Doc. Math. (Online first)
DOI 10.4171/DM/1069

Published by EMS Press

Costratification and actions of tensor-triangulated categories

Charalampos Verasdanis

Abstract. We develop the theory of costratification in the setting of relative tensor-triangular geometry, in the sense of Stevenson, providing a unified approach to classification results of Neeman and Benson–Iyengar–Krause. In addition, we introduce and study prime localizing submodules and prime colocalizing hom-submodules, in the first case, generalizing objectwise-prime localizing tensor-ideals. We apply our results to show that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified. An application of the results proved in this paper, which appears in separate work, is that the singularity category of a locally hypersurface ring (more generally a noetherian separated scheme with hypersurface singularities) is costratified.

Introduction

The theory of cosupport and costratification in tensor-triangulated categories was initiated by Bensor–Iyengar–Krause [12], inspired by the classification of colocalizing subcategories of the derived category of a commutative noetherian ring by Neeman [21]. Their main application was the classification of Hom-closed colocalizing subcategories of the stable module category of a finite group. Compared to the theory of support and stratification [7, 9–11] (which is related to the classification of localizing subcategories) the theory of costratification has not been explored to the same extent. In this paper, we develop the theory of costratification (in the more general context of relative tensor-triangular geometry [22]) and thereby provide a unification of the aforementioned classifications. In addition, we generalize Neeman's classification [21] to derived categories of noetherian separated schemes.

It should be noted that Barthel–Castellana–Heard–Sanders have independently developed a theory of costratification in tensor-triangular geometry [6]. On the one hand, our results encompass a broader class of triangulated categories, i.e., those that are not necessarily tensor-triangulated but are endowed with an action of a tensor-triangulated category. Most importantly, singularity categories of commutative noetherian rings [15, 24]. Applications in this context appear separately in [31]. We also relate the concepts of prime localizing submodules and prime colocalizing hom-submodules with stratification and costratification. On the other hand, by focusing on the tensor-triangulated case, the

---

Mathematics Subject Classification 2020: 18G80 (primary); 18F99 (secondary).
Keywords: costratification, cosupport, relative tensor-triangular geometry.

article [6] establishes, for example, descent results for costratification and also provides a variety of examples and applications.

Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category and let $\mathcal{K}$ be a compactly generated triangulated category endowed with an action of $\mathcal{T}$; see Section 1 for details. A special case is $\mathcal{K} = \mathcal{T}$ with the action given by the tensor product of $\mathcal{T}$.

Section 1 consists of preliminary material – including a very brief account of the Balmer–Favi support [5] and the smashing spectrum and the small smashing support [1] – and establishes concepts and basic lemmas that we will be using throughout the paper.

In Section 2, we introduce the notion of a good support–cosupport pair on $\mathcal{T}$ taking values in a space S, which induces a support–cosupport pair on $\mathcal{K}$. For example, one could take the small smashing support–cosupport or the Balmer–Favi support–cosupport or the BIK support–cosupport. The reason we do not fix a specific support–cosupport from the above list is for conceptual clarity. The notion we will most be concerned with is that of costratification. There are dual results regarding stratification but for brevity we will not mention these in the introduction. We say that $\mathcal{K}$ is *costratified* (with respect to a specified support–cosupport) if there is a bijective correspondence between subsets of a certain subspace of S and the collection of colocalizing hom-submodules of $\mathcal{K}$; see Definition 2.13 for the precise statement. Our first main result is the following.

Theorem 2.22. *Let* $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ *be a good support-cosupport pair on* $\mathcal{T}$*. Then* $\mathcal{K}$ *is costratified with respect to* (s_Γ, c_Γ) *if and only if* $\mathcal{K}$ *satisfies the colocal-to-global principle and cominimality.*

These two conditions, the colocal-to-global principle and cominimality, are in a sense dual to the more well-established local-to-global principle and minimality, with the two latter introduced in [11] and studied further in [7, 22, 25, 26]. In the generality of relative tensor-triangular geometry, we are able to prove that the local-to-global principle implies the colocal-to-global principle; see Corollary 2.28. If one restricts to the case $\mathcal{K} = \mathcal{T}$, i.e., classical tensor-triangular geometry, then the converse has been shown to hold in [6]. In our setting, their techniques cannot be reproduced and thus it remains unclear whether the colocal-to-global principle implies the local-to-global principle.

In Section 3, we introduce and study prime localizing submodules and prime colocalizing hom-submodules of $\mathcal{K}$, the former specializing to the class of objectwise-prime localizing tensor-ideals when $\mathcal{K} = \mathcal{T}$; see [1, 30]. Our main results are Theorem 3.11 and Theorem 3.12, where we prove that if $\mathcal{K}$ is (co)stratified, then there is a bijective correspondence between points of a certain subspace of S and prime (co)localizing (hom-)submodules of $\mathcal{K}$, obtaining a complete description of the latter as precisely the kernels of the associated action (relative internal-hom) functors. In addition, we analyze the relation of prime localizing submodules and prime colocalizing hom-submodules with the Action Formula (which generalizes the Tensor Product Formula) and the Internal-Hom Formula.

In Section 4, we show that checking costratification of $\mathcal{K}$ can be reduced to checking costratification of certain smashing localizations of $\mathcal{K}$. In the highest generality, our result is stated in Theorem 4.4 but in the case where $\mathcal{K} = \mathcal{T}$ and one fixes the support–cosupport theory given by the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ and the Balmer–Favi support, we have the following result which summarizes Corollaries 4.7 and 4.8.

Theorem. *Assume that $\mathcal{T}$ satisfies the colocal-to-global principle.*

(a) *$\mathcal{T}$ is costratified if and only if $\mathcal{T}/\operatorname{loc}^{\otimes}(\mathfrak{p})$ is costratified, for all $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.*

(b) *Suppose $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) = \bigcup_{j\in J} U_j$ is a cover of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ by complements of Thomason subsets. If $\mathcal{T}(U_j)$ satisfies cominimality, for all $j \in J$, then $\mathcal{T}$ is costratified.*

We also obtain the following generalization of Corollary 4.8 by replacing $\mathcal{T}$ with a $\mathcal{T}$-module $\mathcal{K}$.

Theorem 4.9. *Suppose that $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) = \bigcup_{j\in J} U_j$ is a cover of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ by complements of Thomason subsets. If $\mathcal{K}(U_j)$ (as a $\mathcal{T}(U_j)$-module) satisfies cominimality, for all $j \in J$, then $\mathcal{K}$ satisfies cominimality. If, moreover, $\mathcal{K}$ satisfies the colocal-to-global principle, then $\mathcal{K}$ is costratified.*

Finally, in Section 5, we present our application: Using the general machinery developed throughout, we first give in Theorem 5.5 a more streamlined proof of Neeman's classification of the colocalizing subcategories of the derived category of a commutative noetherian ring and then we combine Theorem 5.5 with Corollary 4.8 to show the following.

Theorem 5.9. *Let X be a noetherian separated scheme. Then the derived category of quasi-coherent sheaves $\mathrm{D}(X)$ is costratified.*

1. Preliminaries

1.1. Actions and basic lemmas

Throughout, $\mathcal{T} = (\mathcal{T}, \otimes, 1)$ will be a rigidly-compactly generated tensor-triangulated category (big tt-category) as in [5], i.e., $\mathcal{T}$ is a tensor-triangulated category with arbitrary coproducts and it is generated by its compact objects as a localizing subcategory. Moreover, the (essentially small) subcategory $\mathcal{T}^{\mathrm{c}}$ of compact objects of $\mathcal{T}$ is a tensor-triangulated subcategory and the rigid objects of $\mathcal{T}$ coincide with the compact objects. The internal-hom functor of $\mathcal{T}$ will be denoted by $[-,-]\colon \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \to \mathcal{T}$.

Let $\mathcal{K}$ be a compactly generated triangulated category and let $*\colon \mathcal{T} \times \mathcal{K} \to \mathcal{K}$ be an action of $\mathcal{T}$ on $\mathcal{K}$, in the sense of [22]. In short, $*\colon \mathcal{T} \times \mathcal{K} \to \mathcal{K}$ is a coproduct-preserving triangulated functor in each variable such that there exist natural (in all variables) isomorphisms $\alpha_{X,Y,A}\colon X * (Y * A) \xrightarrow{\cong} (X \otimes Y) * A$ and $l_A\colon 1 * A \xrightarrow{\cong} A$, $\forall X, Y \in \mathcal{T}$, $\forall A \in \mathcal{K}$. The natural isomorphism α is called the *associator* and the natural isomorphism l is called the *unitor*. There is also a host of coherence conditions that need to be satisfied; we refer the reader to the aforementioned source for details. We call $\mathcal{K} = (\mathcal{K}, *)$ a $\mathcal{T}$-*module*.

By definition, for every object $X \in \mathcal{T}$, the functor $X * -\colon \mathcal{K} \to \mathcal{K}$ is a coproduct-preserving triangulated functor. Hence, by Brown representability, $X * -$ admits a right adjoint $[X,-]_*\colon \mathcal{K}\to\mathcal{K}$. Assembling these right adjoints yields a functor

$$[-,-]_*\colon \mathcal{T}^{\mathrm{op}}\times\mathcal{K}\to\mathcal{K}$$

that we call the *relative internal-hom*. Since $[1,-]_*$ is the right adjoint of $1*-\cong \mathrm{Id}_{\mathcal{K}}$, it holds that $[1,-]_*\cong\mathrm{Id}_{\mathcal{K}}$. Specifically, the composite

$$m := \mathrm{Id}_{\mathcal{K}}\to[1,1*-]_*\xrightarrow{[1,l]_*}[1,-]_*,$$

where the first map is the unit of adjunction, is a natural isomorphism (which we call the *hom-unitor*).

Lemma 1.1. *Let $X,Y\in\mathcal{T}$ and $A\in\mathcal{K}$. Then there exists a natural (in all variables) isomorphism $\beta_{X,Y,A}\colon[X\otimes Y,A]_*\xrightarrow{\cong}[X,[Y,A]_*]_*$ called the hom-associator.*

Proof. Let $B\in\mathcal{K}$. By the adjunction between the action and the relative internal-hom and the relation $(X\otimes Y)*B\cong(Y\otimes X)*B\cong Y*(X*B)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}\big(B,[X\otimes Y,A]_*\big)&\cong\mathrm{Hom}_{\mathcal{K}}\big((X\otimes Y)*B,A\big)\\&\cong\mathrm{Hom}_{\mathcal{K}}\big((Y\otimes X)*B,A\big)\\&\cong\mathrm{Hom}_{\mathcal{K}}\big(Y*(X*B),A\big)\\&\cong\mathrm{Hom}_{\mathcal{K}}\big(X*B,[Y,A]_*\big)\\&\cong\mathrm{Hom}_{\mathcal{K}}\big(B,[X,[Y,A]_*]_*\big).\end{aligned}$$

Consequently, for $B=[X\otimes Y,A]_*$, the image of the identity morphism on $[X\otimes Y,A]_*$ under the above series of isomorphisms gives a natural (in all variables) isomorphism $\beta_{X,Y,A}\colon[X\otimes Y,A]_*\xrightarrow{\cong}[X,[Y,A]_*]_*$. ■

Notation 1.2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two $\mathcal{T}$-modules. For $i=1,2$, the relative internal-hom of $\mathcal{K}_i$ will be denoted by $[-,-]_i$. Let $X,Y\in\mathcal{T}$ and $A\in\mathcal{K}_i$. The associator and unitor natural isomorphisms will be denoted by $\alpha^i_{X,Y,A}$ and l^i_A, respectively. The hom-associator and hom-unitor natural isomorphisms will be denoted by $\beta^i_{X,Y,A}$ and m^i_A, respectively. The unit and the counit of the action-hom adjunction will be denoted by $u^i_{X,A}\colon A\to[X,X*_iA]_i$ and $c^i_{X,A}\colon X*_i[X,A]_i\to A$, respectively. We denote by $\sigma_{X,Y}\colon X\otimes Y\xrightarrow{\cong}Y\otimes X$ the symmetry natural isomorphism. We denote by $c_{X,-}\colon X\otimes[X,-]\to\mathrm{Id}_{\mathcal{T}}$ the counit of the adjunction $X\otimes-\dashv[X,-]$. Set $X^\vee:=[X,1]$ and define the morphism $\mathrm{ev}^i_{X,A}\colon X*_iA\to[X^\vee,A]_i$ as the following composite:

$$X*_iA\xrightarrow{u^i_{X^\vee,X*_iA}}\big[X^\vee,X^\vee*_i(X*_iA)\big]_i\xrightarrow[\cong]{[X^\vee,\alpha^i_{X^\vee,X,A}]_i}\big[X^\vee,(X^\vee\otimes X)*_iA\big]_i$$

$$\xrightarrow[\cong]{[X^\vee,\sigma_{X^\vee,X}*_iA]_i}\big[X^\vee,(X\otimes X^\vee)*_iA\big]_i\xrightarrow{[X^\vee,c_{X,1}*_iA]_i}[X^\vee,1*_iA]_i\xrightarrow[\cong]{[X^\vee,l^i_A]_i}[X^\vee,A]_i.$$

Definition 1.3. A functor $F\colon \mathcal{K}_1 \to \mathcal{K}_2$, between $\mathcal{T}$-modules $\mathcal{K}_1$ and $\mathcal{K}_2$, is called *action-preserving* if there is a natural isomorphism $\phi\colon F(- *_1 -) \to - *_2 F(-)$ between the functors $F(- *_1 -)$, $- *_2 F(-)\colon \mathcal{T} \times \mathcal{K}_1 \to \mathcal{K}_2$ such that, for all $X, Y \in \mathcal{T}$ and for all $A \in \mathcal{K}_1$, the following diagrams commute:

$$\begin{tikzcd}
F\big(X *_1 (Y *_1 A)\big) \arrow[r, "\phi_{X,Y *_1 A}"] \arrow[d, "F\alpha^1_{X,Y,A}"'] & X *_2 F(Y *_1 A) \arrow[r, "X *_2 \phi_{Y,A}"] & X *_2 (Y *_2 FA) \arrow[d, "\alpha^2_{X,Y,FA}"] \\
F\big((X \otimes Y) *_1 A\big) \arrow[rr, "\phi_{X \otimes Y,A}"] & & (X \otimes Y) *_2 FA,
\end{tikzcd}$$

$$\begin{tikzcd}
F(1 *_1 A) \arrow[r, "\phi_{1,A}"] \arrow[d, "Fl^1_A"'] & 1 *_2 FA \arrow[dl, "l^2_{FA}"] \\
FA. &
\end{tikzcd}$$

Definition 1.4. A functor $G\colon \mathcal{K}_2 \to \mathcal{K}_1$, between $\mathcal{T}$-modules $\mathcal{K}_2$ and $\mathcal{K}_1$, is called hom-*preserving* if there is a natural isomorphism $\psi\colon [-, G(-)]_1 \to G[-,-]_2$ between the functors $[-, G(-)]_1$, $G[-,-]_2\colon \mathcal{T}^{\mathrm{op}} \times \mathcal{K}_2 \to \mathcal{K}_1$ such that, for all $X, Y \in \mathcal{T}$ and for all $B \in \mathcal{K}_2$, the following diagrams commute:

$$\begin{tikzcd}
\big[X, [Y, GB]_1\big]_1 \arrow[r, "{[X, \psi_{Y,B}]_1}"] \arrow[d, "\beta^1_{X,Y,GB}"'] & \big[X, G[Y, B]_2\big]_1 \arrow[r, "{\psi_{X,[Y,B]_2}}"] & G\big[X, [Y, B]_2\big]_2 \arrow[d, "G\beta^2_{X,Y,B}"] \\
{[X \otimes Y, GB]_1} \arrow[rr, "\psi_{X \otimes Y,B}"] & & G[X \otimes Y, B]_2,
\end{tikzcd}$$

$$\begin{tikzcd}
GB \arrow[d, "m^1_{GB}"'] \arrow[dr, "Gm^2_B"] & \\
{[1, GB]_1} \arrow[r, "\psi_{1,B}"'] & G[1, B]_2.
\end{tikzcd}$$

Lemma 1.5. *Let $\mathcal{T}$ and $\mathcal{K}$ be triangulated categories with $\mathcal{T}$ compactly generated and let $F_1, F_2\colon \mathcal{T} \to \mathcal{K}$ be coproduct-preserving triangulated functors (or contravariant triangulated functors that send coproducts to products). If there is a natural transformation $\theta\colon F_1 \to F_2$ such that θ_x is an isomorphism, for all $x \in \mathcal{T}^{\mathrm{c}}$, then θ is a natural isomorphism.*

Proof. The subcategory $\mathcal{X} = \{X \in \mathcal{T} \mid \theta_{\Sigma^n X} \text{ is an isomorphism } \forall n \in \mathbb{Z}\}$ is a localizing subcategory of $\mathcal{T}$ that contains $\mathcal{T}^{\mathrm{c}}$. Consequently, $\mathcal{X} = \mathcal{T}$ and this proves the statement. ■

Lemma 1.6. *Let $F\colon \mathcal{K}_1 \to \mathcal{K}_2$ be a coproduct and action-preserving triangulated functor between $\mathcal{T}$-modules and let $G\colon \mathcal{K}_2 \to \mathcal{K}_1$ be the right adjoint to F. Then G is* hom-*preserving. If G is coproduct-preserving, then G is action-preserving. If F is product-preserving, then F is* hom-*preserving.*

Proof. We denote by $\eta\colon \mathrm{Id}_{\mathcal{K}_1} \to GF$ and $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{K}_2}$ the unit and the counit, respectively, of the adjunction $F \dashv G$. Let $A \in \mathcal{K}_1$, $B \in \mathcal{K}_2$ and $X \in \mathcal{T}$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{K}_1}\big(A, [X, GB]_1\big) &\cong \mathrm{Hom}_{\mathcal{K}_1}(X * A, GB)\\
&\cong \mathrm{Hom}_{\mathcal{K}_2}\big(F(X * A), B\big)\\
&\cong \mathrm{Hom}_{\mathcal{K}_2}(X * FA, B)\\
&\cong \mathrm{Hom}_{\mathcal{K}_2}\big(FA, [X, B]_2\big)\\
&\cong \mathrm{Hom}_{\mathcal{K}_1}\big(A, G[X, B]_2\big).
\end{aligned}$$

For $A = [X, GB]_1$, the image of the identity morphism on $[X, GB]_1$ under the above series of isomorphisms provides a natural (in both variables) isomorphism

$$\psi_{X,B}\colon [X, GB]_1 \to G[X, B]_2$$

that satisfies the conditions of Definition 1.4, showing that G is hom-preserving. More precisely, $\psi_{X,B}$ is the following composite:

$$\begin{aligned}
&[X, GB]_1 \xrightarrow{\eta_{[X,GB]_1}} GF[X, GB]_1 \xrightarrow{Gu^2_{X,F[X,GB]_1}} G\big[X, X *_2 F[X, GB]_1\big]_2\\
&\xrightarrow{G[X,\phi^{-1}_{X,[X,GB]_1}]_2} G\big[X, F\big(X *_1 [X, GB]_1\big)\big]_2 \xrightarrow{G[X,F(c^1_{X,GB})]_2} G[X, FGB]_2\\
&\xrightarrow{G[X,\varepsilon_B]_2} G[X, B]_2.
\end{aligned}$$

Now suppose that G preserves coproducts. We define a natural transformation

$$\xi_{X,B}\colon X *_1 GB \to G(X *_2 B)$$

as the composite

$$X *_1 GB \xrightarrow{\eta_{X *_1 GB}} GF(X *_1 GB) \xrightarrow{G\phi_{X,GB}} G(X *_2 FGB) \xrightarrow{G(X *_2 \varepsilon_B)} G(X *_2 B).$$

We claim that the square

$$\begin{tikzcd}
X *_1 GB \arrow[r, "\xi_{X,B}"] \arrow[d, "\mathrm{ev}^1_{X,GB}"'] & G(X *_2 B) \arrow[d, "G\mathrm{ev}^2_{X,B}"] \\
{[X^\vee, GB]_1} \arrow[r, "\psi_{X^\vee,B}", "\cong"'] & G[X^\vee, B]_2
\end{tikzcd} \tag{1.7}$$

commutes. First, square (1.7) can be expanded as follows:

$$\begin{tikzcd}
X *_1 GB \arrow[r, "\eta_{X *_1 GB}"] \arrow[d, "\mathrm{ev}^1_{X,GB}"'] \arrow[dr, phantom, "(1)"] & GF(X *_1 GB) \arrow[r, "G\phi_{X,GB}"] \arrow[d, "GF\mathrm{ev}^1_{X,GB}"'] & G(X *_2 FGB) \arrow[r, "G(X *_2 \varepsilon_B)"] \arrow[d, "G\mathrm{ev}^2_{X,FGB}"] \arrow[dr, phantom, "(3)"] & G(X *_2 B) \arrow[d, "G\mathrm{ev}^2_{X,B}"] \\
{[X^\vee, GB]_1} \arrow[r, "\eta_{[X^\vee,GB]_1}"] & GF[X^\vee, GB]_1 \arrow[d, "Gu^2_{X^\vee, F[X^\vee,GB]_1}"'] \arrow[r, phantom, "(2)"] & G[X^\vee, FGB]_2 \arrow[r, "{G[X^\vee,\varepsilon_B]_2}"] & G[X^\vee, B]_2 \\
& G[X^\vee, X^\vee *_2 F[X^\vee, GB]_1]_2 \arrow[r, "{G[X^\vee,\phi^{-1}_{X^\vee,[X^\vee,GB]_1}]_2}"'] & G[X^\vee, F(X^\vee *_1 [X^\vee, GB]_1)]_2 \arrow[u, "{G[X^\vee,F(c^1_{X^\vee,GB})]_2}"'] &
\end{tikzcd}$$

where square (1) commutes by naturality of η and square (3) commutes by naturality of ev^2_X. Therefore, in order to show that (1.7) commutes, it suffices to show that diagram (2) commutes. We will prove this slightly more generally. We claim that the following diagram commutes:

$$\begin{tikzcd}
F(X *_1 A) \arrow[rr, "\phi_{X,A}"] \arrow[d, "F\mathrm{ev}^1_{X,A}"'] & & X *_2 FA \arrow[d, "\mathrm{ev}^2_{X,FA}"] \\
F[X^\vee, A]_1 \arrow[d, "u^2_{X^\vee, F[X^\vee,A]_1}"'] & & {[X^\vee, FA]_2} \\
{[X^\vee, X^\vee *_2 F[X^\vee, A]_1]_2} \arrow[rr, "{[X^\vee,\phi^{-1}_{X^\vee,[X^\vee,A]_1}]_2}"'] & & {[X^\vee, F(X^\vee *_1 [X^\vee, A]_1)]_2.} \arrow[u, "{[X^\vee,F(c^1_{X^\vee,A})]_2}"']
\end{tikzcd} \tag{1.8}$$

Set

$$\begin{aligned}
f_1 &= u^2_{X^\vee, F[X^\vee, X^\vee *_1 (X *_1 A)]_1}, & g_1 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, X^\vee *_1 (X *_1 A)]_1}]_2,\\
f_2 &= u^2_{X^\vee, F[X^\vee, (X^\vee \otimes X) *_1 A]_1}, & g_2 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, (X^\vee \otimes X) *_1 A]_1}]_2,\\
f_3 &= u^2_{X^\vee, F[X^\vee, (X \otimes X^\vee) *_1 A]_1}, & g_3 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, (X \otimes X^\vee) *_1 A]_1}]_2,\\
f_4 &= u^2_{X^\vee, F[X^\vee, 1 *_1 A]_1}, & g_4 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, 1 *_1 A]_1}]_2,\\
f_5 &= u^2_{X^\vee, F[X^\vee, A]_1}, & g_5 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, A]_1}]_2,
\end{aligned}$$

$$\begin{aligned}
h_1 &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X *_1 A}]_2 \circ [X^\vee, Fc^1_{X^\vee *_1 (X *_1 A)}]_2,\\
h_2 &= [X^\vee, \phi_{X^\vee \otimes X, A}]_2 \circ [X^\vee, Fc^1_{X^\vee, (X^\vee \otimes X) *_1 A}]_2,\\
h_3 &= [X^\vee, \phi_{X \otimes X^\vee, A}]_2 \circ [X^\vee, Fc^1_{X^\vee, (X \otimes X^\vee) *_1 A}]_2,\\
h_4 &= [X^\vee, \phi_{1,A}]_2 \circ [X^\vee, Fc^1_{X^\vee, 1 *_1 A}]_2,\\
h_5 &= \left[X^\vee, F(c^1_{X^\vee, A})\right]_2,
\end{aligned}$$

and expand diagram (1.8) as below:

$$\begin{tikzcd}
F(X *_1 A) \arrow[rrr, "\phi_{X,A}", "\cong"'] \arrow[d, "Fu^1_{X^\vee, X *_1 A}"'] & & & X *_2 FA \arrow[d, "u^2_{X^\vee, X *_2 FA}"] \\
F[X^\vee, X^\vee *_1 (X *_1 A)]_1 \arrow[r, "f_1"] \arrow[d, "{F[X^\vee, \alpha^1_{X^\vee,X,A}]_1}"'] \arrow[dr, phantom, "(1)"] & {[X^\vee, X^\vee *_2 F[X^\vee, X^\vee *_1 (X *_1 A)]_1]_2} \arrow[r, "g_1"] \arrow[d, "{[X^\vee, X^\vee *_2 F[X^\vee, \alpha^1_{X^\vee,X,A}]_1]_2}"] \arrow[dr, phantom, "(2)"] & {[X^\vee, F(X^\vee *_1 [X^\vee, X^\vee *_1 (X *_1 A)]_1)]_2} \arrow[r, "h_1"] \arrow[d, "{[X^\vee, F(X^\vee *_1 [X^\vee, \alpha^1_{X^\vee,X,A}]_1)]_2}"] \arrow[dr, phantom, "(3)"] & {[X^\vee, X^\vee *_2 (X *_2 FA)]_2} \arrow[d, "{[X^\vee, \alpha^2_{X^\vee,X,FA}]_2}"] \\
F[X^\vee, (X^\vee \otimes X) *_1 A]_1 \arrow[r, "f_2"] \arrow[d, "{F[X^\vee, \sigma_{X^\vee,X} *_1 A]_1}"'] \arrow[dr, phantom, "(4)"] & {[X^\vee, X^\vee *_2 F[X^\vee, (X^\vee \otimes X) *_1 A]_1]_2} \arrow[r, "g_2"] \arrow[d, "{[X^\vee, X^\vee *_2 F[X^\vee, \sigma_{X^\vee,X} *_1 A]_1]_2}"] \arrow[dr, phantom, "(5)"] & {[X^\vee, F(X^\vee *_1 [X^\vee, (X^\vee \otimes X) *_1 A]_1)]_2} \arrow[r, "h_2"] \arrow[d, "{[X^\vee, F(X^\vee *_1 [X^\vee, \sigma_{X^\vee,X} *_1 A]_1)]_2}"] \arrow[dr, phantom, "(6)"] & {[X^\vee, (X^\vee \otimes X) *_2 FA]_2} \arrow[d, "{[X^\vee, \sigma_{X^\vee,X} *_2 FA]_2}"] \\
F[X^\vee, (X \otimes X^\vee) *_1 A]_1 \arrow[r, "f_3"] \arrow[d, "{F[X^\vee, c_{X,1} *_1 A]_1}"'] \arrow[dr, phantom, "(7)"] & {[X^\vee, X^\vee *_2 F[X^\vee, (X \otimes X^\vee) *_1 A]_1]_2} \arrow[r, "g_3"] \arrow[d, "{[X^\vee, X^\vee *_2 F[X^\vee, c_{X,1} *_1 A]_1]_2}"] \arrow[dr, phantom, "(8)"] & {[X^\vee, F(X^\vee *_1 [X^\vee, (X \otimes X^\vee) *_1 A]_1)]_2} \arrow[r, "h_3"] \arrow[d, "{[X^\vee, F(X^\vee *_1 [X^\vee, c_{X,1} *_1 A]_1)]_2}"] \arrow[dr, phantom, "(9)"] & {[X^\vee, (X \otimes X^\vee) *_2 FA]_2} \arrow[d, "{[X^\vee, c_{X,1} *_2 FA]_2}"] \\
F[X^\vee, 1 *_1 A]_1 \arrow[r, "f_4"] \arrow[d, "{F[X^\vee, l^1_A]_1}"'] \arrow[dr, phantom, "(10)"] & {[X^\vee, X^\vee *_2 F[X^\vee, 1 *_1 A]_1]_2} \arrow[r, "g_4"] \arrow[d, "{[X^\vee, X^\vee *_2 F[X^\vee, l^1_A]_1]_2}"] \arrow[dr, phantom, "(11)"] & {[X^\vee, F(X^\vee *_1 [X^\vee, 1 *_1 A]_1)]_2} \arrow[r, "h_4"] \arrow[d, "{[X^\vee, F(X^\vee *_1 [X^\vee, l^1_A]_1)]_2}"] \arrow[dr, phantom, "(12)"] & {[X^\vee, 1 *_2 FA]_2} \arrow[d, "{[X^\vee, l^2_{FA}]_2}"] \\
F[X^\vee, A]_1 \arrow[r, "f_5"] & {[X^\vee, X^\vee *_2 F[X^\vee, A]_1]_2} \arrow[r, "g_5"] & {[X^\vee, F(X^\vee *_1 [X^\vee, A]_1)]_2} \arrow[r, "h_5"] & {[X^\vee, FA]_2.}
\end{tikzcd}$$

It is clear that the squares (1), (2), . . . , (12) commute. Thus, it remains to show that $u^2_{X^\vee, X *_2 FA} \circ \phi_{X,A} = h_1 \circ g_1 \circ f_1 \circ Fu^1_{X^\vee, X *_1 A}$. Indeed,

$$\begin{aligned}
h_1 \circ g_1 \circ f_1 \circ Fu^1_{X^\vee, X *_1 A} &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X *_1 A}]_2 \\
&\quad \circ [X^\vee, Fc^1_{X^\vee *_1 (X *_1 A)}]_2 \circ [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, X^\vee *_1 (X *_1 A)]_1}]_2 \\
&\quad \circ u^2_{X^\vee, F[X^\vee, X^\vee *_1 (X *_1 A)]_1} \circ Fu^1_{X^\vee, X *_1 A} \\
&= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X *_1 A}]_2 \\
&\quad \circ [X^\vee, Fc^1_{X^\vee *_1 (X *_1 A)}]_2 \circ [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, X^\vee *_1 (X *_1 A)]_1}]_2 \\
&\quad \circ [X^\vee, X^\vee *_2 Fu^1_{X^\vee, X *_1 A}]_2 \circ u^2_{X^\vee, F(X *_1 A)} \\
&= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X *_1 A}]_2 \\
&\quad \circ [X^\vee, Fc^1_{X^\vee *_1 (X *_1 A)}]_2 \circ [X^\vee, F(X^\vee *_1 u^1_{X^\vee, X *_1 A})]_2 \\
&\quad \circ [X^\vee, \phi^{-1}_{X^\vee, X *_1 A}]_2 \circ u^2_{X^\vee, F(X *_1 A)} \\
&= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X *_1 A}]_2 \\
&\quad \circ [X^\vee, \phi^{-1}_{X^\vee, X *_1 A}]_2 \circ u^2_{X^\vee, F(X *_1 A)} \\
&= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ u^2_{X^\vee, F(X *_1 A)} \\
&= u^2_{X^\vee, X *_2 FA} \circ \phi_{X,A}.
\end{aligned}$$

We conclude that diagram (1.8) commutes and, as a result, square (1.7) commutes. If $X \in \mathcal{T}^c$, then by [22, Lemma 4.6], $\mathrm{ev}^i_{X,-}$ is an isomorphism. Consequently, the restriction of $\xi_{-,B} \colon - *_1 GB \to G(- *_2 B)$ to the compact objects of $\mathcal{T}$ is a natural isomorphism.

Since the triangulated functors $- *_1 GB$ and $G(- *_2 B)$ are coproduct-preserving, it follows by Lemma 1.5 that $\xi_{-,B}$ is a natural isomorphism. It is easy to verify that the conditions of Definition 1.3 are satisfied. We conclude that G is action-preserving.

The proof that if F preserves products, then F is hom-preserving is similar and left to the interested reader. ∎

Let $\mathcal{K}$ be a $\mathcal{T}$-module. A subcategory $\mathcal{L} \subseteq \mathcal{K}$ is called a *localizing submodule* if $\mathcal{L}$ is a localizing subcategory such that $X * A \in \mathcal{L}$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{L}$. The collection of localizing submodules of $\mathcal{K}$ is denoted by $\operatorname{Loc}^*(\mathcal{K})$. A subcategory $\mathcal{C} \subseteq \mathcal{K}$ is called a *colocalizing* hom-*submodule* if $\mathcal{C}$ is a colocalizing subcategory such that $[X, A]_* \in \mathcal{C}$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{C}$. The collection of colocalizing hom-submodules of $\mathcal{K}$ is denoted by $\operatorname{Coloc}^{\text{hom}}(\mathcal{K})$. Let A be an object of $\mathcal{K}$. The localizing (resp. colocalizing) submodule of $\mathcal{K}$ generated (resp. cogenerated) by A, i.e., the smallest localizing (resp. colocalizing) submodule of $\mathcal{K}$ that contains A, is denoted by $\operatorname{loc}^*(A)$ (resp. $\operatorname{coloc}^{\text{hom}}(A)$). Specializing to the case $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, we obtain the notions of localizing tensor-ideal and colocalizing left hom-ideal.

We define the *annihilator* of $\mathcal{K}$ in $\mathcal{T}$ as the subcategory

$$\operatorname{Ann}_{\mathcal{T}}(\mathcal{K}) := \{X \in \mathcal{T} \mid X * A = 0,\ \forall A \in \mathcal{K}\} \subseteq \mathcal{T},$$

which is equal to $\bigcap_{A \in \mathcal{K}} \operatorname{Ker}(- * A)$ and hence a localizing tensor-ideal of $\mathcal{T}$. In case $\operatorname{Ann}_{\mathcal{T}}(\mathcal{K}) = 0$ (for instance, when $\mathcal{K} = \mathcal{T}$ and $* = \otimes$) $\mathcal{K}$ is called a *conservative* $\mathcal{T}$-module.

Lemma 1.9. *Let $\mathcal{K}_1, \mathcal{K}_2$ be two $\mathcal{T}$-modules and let $\mathcal{A} \subseteq \operatorname{Ob}(\mathcal{K}_1)$ and $\mathcal{B} \subseteq \operatorname{Ob}(\mathcal{K}_2)$.*

(a) *If $F\colon \mathcal{K}_1 \to \mathcal{K}_2$ is a coproduct and action-preserving triangulated functor, then $F(\operatorname{loc}^*(\mathcal{A})) \subseteq \operatorname{loc}^*(F\mathcal{A})$.*

(b) *If $F\colon \mathcal{K}_1^{\text{op}} \to \mathcal{K}_2$ is a triangulated functor that sends coproducts to products and $F(X * A) \cong [X, FA]_*$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{K}_1$, then $F(\operatorname{loc}^*(\mathcal{A})) \subseteq \operatorname{coloc}^{\text{hom}}(F\mathcal{A})$.*

(c) *If $G\colon \mathcal{K}_2 \to \mathcal{K}_1$ is a product and* hom-*preserving triangulated functor, then it holds that $G(\operatorname{coloc}^{\text{hom}}(\mathcal{B})) \subseteq \operatorname{coloc}^{\text{hom}}(G\mathcal{B})$.*

Proof. We will prove (a). The subcategory $\mathcal{X} = \{A \in \mathcal{K}_1 \mid FA \in \operatorname{loc}^*(F\mathcal{A})\}$ is a localizing submodule of $\mathcal{K}_1$ that contains $\mathcal{A}$. Therefore, $\mathcal{X}$ contains $\operatorname{loc}^*(\mathcal{A})$, proving the statement. The proofs of (b) and (c) are similar. ∎

1.2. Balmer–Favi support

Let $\operatorname{Spc}(\mathcal{T}^c)$ be the Balmer spectrum; see [2]. Then a point $\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^c)$ is called *visible* (*weakly visible* in [7]) if there exist Thomason subsets V, W of $\operatorname{Spc}(\mathcal{T}^c)$ such that $\{\mathfrak{p}\} = V \cap (\operatorname{Spc}(\mathcal{T}^c) \setminus W)$ [5, 22]. In particular, if $\operatorname{Spc}(\mathcal{T}^c)$ is noetherian, then every point of $\operatorname{Spc}(\mathcal{T}^c)$ is visible. The subsets V and W correspond to thick tensor-ideals $\mathcal{T}^c_V$, $\mathcal{T}^c_W$ of compact objects. Let $\mathcal{T}_V = \operatorname{loc}^{\otimes}(\mathcal{T}^c_V)$ and $\mathcal{T}_W = \operatorname{loc}^{\otimes}(\mathcal{T}^c_W)$. (It should be noted that the localizing subcategories generated by $\mathcal{T}^c_V$ and $\mathcal{T}^c_W$ are already tensor-ideals.) Since the ideals $\mathcal{T}_V$ and $\mathcal{T}_W$ are compactly generated, they are smashing ideals [17]. Therefore, they have associated left and right idempotents e_V, f_V and e_W, f_W, respectively. Let $g_{\mathfrak{p}} = e_V \otimes f_W$. Then the objects $\{g_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{\mathrm{c}}) \text{ and } \mathfrak{p} \text{ is visible}\}$ are pairwise-orthogonal tensor-idempotents. Let X be an object of $\mathcal{T}$ and assume that all points of $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ are visible. Then the *Balmer–Favi* support of X is $\operatorname{Supp}(X) = \{\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{\mathrm{c}}) \mid g_{\mathfrak{p}} \otimes X \neq 0\}$.

Lemma 1.10 ([5, Proposition 7.17]). *Assuming that all points of* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ *are visible, the map*

$$\operatorname{Supp}(-)\colon \operatorname{Ob}(\mathcal{T}) \to \mathcal{P}\big(\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})\big), \quad X \mapsto \operatorname{Supp}(X)$$

satisfies the following properties:

(a) $\operatorname{Supp}(0) = \varnothing$ *and* $\operatorname{Supp}(1) = \operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$.

(b) $\operatorname{Supp}(\coprod_{i \in I} X_i) = \bigcup_{i \in I} \operatorname{Supp}(X_i)$.

(c) $\operatorname{Supp}(\Sigma X) = \operatorname{Supp}(X)$.

(d) $\operatorname{Supp}(Y) \subseteq \operatorname{Supp}(X) \cup \operatorname{Supp}(Z)$, *for any triangle* $X \to Y \to Z$.

(e) $\operatorname{Supp}(X \otimes Y) \subseteq \operatorname{Supp}(X) \cap \operatorname{Supp}(Y)$.

(f) $\operatorname{Supp}(x \otimes y) = \operatorname{Supp}(x) \cap \operatorname{Supp}(y), \ \forall x, y \in \mathcal{T}^{\mathrm{c}}$.

Let $V \subseteq \operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ be a Thomason subset with complement U. The quotient $\mathcal{T}/\mathcal{T}_V$ is denoted by $\mathcal{T}(U)$. Since $\mathcal{T}_V$ is a smashing ideal, $\mathcal{T}(U)$ is a big tt-category and moreover, $\operatorname{Spc}(\mathcal{T}(U)^{\mathrm{c}}) \cong U$; see [4, Proposition 1.11].

1.3. Smashing support

Following [1], we briefly recall some facts concerning the smashing spectrum of a big tt-category $\mathcal{T}$. We denote by $\mathrm{S}^{\otimes}(\mathcal{T})$ the lattice of smashing tensor-ideals of $\mathcal{T}$. Then, under the hypothesis that $\mathrm{S}^{\otimes}(\mathcal{T})$ is a spatial frame, there is a space $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ associated with $\mathrm{S}^{\otimes}(\mathcal{T})$ via Stone duality, called the *smashing spectrum* of $\mathcal{T}$ that consists of the *meet-prime smashing ideals* of $\mathcal{T}$, i.e., those smashing ideals P such that $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq P \Rightarrow \mathcal{S}_1 \subseteq P$ or $\mathcal{S}_2 \subseteq P$, $\forall \mathcal{S}_1, \mathcal{S}_2 \in \mathrm{S}^{\otimes}(\mathcal{T})$. The open subsets of $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ stand in bijection with the smashing tensor-ideals and are of the form $U_{\mathcal{S}} = \{P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \mathcal{S} \not\subseteq P\}$, where $\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T})$. The closed subsets of $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ are of the form $V_{\mathcal{S}} = \{P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \mathcal{S} \subseteq P\}$. A point $P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ is called *locally closed* if $\{P\} = U_{\mathcal{S}} \cap V_{\mathcal{R}}$, for some smashing ideals $\mathcal{S}, \mathcal{R}$. If P is locally closed, then the ideal $\mathcal{R}$ can be replaced by P in the sense that $\{P\} = U_{\mathcal{S}} \cap V_P$. Each smashing ideal $\mathcal{S}$ corresponds to a left idempotent $e_{\mathcal{S}}$ and a right idempotent $f_{\mathcal{S}}$, which are the images of the tensor-unit of $\mathcal{T}$ under the associated acyclization and localization functors, respectively; see [5]. If P is locally closed and $\{P\} = U_{\mathcal{S}} \cap V_P$, then the *Rickard idempotent* associated with P is $\Gamma_P = e_{\mathcal{S}} \otimes f_P$. If every point of $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ is locally closed, then $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ is called T_D. Let X be an object of $\mathcal{T}$. The *big smashing support* of X is the subset $\operatorname{supp}^{\mathrm{s}}(X) = \{P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) \mid X \notin P\}$. If $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, then the *small smashing support* of X is $\operatorname{Supp}^{\mathrm{s}}(X) = \{P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \Gamma_P \otimes X \neq 0\}$. It holds that $\operatorname{Supp}^{\mathrm{s}}(X) \subseteq \operatorname{supp}^{\mathrm{s}}(X)$, with the two being equal if $X \in \mathcal{T}^{\mathrm{c}}$.

Hypothesis 1.11. Throughout the paper, whenever we state results concerning the smashing spectrum $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, we will always assume that the frame $\mathrm{S}^{\otimes}(\mathcal{T})$ of smashing ideals of $\mathcal{T}$ is a spatial frame.

Lemma 1.12 ([1, Lemma 3.2.8]). *The map*

$$\mathrm{supp}^{\mathrm{s}}(-)\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}\big(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})\big), \quad X \mapsto \mathrm{supp}^{\mathrm{s}}(X)$$

satisfies the following properties:

(a) $\mathrm{supp}^{\mathrm{s}}(0) = \varnothing$ *and* $\mathrm{supp}^{\mathrm{s}}(1) = \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.

(b) $\mathrm{supp}^{\mathrm{s}}(\coprod_{i\in I} X_i) = \bigcup_{i\in I} \mathrm{supp}^{\mathrm{s}}(X_i)$.

(c) $\mathrm{supp}^{\mathrm{s}}(\Sigma X) = \mathrm{supp}^{\mathrm{s}}(X)$.

(d) $\mathrm{supp}^{\mathrm{s}}(Y) \subseteq \mathrm{supp}^{\mathrm{s}}(X) \cup \mathrm{supp}^{\mathrm{s}}(Z)$, *for any triangle* $X \to Y \to Z$.

(e) $\mathrm{supp}^{\mathrm{s}}(X \otimes Y) \subseteq \mathrm{supp}^{\mathrm{s}}(X) \cap \mathrm{supp}^{\mathrm{s}}(Y)$.

(f) $\mathrm{supp}^{\mathrm{s}}(x \otimes y) = \mathrm{supp}^{\mathrm{s}}(x) \cap \mathrm{supp}^{\mathrm{s}}(y)$, $\forall x, y \in \mathcal{T}^{\mathrm{c}}$.

Assuming that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is* T_D*, the same properties are satisfied by the map*

$$\mathrm{Supp}^{\mathrm{s}}(-)\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})), \quad X \mapsto \mathrm{Supp}^{\mathrm{s}}(X).$$

The map $\psi\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$, where $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ denotes the Hochster dual of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$, that sends $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ to $P \cap \mathcal{T}^{\mathrm{c}}$ is a surjective continuous map and according to [1, Corollary 5.1.5], ψ is a homeomorphism if and only if $\mathcal{T}$ satisfies the Telescope Conjecture (meaning that every smashing ideal is generated by compact objects). For further discussion on the smashing spectrum and related concerns, see also [30].

2. Stratification–costratification

Fix a big tt-category $\mathcal{T}$ and a $\mathcal{T}$-module $\mathcal{K}$. We always assume that $\mathcal{K}$ is compactly generated. Let us recall some well-known facts concerning Brown–Comenetz duals of compact objects. Let x be a compact object of $\mathcal{T}$. Then $H_x := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathcal{T}}(x,-), \mathbb{Q}/\mathbb{Z})\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is a cohomological functor that sends coproducts to products. So, by Brown Representability, H_x is representable. The representing object of H_x is denoted by Ix and is called the *Brown–Comenetz dual* of x. The functor $\mathrm{Hom}_{\mathcal{T}}(-, Ix)|_{\mathcal{T}^{\mathrm{c}}}$ is an injective object of $\mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$, the abelian category of additive functors $\{\mathcal{T}^{\mathrm{c}}\}^{\mathrm{op}} \to \mathrm{Ab}$; see [20]. Hence, by [13, Theorem 1.8] (see also [8, Theorem 8.6]) Ix is pure-injective. Choosing a skeleton for the subcategory of compact objects, the product of the associated Brown–Comenetz duals is denoted by I. Being a product of pure-injective objects, I is also pure-injective. Using the fact that $\mathcal{T}$ is compactly generated, one can easily check that I is a cogenerator of $\mathcal{T}$ in the sense that $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n I) = 0$, $\forall n \in \mathbb{Z}$ implies that $X = 0$. It holds that $\mathcal{T} = \mathrm{coloc}(I)$. This follows from the fact that the Brown–Comenetz duals of the compact objects form a perfect cogenerating set for $\mathcal{T}$; see [14]. We use the symbol $I_{\mathcal{T}}$ if there is any possibility for confusion. Similarly, $\mathcal{K}$ has a pure-injective cogenerator $I_{\mathcal{K}}$.

2.1. Support–cosupport

Fix a topological space S.

Definition 2.1 (See also [7, Definition 7.1]). A *support data for* $\mathcal{T}$ *with values in* S is a map $\mathrm{s}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ that satisfies the following properties:

(a) $\mathrm{s}(0) = \varnothing$ and $\mathrm{s}(1) = S$.

(b) $\mathrm{s}(\coprod X_i) = \bigcup \mathrm{s}(X_i)$.

(c) $\mathrm{s}(\Sigma X) = \mathrm{s}(X)$.

(d) $\mathrm{s}(Y) \subseteq \mathrm{s}(X) \cup \mathrm{s}(Z)$, for any triangle $X \to Y \to Z$.

(e) $\mathrm{s}(X \otimes Y) \subseteq \mathrm{s}(X) \cap \mathrm{s}(Y)$.

Definition 2.2. A *cosupport data for* $\mathcal{T}$ *with values in* S is a map $\mathrm{c}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ that satisfies the following properties:

(a) $\mathrm{c}(0) = \varnothing$ and $\mathrm{c}(I) = S$.

(b) $\mathrm{c}(\prod X_i) = \bigcup \mathrm{c}(X_i)$.

(c) $\mathrm{c}(\Sigma X) = \mathrm{c}(X)$.

(d) $\mathrm{c}(Y) \subseteq \mathrm{c}(X) \cup \mathrm{c}(Z)$, for any triangle $X \to Y \to Z$.

(e) $\mathrm{c}([X, Y]) \subseteq \mathrm{c}([X, I]) \cap \mathrm{c}(Y)$.

Remark 2.3. Let $\mathrm{c}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ be a cosupport data. It is a straightforward verification, using the properties of $[-, I]$, that setting $\mathrm{s}(X) = \mathrm{c}([X, I])$ gives rise to a support data $\mathrm{s}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$.

Definition 2.4. Let $\mathrm{c}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ be a cosupport data. The support data $\mathrm{s}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ defined by $\mathrm{s}(X) = \mathrm{c}([X, I])$ is called *the support data induced by* c. We say that (s, c) is a *support–cosupport pair*.

Lemma 2.5. *Let* $\Gamma\colon S \to \mathrm{Ob}(\mathcal{T})$ *be a map such that* $\Gamma_s := \Gamma(s) \neq 0$, $\forall s \in S$. *Then the maps*

$$\begin{aligned} \mathrm{s}_\Gamma\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S), &\quad \mathrm{s}_\Gamma(X) = \{s \in S \mid \Gamma_s \otimes X \neq 0\}, \\ \mathrm{c}_\Gamma\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S), &\quad \mathrm{c}_\Gamma(X) = \{s \in S \mid [\Gamma_s, X] \neq 0\} \end{aligned}$$

are a support and cosupport data, respectively. Moreover, s_Γ *is induced by* c_Γ.

Proof. That s_Γ and c_Γ are a support and cosupport data, respectively, follows from the fact that $\Gamma_s \otimes -$ is a coproduct-preserving triangulated functor and $[\Gamma_s, -]$ is a product-preserving triangulated functor. Let $X \in \mathcal{T}$. The claim that s_Γ is induced by c_Γ follows from the isomorphism $[\Gamma_s, [X, I]] \cong [\Gamma_s \otimes X, I]$ and the fact that I is a cogenerator of $\mathcal{T}$. ∎

Definition 2.6. A support–cosupport pair $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ is called *good* if it is induced by a map $\Gamma\colon S \to \mathrm{Ob}(\mathcal{T})$ such that $\Gamma_s \otimes \Gamma_r = 0$, $\forall s \neq r$ and $\Gamma_s \otimes \Gamma_s \cong \Gamma_s \neq 0$, $\forall s \in S$.

Remark 2.7. An important feature of a good support–cosupport pair $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ is the following equality: $\mathrm{s}_\Gamma(\Gamma_s) = c_\Gamma([\Gamma_s, I]) = \{s\}, \forall s \in S$.

Example 2.8. The following are good support–cosupport pairs on $\mathcal{T}$:

(a) Assuming that the frame of smashing ideals of $\mathcal{T}$ is a spatial frame, the big smashing support–cosupport:

$$\begin{aligned} \mathrm{supp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid f_P \otimes X \neq 0\}, \\ \mathrm{cosupp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [f_P, X] \neq 0\}. \end{aligned}$$

Since $\mathrm{Ker}(f_P \otimes -) = P$, it holds that $P \in \mathrm{supp}^{\mathrm{s}}(X)$ if and only if $X \notin P$. Using the equality $P^\perp = \mathrm{Im}(f_P \otimes -)$, one can deduce that $\mathrm{Ker}[f_P, -] = (P^\perp)^\perp$, so $P \in \mathrm{cosupp}^{\mathrm{s}}(X)$ if and only if $X \notin (P^\perp)^\perp$.

(b) Assuming further that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, the small smashing support–cosupport:

$$\begin{aligned} \mathrm{Supp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \Gamma_P \otimes X \neq 0\}, \\ \mathrm{Cosupp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [\Gamma_P, X] \neq 0\}. \end{aligned}$$

(c) If every point of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is visible, the Balmer–Favi support–cosupport:

$$\begin{aligned} \mathrm{Supp}(X) &= \{\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid g_{\mathfrak{p}} \otimes X \neq 0\}, \\ \mathrm{Cosupp}(X) &= \{\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid [g_{\mathfrak{p}}, X] \neq 0\}. \end{aligned}$$

(d) If R is a graded commutative noetherian ring and $\mathcal{T}$ is R-linear, the BIK support–cosupport:

$$\begin{aligned} \mathrm{supp}_R(X) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \Gamma_{\mathfrak{p}}1 \otimes X \neq 0\}, \\ \mathrm{cosupp}_R(X) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid [\Gamma_{\mathfrak{p}}1, X] \neq 0\}. \end{aligned}$$

See [11, 12].

Remark 2.9. Suppose that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ with associated idempotent $\Gamma_P = e_{\mathcal{S}} \otimes f_P$. If X is an object of $\mathcal{T}$ such that $P \in \mathrm{Cosupp}^{\mathrm{s}}(X)$, then $0 \neq [\Gamma_P, X] = [e_{\mathcal{S}} \otimes f_P, X] \cong [e_{\mathcal{S}}, [f_P, X]]$. Hence, $[f_P, X] \neq 0$. In other words, $P \in \mathrm{cosupp}^{\mathrm{s}}(X)$. This shows that $\mathrm{Cosupp}^{\mathrm{s}}(X) \subseteq \mathrm{cosupp}^{\mathrm{s}}(X)$, $\forall X \in \mathcal{T}$.

Proposition 2.10. *Assuming that* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is* T_D*, the small and big smashing cosupports coincide, i.e.,* $\mathrm{Cosupp}^{\mathrm{s}}(X) = \mathrm{cosupp}^{\mathrm{s}}(X)$*,* $\forall X \in \mathcal{T}$*, if and only if every point of* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is closed, i.e.,* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is* T_1.

Proof. Since $\mathrm{Supp}^{\mathrm{s}}(-) = \mathrm{Cosupp}^{\mathrm{s}}([-, I])$ and $\mathrm{supp}^{\mathrm{s}}(-) = \mathrm{cosupp}^{\mathrm{s}}([-, I])$, if the small and big smashing cosupports coincide, then so do the small and big smashing supports. By [30, Lemma 4.28], it follows that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_1. Conversely, if $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_1 and $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, then $V_P = \{P\}$. This implies that $\Gamma_P = f_P$. So, $[\Gamma_P, X] = 0$ if and only if $[f_P, X] = 0$, for all $X \in \mathcal{T}$. Hence, $\mathrm{Cosupp}^{\mathrm{s}} = \mathrm{cosupp}^{\mathrm{s}}$. ∎

Remark 2.11. Assume that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and consider the small smashing support–cosupport. Then $\mathrm{Cosupp}^{\mathrm{s}}(1) = \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [\Gamma_P, 1] \neq 0\}$. There are many cases where $\mathrm{Cosupp}^{\mathrm{s}}(1) \neq \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. For instance,

$$\mathrm{Cosupp}^{\mathrm{s}}(\mathbb{Z}_p) = \big\{(p)\big\} \neq \mathrm{Spc}^{\mathrm{s}}\big(\mathrm{D}(\mathbb{Z}_p)\big) \cong \mathrm{Spec}(\mathbb{Z}_p) = \big\{(0), (p)\big\}.$$

For more examples and results concerning the cosupport in derived categories of commutative noetherian rings, see [29].

Let (s_Γ, c_Γ) be a support–cosupport pair on $\mathcal{T}$ induced by a map $\Gamma\colon S \to \mathrm{Ob}(\mathcal{T})$ and define the maps

$$s_\Gamma^*\colon \mathrm{Ob}(\mathcal{K}) \to \mathcal{P}(S), \quad s_\Gamma^*(A) = \{s \in S \mid \Gamma_s * A \neq 0\},$$
$$c_\Gamma^*\colon \mathrm{Ob}(\mathcal{K}) \to \mathcal{P}(S), \quad c_\Gamma^*(A) = \{s \in S \mid [\Gamma_s, A]_* \neq 0\}.$$

Lemma 2.12. *The maps s_Γ^* and c_Γ^* satisfy the following properties:*

(a) $s_\Gamma^*(0) = \varnothing$.
(b) $s_\Gamma^*(\coprod A_i) = \bigcup s_\Gamma^*(A_i)$.
(c) $s_\Gamma^*(\Sigma A) = s_\Gamma^*(A)$.
(d) $s_\Gamma^*(B) \subseteq s_\Gamma^*(A) \cup s_\Gamma^*(C)$, *for any triangle $A \to B \to C$ of $\mathcal{K}$.*
(e) $s_\Gamma^*(X * A) \subseteq s_\Gamma(X) \cap s_\Gamma^*(A)$.
(f) $c_\Gamma^*(0) = \varnothing$.
(g) $c_\Gamma^*(\prod A_i) = \bigcup c_\Gamma^*(A_i)$.
(h) $c_\Gamma^*(\Sigma A) = c_\Gamma^*(A)$.
(i) $c_\Gamma^*(B) \subseteq c_\Gamma^*(A) \cup c_\Gamma^*(C)$, *for any triangle $A \to B \to C$ of $\mathcal{K}$.*
(j) $c_\Gamma^*([X, A]_*) \subseteq c_\Gamma([X, I_\mathcal{T}]) \cap c_\Gamma^*(A)$.

Proof. The argument is essentially the same as the one given in Lemma 2.5. The property $c_\Gamma^*([X, A]_*) \subseteq c_\Gamma([X, I_\mathcal{T}]) \cap c_\Gamma^*(A)$ follows from Lemma 1.1. ∎

2.2. The (co)local-to-global principle and (co)minimality

We define two pairs of inclusion-preserving maps

$$\begin{tikzcd} \mathcal{P}(S) \arrow[r, shift left, "\tau_{s_\Gamma^*}"] & \mathrm{Loc}^*(\mathcal{K}) \arrow[l, shift left, "\sigma_{s_\Gamma^*}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{P}(S) \arrow[r, shift left, "\tau_{c_\Gamma^*}"] & \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K}) \arrow[l, shift left, "\sigma_{c_\Gamma^*}"] \end{tikzcd}$$

by the formulas

$$\tau_{s_\Gamma^*}(W) = \big\{A \in \mathcal{K} \mid s_\Gamma^*(A) \subseteq W\big\} \quad \text{and} \quad \sigma_{s_\Gamma^*}(\mathcal{L}) = \bigcup_{A \in \mathcal{L}} s_\Gamma^*(A),$$
$$\tau_{c_\Gamma^*}(W) = \big\{A \in \mathcal{K} \mid c_\Gamma^*(A) \subseteq W\big\} \quad \text{and} \quad \sigma_{c_\Gamma^*}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} c_\Gamma^*(A).$$

It is clear from the properties of s_{Γ}^{*} and c_{Γ}^{*} that the maps $\tau_{s_{\Gamma}^{*}}, \sigma_{s_{\Gamma}^{*}}, \tau_{c_{\Gamma}^{*}}, \sigma_{c_{\Gamma}^{*}}$ are well defined. Moreover, $\operatorname{Im} \sigma_{s_{\Gamma}^{*}} \subseteq \mathcal{P}(\sigma_{s_{\Gamma}^{*}}(\mathcal{K}))$ and $\operatorname{Im} \sigma_{c_{\Gamma}^{*}} \subseteq \mathcal{P}(\sigma_{c_{\Gamma}^{*}}(\mathcal{K}))$. In fact, $\sigma_{s_{\Gamma}^{*}}(\mathcal{K}) = \sigma_{c_{\Gamma}^{*}}(\mathcal{K}) = c_{\Gamma}^{*}(I_{\mathcal{K}})$. The first equality follows from the adjunction $\Gamma_s * - \dashv [\Gamma_s, -]_*$. The second equality is a special case of Lemma 2.19 using the fact that $\mathcal{K} = \operatorname{coloc}^{\mathrm{hom}}(I_{\mathcal{K}})$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then $\Gamma_s * - \neq 0$, $\forall s \in S$. Hence, in this case, $\sigma_{s_{\Gamma}^{*}}(\mathcal{K}) = S$. For $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, we obtain the maps $\tau_{s_{\Gamma}}, \sigma_{s_{\Gamma}}, \tau_{c_{\Gamma}}, \sigma_{c_{\Gamma}}$.

Definition 2.13. Let $\mathcal{K}$ be a $\mathcal{T}$-module.

(a) $\mathcal{K}$ is *stratified* by Γ if $\tau_{s_{\Gamma}^{*}}$ and $\sigma_{s_{\Gamma}^{*}}$, between $\mathcal{P}(c_{\Gamma}^{*}(I_{\mathcal{K}}))$ and $\operatorname{Loc}^{*}(\mathcal{K})$, are mutually inverse bijections.

(b) $\mathcal{K}$ is *costratified* by Γ if $\tau_{c_{\Gamma}^{*}}$ and $\sigma_{c_{\Gamma}^{*}}$, between $\mathcal{P}(c_{\Gamma}^{*}(I_{\mathcal{K}}))$ and $\operatorname{Coloc}^{\mathrm{hom}}(\mathcal{K})$, are mutually inverse bijections.

Remark 2.14. Since we will always work with a fixed support–cosupport pair induced by a map $\Gamma\colon S \to \operatorname{Ob}(\mathcal{T})$, we will omit the reference to Γ in Definition 2.13 and say "$\mathcal{K}$ is stratified" and "$\mathcal{K}$ is costratified", respectively. We will mention explicit support–cosupport pairs where appropriate.

Example 2.15. Let R be a commutative noetherian ring. Then the singularity category $\mathrm{S}(R) = \mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} R)$, which is the homotopy category of acyclic complexes of injective R-modules, is a compactly generated triangulated category; see [15]. In [24], it was shown that the action $*\colon \mathrm{D}(R) \times \mathrm{S}(R) \to \mathrm{S}(R)$ defined by $X * A = \widetilde{X} \otimes_R A$, where $\widetilde{X}$ is a K-flat resolution of X, stratifies $\mathrm{S}(R)$ when R is a locally hypersurface ring (more generally a noetherian separated scheme with hypersurface singularities). Using the results proved in the sequel, in [31] we prove that if R is a locally hypersurface ring (more generally a noetherian separated scheme with hypersurface singularities) then $\mathrm{S}(R)$ is costratified.

Definition 2.16 (For (a), see [22, Definition 6.1]). Let $\mathcal{K}$ be a $\mathcal{T}$-module.

(a) $\mathcal{K}$ satisfies the *local-to-global principle* if

$$\operatorname{loc}^{*}(A) = \operatorname{loc}^{*}(\Gamma_s * A \mid s \in S), \quad \forall A \in \mathcal{K}.$$

(b) $\mathcal{K}$ satisfies *minimality* if, for all $s \in S$, $\operatorname{loc}^{*}(\Gamma_s * A \mid A \in \mathcal{K})$ is a minimal element of $\operatorname{Loc}^{*}(\mathcal{K})$ in the sense that it does not contain any non-zero proper localizing submodule of $\mathcal{K}$.

(i) $\mathcal{K}$ satisfies the *colocal-to-global principle* if

$$\operatorname{coloc}^{\mathrm{hom}}(A) = \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in S\big), \quad \forall A \in \mathcal{K}.$$

(ii) $\mathcal{K}$ satisfies *cominimality* if, for all $s \in S$, $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ is a minimal element of $\operatorname{Coloc}^{\mathrm{hom}}(\mathcal{K})$ in the sense that it does not contain any non-zero proper colocalizing hom-submodule of $\mathcal{K}$.

Remark 2.17. Let $X \in \mathcal{T}$. Since $\mathcal{K} = \operatorname{coloc}(I_{\mathcal{K}}) = \operatorname{coloc}^{\mathrm{hom}}(I_{\mathcal{K}})$, by Lemma 1.9 for the functor $[X, -]_*$, we have $\operatorname{coloc}^{\mathrm{hom}}([X, I_{\mathcal{K}}]_*) = \operatorname{coloc}^{\mathrm{hom}}([X, A]_* \mid A \in \mathcal{K})$. In particular, $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*) = \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid A \in \mathcal{K})$, $\forall s \in S$.

Remark 2.18. It is clear from the definition of s_Γ^* and c_Γ^* that

$$\begin{gathered}\operatorname{loc}^*(\Gamma_s * A \mid s \in S) = \operatorname{loc}^*\big(\Gamma_s * A \mid s \in s_\Gamma^*(A)\big),\\ \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in S\big) = \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in c_\Gamma^*(A)\big).\end{gathered}$$

In addition, if $\mathcal{K}$ satisfies the local-to-global (resp. colocal-to-global) principle, then s_Γ^* (resp. c_Γ^*) detects vanishing, i.e., $s_\Gamma^*(A) = \varnothing \Rightarrow A = 0$ and similarly for c_Γ^*. For the case $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, it holds that codetection implies detection, since $\varnothing = s_\Gamma(X) = c_\Gamma([X, I])$ implies $[X, I] = 0$, so $X = 0$.

For the rest of the section, fix a good support–cosupport pair (s_Γ, c_Γ) on $\mathcal{T}$.

Lemma 2.19. *Let $\mathcal{A}$ be a collection of objects of $\mathcal{K}$. Then we have the following equalities of subsets of S:*

(a) $\sigma_{s_\Gamma^*}(\operatorname{loc}^*(\mathcal{A})) = \bigcup_{A\in\mathcal{A}} s_\Gamma^*(A)$.

(b) $\sigma_{c_\Gamma^*}(\operatorname{coloc}^{\mathrm{hom}}(\mathcal{A})) = \bigcup_{A\in\mathcal{A}} c_\Gamma^*(A)$.

Proof. We will prove the case of $\sigma_{c_\Gamma^*}$. Let s be an element of S. Then

$$\begin{aligned} s \notin \bigcup_{A\in\mathcal{A}} c_\Gamma^*(A) &\Leftrightarrow \mathcal{A} \subseteq \operatorname{Ker}[\Gamma_s, -]_* \\ &\Leftrightarrow \operatorname{coloc}^{\mathrm{hom}}(\mathcal{A}) \subseteq \operatorname{Ker}[\Gamma_s, -]_* \\ &\Leftrightarrow s \notin \bigcup_{A\in\operatorname{coloc}^{\mathrm{hom}}(\mathcal{A})} c_\Gamma^*(A) = \sigma_{c_\Gamma^*}\big(\operatorname{coloc}^{\mathrm{hom}}(\mathcal{A})\big). \end{aligned}$$ ■

Remark 2.20. If (s_Γ, c_Γ) is a good support-cosupport pair on $\mathcal{T}$, then $c_\Gamma^*([\Gamma_s, A]_*) \subseteq \{s\}$. Hence, if $[\Gamma_s, A]_* \neq 0$ (i.e., $s \in c_\Gamma^*(A)$) then $c_\Gamma^*([\Gamma_s, A]_*) = \{s\}$. In particular, if $s \in c_\Gamma^*(I_\mathcal{K})$, then $c_\Gamma^*([\Gamma_s, I_\mathcal{K}]_*) = \{s\}$.

Lemma 2.21. *It holds that $\sigma_{s_\Gamma^*} \circ \tau_{s_\Gamma^*} = \mathrm{Id}$ and $\sigma_{c_\Gamma^*} \circ \tau_{c_\Gamma^*} = \mathrm{Id}$, where both composites are restricted to $\mathcal{P}(c_\Gamma^*(I_\mathcal{K}))$. In particular, the respective restrictions of $\tau_{s_\Gamma^*}$ and $\tau_{c_\Gamma^*}$ are injective, while $\sigma_{s_\Gamma^*}$ and $\sigma_{c_\Gamma^*}$ are surjective.*

Proof. We will prove that $\sigma_{c_\Gamma^*} \circ \tau_{c_\Gamma^*} = \mathrm{Id}$ (restricted to $\mathcal{P}(c_\Gamma^*(I_\mathcal{K}))$). Let W be a subset of $c_\Gamma^*(I_\mathcal{K})$. Clearly $(\sigma_{c_\Gamma^*} \circ \tau_{c_\Gamma^*})(W) \subseteq W$, since $(\sigma_{c_\Gamma^*} \circ \tau_{c_\Gamma^*})(W) = \bigcup_{c_\Gamma^*(A)\subseteq W} c_\Gamma^*(A)$. Let s be an element of W. Then $s \in c_\Gamma^*([\Gamma_s, I_\mathcal{K}]_*) = \{s\} \subseteq W$. Therefore, $s \in (\sigma_{c_\Gamma^*} \circ \tau_{c_\Gamma^*})(W)$, completing the proof. ■

Theorem 2.22. *Let (s_Γ, c_Γ) be a good support-cosupport pair on $\mathcal{T}$.*

(a) *$\mathcal{K}$ is stratified with respect to (s_Γ, c_Γ) if and only if $\mathcal{K}$ satisfies the local-to-global principle and minimality.*

(b) *$\mathcal{K}$ is costratified with respect to (s_Γ, c_Γ) if and only if $\mathcal{K}$ satisfies the colocal-to-global principle and cominimality.*

Proof. We will only prove (b), since (a) is proved analogously. Suppose that $\mathcal{K}$ is costratified. Then $\sigma_{c_\Gamma^*}$ is injective. Let A be an object of $\mathcal{K}$. Then

$$\begin{aligned}\sigma_{c_\Gamma^*}\big(\operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in c_\Gamma^*(A)\big)\big) &= \bigcup_{s \in c_\Gamma^*(A)} c_\Gamma^*\big([\Gamma_s, A]_*\big)\\ &= \bigcup_{s \in c_\Gamma^*(A)} \{s\}\\ &= c_\Gamma^*(A)\\ &= \sigma_{c_\Gamma^*}\big(\operatorname{coloc}^{\mathrm{hom}}(A)\big),\end{aligned}$$

where the first and last equalities are due to Lemma 2.19. Since $\sigma_{c_\Gamma^*}$ is injective, it follows that

$$\operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in c_\Gamma^*(A)\big) = \operatorname{coloc}^{\mathrm{hom}}(A).$$

Thus, $\mathcal{K}$ satisfies the colocal-to-global principle. In particular, c_Γ^* detects vanishing.

Let s be an element of $c_\Gamma^*(I_{\mathcal{K}})$ and A a non-zero object in $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$. Then $\varnothing \neq c_\Gamma^*(A) \subseteq c_\Gamma^*([\Gamma_s, I_{\mathcal{K}}]_*) = \{s\}$. Therefore, $c_\Gamma^*(A) = c_\Gamma^*([\Gamma_s, I_{\mathcal{K}}]_*)$. Since $\sigma_{c_\Gamma^*}$ is injective, $\operatorname{coloc}^{\mathrm{hom}}(A) = \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$. Hence, $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ is minimal.

Suppose that $\mathcal{K}$ satisfies the colocal-to-global principle and cominimality. Let $\mathcal{C} \in \operatorname{Coloc}^{\mathrm{hom}}(\mathcal{K})$. Clearly, $\mathcal{C} \subseteq (\tau_{c_\Gamma^*} \circ \sigma_{c_\Gamma^*})(\mathcal{C})$. Let $A \in (\tau_{c_\Gamma^*} \circ \sigma_{c_\Gamma^*})(\mathcal{C})$, i.e., $c_\Gamma^*(A) \subseteq \sigma_{c_\Gamma^*}(\mathcal{C})$. Then

$$\begin{aligned}\operatorname{coloc}^{\mathrm{hom}}(A) &= \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in c_\Gamma^*(A)\big)\\ &\subseteq \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, I_{\mathcal{K}}]_* \mid s \in c_\Gamma^*(A)\big)\\ &\subseteq \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, I_{\mathcal{K}}]_* \mid s \in \sigma_{c_\Gamma^*}(\mathcal{C})\big)\\ &\subseteq \mathcal{C}.\end{aligned}$$

The first equality is due to the colocal-to-global principle. The first containment relation follows from Remark 2.17, while the second containment relation is clear. For the third containment, if $s \in \sigma_{c_\Gamma^*}(\mathcal{C})$, then there exists an object $B \in \mathcal{C}$ such that $[\Gamma_s, B]_* \neq 0$. Since $[\Gamma_s, B]_* \in \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ and the latter is minimal, we have $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*) = \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, B]_*) \subseteq \mathcal{C}$. We infer that $A \in \mathcal{C}$, proving that $(\tau_{c_\Gamma^*} \circ \sigma_{c_\Gamma^*})(\mathcal{C}) = \mathcal{C}$. So, $\sigma_{c_\Gamma^*}$ is injective and thus, $\sigma_{c_\Gamma^*}$ is bijective. This shows that $\mathcal{K}$ is costratified. ■

Remark 2.23. Theorem 2.22 (b) could be stated slightly more generally, replacing a good support–cosupport pair (s_Γ, c_Γ), in the sense of Definition 2.6, with one that satisfies the property stated in Remark 2.20, i.e., if A is an object of $\mathcal{K}$ such that $[\Gamma_s, A]_* \neq 0$, then $c_\Gamma^*([\Gamma_s, A]_*) = \{s\}$. Similarly, the analogous property for Theorem 2.22 (a) is: if A is an object of $\mathcal{K}$ such that $\Gamma_s * A \neq 0$, then $s_\Gamma^*(\Gamma_s * A) = \{s\}$. This observation will be useful in Section 5, where we consider the support–cosupport for objects of the derived category of a commutative noetherian ring defined by the residue fields.

2.3. Local-to-global implies colocal-to-global

Let (s_Γ, c_Γ) be a (not necessarily good) support–cosupport pair on $\mathcal{T}$. For our next result, we need an additional assumption.

Hypothesis 2.24. We further assume that the relative internal-hom of $\mathcal{K}$ is a triangulated functor in the first variable, i.e., $[-, A]_*\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{K}$ preserves triangles, for all $A \in \mathcal{K}$. This is true, e.g., if $\mathcal{K}$ satisfies a formulation of May's TC3 axiom [16] replacing the tensor product of $\mathcal{T}$ with the action of $\mathcal{T}$ on $\mathcal{K}$. The proof of [18, Theorem C.1] goes through verbatim. Our assumption is satisfied by all known examples.

One could, of course, incorporate Hypothesis 2.24 in the definition of a $\mathcal{T}$-module. We decided to state it as an extra hypothesis because the abundance of examples satisfying it, i.e., all known examples, indicate that it is a property satisfied by every $\mathcal{T}$-module (even though a proof has not been discovered yet).

Lemma 2.25. *Suppose that* $\mathcal{T} = \mathrm{loc}^{\otimes}(G)$. *Then the following hold:*

(a) $\mathrm{loc}^*(A) = \mathrm{loc}^*(G * A)$, $\forall A \in \mathcal{K}$.

(b) $\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([G, A]_*)$, $\forall A \in \mathcal{K}$ *(under Hypothesis 2.24).*

Proof. We will prove (b). The inclusion $\mathrm{coloc}^{\mathrm{hom}}([G, A]_*) \subseteq \mathrm{coloc}^{\mathrm{hom}}(A)$ is clear. Since $\mathcal{T} = \mathrm{loc}^{\otimes}(G)$, it holds that $1 \in \mathrm{loc}^{\otimes}(G)$. By Lemma 1.9 for the functor $[-, A]_*$, it follows that $A \cong [1, A]_* \in \mathrm{coloc}^{\mathrm{hom}}([G, A]_*)$. Therefore, $\mathrm{coloc}^{\mathrm{hom}}(A) \subseteq \mathrm{coloc}^{\mathrm{hom}}([G, A]_*)$. The proof of (a) is analogous. ∎

Remark 2.26. An easy generalization of Lemma 2.25 is the following: If $\mathcal{T} = \mathrm{loc}^{\otimes}(\mathcal{G})$, for a collection of objects $\mathcal{G}$, then

$$\forall A \in \mathcal{K}\colon \mathrm{loc}^*(A) = \mathrm{loc}^*(G * A \mid G \in \mathcal{G}) \ \text{ and } \ \mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}\big([G, A]_* \mid G \in \mathcal{G}\big).$$

Proposition 2.27 (See also [22, Proposition 6.8]). *If $\mathcal{T}$ satisfies the local-to-global principle, then $\mathcal{K}$ satisfies the local-to-global principle and (under Hypothesis 2.24) the colocal-to-global principle.*

Proof. Since $\mathcal{T}$ satisfies the local-to-global principle, it holds that $\mathcal{T} = \mathrm{loc}^{\otimes}(\Gamma_s \mid s \in S)$. Hence, by Remark 2.26, it follows that $\mathrm{loc}^*(A) = \mathrm{loc}^*(\Gamma_s * A \mid s \in S)$ and $\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in S)$, for all $A \in \mathcal{K}$. This proves the statement. ∎

Corollary 2.28. *Under Hypothesis 2.24 for the case $\mathcal{K} = \mathcal{T}$, if $\mathcal{T}$ satisfies the local-to-global principle, then $\mathcal{T}$ satisfies the colocal-to-global principle.*

Example 2.29. Let R be a graded commutative noetherian ring such that $\mathcal{T}$ is R-linear and consider the BIK support–cosupport $(\mathrm{supp}_R, \mathrm{cosupp}_R)$, which takes values in $\mathrm{supp}_R(1) \subseteq \mathrm{Spec}(R)$ – this may not be an equality. As explained in [7, Corollary 7.11], if $\mathcal{T}$ is stratified in the sense of BIK, then $\mathrm{supp}_R(1)$ is homeomorphic to $\mathrm{Spc}(\mathcal{T}^c)$ and the BIK support is identified with the Balmer–Favi support under this homeomorphism. It then follows that

$\mathcal{T}$ is stratified by the Balmer–Favi support. Now since the tensor-idempotents $\Gamma_{\mathfrak{p}}1$ (defining the BIK support) and the tensor-idempotents $g_{\mathfrak{p}}$ (defining the Balmer–Favi support) have the same support (which is $\{\mathfrak{p}\}$) it follows that $\mathrm{loc}^{\otimes}(\Gamma_{\mathfrak{p}}1) = \mathrm{loc}^{\otimes}(g_{\mathfrak{p}})$. Applying Lemma 1.9 for the functor $[-, I]$ (taking into account Hypothesis 2.24) it follows that $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_{\mathfrak{p}}1, I]) = \mathrm{coloc}^{\mathrm{hom}}([g_{\mathfrak{p}}, I])$. By Corollary 2.28, $\mathcal{T}$ satisfies the colocal-to-global principle with respect to the Balmer–Favi support. Taking into account Theorem 2.22, we conclude that if $\mathcal{T}$ is BIK-stratified, then: $\mathcal{T}$ is Balmer–Favi-costratified if and only if $\mathcal{T}$ is BIK-costratified if and only if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_{\mathfrak{p}}1, I])$ is minimal, for all $\mathfrak{p} \in \mathrm{supp}_R(1)$. If $\mathcal{T} = \underline{\mathrm{Mod}}(kG)$ is the stable module category of the group algebra of a finite group G, then $\mathcal{T}$ is BIK-costratified by the canonical action of $H^*(G, k)$; see [12, Theorem 11.13]. We infer that $\underline{\mathrm{Mod}}(kG)$ is Balmer–Favi-costratified.

3. Prime submodules

In this section we introduce the classes of prime localizing submodules and hom-prime colocalizing submodules of a given $\mathcal{T}$-module $\mathcal{K}$. The class of prime localizing submodules generalizes the class of objectwise-prime localizing tensor-ideals [1,30] in the context of relative tensor-triangular geometry, while the class of hom-prime colocalizing submodules specializes to the class of hom-prime colocalizing left hom-ideals if $\mathcal{K} = \mathcal{T}$.

3.1. Prime localizing and colocalizing submodules

As before, $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ will be a good support-cosupport pair on $\mathcal{T}$ with values in a space S. Given $\mathcal{L} \in \mathrm{Loc}^*(\mathcal{K})$ and $\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$, we define two subcategories of $\mathcal{T}$ as follows:

$$\mathcal{L}^{\otimes \mathrm{L}} = \{X \in \mathcal{T} \mid X * \mathcal{K} \subseteq \mathcal{L}\},$$
$$\mathcal{C}^{\otimes \mathrm{C}} = \{X \in \mathcal{T} \mid [X, \mathcal{K}]_* \subseteq \mathcal{C}\},$$

where $X * \mathcal{K} := \mathrm{loc}^*(X * A \mid A \in \mathcal{K})$ and $[X, \mathcal{K}]_* := \mathrm{coloc}^{\mathrm{hom}}([X, A]_* \mid A \in \mathcal{K})$, with the latter also being equal to $\mathrm{coloc}^{\mathrm{hom}}([X, I_{\mathcal{K}}]_*)$. Evidently, if $\mathcal{L}_1 \subseteq \mathcal{L}_2$, then $\mathcal{L}_1^{\otimes \mathrm{L}} \subseteq \mathcal{L}_2^{\otimes \mathrm{L}}$ and if $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_1^{\otimes \mathrm{C}} \subseteq \mathcal{C}_2^{\otimes \mathrm{C}}$.

Remark 3.1. Clearly, $\mathcal{L}^{\otimes \mathrm{L}}$ is a localizing tensor-ideal of $\mathcal{T}$ and $\mathcal{C}^{\otimes \mathrm{C}}$ is closed under coproducts, suspensions and the tensor product. Under Hypothesis 2.24, $\mathcal{C}^{\otimes \mathrm{C}}$ is also closed under cones and so, $\mathcal{C}^{\otimes \mathrm{C}}$ is a localizing tensor-ideal of $\mathcal{T}$.

Remark 3.2. If $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, then $\mathcal{L}^{\otimes \mathrm{L}} = \mathcal{L}$. The inclusion $\mathcal{L}^{\otimes \mathrm{L}} \subseteq \mathcal{L}$ follows from the equality $X \otimes \mathcal{T} = \mathrm{loc}^{\otimes}(X)$, while the inclusion $\mathcal{L} \subseteq \mathcal{L}^{\otimes \mathrm{L}}$ holds because $\mathcal{L}$ is a tensor-ideal.

Definition 3.3. Let $\mathcal{K}$ be a $\mathcal{T}$-module.

(a) A proper localizing submodule $\mathcal{L} \subsetneq \mathcal{K}$ is called *prime* if $X * A \in \mathcal{L}$ implies $X \in \mathcal{L}^{\otimes \mathrm{L}}$ or $A \in \mathcal{L}$.

(b) A proper colocalizing hom-submodule $\mathcal{C} \subsetneq \mathcal{K}$ is called hom-*prime* if $[X, A]_* \in \mathcal{C}$ implies $X \in \mathcal{C}^{\otimes \mathrm{C}}$ or $A \in \mathcal{C}$.

Remark 3.4. If $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, then the notion of prime localizing submodule recovers the notion of objectwise-prime localizing tensor-ideal; see Remark 3.2. The notion of hom-prime colocalizing hom-submodule provides the notion of hom-prime colocalizing left hom-ideal.

Lemma 3.5. *Let $\mathcal{L}$ be a prime localizing submodule of $\mathcal{K}$ and let $\mathcal{C}$ be a* hom*-prime colocalizing submodule of $\mathcal{K}$. Then $\mathcal{L}^{\otimes \mathrm{L}}$ and $\mathcal{C}^{\otimes \mathrm{C}}$ are objectwise-prime, in the sense that if $X \otimes Y \in \mathcal{L}^{\otimes \mathrm{L}}$, then $X \in \mathcal{L}^{\otimes \mathrm{L}}$ or $Y \in \mathcal{L}^{\otimes \mathrm{L}}$ and similarly for $\mathcal{C}^{\otimes \mathrm{C}}$.*

Proof. We will prove that $\mathcal{C}^{\otimes \mathrm{C}}$ is objectwise-prime. The proof for $\mathcal{L}^{\otimes \mathrm{L}}$ is analogous. Let X, Y be objects of $\mathcal{T}$ such that $X \otimes Y \in \mathcal{C}^{\otimes \mathrm{C}}$. Then $[X \otimes Y, A]_* \in \mathcal{C}$, $\forall A \in \mathcal{K}$. By Lemma 1.1, $[X, [Y, A]_*]_* \cong [X \otimes Y, A]_*$. Since $\mathcal{C}$ is hom-prime, $X \in \mathcal{C}^{\otimes \mathrm{C}}$ or $[Y, A]_* \in \mathcal{C}$. This proves that $\mathcal{C}^{\otimes \mathrm{C}}$ is objectwise-prime since the statement $[Y, A]_* \in \mathcal{C}$, $\forall A \in \mathcal{K}$ is equivalent to $Y \in \mathcal{C}^{\otimes \mathrm{C}}$ by definition. ∎

The main result of this section, i.e., Theorem 3.11, is a consequence of the following series of lemmas.

Lemma 3.6. *The following statements hold:*

(a) $\mathrm{Ker}(\Gamma_s \otimes -) \subseteq \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \mathrm{Ker}([\Gamma_s, -]_*)^{\otimes \mathrm{C}}$, $\forall s \in S$.

(b) *If $\mathcal{K}$ is conservative, then* $\mathrm{Ker}(\Gamma_s \otimes -) = \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$, $\forall s \in S$.

(c) *If* $\mathrm{Ker}(\Gamma_s \otimes -) = \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$, *$\forall s \in S$ and s_Γ detects vanishing, then $\mathcal{K}$ is conservative.*

Proof. Let X be an object of $\mathcal{T}$. Then we have $X \in \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$ if and only if $\Gamma_s * (X * A) \cong (\Gamma_s \otimes X) * A = 0$, $\forall A \in \mathcal{K}$, which is equivalent to $(\Gamma_s \otimes X) * - = 0$. Similarly, using the isomorphism $[\Gamma_s \otimes X, -]_* \cong [\Gamma_s, [X, -]_*]_*$, one deduces that $X \in \mathrm{Ker}([\Gamma_s, -]_*)^{\otimes \mathrm{C}}$ if and only if $[\Gamma_s \otimes X, -]_* = 0$. Since $(\Gamma_s \otimes X) * - \dashv [\Gamma_s \otimes X, -]_*$, these two functors are either both the zero functor or none of them is the zero functor. So, $\mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \mathrm{Ker}([\Gamma_s, -]_*)^{\otimes \mathrm{C}}$. From the equality

$$\mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \big\{ X \in \mathcal{T} \mid (\Gamma_s \otimes X) * - = 0 \big\},$$

it immediately follows that $\mathrm{Ker}(\Gamma_s \otimes -) \subseteq \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$. This proves (a).

If $\mathcal{K}$ is conservative and $(\Gamma_s \otimes X) * - = 0$, then $\Gamma_s \otimes X = 0$. Hence, $\mathrm{Ker}(\Gamma_s \otimes -) = \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$. This proves (b).

Let $X \in \mathcal{T}$ such that $X * - = 0$. Then $(\Gamma_s \otimes X) * - \cong X * (\Gamma_s * -) = 0$, $\forall s \in S$. Therefore, $X \in \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$. This implies that $X \in \mathrm{Ker}(\Gamma_s \otimes -)$, $\forall s \in S$, i.e., $\Gamma_s \otimes X = 0$, $\forall s \in S$. Equivalently, $s_\Gamma(X) = \varnothing$. Since s_Γ detects vanishing, we have $X = 0$. This proves (c). ∎

Remark 3.7. The inclusion $\mathrm{Ker}(\Gamma_s \otimes -) \subseteq \mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$ is not an equality in general. Suppose that $c_\Gamma^*(I_\mathcal{K}) \subsetneq S$ and let $s \notin c_\Gamma^*(I_\mathcal{K})$. Then $\mathrm{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \mathcal{T}$ and so equality in the above inclusion would mean that $\mathrm{Ker}(\Gamma_s \otimes -) = \mathcal{T}$, i.e., $\Gamma_s = 0$, which is false.

Lemma 3.8. *Let $\mathcal{L}$ be a prime localizing submodule of $\mathcal{K}$ and let $\mathcal{C}$ be a* hom-*prime colocalizing submodule of $\mathcal{K}$.*

(a) *There is at most one $s \in c_\Gamma^*(I_\mathcal{K})$ such that $\mathcal{L} \subseteq \operatorname{Ker}(\Gamma_s * -)$.*

(b) *There is at most one $s \in c_\Gamma^*(I_\mathcal{K})$ such that $\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_*$.*

Proof. The proof of (a) is similar to the proof of (b), which we prove below.

Let $s \in c_\Gamma^*(I_\mathcal{K})$ and suppose that $\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_*$. Let $r \in S$ such that $r \neq s$ and let $A \in \mathcal{K}$. Then $[\Gamma_s, [\Gamma_r, A]_*]_* = 0 \in \mathcal{C}$. Since $\mathcal{C}$ is hom-prime,

$$\Gamma_s \in \mathcal{C}^{\otimes \mathrm{C}} \subseteq \operatorname{Ker}[\Gamma_s, -]_*^{\otimes \mathrm{C}} = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} \quad \text{or} \quad [\Gamma_r, A]_* \in \mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_*.$$

For the equality $\operatorname{Ker}[\Gamma_s, -]_*^{\otimes \mathrm{C}} = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$, see Lemma 3.6. The former of the two does not hold since $\Gamma_s \in \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$ if and only if $\Gamma_s * - = 0$, but $s \in c_\Gamma^*(I_\mathcal{K})$ which means that $\Gamma_s * - \neq 0$. It follows that $\mathcal{C}$ contains all objects $[\Gamma_r, A]_*$, for $r \neq s$ and $A \in \mathcal{K}$. So, if $\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_r, -]_*$, for $r \neq s$ and $r \in c_\Gamma^*(I_\mathcal{K})$, then $[\Gamma_r, A]_* \in \operatorname{Ker}[\Gamma_r, -]_*$, $\forall A \in \mathcal{K}$. It follows that $[\Gamma_r, -]_* = 0$; thus, $\Gamma_r * - = 0$, which is false since $r \in c_\Gamma^*(I_\mathcal{K})$. ■

Lemma 3.9. *If $\mathcal{K}$ satisfies the colocal-to-global principle, then*

$$\operatorname{Ker}[\Gamma_s, -]_* = \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s\big), \quad \forall s \in S.$$

Analogously, if $\mathcal{K}$ satisfies the local-to-global principle, then

$$\operatorname{Ker}(\Gamma_s * -) = \operatorname{loc}^*(\Gamma_r * A \mid r \neq s,\ A \in \mathcal{K}), \quad \forall s \in S.$$

Proof. Let $r, s \in S$ such that $r \neq s$. Then $[\Gamma_r, I_\mathcal{K}]_* \in \operatorname{Ker}[\Gamma_s, -]_*$. Therefore,

$$\operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s\big) \subseteq \operatorname{Ker}[\Gamma_s, -]_*.$$

Let $A \in \operatorname{Ker}[\Gamma_s, -]_*$. Then $s \notin c_\Gamma^*(A)$. Since $\mathcal{K}$ satisfies the colocal-to-global principle,

$$\begin{aligned}\operatorname{coloc}^{\mathrm{hom}}(A) &= \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_r, A]_* \mid r \in c_\Gamma^*(A)\big)\\ &= \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_r, A]_* \mid r \neq s\big)\\ &\subseteq \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s\big).\end{aligned}$$

See Remark 2.18 for the second equality and Remark 2.17 for the containment relation. Hence, $A \in \operatorname{coloc}^{\mathrm{hom}}([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s)$, completing the proof. The case of $\operatorname{Ker}(\Gamma_s * -)$ is similar and left to the reader. ■

Lemma 3.10. *Let $\mathcal{C} \in \operatorname{Coloc}^{\mathrm{hom}}(\mathcal{K})$. Then*

$$\tau_{c_\Gamma^*}\big(\sigma_{c_\Gamma^*}(\mathcal{C})\big) = \bigcap_{\substack{\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_* \\ s \in c_\Gamma^*(I_\mathcal{K})}} \operatorname{Ker}[\Gamma_s, -]_*.$$

If $\mathcal{K}$ is costratified, then

$$\mathcal{C} = \bigcap_{\substack{\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_* \\ s \in c_\Gamma^*(I_\mathcal{K})}} \operatorname{Ker}[\Gamma_s, -]_*.$$

Analogously, if $\mathcal{L} \in \mathrm{Loc}^*(\mathcal{K})$*, then*

$$\tau_{s^*_\Gamma}\big(\sigma_{s^*_\Gamma}(\mathcal{L})\big) = \bigcap_{\substack{\mathcal{L} \subseteq \mathrm{Ker}(\Gamma_s * -) \\ s \in c^*_\Gamma(I_\mathcal{K})}} \mathrm{Ker}(\Gamma_s * -).$$

If $\mathcal{K}$ *is stratified, then*

$$\mathcal{L} = \bigcap_{\substack{\mathcal{L} \subseteq \mathrm{Ker}(\Gamma_s * -) \\ s \in c^*_\Gamma(I_\mathcal{K})}} \mathrm{Ker}(\Gamma_s * -).$$

Proof. Let $A \in \mathcal{K}$. Then $A \notin \bigcap_{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_*} \mathrm{Ker}[\Gamma_s, -]_*$ if and only if there exists $s \in S$ such that $\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_*$ and $[\Gamma_s, A]_* \neq 0$. Equivalently, $s \notin \sigma_{c^*_\Gamma}(\mathcal{C})$ and $s \in c^*_\Gamma(A)$. In other words, $c^*_\Gamma(A) \nsubseteq \sigma_{c^*_\Gamma}(\mathcal{C})$. Since $\tau_{c^*_\Gamma}(\sigma_{c^*_\Gamma}(\mathcal{C}))$ consists precisely of those $A \in \mathcal{K}$ such that $c^*_\Gamma(A) \subseteq \sigma_{c^*_\Gamma}(\mathcal{C})$, it follows that $\tau_{c^*_\Gamma}(\sigma_{c^*_\Gamma}(\mathcal{C})) = \bigcap_{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_*} \mathrm{Ker}[\Gamma_s, -]_*$. Finally, if $\mathcal{K}$ is costratified, then $\mathcal{C} = \tau_{c^*_\Gamma}(\sigma_{c^*_\Gamma}(\mathcal{C}))$, which proves the statement (the indexing set of the intersection involved in the claimed equalities can be considered to consist of points $s \in c^*_\Gamma(I_\mathcal{K})$ since if $s \notin c^*_\Gamma(I_\mathcal{K})$, then $[\Gamma_s, -]_* = 0$ and so $\mathrm{Ker}[\Gamma_s, -]_* = \mathcal{K}$ so the intersection is not affected). The rest is similar and left to the reader. ■

Theorem 3.11. *Let* $\mathcal{K}$ *be a costratified* $\mathcal{T}$*-module. Then there is a bijective correspondence between* hom*-prime colocalizing submodules of* $\mathcal{K}$ *and points of* $c^*_\Gamma(I_\mathcal{K})$*. A point* $s \in c^*_\Gamma(I_\mathcal{K})$ *is associated with* $\mathrm{Ker}[\Gamma_s, -]_* = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s)$.

Proof. Let $\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$ be hom-prime. Then, by Lemma 3.10,

$$\mathcal{C} = \bigcap_{\substack{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_* \\ s \in c^*_\Gamma(I_\mathcal{K})}} \mathrm{Ker}[\Gamma_s, -]_*.$$

It follows by Lemma 3.8 that $\mathcal{C}$ must be contained in $\mathrm{Ker}[\Gamma_s, -]_*$, for a unique $s \in c^*_\Gamma(I_\mathcal{K})$. Conclusion: $\mathcal{C} = \mathrm{Ker}[\Gamma_s, -]_*$, for a unique $s \in c^*_\Gamma(I_\mathcal{K})$. The equality

$$\mathrm{Ker}[\Gamma_s, -]_* = \mathrm{coloc}^{\mathrm{hom}}\big([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s\big)$$

was proved in Lemma 3.9. ■

Using Lemmas 3.8, 3.9 and 3.10, one can prove with analogous arguments the following.

Theorem 3.12. *Let* $\mathcal{K}$ *be a stratified* $\mathcal{T}$*-module. Then there is a bijective correspondence between prime localizing submodules of* $\mathcal{K}$ *and points of* $c^*_\Gamma(I_\mathcal{K})$*. A point* $s \in c^*_\Gamma(I_\mathcal{K})$ *is associated with* $\mathrm{Ker}(\Gamma_s * -) = \mathrm{loc}^*(\Gamma_r * A \mid r \neq s,\ A \in \mathcal{K})$.

The following observation, which is of independent interest and will not play a role in the sequel, showcases a conceptual similarity between the theory of actions of tensor-triangulated categories and the theory of associated primes of modules over rings. To see this, recall the following result: If R is a ring and M is a non-zero R-module such that for every non-zero submodule $N \subseteq M$, it holds that $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(N)$, then $\mathrm{Ann}_R(M)$ is a prime ideal of R.

Proposition 3.13. *Let $\mathcal{L}$ be a non-zero localizing submodule of $\mathcal{K}$ such that for every non-zero localizing submodule $\mathcal{L}'$ of $\mathcal{K}$ with $\mathcal{L}' \subseteq \mathcal{L}$, it holds that $\mathrm{Ann}_{\mathcal{T}}(\mathcal{L}) = \mathrm{Ann}_{\mathcal{T}}(\mathcal{L}')$. Then $\mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$ is an objectwise-prime localizing ideal of $\mathcal{T}$.*

Proof. Let $X, Y \in \mathcal{T}$ such that $X \otimes Y \in \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$. This means that $(X \otimes Y) * \mathcal{L} = 0$. Suppose that $X \notin \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$, i.e., $X * \mathcal{L} \neq 0$. Then $\mathrm{Ann}_{\mathcal{T}}(X * \mathcal{L}) = \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$. Since $Y * (X * \mathcal{L}) = (X \otimes Y) * \mathcal{L} = 0$, it follows that $Y \in \mathrm{Ann}_{\mathcal{T}}(X * \mathcal{L})$, so $Y \in \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$. ■

3.2. The action and internal-hom formulas

Now we will explore the relation between prime (co)localizing (hom-)submodules and (co)minimality and two properties (the Action Formula and the Internal-Hom Formula) that a support–cosupport pair may or may not satisfy.

Definition 3.14. Let $\mathcal{K}$ be a $\mathcal{T}$-module.

(a) $\mathcal{K}$ satisfies the *Action Formula* (AF) if

$$s^*_\Gamma(X * A) = s_\Gamma(X) \cap s^*_\Gamma(A), \quad \forall X \in \mathcal{T},\ \forall A \in \mathcal{K}.$$

(b) $\mathcal{K}$ satisfies the *Internal-Hom Formula* (IHF) if

$$c^*_\Gamma\big([X, A]_*\big) = c_\Gamma\big([X, I_{\mathcal{T}}]\big) \cap c^*_\Gamma(A), \quad \forall X \in \mathcal{T},\ \forall A \in \mathcal{K}.$$

(Recall that $c_\Gamma([X, I_{\mathcal{T}}]) = s_\Gamma(X)$.)

Lemma 3.15. *Let $\mathcal{K}$ be a $\mathcal{T}$-module.*

(a) *If $\mathcal{K}$ satisfies the Action Formula, then $\mathrm{Ker}(\Gamma_s * -)$ is a prime localizing submodule, $\forall s \in S$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then the converse holds.*

(b) *If $\mathcal{K}$ satisfies the Internal-Hom Formula, then $\mathrm{Ker}[\Gamma_s, -]_*$ is a* hom-*prime colocalizing* hom-*submodule, $\forall s \in S$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then the converse holds.*

Proof. The proof of (a) is similar to the proof of (b), which we prove below.

The Internal-Hom Formula can be restated as follows:

$$[\Gamma_s \otimes X, A]_* = 0 \implies \Gamma_s \otimes X = 0 \quad \text{or} \quad [\Gamma_s, A]_* = 0.$$

The converse implication holds by the definition of cosupport. So, if $[X, A]_* \in \mathrm{Ker}[\Gamma_s, -]_*$, then we have $X \in \mathrm{Ker}(\Gamma_s \otimes -) \subseteq \mathrm{Ker}[\Gamma_s, -]_*^{\otimes\mathrm{C}}$ or $A \in \mathrm{Ker}[\Gamma_s, -]_*$; for the first alternative, see Lemma 3.6. This means that $\mathrm{Ker}[\Gamma_s, -]_*$ is hom-prime. Now if $\mathrm{Ker}[\Gamma_s, -]_*$ is hom-prime and $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then $\mathrm{Ker}(\Gamma_s \otimes -) = \mathrm{Ker}[\Gamma_s, -]_*^{\otimes\mathrm{C}}$. Therefore, if $[\Gamma_s \otimes X, A]_* = 0$, then $\Gamma_s \otimes X = 0$ or $[\Gamma_s, A]_* = 0$, which is precisely the statement of the Internal-Hom Formula. ■

Proposition 3.16. *Let $\mathcal{K}$ be a $\mathcal{T}$-module.*

(a) *If $\mathcal{T}$ satisfies minimality, then $\mathcal{K}$ satisfies the Action Formula and (under Hypothesis 2.24) the Internal-Hom Formula.*

(b) *If $\mathcal{K}$ is a conservative $\mathcal{T}$-module and $\mathcal{K}$ satisfies cominimality, then $\mathcal{K}$ satisfies the Internal-Hom Formula.*

(c) *If $\mathcal{T}$ satisfies the Internal-Hom Formula, then $\mathcal{T}$ satisfies the Action Formula.*

Proof. Let $s \in S$, $X \in \mathcal{T}$, $A \in \mathcal{K}$.

(a) If $s \in s_\Gamma(X) \cap s^*_\Gamma(A)$, then $\Gamma_s \otimes X \neq 0$ and $\Gamma_s * A \neq 0$. Since $\operatorname{loc}^\otimes(\Gamma_s)$ is minimal, it follows that $\Gamma_s \in \operatorname{loc}^\otimes(\Gamma_s \otimes X)$. Hence, $\Gamma_s * A \in \operatorname{loc}^*((\Gamma_s \otimes X) * A)$. Since $\Gamma_s * A \neq 0$, it holds that $\Gamma_s * (X * A) \cong (\Gamma_s \otimes X) * A \neq 0$. In other words, $s \in s^*_\Gamma(X * A)$. Conclusion: $\mathcal{K}$ satisfies AF.

Now suppose that $s \in c_\Gamma([X, I_\mathcal{T}]) \cap c^*_\Gamma(A)$. Then $\Gamma_s \otimes X \neq 0$ and $[\Gamma_s, A]_* \neq 0$ (recall that $c_\Gamma([X, I_\mathcal{T}]) = s_\Gamma(X)$). Since $\operatorname{loc}^\otimes(\Gamma_s)$ is minimal, it follows that $\Gamma_s \in \operatorname{loc}^\otimes(\Gamma_s \otimes X)$. Hence, $[\Gamma_s, A]_* \in \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s \otimes X, A]_*)$. It follows that $[\Gamma_s \otimes X, A]_* \neq 0$, i.e., $s \in c^*_\Gamma([X, A]_*)$. Conclusion: $\mathcal{K}$ satisfies IHF.

(b) Suppose that $s \in c_\Gamma([X, I_\mathcal{T}]) \cap c^*_\Gamma(A)$. Then $\Gamma_s \otimes X \neq 0$ and $[\Gamma_s, A]_* \neq 0$. Aiming for contradiction, assume that $[\Gamma_s \otimes X, A]_* = 0$. Then $A \in \operatorname{Ker}[\Gamma_s \otimes X, -]_*$. Since $0 \neq [\Gamma_s, A]_* \in \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$, it follows by cominimality of $\mathcal{K}$ that

$$\operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, I_\mathcal{K}]_*\big) = \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_*\big) \subseteq \operatorname{coloc}^{\mathrm{hom}}(A) \subseteq \operatorname{Ker}[\Gamma_s \otimes X, -]_*.$$

Thus, $[\Gamma_s \otimes X, I_\mathcal{K}]_* = [\Gamma_s \otimes X, [\Gamma_s, I_\mathcal{K}]_*]_* = 0$. So, $[\Gamma_s \otimes X, -]_* = 0$. By the $(\Gamma_s \otimes X) * - \dashv [\Gamma_s \otimes X, -]_*$ adjunction, it follows that $(\Gamma_s \otimes X) * - = 0$, i.e., $\Gamma_s \otimes X \in \operatorname{Ann}_\mathcal{T}(\mathcal{K})$. Since $\mathcal{K}$ is a conservative $\mathcal{T}$-module, $\Gamma_s \otimes X = 0$, which is a contradiction. Conclusion: $\mathcal{K}$ satisfies IHF.

(c) Let $X, Y \in \mathcal{T}$. Then

$$\begin{aligned} s_\Gamma(X \otimes Y) &= c_\Gamma\big([X \otimes Y, I_\mathcal{T}]\big) \\ &= c_\Gamma\big([X, [Y, I_\mathcal{T}]]\big) \\ &= c_\Gamma\big([X, I_\mathcal{T}]\big) \cap c_\Gamma\big([Y, I_\mathcal{T}]\big) \\ &= s_\Gamma(X) \cap s_\Gamma(Y). \end{aligned}$$

Conclusion: IHF implies AF. ■

Remark 3.17. If $\mathcal{K} = \mathcal{T}$, then the statement of the Action Formula is the following:

$$s_\Gamma(X \otimes Y) = s_\Gamma(X) \cap s_\Gamma(Y), \quad \forall X, Y \in \mathcal{T}.$$

This is known as the Tensor Product Formula (which does not hold in general); see [5,7]. See also [3] for a support theory that does satisfy the Tensor Product Formula. On the other hand, the statement of the Internal-Hom Formula is the following:

$$c_\Gamma\big([X, Y]\big) = s_\Gamma(X) \cap c_\Gamma(Y), \quad \forall X, Y \in \mathcal{T}.$$

For the BIK support, this is equivalent to stratification of $\mathcal{T}$; see [12, Theorem 9.5].

4. Smashing submodules

Let $\mathcal{K}$ be a $\mathcal{T}$-module. Recall our assumption that $\mathcal{K}$ is compactly generated. A *smashing submodule* of $\mathcal{K}$ is a smashing subcategory $\mathcal{M} \subseteq \mathcal{K}$ that is also a submodule. Specifically, the quotient functor $j_{\mathcal{M}} \colon \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is a coproduct-preserving and essentially surjective triangulated functor that has a right adjoint $k_{\mathcal{M}} \colon \mathcal{K}/\mathcal{M} \to \mathcal{K}$ (which is necessarily fully faithful) that preserves coproducts – and products since it is a right adjoint. By Brown representability, $k_{\mathcal{M}}$ has a right adjoint $\ell_{\mathcal{M}} \colon \mathcal{K} \to \mathcal{K}/\mathcal{M}$ (which is necessarily essentially surjective) that preserves products. By the relations $j_{\mathcal{M}} k_{\mathcal{M}} \cong \mathrm{Id} \cong \ell_{\mathcal{M}} k_{\mathcal{M}}$, it follows that $j_{\mathcal{M}}$ and $\ell_{\mathcal{M}}$ take the same values on the image of $k_{\mathcal{M}}$, which is $\mathcal{M}^{\perp}$. The set of smashing submodules of $\mathcal{K}$ is denoted by $\mathrm{S}^*(\mathcal{K})$.

Next we describe the action of $\mathcal{T}$ on $\mathcal{K}/\mathcal{M}$ induced by the action of $\mathcal{T}$ on $\mathcal{K}$. The category $\mathcal{T} \times \mathcal{K}/\mathcal{M}$ is a triangulated category that is the quotient of $\mathcal{T} \times \mathcal{K}$ over $0 \times \mathcal{M}$, with the quotient functor $\mathcal{T} \times \mathcal{K} \to \mathcal{T} \times \mathcal{K}/\mathcal{M}$ being $\mathrm{Id}_{\mathcal{T}} \times j_{\mathcal{M}}$. Since $0 \times \mathcal{M}$ is contained in the kernel of $j_{\mathcal{M}} \circ *$, it follows that $j_{\mathcal{M}} \circ *$ factors through $\mathcal{T} \times \mathcal{K}/\mathcal{M}$ via a functor $* \colon \mathcal{T} \times \mathcal{K}/\mathcal{M} \to \mathcal{K}/\mathcal{M}$. It is straightforward to check that this functor is an action of $\mathcal{T}$ on $\mathcal{K}/\mathcal{M}$. If $X \in \mathcal{T}$ and $A = j_{\mathcal{M}}(B) \in \mathcal{K}/\mathcal{M}$, then $X * A = j_{\mathcal{M}}(X * B)$. The functor $j_{\mathcal{M}} \colon \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is action-preserving. We denote by $[-,-]_* \colon \mathcal{T}^{\mathrm{op}} \times \mathcal{K}/\mathcal{M} \to \mathcal{K}/\mathcal{M}$ the relative internal-hom of $\mathcal{K}/\mathcal{M}$. By Lemma 1.6, $k_{\mathcal{M}}$ is action and hom-preserving and $\ell_{\mathcal{M}}$ is hom-preserving. Moreover, since $I_{\mathcal{K}}$ (the product of the Brown–Comenetz duals of the compact objects of $\mathcal{K}$) is a pure-injective cogenerator of $\mathcal{K}$ and $\ell_{\mathcal{M}}$ is an essentially surjective right adjoint, it follows that $\ell_{\mathcal{M}}(I_{\mathcal{K}})$ is a pure-injective cogenerator of $\mathcal{K}/\mathcal{M}$. In particular, $\mathcal{K}/\mathcal{M} = \mathrm{coloc}(\ell_{\mathcal{M}}(I_{\mathcal{K}}))$.

Now we describe the colocalizing hom-submodules of $\mathcal{K}/\mathcal{M}$. The functor $k_{\mathcal{M}}$ gives a bijective correspondence between the colocalizing subcategories of $\mathcal{K}/\mathcal{M}$ and the colocalizing subcategories of $\mathcal{K}$ contained in $\mathcal{M}^{\perp}$. Since $k_{\mathcal{M}}$ is hom-preserving, this bijection restricts to colocalizing hom-submodules, i.e., the maps

$$\mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K}/\mathcal{M}) \underset{k_{\mathcal{M}}^{-1}}{\overset{k_{\mathcal{M}}}{\rightleftarrows}} \left\{\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K}) \mid \mathcal{C} \subseteq \mathcal{M}^{\perp}\right\} \tag{4.1}$$

are mutually inverse inclusion-preserving bijections. An observation that will be useful in the sequel is that $k_{\mathcal{M}}\, \mathrm{coloc}^{\mathrm{hom}}(j_{\mathcal{M}}(A)) = \mathrm{coloc}^{\mathrm{hom}}(k_{\mathcal{M}} j_{\mathcal{M}}(A))$, $\forall A \in \mathcal{K}$.

Let $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ be a good support–cosupport pair on $\mathcal{T}$. We denote the induced support–cosupport on $\mathcal{K}/\mathcal{M}$ by $(\mathrm{s}_\Gamma^{\mathcal{M}}, \mathrm{c}_\Gamma^{\mathcal{M}})$. Specifically,

$$\begin{aligned} \mathrm{s}_\Gamma^{\mathcal{M}}\big(j_{\mathcal{M}}(A)\big) &= \big\{s \in S \mid \ j_{\mathcal{M}}(\Gamma_s * A) \ \neq 0\big\}, \\ \mathrm{c}_\Gamma^{\mathcal{M}}\big(j_{\mathcal{M}}(A)\big) &= \big\{s \in S \mid \big[\Gamma_s, j_{\mathcal{M}}(A)\big]_* \neq 0\big\}. \end{aligned}$$

Then $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle if

$$\mathrm{coloc}^{\mathrm{hom}}\big(j_{\mathcal{M}}(A)\big) = \mathrm{coloc}^{\mathrm{hom}}\big(\big[\Gamma_s, j_{\mathcal{M}}(A)\big]_* \mid s \in S\big), \quad \forall A \in \mathcal{K}$$

and $\mathcal{K}/\mathcal{M}$ satisfies cominimality if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ is a minimal colocalizing hom-submodule of $\mathcal{K}/\mathcal{M}$, for all $s \in S$. Finally, let $S_{\mathcal{M}} = \{s \in S \mid [\Gamma_s, I_{\mathcal{K}}]_* \in \mathcal{M}^{\perp}\}$.

Proposition 4.2. *Let* $\mathcal{M} \in \mathsf{S}^*(\mathcal{K})$. *The following are equivalent:*

(a) $\mathcal{K}/\mathcal{M}$ *satisfies the colocal-to-global principle.*

(b) $\operatorname{coloc}^{\mathrm{hom}}(B) = \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, B]_* \mid s \in S)$, $\forall B \in \mathcal{M}^{\perp}$.

As a result, if $\mathcal{K}$ *satisfies the colocal-to-global principle, then* $\mathcal{K}/\mathcal{M}$ *satisfies the colocal-to-global principle.*

Proof. Let A be an object of $\mathcal{K}$ and set

$$\begin{aligned}
\mathcal{C}_1 &= \operatorname{coloc}^{\mathrm{hom}}\big(j_{\mathcal{M}}(A)\big),\\
\mathcal{C}_2 &= \operatorname{coloc}^{\mathrm{hom}}\big(\big[\Gamma_s, j_{\mathcal{M}}(A)\big]_* \mid s \in S\big),\\
\mathcal{D}_1 &= \operatorname{coloc}^{\mathrm{hom}}\big(k_{\mathcal{M}} j_{\mathcal{M}}(A)\big),\\
\mathcal{D}_2 &= \operatorname{coloc}^{\mathrm{hom}}\big(\big[\Gamma_s, k_{\mathcal{M}} j_{\mathcal{M}}(A)\big]_* \mid s \in S\big).
\end{aligned}$$

Under the bijection (4.1), $\mathcal{C}_1$ corresponds to $\mathcal{D}_1$, while $\mathcal{C}_2$ corresponds to $\mathcal{D}_2$ (recall that $k_{\mathcal{M}}$ is hom-preserving). So, if $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle, then $\mathcal{C}_1 = \mathcal{C}_2$. Hence, $\mathcal{D}_1 = \mathcal{D}_2$. Since $\operatorname{Im} k_{\mathcal{M}} j_{\mathcal{M}} = \mathcal{M}^{\perp}$, (b) follows. On the other hand, if (b) holds, then $\mathcal{D}_1 = \mathcal{D}_2$. As a result, $\mathcal{C}_1 = \mathcal{C}_2$, i.e., $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle. This proves (a).

If $\mathcal{K}$ satisfies the colocal-to-global principle, then

$$\operatorname{coloc}^{\mathrm{hom}}(A) = \operatorname{coloc}^{\mathrm{hom}}\big([\Gamma_s, A]_* \mid s \in S\big), \quad \forall A \in \mathcal{K},$$

so the equality certainly holds for $A \in \mathcal{M}^{\perp}$. Therefore, $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle by the equivalence (a) $\Leftrightarrow$ (b). ∎

Proposition 4.3. *Suppose that* $s \in S_{\mathcal{M}}$. *Then* $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ *is a minimal colocalizing* hom-*submodule of* $\mathcal{K}$ *if and only if* $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ *is a minimal colocalizing* hom-*submodule of* $\mathcal{K}/\mathcal{M}$.

Proof. Since $s \in S_{\mathcal{M}}$, it holds that $[\Gamma_s, I_{\mathcal{K}}]_* \in \mathcal{M}^{\perp}$. Therefore,

$$\big[\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})\big]_* \cong \ell_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_* \cong j_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_*.$$

So, under the bijection (4.1), $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*) = \operatorname{coloc}^{\mathrm{hom}}(j_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_*)$ corresponds to $\operatorname{coloc}^{\mathrm{hom}}(k_{\mathcal{M}} j_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_*) = \operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$, with the last equality again because $[\Gamma_s, I_{\mathcal{K}}]_* \in \mathcal{M}^{\perp}$. Consequently, $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ is minimal if and only if $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ is minimal. ∎

Combining Proposition 4.2, Proposition 4.3 and Theorem 2.22, we obtain the following result.

Theorem 4.4. *Let* $\{\mathcal{M}_s\}_{s \in S}$ *be a collection of smashing submodules of* $\mathcal{K}$ *such that* $s \in S_{\mathcal{M}_s}$, *for all* $s \in S$. *Then:*

(a) $\mathcal{K}$ *satisfies cominimality if and only if* $\mathcal{K}/\mathcal{M}_s$ *satisfies cominimality, for all* $s \in S$.

(b) *Suppose that $\mathcal{K}$ satisfies the colocal-to-global principle. Then $\mathcal{K}$ is costratified if and only if $\mathcal{K}/\mathcal{M}_s$ is costratified, for all $s \in S$.*

Proof. Since $s \in S_{\mathcal{M}_s}$, for all $s \in S$, by Proposition 4.3 $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ is a minimal colocalizing hom-submodule of $\mathcal{K}$ if and only if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}_s}(I_{\mathcal{K}})]_*)$ is a minimal colocalizing hom-submodule of $\mathcal{K}/\mathcal{M}_s$, for all $s \in S$. In other words, $\mathcal{K}$ satisfies cominimality if and only if $\mathcal{K}/\mathcal{M}_s$ satisfies cominimality, for all $s \in S$. This proves (a). If $\mathcal{K}$ satisfies the colocal-to-global principle, then by Proposition 4.2, it follows that $\mathcal{K}/\mathcal{M}_s$ satisfies the colocal-to-global principle, for all $s \in S$. Statement (b) now follows from (a) and Theorem 2.22. ∎

We will apply Theorem 4.4 to the case $\mathcal{K} = \mathcal{T}$, $* = \otimes$, $S = \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ and $(s_\Gamma, c_\Gamma) = (\mathrm{Supp}^{\mathrm{s}}, \mathrm{Cosupp}^{\mathrm{s}})$ (under Hypothesis 1.11 and provided that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D). In this case, if $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, then $S_P = \{Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [\Gamma_Q, I] \in P^{\perp}\}$. Since $\Gamma_P = e_{\mathcal{S}} \otimes f_P$, for some $\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T})$, and $P^{\perp} = \mathrm{Im}[f_P, -]$, it follows that

$$[\Gamma_P, I] = [e_{\mathcal{S}} \otimes f_P, I] \cong \big[f_P, [e_{\mathcal{S}}, I]\big] \in P^{\perp}.$$

In other words, $P \in S_P$. This leads to the following result.

Corollary 4.5. *Suppose that* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is T_D. Then:*

(a) *$\mathcal{T}$ satisfies cominimality if and only if $\mathcal{T}/P$ satisfies cominimality, for all $P \in$* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.

(b) *Suppose that $\mathcal{T}$ satisfies the colocal-to-global principle. Then $\mathcal{T}$ is costratified if and only if $\mathcal{T}/P$ is costratified, for all $P \in$* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.

Proof. The result is a direct consequence of Theorem 4.4, taking into account the preceding discussion. ∎

Corollary 4.6. *Suppose that* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is T_D and that* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup_{j \in J} V_{\mathcal{S}_j}$ *is a cover of* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *by closed subsets. If $\mathcal{T}/\mathcal{S}_j$ satisfies cominimality, for all $j \in J$, then $\mathcal{T}$ satisfies cominimality. If, moreover, $\mathcal{T}$ satisfies the colocal-to-global principle, then $\mathcal{T}$ is costratified.*

Proof. Let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. Then $P \in V_{\mathcal{S}_j}$, for some $j \in J$. This means that $\mathcal{S}_j \subseteq P$. Let $j_{\mathcal{S}_j} \colon \mathcal{T} \to \mathcal{T}/\mathcal{S}_j$ be the quotient functor. Then $j_{\mathcal{S}_j}(P)$ is a smashing ideal of $\mathcal{T}/\mathcal{S}_j$ such that $(\mathcal{T}/\mathcal{S}_j)/j_{\mathcal{S}_j}(P) \simeq \mathcal{T}/P$. Since $\mathcal{T}/\mathcal{S}_j$ satisfies cominimality, it follows by Corollary 4.5 that $\mathcal{T}/P$ satisfies cominimality. Since this is true for all $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, again by Corollary 4.5, we conclude that $\mathcal{T}$ satisfies cominimality. The “moreover” part follows by Theorem 2.22. ∎

Essentially via the same arguments (left to the reader) one obtains the analogous results for the Balmer spectrum and the Balmer–Favi support. What one needs to note for Corollary 4.8 is that, compared to $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ where the smashing ideals stand in bijection with open subsets of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ (thus closed covers of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ are necessary) the thick ideals of $\mathcal{T}^{\mathrm{c}}$ – and by extension the compactly generated smashing ideals of $\mathcal{T}$ – stand in bijection with Thomason subsets of $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$; hence, a cover by complements of Thomason subsets is what is needed.

Corollary 4.7. *Suppose that every point of* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible. Then:*

(a) $\mathcal{T}$ *satisfies cominimality if and only if* $\mathcal{T}/\operatorname{loc}^{\otimes}(\mathfrak{p})$ *satisfies cominimality, for all* $\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$.

(b) *Suppose that* $\mathcal{T}$ *satisfies the colocal-to-global principle. Then* $\mathcal{T}$ *is costratified if and only if* $\mathcal{T}/\operatorname{loc}^{\otimes}(\mathfrak{p})$ *is costratified, for all* $\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$.

Corollary 4.8. *Suppose that every point of* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible and that* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}}) = \bigcup_{j\in J} U_j$ *is a cover of* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ *by complements of Thomason subsets. If* $\mathcal{T}(U_j)$ *satisfies cominimality, for all* $j \in J$*, then* $\mathcal{T}$ *satisfies cominimality. If, moreover,* $\mathcal{T}$ *satisfies the colocal-to-global principle, then* $\mathcal{T}$ *is costratified.*

In view of applications involving singularity categories of schemes [31], we need a version of Corollary 4.8 for the more general case of a $\mathcal{T}$-module $\mathcal{K}$. Let $\mathcal{S}$ be a compactly generated localizing tensor-ideal of $\mathcal{T}$ and set $\mathcal{M} = \mathcal{S} * \mathcal{K}$. Then $\mathcal{M}$ is a compactly generated localizing submodule of $\mathcal{K}$; see [22, Section 4]. The action of $\mathcal{T}$ on $\mathcal{K}$ induces, as already discussed previously, an action of $\mathcal{T}$ on $\mathcal{K}/\mathcal{M}$. Because of the way $\mathcal{M}$ is defined, it follows that there is an induced action of $\mathcal{T}/\mathcal{S}$ on $\mathcal{K}/\mathcal{M}$ and a colocalizing subcategory of $\mathcal{K}/\mathcal{M}$ is a hom $\mathcal{T}$-submodule if and only if it is a hom $\mathcal{T}/\mathcal{S}$-submodule.

Assuming that every point of $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ is visible, let V be a Thomason subset of $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ and let $U = \operatorname{Spc}(\mathcal{T}^{\mathrm{c}}) \setminus V$ and consider the localizing tensor-ideal $\mathcal{T}_V$ generated by those compact objects of $\mathcal{T}$ whose support is contained in V. By definition, $\mathcal{T}_V$ is compactly generated and hence smashing, so there are associated left and right (respectively) idempotents e_V and f_V such that

$$\mathcal{T}_V = \operatorname{loc}^{\otimes}(e_V) = \operatorname{Ker}(f_V \otimes -) = \operatorname{Im}(e_V \otimes -).$$

We denote by $\mathcal{T}(U)$ the category $\mathcal{T}/\mathcal{T}_V$. It holds that $\operatorname{Spc}(\mathcal{T}(U)^{\mathrm{c}}) \cong U$ and we will treat this homeomorphism as an identification. Let $\mathcal{K}_V = \mathcal{T}_V * \mathcal{K}$ and let $\mathcal{K}(U) = \mathcal{K}/\mathcal{K}_V$. By the previous paragraph, $\mathcal{K}_V$ is a compactly generated localizing submodule of $\mathcal{K}$ and there is an induced action of $\mathcal{T}(U)$ on $\mathcal{K}(U)$ such that a colocalizing subcategory of $\mathcal{K}(U)$ is a hom $\mathcal{T}$-submodule if and only if it is a hom $\mathcal{T}(U)$-submodule. Further, $\mathcal{K}_V = \operatorname{Im}(e_V * -) = \operatorname{Ker}(f_V * -)$ and $\mathcal{K}_V^{\perp} = \operatorname{Im}[e_V, -]_* = \operatorname{Ker}[f_V, -]_*$. By this last observation, it follows that $S_{\mathcal{K}_V} := \{\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{\mathrm{c}}) \mid [g_{\mathfrak{p}}, I_{\mathcal{K}}]_* \in \mathcal{K}_V^{\perp}\} = U$.

The following result is the analogue of [22, Theorem 8.11] for colocalizing hom-submodules.

Theorem 4.9. *Suppose that every point of* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible and that* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}}) = \bigcup_{j\in J} U_j$ *is a cover of* $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ *by complements of Thomason subsets. Let* V_j *be the complement of* U_j*. If* $\mathcal{K}(U_j)$ *(as a* $\mathcal{T}(U_j)$*-module) satisfies cominimality, for all* $j \in J$*, then* $\mathcal{K}$ *satisfies cominimality. If, moreover,* $\mathcal{K}$ *satisfies the colocal-to-global principle, then* $\mathcal{K}$ *is costratified.*

Proof. If $\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^c)$, then there exists $j_{\mathfrak{p}} \in J$ such that $\mathfrak{p} \in U_{j_{\mathfrak{p}}}$. Fix such a $j_{\mathfrak{p}} \in J$, for each $\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^c)$. Then we have a collection $\{\mathcal{K}_{V_{j_{\mathfrak{p}}}}\}_{\mathfrak{p}\in\operatorname{Spc}(\mathcal{T}^c)}$ of smashing submodules of $\mathcal{K}$ such that $\mathfrak{p} \in S_{\mathcal{K}_{V_{j_{\mathfrak{p}}}}}$ since the latter is equal to $U_{j_{\mathfrak{p}}}$. The result now follows by an immediate application of Theorem 4.4. ∎

5. Derived categories of noetherian rings and schemes

Throughout, R will denote a commutative noetherian ring. In the article [21], Neeman proved that there is a bijective correspondence between colocalizing subcategories of $\mathrm{D}(R)$ and subsets of $\operatorname{Spec}(R)$. In this section, we give a more streamlined proof of Neeman's theorem by using the general machinery we developed; specifically Theorem 2.22 and Corollary 2.28. As a direct consequence, we obtain a complete description of the RHom-prime colocalizing subcategories of $\mathrm{D}(R)$ in terms of the residue fields. Further, using Corollary 4.8, we prove that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified.

Remark 5.1. Let X be a quasi-compact separated scheme. By [18, Proposition C.13], $\mathrm{D}(X)$ the derived category of quasi-coherent sheaves over X satisfies Hypothesis 2.24 (with $\mathrm{D}(R)$ being the special case $X = \operatorname{Spec}(R)$). In particular, this allows us to apply Corollary 2.28 later.

5.1. Noetherian rings

We will use the cosupport taking values in $\operatorname{Spec}(R)$ defined by the residue fields $k(\mathfrak{p})$. More specifically, if $X \in \mathrm{D}(R)$, then $\operatorname{Cosupph}(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{RHom}_R(k(\mathfrak{p}), X) \neq 0\}$. We use the notation Cosupph to avoid conflict with the Balmer–Favi cosupport. Note that since $\mathrm{D}(R)$ is generated by its tensor-unit, every colocalizing subcategory of $\mathrm{D}(R)$ is a left RHom-ideal. We denote by I_R the cogenerator of $\mathrm{D}(R)$ that is the product of the Brown-Comenetz duals of the compact objects.

Lemma 5.2. *Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then, for all $X \in \mathrm{D}(R)$, there exist sets J_i such that*

$$\operatorname{RHom}_R\big(k(\mathfrak{p}), X\big) \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(J_i)} \cong \prod_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(J_i)}.$$

The same holds for the complex $\operatorname{RHom}_R(X, k(\mathfrak{p}))$.

Proof. Let E be a K-injective resolution of X. Then $\operatorname{RHom}_R(k(\mathfrak{p}), X)$ is the Hom-complex $\operatorname{Hom}_R(k(\mathfrak{p}), E)$. This is a complex of $k(\mathfrak{p})$-vector spaces, therefore it must be quasi-isomorphic to its cohomology complex with zero differential (which also has $k(\mathfrak{p})$-vector spaces as terms; thus coproducts of copies of $k(\mathfrak{p})$). For $\operatorname{RHom}_R(X, k(\mathfrak{p}))$, pick a K-projective resolution of X instead of a K-injective resolution and argue in an identical manner. The isomorphism between the coproduct and the product in the statement holds because this is a coproduct of suspensions of stalk complexes. ∎

Lemma 5.3. *Let* X *be an object of* $\mathrm{D}(R)$ *such that* $\mathrm{RHom}_R(k(\mathfrak{p}),X)\neq 0$. *Then it holds that* $\mathrm{coloc}(k(\mathfrak{p}))=\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}),X))\subseteq\mathrm{coloc}(X)$.

Proof. It holds that $\mathrm{RHom}_R(k(\mathfrak{p}),X)\cong\prod_{i\in\mathbb{Z}}\Sigma^i k(\mathfrak{p})^{(J_i)}$. Since $k(\mathfrak{p})^{(J_i)}\hookrightarrow k(\mathfrak{p})^{J_i}$ is a map of $k(\mathfrak{p})$-vector spaces, it must split. So, $k(\mathfrak{p})^{(J_i)}$ is a summand of $k(\mathfrak{p})^{J_i}$. This implies that $k(\mathfrak{p})^{(J_i)}\in\mathrm{coloc}(k(\mathfrak{p}))$ and consequently, $\prod_{i\in\mathbb{Z}}\Sigma^i k(\mathfrak{p})^{(J_i)}\in\mathrm{coloc}(k(\mathfrak{p}))$. Thus, $\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}),X))\subseteq\mathrm{coloc}(k(\mathfrak{p}))$. By the fact that $k(\mathfrak{p})$ is a summand of $\mathrm{RHom}_R(k(\mathfrak{p}),X)$, it follows that $\mathrm{coloc}(k(\mathfrak{p}))\subseteq\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}),X))$. Since $\mathrm{D}(R)$ is generated by its tensor-unit, every colocalizing subcategory of $\mathrm{D}(R)$ is a left RHom-ideal. Hence, $\mathrm{RHom}_R(k(\mathfrak{p}),X)\in\mathrm{coloc}(X)$. This completes the proof. ∎

Proposition 5.4. *The category* $\mathrm{D}(R)$ *satisfies the colocal-to-global principle (in particular,* Cosupph *detects vanishing) and* $\mathrm{Cosupph}(k(\mathfrak{p}))=\{\mathfrak{p}\}$, *for each* $\mathfrak{p}\in\mathrm{Spec}(R)$.

Proof. Since $\mathrm{D}(R)$ satisfies the local-to-global principle [19], by Corollary 2.28, $\mathrm{D}(R)$ satisfies the colocal-to-global principle and, by Remark 2.18, Cosupph detects vanishing. Hence, $\mathrm{Cosupph}(k(\mathfrak{p}))\neq\varnothing$. Let $\mathfrak{q}\in\mathrm{Spec}(R)$ such that $\mathfrak{p}\neq\mathfrak{q}$. By Lemma 5.2, it holds that $\mathrm{RHom}_R(k(\mathfrak{p}),k(\mathfrak{q}))$ is quasi-isomorphic to a complex whose terms are of the form $k(\mathfrak{p})^{(I)}\cong k(\mathfrak{q})^{(J)}$ and these are both $k(\mathfrak{p})$ and $k(\mathfrak{q})$-vector spaces. Since $\mathfrak{p}\neq\mathfrak{q}$, this can only happen if the indexing sets I and J are empty. Hence, $\mathrm{RHom}_R(k(\mathfrak{p}),k(\mathfrak{q}))=0$. Consequently, $\mathrm{Cosupph}(k(\mathfrak{p}))=\{\mathfrak{p}\}$. ∎

Theorem 5.5 ([21]). *Let* R *be a commutative noetherian ring. Then* $\mathrm{D}(R)$ *is costratified.*

Proof. Let $\mathfrak{p}\in\mathrm{Spec}(R)$ and let X be a non-zero object in $\mathrm{coloc}(k(\mathfrak{p}))$. Then $\mathrm{coloc}(X)\subseteq\mathrm{coloc}(k(\mathfrak{p}))$. By Proposition 5.4, $\mathrm{D}(R)$ satisfies the colocal-to-global principle, Cosupph detects vanishing and $\mathrm{Cosupph}(k(\mathfrak{p}))=\{\mathfrak{p}\}$. By Lemma 2.19, it follows that $\mathrm{Cosupph}(X)=\{\mathfrak{p}\}$, i.e., $\mathrm{RHom}_R(k(\mathfrak{p}),X)\neq 0$. As a result, by Lemma 5.3, $\mathrm{coloc}(X)=\mathrm{coloc}(k(\mathfrak{p}))$. So, $\mathrm{coloc}(k(\mathfrak{p}))$ is a minimal colocalizing subcategory. Moreover, Lemma 5.3 implies that $\mathrm{coloc}(k(\mathfrak{p}))=\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}),I_R))$ and so, $\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}),I_R))$ is minimal. In conclusion, $\mathrm{D}(R)$ satisfies both the colocal-to-global principle and cominimality; so, Theorem 2.22 implies that $\mathrm{D}(R)$ is costratified; see also Remark 2.23. ∎

Theorem 5.6. *The* RHom*-prime colocalizing subcategories of* $\mathrm{D}(R)$ *correspond to points of* $\mathrm{Spec}(R)$. *The correspondence associates* $\mathfrak{p}\in\mathrm{Spec}(R)$ *with* $\mathrm{Ker}\,\mathrm{RHom}_R(k(\mathfrak{p}),-)=\mathrm{coloc}(k(\mathfrak{q})\mid\mathfrak{q}\neq\mathfrak{p})$.

Proof. Since $\mathrm{D}(R)$ is costratified, and clearly a conservative $\mathrm{D}(R)$-module, Theorem 3.11 implies that the RHom-prime colocalizing subcategories of $\mathrm{D}(R)$ are precisely of the form $\mathrm{Ker}\,\mathrm{RHom}_R(k(\mathfrak{p}),-)=\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{q}),I_R)\mid\mathfrak{q}\neq\mathfrak{p})$ and the claimed equality is due to Lemma 5.3. ∎

Remark 5.7. One could also choose to work with the Balmer–Favi support (or the smashing support since $\mathrm{D}(R)$ satisfies the Telescope Conjecture [19]; see also [30, Lemma 7.2] and [1, Section 6]). There is a homeomorphism between $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R))$ and $\mathrm{Spec}(R)$ [19]

that we can use to express the Balmer–Favi support–cosupport via $\operatorname{Spec}(R)$. For each $\mathfrak{p} \in \operatorname{Spec}(R)$, the Balmer–Favi idempotent associated with $\mathfrak{p}$ is $g_{\mathfrak{p}} = K_{\infty}(\mathfrak{p}) \otimes R_{\mathfrak{p}}$, where $K_{\infty}(\mathfrak{p})$ is the stable Koszul complex and $R_{\mathfrak{p}}$ is the localization of R at $\mathfrak{p}$. The objects $g_{\mathfrak{p}}$ are orthogonal tensor-idempotents, so they define a support–cosupport pair: Let $X \in \mathrm{D}(R)$. Then

$$\operatorname{Supp}(X) = \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid g_{\mathfrak{p}} \otimes X \neq 0\right\},$$
$$\operatorname{Cosupp}(X) = \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{RHom}_R(g_{\mathfrak{p}}, X) \neq 0\right\}.$$

It holds that $\operatorname{loc}(g_{\mathfrak{p}}) = \operatorname{loc}(k(\mathfrak{p}))$ [27, Lemma 3.22]. Therefore,

$$\operatorname{coloc}\left(\operatorname{RHom}_R(g_{\mathfrak{p}}, X)\right) = \operatorname{coloc}\left(\operatorname{RHom}_R\left(k(\mathfrak{p}), X\right)\right) = \operatorname{coloc}\left(k(\mathfrak{p})\right),$$

with the last equality by Lemma 5.3 (provided that $\operatorname{RHom}_R(k(\mathfrak{p}), X) \neq 0$). Since $\mathrm{D}(R)$ is stratified by the Balmer–Favi support [7, Theorem 5.8], in particular it satisfies the local-to-global principle, $\mathrm{D}(R)$ must also satisfy the colocal-to-global principle; see Corollary 2.28. The equality $\operatorname{coloc}(\operatorname{RHom}_R(g_{\mathfrak{p}}, X)) = \operatorname{coloc}(k(\mathfrak{p}))$ shows that $\mathrm{D}(R)$ satisfies cominimality with respect to the Balmer–Favi support–cosupport. Therefore, by Theorem 2.22, $\mathrm{D}(R)$ is costratified with respect to the Balmer–Favi support–cosupport.

Example 5.8 ([23]). We include an example of a category that is not costratified. Let R be an absolutely flat ring that is not semi-artinian. Then there exists a superdecomposable injective R-module E. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $\operatorname{RHom}_R(k(\mathfrak{p}), E) = \operatorname{Hom}_R(k(\mathfrak{p}), E)$. If there was a non-zero map $k(\mathfrak{p}) \to E$, then (as $k(\mathfrak{p})$ is simple and injective since R is absolutely flat) $k(\mathfrak{p})$ would have to be a summand of E, which leads to a contradiction. This shows that $\operatorname{RHom}_R(k(\mathfrak{p}), E) = 0$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, i.e., $\operatorname{Cosupph}(E) = \varnothing$; showcasing the failure of the cosupport to detect vanishing and consequently, the failure of the colocal-to-global principle. As a result, the local-to-global principle cannot hold either, since it implies the colocal-to-global principle.

5.2. Noetherian schemes

Let X be a noetherian separated scheme and denote by $\mathrm{D}(X)$ the derived category of quasi-coherent sheaves over X. Then $\mathrm{D}(X)$ is a big tt-category whose subcategory of compact objects is $\mathrm{D}^{\mathrm{perf}}(X)$ the subcategory of perfect complexes. The Balmer spectrum of $\mathrm{D}(X)$ is homeomorphic to the underlying space of X [28]. The notion of support we consider is the Balmer–Favi support.

Theorem 5.9. *Let X be a noetherian separated scheme. Then $\mathrm{D}(X)$ is costratified.*

Proof. By [22, Corollary 8.13], $\mathrm{D}(X)$ is stratified. In particular, $\mathrm{D}(X)$ satisfies the local-to-global principle. Hence, by Corollary 2.28, $\mathrm{D}(X)$ satisfies the colocal-to-global principle. So, it suffices to prove cominimality. Let $\{U_i\}_{i \in I}$ be an open affine cover of X. As X is noetherian, any open subset of X is quasi-compact, so its complement is Thomason. The corresponding smashing localization $\mathrm{D}(X)(U_i)$ is equivalent to $\mathrm{D}(U_i)$. The latter is costratified (in particular it satisfies cominimality) by Theorem 5.5. The result follows by Corollary 4.8. ■

Acknowledgments. I am grateful to Greg Stevenson for enlightening conversations and to Paul Balmer for his interest and suggestions that led to improvements.

References

[1] S. Balchin and G. Stevenson, Big categories, big spectra. [v1] 2021, [v4] 2023, arXiv:2109.11934v4

[2] P. Balmer, The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.* **588** (2005), 149–168 Zbl 1080.18007 MR 2196732

[3] P. Balmer, Homological support of big objects in tensor-triangulated categories. *J. Éc. polytech. Math.* **7** (2020), 1069–1088 Zbl 1454.18005 MR 4136434

[4] P. Balmer and G. Favi, Gluing techniques in triangular geometry. *Q. J. Math.* **58** (2007), no. 4, 415–441 Zbl 1130.18006 MR 2371464

[5] P. Balmer and G. Favi, Generalized tensor idempotents and the telescope conjecture. *Proc. Lond. Math. Soc. (3)* **102** (2011), no. 6, 1161–1185 Zbl 1220.18009 MR 2806103

[6] T. Barthel, N. Castellana, D. Heard, and B. Sanders, Cosupport in tensor triangular geometry. 2023, arXiv:2303.13480v1

[7] T. Barthel, D. Heard, and B. Sanders, Stratification in tensor triangular geometry with applications to spectral Mackey functors. *Camb. J. Math.* **11** (2023), no. 4, 829–915 Zbl 1524.18032 MR 4650265

[8] A. Beligiannis, Relative homological algebra and purity in triangulated categories. *J. Algebra* **227** (2000), no. 1, 268–361 Zbl 0964.18008 MR 1754234

[9] D. Benson, S. B. Iyengar, and H. Krause, Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)* **41** (2008), no. 4, 573–619 Zbl 1171.18007 MR 2489634

[10] D. J. Benson, S. B. Iyengar, and H. Krause, Stratifying modular representations of finite groups. *Ann. of Math. (2)* **174** (2011), no. 3, 1643–1684 Zbl 1261.20057 MR 2846489

[11] D. J. Benson, S. B. Iyengar, and H. Krause, Stratifying triangulated categories. *J. Topol.* **4** (2011), no. 3, 641–666 Zbl 1239.18013 MR 2832572

[12] D. J. Benson, S. B. Iyengar, and H. Krause, Colocalizing subcategories and cosupport. *J. Reine Angew. Math.* **673** (2012), 161–207 Zbl 1271.18012 MR 2999131

[13] H. Krause, Smashing subcategories and the telescope conjecture—an algebraic approach. *Invent. Math.* **139** (2000), no. 1, 99–133 Zbl 0937.18013 MR 1728877

[14] H. Krause, A Brown representability theorem via coherent functors. *Topology* **41** (2002), no. 4, 853–861 Zbl 1009.18010 MR 1905842

[15] H. Krause, The stable derived category of a Noetherian scheme. *Compos. Math.* **141** (2005), no. 5, 1128–1162 Zbl 1090.18006 MR 2157133

[16] J. P. May, The additivity of traces in triangulated categories. *Adv. Math.* **163** (2001), no. 1, 34–73 Zbl 1007.18012 MR 1867203

[17] H. Miller, Finite localizations. *Bol. Soc. Mat. Mexicana (2)* **37** (1992), no. 1-2, 383–389 Zbl 0852.55015 MR 1317588

[18] D. Murfet, *The mock homotopy category of projectives and Grothendieck duality*. Ph.D. thesis, Australian National University, Canberra, 2007

[19] A. Neeman, The chromatic tower for $D(R)$. *Topology* **31** (1992), no. 3, 519–532 Zbl 0793.18008 MR 1174255

[20] A. Neeman, Brown representability for the dual. *Invent. Math.* **133** (1998), no. 1, 97–105 Zbl 0906.18002 MR 1626473

[21] A. Neeman, Colocalizing subcategories of $\mathbf{D}(R)$. *J. Reine Angew. Math.* **653** (2011), 221–243 Zbl 1221.13030 MR 2794632

[22] G. Stevenson, Support theory via actions of tensor triangulated categories. *J. Reine Angew. Math.* **681** (2013), 219–254 Zbl 1280.18010 MR 3181496

[23] G. Stevenson, Derived categories of absolutely flat rings. *Homology Homotopy Appl.* **16** (2014), no. 2, 45–64 Zbl 1316.13022 MR 3234500

[24] G. Stevenson, Subcategories of singularity categories via tensor actions. *Compos. Math.* **150** (2014), no. 2, 229–272 Zbl 1322.18004 MR 3177268

[25] G. Stevenson, The local-to-global principle for triangulated categories via dimension functions. *J. Algebra* **473** (2017), 406–429 Zbl 1428.18023 MR 3591157

[26] G. Stevenson, Filtrations via tensor actions. *Int. Math. Res. Not. IMRN* **2018** (2018), no. 8, 2535–2558 Zbl 1410.18015 MR 3801492

[27] G. Stevenson, A tour of support theory for triangulated categories through tensor triangular geometry. In *Building bridges between algebra and topology*, pp. 63–101, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2018 Zbl 1400.18018 MR 3793858

[28] R. W. Thomason, The classification of triangulated subcategories. *Compositio Math.* **105** (1997), no. 1, 1–27 Zbl 0873.18003 MR 1436741

[29] P. Thompson, Cosupport computations for finitely generated modules over commutative noetherian rings. *J. Algebra* **511** (2018), 249–269 Zbl 1439.13035 MR 3834773

[30] C. Verasdanis, Stratification and the smashing spectrum. *Math. Z.* **305** (2023), no. 4, article no. 54 Zbl 1524.18026 MR 4659914

[31] C. Verasdanis, Colocalizing subcategories of singularity categories. *J. Algebra* **662** (2025), 608–624 Zbl 1551.18020 MR 4795668

Communicated by Henning Krause

Received 10 April 2025; revised 14 July 2025.

Charalampos Verasdanis
Department of Mathematics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece;
Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague, Czech Republic;
cverasd@math.auth.gr, verasdanis@math.cas.cz
Author IDs: zbMATH verasdanis.charalampos MR 1584098 ORCID 0009-0009-1473-7664
