Doc. Math. (Online first)
DOI 10.4171/DM/1062



# The Zappa–Szép product of twisted groupoids

Anna Duwenig and Boyu Li

**Abstract.** We define and study the external and the internal Zappa–Szép product of twists over groupoids. We determine when a pair $(\Sigma_1, \Sigma_2)$ of twists over a matched pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids gives rise to a *Zappa–Szép twist* $\Sigma$ over the Zappa–Szép product $\mathcal{G}_1 \bowtie \mathcal{G}_2$. We prove that the resulting (reduced and full) twisted groupoid C*-algebra of the Zappa–Szép twist $\Sigma \to \mathcal{G}_1 \bowtie \mathcal{G}_2$ is a C*-blend of its subalgebras corresponding to the subtwists $\Sigma_i \to \mathcal{G}_i$. Using Kumjian–Renault theory, we then prove a converse: Any C*-blend in which the intersection of the three algebras is a Cartan subalgebra in all of them, arises as the reduced twisted groupoid C*-algebras from such a Zappa–Szép twist $\Sigma \to \mathcal{G}_1 \bowtie \mathcal{G}_2$ of two twists $\Sigma_1 \to \mathcal{G}_1$ and $\Sigma_2 \to \mathcal{G}_2$.

## 1. Introduction

The Zappa–Szép product originated from the study of composition and decomposition of groups [36]. A group $K$ is called an *internal Zappa–Szép product* of its two subgroups $G$ and $H$ if

$$(1)\ K = G \cdot H, \quad \text{and} \quad (2)\ G \cap H \text{ is the trivial group } \{e\}.$$

In this case, every element in $K$ can be written uniquely as a product $gh$ for some $g \in G$ and $h \in H$. In particular, if we take any pair $(h', g') \in H \times G$, then there must exist a unique pair $(g, h) \in G \times H$ such that $h'g' = gh$ in $K$. This induces a right group action by $G$ on the space $H$ by $(h', g') \mapsto h' \triangleleft g' := h$ and a left group action of $H$ on the space $G$ by $(h', g') \mapsto h' \triangleright g' := g$. Since the group multiplication of $K$ is associative, these two actions of its subgroups $G$ and $H$ on each other interact with one-another 'in a compatible way'. For example, if $h \in H$ acts on a product $g_1 g_2$ of elements of $G$, then

$$h \triangleright (g_1 g_2) = (h \triangleright g_1)\big([h \triangleleft g_1] \triangleright g_2\big), \tag{1.1}$$

and likewise for the action $(h_1 h_2) \triangleleft g$ of $g \in G$ on a product $h_1 h_2 \in H$. Conversely, any pair of such compatible actions of two groups $G$ and $H$ on one-another, allows us to equip the Cartesian product $G \times H$ with a group multiplication that encodes these actions; this group is denoted $G \bowtie H$ and is called the *external Zappa–Szép product*. A semi-direct product $G \rtimes H$ is a special case of a Zappa–Szép product: in this instance, the subgroup $G$ is normal in the product, it acts trivially on the subgroup $H$, and the action of $H$ on $G$ is by group homomorphisms.

*Mathematics Subject Classification 2020:* 46L05 (primary); 22A22, 46L55 (secondary).
*Keywords:* Zappa–Szép product, twisted groupoid, Cartan subalgebra, C*-blends.

We are interested in extending the notion of Zappa–Szép products to operator algebras. There is a long history in the study of operator algebras of encoding group actions and their dynamics as linear operators on Hilbert spaces. For example, to the continuous action of a topological group $G$ on a space $X$, one can associate a crossed product C*-algebra $C_0(X) \rtimes G$ which can be understood as a non-commutative replacement of the quotient space $X/G$ which, as a topological space, might be rather ill-behaved. Properties of the group dynamics are often reflected in the properties of the corresponding operator algebra and vice versa. Zappa–Szép products encode two-way actions between two structures, and they have been studied in various algebraic contexts [1, 3, 7, 8, 16, 19–23, 27, 28, 35]. Our recent study of the Zappa–Szép product of Fell bundles by a groupoid [12] has resulted in vastly general imprimitivity theorems [14] that extend many classical results arising from group dynamics.

This paper continues our quest to study Zappa–Szép products in C*-algebras. Our main goal is to define the Zappa–Szép product of twisted groupoids. Such groupoids arise naturally in the study of C*-algebras. For instance, Renault [33] showed that any C*-algebra with a Cartan subalgebra can be realized as a twisted groupoid C*-algebra, and recent work of X. Li [24] showed that every simple classifiable C*-algebra in Elliot's classification program has such a Cartan subalgebra and hence a twisted groupoid model.

Let us now explain the content of this paper in more detail. Suppose we are given two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ that share the same unit space $\mathcal{U}$, and suppose these groupoids act on one another in a 'compatible way' akin to the compatibility assumption at (1.1) in the group case; to be precise, they satisfy conditions (ZS1)–(ZS9) in our preliminary Section 2.2. The pair $(\mathcal{G}_1, \mathcal{G}_2)$ is called *a matched pair* in this situation. Then it is well known that their fibred product

$$\mathcal{G}_1 \,{}_s\!\times_r \mathcal{G}_2 = \big\{(x, g) \in \mathcal{G}_1 \times \mathcal{G}_2 : s(x) = r(g)\big\}$$

can be given the structure of a groupoid whose multiplication encodes both actions and with respect to which the canonical maps $\mathcal{G}_i \to \mathcal{G}_1 \,{}_s\!\times_r \mathcal{G}_2$ $(i = 1, 2)$ are injective groupoid homomorphisms. The resulting groupoid is called the *Zappa–Szép product* of $\mathcal{G}_1$ and $\mathcal{G}_2$ and is denoted by $\mathcal{G}_1 \bowtie \mathcal{G}_2$. The conditions that need to be satisfied for this to work boil down to the existence of a certain type of homeomorphism

$$\mathcal{G}_2 \,{}_s\!\times_r \mathcal{G}_1 \to \mathcal{G}_1 \,{}_s\!\times_r \mathcal{G}_2, \quad (g, x) \mapsto (g \triangleright x, g \triangleleft x). \tag{1.2}$$

The question we are concerned with in Section 3 is the following. Suppose $\Sigma_1$ and $\Sigma_2$ are twists over $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively; that is, we have central groupoid extensions

$$\mathbb{T} \times \mathcal{U} \xrightarrow{j_i} \Sigma_i \xrightarrow{\pi_i} \mathcal{G}_i,$$

which, by abuse of notation, we often simply write as $\Sigma_i \to \mathcal{G}_i$. Under what conditions can one build a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$ out of them?

Clearly, the space $\Sigma_1 \,{}_s\!\times_r \Sigma_2$ will not be the right candidate: since $\Sigma_i$ is locally homeomorphic to $\mathbb{T} \times \mathcal{G}_i$, the fibred product $\Sigma_1 \,{}_s\!\times_r \Sigma_2$ is locally homeomorphic to

$(\mathbb{T} \times \mathcal{G}_1)\ {}_s\times_r (\mathbb{T} \times \mathcal{G}_2) = \mathbb{T}^2 \times (\mathcal{G}_1 \bowtie \mathcal{G}_2)$ instead of $\mathbb{T} \times (\mathcal{G}_1 \bowtie \mathcal{G}_2)$. This crude argument already shows that one has to quotient out a copy of $\mathbb{T}$ of the fibred product of $\Sigma_1$ and $\Sigma_2$ in order to have hopes of obtaining a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$. So let us denote the quotient of $\Sigma_1\ {}_s\times_r \Sigma_2$ by the canonical diagonal $\mathbb{T}$-action by $\Sigma_1 *_{\mathbb{T}} \Sigma_2$.

To be able to give $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ the structure of a groupoid, we must assume the existence of a so-called *factorization rule*: a certain type of homeomorphism

$$\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2.$$

As is visible from (1.2), $\Phi$ is, in its essence, a replacement for two-way actions of $\Sigma_1$ and $\Sigma_2$ on each other. In the presence of such a factorization rule, the space $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ can be given the structure of a topological groupoid which depends on $\Phi$ (Proposition 3.14); we denote the groupoid by $\Sigma_1 \bowtie_\Phi \Sigma_2$. It is a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$ (Theorem 3.18) and we hence call it the *external Zappa–Szép twist*. We prove that, in the case that one of the two twists is trivial, then $\Sigma_1 \bowtie_\Phi \Sigma_2$ is canonically isomorphic to the Zappa–Szép product of a Fell line bundle by a groupoid as constructed in [12] (Example 3.21), which shows that our construction behaves as expected with regard to known constructions in the literature.

In Section 4, we introduce a notion of *internal* Zappa–Szép product of twists. Suppose we are given a twist $\Sigma \to \mathcal{G}$, and let $\Sigma_1, \Sigma_2$ be subgroupoids of $\Sigma$. The obvious twist analogue of Condition (1) in the setting of group Zappa–Szép products, is the condition that $\Sigma = \Sigma_1 \cdot \Sigma_2$. If we understand Condition (2) as the intersection of $G$ and $H$ being as small as possible, then in the case of twists, it translates to $\Sigma_1 \cap \Sigma_2 = \mathbb{T} \times \mathcal{U}$. Given both of these assumptions and that the subgroupoids $\Sigma_1$ and $\Sigma_2$ are closed, then we say that $\Sigma$ is an *internal Zappa–Szép product* of $\Sigma_1$ and $\Sigma_2$. It turns out that every such internal Zappa–Szép twist $\Sigma$ gives rise to a factorization rule $\Phi$, meaning that it is also an external Zappa–Szép twist. Moreover, the converse is also true: every external Zappa–Szép twist is an internal Zappa–Szép twist (Theorem 4.3).

The Zappa–Szép product of twisted groupoids occurs naturally in many contexts. In Section 5, we study the setting of twists induced by continuous, normalized, $\mathbb{T}$-valued 2-cocycles. Any such map $c$ on a groupoid $\mathcal{G}$ gives rise to a twist $\Sigma_c$ over $\mathcal{G}$ in a canonical way. Given 2-cocycles $c_1, c_2$ on the matched pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids, the pair $(\Sigma_{c_1}, \Sigma_{c_2})$ of twists allows a factorization rule if and only if the pair $(c_1, c_2)$ of 2-cocycles allows what we call *cocycle connector*, and in this situation, the external Zappa–Szép twist is also induced by a 2-cocycle (Proposition 5.5). We then briefly study the internal Zappa–Szép product of 2-cocycles (Proposition 5.8).

In our final Section 6, we move our results into the realm of C*-algebras. We first show that the (reduced or full) groupoid C*-algebra of the Zappa–Szép groupoid $\mathcal{G}_1 \bowtie \mathcal{G}_2$ twisted by the Zappa–Szép twist $\Sigma_1 \bowtie_\Phi \Sigma_2$, is a C*-blend of the twisted C*-algebras of its components $\Sigma_1 \to \mathcal{G}_1$ and $\Sigma_2 \to \mathcal{G}_2$ (Theorem 6.6). This result, applied to effective groupoids, then motivates the following converse: suppose $A$ is a C*-algebra that has two subalgebras $A_1$, $A_2$ such that

(a) $A_1 \cdot A_2$ is dense in $A$, and (b) $A_1 \cap A_2$ is a Cartan subalgebra in $A_1$, $A_2$, and $A$.

Kumjian–Renault theory then tells us that each of the three algebras $A$, $A_j$ ($j = 1, 2$) corresponds to a twisted groupoid $\Sigma \to \mathcal{G}$ and $\Sigma_j \to \mathcal{G}_j$, respectively, with $\mathcal{G}, \mathcal{G}_j$ effective. We prove that $\Sigma \to \mathcal{G}$ is precisely an internal Zappa–Szép product of $\Sigma_1 \to \mathcal{G}_1$ and $\Sigma_2 \to \mathcal{G}_2$ (Theorem 6.14). We would like to point out that Conditions (a) and (b) closely resemble Conditions (1) and (2), respectively, in an internal Zappa–Szép product. Therefore, this can be viewed as an intrinsic notion of internal Zappa–Szép product of C*-algebras.

Our work extends the Zappa–Szép product construction to the much broader twisted groupoid context, which is applicable to a wider range of C*-algebras. Our notion of factorization rule brings a key insight that allows us to define the external Zappa–Szép product without using actions. This is a crucial step towards our ongoing research on the Zappa–Szép product of Fell bundles.

## 2. Preliminaries

### 2.1. Groupoid and twists

Throughout this paper, we assume that our groupoids, denoted by letters such as $\mathcal{G}$, are locally compact and Hausdorff. We often denote its unit space $\mathcal{G}^{(0)}$ by $\mathcal{U}$. A groupoid is called *r-discrete* if the unit space $\mathcal{U}$ is open in $\mathcal{G}$. The range and source maps of $\mathcal{G}$ are denoted by $r, s \colon \mathcal{G} \to \mathcal{U}$, respectively, and $\mathcal{G}$ is called *étale* if these maps are local homeomorphisms.

Given $u \in \mathcal{U}$, denote $\mathcal{G}u = \{g \in \mathcal{G} : s(g) = u\}$ and $u\mathcal{G} = \{g \in \mathcal{G} : r(g) = u\}$. The isotropy group at $u$ is defined by $\mathcal{G}^u_u = u\mathcal{G} \cap \mathcal{G}u$, and the isotropy bundle by $\mathrm{Iso}(G) = \bigcup_{u\in\mathcal{U}} \mathcal{G}^u_u$. A groupoid is called *principal* if $\mathrm{Iso}(G) = \mathcal{U}$, and *effective* if the interior of $\mathrm{Iso}(G)$ is $\mathcal{U}$.

Twists over groupoids were first introduced in [17]. Here, we will give the definition as stated in [9, Definition 2.1].

**Definition 2.1.** Let $\mathcal{G}$ be a (locally compact Hausdorff) groupoid with unit space $\mathcal{U}$, and regard $\mathbb{T} \times \mathcal{U}$ as a trivial group bundle with fibres $\mathbb{T}$. A *twist* $(\Sigma, \jmath, \pi)$ over $\mathcal{G}$ consists of a locally compact Hausdorff groupoid $\Sigma$ and groupoid homomorphisms $\jmath, \pi$ such that

$$\mathbb{T} \times \mathcal{U} \xrightarrow{\jmath} \Sigma \xrightarrow{\pi} \mathcal{G}$$

is a central groupoid extension, which means that

(T1) $\jmath \colon \mathbb{T} \times \mathcal{U} \to \pi^{-1}(\mathcal{U})$ is a homeomorphism, where $\pi^{-1}(\mathcal{U})$ has the subspace topology from $\Sigma$, and the map $\jmath$ satisfies $\jmath(1, \pi(u)) = u$ for all $u \in \Sigma^{(0)}$;

(T2) $\pi$ is a continuous, open surjection; and

(T3) $z \cdot e := \jmath(z, \pi(r(e)))e$ equals $e\jmath(z, \pi(s(e)))$ for all $e \in \Sigma$ and $z \in \mathbb{T}$.

As explained in [9, p. 5], it follows that $\Sigma$ is locally trivial, that $\pi$ is proper, and that we can identify $\Sigma^{(0)}$ with $\mathcal{U}$ via $\pi$.

### 2.2. Zappa–Szép products of groupoids

The Zappa–Szép product of groupoids was first introduced in [2] and its application in groupoid C*-algebras was further explored in [7]. Here, we will recall its construction.

Given two groupoids $\mathcal{G}_1, \mathcal{G}_2$ with the same unit space $\mathcal{G}_1^{(0)} = \mathcal{G}_2^{(0)} = \mathcal{U}$, define

$$\mathcal{G}_2 \ {}_s\!\times_r \mathcal{G}_1 = \big\{(g,x) \in \mathcal{G}_2 \times \mathcal{G}_1 : s(g) = r(x)\big\}.$$

We say that $(\mathcal{G}_1, \mathcal{G}_2)$ is a *matched pair of groupoids* if there are two continuous maps

$$\begin{aligned} \text{\textvisiblespace} \triangleright \text{\textvisiblespace} \colon \mathcal{G}_2 \ {}_s\!\times_r \mathcal{G}_1 &\to \mathcal{G}_1, \quad (g,x) \mapsto g \triangleright x, \\ \text{\textvisiblespace} \triangleleft \text{\textvisiblespace} \colon \mathcal{G}_2 \ {}_s\!\times_r \mathcal{G}_1 &\to \mathcal{G}_2, \quad (g,x) \mapsto g \triangleleft x, \end{aligned}$$

satisfying the following properties (ZS1)–(ZS9),[1] where $(h,g) \in \mathcal{G}_2^{(2)}$ and $(x,y) \in \mathcal{G}_1^{(2)}$ are such that $s(g) = r(x)$.

(ZS1) $(hg) \triangleright x = h \triangleright (g \triangleright x)$.

(ZS2) $r(g \triangleright x) = r(g)$.

(ZS3) $r(x) \triangleright x = x$.

(ZS4) $g \triangleleft (xy) = (g \triangleleft x) \triangleleft y$.

(ZS5) $s(g \triangleleft x) = s(x)$.

(ZS6) $g \triangleleft s(g) = g$.

(ZS7) $s(g \triangleright x) = r(g \triangleleft x)$.

(ZS8) $g \triangleright (xy) = (g \triangleright x)([g \triangleleft x] \triangleright y)$.

(ZS9) $(hg) \triangleleft x = (h \triangleleft [g \triangleright x])(g \triangleleft x)$.

Properties (ZS1)–(ZS3) mean that $\triangleright$ is a left $\mathcal{G}_2$ action on the space $\mathcal{G}_1$ with momentum map $r\colon \mathcal{G}_1 \to \mathcal{U} = \mathcal{G}_2^{(0)}$; properties (ZS4)–(ZS6) mean that $\triangleleft$ is a right $\mathcal{G}_1$ action on the space $\mathcal{G}_2$ with momentum map $s\colon \mathcal{G}_2 \to \mathcal{U} = \mathcal{G}_1^{(0)}$; and the remaining properties (ZS7)–(ZS9) are needed to turn $\mathcal{G}_1 \ {}_s\!\times_r \mathcal{G}_2$ into a groupoid, explained in more detail below. By [7, Lemma 4], the following additional properties hold:

(ZS10) $g \triangleright s(g) = r(g)$.

(ZS11) $r(x) \triangleleft x = s(x)$.

(ZS12) $(g \triangleright x)^{-1} = (g \triangleleft x) \triangleright x^{-1}$.

(ZS13) $(g \triangleleft x)^{-1} = g^{-1} \triangleleft (g \triangleright x)$.

In [7], the maps $\triangleright$ and $\triangleleft$ are called the *action map* and the *restriction map*, respectively. We have adopted the notation $g \triangleright x$ and $g \triangleleft x$ from [2] in place of writing $g \cdot x$ and $g|_x$, respectively, to avoid confusion in many computations.

[1] We point out that we are following the numbering in [12], which does not agree with that in [7].

With these assumptions, one can put a groupoid structure on the fibred product space $\mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$ by defining the multiplication

$$(y,g)(x,h) = \big(y(g \triangleright x), (g \triangleleft x)h\big), \quad \text{if } s(g) = r(x), \tag{2.1}$$

and inverse

$$(y,g)^{-1} = (g^{-1} \triangleright y^{-1}, g^{-1} \triangleleft y^{-1}).$$

This is called the *Zappa–Szép product groupoid*, which is denoted by $\mathcal{G}_1 \bowtie \mathcal{G}_2$.

Recall from [7, Corollary 8] that the pair $(\mathcal{G}_2, \mathcal{G}_1)$ is also matched and that the map

$$\Psi\colon \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 \to \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2, \quad (g,x) \mapsto (g \triangleright x, g \triangleleft x) \tag{2.2}$$

is a groupoid isomorphism $\mathcal{G}_2 \bowtie \mathcal{G}_1 \cong \mathcal{G}_1 \bowtie \mathcal{G}_2$. The map $\Psi$ is called a factorization rule in [25, Definition 3.3] where such maps are used to define the Zappa–Szép product of matched categories since one can phrase conditions (ZS1)–(ZS9) entirely in terms of this map (Lemma 2.2 below). While this rephrasing is a bit unwieldy, it serves as our motivation for Definition 3.5 later. In the following, we let $\bullet$ denote the left-action of $\mathcal{G}_1$ (resp. right-action of $\mathcal{G}_2$) on the space $\mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$ given by translation, and we let $m_i$ denote the multiplication map of the groupoid $\mathcal{G}_i$.

**Lemma 2.2** (cf. [25, Lemma 3.4]). *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two groupoids with the same unit space $\mathcal{U}$. Then $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair of groupoids if and only if there exists a homeomorphism*

$$\Psi\colon \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 \to \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$$

*satisfying the following properties, where $(g,x) \in \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1$ is arbitrary, where we write the element $\Psi(g,x)$ as $(y,h)$, and where the letters "FG" stand for "factorization of groupoids."*

(FG1) *$r(g) = r(y)$ and $s(x) = s(h)$.*

(FG2) *$\Psi(r(x),x) = (x, s(x))$ and $\Psi(g, s(g)) = (r(g), g)$.*

(FG3) *If $x' \in s(x)\mathcal{G}_1$, then $\Psi(g, xx') = y \bullet \Psi(h, x')$. In other words, the diagram*

$$\mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1^{(2)} \xrightarrow{\mathrm{id}\times m_1} \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 \xrightarrow{\Psi} \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$$
$$\mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1^{(2)} \xrightarrow{\Psi\times\mathrm{id}} \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 \xrightarrow{\mathrm{id}\times\Psi} \mathcal{G}_1^{(2)}\ {}_s\!\times_r \mathcal{G}_2 \xrightarrow{m_1\times\mathrm{id}} \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$$

*commutes.*

(FG4) *If $g' \in \mathcal{G}_2 r(g)$, then $\Psi(g'g, x) = \Psi(g', y) \bullet h$. In other words, the diagram*

$$\mathcal{G}_2^{(2)}\ {}_s\!\times_r \mathcal{G}_1 \xrightarrow{m_2\times\mathrm{id}} \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 \xrightarrow{\Psi} \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$$
$$\mathcal{G}_2^{(2)}\ {}_s\!\times_r \mathcal{G}_1 \xrightarrow{\mathrm{id}\times\Psi} \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2 \xrightarrow{\Psi\times\mathrm{id}} \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2^{(2)} \xrightarrow{\mathrm{id}\times m_2} \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$$

*commutes.*

*In this setting,* $\Psi$ *is called a* factorization rule*, and the relationship between the groupoid multiplication of* $\mathcal{G}_1 \bowtie \mathcal{G}_2$ *and the map* $\Psi$ *is determined by*

$$(y, g)(x, h) = y \bullet \Psi(g, x) \bullet h. \tag{2.3}$$

*In other words, if* $m$ *denotes the multiplication map of* $\mathcal{G}_1 \bowtie \mathcal{G}_2$*, then the diagram*

$$\begin{array}{ccc} (\mathcal{G}_1 \bowtie \mathcal{G}_2)^{(2)} & \xrightarrow{\quad m \quad} & \mathcal{G}_1 \bowtie \mathcal{G}_2 \\ \big\downarrow{\scriptstyle \mathrm{id}\times\Psi\times\mathrm{id}} & & \big\uparrow{\scriptstyle m_1 \times m_2} \\ & (\mathcal{G}_1 \; {}_s{\times}_r \; \mathcal{G}_1) \; {}_s{\times}_r \; (\mathcal{G}_2 \; {}_s{\times}_r \; \mathcal{G}_2) & \end{array}$$

*commutes.*

*Proof.* If $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair and we let $\Psi$ be the map in (2.2), then

- (ZS2) and (ZS5) imply (FG1);
- (ZS3) and (ZS11) imply the first equality in (FG2);
- (ZS6) and (ZS10) imply the second equality in (FG2);
- (ZS4) and (ZS8) imply (FG3); and
- (ZS1) and (ZS9) imply (FG4).

Conversely, assume we have a map $\Psi$ satisfying the factorization rules above. We then let $g \triangleright x \in \mathcal{G}_1$ and $g \triangleleft x \in \mathcal{G}_2$ be the unique elements such that $\Psi(g, x) = (g \triangleright x, g \triangleleft x)$. Since $\Psi$ is a homeomorphism, the maps $\triangleright$ and $\triangleleft$ are well defined and continuous, and the fact that $\mathcal{G}_1 \; {}_s{\times}_r \; \mathcal{G}_2$ is the codomain of $\Psi$ implies (ZS7). The other properties (ZS1)–(ZS6), (ZS8), (ZS9) follow as in the list above, just with reverse implication. Equation (2.3) now follows immediately from Equation (2.1). ∎

Using [7, Proposition 7] (see also [25, Proposition 3.9]), we can furthermore conclude:

**Lemma 2.3.** *If* $(\mathcal{G}_1, \mathcal{G}_2)$ *is a matched pair of groupoids, then for* $i \neq j$*, the multiplication maps* $\mathcal{G}_i \; {}_s{\times}_r \; \mathcal{G}_j \to \mathcal{G}_1 \bowtie \mathcal{G}_2$ *are bijections, and if* $(g, x) \in \mathcal{G}_2 \; {}_s{\times}_r \; \mathcal{G}_1$*, then*

$$gx = (g \triangleright x)(g \triangleleft x). \tag{2.4}$$

The groupoids $\mathcal{G}_i$ embed naturally inside their Zappa–Szép product $\mathcal{G}_1 \bowtie \mathcal{G}_2$:

**Lemma 2.4** (cf. [25, Lemma 3.5]). *Suppose* $(\mathcal{G}_1, \mathcal{G}_2)$ *is a matched pair of groupoids. The maps*

$$\begin{array}{rl} \mathcal{G}_1 \to & \mathcal{G}_1 \bowtie \mathcal{G}_2 \\ x \mapsto & \big(x, s(x)\big) \end{array} \quad \text{and} \quad \begin{array}{rl} \mathcal{G}_2 \to & \mathcal{G}_1 \bowtie \mathcal{G}_2 \\ g \mapsto & \big(r(g), g\big) \end{array} \tag{2.5}$$

*are injective, continuous, closed groupoid homomorphisms; in particular, their images are closed subgroupoids of* $\mathcal{G}_1 \bowtie \mathcal{G}_2$*. If one of* $\mathcal{G}_1$*,* $\mathcal{G}_2$ *is* $r$*-discrete, then the image of the other is open in* $\mathcal{G}_1 \bowtie \mathcal{G}_2$*. The Zappa–Szép product* $\mathcal{G}_1 \bowtie \mathcal{G}_2$ *is* $r$*-discrete (respectively étale) if and only if both* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are* $r$*-discrete (respectively étale), in which case both of their images are open in* $\mathcal{G}_1 \bowtie \mathcal{G}_2$*.*

*Proof.* It is clear that the maps are injective, continuous, and homomorphic. To see that the maps are closed, note that the image of $X \subseteq \mathcal{G}_1$ is exactly $X \ {}_s\!\times_r \mathcal{U}$ and that of $Y \subseteq \mathcal{G}_2$ is $\mathcal{U} \ {}_s\!\times_r Y$. Since $\mathcal{U}$ is closed in $\mathcal{G}_2$ and $\mathcal{G}_1$, it follows that the homomorphisms map closed sets to closed sets. In particular, the maps are homeomorphisms onto their images.

Let us prove that $\mathcal{G}_1$ is open if $\mathcal{G}_2$ is $r$-discrete; one proves the other case analogously. Assume that $\{k_\lambda\}$ is a net in $\mathcal{G}_1 \bowtie \mathcal{G}_2$ that converges to $(x, s(x))$. By continuity of the projection map $p_2: \mathcal{G}_1 \bowtie \mathcal{G}_2 \to \mathcal{G}_2$, this implies that $p_2(k_\lambda) \to s(x)$. Since $\mathcal{G}_2$ is $r$-discrete, there exists $\lambda_0$ such that $p_2(k_\lambda) \in \mathcal{G}_2^{(0)}$ for all $\lambda \geq \lambda_0$; in particular, $k_\lambda = (x_\lambda, s(x_\lambda))$ for some $x_\lambda \in \mathcal{G}_1$.

The claim about étaleness was already proven in [7, Proposition 9]. For $r$-discreteness, note that $(\mathcal{G}_1 \bowtie \mathcal{G}_2)^{(0)} = (\mathcal{U} \times \mathcal{U}) \cap \mathcal{G}_1 \bowtie \mathcal{G}_2$, so if both $\mathcal{G}_1$ and $\mathcal{G}_2$ are $r$-discrete, then so is their Zappa–Szép product. Conversely, suppose $\mathcal{G}_1 \bowtie \mathcal{G}_2$ is $r$-discrete. If $\{x_\lambda\}$ is a net in, say, $\mathcal{G}_1$ that converges to $u \in \mathcal{U}$, then $\{(x_\lambda, s(x_\lambda))\}$ converges to the unit $(u, s(u)) = (u, u)$ in $\mathcal{G}_1 \bowtie \mathcal{G}_2$. Since $(\mathcal{G}_1 \bowtie \mathcal{G}_2)^{(0)}$ is open, the net $\{(x_\lambda, s(x_\lambda))\}$ must then eventually be contained in it, meaning that $x_\lambda \in \mathcal{U}$, proving that $\mathcal{U}$ is open in $\mathcal{G}_1$. The proof for $\mathcal{G}_2$ is identical. ∎

We therefore may identify $\mathcal{G}_i$ with its image in $\mathcal{G}_1 \bowtie \mathcal{G}_2$, and usually, we simply write $x$ and $g$ instead of $(x, s(x))$ and $(r(g), g)$, respectively.

**Corollary 2.5.** *Suppose $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair of groupoids. If $\mathcal{G}_1 \bowtie \mathcal{G}_2$ is effective and $r$-discrete, then so are $\mathcal{G}_1$ and $\mathcal{G}_2$. If $\mathcal{G}_1 \bowtie \mathcal{G}_2$ is principal, then so are $\mathcal{G}_1$ and $\mathcal{G}_2$.*

*Proof.* Principality is inherited by any subgroupoid, and effectiveness by any *open* subgroupoid. ∎

Note that a converse of Corollary 2.5 is clearly false: if, say, both $\mathcal{G}_1$ and $\mathcal{G}_2$ are the full equivalence relation on a set $X$ with at least two distinct elements $x, y$ (so the groupoids are principal), then the Zappa–Szép product for the trivial actions, i.e., $\mathcal{G}_1 \ {}_s\!\times_r \mathcal{G}_2$, contains $((x, y), (y, x))$, which is a non-unit isotropy point.

## 3. The external Zappa–Szép product of two twists

For this section, we assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two groupoids with the same unit space $\mathcal{U}$ and that, over each of these groupoids, we have a twist

$$\mathbb{T} \times \mathcal{U} \xrightarrow{\jmath_i} \Sigma_i \xrightarrow{\pi_i} \mathcal{G}_i.$$

We identify $\Sigma_i^{(0)}$ with $\mathcal{U}$ via $\pi_i$.

We will prove that certain conditions on the pair $(\Sigma_1, \Sigma_2)$ of twists imply that $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair of groupoids as in Section 2.2 and that they allow us to define a new twist $\Sigma_1 \bowtie_\Phi \Sigma_2$ over the Zappa–Szép product groupoid $\mathcal{G}_1 \bowtie \mathcal{G}_2$. Notice that each $\Sigma_i$ carries a copy of $\mathbb{T}$, so their product $\Sigma_i \ {}_s\!\times_r \Sigma_j$ has one too many copies of $\mathbb{T}$ in order to be a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$. We thus first need to quotient out a copy of $\mathbb{T}$ from $\Sigma_i \ {}_s\!\times_r \Sigma_j$: We define a

$\mathbb{T}$-action on $\Sigma_i \,{}_s{\times}_r\, \Sigma_j$ by $z \cdot (e, f) := (z \cdot e, \bar{z} \cdot f)$, and we denote the resulting quotient space by $\Sigma_i *_{\mathbb{T}} \Sigma_j$ whose elements are written as $e \,{}_i{*}_j\, f$. Notice that the balancing from the $\mathbb{T}$-action means that $e \,{}_i{*}_j\, f = (ze) \,{}_i{*}_j\, (\bar{z} f)$.

A very convenient tool in the proofs to come is the following.

**Lemma 3.1.** *For $i, j \in \{1, 2\}$ with $i \neq j$, the quotient map $q_{i,j} \colon \Sigma_i \,{}_s{\times}_r\, \Sigma_j \to \Sigma_i *_{\mathbb{T}} \Sigma_j$ is open and the space $\Sigma_i *_{\mathbb{T}} \Sigma_j$ is locally compact Hausdorff with a $\mathbb{T}$-action given by $z \cdot (e \,{}_i{*}_j\, f) := (z \cdot e) \,{}_i{*}_j\, f$.*

*Proof.* A basic open set of $\Sigma_i \,{}_s{\times}_r\, \Sigma_j$ is of the form

$$U \,{}_s{\times}_r\, V = \big\{(e, f) \in \Sigma_i \,{}_s{\times}_r\, \Sigma_j : e \in U,\ f \in V\big\},$$

for open sets $U \subseteq \Sigma_i$, $V \subseteq \Sigma_j$. We have

$$\begin{aligned} q_{i,j}^{-1}\big(q_{i,j}(U \,{}_s{\times}_r\, V)\big) &= \big\{(z \cdot e, \bar{z} \cdot f) \in \Sigma_i \,{}_s{\times}_r\, \Sigma_j : e \in U,\ f \in V,\ z \in \mathbb{T}\big\} \\ &= \bigcup_{z \in \mathbb{T}} (z \cdot U) \,{}_s{\times}_r\, (\bar{z} \cdot V). \end{aligned}$$

Since $\mathbb{T}$ acts by homeomorphisms on $\Sigma_i$ and $\Sigma_j$, each of the sets $z \cdot U$ and $\bar{z} \cdot V$ is open, proving that $q_{i,j}^{-1}(q_{i,j}(U \,{}_s{\times}_r\, V))$ is the union of basic open sets and hence itself open. Since $\Sigma_i *_{\mathbb{T}} \Sigma_j$ carries the quotient space topology induced by the quotient map $q_{i,j}$, this proves that $q_{i,j}(U \,{}_s{\times}_r\, V)$ is open in $\Sigma_i *_{\mathbb{T}} \Sigma_j$. As $U, V$ were arbitrary, this proves that $q_{i,j}$ is open.

Since $\mathbb{T}$ is compact and $\Sigma_i \,{}_s{\times}_r\, \Sigma_j$ is locally compact Hausdorff, the quotient space is locally compact Hausdorff [26, Chapter 4, Section 31, Example 8].

Since $\mathbb{T}$-acts continuously on $\Sigma_1$ and since the quotient map is continuous and open, it is immediate that $\mathbb{T}$ acts continuously on $\Sigma_1 *_{\mathbb{T}} \Sigma_2$. ∎

Next, we show that the twists $\Sigma_1$ and $\Sigma_2$ that we started with naturally embed into the quotient space $\Sigma_1 *_{\mathbb{T}} \Sigma_2$, so that we can think of them as subspaces.

**Lemma 3.2.** *The maps*

$$\iota_{1,2}^{i} \colon \Sigma_i \to \Sigma_1 *_{\mathbb{T}} \Sigma_2 \quad \textit{and} \quad \iota_{2,1}^{i} \colon \Sigma_i \to \Sigma_2 *_{\mathbb{T}} \Sigma_1$$

*given by*

$$\begin{aligned} \iota_{1,2}^{1}(e) &= e \,{}_1{*}_2\, s(e), \qquad & \iota_{2,1}^{1}(e) &= r(e) \,{}_2{*}_1\, e, \\ \iota_{1,2}^{2}(f) &= r(f) \,{}_1{*}_2\, f, \qquad & \iota_{2,1}^{2}(f) &= f \,{}_2{*}_1\, s(f) \end{aligned}$$

*are embeddings whose images are closed.*

*Proof.* We will do the proof for $\iota_{1,2}^{1}$ only. If $\iota_{1,2}^{1}(e') = \iota_{1,2}^{1}(e)$, then there exists $z \in \mathbb{T}$ such that

$$\big(e', s(e')\big) = \big(z \cdot e, \bar{z} \cdot s(e)\big).$$

But $e' = z \cdot e$ implies $s(e') = s(z \cdot e) = s(e)$, so the fact that we also have $s(e') = \bar{z} \cdot s(e)$ implies $z = 1$ and hence $e' = e$, which proves that $\iota^1_{1,2}$ is injective. Continuity is obvious, seeing that the map is a concatenation of continuous maps.

To see that the image is closed, assume that a net $\{\iota^1_{1,2}(e_\lambda)\}$ converges in $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ to, say, $q_{1,2}(e, f) = e\,{}_1{*}_2\,f$. Since the quotient map $q_{1,2}$ is open (Lemma 3.1) and since we can without loss of generality pass to a subnet, there exists a net $\{(e'_\lambda, f_\lambda)\}$ in $\Sigma_1\,{}_s{\times}_r\,\Sigma_2$ such that $e'_\lambda\,{}_1{*}_2\,f_\lambda = \iota^1_{1,2}(e_\lambda)$, $e'_\lambda \to e$, and $f_\lambda \to f$. From $e'_\lambda\,{}_1{*}_2\,f_\lambda = \iota^1_{1,2}(e_\lambda)$ we conclude that there exists $z_\lambda \in \mathbb{T}$ such that $(z_\lambda \cdot e'_\lambda, \overline{z_\lambda} \cdot f_\lambda) = (e_\lambda, s(e_\lambda))$ in $\Sigma_1\,{}_s{\times}_r\,\Sigma_2$. Since $\mathbb{T}$ is compact and since we can pass to a subnet, we can without loss of generality assume that $\{z_\lambda\}$ converges to some $z \in \mathbb{T}$. Since $e'_\lambda \to e$, this means that

$$f_\lambda = z_\lambda \cdot s(e_\lambda) = z_\lambda \cdot s(z_\lambda \cdot e'_\lambda) \to z \cdot s(z \cdot e) = z \cdot s(e).$$

Since limits are unique and $f_\lambda \to f$, we conclude that $f = z \cdot s(e)$, and so

$$e\,{}_1{*}_2\,f = (z \cdot e)\,{}_1{*}_2\,(\bar{z} \cdot f) = (z \cdot e)\,{}_1{*}_2\,s(e) = \iota^1_{1,2}(z \cdot e) \in \iota^1_{1,2}(\Sigma_1).$$

We have shown that the limit $e\,{}_1{*}_2\,f$ of the net $\{\iota^1_{1,2}(e_\lambda)\}$ lies in $\iota^1_{1,2}(\Sigma_1)$, i.e., the image of $\iota^1_{1,2}$ is closed.

To see that $\iota^1_{1,2}$ is open onto its image, we can adjust the above proof: this time, the limit of $\{\iota^1_{1,2}(e_\lambda)\}$ is allowed to be chosen of the form $e\,{}_1{*}_2\,s(e)$, i.e., the above scalar $z$ can be set to 1, so that $e_\lambda = z_\lambda \cdot e'_\lambda \to z \cdot e = e$, proving that $\iota^1_{1,2}$ is open onto its image by Fell's criterion [37, Proposition 1.1]. ∎

If $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair, so that the set $\mathcal{G}_1\,{}_s{\times}_r\,\mathcal{G}_2$ can be regarded as the Zappa–Szép product groupoid $\mathcal{G}_1 \bowtie \mathcal{G}_2$, then in order for $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ to have any hopes of becoming a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$, we need there to be a projection map. That is the content of the following lemma.

**Lemma 3.3.** *The maps*

$$\begin{aligned}\Pi_{1,2}\colon \Sigma_1 *_{\mathbb{T}} \Sigma_2 &\to \mathcal{G}_1\,{}_s{\times}_r\,\mathcal{G}_2 \\ e\,{}_1{*}_2\,f &\mapsto \big(\pi_1(e), \pi_2(f)\big)\end{aligned} \quad \textit{and} \quad \begin{aligned}\Sigma_2 *_{\mathbb{T}} \Sigma_1 &\to \mathcal{G}_2\,{}_s{\times}_r\,\mathcal{G}_1 \\ f\,{}_2{*}_1\,e &\mapsto \big(\pi_2(f), \pi_1(e)\big),\end{aligned}$$

*are well-defined, continuous, open surjections.*

*Proof.* We will do the proof for $\Pi_{1,2}$; by symmetry, the claim for the other map will then follow.

Since $\pi_1, \pi_2$ are $\mathbb{T}$-invariant and since $s(e) = r(f)$, the map $\Pi_{1,2}$ is well defined. Surjectivity follows from surjectivity of $\pi_1$ and $\pi_2$.

Fell's criterion [37, Proposition 1.1] can be used to show that the map $\Pi_{1,2}$ is continuous, using that the quotient map is open by Lemma 3.1 and that the maps $\pi_1, \pi_2$ are continuous. Likewise, $\Pi_{1,2}$ is open since the quotient map is continuous and the maps $\pi_1, \pi_2$ are open. ∎

Note that each $\Sigma_i$ acts continuously on itself by multiplication on both the left and right. Consequently, $\Sigma_1$ acts continuously on the left of the space $\Sigma_1\,{}_s\!\times_r \Sigma_2$ and $\Sigma_2$ acts on the right. These actions factor through the quotient as follows.

**Lemma 3.4.** *The maps* $\rho, \sigma\colon \Sigma_1 *_{\mathbb{T}} \Sigma_2 \to \mathcal{U}$ *given by*

$$\rho(e\,{}_1{*}_2\,f) = r(e) \quad \textit{and} \quad \sigma(e\,{}_1{*}_2\,f) = s(f)$$

*are continuous surjections; if the groupoids* $\mathcal{G}_i$ *are étale, then these maps are open. On* $\Sigma_1 *_{\mathbb{T}} \Sigma_2$*, we define a left* $\Sigma_1$*-action with anchor map* $\rho$ *and a right* $\Sigma_2$*-action with anchor map* $\sigma$ *by*

$$e' \bullet (e\,{}_1{*}_2\,f) = (e'e)\,{}_1{*}_2\,f \quad \textit{respectively} \quad (e\,{}_1{*}_2\,f) \bullet f' = e\,{}_1{*}_2\,(ff').$$

*This makes* $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ *a left* $\Sigma_1$*-space and a right* $\Sigma_2$*-space.*

Similarly, $\Sigma_2 *_{\mathbb{T}} \Sigma_1$ becomes a left $\Sigma_2$-space and a right $\Sigma_1$-space.

*Proof.* We will do the proof for the left action; the proof for the right action is analogous. Since the $\mathbb{T}$-actions on $\Sigma_1, \Sigma_2$ leave range and source invariant, $\rho$ is well defined. It is surjective as $\rho(u\,{}_1{*}_2\,u) = u$ for any $u \in \mathcal{U}$. Since the quotient map

$$\Sigma_1\,{}_s\!\times_r \Sigma_2 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$$

is open and the map $\Sigma_1\,{}_s\!\times_r \Sigma_2 \to \mathcal{U}$, $(e, f) \mapsto r(e)$, is continuous, the map $\rho$ is likewise continuous. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are étale, then their range maps are open; since $\pi_1, \pi_2$ are open, an easy application Fell's criterion [37, Proposition 1.1] then proves that $\rho$ is open.

It is clear that $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ is algebraically a left $\Sigma_1$-space in the sense of [37, Definition 2.1]. The action is continuous because the action of $\Sigma_1$ on $\Sigma_1\,{}_s\!\times_r \Sigma_2$ is continuous and because the quotient map $\Sigma_1\,{}_s\!\times_r \Sigma_2 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ is open by Lemma 3.1. ■

We are now ready to make a key definition: if the pair $(\Sigma_1, \Sigma_2)$ of twists allows a *factorization rule* as in Definition 3.5, then the pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids is matched in the sense of Section 2.2 (Lemma 3.8). Moreover, we can equip $\Sigma_i *_{\mathbb{T}} \Sigma_j$ with the structure of a groupoid (Proposition 3.14) with respect to which it is a twist over the Zappa–Szép product groupoid $\mathcal{G}_1 \bowtie \mathcal{G}_2$ (Theorem 3.18).

**Definition 3.5.** A *factorization rule* for the pair $(\Sigma_1, \Sigma_2)$ of twists is a homeomorphism

$$\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$$

satisfying the following, where we write the arbitrary element $\Phi(f\,{}_2{*}_1\,e)$ as $e'\,{}_1{*}_2\,f'$, and where the letters "FT" stand for "**f**actorization of **t**wists."

(FT0) $\Phi$ is $\mathbb{T}$-equivariant: $\Phi((z \cdot f)\,{}_2{*}_1\,e) = (z \cdot e')\,{}_1{*}_2\,f'$ for all $z \in \mathbb{T}$.

(FT1) $r(f) = r(e')$ and $s(e) = s(f')$.

(FT2) For all $e \in \Sigma_1$ and $f \in \Sigma_2$, we have $\Phi(f\,{}_2{*}_1\,s(f)) = r(f)\,{}_1{*}_2\,f$ and $\Phi(r(e)\,{}_2{*}_1\,e) = e\,{}_1{*}_2\,s(e)$.

(FT3) For $\tilde{e} \in s(e)\Sigma_1$, we have $\Phi(f\,{}_2{*}_1\,(e\tilde{e})) = e' \bullet \Phi(f'\,{}_2{*}_1\,\tilde{e})$.

(FT4) For $\tilde{f} \in \Sigma_2 r(f)$, we have $\Phi((\tilde{f} f)\,{}_2{*}_1\,e) = \Phi(\tilde{f}\,{}_2{*}_1\,e') \bullet f'$.

If such a factorization rule exists, we will call $(\Sigma_1, \Sigma_2)$ a *matched pair of twists*.

**Remark 3.6.** (i) One can verify that, courtesy of Condition (FT0), Conditions (FT3) and (FT4) are well defined.

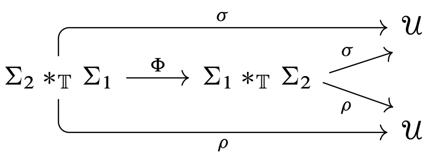
(ii) Condition (FT2) in particular implies for $u \in \mathcal{U} = \mathcal{G}_i^{(0)}$ that $\Phi(u\,{}_2{*}_1\,u) = u\,{}_1{*}_2\,u$.

(iii) In contrast to the factorization rule in [25, Definition 3.3], our map is defined on the quotient space $\Sigma_2 *_{\mathbb{T}} \Sigma_1$ instead of the fibred product $\Sigma_2\,{}_s{\times}_r\,\Sigma_1$.

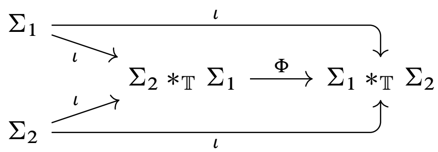
(iv) We will see that twist factorization rules are not unique; see Remark 5.6.

The following lemma is just a restatement of Definition 3.5. In the below diagrams, we write $M_i$ for the map $\Sigma_i *_{\mathbb{T}} \Sigma_i \to \Sigma_i$ induced by the multiplication of the groupoid $\Sigma_i$. We furthermore abuse notation by writing $\iota$ for each of the maps $\iota_{i,j}^k$ defined in Lemma 3.2, and by using the symbols $\rho$ and $\sigma$ from Lemma 3.4 also for the maps $\Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \mathcal{U}$ given by $\rho\colon f\,{}_2{*}_1\,e \mapsto r(f)$ and $\sigma\colon f\,{}_2{*}_1\,e \mapsto s(e)$.

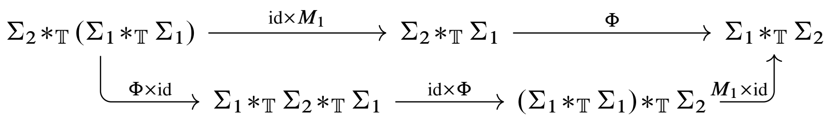
**Lemma 3.7.** *Suppose we are given a pair $(\Sigma_1, \Sigma_2)$ of twists and a homeomorphism $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$. Then $\Phi$ is a factorization rule if and only if it satisfies Condition (FT0) and the following diagrams commute:*



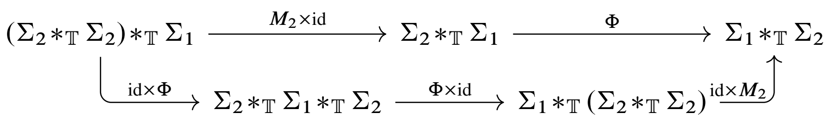

(FT1′)




(FT2′)




(FT3′)




(FT4′)

**Lemma 3.8.** *If $\Phi$ is a twist factorization rule for $(\Sigma_1, \Sigma_2)$, then there is a unique map $\Psi\colon \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 \to \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2$ that makes the diagram*

$$
\begin{array}{ccccc}
\Sigma_2 *_{\mathbb{T}} \Sigma_1 & & \xrightarrow{\quad\Phi\quad} & & \Sigma_1 *_{\mathbb{T}} \Sigma_2 \\
 & \ni & & \in & \\
\Big\downarrow & f\,{}_2{*}_1\, e & & e'\,{}_1{*}_2\, f' & \Big\downarrow \Pi_{1,2} \\
 & \downarrow & & \downarrow & \\
 & \big(\pi_2(f), \pi_1(e)\big) & & \big(\pi_1(e'), \pi_2(f')\big) & \\
\mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1 & & \overset{\Psi}{\dashrightarrow} & & \mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2
\end{array}
\tag{3.1}
$$

*commute. That is,*

$$\Psi\big(\pi_2(f), \pi_1(e)\big) = (\Pi_{1,2} \circ \Phi)(f\,{}_2{*}_1\, e). \tag{3.2}$$

*Moreover, $\Psi$ is a factorization rule for $(\mathcal{G}_1, \mathcal{G}_2)$ as in Lemma 2.2.*

*Proof.* First, we must check that Equation (3.2) determines a well-defined map $\Psi$. If $\tilde{f} \in \Sigma_2$ and $\tilde{e} \in \Sigma_1$ are such that $\pi_2(\tilde{f}) = \pi_2(f)$ and $\pi_1(\tilde{e}) = \pi_1(e)$, then there exist $z, w \in \mathbb{T}$ such that $\tilde{f} = z \cdot f$ and $\tilde{e} = w \cdot e$. By Condition (FT0), this means that

$$\Phi(\tilde{f}\,{}_2{*}_1\, \tilde{e}) = zw \cdot \Phi(f\,{}_2{*}_1\, e).$$

For any $a \in \Sigma_1, b \in \Sigma_2$, we have

$$\Pi_{1,2}\big((z \cdot a)\,{}_1{*}_2\,(w \cdot b)\big) = \big(\pi_1(z \cdot a), \pi_2(w \cdot b)\big) = \big(\pi_1(a), \pi_2(b)\big) = \Pi_{1,2}(a\,{}_1{*}_2\, b).$$

so that

$$\Pi_{1,2}\big(\Phi(\tilde{f}\,{}_2{*}_1\, \tilde{e})\big) = \Pi_{1,2}\big(\Phi(f\,{}_2{*}_1\, e)\big).$$

Thus, $\Psi$ only depends on $(\pi_2(f), \pi_1(e))$, not on $f$ and $e$.

Since $\Phi$ is a homeomorphism, we can exchange the roles of $\Sigma_1$ and $\Sigma_2$ and conclude that the above argument yields a map $\mathcal{G}_1\ {}_s\!\times_r \mathcal{G}_2 \to \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1$ that is clearly inverse to $\Psi$. Since all maps of Diagram (3.1) are open and continuous (see Lemma 3.3), we conclude that $\Psi$ is continuous and open (by an application of Fell's criterion [37, Proposition 1.1]).

Now fix $(\pi_2(f), \pi_1(e)) = (g, x) \in \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1$. If we let $e'\,{}_1{*}_2\, f' \in \Sigma_1 *_{\mathbb{T}} \Sigma_2$ be such that $\Phi(f\,{}_2{*}_1\, e) = e'\,{}_1{*}_2\, f'$, then $\Psi(g, x) = (\pi_1(e'), \pi_2(f'))$. Since

$$r(g) = r(f) \overset{\text{(FT1)}}{=} r(e') \quad \text{and} \quad s(x) = s(e) \overset{\text{(FT1)}}{=} s(f'),$$

Condition (FG1) holds. We have

$$\Psi\big(r(e), \pi_1(e)\big) = (\Pi_{1,2} \circ \Phi)\big(r(e)\,{}_2{*}_1\, e\big) \overset{\text{(FT2)}}{=} \Pi_{1,2}\big(e\,{}_1{*}_2\, s(e)\big) = \big(\pi_1(e), s(e)\big)$$

and

$$\Psi\big(\pi_2(f), s(f)\big) = (\Pi_{1,2} \circ \Phi)\big(f\,{}_2{*}_1\, s(f)\big) \overset{\text{(FT2)}}{=} \Pi_{1,2}\big(r(f)\,{}_1{*}_2\, f\big) = \big(r(f), \pi_2(f)\big),$$

so $\Psi$ satisfies Condition (FG2). If $\pi_1(\tilde{e}) \in s(e)\mathcal{G}_1$, then

$$\begin{aligned}\Psi\big(\pi_2(f), \pi_1(e\tilde{e})\big) &= (\Pi_{1,2} \circ \Phi)\big(f\ {}_2{*}_1\ (e\tilde{e})\big) \overset{\text{(FT3)}}{=} \Pi_{1,2}\big(e' \bullet \Phi(f'\ {}_2{*}_1\ \tilde{e})\big) \\ &= \pi_1(e') \bullet (\Pi_{1,2} \circ \Phi)(f'\ {}_2{*}_1\ \tilde{e}) = \pi_1(e') \bullet \Psi\big(\pi_2(f'), \pi_1(\tilde{e})\big),\end{aligned}$$

proving that Condition (FG3) holds. Likewise, Condition (FT4) implies Condition (FG4). Therefore, $\Psi$ is a factorization rule for $(\mathcal{G}_1, \mathcal{G}_2)$. ∎

**Remark 3.9.** Since $\Psi$ is a factorization rule, it follows from Lemma 2.2 that $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair and from Equation (2.2) that the two-way actions $\triangleright$ and $\triangleleft$ are encoded in the formula

$$\Psi(g, x) = (g \triangleright x, g \triangleleft x). \tag{3.3}$$

If we think of $\mathcal{G}_1 \bowtie \mathcal{G}_2$ as an internal Zappa–Szép product, then the map $\Pi_{1,2}$ from Lemma 3.3 can now be written as

$$\Pi_{1,2}\colon \Sigma_1 *_{\mathbb{T}} \Sigma_2 \to \mathcal{G}_1 \bowtie \mathcal{G}_2, \quad e\ {}_1{*}_2\ f \mapsto \pi_1(e)\pi_2(f),$$

and we furthermore get a second map

$$\Pi_{2,1}\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \mathcal{G}_1 \bowtie \mathcal{G}_2, \quad f\ {}_2{*}_1\ e \mapsto \pi_2(f)\pi_1(e).$$

Considering how $\Psi$ was defined (Equation (3.2)), we can thus rewrite Equation (3.3) as

$$(\Pi_{1,2} \circ \Phi)(f\ {}_2{*}_1\ e) = \big(\pi_2(f) \triangleright \pi_1(e)\big)\big(\pi_2(f) \triangleleft \pi_1(e)\big) = \Pi_{2,1}(f\ {}_2{*}_1\ e), \tag{3.4}$$

so that $\Pi_{2,1}$ is a continuous, open surjection.

Since multiplication in $\mathcal{G}_1 \bowtie \mathcal{G}_2$ entirely captures the factorization rule $\Psi$ (see Equation (2.3)), we can now deduce the following.

**Corollary 3.10.** *Suppose we are given a matched pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids with factorization rule $\Psi$, and a matched pair $(\Sigma_1, \Sigma_2)$ of twists with factorization rule $\Phi$. Then $\Phi$ induces $\Psi$ (via Lemma 3.8) if and only if $\Pi_{1,2} \circ \Phi = \Pi_{2,1}$.*

The above leads to the following definition.

**Definition 3.11.** Given a matched pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids and two twists $\Sigma_i \to \mathcal{G}_i$, we say that a twist factorization rule $\Phi$ *covers* $\mathcal{G}_1 \bowtie \mathcal{G}_2$ if $\Phi$ satisfies Equation (3.4); in other words, the given factorization rule for $(\mathcal{G}_1, \mathcal{G}_2)$ coincides with the rule constructed from $\Phi$ in Lemma 3.8.

Even though it is not defined as a two-way action as in the case of the external Zappa–Szép product of groupoids, a twist factorization rule $\Phi$ captures the essence of a Zappa–Szép product structure, in full analogy to how the map $\Psi$ defined at (2.2) and its properties entirely capture the data needed for the Zappa–Szép product $\mathcal{G}_1 \bowtie \mathcal{G}_2$ (see Lemma 2.2). The following example makes this more precise.

**Example 3.12.** Consider the trivial twists $\Sigma_i = \mathbb{T} \times \mathcal{G}_i$ and assume that $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair of groupoids. We want to find a factorization rule $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ for the pair $(\Sigma_1, \Sigma_2)$ that covers $\mathcal{G}_1 \bowtie \mathcal{G}_2$. Let us denote the given factorization rule of $\mathcal{G}_1 \bowtie \mathcal{G}_2$ by $\Psi$. Since $\Phi$ must satisfy (FT0) ($\mathbb{T}$-equivariance) and Equation (3.4) (see Corollary 3.10 and Remark 3.9), there is not that much choice: $\Phi$ must be of the form

$$\Phi\big((z,g)\,{}_2{*}_1\,(w,x)\big) = (w,y)\,{}_1{*}_2\,\big(z\varphi(g,x),h\big) \quad \text{where } (y,h) = \Psi(g,x) \tag{3.5}$$

for some map $\varphi\colon \mathcal{G}_2\,{}_s{\times}_r\,\mathcal{G}_1 \to \mathbb{T}$. We will quickly argue here that $\varphi \equiv 1$ does the trick. (In Section 5, we will study the more general but still special case where the twists $\Sigma_i$ are induced by 2-cocycles, and we will see in Proposition 5.5 that the choice $\varphi \equiv 1$ is not the only one.)

Recall that $\Psi$ satisfies (FG2), i.e., $\Psi(r(x),x) = (x, s(x))$ and $\Psi(g, s(g)) = (r(g), g)$ for all $x \in \mathcal{G}_1$ and $g \in \mathcal{G}_2$; this means exactly that $\Phi$ satisfies Condition (FT2). Clearly, Condition (FG3) of $\Psi$ corresponds to (FT3) of $\Phi$, and likewise (FG4) to (FT4).

**Example 3.13** (One Fell line bundle). Suppose $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair of groupoids with unit space $\mathcal{U}$ and that $\Sigma_1$ is a twist over $\mathcal{G}_1$; let $L_1 = \mathbb{C} \times_{\mathbb{T}} \Sigma_1$ be the associated line bundle whose elements we write as

$$[\lambda, e] := \big\{(z\lambda, \bar{z}\cdot e) : z \in \mathbb{T}\big\}. \tag{3.6}$$

Recall that $L_1$ is a Fell bundle over the groupoid $\mathcal{G}_1$ with projection map $q_1\colon [\lambda,e] \mapsto \pi(e)$; we let $r_{L_1} = r_{\mathcal{G}_1} \circ q_1$ and $p\colon L_1^{\times} \to \Sigma_1$ the map that sends $[\lambda, e]$ for $\lambda > 0$ to $e$ or, more generally, $p([\lambda,e]) = \mathrm{Ph}(\lambda)\cdot e$, where $\mathrm{Ph}(\lambda) := \lambda/|\lambda|$ is the phase of $\lambda \neq 0$.

If we equip $\mathcal{G}_2$ with the trivial twist $\Sigma_2 = \mathbb{T} \times \mathcal{G}_2$, then a factorization rule

$$\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$$

as in Definition 3.5 contains the same data as a $(\mathcal{G}_1, \mathcal{G}_2)$*-compatible* $\mathcal{G}_2$*-action* on $L_1$

$$\_\triangleright\_\colon \mathcal{G}_2\,{}_s{*}_\rho\,L_1 \to L_1, \quad \big(g,[\lambda,e]\big) \mapsto g \triangleright [\lambda, e],$$

in the sense of [12, Definition 3.1] (which also can be defined for non-étale groupoids; see [14, Definition 4.1]). Indeed, given $\Phi$, we let

$$g \triangleright [\lambda, e] := [\lambda, f] \quad \text{where } f \text{ is such that } f\,{}_1{*}_2\,\big(1, g \triangleleft \pi_1(e)\big) = \Phi\big((1,g)\,{}_2{*}_1\,e\big),$$

and conversely, given a $(\mathcal{G}_1, \mathcal{G}_2)$-compatible $\mathcal{G}_2$-action $\triangleright$ on $L_1$, we let

$$\Phi\colon \quad \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2, \quad (z,g)\,{}_2{*}_1\,e \mapsto p\big(g \triangleright [1,e]\big)\,{}_1{*}_2\,\big(z, g \triangleleft \pi_1(e)\big).$$

A detailed proof of these claims can be found in Appendix A.

We now put a groupoid structure on the quotient space $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ using the factorization rule $\Phi$.

**Proposition 3.14.** *Given a pair* $(\Sigma_1, \Sigma_2)$ *of matched twists as in Definition 3.5 with factorization rule* $\Phi$*, the following makes* $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ *a locally compact Hausdorff groupoid which we will denote by* $\Sigma_1 \bowtie_\Phi \Sigma_2$*: with* $\rho$ *and* $\sigma$ *as in Lemma 3.4, we declare the composable pairs to be*

$$(\Sigma_1 \bowtie_\Phi \Sigma_2)^{(2)} = \big\{(\xi, \eta) : \sigma(\xi) = \rho(\eta)\big\}.$$

*For an element* $e\,{}_1{*}_2\, f$*, we define its inverse as*

$$(e\,{}_1{*}_2\, f)^{-1} = \Phi(f^{-1}\,{}_2{*}_1\, e^{-1}). \tag{3.7}$$

*Given a second element* $e'\,{}_1{*}_2\, f'$ *such that* $\sigma(e\,{}_1{*}_2\, f) = \rho(e'\,{}_1{*}_2\, f')$*, define*

$$(e\,{}_1{*}_2\, f)(e'\,{}_1{*}_2\, f') = e \bullet \Phi(f\,{}_2{*}_1\, e') \bullet f'. \tag{3.8}$$

The reader should compare Equation (3.8) with Equation (2.3), which shows the relationship between the product in $\mathcal{G}_1 \bowtie \mathcal{G}_2$ and the factorization rule $\Psi$.

**Remark 3.15.** We chose the notation $\Sigma_1 \bowtie_\Phi \Sigma_2$ in place of $\Sigma_1 \bowtie \Sigma_2$ since the latter notation could be confused with a Zappa–Szép product of the *groupoids* $\Sigma_1$ and $\Sigma_2$, forgetting their additional structure as twists over $\mathcal{G}_1, \mathcal{G}_2$, and since the groupoid structure highly depends on $\Phi$ (see Remark 5.6).

Before we prove Proposition 3.14, we first show that the multiplication map in Equation (3.8) can be written in some other forms.

**Lemma 3.16.** *The multiplication map in Equation* (3.8) *can also be written as*

$$\begin{aligned}(e\,{}_1{*}_2\, f)(e'\,{}_1{*}_2\, f') &= e \bullet \Phi\big(f \bullet \Phi^{-1}(e'\,{}_1{*}_2\, f')\big)\\ &= \Phi\big(\Phi^{-1}(e\,{}_1{*}_2\, f) \bullet e'\big) \bullet f'.\end{aligned}$$

*Proof.* Suppose $\Phi^{-1}(e'\,{}_1{*}_2\, f') = a\,{}_2{*}_1\, b$. We have by Condition (FT3) that

$$\Phi(f\,{}_2{*}_1\, e') \bullet f' = \Phi(fa\,{}_2{*}_1\, b).$$

Therefore,

$$e \bullet \Phi(f\,{}_2{*}_1\, e') \bullet f' = e \bullet \Phi\big(f \bullet (a\,{}_2{*}_1\, b)\big) = e \bullet \Phi\big(f \bullet \Phi^{-1}(e'\,{}_2{*}_1\, f')\big).$$

The other equality follows similarly. ∎

*Proof of Proposition 3.14.* It is clear that the inversion formula at (3.7) is well defined. To see that the multiplication formula at (3.8) is well defined, just note that Condition (FT0) implies that for all $z, w \in \mathbb{T}$,

$$\begin{aligned}[z \cdot e] \bullet \Phi\big([\bar{z} \cdot f]\,{}_2{*}_1\, e'\big) \bullet f' &= e \bullet \Phi(f\,{}_2{*}_1\, e') \bullet f'\\ &= e \bullet \Phi\big(f\,{}_2{*}_1\, [w \cdot e']\big) \bullet [\overline{w} \cdot f'].\end{aligned}$$

Moreover, we can recover the range of an element $e\,{}_1{*}_2\,f \in \Sigma_1 \bowtie_\Phi \Sigma_2$ as

$$\begin{aligned}(e\,{}_1{*}_2\,f)(e\,{}_1{*}_2\,f)^{-1} &= (e\,{}_1{*}_2\,f)\Phi(f^{-1}\,{}_2{*}_1\,e^{-1}) \\ &= e \bullet \Phi\big(f \bullet \Phi^{-1}\Phi(f^{-1}\,{}_2{*}_1\,e^{-1})\big) && \text{by Lemma 3.16} \\ &= e \bullet \Phi(ff^{-1}\,{}_2{*}_1\,e^{-1}) \\ &= e \bullet \Phi\big(s(e)\,{}_2{*}_1\,e^{-1}\big) && \text{since } s(e) = r(f) \\ &= e \bullet \big(e^{-1}\,{}_2{*}_1\,r(e)\big) && \text{by Condition (FT2)} \\ &= r(e)\,{}_2{*}_1\,r(e), && (3.9)\end{aligned}$$

and its source as

$$\begin{aligned}(e\,{}_1{*}_2\,f)^{-1}(e\,{}_1{*}_2\,f) &= \Phi(f^{-1}\,{}_2{*}_1\,e^{-1})(e\,{}_1{*}_2\,f) \\ &= \Phi\big(\Phi^{-1}\Phi(f^{-1}\,{}_2{*}_1\,e^{-1}) \bullet e\big) \bullet f && \text{by Lemma 3.16} \\ &= \Phi\big(f^{-1}\,{}_2{*}_1\,r(f)\big) \bullet f && \text{since } s(e) = r(f) \\ &= \big(s(f)\,{}_1{*}_2\,f^{-1}\big) \bullet f && \text{by Condition (FT2)} \\ &= s(f)\,{}_1{*}_2\,s(f). && (3.10)\end{aligned}$$

In other words, the unit space of $\Sigma_1 \bowtie_\Phi \Sigma_2$ is given by

$$\begin{aligned}(\Sigma_1 \bowtie_\Phi \Sigma_2)^{(0)} &= \big\{r(e)\,{}_2{*}_1\,r(e) : e\,{}_1{*}_2\,f \in \Sigma_1 *_{\mathbb{T}} \Sigma_2\big\} \\ &= \big\{s(f)\,{}_1{*}_2\,s(f) : e\,{}_1{*}_2\,f \in \Sigma_1 *_{\mathbb{T}} \Sigma_2\big\}.\end{aligned}$$

To prove that multiplication is associative, we take $e_1, e_2, e_3 \in \Sigma_1$ and $f_1, f_2, f_3 \in \Sigma_2$ with appropriate ranges and sources so that the product $(e_1\,{}_1{*}_2\,f_1)(e_2\,{}_1{*}_2\,f_2)(e_3\,{}_1{*}_2\,f_3)$ makes sense. Denote $\Phi(f_1\,{}_2{*}_1\,e_2) = a_1\,{}_1{*}_2\,b_1$ and $\Phi(f_2\,{}_2{*}_1\,e_3) = a_2\,{}_1{*}_2\,b_2$. We have that

$$\begin{aligned}\big((e_1\,{}_1{*}_2\,f_1)(e_2\,{}_1{*}_2\,f_2)\big)(e_3\,{}_1{*}_2\,f_3) &= (e_1a_1\,{}_1{*}_2\,b_1f_2)(e_3\,{}_1{*}_2\,f_3) \\ &= e_1a_1 \bullet \Phi(b_1f_2\,{}_2{*}_1\,e_3) \bullet f_3 \\ &= e_1a_2 \bullet \big(\Phi(b_1\,{}_2{*}_1\,a_2) \bullet b_2\big) \bullet f_3 \quad \text{by (FT4)},\end{aligned}$$

where the last equality follows from Condition (FT4) using $\Phi(f_2\,{}_2{*}_1\,e_3) = a_2 * b_2$. On the other hand, a similar computation shows that

$$\begin{aligned}(e_1\,{}_1{*}_2\,f_1)\big((e_2\,{}_1{*}_2\,f_2)(e_3\,{}_1{*}_2\,f_3)\big) &= (e_1\,{}_1{*}_2\,f_1)(e_2a_2\,{}_1{*}_2\,b_2f_3) \\ &= e_1 \bullet \Phi(f_1\,{}_2{*}_1\,e_2a_2) \bullet b_2f_3 \\ &= e_1 \bullet \big(a_2 \bullet \Phi(b_1\,{}_2{*}_1\,a_2)\big) \bullet b_2f_3.\end{aligned}$$

Both of these equal $(e_1a_2) \bullet \Phi(b_1\,{}_2{*}_1\,a_2) \bullet (b_2f_3)$, and thus the multiplication is associative.

We have already seen in Lemma 3.1 that the space is locally compact Hausdorff. Continuity of the inversion and multiplication maps follows directly from openness and continuity of the quotient map, from continuity of the inversion in $\Sigma_1$ and $\Sigma_2$ respectively of the actions in Lemma 3.4, and from continuity of $\Phi$. ∎

Now that we have a groupoid structure on $\Sigma_1 \bowtie_\Phi \Sigma_2$, we must check that our inclusion maps of Lemma 3.2 are homomorphisms.

**Lemma 3.17.** *In the setting of Proposition 3.14, the maps $\iota^1_{1,2}$ and $\iota^2_{1,2}$ of Lemma 3.2 are groupoid homomorphisms and the unit space of the groupoid $\Sigma_1 \bowtie_\Phi \Sigma_2$ is homeomorphic to $\mathcal{U}$. With this identification, range and source map are given by the maps from Lemma 3.4, i.e.,*

$$r(e\,{}_1{*}_2\, f) = r(e) \quad \text{and} \quad s(e\,{}_1{*}_2\, f) = s(f). \tag{3.11}$$

*Proof.* If $(e, e') \in \Sigma_1^{(2)}$, then

$$\begin{aligned}\iota^1_{1,2}(e)\iota^1_{1,2}(e') &= \big(e\,{}_1{*}_2\, s(e)\big)\big(e'\,{}_1{*}_2\, s(e')\big) \overset{(3.8)}{=} e \bullet \Phi\big(s(e)\,{}_2{*}_1\, e'\big) \bullet s(e)\\ &= e \bullet \Phi\big(\iota^1_{2,1}(e')\big) \bullet s(e) \overset{(FT2)}{=} e \bullet \big(e'\,{}_1{*}_2\, s(e')\big) \bullet s(e) = \iota^1_{1,2}(ee'),\end{aligned}$$

and

$$\iota^1_{1,2}(e)^{-1} = \big(e\,{}_1{*}_2\, s(e)\big)^{-1} \overset{(3.7)}{=} \Phi\big(s(e)\,{}_2{*}_1\, e^{-1}\big) \overset{(FT2)}{=} \iota^1_{1,2}(e^{-1}).$$

One likewise proves that $\iota^2_{1,2}$ is a homomorphism.

The maps $\iota^i_{1,2}\colon \Sigma_i \to \Sigma_1 \bowtie_\Phi \Sigma_2$ clearly coincide on $\mathcal{U} = \Sigma_1^{(0)} = \Sigma_2^{(0)}$. The resulting map $u \mapsto u\,{}_1{*}_2\, u \in \Sigma_1 \bowtie_\Phi \Sigma_2$ has inverse the map $\Pi_{1,2}$ from Lemma 3.3 restricted to $(\Sigma_1 \bowtie_\Phi \Sigma_2)^{(0)}$, which is continuous. This proves that $u \mapsto u\,{}_1{*}_2\, u \in \Sigma_1 \bowtie_\Phi \Sigma_2$ is an embedding, i.e., a homeomorphism onto its image; we can henceforth identify the unit space of $\Sigma_1 \bowtie_\Phi \Sigma_2$ with $\mathcal{U}$. The claim about the range and source maps now follows from Equations (3.9) and (3.10). ■

We now prove that $\Sigma_1 \bowtie_\Phi \Sigma_2$ is a twist over the Zappa–Szép product $\mathcal{G}_1 \bowtie \mathcal{G}_2$.

**Theorem 3.18.** *Given a pair $(\Sigma_1, \Sigma_2)$ of matched twists with factorization rule $\Phi$ as in Definition 3.5 over a pair $(\mathcal{G}_1, \mathcal{G}_2)$, the sequence*

$$\mathbb{T} \times \mathcal{U} \overset{\jmath}{\to} \Sigma_1 \bowtie_\Phi \Sigma_2 \overset{\Pi_{1,2}}{\to} \mathcal{G}_1 \bowtie \mathcal{G}_2,$$

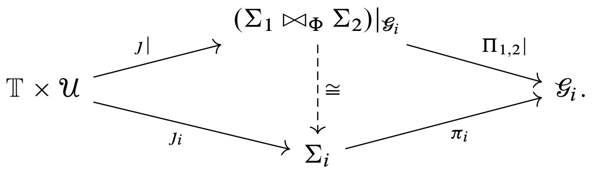
*where*

$$\jmath(z, u) := z \cdot (u\,{}_1{*}_2\, u) = \jmath_1(1, u)\,{}_1{*}_2\, \jmath_2(z, u) = \jmath_1(z, u)\,{}_1{*}_2\, \jmath_2(1, u),$$

*makes the groupoid $\Sigma_1 \bowtie_\Phi \Sigma_2$ a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$. We call it the* external Zappa–Szép twist.

**Remark 3.19.** In the setting where $\Sigma_1$ and $\Sigma_2$ are both trivial and the factorization rule is the map at (3.5) (see Example 3.12), the groupoid we recover is (canonically isomorphic to) the Cartesian product $\mathbb{T} \times (\mathcal{G}_1 \bowtie \mathcal{G}_2)$.

**Remark 3.20.** By construction, the restriction $(\Sigma_1 \bowtie_\Phi \Sigma_2)|_{\mathcal{G}_i} := \Pi^{-1}_{1,2}(\mathcal{G}_i)$ of the external Zappa–Szép twist to one of the closed subgroupoids $\mathcal{G}_i$ of $\mathcal{G}_1 \bowtie \mathcal{G}_2$ is canonically isomorphic to $\Sigma_i$. To be precise, in the following diagram, the dashed arrow is the obvious map onto the $i$-th component, it is a groupoid isomorphism, and the diagram commutes.




In particular, even though $\mathcal{G}_i$ does not need to be open in $\mathcal{G}_1 \bowtie \mathcal{G}_2$ (see Lemma 2.4), the restriction of the Zappa–Szép twist to $\mathcal{G}_i$ is a twist.

*Proof of Theorem* 3.18. We have seen in Proposition 3.14 that $\Sigma_1 \bowtie_\Phi \Sigma_2$ is a locally compact Hausdorff groupoid and in Lemma 3.3 that $\Pi_{1,2}$ is a continuous, open surjection. To see that it is a homomorphism, note first that since $\pi_1, \pi_2$ are homomorphisms, we have

$$\begin{aligned}\Pi_{1,2}\big(a \bullet (e \,{}_1{*}_2\, f) \bullet b\big) &= \Pi_{1,2}\big((ae) \,{}_1{*}_2\, (fb)\big) = \pi_1(ae)\,\pi_2(fb)\\ &= \pi_1(a)\pi_1(e)\,\pi_2(f)\pi_2(b) = \pi_1(a)\,\Pi_{1,2}(e \,{}_1{*}_2\, f)\,\pi_2(b),\end{aligned}$$

which implies

$$\begin{aligned}&\Pi_{1,2}(e \,{}_1{*}_2\, f)\Pi_{1,2}(e' \,{}_1{*}_2\, f')\\ &\quad= \big[\pi_1(e)\pi_2(f)\big]\big[\pi_1(e')\pi_2(f')\big] && \text{by definition of } \Pi_{1,2}\\ &\quad= \pi_1(e)\,\Pi_{2,1}(f \,{}_2{*}_1\, e')\,\pi_2(f') && \text{by definition of } \Pi_{2,1}\\ &\quad= \pi_1(e)\,(\Pi_{1,2} \circ \Phi)(f \,{}_2{*}_1\, e')\,\pi_2(f') && \text{by (3.4)}\\ &\quad= \Pi_{1,2}\big(e \bullet \Phi(f \,{}_2{*}_1\, e') \bullet f'\big) && \text{by the above}\\ &\quad= \Pi_{1,2}\big((e \,{}_1{*}_2\, f)(e' \,{}_1{*}_2\, f')\big) && \text{by (3.8),}\end{aligned}$$

as needed.

Since the $\mathbb{T}$-action on $\Sigma_1 \bowtie_\Phi \Sigma_2$ is continuous (Lemma 3.1), the injective map $\jmath$ is continuous. To see that $\jmath$ is a homomorphism, we first note that for appropriate $u, v \in \mathcal{U}$ and all $z \in \mathbb{T}$,

$$\begin{aligned}\jmath_1(z,u) \bullet (e \,{}_1{*}_2\, f) &= \big(\jmath_1(z,u)e\big) \,{}_1{*}_2\, f = (z \cdot e) \,{}_1{*}_2\, f\\ &= e \,{}_1{*}_2\, (z \cdot f) = e \,{}_1{*}_2\, \big(f\, \jmath_2(z,v)\big) = (e \,{}_1{*}_2\, f) \bullet \jmath_2(z,v),\end{aligned}$$

which are all equal to $z \cdot (e \,{}_1{*}_2\, f)$ as defined in Lemma 3.1. Therefore,

$$\begin{aligned}\big([z_1 \cdot u] \,{}_1{*}_2\, u\big)\big([z_2 \cdot u] \,{}_1{*}_2\, u\big) &= \jmath_1(z_1,u) \bullet \Phi\big(u \,{}_2{*}_1\, [z_2 \cdot u]\big) \bullet \jmath_2(1,u)\\ &= z_1 \cdot \Phi\big(u \,{}_2{*}_1\, [z_2 \cdot u]\big) && \text{by the above}\\ &= z_1 \cdot \big[z_2 \cdot \Phi(u \,{}_2{*}_1\, u)\big] && \text{by (FT0)}\\ &= [z_1 z_2] \cdot (u \,{}_1{*}_2\, u) && \text{by (FT2).}\end{aligned}$$

Next, we show that the range of $\jmath$ is $\Pi_{1,2}^{-1}(\mathcal{U})$. Since $\Pi_{1,2}(\jmath(z,u)) = \Pi_{1,2}([z\cdot u]\,{}_1{*}_2\,u) = \pi_1(z\cdot u)\pi_2(u) = u$, we see that $\jmath(\mathbb{T}\times\mathcal{U}) \subseteq \Pi_{1,2}^{-1}(\mathcal{U})$. Conversely, if $e\,{}_1{*}_2\,f \in \Pi_{1,2}^{-1}(\mathcal{U})$, then $\pi_1(e)\pi_2(f) \in \mathcal{U}$, so $\pi_1(e) = \pi_2(f)^{-1}$. Since $\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{U}$ in $\mathcal{G}_1 \bowtie \mathcal{G}_2$, this implies $\pi_1(e) = \pi_2(f) = u \in \mathcal{U}$. Since $\Sigma_1, \Sigma_2$ are twists, this implies that there exist $z_1, z_2 \in \mathbb{T}$ such that $e = \jmath_1(z_1,u)$ and $f = \jmath_2(z_2,u)$. Therefore, $e\,{}_1{*}_2\,f = \jmath(z_1z_2,u)$, which proves that $\jmath(\mathbb{T}\times\mathcal{U}) \supseteq \Pi_{1,2}^{-1}(\mathcal{U})$.

It remains to show that the injective map $\jmath$ is a homeomorphism onto its image. So suppose $\{\jmath(z_\lambda,u_\lambda)\}$ is a net converging to $\jmath(z,u)$ in $\Sigma_1 \bowtie_\Phi \Sigma_2$; since it suffices for us to show that a subnet of $\{(z_\lambda,u_\lambda)\}$ converges to $(z,u)$ in $\mathbb{T}\times\mathcal{U}$, we will frequently pass to subnets in the following argument (or, in other words, without loss of generality assume that the subnet we pass to is the net we started with). Since the quotient map is open (Lemma 3.1), (a subnet of) $\{\jmath(z_\lambda,u_\lambda)\}$ allows a lift to a net $\{(e_\lambda,f_\lambda)\}$ in $\Sigma_1\,{}_s{\times}_r\,\Sigma_2$ such that $e_\lambda \to \jmath_1(z,u)$ and $f_\lambda \to \jmath_2(1,u)$. Being a lift means that

$$e_\lambda\,{}_1{*}_2\,f_\lambda = \jmath(z_\lambda,u_\lambda) = \jmath_1(z_\lambda,u_\lambda)\,{}_1{*}_2\,\jmath_2(1,u_\lambda)$$

for all $\lambda$, so the definition of the equivalence relation says that there exists $w_\lambda \in \mathbb{T}$ such that

$$e_\lambda = \jmath_1(\overline{w_\lambda}z_\lambda,u_\lambda) \quad\text{and}\quad f_\lambda = \jmath_2(w_\lambda,u_\lambda).$$

Since $\jmath_1$ and $\jmath_2$ are embeddings, the convergences $f_\lambda \to \jmath_2(1,u)$ and $e_\lambda \to \jmath_1(z,u)$ thus imply that $(w_\lambda,u_\lambda) \to (1,u)$ and $(\overline{w_\lambda}z_\lambda,u_\lambda) \to (z,u)$ in $\mathbb{T}\times\mathcal{U}$. We conclude that (a subnet of) $\{(z_\lambda,u_\lambda)\}$ converges to $(z,u)$, proving that $\jmath$ is an embedding. ∎

**Example 3.21** (One Fell line bundle; continuation of Example 3.13). Suppose again that $(\mathcal{G}_1,\mathcal{G}_2)$ is a matched pair of groupoids with unit space $\mathcal{U}$, and that $\Sigma_1$ is a twist over $\mathcal{G}_1$ with associated Fell line bundle $L_1 = \mathbb{C}\times_{\mathbb{T}}\Sigma_1$. If we let $\Sigma_2 = \mathbb{T}\times\mathcal{G}_2$ be the trivial twist, then we have seen that a factorization rule $\Phi$ for $(\Sigma_1,\Sigma_2)$ can be translated into a $(\mathcal{G}_1,\mathcal{G}_2)$-compatible $\mathcal{G}_2$-action $\triangleright$ on $L_1$. By [12, Proposition 3.5], such an action gives rise to a Zappa–Szép product Fell bundle $\mathcal{B} = (q_{\mathcal{B}}\colon L_1 \bowtie \mathcal{G}_2 \to \mathcal{G}_1 \bowtie \mathcal{G}_2)$ that encodes the action; we claim that this bundle $\mathcal{B}$ is (canonically isomorphic to) the Fell line bundle $L$ that is associated to the Zappa–Szép twist $\Sigma_1 \bowtie_\Phi \Sigma_2$, whose elements are of the form $[\lambda, e\,{}_1{*}_2\,(z,g)]$ for $\lambda\in\mathbb{C}$, $e\in\Sigma_1$, $z\in\mathbb{T}$, and $g\in\mathcal{G}_2$.

If we denote the elements of $L_1$ by $[\lambda,e]$, as before in Equation (3.6), then the total space of $\mathcal{B}$ is defined as

$$L_1 \bowtie \mathcal{G}_2 := \big\{\big([\lambda,e],g\big) \in L_1\times\mathcal{G}_2 : s_{\Sigma_1}(e) = r_{\mathcal{G}_2}(g)\big\}$$

with the subspace topology of $L_1\times\mathcal{G}_2$; the projection map $q_{\mathcal{B}}$ maps $([\lambda,e],g)$ to $(\pi_1(e),g) \in \mathcal{G}_1 \bowtie \mathcal{G}_2$; the Fell bundle multiplication is given by

$$\big([\lambda,e],g\big)\big([\lambda',e'],g'\big) := \big([\lambda,e]\big(g \triangleright [\lambda',e']\big), \big(g \triangleleft \pi_1(e')\big)g'\big), \tag{3.12}$$

where $s_{\mathcal{G}_1}(g) = r_{\Sigma_1}(e')$; and the $*$-operation is given by

$$\big([\lambda,e],g\big)^* := \big(g^{-1} \triangleright [\lambda,e]^*, g^{-1} \triangleleft \pi_1(e)^{-1}\big).$$

We show in Appendix A that the map

$$\Omega\colon L = \mathbb{C} \times_{\mathbb{T}} (\Sigma_1 \bowtie_{\Phi} \Sigma_2) \to L_1 \bowtie \mathcal{G}_2, \quad \big[\lambda, e\,{}_1{*}_2\,(z,g)\big] \mapsto \big([\lambda z, e], g\big), \tag{3.13}$$

is an isomorphism of Fell bundles.

We would like to point out that the paper at hand was motivated by our pursuit of the Zappa–Szép product of two Fell bundles. Definition 3.5 suggests that an external Zappa–Szép product should be thought of in terms of the factorization rule. An analogous map can be defined in the more general setting of Fell bundles. We shall leave the construction of the Zappa–Szép product of two Fell bundles to future work.

## 4. The internal Zappa–Szép product of two twists

We now turn our attention to the other face of a Zappa–Szép product: the internal product.

**Definition 4.1.** An *internal Zappa–Szép structure* for a twist $(\Sigma, \jmath, \pi)$ over a groupoid $\mathcal{G}$ with unit space $\mathcal{U}$ is a tuple $(\Sigma_1, \Sigma_2)$ of two closed subgroupoids of $\Sigma$ such that

(I1) $\Sigma = \Sigma_1 \cdot \Sigma_2$ and

(I2) $\Sigma_1 \cap \Sigma_2 = \jmath(\mathbb{T} \times \mathcal{U})$.

Compared with the internal Zappa–Szép product of groups (see also [7] for groupoids), Condition (I1) is a natural replacement of the condition that a group is a product of its two subgroups (see Condition (1) in the introduction), and Condition (I2) is the analogue of the intersection of the subgroups being trivial (Condition (2) *loc. cit.*). Indeed, in order for each $\Sigma_i$ to be a twist itself, it must contain a copy of $\jmath(\mathbb{T} \times \mathcal{U})$, so this is the smallest intersection one can impose on these two subgroupoids $\Sigma_i$.

We first outline several properties of the internal Zappa–Szép structure.

**Lemma 4.2.** *Suppose $(\Sigma_1, \Sigma_2)$ is an internal Zappa–Szép structure for a twist $(\Sigma, \jmath, \pi)$ over a groupoid $\mathcal{G}$. Then the following hold for $i = 1, 2$.*

(I3) *$(\Sigma_2, \Sigma_1)$ is also an internal Zappa–Szép structure for $(\Sigma, \jmath, \pi)$;*

(I4) *$\Sigma_i = \pi^{-1}(\pi(\Sigma_i))$ and $\Sigma_i^{(0)} = \mathcal{U}$;*

(I5) *$\mathcal{G}_i := \pi(\Sigma_i)$ is a closed subgroupoid of $\mathcal{G}$;*

(I6) *$\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$;*

(I7) *the triple $(\Sigma_i, \jmath, \pi|_{\Sigma_i})$ is a twist over $\mathcal{G}_i$;*

(I8) *the map $\Sigma_i \ {}_s{\times}_r \ \Sigma_j \to \Sigma, (e, f) \mapsto ef$, is open.*

*For $(f, e) \in \Sigma_2 \ {}_s{\times}_r \ \Sigma_1$, we further have the following.*

(I9) *There exist $(e', f') \in \Sigma_1 \ {}_s{\times}_r \ \Sigma_2$ such that $fe = e'f'$.*

(I10) *The choice of $(e', f')$ in (I9) has only one degree of freedom; to be more precise,*

$$\big\{(e'', f'') \in \Sigma_1 \ {}_s{\times}_r \ \Sigma_2 : fe = e''f''\big\} = \big\{(z \cdot e', \bar{z} \cdot f') : z \in \mathbb{T}\big\}.$$

*Proof.* (I3) For $e \in \Sigma$, use (I1) to write $e^{-1} = e_1 e_2$ for some $e_i \in \Sigma_i$, so that $e = e_2^{-1} e_1^{-1}$ and hence $\Sigma = \Sigma_2 \cdot \Sigma_1$.

(I4) The containment $\Sigma_i \subseteq \pi^{-1}(\pi(\Sigma_i))$ is obvious. For the reverse containment, note that (I2) implies $\jmath(\mathbb{T} \times \mathcal{U}) \subseteq \Sigma_i$, so in particular, $\Sigma_i^{(0)} = \Sigma^{(0)}$. Since $\Sigma_i$ is a subgroupoid, we further conclude for any $e \in \Sigma_i$ and $z \in \mathbb{T}$ that $z \cdot e = \jmath(z, r(e))e \in \Sigma_i$, which proves $\Sigma_i \supseteq \pi^{-1}(\pi(\Sigma_i))$.

(I5) Since $\pi$ is a groupoid homomorphism, $\mathcal{G}_i$ is a subgroupoid. Now suppose $\{g_\lambda\}$ is a net in $\mathcal{G}_i$ that converges to $\pi(e) = g$ in $\mathcal{G}$. Since $\pi$ is open, Fell's criterion [37, Proposition 1.1] says that (a subnet of) $\{g_\lambda\}$ can be lifted under $\pi$ to a net $\{e_\lambda\}$ in $\Sigma$ such that $e_\lambda \to e$. As $\pi(e_\lambda) = g_\lambda \in \mathcal{G}_i$, we have $e_\lambda \in \pi^{-1}(\mathcal{G}_i) = \Sigma_i$ by (I4). Since $\Sigma_i$ is closed by assumption, the limit $e$ of the net must be contained in $\Sigma_i$, which proves that $\pi(e) = g$ is contained in $\pi(\Sigma_i) = \mathcal{G}_i$, as claimed. (Coincidentally, this proof also shows that $\pi_i := \pi|_{\Sigma_i} : \Sigma_i \to \mathcal{G}_i$ is an open map, which we require for (I7).)

(I6) Since $\mathcal{G}_i$ is a closed subgroupoid of the locally compact Hausdorff groupoid $\mathcal{G}$ by (I5), it is itself locally compact Hausdorff. Since $\pi$ is a groupoid homomorphism, it follows from Condition (I1) that

$$\mathcal{G} = \pi(\Sigma) = \pi(\Sigma_1 \cdot \Sigma_2) = \pi(\Sigma_1) \cdot \pi(\Sigma_2) = \mathcal{G}_1 \cdot \mathcal{G}_2.$$

Note that

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \pi(\Sigma_1) \cap \pi(\Sigma_2) = \pi(\Sigma_1 \cap \Sigma_2) \overset{\text{(I2)}}{=} \pi\big(\jmath(\mathbb{T} \times \mathcal{U})\big) \overset{\text{(T1)}}{=} \mathcal{U}. \tag{4.1}$$

This implies that every element $k \in \mathcal{G}$ has a *unique* decomposition as a product $k = xg$ with $x \in \mathcal{G}_1, g \in \mathcal{G}_2$. By [7, Proposition 7], $\mathcal{G}$ is therefore the internal Zappa–Szép product $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$.

(I7) By the same argument as in (I6), $\Sigma_i$ is a locally compact Hausdorff groupoid. The maps $\jmath_i := \jmath : \mathbb{T} \times \mathcal{U} \to \Sigma_i \subseteq \Sigma$ and $\pi_i$ are clearly continuous groupoid homomorphisms. Since

$$\pi_i^{-1}(\mathcal{U}) = \pi^{-1}(\mathcal{U}) \cap \Sigma_i \overset{\text{(T1)}}{=} \jmath(\mathbb{T} \times \mathcal{U}) \cap \Sigma_i \overset{\text{(I2)}}{=} \jmath(\mathbb{T} \times \mathcal{U}) \overset{\text{(T1)}}{=} \pi^{-1}(\mathcal{U}),$$

we conclude that the subspace topology that $\pi_i^{-1}(\mathcal{U}) = \pi^{-1}(\mathcal{U})$ inherits from $\Sigma_i$ is identical to the topology that it inherits from $\Sigma$. In other words, $\jmath_i$ satisfies (T1) because $\jmath$ does. Furthermore, (T3) clearly holds as well by assumption on $(\Sigma, \jmath, \pi)$. Since we have seen in the proof of (I5) that $\pi_i$ is open, we have shown all conditions listed in Definition 2.1.

(I8) Suppose that $\{\sigma_\lambda\}$ is a net in $\Sigma$ that converges to $ef$ for $e \in \Sigma_i, f \in \Sigma_j$; it suffices to lift a subnet of $\{\sigma_\lambda\}$ under the multiplication map to nets $\{e'_\lambda\}$ in $\Sigma_1$ and $\{f'_\lambda\}$ in $\Sigma_2$ such that $e'_\lambda \to e$ and $f'_\lambda \to f$. By (I1) (applied to either $(\Sigma_1, \Sigma_2)$ or $(\Sigma_2, \Sigma_1)$, using (I3)), for every $\lambda$, there exists $e_\lambda \in \Sigma_i$ and $f_\lambda \in \Sigma_j$ such that $\sigma_\lambda = e_\lambda f_\lambda$. Since $\pi$ is a continuous groupoid homomorphism, we have

$$\pi(e_\lambda)\pi(f_\lambda) = \pi(\sigma_\lambda) \to \pi(ef) = \pi(e)\pi(f)$$

in $\mathcal{G}$. As $\pi(e_\lambda) \in \mathcal{G}_i$ and $\pi(f_\lambda) \in \mathcal{G}_i$, this implies together with (I6) that

$$\pi(e_\lambda) \to \pi(e) \quad \text{and} \quad \pi(f_\lambda) \to \pi(f).$$

Since $\pi$ is open and since we can pass to a subnet, there exist without loss of generality $z_\lambda, w_\lambda \in \mathbb{T}$ such that

$$z_\lambda \cdot e_\lambda \to e \quad \text{and} \quad w_\lambda \cdot f_\lambda \to f.$$

By compactness of $\mathbb{T}$ (and since we can pass to subnets), we can without loss of generality assume that $z_\lambda \to z$ and $w_\lambda \to w$ for some $z, w \in \mathbb{T}$; in particular,

$$e'_\lambda := z \cdot e_\lambda = (z\overline{z_\lambda}) \cdot (z_\lambda \cdot e_\lambda) \to (z\bar{z}) \cdot e = e$$

and

$$f'_\lambda := w \cdot f_\lambda = (w\overline{w_\lambda}) \cdot (w_\lambda \cdot f_\lambda) \to (w\bar{w}) \cdot f = f,$$

so that

$$ef \leftarrow (z \cdot e_\lambda)(w \cdot f_\lambda) = (zw) \cdot \sigma_\lambda \to (zw) \cdot ef.$$

In particular, $zw = 1$ as $\Sigma$ is Hausdorff. By (I4), the net $\{(e'_\lambda, f'_\lambda)\}$ is in $\Sigma_i \,{}_s\!\times_r \Sigma_j$, converges to $(e, f)$, and since $zw = 1$, it satisfies

$$e'_\lambda f'_\lambda = (z \cdot e_\lambda)(w \cdot f_\lambda) = (zw) \cdot (e_\lambda f_\lambda) = \sigma_\lambda.$$

This proves that the surjective map $\Sigma_i \,{}_s\!\times_r \Sigma_j \to \Sigma$, $(e, f) \mapsto ef$, is open.

(I9) Since $(f, e) \in \Sigma_2 \,{}_s\!\times_r \Sigma_1$, we see that $\pi(fe)$ is an element of $\mathcal{G}_2 \cdot \mathcal{G}_1$. Since $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$, there exist unique $g_i \in \mathcal{G}_i$ such that $\pi(fe) = g_1 g_2$. We can let $f_i \in \Sigma_i$ be any lifts under $\pi_i$ of $g_i$. Since $\pi(fe) = g_1 g_2 = \pi(f_1 f_2)$, there exists a unique $z \in \mathbb{T}$ such that $fe = (z \cdot f_1) f_2$. Note that $e' := z \cdot f_1 \in \Sigma_1$ and $f' := f_2 \in \Sigma_2$, so we have written $fe$ as a product of an element in $\Sigma_1$ and $\Sigma_2$, as claimed.

(I10) We clearly have $\supseteq$. For the converse containment, assume $fe = e'' f''$. Then $e'' f'' = e' f'$, so that $e'^{-1} e'' = f' f''^{-1}$ is an element of $\Sigma_1 \cap \Sigma_2$. By (I2), this means that there exists $z \in \mathbb{T}$ such that $z \cdot e' = e''$ and $z \cdot f'' = f'$. ∎

As in the case of any construction of a Zappa–Szép product (e.g., [7, Proposition 7] and [25, Proposition 3.13]), the external and internal Zappa–Szép structures are, in fact, two sides of the same coin. The same is true for our Zappa–Szép twists, as is summarized in the following main theorem of this section.

**Theorem 4.3.** *There is a one-to-one correspondence between internal Zappa–Szép structures (Definition 4.1) and matched twists (Definition 3.5). To be more precise:*

(1) *A pair $(\Sigma_1, \Sigma_2)$ of matched twists with factorization rule $\Phi$ is an internal Zappa–Szép structure for the external Zappa–Szép twist $(\Sigma_1 \bowtie_\Phi \Sigma_2, \jmath, \Pi_{1,2})$ of Theorem 3.18.*

(2) *For any internal Zappa–Szép structure* $(\Sigma_1, \Sigma_2)$ *of a twist* $(\Sigma, \jmath, \pi)$ *over a groupoid* $\mathcal{G}$*, the map* $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ *determined by*

$$\Phi(f\,{}_2{*}_1\,e) = e'\,{}_1{*}_2\,f' \quad \textit{whenever } fe = e'f' \textit{ in } \Sigma,$$

*is well defined. It is furthermore the unique factorization rule for* $(\Sigma_1, \Sigma_2)$ *that covers* $\mathcal{G} = \pi(\Sigma_1) \bowtie \pi(\Sigma_2)$ *such that the map*

$$\Sigma_1 \bowtie_\Phi \Sigma_2 \to \Sigma, \quad e\,{}_1{*}_2\,f \mapsto ef,$$

*is an isomorphism of twists, meaning that the following diagram commutes.*

$$\begin{array}{ccccccc} & & \Sigma_1 \bowtie_\Phi \Sigma_2 & \xrightarrow{\Pi_{1,2}} & \pi(\Sigma_1) \bowtie \pi(\Sigma_2) & \ni & (x, g) \\ & \overset{\jmath}{\nearrow} & \Big\downarrow & & \Big\downarrow \cong & & \Big\downarrow \\ \mathbb{T} \times \mathcal{U} & \underset{\jmath}{\searrow} & \Sigma & \xrightarrow[\pi]{} & \mathcal{G} & \ni & xg \end{array}$$

In Part (1), one could be pedantic and say that $(\iota^1_{1,2}(\Sigma_1), \iota^2_{1,2}(\Sigma_2))$ is the internal Zappa–Szép structure for $(\Sigma_1 \bowtie_\Phi \Sigma_2, \jmath, \Pi_{1,2})$, but courtesy of Lemmas 3.2 and 3.17, we can identify $\Sigma_i$ with its homeomorphic, closed image $\iota^i_{1,2}(\Sigma_i)$.

*Proof.* (1) Suppose $(\Sigma_1, \Sigma_2)$ is a matched pair of twists with factorization rule $\Phi$. By Lemmas 3.2 and 3.17, $\Sigma_1, \Sigma_2$ are (homeomorphic to) closed subgroupoids of $\Lambda := \Sigma_1 \bowtie_\Phi \Sigma_2$. Since

$$\begin{aligned} e\,{}_1{*}_2\,f &= e \bullet \big(s(e)\,{}_1{*}_2\,r(f)\big) \bullet f = e \bullet \Phi\big(s(e)\,{}_2{*}_1\,r(f)\big) \bullet f \quad \text{by Remark 3.6} \\ &= \big(e\,{}_1{*}_2\,s(e)\big)\big(r(f)\,{}_1{*}_2\,f\big) = \iota^1_{1,2}(e)\iota^2_{1,2}(f), \end{aligned}$$

we see that $\Lambda = \iota^1_{1,2}(\Sigma_1)\iota^2_{1,2}(\Sigma_2)$, i.e., Condition (I1) is satisfied. Moreover, an element of $\Lambda$ is in $\iota^1_{1,2}(\Sigma_1) \cap \iota^2_{1,2}(\Sigma_2)$ if it can be written both as $e\,{}_1{*}_2\,s(e)$ for some $e \in \Sigma_1$ and as $r(e)\,{}_1{*}_2\,f$ for some $f \in \Sigma_2$; in particular, there exists $z \in \mathbb{T}$ such that $(e, s(e)) = (z \cdot r(e), \bar{z} \cdot f)$. We conclude that the ranges and sources of $e$ and $f$ are all the same element, say $u$, so that we can write $e = z \cdot u = f$ and conclude that the element we started with can be written as $e\,{}_1{*}_2\,s(e) = z \cdot (u\,{}_1{*}_2\,u) = \jmath(z, u)$. This proves $\iota^1_{1,2}(\Sigma_1) \cap \iota^2_{1,2}(\Sigma_2) \subseteq \jmath(\mathbb{T} \times \mathcal{U})$, i.e., Condition (I2).

(2) Suppose $(\Sigma_1, \Sigma_2)$ is an internal Zappa–Szép structure of a twist $(\Sigma, \jmath, \pi)$. Because of (I9), we may use the Axiom of Choice to define a map $\Phi_0\colon \Sigma_2\,{}_s{\times}_r\,\Sigma_1 \to \Sigma_1\,{}_s{\times}_r\,\Sigma_2$ by letting it send a tuple $(f, e)$ to any element $(e', f')$ for which $fe = e'f'$ in $\Sigma$. Now, if $\Phi(z \cdot f, \bar{z} \cdot e) = (\widetilde{\sigma}_1, \widetilde{\sigma}_2)$, then

$$\widetilde{\sigma}_1\widetilde{\sigma}_2 = (z \cdot f)(\bar{z} \cdot e) = fe = e'f',$$

so that $e'^{-1}\widetilde{\sigma}_1 = f'\widetilde{\sigma}_2^{-1} \in \Sigma_1 \cap \Sigma_2$. By (I2), this means there exists $w \in \mathbb{T}$ such that $w \cdot e' = \widetilde{\sigma}_1$ and $w \cdot \widetilde{\sigma}_2 = f'$. In other words, $\widetilde{\sigma}_1\,{}_1{*}_2\,\widetilde{\sigma}_2 = e'\,{}_1{*}_2\,f'$ in $\Sigma_1 *_{\mathbb{T}} \Sigma_2$. This proves that $\Phi_0$ induces a map

$$\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2.$$

Thanks to (I9), this map is injective. And while $\Phi_0$ was not unique, this map is.

To see that $\Phi$ is continuous, assume that $\{\xi_\lambda\}$ is a net in $\Sigma_2 *_{\mathbb{T}} \Sigma_1$ that converges to $\xi$; it suffices to show that a subnet of $\{\Phi(\xi_\lambda)\}$ converges to $\Phi(\xi)$. Pick any $(e, f) \in \Sigma_2 \,{}_s\!\times_r \Sigma_1$ such that $\xi = e \,{}_2{*}_1 f$. Since the quotient map $q_{2,1}$ is open (Lemma 3.1) and since we can pass to subnets, we can without loss of generality assume that there exists a net $\{(e_\lambda, f_\lambda)\}$ in $\Sigma_2 \,{}_s\!\times_r \Sigma_1$ such that $\xi_\lambda = e_\lambda \,{}_2{*}_1 f_\lambda$ and $(e_\lambda, f_\lambda) \to (e, f)$; in particular $e_\lambda f_\lambda \to ef$. Now pick any $(\sigma, \mu) \in \Sigma_1 \,{}_s\!\times_r \Sigma_2$ such that $ef = \sigma\mu$. Since the multiplication map $\Sigma_1 \,{}_s\!\times_r \Sigma_2 \to \Sigma$ is surjective by (I1) and open by (I8) and since we can, again, pass to subnets, can without loss of generality lift the convergent net $\{e_\lambda f_\lambda\}$, i.e., there exists a net $\{(\sigma_\lambda, \mu_\lambda)\}$ in $\Sigma_1 \,{}_s\!\times_r \Sigma_2$ such that $(\sigma_\lambda, \mu_\lambda) \to (\sigma, \mu)$ and $\sigma_\lambda\mu_\lambda = e_\lambda f_\lambda$. The latter means exactly that $\Phi(\xi_\lambda) = \sigma_\lambda \,{}_1{*}_2\, \mu_\lambda$, and the former implies that $\Phi(\xi_\lambda) \to \sigma \,{}_1{*}_2\, \mu = \Phi(\xi)$. This proves that $\Phi$ is continuous.

Since $(\Sigma_2, \Sigma_1)$ is also a matched pair by (I3), the same argument yields an injective continuous map $\Sigma_1 *_{\mathbb{T}} \Sigma_2 \to \Sigma_2 *_{\mathbb{T}} \Sigma_1$ which, by construction, is inverse to $\Phi$, so $\Phi$ is a homeomorphism. We claim that $\Phi$ is a factorization rule, i.e., satisfies the conditions in Definition 3.5.

If $\Phi(e_2 \,{}_2{*}_1 e_1) = \sigma_1 \,{}_1{*}_2 \sigma_2$, then $e_2 e_1 = \sigma_1\sigma_2$ by definition of $\Phi$. This implies $(z \cdot e_2)e_1 = (z \cdot \sigma_1)\sigma_2$, so that $\Phi((z \cdot e_2) \,{}_2{*}_1\, e_1) = (z \cdot \sigma_1) \,{}_1{*}_2\, \sigma_2 = z \cdot \Phi(e_2 \,{}_2{*}_1\, e_1)$, proving Condition (FT0). Condition (FT1) follows from $r(f) = r(fe) = r(e'f') = r(e')$ and $s(e) = s(fe) = s(e'f') = s(f')$. Since $es(e) = e = r(e)e$, we further get Condition (FT2). Lastly, if $f_i \in \Sigma_i$ with appropriate range and source, then since $(e_2e_1)f_1 = \sigma_1(\sigma_2 f_1)$ and $f_2(e_2e_1) = (f_2\sigma_1)\sigma_2$, we conclude that

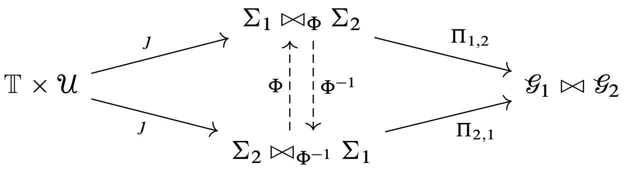
$$\Phi(e_2 \,{}_2{*}_1\, e_1 f_1) = \sigma_1 \bullet \Phi(\sigma_2 \,{}_2{*}_1\, f_1) \quad \text{and} \quad \Phi(f_2 e_2 \,{}_2{*}_1\, e_1) = \Phi(f_2 \,{}_2{*}_1\, \sigma_1) \bullet \sigma_2,$$

which proves Conditions (FT3) and (FT4), so that $\Phi$ is indeed a factorization rule and $(\Sigma_1, \Sigma_2)$ is a matched pair of twists.

It remains to prove the claim about the map $e \,{}_1{*}_2\, f \mapsto ef$. It is well defined since centrality of the $\mathbb{T}$-action on $\Sigma$ implies $e_1e_2 = (z \cdot e_1)(\bar{z} \cdot e_2)$. It is now easy to check that the map is a groupoid homomorphism. It is a bijection by Lemma 4.2 (I10) (the only degree of freedom was the choice of $\mathbb{T}$). The isomorphism is continuous (respectively open) since the quotient map is open (respectively continuous) and the map $\Sigma_1 \,{}_s\!\times_r \Sigma_2 \to \Sigma$, $(e, f) \mapsto ef$, is continuous (respectively open by (I8)). It is obvious that the diagram commutes since $\pi$ is a homomorphism. ∎

As mentioned earlier, the Zappa–Szép product $\mathcal{G}_1 \bowtie \mathcal{G}_2$ of two groupoids is canonically isomorphic to $\mathcal{G}_2 \bowtie \mathcal{G}_1$. A similar results holds for twists:

**Corollary 4.4** (cf. [7, Corollary 9]). *Suppose $(\Sigma_1, \Sigma_2)$ is a matched pair of twists with factorization rule $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$. Then $(\Sigma_2, \Sigma_1)$ is a matched pair of twists* *with factorization rule $\Phi^{-1}$. Moreover, $\Phi$ and $\Phi^{-1}$ are groupoid isomorphisms that make the diagram*




*commute. In other words, the associated external Zappa–Szép twists are isomorphic as twists.*

*Proof.* Let us write $\Sigma := \Sigma_1 \bowtie_\Phi \Sigma_2$. Since $(\Sigma_1, \Sigma_2)$ is a matched pair of twists, it follows from Theorem 4.3 (1) that $(\Sigma_1, \Sigma_2)$ is an internal Zappa–Szép structure for $(\Sigma, \jmath, \Pi_{1,2})$. Thus by (I3), $(\Sigma_2, \Sigma_1)$ is also an internal Zappa–Szép structure for $(\Sigma, \jmath, \Pi_{1,2})$. By Theorem 4.3 (2), this implies that $(\Sigma_2, \Sigma_1)$ is a matched pair; it remains to find its factorization rule.

Since the multiplication of $\Sigma$ given at (3.8) is constructed exactly such that

$$\iota^2_{1,2}(f)\iota^1_{1,2}(e_1) = \big(r(f)\,{}_1{*}_2\, f\big)\big(e\,{}_1{*}_2\, s(e)\big) = \Phi(f\,{}_2{*}_1\, e),$$

we can conclude two things: Since the factorization rule $\Phi'\colon \Sigma_1 *_\mathbb{T} \Sigma_2 \to \Sigma_2 *_\mathbb{T} \Sigma_1$ associated to the matched pair $(\Sigma_2, \Sigma_1)$ is determined by sending $\sigma\,{}_1{*}_2\,\mu \in \Sigma_1 *_\mathbb{T} \Sigma_2$ to the unique element $f\,{}_1{*}_2\,e \in \Sigma_2 *_\mathbb{T} \Sigma_1$ for which $\sigma\mu = fe$ in $\Sigma$, it follows from $\sigma\mu = fe = \Phi(f\,{}_2{*}_1\,e)$ that $\Phi'$ must indeed be given by $\Phi^{-1}$. Secondly, the isomorphism from $\Sigma_2 *_\mathbb{T} \Sigma_1$ to $\Sigma = \Sigma_1 \bowtie_\Phi \Sigma_2$ that Theorem 4.3 (2) gives us, is exactly the map $\Phi$. ■

## 5. Zappa–Szép product of 2-cocycles

One natural way of constructing twists over groupoids is via 2-cocycles. Recall that a *(normalized, $\mathbb{T}$-valued) 2-cocycle* on a groupoid $\mathcal{G}$ is a continuous map $c\colon \mathcal{G}^{(2)} \to \mathbb{T}$ such that $c(r(g), g) = 1 = c(g, s(g))$ and such that the *cocycle identity* holds:

$$c(g, hk)c(h, k) = c(gh, k)c(g, h) \quad \text{for all } (g, h, k) \in \mathcal{G}^{(3)}. \tag{5.1}$$

Such a 2-cocycle defines a twist $\Sigma_c$ as follows. As a topological space, $\Sigma_c$ is just $\mathbb{T} \times \mathcal{G}$, and its composition and inversion is defined for $(g, h) \in \mathcal{G}^{(2)}$ and $z, w \in \mathbb{T}$ by

$$(z, g)(w, h) = \big(c(g, h)zw, gh\big) \quad \text{and} \quad (z, g)^{-1} = \big(\overline{z\,c(g, g^{-1})}, g^{-1}\big).$$

If we let $\mathcal{U} = \mathcal{G}^{(0)}$, then the central groupoid extension is given by

$$\mathbb{T} \times \mathcal{U} \xrightarrow{\text{incl}} \Sigma_c \xrightarrow{\text{pr}_2} \mathcal{G}.$$

In this section, we study the Zappa–Szép product of two twisted groupoids arising from 2-cocycles. We start with a matched pair $(\mathcal{G}_1, \mathcal{G}_2)$ of groupoids, each of which carries a continuous 2-cocycle $c_i$.

**Definition 5.1.** We call a continuous map $\varphi\colon \mathscr{G}_2 \,{}_s\!\times_r \mathscr{G}_1 \to \mathbb{T}$ a *cocycle connector for the pair* $(c_1, c_2)$ if the following hold.

(CC1) For all $g \in \mathscr{G}_2$ and $x \in \mathscr{G}_1$, $\varphi(r(x), x) = 1 = \varphi(g, s(g))$.

(CC2) For all $g \in \mathscr{G}_2$ and $(x, y) \in \mathscr{G}_1^{(2)}$ such that $s(g) = r(x)$, we have

$$\varphi(g, xy)c_1(x, y) = \varphi(g, x)\varphi(g \triangleleft x, y)c_1\big(g \triangleright x, (g \triangleleft x) \triangleright y\big).$$

(CC3) For all $(g, h) \in \mathscr{G}_2^{(2)}$ and $x \in \mathscr{G}_1$ such that $s(h) = r(x)$, we have

$$\varphi(gh, x)c_2(g, h) = \varphi(h, x)\varphi(g, h \triangleright x)c_2\big(g \triangleleft (h \triangleright x), h \triangleleft x\big).$$

Definition 5.1 should be compared to [25, Definition 7.12]; a cocycle connector $\varphi$ corresponds in spirit exactly to their map $\varphi_{1,1}$, even if their set-up is rather different from ours.

**Remark 5.2** (Sanity checks). The reader is encouraged to check that the components in Conditions (CC2) and (CC3) make sense. For (CC2), for example, one requires Conditions (ZS8), (ZS2), and (ZS7) of the matched pair $(\mathscr{G}_1, \mathscr{G}_2)$ as listed in Section 2.2.

**Example 5.3.** If $c_1, c_2$ are the trivial 2-cocycles, so constant 1, then the constant map $\varphi(g, x) = 1$ is always a cocycle connector.

**Example 5.4.** Let $\mathscr{G}_1 = \mathscr{G}_2 = \mathbb{Z}$ act trivially on each other, and fix $\theta \in \mathbb{R}$. The map $\varphi\colon \mathscr{G}_2 \times \mathscr{G}_1 \to \mathbb{T}$ given by $(m, n) \mapsto \mathrm{e}^{2\pi i\theta mn}$ is a cocycle connector for the trivial 2-cocycles on $\mathscr{G}_1$ and $\mathscr{G}_2$.

We now prove that matched twists from 2-cocycles come precisely from cocycle connectors.

**Proposition 5.5** (External Zappa–Szép twist for 2-cocycles). *Suppose* $(\mathscr{G}_1, \mathscr{G}_2)$ *is a matched pair of groupoids with continuous 2-cocycles* $c_i$. *Then* $(\Sigma_{c_1}, \Sigma_{c_2})$ *is a matched pair of twists (Definition 3.5) if and only if there exists a cocycle connector for* $(c_1, c_2)$ *(Definition 5.1). Indeed, the factorization rule* $\Phi\colon \Sigma_{c_2} *_{\mathbb{T}} \Sigma_{c_1} \to \Sigma_{c_1} *_{\mathbb{T}} \Sigma_{c_2}$ *that covers* $\mathscr{G}_1 \bowtie \mathscr{G}_2$ *and the cocycle connector* $\varphi\colon \mathscr{G}_2 \,{}_s\!\times_r \mathscr{G}_1 \to \mathbb{T}$ *are related by the formula*

$$\Phi\big((1, g) \,{}_2{*}_1\, (1, x)\big) = (1, g \triangleright x) \,{}_1{*}_2\, \big(\varphi(g, x), g \triangleleft x\big) \tag{5.2}$$

*for all* $(g, x) \in \mathscr{G}_2 \,{}_s\!\times_r \mathscr{G}_1$. *In this case, the map*

$$c\colon (\mathscr{G}_1 \bowtie \mathscr{G}_2)^{(2)} \to \mathbb{T}, \quad c\big((x_1, g_1), (x_2, g_2)\big) = c_1(x_1, g_1 \triangleright x_2)\,\varphi(g_1, x_2)\,c_2(g_1 \triangleleft x_2, g_2),$$

*is a 2-cocycle on* $\mathscr{G}_1 \bowtie \mathscr{G}_2$, *and the twist* $\Sigma_c$ *it induces is canonically isomorphic to the external Zappa–Szép twist* $\Sigma_{c_2} \bowtie_\Phi \Sigma_{c_1}$ *from Theorem 3.18.*

**Remark 5.6.** Example 5.3 shows that cocycle connectors are very much not unique, and that different choices of cocycle connectors can lead to non-isomorphic twists. Consequently, Proposition 5.5 shows that there is likewise no uniqueness of twist factorization rules, even if the twists $\Sigma_1, \Sigma_2$ are trivial and even if the two-way actions of $\mathcal{G}_1$ and $\mathcal{G}_2$ on one another are fixed.

*Proof.* Write $\Sigma_i = \Sigma_{c_i}$, and suppose first that $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ is a factorization rule for the pair $(\Sigma_1, \Sigma_2)$. For any fixed $(g, x) \in \mathcal{G}_2\ {}_s\!\times_r \mathcal{G}_1$, it follows from Equation (3.4) in Remark 3.9 (see also Example 3.12) that there exist $z_1, z_1 \in \mathbb{T}$ such that

$$\Phi\big((1, g)\,{}_2{*}_1\,(1, x)\big) = (z_1, g \triangleright x)\,{}_1{*}_2\,(z_2, g \triangleleft x). \tag{5.3}$$

Since $(z_1, g \triangleright x) = z_1 \cdot (1, g \triangleright x)$, the balancing in $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ allows us to turn Equation (5.2) into Equation (5.3) by defining $\varphi(g, x) := z_1 z_2$. Note that $\varphi$ is well defined: the elements $(1, x)$ and $(1, g \triangleright x)$ of $\mathcal{G}_1$ have no other representatives with 1 in the first component, so that $g, x$ and the product $z_1 z_2$ are uniquely determined.

Observe that, because of Condition (FT0), we can generalize Equation (5.2) to

$$\Phi\big((z, g)\,{}_2{*}_1\,(w, x)\big) = (z, g \triangleright x)\,{}_1{*}_2\,\big(\varphi(g, x)w, g \triangleleft x\big) \tag{5.4}$$

To see that $\varphi$ is continuous, let

$$u := r_{\Sigma_1 \bowtie_\Phi \Sigma_2}\big((1, g \triangleright x)\,{}_1{*}_2\,(1, g \triangleleft x)\big).$$

Using that $\Sigma_1 \bowtie_\Phi \Sigma_2$ is a twist over $\mathcal{G}_1 \bowtie \mathcal{G}_2$, we rewrite Equation (5.3) to

$$\jmath\big(\varphi(g, x), u\big) = \Phi\big((1, g)\,{}_2{*}_1\,(1, x)\big)\big[(1, g \triangleright x)\,{}_1{*}_2\,(1, g \triangleleft x)\big]^{-1}.$$

In other words, if $\mathrm{pr}_{\mathbb{T}}\colon \mathbb{T} \times \mathcal{U} \to \mathbb{T}$ denotes the map $(z, u) \mapsto z$, then

$$\varphi(g, x) = (\mathrm{pr}_{\mathbb{T}} \circ \jmath^{-1})\Big(\Phi\big((1, g)\,{}_2{*}_1\,(1, x)\big)\big[(1, g \triangleright x)\,{}_1{*}_2\,(1, g \triangleleft x)\big]^{-1}\Big).$$

This shows that $\varphi$ is built from an array of continuous maps (multiplication and inversion of $\Sigma_1 \bowtie_\Phi \Sigma_2$ are continuous and $\jmath$ is a homeomorphism onto its image; Proposition 3.14), so that $\varphi$ is itself continuous.

Now take $g \in (\mathcal{G}_2)^u_v$. Then

$$\varphi(u, g) = 1 \iff \Phi\big((1, g)\,{}_2{*}_1\,(1, u)\big) = (1, g \triangleright u)\,{}_1{*}_2\,(1, g \triangleleft u).$$

With (ZS10) and (ZS6), we rewrite the right-hand condition to arrive at:

$$\varphi(u, g) = 1 \iff \Phi\big((1, g)\,{}_2{*}_1\,(1, u)\big) = (1, v)\,{}_1{*}_2\,(1, g). \tag{5.5}$$

The right-hand equality is indeed satisfied by Condition (FT2) for $i = 2$, so the first half of (CC1) holds. The second half follows analogously from Condition (FT2) for $i = 1$.

To see that (CC2) holds, fix $g \in \mathcal{G}_2$ and $(x, y) \in \mathcal{G}_1^{(2)}$ such that $s(g) = r(x)$; if we let $z := c_1(g \triangleright x, [g \triangleleft x] \triangleright y)$ and $w := c_1(x, y)$, then we must verify that $\varphi(g, xy)w = \varphi(g, x)\varphi(g \triangleleft x, y)z$. By Equation (5.4), we have

$$\Phi\big((1, g)\,{}_2{*}_1\,(1, x)\big) = (1, g \triangleright x)\,{}_1{*}_2\,\big(\varphi(g, x), g \triangleleft x\big),$$

so it follows from Condition (FT3) that

$$\Phi\big((1,g)\,{}_2{*}_1\,(1,x)(1,y)\big) = (1, g \triangleright x) \bullet \Phi\big(\big(\varphi(g,x), g \triangleleft x\big)\,{}_2{*}_1\,(1,y)\big). \tag{5.6}$$

We will compute both sides of this equation. For the left-hand side, we use in the first step that $w \in \mathbb{T}$ is such that $(1,x)(1,y) = (w, xy)$ and we use (ZS8) in the last step:

$$\begin{aligned}\Phi\big((1,g)\,{}_2{*}_1\,(1,x)(1,y)\big) &= \Phi\big((1,g)\,{}_2{*}_1\,(w,xy)\big)\\ &= (1, g \triangleright xy)\,{}_1{*}_2\,\big(\varphi(g,xy)w, g \triangleleft xy\big) \quad \text{by (5.2)}\\ &= \big(1, (g \triangleright x)\big([g \triangleleft x] \triangleright y\big)\big)\,{}_1{*}_2\,\big(\varphi(g,xy)w, g \triangleleft xy\big).\end{aligned} \tag{5.7}$$

For right-hand side of Equation (5.6), we first write

$$\begin{aligned}\Phi\big((1, g \triangleleft x)\,{}_2{*}_1\,(1,y)\big) &\overset{(5.4)}{=} \big(1, [g \triangleleft x] \triangleright y\big)\,{}_1{*}_2\,\big(\varphi(g \triangleleft x, y), [g \triangleleft x] \triangleleft y\big)\\ &\overset{(ZS4)}{=} \big(1, [g \triangleleft x] \triangleright y\big)\,{}_1{*}_2\,\big(\varphi(g \triangleleft x, y), g \triangleleft xy\big),\end{aligned}$$

so that

$$\Phi\big(\big(\varphi(g,x), g \triangleleft x\big)\,{}_2{*}_1\,(1,y)\big) = \big(\varphi(g,x), [g \triangleleft x] \triangleright y\big)\,{}_1{*}_2\,\big(\varphi(g \triangleleft x, y), g \triangleleft xy\big). \tag{5.8}$$

The definition of the multiplication in $\Sigma_2 = \Sigma_{c_2}$ and the definition of $z$ tell us that

$$\big(1, g \triangleright x\big)\big(\varphi(g,x), [g \triangleleft x] \triangleright y\big) = \big(z\varphi(g,x), (g \triangleright x)\big([g \triangleleft x] \triangleright y\big)\big).$$

Combining this with Equation (5.8), the right-hand side of Equation (5.6) becomes

$$\begin{aligned}&\big(z\varphi(g,x), (g \triangleright x)\big([g \triangleleft x] \triangleright y\big)\big)\,{}_1{*}_2\,\big(\varphi(g \triangleleft x, y), g \triangleleft xy\big)\\ &\qquad = \big(1, (g \triangleright x)\big([g \triangleleft x] \triangleright y\big)\big)\,{}_1{*}_2\,\big(z\varphi(g,x)\varphi(g \triangleleft x, y), g \triangleleft xy\big).\end{aligned}$$

Comparing this to the left-hand side of Equation (5.6), which we have computed in line (5.7), we conclude that we must have

$$\varphi(g,xy)w = z\varphi(g,x)\varphi(g \triangleleft x, y),$$

which is exactly Condition (CC2). One shows Condition (CC3) analogously, invoking Condition (FT4) in place of (FT3).

Conversely, assume that we have a cocycle connector $\varphi$. We define $\Phi$ by the formula at (5.4); we get Condition (FT0) ($\mathbb{T}$-equivariance) for free. The map

$$\mathcal{G}_2\ {}_s{\times}_r\ \mathcal{G}_1 \to \mathcal{G}_1\ {}_s{\times}_r\ \mathcal{G}_2, \quad (g,x) \mapsto (g \triangleright x, g \triangleleft x),$$

has inverse given by

$$f\colon \mathcal{G}_1\ {}_s{\times}_r\ \mathcal{G}_2 \to \mathcal{G}_2\ {}_s{\times}_r\ \mathcal{G}_1, \quad (y,h) \mapsto \big([h^{-1} \triangleleft y^{-1}]^{-1}, [h^{-1} \triangleright y^{-1}]^{-1}\big).$$

If we let $\mathrm{pr}_{\mathcal{G}_j}$ denote the projection map $\mathcal{G}_1 \bowtie \mathcal{G}_2 \to \mathcal{G}_j$, then it is easy to check that

$$\Phi^{-1}\big((z,y)\,{}_1{*}_2\,(w,h)\big) = \big(z, \mathrm{pr}_{\mathcal{G}_2}\big(f(y,h)\big)\big)\,{}_2{*}_1\,\big(\overline{\varphi\big(f(y,h)\big)}w, \mathrm{pr}_{\mathcal{G}_1}\big(f(y,h)\big)\big).$$

As a concatenation of continuous maps, we see that $\Phi$ and $\Phi^{-1}$ are continuous, so $\Phi$ is a homeomorphism.

Condition (FT1) follows directly from the definition of $\Phi$ from $\varphi$ (Equation (5.4)) and from the Zappa–Szép product conditions on $\mathcal{G}_1 \bowtie \mathcal{G}_2$:

$$r_{\Sigma_2}(z,g) = r_{\mathcal{G}_2}(g) \overset{\text{(ZS2)}}{=} r_{\mathcal{G}_1}(g \triangleright x) = r_{\Sigma_1}(z, g \triangleright x)$$

and

$$s_{\Sigma_1}(w,x) = s_{\mathcal{G}_1}(x) \overset{\text{(ZS5)}}{=} s_{\mathcal{G}_2}(g \triangleleft x) = s_{\Sigma_2}\big(\varphi(g,x)w, g \triangleleft x\big).$$

For (FT2), we compute

$$\begin{aligned}(\Phi \circ \iota^1_{2,1})(w,x) &= \Phi\big(\big(1, r(x)\big)\,{}_2{*}_1\,(w,x)\big)\\ &= \big(1, r(x) \triangleright x\big)\,{}_1{*}_2\,\big(\varphi\big(r(x),x\big)w, r(x) \triangleleft x\big)\\ &\overset{(\dagger)}{=} (1,x)\,{}_1{*}_2\,\big(w, s(x)\big) = (w,x)\,{}_1{*}_2\,\big(1, s(x)\big) = \iota^1_{1,2}(w,x),\end{aligned}$$

where ($\dagger$) follows from (ZS3), (CC1), (ZS11). Likewise

$$(\Phi \circ \iota^2_{2,1})(z,g) = \iota^2_{1,2}(z,g).$$

For (FT3), we must prove that Equation (5.6) holds. We amend the computation we have done in Equation (5.7), where as before $w := c_1(x,y)$ and $z := c_1(g \triangleright x, [g \triangleleft x] \triangleright y)$:

$$\begin{aligned}\Phi\big((1,g)\,{}_2{*}_1\,(1,x)(1,y)\big) &\overset{(5.7)}{=} \big(1, (g \triangleright x)([g \triangleleft x] \triangleright y)\big)\,{}_1{*}_2\,\big(\varphi(g,xy)w, g \triangleleft xy\big)\\ &= (1, g \triangleright x) \bullet \big(\bar{z}, [g \triangleleft x] \triangleright y\big)\,{}_1{*}_2\,\big(\varphi(g,xy)w, g \triangleleft xy\big).\end{aligned}$$

To conclude Condition (FT3), it now suffices to prove

$$\Phi\big(\big(\varphi(g,x), g \triangleleft x\big)\,{}_2{*}_1\,(1,y)\big) = \big(\bar{z}, [g \triangleleft x] \triangleright y\big)\,{}_1{*}_2\,\big(\varphi(g,xy)w, g \triangleleft xy\big).$$

Since $\varphi(g,x)\varphi(g \triangleleft x, y) = \varphi(g,xy)w\bar{z}$ by (CC2) and since $\Phi$ is $\mathbb{T}$-equivariant, this is equivalent to showing

$$\Phi\big((1, g \triangleleft x)\,{}_2{*}_1\,(1,y)\big) = \big(1, [g \triangleleft x] \triangleright y\big)\,{}_1{*}_2\,\big(\varphi(g \triangleleft x, y), g \triangleleft xy\big).$$

But this is exactly the definition of $\Phi$ at (5.4) once one realizes that $g \triangleleft xy = (g \triangleleft x) \triangleleft y$ by (ZS4). Condition (FT4) is shown analogously, using (CC3). This concludes our proof that $\Phi$ is a factorization rule.

Since the continuous map

$$\mathfrak{s}\colon \mathcal{G}_1 \bowtie \mathcal{G}_2 \to \Sigma_1 \bowtie_\Phi \Sigma_2, \quad (x,g) \mapsto (1,x)\,{}_1{*}_2\,(1,g),$$

satisfies $(\Pi_{1,2} \circ \mathfrak{s})(x, g) = (x, g)$, it is a continuous section of the twist $\Sigma_1 \bowtie_\Phi \Sigma_2$, so this means that the function $c' \colon (\mathcal{G}_1 \bowtie \mathcal{G}_2)^{(2)} \to \mathbb{T}$ given for $(\xi_1, \xi_2) \in (\mathcal{G}_1 \bowtie \mathcal{G}_2)^{(2)}$ by

$$c(\xi_1, \xi_2) = \mathfrak{s}(\xi_1)\,\mathfrak{s}(\xi_2)\,\mathfrak{s}(\xi_2\xi_2)^{-1}$$

is a 2-cocycle on $\mathcal{G}_1 \bowtie \mathcal{G}_2$ and that $\Sigma_1 \bowtie_\Phi \Sigma_2 \cong \Sigma_c$ (cf. [17, Fact 4.1], [31, Proposition I.1.14], [34, Remark 11.1.6]). We compute for $\xi_i = (x_i, g_i)$:

$$\begin{aligned}
\mathfrak{s}(\xi_1)\,\mathfrak{s}(\xi_2) &= (1, x_1) \bullet \Phi\big((1, g_1)\,{}_2{*}_1\,(1, x_2)\big) \bullet (1, g_2)\\
&= (1, x_1) \bullet (1, g_1 \triangleright x_2)\,{}_1{*}_2\,\big(\varphi(g_1, x_2), g_1 \triangleleft x_2\big) \bullet (1, g_2) \quad \text{by (5.2)}\\
&= c_1(x_1, g_1 \triangleright x_2)\,\varphi(g_1, x_2)\,c_2(g_1 \triangleleft x_2, g_2)\\
&\quad \cdot \big(1, x_1[g_1 \triangleright x_2]\big)\,{}_1{*}_2\,\big(1, [g_1 \triangleleft x_2]g_2\big)\\
&= c_1(x_1, g_1 \triangleright x_2)\,\varphi(g_1, x_2)\,c_2(g_1 \triangleleft x_2, g_2) \cdot \mathfrak{s}(\xi_1\xi_2),
\end{aligned}$$

which proves that $c$ has the claimed form. ∎

**Example 5.7.** Using Example 5.4, we recover a very famous twisted group: By Proposition 5.5, it follows that

$$c \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}, \quad c(\vec{m}, \vec{n}) = \mathrm{e}^{2\pi i \theta m_2 n_1},$$

is a 2-cocycle on $\mathbb{Z}^2$. It is well known that the twisted group C$^*$-algebra associated to this 2-cocycle is the rotation algebra, $A_\theta$.

In the case of an internal Zappa–Szép twist, the cocycle connector can be recovered from the 2-cocycle directly:

**Proposition 5.8** (Internal Zappa–Szép twist for 2-cocycles). *Suppose $c$ is a 2-cocycle on a groupoid $\mathcal{G}$ such that $(\Sigma_c, \mathcal{G})$ has an internal Zappa–Szép product structure $(\Sigma_1, \Sigma_2)$ as in Definition 4.1; let $\mathcal{G}_i := \mathrm{pr}_2(\Sigma_i) \leq \mathcal{G}$. Then $c_i := c|_{\mathcal{G}_i}$ is a 2-cocycle on $\mathcal{G}_i$, and the map $\varphi \colon \mathcal{G}_2\, {}_s{\times}_r\, \mathcal{G}_1 \to \mathbb{T}$ given by*

$$\varphi(g, x) = c(g, x)\overline{c(g \triangleright x, g \triangleleft x)}$$

*is a cocycle connector for the pair $(c_1, c_2)$.*

If we want to think of $\mathcal{G}_1 \bowtie \mathcal{G}_2$ as an external Zappa–Szép product, then the above displayed equation should be written as

$$\varphi(g, x) = c\big((r(g), g), (x, s(x))\big)\overline{c\big((g \triangleright x, s(g)), (r(x), g \triangleleft x)\big)}.$$

*Proof.* The fact that $c_i$ is a 2-cocycle on $\mathcal{G}_i$ is obvious. The map $\varphi$ is continuous as concatenation of continuous maps; it remains to show the algebraic conditions listed in Definition 5.1.

(CC1) Follows immediately since $c$ is normalized and since $r(x) \triangleleft x = s(x)$ by (ZS11) and $g \triangleright s(g) = r(g)$ by (ZS10).

(CC2) Fix $g \in \mathcal{G}_2$ and $(x, y) \in \mathcal{G}_1^{(2)}$ such that $s(g) = r(x)$. We compute

$$c_1(x, y)\varphi(g, xy) = c(x, y)c(g, xy)\overline{c\big(g \triangleright [xy], g \triangleleft [xy]\big)}$$

which, by the cocycle identity (5.1) (applied to the first two factors) and by (ZS8) (applied to the third factor) equals

$$= \big[c(g, x)c(gx, y)\big]\overline{c\big((g \triangleright x)\big([g \triangleleft x] \triangleright y\big), g \triangleleft [xy]\big)}.$$

We compute the last factor, using the cocycle identity yet again:

$$\begin{aligned} &c\big((g \triangleright x)\big([g \triangleleft x] \triangleright y\big), g \triangleleft [xy]\big) \\ &\quad = c\big(g \triangleright x, \big([g \triangleleft x] \triangleright y\big)\big(g \triangleleft [xy]\big)\big)c\big([g \triangleleft x] \triangleright y, g \triangleleft [xy]\big)\overline{c\big(g \triangleright x, [g \triangleleft x] \triangleright y\big)} \end{aligned}$$

which, since $g \triangleleft [xy] = (g \triangleleft x) \triangleleft y$ and thus $([g \triangleleft x] \triangleright y)(g \triangleleft [xy]) = [g \triangleleft x]y$ by Lemma 2.3, equals

$$= c\big(g \triangleright x, [g \triangleleft x]y\big)c\big([g \triangleleft x] \triangleright y, g \triangleleft [xy]\big)\overline{c\big(g \triangleright x, [g \triangleleft x] \triangleright y\big)}.$$

Again, we take the very first factor and apply the cocycle identity, followed by Lemma 2.3:

$$\begin{aligned} c\big(g \triangleright x, [g \triangleleft x]y\big) &= c\big((g \triangleright x)[g \triangleleft x], y\big)c(g \triangleright x, g \triangleleft x)\overline{c(g \triangleleft x, y)} \\ &= c(gx, y)c(g \triangleright x, g \triangleleft x)\overline{c(g \triangleleft x, y)}. \end{aligned}$$

Combining all of these computations, we arrive at:

$$\begin{aligned} c_1(x, y)\varphi(g, xy) = c(g, x)c(gx, y)&\overline{c(gx, y)c(g \triangleright x, g \triangleleft x)} \\ &\cdot c(g \triangleleft x, y)\overline{c\big([g \triangleleft x] \triangleright y, g \triangleleft [xy]\big)}c\big(g \triangleright x, [g \triangleleft x] \triangleright y\big). \end{aligned}$$

The first row collapses to $\varphi(g, x)$, so that

$$\begin{aligned} c_1(x, y)\varphi(g, xy) &= \varphi(g, x)c(g \triangleleft x, y)\overline{c\big([g \triangleleft x] \triangleright y, g \triangleleft [xy]\big)}c\big(g \triangleright x, [g \triangleleft x] \triangleright y\big) \\ &= \varphi(g, x)\varphi(g \triangleleft x, y)c_1\big(g \triangleright x, (g \triangleleft x) \triangleright y\big), \end{aligned}$$

as needed. One proves (CC3) analogously. ■

## 6. Cartan pairs and C*-blends

Twisted groupoid algebras arise naturally from C*-algebras with Cartan subalgebras due to the work of Renault [33]. There is an abundance of C*-algebras with Cartan subalgebras thanks to the recent work of Xin Li [24], where he proved that every simple classifiable C*-algebra has a Cartan subalgebra. In this section, we will apply our results to twisted groupoids arising from Cartan pairs.

One of our original motivations for this paper was to explore an internal Zappa–Szép product structure of C*-algebras, similar to the internal Zappa–Szép product of groups. The reader should recall the two ingredients (1) and (2) from the introduction for a group $K$ to be the internal Zappa–Szép product of two subgroups $G$, $H$. Given a C*-algebra $A$ with two subalgebras $A_1, A_2$, a good C*-analogue for $A$ being "a product of $A_1$ and $A_2$" is precisely the notion of a C*-*blend* due to Exel [15, Definition 2.1]. Viewed this way, [7, Theorem 13] is not surprising: the full groupoid C*-algebra $C^*(\mathscr{G}_1 \bowtie \mathscr{G}_2)$ of the Zappa–Szép product of a matched pair $(\mathscr{G}_1, \mathscr{G}_2)$ is a C*-blend of the individual full C*-algebras $C^*(\mathscr{G}_1)$ and $C^*(\mathscr{G}_2)$. In our first result of this section, Theorem 6.6, we generalize this result to twisted groupoid C*-algebras, both full and reduced.

In the last part of this section, we are then concerned with a partial converse: Given a C*-blend, what properties are needed to realize the large algebra $A$ as "a Zappa–Szép product" of its two subalgebras $A_1, A_2$? At least if we want the algebras in question to be (reduced twisted) groupoid C*-algebras, then in view of Kumjian–Renault theory, a natural replacement of Condition (2) in this setting is that $D = A_1 \cap A_2$ be a Cartan subalgebra in all of $A_1, A_2, A$. Consequently, in Theorem 6.14, we prove that the twisted Weyl groupoid associated to the Cartan pair $(A, D)$ is exactly the Zappa–Szép product of the twisted Weyl groupoids associated to $(A_1, D)$ and $(A_2, D)$. We shall start with a brief review of twisted groupoid C*-algebras and C*-blends. We refer the reader to [37] for a detailed discussion on groupoid C*-algebras, and to [15] for more information regarding C*-blends.

### 6.1. Twisted groupoid C*-algebras

Suppose $\mathscr{G}$ is a (locally compact Hausdorff) étale groupoid, and let $\Sigma$ be a twist over $\mathscr{G}$ as in Definition 2.1. Then we can give $\Sigma$ a canonical Haar system induced by the system of counting measures on $\mathscr{G}$ and the Lebesgue measure on $\mathbb{T}$. On

$$C_c(\mathscr{G}; \Sigma) := \big\{ f \in C_c(\Sigma) : f(z \cdot e) = z f(g), \text{ for all } z \in \mathbb{T},\ e \in \Sigma \big\},$$

this allows us to define a $*$-algebra structure by the convolution formula

$$f * g(e') = \sum_{\pi(e) \colon s(e') = s(e)} f(e' e^{-1}) g(e)$$

and the involution $f^*(e) = \overline{f(e^{-1})}$.

Aside from the supremum norm $\|\cdot\|_\infty$, there are two other natural choices of norms on $C_c(\mathscr{G}; \Sigma)$: For the *reduced* norm $\|\cdot\|_{\mathrm{r}}$, we first define for $u \in \mathscr{G}^{(0)}$ the representation $\theta_u$ of $C_c(\mathscr{G}; \Sigma)$ on $L^2(\mathscr{G}u; \Sigma u)$ by

$$\big(\theta_u(f)\xi\big)(e') = \sum_{\pi(e) \colon s(e') = s(e)} f(e' e^{-1}) \xi(e),$$

and we then let

$$\|f\|_{\mathrm{r}} := \sup_{u \in \mathscr{G}^{(0)}} \big\|\theta_u(f)\big\|.$$

Completing $C_c(\mathcal{G};\Sigma)$ in $\|\cdot\|_{\mathrm{r}}$ yields the *reduced twisted groupoid* C*-*algebra* $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\Sigma)$. The other norm we consider is the *universal* or *full* norm $\|\cdot\|_{\mathrm{u}}$ on $C_c(\mathcal{G};\Sigma)$:

$$\|f\|_{\mathrm{u}} := \sup\left\{\left\|\theta(f)\right\| : \theta \text{ is a } *\text{-representation of } C_c(\mathcal{G};\Sigma)\right\}.$$

The completion of $C_c(\mathcal{G};\Sigma)$ in $\|\cdot\|_{\mathrm{u}}$ yields the *full twisted groupoid* C*-*algebra* $\mathrm{C}^*(\mathcal{G};\Sigma)$.

Many of our arguments in this section will happen on the level of the dense $*$-subalgebra $C_c(G;\Sigma)$. The following lemma gives a (well-known) description of it. In the following, we write

$$\operatorname{supp}^\circ(f) := \left\{e \in \Sigma : f(e) \neq 0\right\}$$

for the open support of a function $f\colon \Sigma \to \mathbb{C}$.

**Lemma 6.1.** *Suppose $\Sigma$ is a twist over a groupoid $\mathcal{G}$ and suppose that $\mathfrak{U}$ is a base for the topology of $\mathcal{G}$. Then*

$$C_c(\mathcal{G};\Sigma) = \operatorname{span}\left\{f \in C_c(\mathcal{G};\Sigma) : \pi\left(\operatorname{supp}^\circ(f)\right) \subseteq U \text{ for some } U \in \mathfrak{U}\right\}.$$

*In particular, the span of such functions $f$ is dense both in the reduced $C^*_{\mathrm{r}}(\mathcal{G};\Sigma)$ and in the full* $\mathrm{C}^*(\mathcal{G};\Sigma)$ *twisted groupoid* C*-*algebra.*

*Proof.* Our proof is similar to that in [34, Lemma 9.1.3], with changes accounting for the twist. For any $f \in C_c(\mathcal{G};\Sigma)$, we can cover the compact set $\pi(\operatorname{supp}(f))$ with finitely many elements $U_1,\ldots,U_n$ of $\mathfrak{U}$. Let $\{h_j\}_{j=1}^n \subseteq C_c(\mathcal{G})$ be a partition of unity subordinate to $\{U_j\}_{j=1}^n$ and define $f^j(e) := h_j(\pi(e))f(e)$. As a pointwise product of continuous, compactly supported functions, $f^j$ is continuous and compactly supported. Since $\pi(\operatorname{supp}^\circ(f')) \subseteq \operatorname{supp}^\circ(h_j) \subseteq U_j \in \mathfrak{U}$, and since $f$ is $\mathbb{T}$-equivariant and $h_j \circ \pi$ is $\mathbb{T}$-invariant, we conclude $f^j \in C_c(\mathcal{G};\Sigma)$. Lastly, since we picked a partition of unity, we have $\sum_j f_j = f$, which finishes our claim. ■

One reason why the above is so helpful is when it is used in conjunction with the following, which greatly simplifies some computations.

**Lemma 6.2** (see [34, Theorem 11.1.11]). *Suppose $\Sigma$ is a twist over an étale groupoid $\mathcal{G}$. If $f \in C_c(\mathcal{G};\Sigma)$ is such that $\pi(\operatorname{supp}^\circ(f))$ is a bisection, then*

$$\|f\|_\infty = \|f\|_{\mathrm{r}} = \|f\|_{\mathrm{u}}.$$

There is one last tool that we will need later to be able to turn elements of $C_c(\Sigma)$ into elements of $C_c(\mathcal{G};\Sigma)$; we thank Dana P. Williams for pointing us to this lemma which is readily checked.

**Lemma 6.3** (cf. [32, Lemme 3.3]). *Suppose $\Sigma$ is a twist over a groupoid $\mathcal{G}$. For $f \in C_c(\Sigma)$, define $T(f)\colon \Sigma \to \mathbb{C}$ by*

$$T(f)(e) = \int_{\mathbb{T}} \bar{z}\, f(z\cdot e)\,\mathrm{d}z.$$

*Then $T$ is a $\|\cdot\|_\infty$-decreasing, linear, idempotent, surjective map $C_c(\Sigma) \to C_c(\mathcal{G};\Sigma)$.*

### 6.2. C*-blend

We begin with some basic background on C*-blends and prove that the "Zappa–Szép-twisted" C*-algebra of $\Sigma_1 \bowtie_\Phi \Sigma_2 \to \mathcal{G}_1 \bowtie \mathcal{G}_2$ can naturally be realized as a C*-blend of the twisted C*-algebras of the individual twists $\Sigma_j \to \mathcal{G}_j$. In the following, we use the symbol $\odot$ to denote the algebraic tensor product.

**Definition 6.4** ([15, Definition 3.1]). A quintuple $(B_1, \iota_1; B_2, \iota_2; A)$ is a C*-*blend* if

(B1) $A, B_1, B_2$ are C*-algebras;

(B2) each $\iota_j : B_j \to \mathcal{M}(A)$ is a *-homomorphism; and

(B3) the range of the map

$$\iota_1 \odot \iota_2 : B_1 \odot B_2 \to \mathcal{M}(A), \quad b_1 \odot b_2 \mapsto \iota_1(b_1)\iota_2(b_2),$$

is contained and dense in $A$.

If $\iota_1 \odot \iota_2$ is injective, the C*-blend is called a C*-*alloy*. We call the C*-blend *austere* if both maps $\iota_1, \iota_2$ are injective with image contained in $A$. When the maps $\iota_1, \iota_2$ are understood, we may drop them and write the quintuple simply as a triple $(B_1; B_2; A)$.

If $(\mathcal{G}_1, \mathcal{G}_2)$ is a pair of matched $r$-*discrete* groupoids, then it follows from Lemma 2.4 that the subgroupoid $\mathcal{G}_i$ of $\mathcal{G}_1 \bowtie \mathcal{G}_2$ is not only closed but also open; in particular, we have a well-defined map $C_c(\mathcal{G}_i) \to C_c(\mathcal{G}_1 \bowtie \mathcal{G}_2)$ that extends functions by 0 outside of $\mathcal{G}_i$. In the case of étale groupoids, this inclusion can be extended to the associated reduced groupoid C*-algebras, even in the twisted setting:

**Lemma 6.5.** *Suppose $(\mathcal{G}_1, \mathcal{G}_2)$ is a pair of matched* étale *groupoids. Suppose further that $(\Sigma_1, \Sigma_2)$ is a matched pair of twists with factorization rule $\Phi$ that covers $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$; let $\Sigma = \Sigma_1 \bowtie_\Phi \Sigma_2$ be the external Zappa–Szép twist (Theorem 3.18). For $i = 1, 2$, the map*

$$C_c(\mathcal{G}_i; \Sigma_i) \to C_c(\mathcal{G}; \Sigma)$$

*that extends functions by* 0*, extends to an injective $*$-homomorphism*

$$\iota_i^{\mathrm{r}} : C_{\mathrm{r}}^*(\mathcal{G}_i; \Sigma_i) \to C_{\mathrm{r}}^*(\mathcal{G}; \Sigma)$$

*of reduced twisted groupoid* C*-*algebras and to a $*$-homomorphism*

$$\iota_i^{\mathrm{u}} : \mathrm{C}^*(\mathcal{G}_i; \Sigma_i) \to \mathrm{C}^*(\mathcal{G}; \Sigma)$$

*of their full counterparts.*

*Proof.* Lemma 2.4 implies that $\mathcal{G}_i$ is an open subgroupoid of $\mathcal{G}$ since $\mathcal{G}_j$ is $r$-discrete. Furthermore, by Condition (I4), we know that $\Sigma_i = \pi^{-1}(\mathcal{G}_i)$. The claim about reduced C*-algebras now follows from an application of [5, Lemma 2.7]. For the full C*-algebras, the claim follows directly from the universal property of $\mathrm{C}^*(\mathcal{G}_i; \Sigma_i)$ with respect to $*$-representations of $C_c(\mathcal{G}_i; \Sigma_i)$. ∎

**Theorem 6.6.** *Suppose* $(\mathcal{G}_1, \mathcal{G}_2)$ *is a pair of matched* étale *groupoids and that* $(\Sigma_1, \Sigma_2)$ *is a matched pair of twists with factorization rule* $\Phi$ *that covers* $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$. *With* $\iota_1^{\mathrm{r}}, \iota_2^{\mathrm{r}}$ *and* $\iota_1^{\mathrm{u}}, \iota_2^{\mathrm{u}}$ *the maps from Lemma 6.5, the quintuples*

$$\big(\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_1; \Sigma_1), \iota_1^{\mathrm{r}}; \mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_2; \Sigma_2), \iota_2^{\mathrm{r}}; \mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_1 \bowtie \mathcal{G}_2; \Sigma_1 \bowtie_\Phi \Sigma_2)\big)$$

*and*

$$\big(\mathrm{C}^*(\mathcal{G}_1; \Sigma_1), \iota_1^{\mathrm{u}}; \mathrm{C}^*(\mathcal{G}_2; \Sigma_2), \iota_2^{\mathrm{u}}; \mathrm{C}^*(\mathcal{G}_1 \bowtie \mathcal{G}_2; \Sigma_1 \bowtie_\Phi \Sigma_2)\big)$$

*are* $\mathrm{C}^*$*-blends. Moreover, the blend of reduced* $\mathrm{C}^*$*-algebras is austere.*

**Remark 6.7.** Because of Theorem 4.3, we could have likewise said that any *internal* Zappa–Szép structure $(\Sigma_1, \Sigma_2)$ of a twist $\Sigma$ gives rise to a $\mathrm{C}^*$-blend.

**Remark 6.8.** Applying Theorem 6.6 to the situation of trivial twists (see Remark 3.19), we recover [7, Theorem 13]: the quintuple

$$\big(\mathrm{C}^*(\mathcal{G}_1), \iota_1^{\mathrm{u}}; \mathrm{C}^*(\mathcal{G}_2), \iota_2^{\mathrm{u}}; \mathrm{C}^*(\mathcal{G}_1 \bowtie \mathcal{G}_2)\big)$$

is a $\mathrm{C}^*$-blend.

**Remark 6.9.** If the unit space $\mathcal{U}$ is not a point, then the $\mathrm{C}^*$-blend of Theorem 6.6 is not a $\mathrm{C}^*$-alloy. Indeed, take two distinct points $u_1, u_2$ in $\mathcal{U}$. Use Urysohn's lemma to get functions $f_1, f_2 \in C_0(\mathcal{U})$ with $f_i(u_i) = 1$ and $f_i(u_j) = 0$ for $j \neq i$. Then $f_1 f_2 = f_2 f_1$ in $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}; \Sigma)$ even though $f_1 \odot f_2 \neq f_2 \odot f_1$ in $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_1; \Sigma_1) \odot \mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_2; \Sigma_2)$.

**Example 6.10.** One of the motivating examples of a $\mathrm{C}^*$-blend is the crossed product: if $(A, \alpha, \Gamma)$ is a $\mathrm{C}^*$-dynamical system, i.e., $\alpha$ denotes an action of a locally compact group $\Gamma$ on a $\mathrm{C}^*$-algebra $A$, then

$$\big(A; \mathrm{C}^*_{\mathrm{r}}(\Gamma); A \rtimes_{\alpha,r} \Gamma\big) \quad \text{and} \quad \big(A; \mathrm{C}^*(\Gamma); A \rtimes_\alpha \Gamma\big) \tag{6.1}$$

are $\mathrm{C}^*$-blends. See [15, Proposition 3.4] for more details. (Exel proves it for the full $\mathrm{C}^*$-algebras but points out that it likewise holds for the reduced.)

We can, of course, see some subexamples of this appear in our Theorem 6.6. For instance, for a fixed $\theta \in \mathbb{R}$, define the 2-cocycle $c_\theta$ on the group $\mathcal{G} = \mathbb{Z}^2$ by

$$c_\theta\big((m_1, m_2), (n_1, n_2)\big) = \mathrm{e}^{2\pi i \theta m_2 n_1}.$$

Using Examples 5.4 and 5.7, Theorem 6.6 yields that

$$\big(\mathrm{C}^*(\mathbb{Z} \times \{0\}); \mathrm{C}^*(\{0\} \times \mathbb{Z}); \mathrm{C}^*(\mathbb{Z}^2, c_\theta)\big)$$

is a $\mathrm{C}^*$-blend. To put this into the framework of crossed products, we identify $\mathrm{C}^*(\mathbb{Z}^2, c_\theta)$ with the crossed product $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ of $\mathbb{Z}$ acting on $C(\mathbb{T})$ by rotation by $\theta$: the $*$-isomorphism is determined by mapping the generators $\delta_{(1,0)}$ and $\delta_{(0,1)}$ of $\mathrm{C}^*_{\mathrm{r}}(\mathbb{Z}^2, c_\theta) = \mathrm{C}^*(\mathbb{Z}^2, c_\theta)$ to the generators $z\delta_0$ and $\mathrm{const}_1\, \delta_1$ of $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$, respectively. The above $\mathrm{C}^*$-blend then turns into

$$\big(C(\mathbb{T}); \mathrm{C}^*(\mathbb{Z}); C(\mathbb{T}) \rtimes_\theta \mathbb{Z}\big),$$

which is an example of a $\mathrm{C}^*$-blend as in (6.1).

*Proof of Theorem* 6.6. Our proof will follow the ideas in [7, Theorem 13], making the necessary (and quite subtle) adjustments to account for the twist.

Write $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$ and $\Sigma = \Sigma_1 \bowtie_\Phi \Sigma_2$. By definition, the maps $\iota_i^{\mathrm{r}}$ and $\iota_i^{\mathrm{u}}$ coincide when restricted to the subalgebra of compactly supported functions: they extend functions on the subgroupoid $\Sigma_i$ by 0 to functions on $\Sigma$. Let us therefore simply write

$$\iota_i := \iota_i^{\mathrm{r}}|_{C_c(\mathcal{G}_i;\Sigma_i)} = \iota_i^{\mathrm{u}}|_{C_c(\mathcal{G}_i;\Sigma_i)} \colon C_c(\mathcal{G}_i;\Sigma_i) \to C_c(\mathcal{G};\Sigma),$$

independently of whether we regard its codomain $C_c(\mathcal{G};\Sigma)$ as a subalgebra of $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\Sigma)$ or of $\mathrm{C}^*(\mathcal{G};\Sigma)$. We have seen in Lemma 6.5 that the maps $\iota_1^{\mathrm{r}}, \iota_2^{\mathrm{r}}$ are injective with codomain $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\Sigma)$, and that the codomain of $\iota_1^{\mathrm{u}}, \iota_2^{\mathrm{u}}$ is $\mathrm{C}^*(\mathcal{G};\Sigma)$. Thus, it suffices to show that the range of $\iota_1 \odot \iota_2$ is dense in $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\Sigma)$ and in $\mathrm{C}^*(\mathcal{G};\Sigma)$.

By construction, the element $f_1 \circledast f_2 := (\iota_1 \odot \iota_2)(f_1 \odot f_2)$ for $f_i \in C_c(\mathcal{G}_i;\Sigma_i)$ is given by

$$f_1 \circledast f_2\colon \quad e_1 e_2 \mapsto f_1(e_1) f_2(e_2)$$

of $C_c(\mathcal{G};\Sigma)$. We will bootstrap this to more general functions than equivariant ones:

For $n \in \mathbb{N}^\times$, let $C_c(\mathcal{G}_i;\Sigma_i;n)$ be the subspace of $C_c(\Sigma_i)$ consisting of elements $f_i$ for which $f_i(z \cdot e) = z^n f_i(e)$ for all $z \in \mathbb{T}$ and all $e \in \Sigma_i$. Given $f_i \in C_c(\mathcal{G}_i;\Sigma_i;n)$ (for $i = 1, 2$ and the same $n$), consider the continuous and compactly supported function $\Sigma_1 \,{}_s{\times}_r\, \Sigma_2 \to \mathbb{C}$ defined by $(e_1, e_2) \mapsto f_1(e_1) f_2(e_2)$. Since

$$f_1(z \cdot e_1) f_2(\bar{z} \cdot e_2) = z^n f_1(e_1)\, \bar{z}^n f_2(e_2) = f_1(e_1) f_2(e_2),$$

the function factors through a map $\Sigma_1 *_{\mathbb{T}} \Sigma_2 \to \mathbb{C}$; it is continuous and compactly supported since the quotient map $\Sigma_1 \,{}_s{\times}_r\, \Sigma_2 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ is open and continuous. Since the space $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ is homeomorphic to $\Sigma$ by Theorem 4.3 (1), we conclude that the map $f_1 \circledast f_2\colon \Sigma \to \mathbb{T}$ given by $e_1 e_2 \mapsto f_1(e_1) f_2(e_2)$ is an element of $C_c(\Sigma)$. We claim that these elements form a uniformly dense spanning set of the commutative $\mathrm{C}^*$-algebra $C_0(\Sigma)$.

Pick any $e \in \Sigma$. By Theorem 4.3 (1) and Condition (I1), we know that $e = e_1 e_2$ for some $e_i \in \Sigma_i$. Thanks to Lemma 6.3, we may pick $f_i \in C_c(\mathcal{G}_i;\Sigma_i)$ with $f_i(e_i) \neq 0$. Then $f_1 \circledast f_2$ satisfies $(f_1 \circledast f_2)(e) = f_1(e_1) f_2(e_2) \neq 0$, meaning that these functions separate the points of $\Sigma$. By Stone–Weierstrass, they thus generate a uniformly dense $*$-subalgebra of $C_0(\Sigma)$. For $f_i, f_i' \in C_c(\mathcal{G}_i;\Sigma_i)$, we have

$$(f_1 \circledast f_2) \cdot (f_1' \circledast f_2') = (f_1 \cdot f_1') \circledast (f_2 \cdot f_2').$$

As $f_i \cdot f_i' \in C_c(\mathcal{G}_i;\Sigma_i;2)$, we conclude by induction that the set

$$X := \bigcup_{n \in \mathbb{N}^\times} \big\{ f_1 \circledast f_2 : f_i \in C_c(\mathcal{G}_i;\Sigma_i;n) \big\}$$

is closed not just under the involution but also under the (pointwise) multiplication of $C_0(\Sigma)$, so its span is a uniformly dense $*$-subalgebra of $C_0(\Sigma)$. To use span$(X)$ to approximate elements of $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\Sigma)$ in the reduced norm and elements of $\mathrm{C}^*(\mathcal{G};\Sigma)$ in the full norm, we next must argue that it suffices to consider elements supported in preimages of bisections.

Since $\mathcal{G}_i$ is locally compact Hausdorff and étale, its topology has a basis of precompact open bisections. In particular,

$$\mathfrak{U} = \left\{U_1U_2 : \overline{U_i} \subseteq V_i \subseteq \mathcal{G}_i \text{ and } U_i, V_i \text{ are precompact open bisections}\right\} \tag{6.2}$$

is a base for the topology of $\mathcal{G}$. Thus, by Lemma 6.1,

$$\operatorname{span}\left\{f \in C_c(\mathcal{G};\Sigma) : \pi\big(\operatorname{supp}^\circ(f)\big) \subseteq U \text{ for some } U \in \mathfrak{U}\right\}$$

is dense in $C^*_{\mathrm{r}}(\mathcal{G};\Sigma)$ and in $\mathrm{C}^*(\mathcal{G};\Sigma)$. To prove that $\operatorname{ran}(\iota_1 \odot \iota_2)$ is dense in $C^*_{\mathrm{r}}(\mathcal{G};\Sigma)$ and in $\mathrm{C}^*(\mathcal{G};\Sigma)$, it therefore suffices to approximate such dense spanning elements in the reduced, respectively full, norm.

So suppose we are given $f \in C_c(\mathcal{G};\Sigma)$ with $\pi(\operatorname{supp}^\circ(f)) \subseteq U_1U_2$, where $U_i$ is an open bisection of $\mathcal{G}_i$ whose closure is contained in another open bisection $V_i$. Fix $\varepsilon > 0$. If an element $g \in C_c(\mathcal{G};\Sigma)$ satisfies $\pi(\operatorname{supp}^\circ(g)) \subseteq V_1V_2$ and $\|f - g\|_\infty < \varepsilon$, then Lemma 6.2 implies both $\|f - g\|_{\mathrm{r}} < \varepsilon$ and $\|f - g\|_{\mathrm{u}} < \varepsilon$; thus, if we can find such a $g$ in the range of $\iota_1 \odot \iota_2$, we are done.

Since span$(X)$ is uniformly dense in $C_0(\Sigma)$, there exist finitely many $n_j \in \mathbb{N}^\times$ and finite collections $\{\tilde{f}_i^j\}_j \subseteq C_c(\mathcal{G}_i;\Sigma_i;n_j)$ for $i = 1, 2$ such that

$$\Big\| f - \sum_j \tilde{f}_1^j \circledast \tilde{f}_2^j \Big\|_\infty < \varepsilon. \tag{6.3}$$

To construct the element $g \in \operatorname{ran}(\iota_1 \odot \iota_2)$, we will now cut down the functions $\tilde{f}_i^j$ so that they are supported in $\pi^{-1}(V_1V_2)$.

Let $h_i \in C_c(\mathcal{G}_i, [0,1])$ be identically 1 on $U_i$ and zero outside of $V_i$, and consider the pointwise product $f_i^j := (h_i \circ \pi_i) \cdot \tilde{f}_1^j$. By construction, the open support of this continuous function satisfies

$$\operatorname{supp}^\circ(f_i^j) \subseteq \pi^{-1}(V_i) \cap \operatorname{supp}^\circ(\tilde{f}_i^j),$$

and is hence precompact by compactness of $\operatorname{supp}(\tilde{f}_i^j)$. Since $h_i \circ \pi_i$ is $\mathbb{T}$-invariant, we conclude that $f_i^j$ is, like $\tilde{f}_i^j$, an element of $C_c(\mathcal{G}_i;\Sigma_i;n_j)$. Lastly, since $\pi(\operatorname{supp}^\circ(f)) \subseteq U_1U_2$, it is easy to see that Equation (6.3) implies

$$\Big\| f - \sum_j f_1^j \circledast f_2^j \Big\|_\infty < \varepsilon. \tag{6.4}$$

Now that we have adjusted the supports, the next step in our hunt for $g$ in the range of $\iota_1 \odot \iota_2$ is to make the functions $f_i^j$ $\mathbb{T}$-equivariant. To this end, we will use the map $T$ from Lemma 6.3.

Since $f$ is $\mathbb{T}$-equivariant, we have $f = T(f)$, so that for any $e = e_1e_2 \in \Sigma$,

$$\begin{aligned}\Big|\Big(f - \sum_j T(f_1^j) \circledast f_2^j\Big)(e)\Big| &= \Big|T(f)(e) - \sum_j T(f_1^j)(e_1)\, f_2^j(e_2)\Big| \\ &= \Big|\int_{\mathbb{T}} zf(z\cdot e)\,\mathrm{d}z - \sum_j \Big(\int_{\mathbb{T}} zf_1^j(z\cdot e_1)\,\mathrm{d}z\Big) f_2^j(e_2)\Big| \\ &\leq \int_{\mathbb{T}} \Big|f(z\cdot e) - \sum_j f_1^j(z\cdot e_1) f_2^j(e_2)\Big|\,\mathrm{d}z \\ &= \int_{\mathbb{T}} \Big|\Big[f - \sum_j f_1^j \circledast f_2^j\Big](z\cdot e)\Big|\,\mathrm{d}z < \varepsilon \quad \text{by (6.4)},\end{aligned}$$

and a second application of the same trick then shows that we have

$$\Big\|f - \sum_j T(f_1^j) \circledast T(f_2^j)\Big\|_\infty < \varepsilon.$$

As $T(f_i^j) \in C_c(\mathcal{G}_i; \Sigma_i)$, the element

$$g := \sum_j T(f_1^j) \circledast T(f_2^j)$$

is in the range of $\iota_1 \odot \iota_2$. Since

$$\operatorname{supp}^\circ\big(T(f_i^j)\big) \subseteq \mathbb{T}\cdot \operatorname{supp}^\circ(f_i^j) \subseteq \pi^{-1}(V_i),$$

we have $\pi(\operatorname{supp}^\circ(g)) \subseteq V_1V_2$, and so as argued earlier, it follows from $\|f - g\|_\infty < \varepsilon$ that $\|f - g\|_{\mathrm{r}} < \varepsilon$ and $\|f - g\|_{\mathrm{u}} < \varepsilon$ which finishes our proof that the range of $\iota_1 \odot \iota_2$ is dense, both in $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}; \Sigma)$ and in $\mathrm{C}^*(\mathcal{G}; \Sigma)$. ∎

Consider the statement in Theorem 6.6 about reduced twisted groupoid $\mathrm{C}^*$-algebras in the situation where $\mathcal{G}_1 \bowtie \mathcal{G}_2$ is effective. By Corollary 2.5, $\mathcal{G}_1, \mathcal{G}_2$ are effective as well, so by Theorem 5.2 of [33], the three $\mathrm{C}^*$-algebras $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_1; \Sigma_1)$, $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_2; \Sigma_2)$, and $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_1 \bowtie \mathcal{G}_2; \Sigma_1 \bowtie_\Phi \Sigma_2)$ all share the same Cartan subalgebra, $C_0(\mathcal{U})$. Note that this Cartan is exactly the intersection of the two subalgebras $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_1; \Sigma_1)$ and $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_2; \Sigma_2)$. It turns out that this is no coincidence, as we will see in Theorem 6.14.

### 6.3. Cartan subalgebras and Weyl groupoids

Let us start by recalling some basic definitions of Cartan subalgebras and the associated Weyl groupoid and twist.

**Definition 6.11** ([33, Definition 5.1], [29, Definition 2.8]). Let $A$ be a $\mathrm{C}^*$-algebra. A subalgebra $D$ is called a *Cartan subalgebra* if

(1) $D$ is maximal abelian inside $A$;

(2) there exists a faithful conditional expectation from $A$ to $D$; and

(3) the normalizer $N_A(D) = \{n \in A : ndn^*, n^*dn \in D \text{ for all } d \in D\}$ generates $A$ as a $\mathrm{C}^*$-algebra.

By [33, Theorem 5.2] (see also [30, Theorem 1.2] or [18, Corollary 7.6] for the non-separable case), any Cartan pair $(A, D)$ can be realized as $(C_r^*(\mathcal{G}; \Sigma), C_0(\mathcal{U}))$, where $\Sigma \to \mathcal{G}$ is a twisted groupoid with unit space $\mathcal{U}$ and $\mathcal{G}$ is étale and effective. Note that, in particular, the space $\mathcal{U}$ is exactly the Gelfand spectrum $\mathcal{U} = \widehat{D}$. The groupoids $\mathcal{G}$ and $\Sigma$ are called the *Weyl groupoid* and the *Weyl twist*, respectively. We will recall how to construct this Weyl pair $\Sigma \to \mathcal{G}$ from $(A, D)$; one can refer to [33] for more detailed discussions.

Fix $n \in N_A(D)$. Since $D$ contains an approximate identity for $A$ [29, Theorem 2.6] and is closed, $n^*n$ is an element of $D \cong C_0(\widehat{D})$. In particular, for $x \in \widehat{D}$, we may consider $n^*n(x) \in \mathbb{C}$ and define

$$\mathrm{dom}(n) := \big\{x \in \widehat{D} : n^*n(x) > 0\big\}.$$

By [17], there exists a unique homeomorphism $\alpha_n \colon \mathrm{dom}(n) \to \mathrm{dom}(n^*)$ such that for all $d \in D$ and $x \in \mathrm{dom}(n)$,

$$n^*dn(x) = d\big(\alpha_n(x)\big)n^*n(x).$$

Write

$$E := \big\{(n, x) : n \in N_A(D),\ x \in \mathrm{dom}(n)\big\} \tag{6.5}$$

and define the following equivalence relations[2] $\sim$ and $\approx$ on $E$:

$$(n, x) \sim (m, y) \quad \text{if } x = y \text{ and there exist } d, d' \in \widehat{D} \text{ such that } d(x), d'(x) \neq 0 \text{ and } nd = md';$$

$$(n, x) \approx (m, y) \quad \text{if } x = y \text{ and there exist } d, d' \in \widehat{D} \text{ such that } d(x), d'(x) > 0 \text{ and } nd = md'.$$

The Weyl twist $\Sigma$ of $(A, D)$ is defined as the quotient space $E/{\approx}$. We denote the equivalence class of $(n, x) \in E$ under $\approx$ by $[\![n, x]\!]$, and give $\Sigma$ the structure maps

$$[\![m, \alpha_n(x)]\!]\,[\![n, x]\!] = [\![mn, x]\!] \quad \text{and} \quad [\![n, x]\!]^{-1} = [\![n^*, \alpha_n(x)]\!].$$

The basic open sets for the topology are the sets $W_\Sigma(n, U, V)$, indexed by $n \in N_A(D)$ and open sets $U \subseteq \mathbb{T}$ and $V \subseteq \widehat{D}$ and defined by

$$W_\Sigma(n, U, V) = \big\{[\![zn, x]\!] : z \in U,\ x \in V \cap \mathrm{dom}(n)\big\}.$$

Similarly, the Weyl groupoid $\mathcal{G}$ is defined as the quotient space $E/{\sim}$; its elements are denoted $[n, x]$, and its structure maps are given by the same formulas as those for $\Sigma$, replacing double-brackets with regular brackets. We give $\mathcal{G}$ the topology that makes the

[2] See [10, Proposition 2.2], which proves that the definition of $\sim$ given here indeed coincides with that given in [33].

surjective map

$$\Sigma \xrightarrow{\pi} \mathcal{G}, \quad [\![n,x]\!] \mapsto [n,x],$$

continuous; it is automatically open. Note that the source of an element $[\![n,x]\!]$ of $\Sigma$ is given by

$$s\big([\![n,x]\!]\big) = [\![n^*n, x]\!].$$

As $n^*n(x) > 0$, we see that the unit space $\mathcal{U}$ of $\Sigma$ can be identified with $\widehat{D}$. More precisely, for $x \in \widehat{D}$, choose any $f_x \in D$ with $f_x(x) > 0$. Then the map $x \mapsto [\![f_x, x]\!]$ is a homeomorphism $\widehat{D} \to \mathcal{U}$. To realize $\Sigma$ as a twist

$$\mathbb{T} \times \mathcal{U} \xrightarrow{\jmath} \Sigma \xrightarrow{\pi} \mathcal{G}$$

over $\mathcal{G}$, we let $\jmath(z,x) = [\![f,x]\!]$ where $f \in D$ is any element such that $f(x) = z$.

**Remark 6.12.** When describing a topology base for $\Sigma$, we do not actually need the leeway of an open neighborhood $V$ around $x \in \mathrm{dom}(n)$, as long as $n$ is allowed to vary over all of $N_A(D)$. Indeed, for any $f \in D$ and $n \in N_A(D)$, the element $fn$ is also a normalizer. Since $[\![zn,x]\!] = [\![z(nf),x]\!]$ for any $f \in D$ with $f(x) > 0$, any element $f \in C_0(\widehat{D},[0,1])$ with $\mathrm{supp}^\circ(f) = V$ yields

$$W_\Sigma(n,U,V) = \big\{[\![z(nf),x]\!] : z \in U,\ x \in V \cap \mathrm{dom}(n) = \mathrm{dom}(nf)\big\} = W_\Sigma(nf,U,\widehat{D}).$$

We will soon need the following easy corollary of [11, Proposition 4.1].

**Lemma 6.13.** *If $(A,D)$ is a Cartan pair and a set $N \subseteq N_A(D)$ densely spans $A$, then every element of the Weyl twist can be written as $[\![zn,x]\!]$ for some $z \in \mathbb{T}$, $n \in N$, and $x \in \mathrm{dom}(n)$.*

*Proof.* Take any element $e$ of the Weyl twist. It was shown in [6] that the element $\pi(e)$ of the associated Weyl groupoid can be written as $[n,x]$ for some $n \in N$. As $\pi(e) = [n,x] = \pi([\![n,x]\!])$, there exists $z \in \mathbb{T}$ such that $e = z \cdot [\![n,x]\!] = [\![zn,x]\!]$. ∎

The main goal of this section is to prove the following theorem which can be understood as a partial converse to Theorem 6.6.

**Theorem 6.14.** *Suppose that $(A_1;A_2;A)$ is an austere C*-blend. Assume that $D := A_1 \cap A_2$ is a Cartan subalgebra of $A_1, A_2$, and $A$; let $\Sigma_i \to \mathcal{G}_i$ denote the Weyl pair of the Cartan pair $(A_i, D)$, and $\Sigma \to \mathcal{G}$ that of $(A,D)$. Then $(\Sigma_1,\Sigma_2)$ is an internal Zappa–Szép structure for $\Sigma$. Moreover, the map $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ given for $m_j \in N_j$ and $x \in \mathrm{dom}(m_2m_1)$ by*

$$\Phi\big([\![m_2, \alpha_{m_1}(x)]\!]\ {}_2{*}_1\ [\![m_1,x]\!]\big) = [\![n_1, \alpha_{n_2}(x)]\!]\ {}_1{*}_2\ [\![n_2,x]\!] \quad \text{where } m_2m_1g = n_1n_2$$

*for some $n_j \in N_j$ and $g \in D$ with $g(x) > 0$, is the unique factorization rule such that $\Sigma \cong \Sigma_1 \bowtie_\Phi \Sigma_2$ as twists.*

**Notation.** In the following, we will work in the setting of Theorem 6.14. We will let $\mathcal{U} = \widehat{D}$ denote the Gelfand spectrum of $D$. As in (6.5), we let $E$ be the set of representatives for the elements of the Weyl pair

$$\mathbb{T} \times \mathcal{U} \xrightarrow{J} \Sigma \xrightarrow{\pi} \mathcal{G}$$

associated to the specific Cartan pair $(A, D)$ from Theorem 6.14.

If we add a subscript-$j$ ($j = 1, 2$) to any item, then we mean the corresponding item for the Cartan pair $(A_j, D)$. Lastly, we let $N_j = N_{A_j}(D)$ be the normalizers in $A_j$. Note that for all six groupoids that we are considering, the unit space can be identified with $\mathcal{U}$.

Since the C*-blend is assumed to be austere, we can without loss of generality identify $A_j$ with a sub-C*-algebra of $A$. Thus $N_j \subseteq N_A(D)$, and so $E_j \subseteq E$. We will prove that this implies that the Weyl groupoid and twist of $(A_j, D)$ are canonically isomorphic to clopen subgroupoids of the Weyl groupoid $\mathcal{G}$ respectively the Weyl twist $\Sigma$ of $(A, D)$.

**Lemma 6.15.** *In the setting of Theorem 6.14, the inclusion $E_j \subseteq E$ ($j = 1, 2$) factors through injective, continuous, open groupoid homomorphisms $\iota_j^{\Sigma}\colon \Sigma_j \to \Sigma$ and $\iota_j^{\mathcal{G}}\colon \mathcal{G}_j \to \mathcal{G}$, and we have $\iota_j^{\Sigma}(\Sigma_j) = \pi^{-1}(\iota_j^{\mathcal{G}}(\mathcal{G}_j))$.*

*Proof.* From the definition of the equivalence relation that gives rise to $\Sigma$ and $\Sigma_j$, it is clear that two elements of $E_j$ are equivalent in $E$ if and only if they are equivalent in $E_j$. In other words, the inclusion $E_j \subseteq E$ factors through a map

$$\iota := \iota_j^{\Sigma}\colon \Sigma_j \to \Sigma, \quad [\![n, x]\!]_j \mapsto [\![n, x]\!].$$

Since $N_j \subseteq N_A(D)$, we have $[\![n, x]\!]_j \subseteq [\![n, x]\!]$; the map $\iota$ is nevertheless injective. Since the product and inversion of $\Sigma_j$ is exactly the product resp. inversion of $\Sigma$ when restricted to $\Sigma_j$, we conclude that $\iota$ is a homomorphism.

To see continuity, fix a basic open set $W = W_{\Sigma}(n, U, \widehat{D})$ of $\Sigma$ (see Remark 6.12), i.e.,

$$W = \left\{[\![zn, x]\!] : z \in U, x \in \mathrm{dom}(n)\right\}$$

for some fixed $n \in N_A(D)$, $U \subseteq \mathbb{T}$ open. Suppose that $[\![m, x_0]\!]_j \in \iota^{-1}(W)$, so $m \in N_j$ and $x_0 \in \mathrm{dom}(n) \cap \mathrm{dom}(m)$, and there exist $z_0 \in U$ and $f, g \in D$ with

$$f(x_0), g(x_0) > 0 \quad \text{and} \quad (z_0 n) f = mg. \tag{6.6}$$

By replacing $g$ with $gg^*$ and $f$ with $fg^*$, we can without loss of generality assume that $g \geq 0$ everywhere.

Since the function

$$\mathbb{T} \times \mathrm{supp}^{\circ}(f) \to \mathbb{T}, \quad (z, x) \mapsto zz_0 f(x)/\left|f(x)\right|,$$

is continuous, the set

$$\left\{(z, x) \in \mathbb{T} \times \mathrm{supp}^{\circ}(f) : zz_0 f(x)/\left|f(x)\right| \in U\right\} \cap \mathbb{T} \times \left(\mathrm{supp}^{\circ}(g) \cap \mathrm{dom}(n)\right)$$

is open. It furthermore contains $(1, x_0)$, as $f(x_0) > 0$ and $z_0 \in U$; in particular, there exist open neighborhoods $U' \subseteq \mathbb{T}$ of 1 and

$$V' \subseteq \operatorname{supp}^\circ(f) \cap \operatorname{supp}^\circ(g) \cap \operatorname{dom}(n) \subseteq \mathcal{U}$$

of $x_0$ such that, whenever $(z, x) \in U' \times V'$, then $zz_0 f(x)/|f(x)| \in U$. Fix any such $(z, x)$ and let $f' := (\overline{f(x)}/|f(x)|)f$ and $z_1 := zz_0 f(x)/|f(x)|$. By choice of $U' \times V'$, $z_0$ is an element of $U$. We have

$$z_1 n f' = \big(zz_0 f(x)/\big|f(x)\big|\big) n \big(\overline{f(x)}/\big|f(x)\big|\big) f = (zz_0) n f \overset{(6.6)}{=} zmg.$$

and since $g(x), f'(x) > 0$, this implies

$$[\![zm, x]\!] = [\![z_1 n, x]\!].$$

As $z_1 \in U$, this shows that $[\![zm, x]\!] \in W$. In summary, we have shown that

$$\big\{[\![zm, x]\!]_j : z \in U', x \in V' \cap \operatorname{dom}(m)\big\} \subseteq \iota^{-1}(W).$$

The set on the left-hand side is a basic open set of $\Sigma_j$, and since $U'$ is a neighborhood of $1 \in \mathbb{T}$ and $V'$ is one for $x_0 \in \widehat{D}$, this basic open set contains $[\![m, x_0]\!]_j$. Since $[\![m, x_0]\!]_j$ was arbitrary in $\iota^{-1}(W)$, this proves that $\iota^{-1}(W)$ is open in $\Sigma_j$, i.e., $\iota$ is continuous.

To see that $\iota$ is an open map, just note that

$$\iota\big(\big\{[\![zn, x]\!]_j : z \in U, x \in \operatorname{dom}(n)\big\}\big) = \big\{[\![zn, x]\!] : z \in U, x \in \operatorname{dom}(n)\big\},$$

meaning that basic open sets are mapped to basic open sets.

The same proofs go through for the map $[n, x]_j \mapsto [n, x]$ on the level of Weyl groupoids, so this map is likewise an injective, continuous, open groupoid homomorphism. (Alternatively, one can use that the surjective maps from the Weyl twists down to the Weyl groupoids are open maps, and then invoke the above.)

For the last claim, take an arbitrary element $[\![n, x]\!]$ of $\pi^{-1}(\iota^{\mathcal{G}}(\mathcal{G}_j))$. Then there exist $m \in N_j$ and $f, g \in D$ with $f(x) \neq 0 \neq g(x)$ such that $nf = mg$. If we let $m' = g(x)/f(x)m$, then

$$m' \frac{g}{g(x)} = \left(\frac{g(x)}{f(x)} m\right) \frac{g}{g(x)} = m \frac{g}{f(x)} = n \frac{f}{f(x)}. \tag{6.7}$$

Since $N_j$ is closed under scalar multiplication, we have $m' \in N_j$, so that Equation (6.7) implies $[\![n, x]\!] = [\![m', x]\!] \in \iota(\Sigma_j)$ which proves $\pi^{-1}(\iota_j^{\mathcal{G}}(\mathcal{G}_j)) \subseteq \iota_j^{\Sigma}(\Sigma_j)$. The other inclusion is obvious. ∎

Since $\iota^{\mathcal{G}}$ and $\iota^{\Sigma}$ are embeddings of topological groupoids, we will from now on identify $\mathcal{G}_j$ and $\Sigma_j$ with their images in $\mathcal{G}$ and $\Sigma$, respectively. To see that $\mathcal{G}$ is indeed the Zappa–Szép product of its subgroupoids $\mathcal{G}_1$ and $\mathcal{G}_2$, we first check that their intersection is trivial.

**Lemma 6.16.** *In the setting of Theorem 6.14, the intersection of the subgroupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{G}$ is the unit space $\mathcal{U}$.*

*Proof.* Assume that $\gamma \in \mathcal{G}_1 \cap \mathcal{G}_2$, so we may write $\gamma = [n_1, x] = [n_2, x]$ with $n_j \in N_j$ and $x \in \operatorname{dom}(n_1) \cap \operatorname{dom}(n_2)$. The equality says that there exists $f_j \in D$ with $f_j(x) \neq 0$ and $n_1 f_1 = n_2 f_2$. Note that the left-hand side is an element of $A_1$ and the right-hand side is an element of $A_2$, so $n_1 f_1 \in A_1 \cap A_2 = D$, and since $x \in \operatorname{dom}(n_1)$, this element of $D$ satisfies $|(n_1 f_1)(x)|^2 = (n_1^* n_1)(x)|f_1(x)|^2 \neq 0$. To sum up, we have found an element $f_1 \in D$ with $f_1(x) \neq 0$ and such that $n_1 f_1 \in D$ does not vanish at $x$, which means exactly that $\gamma = [n_1, x] \in \mathcal{U}$. ∎

*Proof of Theorem* 6.14. Since the inclusion maps are open by Lemma 6.15, the subgroupoids $\mathcal{G}_j$ of $\mathcal{G}$ are open. Since $\pi^{-1}(\mathcal{G}_j) = \Sigma_j$ by Lemma 6.15, we conclude that $\Sigma_j$ is likewise open. Once we have shown that $(\Sigma_1, \Sigma_2)$ is an internal Zappa–Szép structure for $\Sigma$, then the remaining claims of Theorem 6.14 follow from an application of Theorem 4.3 (2).

Since $D$ is Cartan in $A_j$, we know that $N_j$ densely spans $A_j$. Combining this with Assumption (B3) and a (finicky but easy) $\varepsilon/k$-argument, we conclude that the set

$$N := \{n_1 n_2 : m \in N_j\}$$

densely spans $A$. Since $N_j \subseteq N_A(D)$ and since $N_A(D)$ is closed under multiplication, we conclude that $N \subseteq N_A(D)$. Since $N_j$ is closed under scalar multiplication, it now follows from Lemma 6.13 that any element of $\Sigma$ can be represented as $[\![n_1 n_2, x]\!]$ for some $n_j \in N_j$. Fix one such element.

Since $n_1$ and $n_2$ are both normalizers of $D$, the fact that $[\![n_1 n_2, x]\!] \in \Sigma$ in particular means that

$$x \in \operatorname{dom}(n_1 n_2) = \big\{y \in \operatorname{dom}(n_2) : \alpha_{n_2}(y) \in \operatorname{dom}(n_1)\big\}, \tag{6.8}$$

so we can write

$$[\![n_1 n_2, x]\!] = [\![n_1, \alpha_{n_2}(x)]\!]\,[\![n_2, x]\!].$$

This shows that we can decompose the arbitrary element $[\![n_1 n_2, x]\!]$ of $\Sigma$ into the product of elements $[\![n_1, \alpha_{n_2}(x)]\!] \in \Sigma_1$ and $[\![n_2, x]\!] \in \Sigma_2$, i.e., Condition (I1) holds.

Since $(\Sigma, \jmath, \pi)$ is a short exact sequence, Condition (I2) is equivalent to showing that $\pi(\Sigma_1) \cap \pi(\Sigma_2) \subseteq \mathcal{U}$, which we have done in Lemma 6.16. It remains to show that $\Sigma_i$ is closed in $\Sigma$; we will do so for $i = 1$, as the other proof is analogous. In a nutshell, our proof uses Condition (I2) and the fact that $\jmath(\mathbb{T} \times \mathcal{U}) = \pi^{-1}(\mathcal{U})$ is closed in $\Sigma$.

Assume $\{[\![n_\lambda, x_\lambda]\!]\}$ is a net of elements in $\Sigma_1$ that converges to an element $[\![n_1 n_2, x]\!]$ in $\Sigma$, where $n_i \in N_i$. Then for any fixed neighborhoods $U \subset \mathbb{T}$ of 1 and $W \subset \operatorname{dom}(n_1 n_2)$ of $x$, we must have for large $\lambda$ that

$$[\![n_\lambda, x_\lambda]\!] \in \big\{[\![z n_1 n_2, y]\!] : z \in U, y \in W\big\}. \tag{6.9}$$

In particular, $x_\lambda \to x \in \mathrm{dom}(n_1 n_2)$, so without loss of generality we have $x_\lambda \in W \subset \mathrm{dom}(n_1 n_2)$ for all $\lambda$; in particular by (6.8),

$$x_\lambda, x \in \mathrm{dom}(n_2) \cap \alpha_{n_2}^{-1}\big(\mathrm{dom}(n_1)\big). \tag{6.10}$$

By (6.9), there exist $f_\lambda, g_\lambda \in D$ with $f_\lambda(x_\lambda) > 0$, $g_\lambda(x_\lambda) \in U$ and $n_\lambda f_\lambda = n_1 n_2 g_\lambda$. Choose an open set $V$ around $\alpha_{n_2}(x)$ whose closure is contained entirely in $\mathrm{dom}(n_2^*)$; in particular, $\alpha_{n_2}^{-1}(V)$ is a neighborhood around $x$, so without loss of generality, the entire net $\{x_\lambda\}$ lies in $\alpha_{n_2}^{-1}(V)$. Let $h \in D$ be a $[0,1]$-valued function such that $h|_V \equiv 1$ and $\mathrm{supp}^\circ(h) \subseteq \mathrm{dom}(n_2^*)$. By choice of $f_\lambda$ and $g_\lambda$, we have

$$hn_1^*(n_\lambda f_\lambda) = hn_1^*(n_1 n_2 g_\lambda) = (hn_1^* n_1) n_2 g_\lambda. \tag{6.11}$$

Since $hn_1^* n_1 \in D$ has support in $V \subseteq \mathrm{dom}(n_2^*)$, it follows from [11, Lemma 4.2] that $k := (hn_1^* n_1) \circ \alpha_{n_2}$ is a well-defined element of $D$ and that $(hn_1^* n_1) n_2 = n_2 k$. Since $x_\lambda \in \alpha_{n_2}^{-1}(V \cap \mathrm{dom}(n_1))$, we have

$$h\big(\alpha_{n_2}(x_\lambda)\big) = 1 \tag{6.12}$$

and $(n_1^* n_1)(\alpha_{n_2}(x_\lambda)) > 1$, so that $k(x_\lambda) > 0$. As $n_\lambda \in N_1$, the element $m_\lambda := hn_1^* n_\lambda$ is in $N_1$, and we have

$$\mathrm{dom}(m_\lambda) = \big\{y \in \mathrm{dom}(n_\lambda) : \alpha_{n_\lambda}(y) \in \mathrm{dom}(n_1^*) \cap \alpha_{n_1}\big(\mathrm{supp}^\circ(h)\big)\big\}.$$

Note that $x_\lambda \in \mathrm{dom}(n_\lambda)$ satisfies by Equations (6.10) and (6.12)

$$\alpha_{n_1 n_2}(x_\lambda) = \alpha_{n_1}\big(\alpha_{n_2}(x_\lambda)\big) \in \mathrm{dom}(n_1^*) \cap \alpha_{n_1}\big(\mathrm{supp}^\circ(h)\big).$$

Furthermore, it follows from $n_\lambda f_\lambda = n_1 n_2 g_\lambda$ and $f_\lambda(x_\lambda) \neq 0 \neq g_\lambda(x_\lambda)$ that $\alpha_{n_\lambda}(x_\lambda) = \alpha_{n_1 n_2}(x_\lambda)$. We thus conclude that $x_\lambda \in \mathrm{dom}(m_\lambda)$. By Equation (6.11) and choice of $k$,

$$m_\lambda f_\lambda = hn_1^* n_\lambda f_\lambda = (hn_1^* n_1) n_2 g_\lambda = n_2 k g_\lambda.$$

Since $f_\lambda(x_\lambda) > 0$ and $k(x_\lambda) > 0$ and since $g_\lambda(x_\lambda) \in U$, this shows that

$$\xi_\lambda := [\![m_\lambda, x_\lambda]\!] = \big[\!\big[g_\lambda(x_\lambda) n_2, x_\lambda\big]\!\big] \in \big\{[\![z n_2, y]\!] : z \in U, y \in W\big\}.$$

Since $U$ and $W$ were arbitrary neighborhoods around 1 and $x$, respectively, we have shown that the net $\{\xi_\lambda\}$ converges to $[\![n_2, x]\!]$, an element of $\Sigma_2$. Since we have shown that $\Sigma_2$ is open (see the first paragraph of this proof), we must eventually have that $\xi_\lambda \in \Sigma_2$. Since $\xi_\lambda$ also lies in $\Sigma_1$ by construction, it follows from (I2) that $\xi_\lambda \in \jmath(\mathbb{T} \times \mathcal{U})$. Since $\jmath(\mathbb{T} \times \mathcal{U}) = \pi^{-1}(\mathcal{U})$ is closed in $\Sigma$, we conclude that $[\![n_2, x]\!] = \lim_\lambda \xi_\lambda$ lies in $\jmath(\mathbb{T} \times \mathcal{U})$. This means that $\alpha_{n_2}(x) = x$ and

$$\lim_\lambda [\![n_\lambda, x_\lambda]\!] = [\![n_1 n_2, x]\!] = \big[\!\big[n_1, \alpha_{n_2}(x)\big]\!\big] [\![n_2, x]\!] = [\![n_1, x]\!] \in \Sigma_1,$$

as claimed.

Thus, $(\Sigma_1, \Sigma_2)$ is an internal Zappa–Szép structure for $\Sigma$. It now follows from (I6) that $\mathcal{G} = \mathcal{G}_1 \bowtie \mathcal{G}_2$ and from Theorem 4.3 (2) that the map $\Sigma_1 \bowtie_\Phi \Sigma_2 \to \Sigma$ given by $e\,{}_1{*}_2\, f \mapsto ef$ is an isomorphism of twists for a unique factorization rule $\Phi$. It remains to check that $\Phi$ is as claimed.

By Theorem 4.3 (2), we have $\Phi(f\,{}_2{*}_1\, e) = e'\,{}_1{*}_2\, f'$ whenever $fe = e'f'$ in $\Sigma$ for some $e' \in \Sigma_1$ and $f' \in \Sigma_2$. Now, let us write the elements explicitly, say $f = [\![m_2, \alpha_{m_1}(x)]\!]$, $e = [\![m_1, x]\!]$, $e' = [\![n_1, \alpha_{n_2}(x)]\!]$, and $f' = [\![n_2, x]\!]$ for some $m_j, n_j \in N_j$. Then the equality $fe = e'f'$ means that there exists $g, g' \in D$ such that $g(x), g'(x) > 0$ and $m_2 m_1 g = n_1 n_2 g'$. By construction, $n_2' := n_2 g' \in N_2$ satisfies $f' = [\![n_2', x]\!]$, and so $\Phi$ is as claimed. ∎

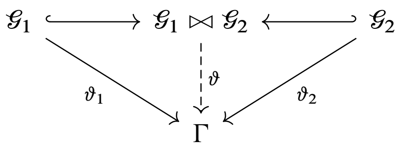
**Corollary 6.17.** *Suppose that* $(A_1; A_2; A)$ *is an austere* C*-*blend and assume that* $D := A_1 \cap A_2$ *is a Cartan subalgebra of* $A$. *Then* $D$ *is Cartan in both* $A_1$ *and* $A_2$ *if and only if there exist conditional expectations* $A \to A_1$ *and* $A \to A_2$.

*Proof.* If there exist conditional expectations $A \to A_j$, then $D$ is Cartan in $A_j$ by [4, Theorem 3.5].

Conversely, suppose $D$ is Cartan in $A_1$ and $A_2$. By Theorem 6.14, the associated Weyl groupoids $\mathcal{G}_1, \mathcal{G}_2$ are subgroupoids of the Weyl groupoid $\mathcal{G}$ of the Cartan pair $(A, D)$, and the Weyl twists $\Sigma_1, \Sigma_2$ are the restrictions $\Sigma|_{\mathcal{G}_1}, \Sigma|_{\mathcal{G}_2}$ of the Weyl twist $\Sigma$ of $(A, D)$. These subgroupoids are clopen by étaleness of $\mathcal{G}$; see Lemma 2.4. By [4, Lemma 3.4], there therefore exist conditional expectations $C^*_r(\mathcal{G}; \Sigma) \to C^*_r(\mathcal{G}_j; \Sigma_j)$ which, by [33, Theorem 5.9], are conditional expectations $A \to A_j$. ∎

**Remark 6.18.** Given a discrete (abelian) group $\Gamma$, we suspect that a $\Gamma$-graded version of Theorem 6.14 can be proved, where Cartan subalgebras are replaced with $\Gamma$-Cartan subalgebras in the sense of [5, Definition 3.9], and groupoids with $\Gamma$-graded groupoids. (See [5, Proposition 5.3. Theorems 4.19 and 6.2] for the correspondence between $\Gamma$-graded twists and $\Gamma$-Cartan pairs of C*-algebras.)

On the groupoid level, it is easy to check that a pair of $\Gamma$-gradings $\vartheta_i : \mathcal{G}_i \to \Gamma$ gives rise to a (necessarily unique) $\Gamma$-grading of $\mathcal{G}_1 \bowtie \mathcal{G}_2$ that makes the diagram




(6.13)

commute, if and only if the pair of gradings satisfies

$$\vartheta_2(g)\vartheta_1(y) = \vartheta_1(g \triangleright y)\vartheta_2(g \triangleleft y) \tag{6.14}$$

in $\Gamma$ for all $(g, y) \in \mathcal{G}_2\, {}_s{\times}_r\, \mathcal{G}_1$. On the level of a '$\Gamma$-graded C*-blend', the grading $\vartheta$ is known to correspond to a $\Gamma$-grading $\{A^\gamma\}_{\gamma \in \Gamma}$ of $A$, but we are unsure what Equation (6.14) translates to in terms of the subalgebras $A_1, A_2$ of $A$. Since the current setting of our theorem is rather technical as-is, we refrained from investigating this idea further.

# A. The Zappa–Szép product of a Fell line bundle and a groupoid

We still owe the reader a proof of the claims in Examples 3.13 and 3.21. Recall that we are in the following setting in both examples: $(\mathcal{G}_1, \mathcal{G}_2)$ is a matched pair of groupoids with unit space $\mathcal{U}$, $\Sigma_2 = \mathbb{T} \times \mathcal{G}_2$ is the trivial twist, and $\Sigma_1$ is a twist over $\mathcal{G}_1$. We let $L_1 = \mathbb{C} \times_{\mathbb{T}} \Sigma_1$ be the associated line bundle.

## A.1. Regarding Example 3.13

We claim that a factorization rule $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ that covers $\mathcal{G}_1 \bowtie \mathcal{G}_2$ contains the same data as a $(\mathcal{G}_1, \mathcal{G}_2)$-compatible $\mathcal{G}_2$-action on $L_1$

$$\llcorner \triangleright \llcorner\colon \mathcal{G}_2 {}_s{*}_\rho L_1 \to L_1, \quad \big(g, [\lambda, e]\big) \mapsto g \triangleright [\lambda, e].$$

So assume first that we are given a factorization rule $\Phi\colon \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$. Given an arbitrary $(g, [\lambda, e]) \in \mathcal{G}_2 {}_s{*}_\rho L_1$, the element $\Phi((1, g)\, {}_2{*}_1\, e)$ of $\Sigma_1 *_{\mathbb{T}} \Sigma_2$ can be written uniquely as $f\, {}_1{*}_2\, (1, h)$ for some $f \in \Sigma_1$ and $h \in \mathcal{G}_2$ thanks to the $\mathbb{T}$-balancing. We may thus let

$$g \triangleright [\lambda, e] := [\lambda, f] \quad \text{where } f \text{ is such that } f\, {}_1{*}_2\, (1, h) = \Phi\big((1, g)\, {}_2{*}_1\, e\big). \tag{App.1}$$

It follows from $\mathbb{T}$-equivariance of $\Phi$ that $\triangleright$ is well defined: given another representative $(\bar{z}\lambda, z \cdot e)$ of $[\lambda, e]$ in $L_1$, we have

$$\Phi\big((1, g)\, {}_2{*}_1\, (z \cdot e)\big) \overset{\text{(FT0)}}{=} z \cdot \big(f\, {}_1{*}_2\, (1, h)\big) = (z \cdot f)\, {}_1{*}_2\, (1, h),$$

so that the equality $[\lambda, f] = [\bar{z}\lambda, z \cdot f]$ in $L_1$ proves that indeed

$$g \triangleright [\bar{z}\lambda, z \cdot e] = g \triangleright [\lambda, e].$$

Continuity of $\Phi$ implies continuity of $\triangleright$.

**Claim 1.** $\triangleright$ *satisfies Conditions* (A1)–(A5) *of* [12, Definition 3.1], *so that it is a* $(\mathcal{G}_1, \mathcal{G}_2)$-*compatible* $\mathcal{G}_2$-*action on the Fell bundle* $L_1$.

*Proof of the claim.* Assume throughout that $\Phi((1, g)\, {}_2{*}_1\, e) = f\, {}_1{*}_2\, (1, h)$. Roughly speaking, the reader can expect the dictionary to be as follows:

- (FT0) makes $\triangleright$ well defined, as we have seen earlier;
- Equation (3.4) yields (A1), i.e., $g \triangleright \llcorner$ is a map from $(L_1)_x$ to $(L_1)_{g \triangleright x}$ (linearity comes for free);
- (FT4) yields (A2), i.e., $g' \triangleright (g \triangleright \llcorner) = g'g \triangleright \llcorner$;
- (FT2) yields (A3), i.e., units act trivially;
- (FT3) yields (A4), i.e., $g$ acts as expected on a product in $L_1$; and lastly
- (FT2) and (FT3) combined yield (A5), i.e., the involution of $g \triangleright [\lambda, e]$ is as expected.

(A1) Take $(g,e) \in \mathcal{G}_2 \,{}_s\!\times_r \Sigma_1$. Because of Equation (3.4), we have for $f \,{}_1\!*_2 (1,h) = \Phi((1,g)\,{}_2\!*_1 e)$ that

$$\pi_1(f)h \overset{(3.4)}{=} g\pi_1(e) \overset{(2.4)}{=} \big(g \triangleright \pi_1(e)\big)\big(g \triangleleft \pi_1(e)\big).$$

Since $\pi_1(f), g \triangleright \pi_1(e) \in \mathcal{G}_1$ and $h, g \triangleleft \pi_1(e) \in \mathcal{G}_2$, the uniqueness of the decomposition of elements in $\mathcal{G}_1 \bowtie \mathcal{G}_2$ implies that

$$\pi_1(f) = g \triangleright \pi_1(e) \quad \text{and} \quad h = g \triangleleft \pi_1(e) \tag{$\clubsuit$}$$

meaning that $g \vDash [\lambda, e]$ is an element of $(L_1)_{g\triangleright\pi_1(e)}$. In other words, $g \vDash \llcorner\!\lrcorner$ is indeed a map $(L_1)_{\pi_1(e)} \to (L_1)_{g\triangleright\pi_1(e)}$. Since $g \vDash \llcorner\!\lrcorner$ only multiplies the $\mathbb{C}$-component by a scalar, it is a linear map, proving Condition (A1).

(A2) Suppose $(g', g) \in \mathcal{G}_2^{(2)}$. Because of Condition (FT4), it follows from the equality $\Phi((1,g)\,{}_2\!*_1 e) = f \,{}_1\!*_2 (1,h)$ that

$$\Phi\big((1,g'g)\,{}_2\!*_1 e\big) = \Phi\big((1,g')\,{}_2\!*_1 f\big) \bullet (1,h).$$

Assume that $f'$ is such that $\Phi((1,g')\,{}_2\!*_1 f) = f' \,{}_1\!*_2 (1,h')$, so that

$$g' \vDash \big(g \vDash [\lambda, e]\big) = g' \vDash [\lambda, f] = [\lambda, f']$$

and

$$\Phi\big((1,g'g)\,{}_2\!*_1 e\big) = f' \,{}_1\!*_2 (1, h'h).$$

The latter implies

$$(g'g) \vDash [\lambda, e] = [\lambda, f'] = g' \vDash \big(g \vDash [\lambda, e]\big),$$

which proves that $g' \vDash (g \vDash \llcorner\!\lrcorner) = (g'g) \vDash \llcorner\!\lrcorner$.

(A3) Assume $g = u \in \mathcal{U}$. Then since $\Phi$ is compatible with the inclusion maps, we have

$$\Phi\big((1,u)\,{}_2\!*_1 u\big) \overset{(FT2)}{=} u \,{}_1\!*_2 (1,u),$$

so that $u \vDash [\lambda, u] = [\lambda, u]$, which proves that $u \vDash \llcorner\!\lrcorner$ is the identity map.

(A4) Suppose that $(e, e') \in \Sigma_2^{(2)}$. Because of Condition (FT3), it follows from

$$\Phi\big((1,g)\,{}_2\!*_1 e\big) = f \,{}_1\!*_2 (1,h)$$

that

$$\Phi\big((1,g)\,{}_2\!*_1 ee'\big) = f \cdot \Phi\big((1,h)\,{}_2\!*_1 e'\big).$$

Assume that $\Phi((1,h)\,{}_2\!*_1 e') = f' \,{}_1\!*_2 (1,h')$, so that

$$\big(g \triangleleft \pi_1(e)\big) \vDash [\lambda', e'] \overset{(\clubsuit)}{=} h \vDash [\lambda', e'] = [\lambda', f']$$

and

$$\Phi\big((1,g)\,{}_2\!*_1 ee'\big) = (ff')\,{}_1\!*_2 (1,h').$$

The latter implies

$$g \blacktriangleright [\lambda\lambda', ee'] = [\lambda\lambda', ff'] = [\lambda, f][\lambda', f'],$$

so that we have shown

$$g \blacktriangleright [\lambda\lambda', ee'] = \big(g \blacktriangleright [\lambda, e]\big)\big(\big(g \triangleleft \pi_1(e)\big) \blacktriangleright [\lambda', e']\big),$$

as needed.

(A5) We have

$$r(g)\,{}_1{*}_2\,(1, g) \overset{\text{(FT2)}}{=} \Phi\big((1, g)\,{}_2{*}_1\, ee^{-1}\big) \overset{\text{(FT3)}}{=} f \bullet \Phi\big((1, h)\,{}_2{*}_1\, e^{-1}\big),$$

or in other words

$$f^{-1}\,{}_1{*}_2\,(1, g) = \Phi\big((1, h)\,{}_2{*}_1\, e^{-1}\big).$$

This explains (†) in the following:

$$\big(g \triangleleft \pi_1(e)\big) \blacktriangleright [\lambda, e]^* \overset{(\clubsuit)}{=} h \blacktriangleright [\overline{\lambda}, e^{-1}] \overset{(\dagger)}{=} [\overline{\lambda}, f^{-1}] = [\lambda, f]^* = \big(g \blacktriangleright [\lambda, e]\big)^*.$$

This concludes the proof of the claim. ∎

Conversely, suppose we are given a $(\mathcal{G}_1, \mathcal{G}_2)$-compatible $\mathcal{G}_2$-action on $L_1$, and let $p\colon {L_1}^\times \to \Sigma_1$ be given by $p([\lambda, e]) = \mathrm{Ph}(\lambda) \cdot e$. Note that, in particular, $p(\mu\xi) = \mathrm{Ph}(\mu) \cdot p(\xi) = p\big(\mathrm{Ph}(\mu)\xi\big)$ for all $\xi \in L_1$ and all $\mu \in \mathbb{C}^\times$, so that linearity of $g \blacktriangleright \text{\textvisiblespace}$ implies for $\xi = [\lambda, e]$ that

$$p(g \blacktriangleright \xi) = p\big(\lambda\big(g \blacktriangleright [1, e]\big)\big) = p\big(\mathrm{Ph}(\lambda)\big(g \blacktriangleright [1, e]\big)\big) = p\big(g \blacktriangleright \big[1, p(\xi)\big]\big). \qquad (\spadesuit)$$

We will furthermore need that

$$g \triangleright \pi_1(e) = g \triangleright q_1\big([\lambda, e]\big) \overset{\text{(A1)}}{=} q_1\big(g \blacktriangleright [\lambda, e]\big) = \pi_1\big(p\big(g \blacktriangleright [\lambda, e]\big)\big). \qquad (\diamondsuit)$$

Define $\Phi_0\colon \Sigma_2\; {}_s{\times}_r\; \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2$ by

$$\Phi_0\big((z, g), e\big) = p\big(g \blacktriangleright [1, e]\big)\,{}_1{*}_2\,\big(z, g \triangleleft \pi_1(e)\big).$$

Note that, since $s_{\mathcal{G}_2}(g) = s_{\Sigma_2}(z, g) = r_{\Sigma_1}(e) = \rho([1, e])$, the element $g \blacktriangleright [1, e]$ of $L_1$ is defined and lies in the fibre over $g \triangleright \pi_1(e)$. In particular,

$$s_{\Sigma_1}\big(p\big(g \blacktriangleright [1, e]\big)\big) = s_{\mathcal{G}_1}\big(g \triangleright \pi_1(e)\big) = r_{\mathcal{G}_2}\big(g \triangleleft \pi_1(e)\big) = r_{\Sigma_2}\big(z, g \triangleleft \pi_1(e)\big),$$

so that $\Phi_0$ indeed takes values in $\Sigma_1 *_{\mathbb{T}} \Sigma_2$. Since $\blacktriangleright$ is linear, (†) in the following holds:

$$w \cdot p\big(g \blacktriangleright [1, e]\big) = p\big(w\big(g \blacktriangleright [1, e]\big)\big) \overset{(\dagger)}{=} p\big(g \blacktriangleright [w, e]\big) = p\big(g \blacktriangleright [1, w \cdot e]\big).$$

This explains (‡) in

$$\begin{aligned}\Phi_0\big((wz, g), e\big) &= p\big(g \blacktriangleright [1, e]\big)\,{}_1{*}_2\,\big(wz, g \triangleleft \pi_1(e)\big)\\ &\overset{(\ddagger)}{=} p\big(g \blacktriangleright [1, w \cdot e]\big)\,{}_1{*}_2\,\big(z, g \triangleleft \pi_1(e)\big) = \Phi_0\big((z, g), w \cdot e\big).\end{aligned}$$

This proves that $\Phi_0$ is constant on $\mathbb{T}$-equivalence classes, i.e., it factors through a map

$$\Phi\colon \quad \Sigma_2 *_{\mathbb{T}} \Sigma_1 \to \Sigma_1 *_{\mathbb{T}} \Sigma_2, \quad (z,g)\,{}_2{*}_1\, e \mapsto p\big(g \vdash [1,e]\big)\,{}_1{*}_2\big(z, g \triangleleft \pi_1(e)\big).$$

Since the quotient map $\Sigma_2\,{}_s{\times}_r\,\Sigma_1 \to \Sigma_2 *_{\mathbb{T}} \Sigma_1$ is open, $\Phi$ is continuous. Moreover, its inverse is given by the unwieldy, but clearly continuous,

$$f\,{}_1{*}_2(w,h) \mapsto \big(w, h \triangleleft \big(h^{-1} \triangleright \pi_1(f^{-1})\big)\big)\,{}_2{*}_1\, p\big(\big(h^{-1} \triangleleft \pi_1(f^{-1})\big) \vdash [1,f]\big).$$

**Claim 2.** *$\Phi$ satisfies Conditions (FT0)–(FT4), so that $(\Sigma_1, \Sigma_2)$ is a matched pair of twists with factorization rule $\Phi$.*

*Proof of the claim.* The dictionary is similar to that in the proof of Claim 1.

(FT0) $\Phi$ is $\mathbb{T}$-equivariant because

$$\begin{aligned} w \cdot \Phi\big((z,g)\,{}_2{*}_1\, e\big) &= w \cdot \big(p\big(g \vdash [1,e]\big)\,{}_1{*}_2\big(z, g \triangleleft \pi_1(e)\big)\big)\\ &= p\big(g \vdash [1, w\cdot e]\big)\,{}_1{*}_2\big(z, g \triangleleft \pi_1(e)\big)\\ &= \Phi\big((z,g)\,{}_2{*}_1\,(w \cdot e)\big). \end{aligned}$$

(FT1) We compute

$$r_{\Sigma_1}\big(p\big(g \vdash [1,e]\big)\big) = (r_{\mathcal{G}_1} \circ \pi_1)\big(p\big(g \vdash [1,e]\big)\big) \overset{(\diamond)}{=} r_{\mathcal{G}_1}\big(g \triangleright \pi_1(e)\big) \overset{\text{(ZS2)}}{=} r(g) = r(z,g)$$

and

$$s_{\Sigma_2}\big(z, g \triangleleft \pi_1(e)\big) = s_{\mathcal{G}_2}\big(g \triangleleft \pi_1(e)\big) \overset{\text{(ZS5)}}{=} s_{\mathcal{G}_1}\big(\pi_1(e)\big) = s_{\Sigma_1}(e).$$

(FT2) For $v := s(g) = r(e)$, we have $g \vdash [1,v] = [1, r(g)]$ by (A1) and by linearity, so that

$$\begin{aligned} \Phi\big((z,g)\,{}_2{*}_1\, v\big) &= p\big(g \vdash [1,v]\big)\,{}_1{*}_2\big(z, g \triangleleft \pi_1(v)\big)\\ &= p\big([1, r(g)]\big)\,{}_1{*}_2(z, g \triangleleft v) = r(g)\,{}_1{*}_2(z,g), \end{aligned}$$

proving that $\iota^2_{1,2} = \Phi \circ \iota^2_{2,1}$. Likewise, $v \triangleleft \pi_1(e) = s(e)$ by (ZS11) and $v \vdash [1,e] = [1,e]$ by (A3), so that

$$\begin{aligned} \Phi\big((1,v)\,{}_2{*}_1\, e\big) &= p\big(v \vdash [1,e]\big)\,{}_1{*}_2\big(z, v \triangleleft \pi_1(e)\big)\\ &= p\big([1,e]\big)\,{}_1{*}_2\big(z, s(e)\big) = e\,{}_1{*}_2\big(z, s(e)\big), \end{aligned}$$

proving that $\iota^1_{1,2} = \Phi \circ \iota^1_{2,1}$.

(FT3) We have

$$\Phi\big(\big(z, g \triangleleft \pi_1(e)\big)\,{}_2{*}_1\, e'\big) = p\big(\big(g \triangleleft \pi_1(e)\big) \vdash [1,e']\big)\,{}_1{*}_2\big(z, \big(g \triangleleft \pi_1(e)\big) \triangleleft \pi_1(e')\big).$$

Since (A4) says that

$$\big(g \vdash [1,e]\big)\big(\big(g \triangleleft \pi_1(e)\big) \vdash [1,e']\big) = g \vdash [1, ee'],$$

we conclude

$$\begin{aligned} p\big(g \vdash [1,e]\big) \bullet \Phi\big((z, g \triangleleft \pi_1(e))\,{}_2{*}_1\, e'\big) &= p\big(g \vdash [1, ee']\big)\,{}_1{*}_2\,\big(z, g \triangleleft \pi_1(ee')\big) \\ &= \Phi\big((z,g)\,{}_2{*}_1\, ee'\big), \end{aligned}$$

as needed.

(FT4) We have

$$\begin{aligned} \Phi\big((w,g')(z,g)\,{}_2{*}_1\, e\big) &= \Phi\big((wz, g'g)\,{}_2{*}_1\, e\big) \\ &= p\big((g'g) \vdash [1,e]\big)\,{}_1{*}_2\,\big(wz, (g'g) \triangleleft \pi_1(e)\big). \end{aligned}$$

On the other hand, since $\pi_1(p(g \vdash [1,e])) = g \triangleright \pi_1(e)$ by Equation (◇), we get

$$\Phi\big((w,g')\,{}_2{*}_1\, p(g \vdash [1,e])\big) = p\big(g' \vdash \big[1, p\big(g \vdash [1,e]\big)\big]\big)\,{}_1{*}_2\,\big(w, g' \triangleleft \big(g \triangleright \pi_1(e)\big)\big)$$

which by Equation (♠) equals

$$= p\big(g' \vdash \big(g \vdash [1,e]\big)\big)\,{}_1{*}_2\,\big(w, g' \triangleleft \big(g \triangleright \pi_1(e)\big)\big).$$

Thus, with (ZS9),

$$\begin{aligned} &\Phi\big((w,g')\,{}_2{*}_1\, p\big(g \vdash [1,e]\big)\big) \bullet \big(z, g \triangleright \pi_1(e)\big) \\ &\quad = p\big((g'g) \vdash [1,e]\big)\,{}_1{*}_2\,\big(wz, (g'g) \triangleleft \pi_1(e)\big) \\ &\quad = \Phi\big((w,g')(z,g)\,{}_2{*}_1\, e\big), \end{aligned}$$

as needed. ■

This finishes the proof of the claims made in Example 3.13.

## A.2. Regarding Example 3.21

We next consider the Fell line bundle $L := \mathbb{C} \times_{\mathbb{T}} (\Sigma_1 \bowtie_\Phi \Sigma_2)$ associated to the Zappa–Szép twist $\Sigma_1 \bowtie_\Phi \Sigma_2$, and the Zappa–Szép product $L_1 \bowtie \mathcal{G}_2$ of the Fell bundle $L_1$ and the groupoid $\mathcal{G}_2$. We claim that the map $\Omega$ given in Equation (3.13) exists and is the claimed isomorphism.

To construct $\Omega$, consider the map

$$\Omega_0\colon \quad \mathbb{C} \times (\Sigma_1 \,{}_s{\times}_r\, \Sigma_2) \to L_1 \bowtie \mathcal{G}_2, \quad \big(\lambda, e, (z,g)\big) \mapsto \big([\lambda z, e], g\big),$$

which clearly satisfies for $w \in \mathbb{T}$,

$$\Omega_0\big(\lambda, e, (wz, g)\big) = \Omega_0\big(w\lambda, e, (z,g)\big) = \Omega_0\big(\lambda, w \cdot e, (z,g)\big),$$

so it factors through a map

$$\Omega\colon \quad L = \mathbb{C} \times_{\mathbb{T}} (\Sigma_1 \bowtie_\Phi \Sigma_2) \to L_1 \bowtie \mathcal{G}_2, \quad \big[\lambda, e\,{}_1{*}_2\,(z,g)\big] \mapsto \big([\lambda z, e], g\big).$$

An application of [13, Propositions A.7 and A.8] shows that $\Omega$ is an isomorphism of upper semi-continuous Banach bundles. To see that $\Omega$ is multiplicative, let $\xi_i = [\lambda_i, e_i\,{}_1{*}_2\,(z_i, g_i)] \in L$ be such that $s_{\mathcal{G}_2}(g_1) = r_{\Sigma_1}(e_2)$, so that

$$\xi_1\xi_2 = \big[\lambda_1\lambda_2, e_1 \bullet \Phi\big((z_1, g_1)\,{}_2{*}_1\,e_2\big) \bullet (z_2, g_2)\big].$$

If $f \in \Sigma_1$ is such that $f\,{}_1{*}_2\,(1, g_1 \triangleleft \pi_1(e_2)) = \Phi((1, g_1)\,{}_2{*}_1\,e_2)$, then the definition of $\vDash$ at (App.1) gives

$$[z_1\lambda_1, e_1]\big(g_1 \vDash [z_2\lambda_2, e_2]\big) \overset{(\text{App.1})}{=} [z_1\lambda_1, e_1][z_2\lambda_2, f] = [z_1z_2\lambda_1\lambda_2, e_1 f]. \tag{App.2}$$

Furthermore,

$$\begin{aligned} e_1 \bullet \Phi\big((z_1, g_1)\,{}_2{*}_1\,e_2\big) \bullet (z_2, g_2) &= (e_1 f)\,{}_1{*}_2\,\big((z_1, g_1 \triangleleft \pi_1(e_2))(z_2, g_2)\big) \\ &= (e_1 f)\,{}_1{*}_2\,\big(z_1z_2, \big(g_1 \triangleleft \pi_1(e_2)\big)g_2\big), \end{aligned}$$

so that

$$\begin{aligned} \Omega(\xi_1\xi_2) &= \Omega\big(\big[\lambda_1\lambda_2, (e_1 f)\,{}_1{*}_2\,\big(z_1z_2, \big(g_1 \triangleleft \pi_1(e_2)\big)g_2\big)\big]\big) \\ &= \big([z_1z_2\lambda_1\lambda_2, e_1 f], \big(g_1 \triangleleft \pi_1(e_2)\big)g_2\big) \\ &= \big([z_1\lambda_1, e_1]\big(g_1 \vDash [z_2\lambda_2, e_2]\big), \big(g_1 \triangleleft \pi_1(e_2)\big)g_2\big) \quad \text{by Eq. (App.2)} \\ &= \big([z_1\lambda_1, e_1], g_1\big)\big([z_2\lambda_2, e_2], g_2\big) \quad \text{by definition of } L_1 \bowtie \mathcal{G}_2 \text{ (Eq. (3.12))} \\ &= \Omega(\xi_1)\Omega(\xi_2). \end{aligned}$$

To see that $\Omega$ is $*$-preserving, we compute

$$\big[\lambda, e\,{}_1{*}_2\,(z, g)\big]^* = \big[\overline{\lambda}, \big(e\,{}_1{*}_2\,(z, g)\big)^{-1}\big] = \big[\overline{\lambda}, \Phi\big((z, g)^{-1}\,{}_2{*}_1\,e^{-1}\big)\big].$$

Thus, if $h \in \Sigma_1$ is such that $h\,{}_1{*}_2\,(1, g^{-1} \triangleleft \pi_1(e^{-1})) = \Phi((1, g^{-1})\,{}_2{*}_1\,e^{-1})$, then again by the definition of $\vDash$ at (App.1), we have

$$g^{-1} \vDash [z\lambda, e]^* = g^{-1} \vDash [\overline{z\lambda}, e^{-1}] = [\overline{z\lambda}, h]. \tag{App.3}$$

Lastly,

$$\begin{aligned} \Omega\big(\big[\lambda, e\,{}_1{*}_2\,(z, g)\big]^*\big) &= \Omega\big(\big[\overline{\lambda}, h\,{}_1{*}_2\,\big(\bar{z}, g^{-1} \triangleleft \pi_1(e^{-1})\big)\big]\big) \\ &= \big([\overline{z\lambda}, h], g^{-1} \triangleleft \pi_1(e^{-1})\big) \\ &= \big(g^{-1} \vDash [z\lambda, e]^*, g^{-1} \triangleleft \pi_1(e)^{-1}\big) \quad \text{by Eq. (App.3)} \\ &= \big([z\lambda, e], g\big)^* = \Omega\big(\big[\lambda, e\,{}_1{*}_2\,(z, g)\big]\big)^*. \end{aligned}$$

This concludes our proof that $\Omega\colon L \to \mathcal{B}$ is an isomorphism of Fell bundles.

**Acknowledgments.** Anna Duwenig thanks Jonathan Taylor and Ying-Fen Lin for helpful discussions. The authors would further like to thank an anonymous referee for their comments.

**Funding.** Anna Duwenig was supported by an FWO Senior Postdoctoral Fellowship (project number 1206124N). Boyu Li was supported by an NSF grant (DMS-2350543).

## References


[1] Z. Afsar, N. Brownlowe, J. Ramagge, and M. F. Whittaker, C*-algebras of self-similar actions of groupoids on higher-rank graphs and their equilibrium states. [v1] 2019, [v2] 2024, arXiv:1910.02472v2

[2] M. Aguiar and N. Andruskiewitsch, Representations of matched pairs of groupoids and applications to weak Hopf algebras. In *Algebraic structures and their representations*, pp. 127–173, Contemp. Math. 376, American Mathematical Society, Providence, RI, 2005 Zbl 1100.16032 MR 2147019

[3] E. Bédos, S. Kaliszewski, J. Quigg, and J. Spielberg, On finitely aligned left cancellative small categories, Zappa-Szép products and Exel-Pardo algebras. *Theory Appl. Categ.* **33** (2018), article no. 42, 1346–1406 Zbl 1415.46036 MR 3909245

[4] J. H. Brown, R. Exel, A. H. Fuller, D. R. Pitts, and S. A. Reznikoff, Intermediate $C^*$-algebras of Cartan embeddings. *Proc. Amer. Math. Soc. Ser. B* **8** (2021), 27–41 Zbl 1471.46054 MR 4199728

[5] J. H. Brown, A. H. Fuller, D. R. Pitts, and S. A. Reznikof, Graded $C^*$-algebras and twisted groupoid $C^*$-algebras. *New York J. Math.* **27** (2021), 205–252 Zbl 1469.46062 MR 4209533

[6] J. H. Brown and E. Gillaspy, Corrigendum to "Cartan subalgebras for non-principal twisted groupoid $C^*$-algebras" [J. Funct. Anal. 279 (6) (2020), article no. 108611]. *J. Funct. Anal.* **282** (2022), no. 6, article no. 109354 Zbl 1477.46056 MR 4364972

[7] N. Brownlowe, D. Pask, J. Ramagge, D. Robertson, and M. F. Whittaker, Zappa-Szép product groupoids and $C^*$-blends. *Semigroup Forum* **94** (2017), no. 3, 500–519 Zbl 1431.22002 MR 3648980

[8] N. Brownlowe, J. Ramagge, D. Robertson, and M. F. Whittaker, Zappa-Szép products of semigroups and their $C^*$-algebras. *J. Funct. Anal.* **266** (2014), no. 6, 3937–3967 Zbl 1286.22002 MR 3165249

[9] K. Courtney, A. Duwenig, M. C. Georgescu, A. an Huef, and M. G. Viola, Alexandrov groupoids and the nuclear dimension of twisted groupoid $C^*$-algebras. *J. Funct. Anal.* **286** (2024), no. 9, article no. 110372 Zbl 1552.46028 MR 4708666

[10] A. Duwenig, E. Gillaspy, and R. Norton, Analyzing the Weyl construction for dynamical Cartan subalgebras. *Int. Math. Res. Not. IMRN* **2022** (2022), no. 20, 15721–15755 Zbl 1509.46032 MR 4498163

[11] A. Duwenig, E. Gillaspy, R. Norton, S. Reznikoff, and S. Wright, Cartan subalgebras for non-principal twisted groupoid $C^*$-algebras. *J. Funct. Anal.* **279** (2020), no. 6, article no. 108611 Zbl 1459.46049 MR 4096726

[12] A. Duwenig and B. Li, The Zappa-Szép product of a Fell bundle and a groupoid. *J. Funct. Anal.* **282** (2022), no. 1, article no. 109268 Zbl 1491.46065 MR 4323510

[13] A. Duwenig and B. Li, Equivalence of Fell bundles is an equivalence relation. *Münster J. Math.* **16** (2023), no. 1, 95–145 Zbl 1521.46029 MR 4563261

[14] A. Duwenig and B. Li, Imprimitivity theorems and self-similar actions on Fell bundles. *J. Funct. Anal.* **288** (2025), no. 2, article no. 110699 Zbl 1562.46052 MR 4811065

[15] R. Exel, Blends and alloys. *C. R. Math. Acad. Sci. Soc. R. Can.* **35** (2013), no. 3, 77–113 Zbl 1303.46041 MR 3136103

[16] R. Exel and E. Pardo, Self-similar graphs, a unified treatment of Katsura and Nekrashevych C*-algebras. *Adv. Math.* **306** (2017), 1046–1129 Zbl 1390.46050 MR 3581326

[17] A. Kumjian, On $C^*$-diagonals. *Canad. J. Math.* **38** (1986), no. 4, 969–1008 Zbl 0627.46071 MR 0854149

[18] B. K. Kwaśniewski and R. Meyer, Noncommutative Cartan C*-subalgebras. *Trans. Amer. Math. Soc.* **373** (2020), no. 12, 8697–8724 Zbl 1473.46075 MR 4177273

[19] M. Laca, I. Raeburn, J. Ramagge, and M. F. Whittaker, Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions. *J. Funct. Anal.* **266** (2014), no. 11, 6619–6661 Zbl 1305.46059 MR 3192463

[20] M. Laca, I. Raeburn, J. Ramagge, and M. F. Whittaker, Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs. *Adv. Math.* **331** (2018), 268–325 Zbl 1392.37007 MR 3804678

[21] B. Li and D. Yang, Zappa-Szép actions of groups on product systems. *J. Operator Theory* **88** (2022), no. 2, 247–274 Zbl 1563.46098 MR 4534897

[22] H. Li and D. Yang, KMS states of self-similar $k$-graph C*-algebras. *J. Funct. Anal.* **276** (2019), no. 12, 3795–3831 Zbl 1421.46041 MR 3957999

[23] H. Li and D. Yang, Self-similar $k$-graph C*-algebras. *Int. Math. Res. Not. IMRN* **2021** (2021), no. 15, 11270–11305 Zbl 1496.46052 MR 4294118

[24] X. Li, Every classifiable simple C*-algebra has a Cartan subalgebra. *Invent. Math.* **219** (2020), no. 2, 653–699 Zbl 1450.46047 MR 4054809

[25] A. Mundey and A. Sims, Homology and twisted $C^*$-algebras for self-similar actions and Zappa-Szép products. *Results Math.* **80** (2025), no. 1, article no. 9 Zbl 07961623 MR 4836749

[26] J. R. Munkres, *Topology*. Prentice Hall, Upper Saddle River, NJ, 2000 Zbl 0951.54001 MR 3728284

[27] V. Nekrashevych, $C^*$-algebras and self-similar groups. *J. Reine Angew. Math.* **630** (2009), 59–123 Zbl 1175.46048 MR 2526786

[28] E. Ortega and E. Pardo, Zappa-Szép products for partial actions of groupoids on left cancellative small categories. *J. Noncommut. Geom.* **17** (2023), no. 4, 1335–1366 Zbl 1537.46058 MR 4653787

[29] D. R. Pitts, Normalizers and approximate units for inclusions of $C^*$-algebras. *Indiana Univ. Math. J.* **72** (2023), no. 5, 1849–1866 Zbl 1541.46035 MR 4671888

[30] A. I. Raad, A generalization of Renault's theorem for Cartan subalgebras. *Proc. Amer. Math. Soc.* **150** (2022), no. 11, 4801–4809 Zbl 1506.46044 MR 4489313

[31] J. Renault, *A groupoid approach to $C^*$-algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl 0433.46049 MR 0584266

[32] J. Renault, Représentation des produits croisés d'algèbres de groupoïdes. *J. Operator Theory* **18** (1987), no. 1, 67–97 Zbl 0659.46058 MR 0912813

[33] J. Renault, Cartan subalgebras in $C^*$-algebras. *Irish Math. Soc. Bull.* (2008), no. 61, 29–63 Zbl 1175.46050 MR 2460017

[34] A. Sims, G. Szabó, and D. Williams, *Operator algebras and dynamics: groupoids, crossed products, and Rokhlin dimension*. Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2020 Zbl 1444.46001 MR 4321941

[35] C. Starling, Boundary quotients of C*-algebras of right LCM semigroups. *J. Funct. Anal.* **268** (2015), no. 11, 3326–3356 Zbl 1343.46055 MR 3336727

[36] J. Szép, On the structure of groups which can be represented as the product of two subgroups. *Acta Sci. Math. (Szeged)* **12** (1950), 57–61 Zbl 0038.01601 MR 0037296

[37] D. P. Williams, *A tool kit for groupoid $C^*$-algebras*. Math. Surveys Monogr. 241, American Mathematical Society, Providence, RI, 2019 Zbl 1456.46002 MR 3969970

Communicated by Wilhelm Winter

Received 25 June 2024; revised 28 April 2025.

**Anna Duwenig**
School of Mathematics and Statistics, UNSW Sydney, High Street, Kensington, Sydney, NSW 2052, Australia; a.duwenig@unsw.edu.au
Author IDs: zbMATH duwenig.anna MR 1378284 ORCID 0000-0001-6042-2561

**Boyu Li**
Department of Mathematical Sciences, New Mexico State University, 1780 E University Ave, Las Cruces, NM 88003, USA; boyuli@nmsu.edu
Author IDs: zbMATH li.boyu MR 1169918 ORCID 0000-0003-2484-1851