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Nef and effective cones of some Quot schemes

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Abstract. Let C be a smooth projective curve over $\mathbb{C}$ of genus $g(C) \geqslant 3$ (respectively, $g(C) = 2$). Fix integers r, k such that $2 \leqslant k \leqslant r-2$, (respectively, $3 \leqslant k \leqslant r-2$). Let $\mathcal{Q} := \mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, k, d)$ be the Quot scheme parametrizing rank k and degree d quotients of the trivial bundle of rank r. Let $\mathcal{Q}_L$ denote the closed subscheme of the Quot scheme parametrizing quotients such that the quotient sheaf has determinant L. It is known that $\mathcal{Q}_L$ is an integral, normal, local complete intersection, locally factorial scheme of Picard rank 2, when $d \gg 0$. In this article we compute the nef cone, effective cone and canonical divisor of this variety when $d \gg 0$. We show this variety is Fano iff $r = 2k + 1$.

1. Introduction

Let C be a smooth projective curve over the field of complex numbers $\mathbb{C}$. Fix integers $0 < k < r$. Let E be a locally free sheaf of rank r on C. Let $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$ denote Grothendieck's Quot scheme parametrizing rank k and degree d quotients of the bundle E. Quot schemes are ubiquitous in algebraic geometry. They are useful in many moduli questions, for example GIT constructions of moduli spaces of vector bundles and sheaves. They have interesting geometric, representation-theoretic and enumerative properties. They admit natural generalisations; for example, moduli spaces of stable maps to homogeneous space bundles. It is therefore natural to investigate some basic and important questions about Quot schemes of bundles over curves.

Let $\mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, k, d)$ denote the Quot scheme parametrizing rank k and degree d quotients of the trivial bundle of rank r. When $C \cong \mathbb{P}^1$, Stromme, in [14], proved that $\mathrm{Quot}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, k, d)$ is a smooth projective variety of Picard rank 2 and computed its nef cone. In [10], the author computed the effective cone of $\mathrm{Quot}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, k, d)$. In [15], the author determined the movable cone of $\mathrm{Quot}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, k, d)$ and the stable base locus decomposition of the effective cone. In [9], the author studied the birational geometry of $\mathrm{Quot}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, k, d)$.

For curves of higher genus, the space $\mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, k, d)$ was studied in [1] and it was proved that when $d \gg 0$, this Quot scheme is irreducible and generically smooth. This was generalized by Popa and Roth [11], who proved the same result for $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$,

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the Quot scheme parametrizing quotients of a vector bundle E. See also [3, 4, 6, 13] for similar results on other variations of this Quot scheme. In [12], the author generalizes the above mentioned result of Popa and Roth to the case of nodal curves.

Following the work of Popa and Roth [11], it is natural to ask if the results of [9, 10, 14, 15] can be generalized to the higher genus case. A more basic question is to know if these Quot schemes are integral, and if one can compute their divisor class group and their Picard group. Motivated by these questions, in [5], the authors prove that for $d \gg 0$, $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d)$ is integral, local complete intersection, normal and locally factorial scheme and compute its Picard group. Consider the determinant map

$$\det : \mathrm{Quot}_{C/\mathbb{C}}(E,k,d) \to \mathrm{Pic}^d(C),$$

defined as follows. It sends a closed point $[E \to F]$ to the determinant line bundle $[\det(F)]$. For $[L] \in \mathrm{Pic}^d(C)$, let us denote the fiber over $[L]$ by $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d)_L$. By definition, $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d)_L$ parametrizes quotients $[E \to F]$ such that rank of F is k and determinant of F is L. With some mild assumptions on the genus of C and k, it was proved in [5] that when $d \gg 0$ the scheme $\mathrm{Quot}_{C/\mathbb{C}}(E,k,d)_L$ is integral, local complete intersection, normal and locally factorial scheme and moreover, its Picard group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. See the next section for precise statements of the main results in [5].

For ease of notation, let $\mathcal{Q}_L$ denote $\mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r},k,d)_L$. Let $\mathrm{N}^1(\mathcal{Q}_L)$ denote the Neron-Severi group of $\mathcal{Q}_L$. Let $\mathrm{Nef}(\mathcal{Q}_L) \subset \mathrm{N}^1(\mathcal{Q}_L) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the cone of nef divisors. Similarly, we have the cone of effective divisors, which we denote by $\mathrm{Eff}(\mathcal{Q}_L)$ and the movable cone of divisors which we denote $\mathrm{Mov}(\mathcal{Q}_L)$. In this article, we compute these cones, thereby generalizing the results in [10, 14, 15]. Note that in the case of $\mathbb{P}^1$, this scheme is just $\mathrm{Quot}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_C^{\oplus r},k,d)$. It is surprising to note the similarity between $\mathcal{Q}_L$ and $\mathrm{Quot}_{\mathbb{P}^1/\mathbb{C}}(\mathcal{O}_C^{\oplus r},k,d)$. In view of this similarity, and the results of [9], it is natural to ask if $\mathcal{Q}_L$ is a Mori dream space.

We now state the main results of this article. Let α and β_{d+g-1} denote the boundaries of the nef cone. For the definitions of these line bundles, see the discussion before (4.3), the discussion before Lemma 4.7 and Remark 4.8. We also compute the class of the canonical divisor of $\mathcal{Q}_L$ in terms of α and β_{d+g-1}. Combining this with Theorem 4.14 we give a necessary and sufficient condition for $\mathcal{Q}_L$ to be Fano. This question, regarding when $\mathcal{Q}_L$ is Fano, was raised by Pieter Belmans in his Blog.[1]

In the following theorem, for the definitions of the curves D_1 and D_2, see the proofs of Propositions 4.4 and 4.9.

Theorem 1.1. *Assume one of the following two holds:*

- $g(C) \geqslant 3$ *and* $2 \leqslant k \leqslant r-2$, *or*
- $g(C) = 2$ *and* $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. *Then we have the following results:*

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[1] See https://pbelmans.ncag.info/blog/2022/11/09/fortnightly-links-160/ visited on 14 April 2026.

(A) $\mathrm{Pic}(\mathcal{Q}_L)$ *is generated by the line bundles* α, β_{d+g-1}. *Both these are globally generated, nef but not ample. In particular,*

$$\mathrm{Nef}(\mathcal{Q}_L) = \mathbb{R}_{\geqslant 0}\alpha + \mathbb{R}_{\geqslant 0}\beta_{d+g-1}.$$

The boundaries of the cone of effective curves are given by the classes of the curves D_1 *and* D_2.

(B) *The effective cone of* $\mathcal{Q}_L$ *is given by*

$$\begin{aligned}\mathrm{Eff}(\mathcal{Q}_L) = \mathbb{R}_{\geqslant 0}&\big(d(k+1)\alpha - k\beta_{d+g-1}\big)\\ &+ \mathbb{R}_{\geqslant 0}\big(-d(r-k-1)\alpha + (r-k)\beta_{d+g-1}\big).\end{aligned}$$

The cone $\mathrm{Mov}(\mathcal{Q}_L) = \mathrm{Eff}(\mathcal{Q}_L)$.

(C) *Let* $\omega_{\mathcal{Q}_L}$ *denote the canonical divisor of* $\mathcal{Q}_L$. *Then*

$$\omega_{\mathcal{Q}_L} = \big[d(r-2k-2) + r(g-1)\big]\alpha + (2k-r)\beta_{d+g-1}.$$

Thus, $\mathcal{Q}_L$ *is Fano iff* $r = 2k+1$.

Statement (A) is proved in Theorem 4.14, statement (B) is proved in Theorem 5.16 and statement (C) is proved in Theorem 6.10.

2. Recollection of some results from [5]

Let C be a smooth projective curve over the field of complex numbers $\mathbb{C}$. We shall denote the genus of C by $g(C)$. Throughout this article we shall assume that $g(C) \geqslant 2$. Let E be a locally free sheaf on C of rank r and degree e. In the latter sections we will be considering only the case $E = \mathcal{O}_C^{\oplus r}$, whence, $e = 0$. Let k be an integer such that $0 < k < r$. Throughout this article

$$\mathrm{Quot}_{C/\mathbb{C}}(E, k, d) \tag{2.1}$$

will denote the Quot scheme of quotients of E of rank k and degree d. There is a map

$$\det\colon \mathrm{Quot}_{C/\mathbb{C}}(E, k, d) \to \mathrm{Pic}^d(C), \tag{2.2}$$

see [5, equation (2.5)]. This map sends a closed point $[E \to F] \in \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$ to $[\det(F)] \in \mathrm{Pic}^d(C)$. Let L be a line bundle on C of degree d and let

$$\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L := \det^{-1}\big([L]\big)$$

be the scheme theoretic fiber over the point $[L] \in \mathrm{Pic}^d(C)$. We recall the main results in [5].

(A) [5, Theorem 3.3, Corollary 3.5]. First consider the case $k = r - 1$. Assume $d > 2g - 2 + e - \mu_{\min}(E)$. There is a locally free sheaf $\mathcal{E}$ on $\mathrm{Pic}^{e-d}(C)$ such that the following holds. We have an isomorphism of schemes over $\mathrm{Pic}^{e-d}(C)$, $\mathbb{P}(\mathcal{E}^\vee) \xrightarrow{\sim} \mathrm{Quot}_{C/\mathbb{C}}(E, r-1, d)$. In particular, under the above assumption on d, the space $\mathrm{Quot}_{C/\mathbb{C}}(E, r-1, d)$ is smooth and $\mathrm{Pic}(\mathrm{Quot}_{C/\mathbb{C}}(E, r-1, d)) \cong \mathrm{Pic}(\mathrm{Pic}^0(C)) \times \mathbb{Z}$.

Next we consider the case when $k \leqslant r - 2$. There is a number $d_0(E, k)$, which depends only on E and k, such that the following statements hold. Let $d \geqslant d_0(E, k)$.

(1) [5, Theorem 6.3] Then $\det\colon \mathrm{Quot}_{C/\mathbb{C}}(E, k, d) \to \mathrm{Pic}^d(C)$ is a flat map. Further, $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)$ is local complete intersection scheme which is an integral and normal variety and is locally factorial.

(2) [5, Theorem 9.1] Let $k = 1$ and $r \geqslant 3$ (the case $k = 1$ and $r = 2$ is dealt with in the case $k = r - 1$ above). Then $\mathrm{Pic}(\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)) \cong \mathrm{Pic}(\mathrm{Pic}^d(C)) \times \mathbb{Z} \times \mathbb{Z}$.

(3) [5, Theorem 8.7] Let $k \geqslant 2$, $g(C) \geqslant 2$. Then $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L$ is a local complete intersection scheme, which is also integral, normal and locally factorial.

(4) [5, Theorem 8.9] Assume one of the following two holds

- $k \geqslant 2$ and $g(C) \geqslant 3$, or
- $k \geqslant 3$ and $g(C) = 2$.

We have isomorphisms

$$\mathrm{Pic}\left(\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L\right) \cong \mathrm{Pic}(M^s_{k,L}) \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}.$$

(5) [5, Theorem 9.1] Let $k = 1$ and $r \geqslant 3$. Then $\mathrm{Pic}(\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L) \cong \mathbb{Z} \times \mathbb{Z}$.

In view of point (4), it becomes a particularly interesting question to investigate the nef cone of the scheme $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L$. The purpose of this article is to investigate this question when E is the trivial bundle of rank r. Before we proceed we mention a few points from [5] which we shall use. The discussion in this paragraph assumes that $d \gg 0$. The "good locus" of the Quot scheme is defined to be the set of points (see [5, Definition 4.4])

$$\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_g := \left\{[E \to F] \mid H^1(E^\vee \otimes F) = 0\right\}.$$

Let A be a locally closed subset of the Quot scheme. Then the good locus of A, denoted A_g is defined to be the subset $A \cap \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_g$. An important property of the good locus is that the morphism det in equation (2.2) restricted to the good locus is a smooth morphism. In particular, taking $A = \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L$, we get the locus $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_{g,L}$. Another subset of the Quot scheme which will be used is the locus of stable quotients, that is,

$$\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)^s := \left\{[E \to F] \mid F \text{ is stable}\right\}.$$

Similarly, we define $\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)^s_L$. We have inclusions

$$\mathrm{Quot}_{C/\mathbb{C}}(E, k, d)^s_L \subset \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_{g,L} \subset \mathrm{Quot}_{C/\mathbb{C}}(E, k, d)_L.$$

If $Y \subset X$ is locally closed, then we denote $\operatorname{codim}(Y, X) = \dim(X) - \dim(Y)$. In the proof of [5, Theorem 8.9], it is proved that

$$\operatorname{codim}\left(\operatorname{Quot}_{C/\mathbb{C}}(E,k,d)_L \setminus \operatorname{Quot}_{C/\mathbb{C}}(E,k,d)^s_L, \operatorname{Quot}_{C/\mathbb{C}}(E,k,d)_L\right) \geqslant 2.$$

3. Notation

Fix a point $[L] \in \operatorname{Pic}^d(C)$. For the remainder of this article we impose the following conventions:

- Denote by $\mathcal{Q} := \operatorname{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, k, d)$ and denote by $\mathcal{Q}_L := \operatorname{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, k, d)_L$.
- Assume one of the following two holds:
  - † $g(C) \geqslant 3$ and $2 \leqslant k \leqslant r-2$, or
  - † $g(C) = 2$ and $3 \leqslant k \leqslant r-2$.
- If $Y \subset X$ is locally closed, then we denote $\operatorname{codim}(Y, X) = \dim(X) - \dim(Y)$.
- For a scheme T, we shall denote by $p_C : C \times T \to C$ the projection. The projection onto the second factor will be denoted by p_2.
- Let
$$0 \to \mathcal{K} \to p_C^* \mathcal{O}_C^{\oplus r} \to \mathcal{F} \to 0 \tag{3.1}$$
denote the universal quotient on $C \times \mathcal{Q}$. We will abuse notation and use the same notation to denote its restriction to $C \times \mathcal{Q}_L$.
- We shall assume that $d \gg 0$. In particular, $d \geqslant d_0(E, k)$.

Remark 3.2. At several places we will use the following easy consequence of cohomology and base change. Let $f : X \to Y$ be a projective morphism and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Suppose $h^1(X_y, \mathcal{F}_y) = 0$ for all closed points $y \in Y$. Then $f_*(\mathcal{F})$ is locally free. Let $g : Y' \to Y$ be a morphism and consider the Cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Then the natural map $g^* f_*(\mathcal{F}) \to f'_* g'^*(\mathcal{F})$ is an isomorphism.

4. Nef cone of $\mathcal{Q}_L$

Recall the universal sequence (3.1) on $C \times \mathcal{Q}_L$. Applying $\wedge^{r-k}$ to the inclusion $\mathcal{K} \subset p_C^* \mathcal{O}_C^{\oplus r}$, we get an inclusion, which sits in a short exact sequence

$$0 \to \wedge^{r-k}\mathcal{K} \to \wedge^{r-k}\left(p_C^* \mathcal{O}_C^{\oplus r}\right) \to \mathcal{F}' \to 0. \tag{4.1}$$

Let $q \in \mathcal{Q}_L$ be a closed point. The restriction of the map in (4.1) to $C \times q$ is the same as restricting the map (3.1) to $C \times q$ and then applying $\wedge^{r-k}$. From this it easily follows that the restriction of (4.1) to $C \times q$ is an inclusion. For each point $q \in \mathcal{Q}_L$, the sheaf $\wedge^{r-k}\mathcal{K}_q := \wedge^{r-k}\mathcal{K}|_{C\times q} \cong L^{-1}$. Thus, it follows that the rank and degree of $\mathcal{F}'_q$ are constant. As $\mathcal{Q}_L$ is an integral scheme, it follows that $\mathcal{F}'$ of (4.1) is flat over $\mathcal{Q}_L$. The quotient

$$\wedge^{r-k}(p_C^*\mathcal{O}_C^{\oplus r}) \to \mathcal{F}'$$

on $C \times \mathcal{Q}_L$ gives rise to a morphism from $\mathcal{Q}_L$ to the quot scheme

$$\mathrm{Quot}_{C/\mathbb{C}}\left(\wedge^{r-k}(\mathcal{O}_C^{\oplus r}), \binom{r}{r-k} - 1, d\right).$$

Moreover, for each q, the cokernel $\mathcal{F}'_q$ has determinant L. It follows that the image of the composite map

$$\mathcal{Q}_L \to \mathrm{Quot}_{C/\mathbb{C}}\left(\wedge^{r-k}(\mathcal{O}_C^{\oplus r}), \binom{r}{r-k} - 1, d\right) \xrightarrow{\det} \mathrm{Pic}^d(C)$$

is, at least set theoretically, the closed point $[L]$. As $\mathcal{Q}_L$ is an integral scheme, it follows that the image lands in the scheme theoretic fiber $\mathrm{Quot}_{C/\mathbb{C}}(\wedge^{r-k}(\mathcal{O}_C^{\oplus r}), \binom{r}{r-k} - 1, d)_L$. By (A) (that is, [5, Theorem 3.3, Corollary 3.5]) it follows that we get a map from

$$f\colon \mathcal{Q}_L \to \mathbb{P}(\mathcal{E}_{L^{-1}}^\vee) \tag{4.2}$$

(note that $e = 0$). For ease of notation we denote $\mathbb{P}(\mathcal{E}_{L^{-1}}^\vee)$ by $\mathbb{P}$. Define

$$\alpha := f^*\mathcal{O}_{\mathbb{P}}(1). \tag{4.3}$$

Proposition 4.4. *The line bundle α is nef but not ample.*

Proof. It is clear that α is nef. To show it is not ample, it suffices to find a curve $D_1 \subset \mathcal{Q}_L$ such that the restriction of α to D_1 is trivial.

We begin with describing the map f in (4.2) in some more detail. Let $p_2\colon C \times \mathcal{Q}_L \to \mathcal{Q}_L$ denote the projection. By the Seesaw theorem, there is a line bundle M on $\mathcal{Q}_L$ such that $\wedge^{r-k}\mathcal{K} \cong p_C^*L^{-1} \otimes p_2^*M$. Tensoring (4.1) with p_C^*L we get the following exact sequence of sheaves on $C \times \mathcal{Q}_L$

$$0 \to p_2^*M \to [\wedge^{r-k}(p_C^*\mathcal{O}_C^{\oplus r})] \otimes p_C^*L \to \mathcal{F}' \otimes p_C^*L \to 0.$$

Applying p_{2*} we get the following exact sequence of sheaves on $\mathcal{Q}_L$

$$0 \to M \to H^0(C, [\wedge^{r-k}(\mathcal{O}_C^{\oplus r})] \otimes L) \otimes \mathcal{O}_{\mathcal{Q}_L} \to p_{2*}(\mathcal{F}' \otimes p_C^*L) \to H^1(C, \mathcal{O}_C) \otimes M \to 0.$$

The last term on the right is 0 as the degree of L is d, which is assumed to be very large. It follows that $p_{2*}(\mathcal{F}' \otimes p_C^*L)$ is locally free. Taking dual of the above sequence, we get a surjection

$$H^0(C, \wedge^{r-k}[\wedge^{r-k}(\mathcal{O}_C^{\oplus r})] \otimes L)^\vee \otimes \mathcal{O}_{\mathcal{Q}_L} \to M^{-1},$$

which defines the map f to

$$\mathbb{P}\big(H^0\big(C, \wedge^{r-k}\big[\wedge^{r-k}\big(\mathcal{O}_C^{\oplus r}\big)\big] \otimes L\big)^\vee\big).$$

It is clear that the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ is M^{-1}. Thus, $\alpha = M^{-1}$.

To construct the curve D_1, fix a closed point $x \in C$ and a subsheaf $K' \subset \mathcal{O}_C^{\oplus r}$ with $\det(K') = L^{-1} \otimes \mathcal{O}_C(x)$. Let $D_1 \subset \mathbb{P}(K'_x)$ be a line in the projective space associated to the vector space K'_x. Let $\iota_x \colon D_1 \to C \times D_1$ denote the map $t \mapsto (x, t)$. On $C \times D_1$ we have the surjection

$$p_C^* K' \to \iota_{x*}\iota_x^* p_C^* K' = \iota_{x*}(K'_x \otimes \mathcal{O}_{D_1}) \to \iota_{x*}\big(\mathcal{O}_{D_1}(1)\big).$$

Let $\widetilde{K}_1$ be the kernel of this surjection. As $\iota_{x*}(\mathcal{O}_{D_1}(1))$ is flat over D_1, it follows that $\widetilde{K}_1$ is flat over D_1. We have the following commutative diagram over $C \times D_1$ in which all three term sequences are exact

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}_1 \arrow[r] \arrow[d] & p_C^*\mathcal{O}^{\oplus r} \arrow[r] \arrow[d, equal] & \mathcal{G}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & p_C^* K' \arrow[r] \arrow[d] & p_C^*\mathcal{O}^{\oplus r} \arrow[r] & p_C^* F' \arrow[r] & 0 \\
 & \iota_{x*}\big(\mathcal{O}_{D_1}(1)\big). & & &
\end{tikzcd} \tag{4.5}$$

It follows easily that for $t \in D_1$, the rank and degree of $\mathcal{G}_{1,t}$ are independent of t. Thus, $\mathcal{G}_1$ is flat over D_1. Note that

$$\begin{aligned}
\wedge^{r-k}\widetilde{K}_1 &\cong \det(p_C^* K') \otimes \det\big(\iota_{x*}\big(\mathcal{O}_{D_1}(1)\big)\big)^{-1} \\
&\cong p_C^*\big(L^{-1} \otimes \mathcal{O}_C(x)\big) \otimes p_C^*\mathcal{O}_C(-x) \\
&\cong p_C^* L^{-1}.
\end{aligned}$$

It easily follows that we get a morphism $D_1 \to \mathcal{Q}_L$, which is an inclusion on closed points.

As $p_C^* L^{-1} = p_C^* L^{-1} \otimes p_2^*\mathcal{O}_{D_1}$, from the description of the morphism f, it is clear that the restriction of $f^*(\mathcal{O}_{\mathbb{P}}(1))$ to D_1 is the trivial bundle. This shows that α is not ample. ∎

Let $\mathcal{Q}_L^s$ denote the locus of quotients $[\mathcal{O}_C^{\oplus r} \to F]$ such that F is a stable bundle. To construct our second nef class, we shall first construct a vector bundle on $\mathcal{Q}_L^s$. The determinant of this gives a line bundle on $\mathcal{Q}_L^s$. Under our hypothesis, it follows that $\operatorname{codim}(\mathcal{Q}_L \setminus \mathcal{Q}_L^s, \mathcal{Q}_L) \geqslant 2$ (see the discussion before equation (8.10) in [5]). Thus, the line bundle constructed on $\mathcal{Q}_L^s$ extends uniquely to a line bundle on $\mathcal{Q}_L$. We will show that this line bundle is nef.

Let M be a line bundle on C of degree m such that

$$\frac{d}{k} + m > 2g - 2. \tag{4.6}$$

Recall the universal sheaf $\mathcal{F}$ from (3.1). Let p_2 denote the projection $C \times \mathcal{Q}^s_L \to \mathcal{Q}^s_L$. Consider the sheaf $p_{2*}(\mathcal{F} \otimes p_C^* M)$ on $\mathcal{Q}^s_L$. For a point $q \in \mathcal{Q}^s_L$, we have $h^1(C, \mathcal{F}_q \otimes M) = 0$ iff $\hom(\mathcal{F}_q \otimes M, \omega_C) = 0$. Assume $\mu_{\min}(\mathcal{F}_q \otimes M) > \mu_{\max}(\omega_C)$, that is, if (4.6) holds. By [8, Lemma 1.3.3], it follows that $\hom(\mathcal{F}_q \otimes M, \omega_C) = 0$. Thus, if m is such that this inequality holds, then it follows easily, using cohomology and base change [7, Theorem 12.11], that the sheaf $p_{2*}(\mathcal{F} \otimes p_C^* M)$ on $\mathcal{Q}^s_L$ is locally free. The determinant of this locally free sheaf gives a line bundle on $\mathcal{Q}^s_L$, which extends uniquely to a line bundle on $\mathcal{Q}_L$. We denote this line bundle by β_M.

Lemma 4.7. *Let M and M' have the same degree m such that* (4.6) *holds. Then β_M is numerically equivalent to $\beta_{M'}$.*

Proof. Let $\mathcal{P}$ be a Poincare bundle on $C \times \mathrm{Pic}^m(C)$. Let p_{ij} denote the projection maps from $C \times \mathcal{Q}^s_L \times \mathrm{Pic}^m(C)$. Consider the sheaf $p_{23*}(\mathcal{F} \otimes p_{13}^* \mathcal{P})$ on $\mathcal{Q}^s_L \times \mathrm{Pic}^m(C)$. For a point $(q, M) \in \mathcal{Q}^s_L \times \mathrm{Pic}^m(C)$, we have $h^1(C, \mathcal{F}_q \otimes M) = 0$. It follows easily that the sheaf $p_{23*}(\mathcal{F} \otimes p_{13}^* \mathcal{P})$ on $\mathcal{Q}^s_L \times \mathrm{Pic}^m(C)$ is locally free. Let us denote the determinant of this sheaf by $\mathcal{R}$. It can be easily seen, for example, using similar reasoning as in [5, Proposition 8.1], that $\mathcal{Q}_L \times \mathrm{Pic}^m(C)$ is locally factorial. It easily follows that the line bundle $\mathcal{R}$ extends uniquely to a line bundle on $\mathcal{Q}_L \times \mathrm{Pic}^m(C)$, which we continue to denote by $\mathcal{R}$.

It is also easily seen, using Remark 3.2, that the restriction of $p_{23*}(\mathcal{F} \otimes p_{13}^* \mathcal{P})$ to $\mathcal{Q}^s_L \times [M]$ equals $p_{2*}(\mathcal{F} \otimes p_C^* M)$. Thus, it easily follows that $\mathcal{R}$ restricted to $\mathcal{Q}_L \times [M]$ equals β_M. Similarly, it follows that $\mathcal{R}$ restricted to $\mathcal{Q}_L \times [M']$ equals $\beta_{M'}$. It easily follows that β_M is numerically equivalent to $\beta_{M'}$. This completes the proof of the lemma. ■

Remark 4.8. In view of the above lemma, when m satisfies (4.6), we shall denote the corresponding numerical class by β_m.

It is easily checked that when $d \gg 0$ then $m = d + g - 1$ satisfies (4.6).

Proposition 4.9. *The class β_{d+g-1} is globally generated and hence nef. This class is not ample.*

Proof. We will show that for any point $q \in \mathcal{Q}_L$, there is a line bundle M on C of degree $d + g - 1$, such that the line bundle β_M on $\mathcal{Q}_L$ has a global section which does not vanish at q. This will show that β_{d+g-1} is globally generated. As a globally generated line bundle is nef, it follows that β_{d+g-1} is nef.

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^r$ given by $t \cdot (a_1, \ldots, a_r) = (a_1, ta_2, \ldots, t^{r-1}a_r)$. This action gives rise to an action of $\mathbb{C}^*$ on $\mathcal{O}_C^{\oplus r}$ and so also on $\mathcal{Q}_L$. Indeed, this action sends an inclusion φ to the inclusion $\varphi \circ t^{-1}$, in the following commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\varphi\circ t^{-1}"] \arrow[d, equal] & \mathcal{O}_C^{\oplus r} \arrow[r] \arrow[d, "t"] & F' \arrow[d] \\
K \arrow[r, "\varphi"] & \mathcal{O}_C^{\oplus r} \arrow[r] & F.
\end{tikzcd}$$

Thus, given any point $q \in \mathcal{Q}_L$, we may find a $\mathbb{C}^*$ equivariant morphism $h\colon \mathbb{C}^* \to \mathcal{Q}_L$ such that $h(1) = q$. Note that for $t \in \mathbb{C}^*$, the kernel sheaf in $h(t)$ is the same as the kernel sheaf in q. The morphism h extends to a morphism $\mathbb{C} \to \mathcal{Q}_L$ and the point $h(0)$ is fixed under the action of $\mathbb{C}^*$ on $\mathcal{Q}_L$. Thus, $h(0)$ is a quotient q_0 whose kernel equals

$$K_0 = \bigoplus_{i=1}^{r-k} \mathcal{O}_C(-D_i),$$

where each D_i is an effective divisor of degree d_i, and the d_i satisfy $\sum_i d_i = d$. See, for example, [2, Section 3]. Let M be a general line bundle of degree $d + g - 1$. Then $M \otimes \mathcal{O}_C(-D_i)$ is a general line bundle of degree $d - d_i + g - 1 \geqslant g - 1$. In particular, $h^1(C, M \otimes \mathcal{O}_C(-D_i)) = 0$. It follows that $h^1(C, M \otimes K_0) = 0$ and so this vanishing holds for the kernels in an open set containing q_0. In particular, it follows that

$$h^1(C, M \otimes K) = 0, \tag{4.10}$$

where K is the kernel of the quotient q we started with.

Since $d \gg 0$, we have $d + g - 1 > 2g - 2$ and so $h^1(C, M) = 0$. If q' is any point in $\mathcal{Q}_L$, then the cohomology long exact sequence of the short exact sequence

$$0 \to \mathcal{K}_{q'} \otimes M \to (\mathcal{O}_C^{\oplus r}) \otimes M \to \mathcal{F}_{q'} \otimes M \to 0$$

shows that $h^1(C, \mathcal{F}_{q'} \otimes M) = 0$. In particular, it follows that the sheaf $p_{2*}(\mathcal{F} \otimes p_C^* M)$ is locally free on all of $\mathcal{Q}_L$. Applying $p_{2*}(- \otimes p_C^* M)$ to (3.1), and using (4.10), it follows that on an open set containing the point q, the map

$$p_{2*}(p_C^*(\mathcal{O}_C^{\oplus r}) \otimes p_C^* M) \to p_{2*}(\mathcal{F} \otimes p_C^* M)$$

is a surjection. Applying $\wedge^k$ we get that the map

$$\wedge^k \big(H^0(C, M)^{\oplus r}\big) \to \beta_M$$

is surjective on an open set containing q. It follows that β_{d+g-1} is globally generated and so nef.

To show that β_{d+g-1} is not ample, we will construct a curve $D_2 \subset \mathcal{Q}_L$ such that $[D_2] \cdot [\beta_{d+g-1}] = 0$. Let D_2 be a line in $\mathbb{P}(H^0(C, L)^\vee)$. Then on $C \times D_2$ we have a short exact sequence

$$0 \to p_C^* L^{-1} \otimes p_2^* \mathcal{O}_{D_2}(-1) \to p_C^* \mathcal{O}_C \to \mathcal{G}_2 \to 0.$$

"Adding" to this, identity maps of the type $p_C^* \mathcal{O}_C^{\oplus l} \to p_C^* \mathcal{O}_C^{\oplus l}$, we get the quotient

$$\begin{aligned} 0 \to \big(p_C^* L^{-1} \otimes p_2^* \mathcal{O}_{D_2}(-1)\big) \oplus \big(p_C^* \mathcal{O}_C^{\oplus(r-k-1)}\big) \\ \to p_C^* \mathcal{O}_C \oplus p_C^* \mathcal{O}_C^{\oplus(r-k-1)} \oplus \big(p_C^* \mathcal{O}_C^{\oplus k}\big) \to \mathcal{G}_2 \oplus \big(p_C^* \mathcal{O}_C^{\oplus k}\big) \to 0. \end{aligned} \tag{4.11}$$

This defines a morphism

$$f' \colon D_2 \to \mathcal{Q}_L,$$

which is clearly injective on closed points. Let M be a line bundle of degree $d+g-1$ such that $h^1(C, L^{-1}\otimes M)=0$. Using Riemann-Roch we get that $h^0(C, L^{-1}\otimes M)=0$. Again, as $d \gg 0$, we have $h^1(C,M)=0$. It easily follows that for each $t \in D_2$,

$$h^1\big(C, \big[\mathcal{G}_{2,t} \oplus \big(\mathcal{O}_C^{\oplus k}\big)\big] \otimes M\big) = 0.$$

It follows easily using Remark 3.2 that $f'^*\beta_M = \det(p_{2*}((\mathcal{G}_2 \oplus p_C^*\mathcal{O}_C^{\oplus k}) \otimes p_C^*M))$. We claim that $p_{2*}((\mathcal{G}_2 \oplus p_C^*\mathcal{O}_C^{\oplus k}) \otimes p_C^*M)$ is the trivial bundle. To see this, note that

$$p_{2*}\big[p_C^*L^{-1} \otimes p_2^*\mathcal{O}_{D_2}(-1) \otimes p_C^*M\big] = 0,$$

and

$$R^1p_{2*}\big[\big\{\big(p_C^*L^{-1} \otimes p_2^*\mathcal{O}_{D_2}(-1)\big) \oplus \big(p_C^*\mathcal{O}_C^{\oplus(r-k-1)}\big)\big\} \otimes p_C^*M\big] = 0.$$

It follows that when we apply $p_{2*}(- \otimes p_C^*M)$ to the sequence (4.11), we get the following exact sequence in which the first two terms are trivial bundles,

$$\begin{aligned} 0 \to H^0(C,M)^{\oplus(r-k-1)} \otimes \mathcal{O}_{D_2} &\to H^0(C,M)^{\oplus r} \otimes \mathcal{O}_{D_2} \\ &\to p_{2*}\big((\mathcal{G}_2 \oplus p_C^*\mathcal{O}_C^{\oplus k}) \otimes p_C^*M\big) \to 0. \end{aligned}$$

Thus, it follows that the last bundle is also trivial, and so $f'^*\beta_M$ is the trivial bundle. This shows that $[D_2]\cdot[\beta_{d+g-1}] = 0$. ■

Lemma 4.12. *We have* $[D_2]\cdot[\alpha] = [D_1]\cdot[\beta_{d+g-1}] = 1$.

Proof. To compute $[D_2]\cdot[\alpha]$ we shall use the description of the map f in the proof of Proposition 4.4. The kernel in the family of sheaves defining D_2 (see (4.11)) is

$$\big(p_C^*L^{-1} \otimes p_2^*\mathcal{O}_{D_2}(-1)\big) \oplus \big(p_C^*\mathcal{O}_C^{\oplus(r-k-1)}\big).$$

Taking $\wedge^{r-k}$ of this sheaf gives the line bundle $p_C^*L^{-1} \otimes p_2^*\mathcal{O}_{D_2}(-1)$. Thus, the pullback of α to D_2 is $\mathcal{O}_{D_2}(1)$. The degree of this line bundle is 1. Thus, $[D_2]\cdot[\alpha]=1$.

Recall the family of quotients (4.5) which defines a morphism $D_1 \to \mathcal{Q}_L$. Using cohomology and base change, it easily follows that the pullback of β_{d+g-1} to D_1 is $\det(p_{2*}(\mathcal{G}_1 \otimes M))$, where M is any line bundle of degree $d+g-1$. From the short exact sequence (see (4.5))

$$0 \to \iota_{x*}\big(\mathcal{O}_{D_1}(1)\big) \to \mathcal{G}_1 \to p_C^*F' \to 0$$

it easily follows that $\det(p_{2*}(\mathcal{G}_1 \otimes M)) = \mathcal{O}_{D_1}(1)$. Thus, $[D_1]\cdot[\beta_{d+g-1}] = 1$. ■

Remark 4.13. As a corollary of $[D_2]\cdot[\alpha]=1$, we see that α and β_{d+g-1} are not numerically equivalent. Thus, it follows that the natural map $\mathrm{Pic}(\mathcal{Q}_L) \to \mathrm{N}^1(\mathcal{Q}_L)$ is an isomorphism.

Putting together the above results, we have the following theorem, which is part (A) of Theorem 1.1.

Theorem 4.14. *Assume one of the following two holds:*

- $g(C) \geqslant 3$ *and* $2 \leqslant k \leqslant r-2$, *or*
- $g(C) = 2$ *and* $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. *Then* $\mathrm{Pic}(\mathcal{Q}_L)$ *is generated by the line bundles* α, β_{d+g-1}. *Both these are globally generated and so nef, but not ample. In particular, they are the boundaries of the nef cone. The boundaries of the cone of effective curves are given by the classes of* D_1 *and* D_2.

Proof. Given an integral curve $C' \subset \mathcal{Q}_L$, let us write $[C'] = a[D_1] + b[D_2]$ in $N_1(\mathcal{Q}_L)$. Intersecting with α and β_{d+g-1} we easily see that $a \geqslant 0$ and $b \geqslant 0$. ■

5. Effective cone

In this section we shall determine the cone of effective divisors in the Picard group of $\mathcal{Q}_L$. Let

$$\mathcal{Q}'_L = \left\{ \left[\mathcal{O}_C^{\oplus r} \xrightarrow{q} F \right] \in \mathcal{Q}^s_L \mid \mathrm{Ker}(q) \text{ is stable} \right\}. \tag{5.1}$$

denote the open set consisting of quotients such that both F and $\mathrm{Ker}(q)$ are stable.

Lemma 5.2. *We have* $\mathrm{codim}(\mathcal{Q}_L \setminus \mathcal{Q}'_L, \mathcal{Q}_L) \geqslant 2$.

Proof. Let $\mathcal{Q}^{\mathrm{tf}}_L \subset \mathcal{Q}_L$ denote the locus of quotients $[\mathcal{O}_C^{\oplus r} \to F]$ such that F is torsion free. Note that $\mathcal{Q}^s_L \subset \mathcal{Q}^{\mathrm{tf}}_L$. For ease of notation, let us denote the Quot scheme $\mathrm{Quot}_{C/\mathbb{C}}(\mathcal{O}_C^{\oplus r}, r-k, d)$ by $\widetilde{\mathcal{Q}}$. There is an isomorphism of schemes $\varphi \colon \mathcal{Q}^{\mathrm{tf}}_L \to \widetilde{\mathcal{Q}}^{\mathrm{tf}}_L$ which sends a quotient

$$\left[\mathrm{Ker}(q) \subset \mathcal{O}_C^{\oplus r} \xrightarrow{q} F \right] \mapsto \left[F^\vee \subset \mathcal{O}_C^{\oplus r} \to \mathrm{Ker}(q)^\vee \right].$$

The open set $\mathcal{Q}'_L$ is precisely the intersection $\mathcal{Q}^s_L \cap \varphi^{-1}(\widetilde{\mathcal{Q}}^s_L)$. By the discussion before equation (8.10) in [5], it follows that $\mathrm{codim}(\mathcal{Q}_L \setminus \mathcal{Q}^s_L, \mathcal{Q}_L) \geqslant 2$. Similarly,

$$\mathrm{codim}\left(\widetilde{\mathcal{Q}}^{\mathrm{tf}}_L \setminus \widetilde{\mathcal{Q}}^s_L, \widetilde{\mathcal{Q}}^{\mathrm{tf}}_L \right) \geqslant 2.$$

Thus, it follows that the codimension of $\mathcal{Q}^{\mathrm{tf}}_L \setminus \varphi^{-1}(\widetilde{\mathcal{Q}}^s_L)$ in $\mathcal{Q}^{\mathrm{tf}}_L$ is at least 2. The lemma now follows easily. ■

We will use the above lemma to write down another curve $D_3 \to \mathcal{Q}_L$ such that the image lies in $\mathcal{Q}'_L$. Let E be a stable bundle of rank $r-k$ with $\det(E) = L^{-1}$. Consider the space $\mathbb{P}(\mathrm{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee)$. The closed points of this space are in bijection with nonzero maps $\mathcal{O}_C^{\oplus r} \to E^\vee$. The locus of points in $\mathbb{P}(\mathrm{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee)$ corresponding to non-surjective maps $\mathcal{O}_C^{\oplus r} \to E^\vee$ has codimension at least 2, see the proof in [5, Lemma 7.12]. Let

$$U_{E^\vee} \subset \mathbb{P}\left(\mathrm{Hom}(\mathcal{O}_C^{\oplus r}, E^\vee)^\vee \right) = \mathbb{P}\left(\mathrm{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee \right)$$

denote the locus parametrizing surjective maps $\mathcal{O}_C^{\oplus r} \to E^\vee$. Let $M^s_{r-k,L}$ denote the moduli space of stable bundles of rank $r-k$ and determinant L. Consider the natural map

$$\pi\colon \widetilde{\mathcal{Q}}^s_L \to M^s_{r-k,L}, \tag{5.3}$$

which sends $[\mathcal{O}_C^{\oplus r} \to F] \mapsto [F]$. The fiber over the point $[F]$ is precisely the set U_F. Let T denote the closed subset $\widetilde{\mathcal{Q}}^s_L \setminus \widetilde{\mathcal{Q}}'_L$.

Lemma 5.4. *For general* $F \in M^s_{r-k,L}$, $\operatorname{codim}(T \cap U_F, U_F) \geqslant 2$.

Proof. If $\operatorname{codim}(T \cap U_F, U_F) \leqslant 1$ for general $[F] \in M^s_{r-k,L}$, then it follows that

$$\dim(T) = \dim(M^s_{r-k,L}) + \dim\big(\pi^{-1}([F])\big) - 1 = \dim(\widetilde{\mathcal{Q}}^s_L) - 1.$$

This contradicts Lemma 5.2, which says that $\operatorname{codim}(T, \widetilde{\mathcal{Q}}^s_L) \geqslant 2$. ■

Thus, it follows that for general E, the locus of points in $U_{E^\vee}$, such that the kernel of $\mathcal{O}_C^{\oplus r} \to E^\vee$ is not stable, has codimension $\geqslant 2$. In other words, if $U' \subset \mathbb{P}(\operatorname{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee)$ denotes the set of points corresponding to inclusions $E \to \mathcal{O}_C^{\oplus r}$ such that the cokernel is torsion free and stable, then for general E,

$$\operatorname{codim}\big(\mathbb{P}\big(\operatorname{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee\big) \setminus U', \mathbb{P}\big(\operatorname{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee\big)\big) \geqslant 2.$$

If $W \subset \mathbb{P}^n$ is a closed subset such that $\operatorname{codim}(W, \mathbb{P}^n) \geqslant 2$, then the general line in $\mathbb{P}^n$ does not meet W. This is easily seen by projecting from a point outside W. Thus, we can find a line $D_3 \subset \mathbb{P}(\operatorname{Hom}(E, \mathcal{O}_C^{\oplus r})^\vee)$, which is completely contained in U'. We get a family of quotients on $C \times D_3$

$$0 \to p_C^* E \otimes p_2^* \mathcal{O}_{D_3}(-1) \to p_C^* \mathcal{O}_C^{\oplus r} \to \mathcal{G}_3 \to 0, \tag{5.5}$$

such that for each $t \in D_3$, the sheaf $\mathcal{G}_{3,t}$ is stable. The above family defines a morphism $D_3 \to \mathcal{Q}_L$, which is injective on closed points. Clearly, the image of D_3 lands in $\mathcal{Q}'_L$.

Lemma 5.6. *We have* $[\alpha] \cdot [D_3] = r - k$ *and* $[\beta_{d+g-1}] \cdot [D_3] = d(r-k-1)$.

Proof. To compute $[\alpha] \cdot [D_3]$ we follow the description of the map f (see (4.2)) given in Proposition 4.4.

$$\wedge^{r-k}\big(p_C^* E \otimes p_2^* \mathcal{O}_{D_3}(-1)\big) = p_C^* L^{-1} \otimes p_2^* \mathcal{O}_{D_3}\big(-(r-k)\big).$$

Thus, it follows that $[\alpha] \cdot [D_3] = r - k$.

Let M be a line bundle of degree $d + g - 1$. By Serre duality, we have,

$$h^1(C, E \otimes M) = h^0(E^\vee \otimes M^\vee \otimes \omega_C).$$

Note E is stable and

$$\mu(E^\vee \otimes M^\vee \otimes \omega_C) = \frac{d}{r-k} - (d + g - 1) + 2g - 2.$$

The slope is < 0 for $d \gg 0$ as $r-k \geqslant 2$. Thus, $h^0(E^\vee \otimes M^\vee \otimes \omega_C) = h^1(C, E\otimes M) = 0$. Thus, applying $p_{2*}(- \otimes p_C^* M)$ to (5.5), we get the short exact sequence

$$0 \to H^0(C, E\otimes M) \otimes \mathcal{O}_{D_3}(-1) \to H^0(C, M)^{\oplus r} \otimes \mathcal{O}_{D_3} \to p_{2*}(\mathcal{G}_3 \otimes p_C^* M) \to 0.$$

It follows that

$$\begin{aligned}[D_3]\cdot[\beta_{d+g-1}] &= \deg\big(p_{2*}(\mathcal{G}_3 \otimes p_C^* M)\big)\\ &= h^0(C, E\otimes M) = \chi(E\otimes M)\\ &= d(r-k-1).\end{aligned}$$

This completes the proof of the lemma. ■

Consider the map $\pi\colon \mathcal{Q}_L^s \to M_{k,L}^s$, where $M_{k,L}^s$ denotes the moduli space parametrizing stable bundles of rank k and determinant L. The map π sends a quotient $[\mathcal{O}_C^{\oplus r} \to F]$ to $[F]$. The fiber of π over the point $[F]$ is the subset $U_F \subset \mathbb{P}(\mathrm{Hom}(\mathcal{O}_C^{\oplus r}, F)^\vee)$ corresponding to surjective maps. Let $U'_F \subset U_F$ be the subset corresponding to maps for which the kernel is also a stable bundle. Arguing as in the construction of the curve D_3, we see that for a general stable bundle F, one has

$$\operatorname{codim}\big(\mathbb{P}\big(\mathrm{Hom}(\mathcal{O}_C^{\oplus r}, F)^\vee\big) \setminus U'_F, \mathbb{P}\big(\mathrm{Hom}(\mathcal{O}_C^{\oplus r}, F)^\vee\big)\big) \geqslant 2.$$

Thus, taking F general stable and taking D_4 to be a general line in $\mathbb{P}(\mathrm{Hom}(\mathcal{O}_C^{\oplus r}, F)^\vee)$, to get a family

$$0 \to \mathcal{K}_4 \to p_C^* \mathcal{O}_C^{\oplus r} \to p_C^* F \otimes p_2^* \mathcal{O}_{D_4}(1) \to 0 \tag{5.7}$$

on $C \times D_4$. Again, this family has the property that for each $t \in D_4$, the sheaf $\mathcal{K}_{4,t}$ is stable.

Lemma 5.8. *We have* $[\alpha]\cdot[D_4] = k$ *and* $[\beta_{d+g-1}]\cdot[D_4] = d(k+1)$.

Proof. Note that $\wedge^{r-k}\mathcal{K}_4 = (p_C^* L \otimes p_2^*\mathcal{O}_{D_4}(k))^{-1}$. Using the description in Proposition 4.4, it follows that $[\alpha]\cdot[D_4] = k$. Let M be a line bundle of degree $d+g-1$. Since

$$p_{2*}\big(p_C^*(F\otimes M) \otimes p_2^*\mathcal{O}_{D_4}(1)\big) = H^0(C, F\otimes M) \otimes \mathcal{O}_{D_4}(1),$$

and $H^1(C, F\otimes M) = 0$, it follows that $[\beta_{d+g-1}]\cdot[D_4] = \chi(F\otimes M) = d(k+1)$. ■

Lemma 5.9. *Let a and b be integers such that $a\alpha + b\beta_{d+g-1}$ is an effective divisor. Then* $ak + bd(k+1) \geqslant 0$.

Proof. Let $Y \subset \mathcal{Q}_L$ be an effective divisor. Then Y cannot contain all the fibers of the map $\pi\colon \mathcal{Q}_L^s \to M_{k,L}^s$. Thus, for general F, the intersection $Y \cap U'_F \subsetneq U'_F$. In particular, Y does not contain the general line in U'_F, that is, Y does not contain D_4. Thus, $[Y]\cdot[D_4] \geqslant 0$. Letting the class of Y to be $a\alpha + b\beta_{d+g-1}$, the lemma follows easily. ■

Let $\mathrm{Pic}(M_{k,L}) = \mathrm{Pic}(M_{k,L}^s) = \mathbb{Z}[\Theta]$, where Θ is the unique ample generator.

Lemma 5.10. *Let* $\lambda_0 := \gcd(k, d(k+1)) = \gcd(k, d)$. *Then* $\pi^*\Theta = \frac{1}{\lambda_0}(d(k+1)\alpha - k\beta_{d+g-1})$.

Proof. Let us write $\pi^*\Theta = a\alpha + b\beta_{d+g-1}$. Clearly, $[\pi^*\Theta]\cdot[D_4] = 0$ as $\pi(D_4) = [F]$. This gives

$$ak + bd(k+1) = 0.$$

Thus, $\pi^*\Theta = \lambda'(-d(k+1)\alpha + k\beta_{d+g-1})$, for some rational number λ'. We claim that $\lambda' \neq 0$. This is clear from the point (4), which says that the pullback of Θ is not trivial.

Note that $[\pi(D_3)]\cdot[\Theta] \geqslant 0$ and so we get $[\pi^*\Theta]\cdot[D_3] \geqslant 0$. Using Lemma 5.6 we get

$$\begin{aligned}
[\pi^*\Theta]\cdot[D_3] &= \lambda'\big(-d(k+1)\alpha + k\beta_{d+g-1}\big)\cdot[D_3]\\
&= \lambda'\big(-d(k+1)(r-k) + kd(r-k-1)\big)\\
&= \lambda'(-dr).
\end{aligned}$$

The condition that $[\pi^*\Theta]\cdot[D_3] \geqslant 0$ now forces that $\lambda' < 0$. Thus, we get that $\pi^*\Theta = \lambda(d(k+1)\alpha - k\beta_{d+g-1})$, where $\lambda > 0$ is a rational number.

We can determine the precise value of λ as follows. First we need to recall some facts from [5]. In the proof of [5, Theorem 8.9], it is shown that there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Pic}\left(M^s_{k,L\otimes\mathcal{O}_C(knP)}\right) \arrow[r] \arrow[d, equal] & \operatorname{Pic}(\mathcal{Q}^s_L) \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \operatorname{Pic}\left(M^s_{k,L\otimes\mathcal{O}_C(knP)}\right) \arrow[r] & \operatorname{Pic}(\mathcal{Q}^s_L) \arrow[r] & \operatorname{Pic}\left(\theta^{-1}\left[F\otimes\mathcal{O}_C(nP)\right]\right) \cong \mathbb{Z} &
\end{tikzcd}$$

in which the rows are exact sequences. We recall the map $\theta\colon \mathcal{Q}^s_L \to M^s_{k,L\otimes\mathcal{O}_C(knP)}$ is defined as $\theta([\mathcal{O}_C^{\oplus r} \to F]) = [F\otimes\mathcal{O}_C(nP)]$. The existence of the above diagram is proved using the same method as described in the second paragraph of the proof of [5, Theorem 7.17]. Fix a point $P \in C$ and let $n \gg 0$ be a fixed integer, as in the discussion in the beginning of Section 7 in [5]. Consider the isomorphism $\delta\colon M^s_{k,L} \to M^s_{k,L\otimes\mathcal{O}_C(knP)}$ given by $[F] \mapsto [F\otimes\mathcal{O}_C(nP)]$. Then $\theta = \delta\circ\pi$ and so $\theta^{-1}([F\otimes\mathcal{O}_C(nP)]) = \pi^{-1}([F]) = U_F$. Thus, the lower row in the above diagram is identified with

$$1 \to \operatorname{Pic}(M^s_{k,L}) \to \operatorname{Pic}(\mathcal{Q}^s_L) \to \operatorname{Pic}(U_F) \cong \mathbb{Z}.$$

Given an element $\gamma \in \operatorname{Pic}(\mathcal{Q}^s_L)$, its image in $\operatorname{Pic}(U_F) \cong \mathbb{Z}$ is the intersection of a general line in U_F with γ, that is, $[\gamma]\cdot[D_4]$. Thus, to compute $\pi^*\Theta$ in terms of α and β_{d+g-1}, we need to compute the generator of the kernel of the map $\mathbb{Z}^{\oplus 2} \xrightarrow{\varphi} \mathbb{Z}$ which sends $\varphi(1,0) = k$ and $\varphi(0,1) = d(k+1)$. Let $\lambda_0 := \gcd(k, d(k+1)) = \gcd(k, d)$. It is easily checked that this generator, that is, $\pi^*\Theta$, equals

$$\pi^*\Theta = \frac{d(k+1)}{\lambda_0}\alpha - \frac{k}{\lambda_0}\beta.$$

This completes the proof of the lemma. ■

Remark 5.11. As a corollary, we also get the following. Since $\mathrm{Pic}(\mathcal{Q}^s_L)$ is generated by α and β_{d+g-1}, it follows that the image of the restriction map $\mathrm{Pic}(\mathcal{Q}^s_L) \to \mathrm{Pic}(U_F)$ is generated by $\gcd(k, d(k+1)) = \gcd(k, d)$.

As a corollary of Lemmas 5.9 and 5.10, we get the following corollary.

Corollary 5.12. *The class $\pi^*\Theta$ is a boundary of the effective cone.*

Proof. It is clear that $\pi^*\Theta$ is an effective divisor. By Lemma 5.9, if $a\alpha + b\beta_{d+g-1}$ is effective, then a and b satisfy the inequality $ak + bd(k+1) \geqslant 0$. It is clear that the coefficients a and b in $\pi^*\Theta$ satisfy the equality $ak + bd(k+1) = 0$. Thus, the corollary follows. ■

Recall the space $\mathcal{Q}'_L$ defined before Lemma 5.2. On $\mathcal{Q}'_L$ we have the map

$$\pi' : \mathcal{Q}'_L \to M^s_{r-k,L},$$

which sends a quotient $[\mathcal{O}_C^{\oplus r} \xrightarrow{q} F]$ to $[\mathrm{Ker}(q)]$. Arguing as in Lemma 5.9, we have the following lemma.

Lemma 5.13. *Let a and b be integers such that $a\alpha + b\beta_{d+g-1}$ is an effective divisor. Then $a(r-k) + bd(r-k-1) \geqslant 0$.*

Proof. The proof is the same as that of Lemma 5.9, except that we use the map π' now. In this case we will have the condition $[Y] \cdot [D_3] \geqslant 0$. The lemma easily follows using Lemma 5.6. ■

Let $\mathrm{Pic}(M_{r-k,L}) = \mathrm{Pic}(M^s_{r-k,L}) = \mathbb{Z}[\Theta']$, where Θ' is the unique ample generator. Similar to Lemma 5.10, we have the following lemma.

Lemma 5.14. *Let $\lambda_1 := \gcd(r-k, d(r-k-1)) = \gcd(r-k, d)$. Then*

$$\pi'^*\Theta' = \frac{1}{\lambda_1}\big(d(r-k-1)\alpha - (r-k)\beta_{d+g-1}\big).$$

Proof. Let us write $\pi'^*\Theta' = a\alpha + b\beta_{d+g-1}$. Recall the curve D_3 defined using the family (5.5). Clearly, $[\pi'^*\Theta'] \cdot [D_3] = 0$ as $\pi'(D_3) = [E]$. This gives

$$a(r-k) + bd(r-k-1) = 0.$$

Thus, $\pi'^*\Theta' = \lambda'(-d(r-k-1)\alpha + (r-k)\beta_{d+g-1})$, for some rational number λ'. As in Lemma 5.10, we have that $\lambda' \neq 0$. Note that $[\pi'(D_4)] \cdot [\Theta'] \geqslant 0$ and so we get $[\pi'^*\Theta'] \cdot [D_4] \geqslant 0$. Using Lemma 5.8 we get

$$\begin{aligned}
[\pi'^*\Theta'] \cdot [D_4] &= \lambda'\big(-d(r-k-1)\alpha + (r-k)\beta_{d+g-1}\big) \cdot [D_4] \\
&= \lambda'\big(-d(r-k-1)k + (r-k)d(k+1)\big) \\
&= \lambda'(dr).
\end{aligned}$$

The condition that $[\pi'^*\Theta'] \cdot [D_4] \geqslant 0$ now forces that $\lambda' > 0$. Thus, we get that $\pi'^*\Theta' = \lambda(-d(r-k-1)\alpha + (r-k)\beta_{d+g-1})$, where $\lambda > 0$ is a rational number. Arguing as in Lemma 5.10, we get

$$\pi'^*\Theta' = \frac{-d(r-k-1)}{\lambda_1}\alpha + \frac{r-k}{\lambda_1}\beta_{d+g-1}.$$

This completes the proof of the lemma. ∎

As a corollary of Lemmas 5.13 and 5.14, we get the following corollary.

Corollary 5.15. *The class $\pi'^*\Theta'$ is a boundary of the effective cone.*

Thus, combining the above results, we have the following theorem, which is part (B) of Theorem 1.1.

Theorem 5.16. *Assume one of the following two holds:*

- $g(C) \geqslant 3$ *and* $2 \leqslant k \leqslant r-2$, *or*
- $g(C) = 2$ *and* $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. The effective cone of $\mathcal{Q}_L$ is spanned by non-negative linear combinations of the classes $d(k+1)\alpha - k\beta_{d+g-1}$ and $-d(r-k-1)\alpha + (r-k)\beta_{d+g-1}$. Further, $\mathrm{Mov}(\mathcal{Q}_L) = \mathrm{Eff}(\mathcal{Q}_L)$.

Proof. Clearly, $\mathrm{Mov}(\mathcal{Q}_L) \subset \mathrm{Eff}(\mathcal{Q}_L)$. Since the boundaries of $\mathrm{Eff}(\mathcal{Q}_L)$, namely $\pi^*\Theta$ and $\pi'^*\Theta'$, define morphisms on the open subset $\mathcal{Q}'_L$, whose complement in $\mathcal{Q}_L$ has codimension $\geqslant 2$, it follows that these boundaries are in $\mathrm{Mov}(\mathcal{Q}_L)$. Thus, equality follows. ∎

6. Canonical divisor

In this section, we shall determine the canonical divisor of $\mathcal{Q}_L$ in terms of α and β.

Let ω_C denote the canonical divisor of C. Consider the open subset $\mathcal{Q}_g \subset \mathcal{Q}$ consisting of quotients $[\mathcal{O}_C^{\oplus r} \to F]$ for which $h^1(C, F) = 0$. If $h^1(C, F) = 0$, then applying $\mathrm{Hom}(-, F)$ to the short exact sequence $0 \to K \to \mathcal{O}_C^{\oplus r} \to F \to 0$, it follows that $\mathrm{ext}^1(K, F) = 0$. Thus, $\mathcal{Q}_g$ is contained in the smooth locus of $\mathcal{Q}$. Recall that $\mathcal{Q}^s$ denoted the open subset consisting of quotients $[\mathcal{O}_C^{\oplus r} \to F]$ for which F is stable. Clearly, $\mathcal{Q}^s \subset \mathcal{Q}_g$ as $d \gg 0$. Using Lemma 2.7 and equation (6.4) in [5], it follows that the morphism $\det\colon \mathcal{Q}_g \to \mathrm{Pic}^d(C)$ is a smooth morphism. It follows that the locus $\mathcal{Q}_{g,L} = \mathcal{Q}_L \cap \mathcal{Q}_g$ is contained in the smooth locus of $\mathcal{Q}_L$. As $\mathrm{codim}(\mathcal{Q}_L \setminus \mathcal{Q}^s_L, \mathcal{Q}_L) \geqslant 2$, it follows that $\mathrm{codim}(\mathcal{Q}_L \setminus \mathcal{Q}_{g,L}, \mathcal{Q}_L) \geqslant 2$. Thus, to determine the canonical divisor of $\mathcal{Q}_L$, it suffices to determine the canonical divisor of $\mathcal{Q}_{g,L}$. As the morphism det is smooth on $\mathcal{Q}_g$, and the canonical divisor of $\mathrm{Pic}^d(C)$ is trivial, it follows easily from the exact sequence (det being the morphism in (2.2))

$$0 \to \det{}^*\Omega_{\mathrm{Pic}^d(C)} \to \Omega_{\mathcal{Q}_g} \to \Omega_{\det} \to 0$$

that

$$\det(\Omega_{\mathcal{Q}_{g,L}}) = \det(\Omega_{\det}|_{\mathcal{Q}_{g,L}}) = \det(\Omega_{\mathcal{Q}_g})|_{\mathcal{Q}_{g,L}}. \tag{6.1}$$

Recall the universal sequence (3.1) on $C \times \mathcal{Q}$. Using the same method as in [14, Theorem 7.1], we may show that the tangent bundle on $\mathcal{Q}_g$ equals $p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F})$. It easily follows that

$$\Omega_{\mathcal{Q}_g}|_{\mathcal{Q}_{g,L}} = \big(p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F})\big)^\vee, \tag{6.2}$$

where we use the same notation to denote the restriction of the sheaves $\mathcal{K}, \mathcal{F}$ to $C \times \mathcal{Q}_{g,L}$. Thus, the canonical divisor of $\mathcal{Q}_{g,L}$ equals the determinant of the locally free sheaf $(p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F}))^\vee$.

To compute the canonical divisor in terms of α and β, we need two curves in $\mathcal{Q}_{g,L}$. One of these is the curve D_1, given by the family (4.5). Let us check that the image of D_1 is contained in $\mathcal{Q}_{g,L}$. Recall from [11, Corollary 6.3] that when $d \gg 0$, and K' is a general stable bundle, then the cokernel of the general inclusion $K' \to \mathcal{O}_C^{\oplus r}$ is a stable bundle. In particular, we may assume that both K' and F' in (4.5) are stable bundles. For each $t \in D_1$, the quotient $\mathcal{G}_{1,t} \cong F' \oplus \mathbb{C}_x$ and so

$$h^1(C, \mathcal{G}_{1,t}) = 0.$$

It follows that the image of D_1 is contained in $\mathcal{Q}_{g,L}$.

Our second curve is the curve D_3 given by the family (5.5). Recall the space $\mathcal{Q}'_L$ from (5.1). We had seen that the image of $D_3 \to \mathcal{Q}_L$ is contained in $\mathcal{Q}'_L$. Also note that $\mathcal{Q}'_L \subset \mathcal{Q}_{g,L}$. Thus, the curves D_1 and D_3 are contained in $\mathcal{Q}_{g,L}$. Next we will compute the degree of the line bundle $\det(p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F}))$ restricted to D_1 and D_3.

Lemma 6.3. *The degree of* $\det(p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F}))$ *restricted to* D_1 *is* $r - 2k$.

Proof. As $\mathrm{ext}^1(K, F) = 0$ for a point $[K \subset \mathcal{O}_C^{\oplus r} \to F] \in \mathcal{Q}_g$, it follows easily that (see (4.5))

$$p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F})|_{D_1} = p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{G}_1).$$

From (4.5) it follows that we have the following short exact sequence on $C \times D_1$

$$0 \to \iota_{x*}\big(\mathcal{O}_{D_1}(1)\big) \to \mathcal{G}_1 \to p_C^* F' \to 0.$$

For ease of notation, let $\mathcal{T}$ denote the sheaf $\iota_{x*}(\mathcal{O}_{D_1}(1))$. Applying $p_{2*}(\widetilde{K}_1^\vee \otimes -)$, we get the short exact sequence

$$0 \to p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{T}) \to p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{G}_1) \to p_{2*}(\widetilde{K}_1^\vee \otimes p_C^* F') \to 0. \tag{6.4}$$

Let us first compute determinant of the sheaf $p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{T})$. Apply $\mathcal{H}om(-, \mathcal{T})$ to the short exact sequence $0 \to \widetilde{K}_1 \to p_C^* K' \to \mathcal{T} \to 0$ (see (4.5)) yields the long exact sequence

$$0 \to \mathcal{H}om(\mathcal{T}, \mathcal{T}) \to \mathcal{T}^{\oplus(r-k)} \to \widetilde{K}_1^\vee \otimes \mathcal{T} \to \mathcal{E}xt^1(\mathcal{T}, \mathcal{T}) \to 0. \tag{6.5}$$

Applying $\mathcal{H}om(-, \mathcal{T})$ to the short exact sequence

$$0 \to p_C^* \mathcal{O}_C(-x) \otimes p_2^* \mathcal{O}_{D_1}(1) \to p_2^* \mathcal{O}_{D_1}(1) \to \mathcal{T} \to 0, \tag{6.6}$$

one easily checks that the sheaves $\mathcal{H}om(\mathcal{T},\mathcal{T})$ and $\mathcal{E}xt^1(\mathcal{T},\mathcal{T})$ are isomorphic to $\iota_{x*}(\mathcal{O}_{D_1})$. As all the sheaves in (6.5) are coherent over D_1, applying p_{2*} we get the following exact sequence of sheaves on D_1

$$0 \to \mathcal{O}_{D_1} \to \mathcal{T}^{\oplus(r-k)} \to p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{T}) \to \mathcal{O}_{D_1} \to 0.$$

From this it follows that

$$\det\big(p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{T})\big) \cong \mathcal{O}_{D_1}(r-k). \tag{6.7}$$

Next let us compute the determinant of the sheaf $p_{2*}(\widetilde{K}_1^\vee \otimes p_C^* F')$. For this we apply $\mathcal{H}om(-, p_C^* F')$ to the short exact sequence

$$0 \to \widetilde{K}_1 \to p_C^* K' \to \mathcal{T} \to 0.$$

We get the following long exact sequence on $C \times D_1$

$$0 \to p_C^* K'^\vee \otimes p_C^* F' \to \widetilde{K}_1^\vee \otimes p_C^* F' \to \mathcal{E}xt^1(\mathcal{T}, p_C^* F') \to 0. \tag{6.8}$$

The last term equals

$$\mathcal{E}xt^1(\mathcal{T}, p_C^* F') \cong \mathcal{E}xt^1(\mathcal{T} \otimes p_C^* F', \mathcal{O}_{C\times D_1}) \cong \mathcal{E}xt^1(\mathcal{T}, \mathcal{O}_{C\times D_1})^{\oplus k}.$$

Applying $\mathcal{H}om(-,\mathcal{O}_{C\times D_1})$ to (6.6) one easily sees that $\mathcal{E}xt^1(\mathcal{T},\mathcal{O}_{C\times D_1}) \cong \iota_{x*}(\mathcal{O}_{D_1}(-1))$. Note that $h^1(K'^\vee \otimes F') = 0$ as both K' and F' are stable. Applying p_{2*} to (6.8), we get the following exact sequence

$$0 \to \mathrm{Hom}(K', F') \to p_{2*}(\widetilde{K}_1^\vee \otimes p_C^* F') \to \mathcal{O}_{D_1}(-1)^{\oplus k} \to 0.$$

It follows that $\det(p_{2*}(\widetilde{K}_1^\vee \otimes p_C^* F')) \cong \mathcal{O}_{D_1}(-k)$. Using this and equation (6.7) in (6.4), we get $\det(p_{2*}(\widetilde{K}_1^\vee \otimes \mathcal{G}_1)) = \mathcal{O}_{D_1}(r-2k)$. ■

Lemma 6.9. *The degree of* $\det(p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F}))$ *restricted to* D_3 *is* $r(d+(r-k)(1-g))$.

Proof. Recall from (5.5) the family parameterized by D_3. It follows that

$$p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F})|_{D_3} = p_{2*}\big(p_C^* E^\vee \otimes p_2^*\mathcal{O}_{D_3}(1) \otimes \mathcal{G}_3\big).$$

Note that as $E^\vee$ is a stable bundle of degree $d \gg 0$, we have $H^1(C, E^\vee) = 0$. Tensoring (5.5) with $p_C^* E^\vee \otimes p_2^*\mathcal{O}_{D_3}(1)$ and applying p_{2*} yields the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(E,E) \otimes \mathcal{O}_{D_3} \to \big[H^0(C,E^\vee) \otimes \mathcal{O}_{D_3}(1)\big]^{\oplus r} \\ \to p_{2*}\big(p_C^* E^\vee \otimes p_2^*\mathcal{O}_{D_3}(1) \otimes \mathcal{G}_3\big) \to \mathrm{Ext}^1(E,E) \otimes \mathcal{O}_{D_3} \to 0. \end{aligned}$$

It follows that

$$\begin{aligned} \det\big(p_{2*}\big(p_C^* E^\vee \otimes p_2^*\mathcal{O}_{D_3}(1) \otimes \mathcal{G}_3\big)\big) \cong \mathcal{O}_{D_3}\big(rh^0(C,E^\vee)\big) = \mathcal{O}_{D_3}\big(r\chi(E^\vee)\big) \\ = \mathcal{O}_{D_3}\big(r\big(d+(r-k)(1-g)\big)\big). \end{aligned}$$

This completes the proof of the lemma. ■

We have the following theorem, which is part (C) of Theorem 1.1.

Theorem 6.10. *Assume one of the following two holds:*

- $g(C) \geqslant 3$ *and* $2 \leqslant k \leqslant r-2$, *or*
- $g(C) = 2$ *and* $3 \leqslant k \leqslant r-2$.

Let $d \gg 0$. *Let* $\omega_{\mathcal{Q}_L}$ *denote the canonical divisor of* $\mathcal{Q}_L$. *Then*

$$\omega_{\mathcal{Q}_L} = \big[d(r-2k-2) + r(g-1)\big]\alpha + (2k-r)\beta_{d+g-1}.$$

In particular, $\mathcal{Q}_L$ *is Fano iff* $r = 2k+1$.

Proof. Let us write $\omega_{\mathcal{Q}_L} = a\alpha + b\beta_{d+g-1}$. Recall $\omega_{\mathcal{Q}_L} = \det(p_{2*}(\mathcal{K}^\vee \otimes \mathcal{F}))^\vee$. It follows from Lemmas 6.3 and 6.9 that

$$\begin{aligned}[\omega_{\mathcal{Q}_L}] \cdot [D_1] &= 2k - r,\\ [\omega_{\mathcal{Q}_L}] \cdot [D_3] &= -r\big(d + (r-k)(1-g)\big).\end{aligned}$$

The proof of Proposition 4.4 shows that $[\alpha] \cdot [D_1] = 0$. Using Lemmas 4.12 and 5.6, we get the following two equations in a and b:

$$\begin{aligned} b &= 2k - r,\\ a(r-k) + bd(r-k-1) &= -r\big(d + (r-k)(1-g)\big).\end{aligned}$$

One easily computes that $a = d(r-2k-2) + r(g-1)$. Thus,

$$\omega_{\mathcal{Q}_L} = \big[d(r-2k-2) + r(g-1)\big]\alpha + (2k-r)\beta_{d+g-1}.$$

For $\mathcal{Q}_L$ to be Fano, we need that $d(r-2k-2) + r(g-1) < 0$ and $2k - r < 0$. Since $d \gg 0$, this happens iff $r - 2k - 2 < 0$ and $2k - r < 0$, that is, iff $2k < r < 2k+2$, that is, iff $r = 2k+1$. ■

This completes the proof of Theorem 1.1.

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References

[1] A. Bertram, G. Daskalopoulos, and R. Wentworth, Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians. *J. Amer. Math. Soc.* **9** (1996), no. 2, 529–571 Zbl 0865.14017 MR 1320154

[2] E. Bifet, F. Ghione, and M. Letizia, On the Abel-Jacobi map for divisors of higher rank on a curve. *Math. Ann.* **299** (1994), no. 4, 641–672 Zbl 0840.14003 MR 1286890

[3] D. Cheong, I. Choe, and G. H. Hitching, Isotropic Quot schemes of orthogonal bundles over a curve. *Internat. J. Math.* **32** (2021), no. 8, article no. 2150047 Zbl 1475.14059 MR 4300437

[4] D. Cheong, I. Choe, and G. H. Hitching, Irreducibility of Lagrangian Quot schemes over an algebraic curve. *Math. Z.* **300** (2022), no. 2, 1265–1289 Zbl 1494.14028 MR 4363776

[5] C. Gangopadhyay and R. Sebastian, Picard groups of some Quot schemes. *Int. Math. Res. Not. IMRN* **2024** (2024), no. 11, 9194–9217 Zbl 1562.14008 MR 4756109

[6] T. Goller, A weighted topological quantum field theory for Quot schemes on curves. *Math. Z.* **293** (2019), no. 3-4, 1085–1120 Zbl 1439.14114 MR 4024579

[7] R. Hartshorne, *Algebraic geometry*. Grad. Texts in Math. 52, Springer, New York, 1977 Zbl 0367.14001 MR 0463157

[8] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*. 2nd edn., Cambridge Math. Lib., Cambridge University Press, Cambridge, 2010 Zbl 1206.14027 MR 2665168

[9] A. Ito, On birational geometry of the space of parametrized rational curves in Grassmannians. *Trans. Amer. Math. Soc.* **369** (2017), no. 9, 6279–6301 Zbl 1368.14011 MR 3660221

[10] S.-Y. Jow, The effective cone of the space of parametrized rational curves in a Grassmannian. *Math. Z.* **272** (2012), no. 3-4, 947–960 Zbl 1264.14012 MR 2995148

[11] M. Popa and M. Roth, Stable maps and Quot schemes. *Invent. Math.* **152** (2003), no. 3, 625–663 Zbl 1024.14015 MR 1988296

[12] P. Rasul, Irreducibility of some Quot schemes on nodal curves. 2024, arXiv:2401.10528v2

[13] P. Rasul and R. Sebastian, Irreducibility and singularities of some nested Quot schemes. *J. Pure Appl. Algebra* **230** (2026), no. 4, article no. 108237

[14] S. A. Strømme, On parametrized rational curves in Grassmann varieties. In *Space curves (Rocca di Papa, 1985)*, pp. 251–272, Lecture Notes in Math. 1266, Springer, Berlin, 1987 Zbl 0625.14027 MR 0908717

[15] K. Venkatram, *Birational geometry of the space of rational curves in homogeneous varieties*. Ph.D. thesis, Massachusetts Institute of Technology, 2011 MR 2992651

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