \in M\}$, $\overline{W} = \{s_1\}$
Lacing number	1
Coxeter dual number	2

Appendix B. Proof in Proposition 3.4

The calculation of (3.5):

$$
\begin{aligned}
&T_-^a T_0^b T_+^d P(E_1(n,l)) \\
&= T_-^a T_0^b T_+^d \Big(\prod_{r=1}^{n} \prod_{s=1}^{l-1} G_{-r-st} \Big) T_+^n \\
&= T_-^a \Big(\prod_{r=1}^{n} \prod_{s=1}^{l-1} G_{-r-st-d} \Big) T_0^b T_+^{d+n} \\
&= T_-^a \Big(\prod_{r=1}^{n} \prod_{s=1}^{l-1} (T_- T_+ - (-r-st-d)T_0 + (-r-st-d)(-r-st-d+1)) \Big) T_0^b T_+^{d+n} \\
&= T_-^a \Big(\prod_{r=1}^{n} \prod_{s=1}^{l-1} (r+st+d)(T_0+r+st+d-1) \Big) T_0^b T_+^{n+d} \mod T_- U(K_0).
\end{aligned}
$$

The calculation of (3.6):

$$
\begin{aligned}
&T_-^a T_0^b T_+^d P(E_2(n,l)) \\
&= T_-^a T_0^b T_+^m \Big(\prod_{i=1}^{p-n} G_{-i} \Big) \prod_{r=0}^{p-n-1} \prod_{s=1}^{q-l} G_{r+st-p+n} \\
&= T_-^a T_0^b T_+^m \prod_{r=0}^{p-n-1} \prod_{s=0}^{q-l} G_{r+st-p+n} \\
&= T_-^a \prod_{r=0}^{p-n-1} \prod_{s=0}^{q-l} G_{r+st-p+n-m} T_0^b T_+^m \\
&= T_-^a \prod_{r=1}^{p-n} \prod_{s=0}^{q-l} G_{st-m-r} T_0^b T_+^m \\
&= T_-^a \prod_{r=1}^{p-n} \prod_{s=0}^{q-l} (st-m-r)(-T_0+st-m-r+1) T_0^b T_+^m \mod T_- U(K_0).
\end{aligned}
$$

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