J. Eur. Math. Soc. 28, 3385–3451 (2026) DOI 10.4171/JEMS/1758

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Geometric Sen theory over rigid analytic spaces

Received August 30, 2022; revised July 11, 2025

Abstract. In this work we develop geometric Sen theory for rigid analytic spaces, generalizing the previous work of Pan for curves. We also extend the axiomatic Sen–Tate formalism of Berger–Colmez to a certain class of locally analytic representations.

Keywords: locally analytic representations, condensed mathematics, Sen theory, p-adic Simpson correspondence.

1. Introduction

Motivation

Let p be a prime number, $\mathrm{Gal}_{\mathbb{Q}_p}$ the absolute Galois group of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. In [17], Lue Pan has introduced a new and powerful tool to compute proétale cohomology of $\widehat{\mathcal{O}}$-modules over rigid analytic spaces based on the axiomatic framework of Sen theory of Berger–Colmez [1, 2]. The objective of this paper is to generalize Pan's method from curves to log smooth adic spaces; our motivation to develop such a theory is to study the Hodge–Tate structure of completed cohomology of Shimura varieties, which is carried out in the author's work [20].

Before discussing the technical aspects of the paper, let us give a summary of the main ideas behind the construction of the geometric Sen operators. The departure point is the classical method of Sen [28–30] which consists in extracting some linear datum from a continuous $\mathbb{C}_p$-linear representation V of $\mathrm{Gal}_{\mathbb{Q}_p}$, called the *Sen module* of V and usually denoted by $\mathbf{D}_{\mathrm{Sen}}(V)$, which carries the same information as V. Explicitly, the construction of Sen goes as follows. Let $\mathbb{Q}_p(\zeta_{p^\infty})$ be the p-cyclotomic algebraic extension of $\mathbb{Q}_p$ obtained by adding all the p-power roots of unity, and let $\mathbb{Q}_p^{\mathrm{cyc}}$ be its p-adic completion. The Galois group of $\mathbb{Q}_p(\zeta_{p^\infty})$ over $\mathbb{Q}_p$ is naturally isomorphic to $\Gamma = \mathbb{Z}_p^\times$ via the cyclotomic character, and we have a short exact sequence of profinite groups

$$1 \to H \to \mathrm{Gal}_{\mathbb{Q}_p} \to \Gamma \to 1.$$

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Mathematics Subject Classification 2020: 14G22 (primary); 14G45, 26E30 (secondary).

Then, given V a finite-dimensional continuous $\mathbb{C}_p$-semilinear representation of $\mathrm{Gal}_{\mathbb{Q}_p}$, the Sen module $\mathbf{D}_{\mathrm{Sen}}(V)$ is the subspace of V consisting of those vectors v which are H-invariant, and for which the Γ-orbit $\Gamma v \subset V$ is a finite-dimensional $\mathbb{Q}_p$-vector space.

The construction of the Sen module can be expressed in a different and more convenient way using the theory of locally analytic representations for which we will follow [18, 19] (though classical versions of this theory have been known for a long time thanks to the work of Lazard, Schneider–Teitelbaum, Emerton and others). Let V be as before. Then $\mathbf{D}_{\mathrm{Sen}}(V)$ is the same as the space $(V^H)^{\Gamma\text{-la}}$ of Γ-locally analytic vectors in V^H; the group Γ is a p-adic Lie group acting on a Banach representation V^H, hence one defines $(V^H)^{\Gamma\text{-la}} \subset V^H$ to be the subspace of vectors v for which the orbit map of Γ is given by a locally analytic function on Γ with values in V^H. Since the Sen module is nothing but the *decompletion by locally analytic vectors* of the action of Γ, it carries a natural action of $\mathrm{Lie}\,\Gamma$ which is a one-dimensional Lie algebra. The action of the generator given by the derivative of the cyclotomic character $\chi^{\mathrm{cyc}} : \Gamma \to \mathbb{Z}_p^\times$ is precisely the (arithmetic) *Sen operator* of $\mathbf{D}_{\mathrm{Sen}}(V)$.

Of course, for the previous construction to be useful one needs to say something about the Sen module of V. This has been worked out by Sen and the conclusion is the following. First, if $V = \mathbb{C}_p$ is the trivial semilinear representation, then $\mathbf{D}_{\mathrm{Sen}}(\mathbb{C}_p) = \mathbb{Q}_p(\zeta_{p^\infty})$ is the algebraic p-cyclotomic extension of $\mathbb{Q}_p$. In particular, Γ acts locally constantly on $\mathbf{D}_{\mathrm{Sen}}(\mathbb{C}_p)$, and hence its Sen operator is trivial. Second, when V is a $\mathbb{C}_p$-semilinear representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ of rank n, $\mathbf{D}_{\mathrm{Sen}}(V)$ is a $\mathbb{Q}_p(\zeta_{p^\infty})$-vector space of rank n, and the natural base change

$$\mathbf{D}_{\mathrm{Sen}}(V) \otimes_{\mathbb{Q}_p(\zeta_{p^\infty})} \mathbb{C}_p \to V$$

is a $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant isomorphism. In other words, V can be fully recovered from the locally analytic Γ-representation $\mathbf{D}_{\mathrm{Sen}}(V)$, which in addition has the same rank as V. This is a huge simplification of the framework, namely, we started with an action of a very complicated group such as $\mathrm{Gal}_{\mathbb{Q}_p}$ in a representation which is semilinear over a very large field $\mathbb{C}_p$, and obtained a locally analytic semilinear representation of the much simpler group $\Gamma \cong \mathbb{Z}_p^\times$ in a vector space over the much smaller field $\mathbb{Q}_p(\zeta_{p^\infty})$. Note that, since $\mathrm{Lie}\,\Gamma$ acts linearly on $\mathbb{Q}_p(\zeta_{p^\infty})$, after taking base change to $\mathbb{C}_p$ the Sen operator can be thought of as a Galois equivariant $\mathbb{C}_p$-linear operator on V itself; by abuse of language we do not make a distinction between the two ways to interpret the Sen operator.

The key techniques that are tacitly used in the previous discussion are *v-descent of vector bundles on perfectoid spaces* [27, Lemma 17.1.8] in order to descend from $\mathbb{C}_p$ to $\mathbb{Q}_p^{\mathrm{cyc}}$, and *Tate's normalized traces* [32] that allow one to decomplete from $\mathbb{Q}_p^{\mathrm{cyc}}$ to $\mathbb{Q}_p(\zeta_{p^\infty})$. A major part of our work is to study an abstract framework for the decompletion by Tate's normalized traces, following [1], in particular proving some permanence properties along étale maps that are useful when considering Sen theory in sheaves over geometric objects. We also study decompletions of a larger class of semilinear representations that we call *relative locally analytic representations* which show up naturally in applications to the theory of p-adic automorphic forms, as has been demonstrated in [17]. This is exclusively the content of the long Section 2.

Going back to our geometric applications, we want to perform an analogue simplification of the theory of continuous semilinear representations after replacing $\mathbb{Q}_p$ by some smooth rigid variety X over $\mathbb{C}_p$. This requires rephrasing the framework in more geometric terms. First, instead of studying $\mathbb{C}_p$-semilinear representations of $\mathrm{Gal}_{\mathbb{Q}_p}$, we will study proétale $\widehat{\mathcal{O}}$-vector bundles over X; when $X = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, both categories are naturally equivalent. In the arithmetic situation we had a preferred algebraic extension of $\mathbb{Q}_p$ thas was used to perform the decompletion, namely the cyclotomic extension. In the geometric context there is not a preferred cover of X that plays a similar role, but it does exist locally after fixing some étale chart $\psi : X \to \mathbb{T}^n_{\mathbb{C}_p}$, where $\mathbb{T}^n_{\mathbb{C}_p} = \mathrm{Spa}(\mathbb{C}_p\langle T_1, \dots, T_n\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T_1, \dots, T_n\rangle)$ is an n-dimensional torus. That is, after fixing the map ψ, one can consider the proétale $\Gamma \cong \mathbb{Z}_p^n$-torsor $X_\infty \to X$ obtained by taking p-power roots of the variables T_i. In this situation one can construct Tate's normalized traces for the completed functions of X_∞, and apply the abstract framework of Sen theory to decomplete the theory of $\widehat{\mathcal{O}}$-modules on X to some locally analytic semilinear representations of Γ over the Γ-smooth functions of X_∞; by taking the derivations of Γ one constructs the so-called *geometric Sen operators* on $\widehat{\mathcal{O}}$-modules over X.

The local geometric decompletion above is well known; see for example [3]. The abstract theory of locally analytic representations also implies that the Sen operators of the local decompletion above can be used to compute proétale cohomology of vector bundles. However, one would like to have an equally powerful global construction obtained by gluing the local ones. Section 3 is dedicated to this gluing process. The key observation, explained by Lue Pan to us, is that the local Sen operators should glue along the descent datum of the sheaf $R^1\nu_*\widehat{\mathcal{O}}$, where $\nu : X_{\text{proét}} \to X_{\text{ét}}$ is the projection from the proétale site to the étale site of X. By [25, Proposition 3.23] we know that $R^1\nu_*\widehat{\mathcal{O}}$ is naturally isomorphic to $\Omega^1_X(-1)$ (where the (-1) is the inverse of the Tate twist). Thus, the geometric Sen operators of an $\widehat{\mathcal{O}}$-module should glue to a suitable *Higgs field* in $X_{\text{proét}}$. Furthermore, the cohomology of this Higgs field should essentially compute the proétale cohomology.

In the next section we explain in more detail the ideas previously presented, stating the main results of the paper, and sketching the construction of the geometric Sen operators for the case of the torus.

Main results

Let $\mathbb{Q}_p^{\text{cyc}}$ be the completion of the p-adic cyclotomic extension of $\mathbb{Q}_p$ and let (C, C^+) be a perfectoid field containing $\mathbb{Q}_p^{\text{cyc}}$. In the following we shall consider almost mathematics with respect to the ideal $\mathfrak{m}_C \subset C^+$ of topologically nilpotent elements, and write "$=^{\text{ae}}$" for a natural equivalence in the category of almost modules (and "$\cong^{\text{ae}}$" for a non-natural isomorphism of almost modules).

Let X be a log smooth fs locally noetherian adic space over (C, C^+) with reduced normal crossing divisors, let $\widehat{\mathcal{O}}_X^{(+)}$ and $\mathcal{O}_X^{(+)}$ denote the structural sheaves of the pro-Kummer-étale and Kummer-étale sites of X respectively, and finally let $\Omega^1_X(\log)$ be the sheaf of log differential forms on X; we refer to [6] for log geometry on adic spaces.

In order to state the main theorem in geometric Sen theory we need the following definition.

Definition 1.0.1. A pro-Kummer-étale $\widehat{\mathcal{O}}_X$-module $\mathcal{F}$ over X is said to be *relative locally analytic ON Banach*[1] if there is a Kummer-étale cover $\{U_j\}_{j\in J}$ of X such that, for all j, the restriction $\mathcal{F}|_{U_j}$ admits a p-adically complete $\widehat{\mathcal{O}}_X^+$-lattice $\mathcal{F}_j^0$, and there is $\varepsilon > 0$ (depending on j) such that $\mathcal{F}_j^0/p^{\varepsilon} \cong^{\mathrm{ae}} \bigoplus_{I_j} \mathcal{O}_X^+/p^{\varepsilon}$ for I_j an index set. Here $p^{\varepsilon} \in K$ is an element of norm $|p^{\varepsilon}| = |p|^{\varepsilon}$.

Remark 1.0.2. The relative locally analytic condition can be considered as a "smallness" condition on the sheaf $\mathcal{F}$ in the sense of Faltings [9]. Examples of relative locally analytic ON $\widehat{\mathcal{O}}_X$-modules are $\widehat{\mathcal{O}}_X$-vector bundles; this can be deduced from v-descent of vector bundles on perfectoid spaces [27, Lemma 17.1.8]. See Example 2.3.8 below.

Remark 1.0.3. A more interesting class of relative locally analytic $\widehat{\mathcal{O}}_X$-modules arises from a pro-Kummer-étale G-torsor $\widetilde{X} \to X$ with G a p-adic Lie group and V a locally analytic representation of G; see the discussion after Theorem 1.0.4. These examples appear naturally in the study of locally analytic vectors of completed cohomology of Shimura varieties; see [17, 20].

On the other hand, the tag "relative locally analytic" for an $\widehat{\mathcal{O}}_X$-module is inspired from a similar property satisfied by locally analytic representations (see Lemma 2.1.5) that informally says that an action of a p-adic Lie group G on a Banach representation V is locally analytic if and only if for all $g \in G$, the action of $g - 1$ has operator norm < 1 (for some fixed norm on V).

Theorem 1.0.4 (Theorem 3.3.2). *Let $\mathcal{F}$ be a relative locally analytic ON Banach $\widehat{\mathcal{O}}_X$-sheaf over X. Then there is an $\widehat{\mathcal{O}}_X$-linear Higgs field*

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1(\log)(-1),$$

(i.e. $\theta_{\mathcal{F}} \wedge \theta_{\mathcal{F}} = 0$) called the geometric Sen operator[2] *satisfying the following properties:*

(1) *The formation of $\theta_{\mathcal{F}}$ is functorial in $\mathcal{F}$.*

(2) *Let $\nu : X_{\text{prokét}} \to X_{\text{két}}$ be the projection from the pro-Kummer-étale site to the Kummer-étale site. Then there is a natural equivalence*

$$R^i \nu_* \mathcal{F} = \nu_* H^i(\theta_{\mathcal{F}}, \mathcal{F}),$$

where $H^i(\theta_{\mathcal{F}}, \mathcal{F})$ is the cohomology of the Higgs complex

$$0 \to \mathcal{F} \xrightarrow{\theta_{\mathcal{F}}} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1(\log)(-1) \to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^i(\log)(-i)$$
$$\to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^d(\log)(-d) \to 0.$$

[1] Where *ON* comes from orthonormalizable.

[2] Here (-1) refers to the inverse of the Tate twist.

(3) *Suppose that* $\theta_{\mathcal{F}} = 0$. *Then* $\nu_*\mathcal{F}$ *is locally in the Kummer-étale topology of* X *an ON Banach* $\mathcal{O}_X$*-module*[3] *and* $\mathcal{F} = \widehat{\mathcal{O}}_X \widehat{\otimes}_{\mathcal{O}_X} \nu_*\mathcal{F}$. *Conversely, for any ON Banach* $\mathcal{O}_X$*-module* $\mathcal{G}$ *the pullback* $\widehat{\mathcal{O}}_X \widehat{\otimes}_{\mathcal{O}_X} \mathcal{G}$ *has trivial Sen operator.*

(4) *If* X *has a form* X' *over a discretely valued field with perfect residue field* (K, K^+) *and* $\mathcal{F}$ *is defined over* X', *then* $\theta_{\mathcal{F}}$ *is Galois equivariant. In particular, we recover the natural splitting*

$$R\nu_*\widehat{\mathcal{O}}_X = \bigoplus_{i=0}^{d} \Omega_X^i(\log)(-i)[-i],$$

deduced from [25, Proposition 3.23] *in the non-log case, and from* [7, Theorem 3.2.4] *in the log case.*

(5) *Let* $f : Y \to X$ *be a morphism of smooth fs log adic spaces over* (C, C^+). *Then there is a commutative diagram*

$$\begin{array}{ccc} f^*\mathcal{F} & \xrightarrow{f^*\theta_{\mathcal{F}}} & f^*\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\Omega_X^1(\log)(-1) \\ & \searrow^{\theta_{f^*\mathcal{F}}} & \downarrow{\scriptstyle \mathrm{id} \otimes f^*} \\ & & f^*\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_Y^1(\log)(-1) \end{array}$$

The property of being a relative locally analytic $\widehat{\mathcal{O}}_X$-module might look a bit mysterious. Nevertheless, these sheaves arise naturally when studying locally analytic vectors of proétale cohomology. Let us explain the context in which they appear. Let G be a compact p-adic Lie group and $\widetilde{X} \to X$ a pro-Kummer-étale G-torsor (e.g. take for X a finite level modular curve and for $\widetilde{X}$ the perfectoid modular curve). Let V be a $\mathbb{Q}_p$-Banach locally analytic representation of G, for example, we can take $V = \mathcal{O}(\mathbb{G})$ for some group affinoid neighbourhood $\mathbb{G}$ of G, endowed with the left regular action. Then V defines a pro-Kummer-étale sheaf $V_{\text{két}}$ over X by descending the G-representation V along the torsor $\widetilde{X} \to X$. By Lemma 2.1.5 below, there is a lattice $V^0 \subset V$ and an open subgroup $G_0 \subset G$ such that G_0 stabilizes V^0, and that the action of G_0 on V^0/p is trivial. Therefore, the pro-Kummer-étale $\widehat{\mathcal{O}}_X$-module $V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ is a relative locally analytic sheaf. Indeed, the restriction of $V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ to $X_{G_0} := \widetilde{X}/G_0$ satisfies the conditions of Definition 1.0.1. In this situation we have a more refined result.

Theorem 1.0.5 (Theorem 3.3.4). *Let* X *be an fs log smooth adic space over* $\mathrm{Spa}(C, C^+)$ *with log structure given by normal crossing divisors. Let* G *be a* p*-adic Lie group and* $\widetilde{X} \to X$ *a pro-Kummer-étale* G*-torsor. Then there is an* $\widehat{\mathcal{O}}_X$*-linear map*[4]

$$\theta_{\widetilde{X}} : \widehat{\mathcal{O}}_X \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\, G)^{\vee}_{\text{két}} \to \widehat{\mathcal{O}}_X(-1) \otimes_{\mathcal{O}_X} \Omega_X^1(\log)$$

[3] This means that locally in the Kummer-étale topology, $\nu_*\mathcal{F}$ admits a Banach basis over $\mathcal{O}_X$.

[4] Here we see $\mathrm{Lie}\, G$ as the adjoint representation of G.

such that $\theta_{\widetilde{X}} \wedge \theta_{\widetilde{X}} = 0$, called the geometric Sen operator of the torsor $\widetilde{X}$. Moreover, for any locally analytic Banach representation V of G, we have a commutative diagram

$$\begin{array}{ccc} V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X & \xrightarrow{d_V \otimes \mathrm{id}_{\widehat{\mathcal{O}}}} & (V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\, G)^{\vee}_{\mathrm{k\acute{e}t}} \\ & \searrow^{\theta_V} & \downarrow{\scriptstyle \mathrm{id}_V \otimes \theta_{\widetilde{X}}} \\ & & (V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X(-1)) \otimes_{\mathcal{O}_X} \Omega^1_X(\log) \end{array}$$

such that $d_V : V \to V \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\, G)^{\vee}$ is the connection induced by derivations, and θ_V is the geometric Sen operator on $V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ of Theorem 1.0.4.

Remark 1.0.6. Let us explain the meaning of the condition $\theta_{\widetilde{X}} \wedge \theta_{\widetilde{X}} = 0$ in Theorem 1.0.5. The map $\theta_{\widetilde{X}}$ is adjoint to a map

$$\theta : \widehat{\mathcal{O}}_X \to (\mathrm{Lie}\, G)_{\mathrm{k\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X(-1) \otimes_{\mathcal{O}_X} \Omega^1_X(\log) =: \mathcal{E}.$$

The condition $\theta_{\widetilde{X}} \wedge \theta_{\widetilde{X}} = 0$ means that $\theta \wedge \theta = 0$ in $(\mathrm{Lie}\, G)_{\mathrm{k\acute{e}t}} \otimes_{\mathbb{Q}_p} \Omega^2_X(\log) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(-2)$ with respect to the Lie bracket in $\mathrm{Lie}\, G$. Equivalently, the associated map $\Omega^1_X(\log)^{\vee} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(1) \to (\mathrm{Lie}\, G)_{\mathrm{k\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ is a morphism of $\widehat{\mathcal{O}}_X$-linear Lie algebras, where $\Omega^1_X(\log)^{\vee}$ has the zero bracket.

Moreover, let $H \to G$ be a morphism of p-adic Lie groups, let Y an fs log smooth adic space over (C, C^+) and let $\widetilde{Y} \to Y'$ be an H-torsor. Suppose we are given a commutative diagram compatible with the group actions

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Then the following square is commutative:

$$\begin{array}{ccc} f^*(\mathrm{Lie}\, G)^{\vee}_{\mathrm{k\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_Y & \xrightarrow{f^*\theta_{\widetilde{X}}} & f^*\Omega^1_X(\log) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y(-1) \\ \downarrow & & \downarrow \\ (\mathrm{Lie}\, H)^{\vee}_{\mathrm{k\acute{e}t}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_Y & \xrightarrow{\theta_{\widetilde{Y}}} & \Omega^1_Y(\log) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y(-1) \end{array}$$

Let $\mathrm{Sen}_{\widetilde{X}} : \Omega^1_X(\log)^{\vee} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(1) \to (\mathrm{Lie}\, G)_{\mathrm{k\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ be the dual of the Sen operator $\theta_{\widetilde{X}}$. As a corollary of the previous theorem we deduce that the locally analytic vectors of the torsor $\widetilde{X}$ satisfy some differential equations.

Corollary 1.0.7 (Corollary 3.2.6). *Let $|\widetilde{X}|$ be the underlying topological space of $\widetilde{X}$ and let $\widehat{\mathcal{O}}_X|_{|\widetilde{X}|}$ be the restriction of $\widehat{\mathcal{O}}_X$ to a sheaf on $|\widetilde{X}|$. Let $\mathcal{O}^{\mathrm{la}}_{\widetilde{X}} \subset \widehat{\mathcal{O}}_X|_{|\widetilde{X}|}$ be the subsheaf consisting of locally analytic sections for the action of G. Then the action of $\mathcal{O}^{\mathrm{la}}_{\widetilde{X}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$ on $\mathcal{O}^{\mathrm{la}}_{\widetilde{X}}$ by derivations vanishes when restricted to the image of the Sen map $\mathrm{Sen}_{\widetilde{X}} : \Omega^1_X(\log)^{\vee} \otimes_{\mathcal{O}_X} \mathcal{O}^{\mathrm{la}}_{\widetilde{X}}(1) \to \mathrm{Lie}\, G \otimes_{\mathbb{Q}_p} \mathcal{O}^{\mathrm{la}}_{\widetilde{X}}$.*

To motivate the construction of the geometric Sen operator over rigid spaces, we recall how the arithmetic Sen operator is defined.

Let $\mathbb{Q}_p(\zeta_{p^\infty})$ be the cyclotomic extension and $\mathbb{Q}_p^{\mathrm{cyc}}$ its p-adic completion, and let $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$ containing $\mathbb{Q}_p(\zeta_{p^\infty})$. We denote $H = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))$ and $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. Let V be a finite-dimensional representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$. The *Sen module* $\mathrm{Sen}(V)$ attached to V is the finite-dimensional $\mathbb{Q}_p(\zeta_p^\infty)$-vector space consisting of the finite Γ-vectors of $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^H$. It turns out that the dimension of $\mathrm{Sen}(V)$ over $\mathbb{Q}_p(\zeta_{p^\infty})$ is equal to $\dim_{\mathbb{Q}_p} V$. Indeed, one has a $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant isomorphism

$$\mathrm{Sen}(V) \otimes_{\mathbb{Q}_p(\zeta_p^\infty)} \mathbb{C}_p = V \otimes_{\mathbb{Q}_p} \mathbb{C}_p.$$

As explained in [2], an equivalent way to construct $\mathrm{Sen}(V)$ is by taking the Γ-locally analytic vectors of $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^H$, in particular $\mathrm{Sen}(V)$ admits an action of $\mathrm{Lie}\,\Gamma$. We let $\theta_{\mathbb{Q}_p} \in \mathrm{Lie}\,\Gamma$ be the natural basis arising from the cyclotomic character $\chi_{\mathrm{cyc}} : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$. Since Γ acts locally constantly on $\mathbb{Q}_p(\zeta_{p^\infty})$ (i.e. for each $v \in \mathbb{Q}_p(\zeta_{p^\infty})$ the orbit $\Gamma \cdot v$ is finite), $\theta_{\mathbb{Q}_p}$ is a $\mathbb{Q}_p(\zeta_{p^\infty})$-linear operator. One defines the Sen operator of $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ to be the $\mathbb{C}_p$-extension of scalars of $\theta_{\mathbb{Q}_p}$.

Summarizing, one can consider $\mathrm{Sen}(V)$ as a decompletion of the semilinear representation $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ by taking locally analytic vectors along the "perfectoid cyclotomic coordinate" $\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p$. The Sen operator is then a "differential operator" obtained from the cyclotomic extension.

Let us now sketch the definition of the geometric Sen operator. For simplicity, we assume that $X = \mathbb{T} = \mathrm{Spa}(C\langle T^{\pm 1}\rangle, C^+\langle T^{\pm 1}\rangle)$ is a one-dimensional torus. Let $\mathcal{F}$ be a relative locally analytic sheaf over $\mathbb{T}$, and suppose in addition that there is a p-adically complete lattice $\mathcal{F}^0 \subset \mathcal{F}$ and $\varepsilon > 0$ such that $\mathcal{F}^0/p^\varepsilon =^{\mathrm{ae}} \bigoplus_I \mathcal{O}_X^+/p^\varepsilon$. We want to compute the (geometric) proétale cohomology $R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbb{T}, \mathcal{F})$. Let

$$\mathbb{T}_n = \mathrm{Spa}(C\langle T^{\pm 1/p^n}\rangle, C^+\langle T^{\pm 1/p^n}\rangle),$$

and let $\mathbb{T}_\infty = \varprojlim_n \mathbb{T}_n$ be the perfectoid torus. Then $\mathbb{T}_\infty$ is a Galois cover of $\mathbb{T}$ with group $\Gamma = \mathbb{Z}_p(1)$. By Scholze's almost acyclicity of $\mathcal{O}_X^+/p$ in affinoid perfectoid spaces [23, Proposition 7.13], one has

$$R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbb{T}, \mathcal{F}) = R\Gamma(\Gamma, \mathcal{F}(\mathbb{T}_\infty)).$$

By the comparison theorem between continuous and locally analytic cohomology (Theorem 2.1.2), one sees that

$$\begin{aligned} R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbb{T}, \mathcal{F}) &= R\Gamma(\Gamma, \mathcal{F}(\mathbb{T}_\infty)^{R\Gamma\text{-la}}) \\ &= R\Gamma\big(\Gamma^{\mathrm{sm}}, R\Gamma(\mathrm{Lie}\,\Gamma, \mathcal{F}(\mathbb{T}_\infty)^{R\Gamma\text{-la}})\big), \end{aligned}$$

where $V^{R\Gamma\text{-la}}$ are the derived locally analytic vectors of a Γ-representation V, $R\Gamma(\mathrm{Lie}\,\Gamma, -)$ is the Lie algebra cohomology (equivalently, the cohomology of a

Chevalley–Eilenberg complex), and $R\Gamma(\Gamma^{\mathrm{sm}}, -)$ is the smooth group cohomology (equivalently, group cohomology for smooth or locally constant cochains); see Section 2.1 for the definitions. In other words, we have divided the problem of computing proétale cohomology into three steps: first, we need to compute the derived locally analytic vectors of $\mathcal{F}(\mathbb{T}_\infty)$; second, we take the Lie algebra cohomology of $\mathcal{F}(\mathbb{T}_\infty)$; and finally, we take the Γ-invariants of a smooth representation.

Let us focus on the first step which seems to be the most subtle. For $n \geq m$ there are normalized traces (we call them *Sen traces*)

$$R_m^n : C\langle T^{\pm 1/p^n}\rangle \to C\langle T^{\pm 1/p^m}\rangle$$

where $R_m^n = \frac{1}{p^{n-m}} \sum_{\sigma \in p^m\Gamma/p^n\Gamma} \sigma$. These extend to Sen traces

$$R_m : C\langle T^{\pm 1/p^\infty}\rangle \to C\langle T^{\pm 1/p^m}\rangle$$

such that, for any $f \in C\langle T^{\pm 1/p^\infty}\rangle$, the sequence $(R_m(f))_m$ converges to f. Furthermore, the tuple $(C\langle T^{\pm 1/p^\infty}\rangle, \Gamma)$ satisfies the Colmez–Sen–Tate axioms of [1].

Let $(\zeta_{p^n})_n$ be a compatible system of primitive p-power roots of unity, and let $\psi : \mathbb{Z}_p \xrightarrow{\sim} \Gamma$ be the induced isomorphism. Using ψ we define the affinoid group $\mathbb{G}_n$ which is a copy of the additive group of radius p^{-n} (i.e. $\mathbb{G}_n(\mathbb{Q}_p) = p^n\mathbb{Z}_p$). We will keep using the expression "$p^n\Gamma$-analytic" instead of $\mathbb{G}_n$-analytic. The following theorem is a generalization of [1, Proposition 3.3.1] to relative locally analytic sheaves; it can be seen as a decompletion theorem à la Kedlaya–Liu [14] using locally analytic vectors.

Theorem 1.0.8 (Theorem 2.4.4). *There exists $n \gg 0$ depending on ε such that*

$$\mathcal{F}(\mathbb{T}_\infty) = C\langle T^{\pm 1/p^\infty}\rangle \hat{\otimes}_{C\langle T^{\pm 1/p^n}\rangle} \mathcal{F}(\mathbb{T}_\infty)^{p^n\Gamma\text{-an}}.$$

Moreover,

$$\mathcal{F}(\mathbb{T}_\infty)^{R\Gamma\text{-la}} = \mathcal{F}(\mathbb{T}_\infty)^{\Gamma\text{-la}} = \varinjlim_m C\langle T^{\pm 1/p^m}\rangle \otimes_{C\langle T^{\pm 1/p^n}\rangle} \mathcal{F}(\mathbb{T}_\infty)^{p^n\Gamma\text{-an}}.$$

The theorem shows that, under certain conditions on $\mathcal{F}$, the derived locally analytic vectors of $\mathcal{F}(\mathbb{T}_\infty)$ are concentrated in degree 0, and all the relevant information is already encoded in the $p^m\Gamma$-analytic vectors for some $m \gg 0$. In particular, we have

$$R\Gamma_{\text{proét}}(\mathbb{T}_\infty, \mathcal{F}) = R\Gamma\big(\Gamma^{\mathrm{sm}}, R\Gamma(\operatorname{Lie}\Gamma, \mathcal{F}(\mathbb{T}_\infty)^{p^n\Gamma\text{-an}})\big). \tag{1.1}$$

Thus, the problem of computing proétale cohomology has been reduced to the problem of computing Lie algebra cohomology. The module $\mathcal{F}(\mathbb{T}_\infty)^{p^n\Gamma\text{-an}}$ is not mysterious at all: it is the Higgs bundle attached to $\mathcal{F}$ via the local p-adic Simpson correspondence of $\mathbb{T}$.

Now, the action of $\operatorname{Lie}\Gamma$ is $C\langle T^{\pm 1/p^n}\rangle$-linear and Γ-equivariant. By extending scalars, it induces a Γ-equivariant $C\langle T^{\pm 1/p^\infty}\rangle$-linear action on $\mathcal{F}(\mathbb{T}_\infty)$. Moreover, if X has a form X' over (K, K^+) and $\mathcal{F}$ is defined over X', this action is Gal_K-equivariant, where Gal_K acts on $\operatorname{Lie}\Gamma$ via the cyclotomic character.

By Proposition 3.1.2, $\mathrm{Lie}\,\Gamma \otimes C\langle T^{\pm 1/p^{\infty}}\rangle = \Omega^{1,\vee}_{\mathbb{T}}(\mathbb{T}) \otimes C\langle T^{\pm 1/p^{\infty}}\rangle(1)$; this identification arises from the well known equivalence $H^1_{\text{proét}}(\mathbb{T}, \widehat{\mathcal{O}}) = \Omega^1_{\mathbb{T}}(-1)$, and the fact that the left term can be computed as

$$H^1(\Gamma, C\langle T^{\pm 1/p^{\infty}}\rangle) = H^1(\mathrm{Lie}\,\Gamma, C\langle T^{\pm 1}\rangle) = (\mathrm{Lie}\,\Gamma)^{\vee} \otimes_{\mathbb{Q}_p} C\langle T^{\pm 1}\rangle.$$

This shows that the action of $\mathrm{Lie}\,\Gamma$ defines an $\widehat{\mathcal{O}}_X$-linear map of proétale sheaves over $\mathbb{T}$,

$$\mathrm{Sen}_{\mathcal{F}} : \Omega^{1,\vee}_{\mathbb{T}} \otimes_{\mathcal{O}_X} \mathcal{F}(1) \to \mathcal{F},$$

or equivalently, it defines the geometric Sen operator

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{\mathbb{T}}(-1).$$

In general, we shall construct the Sen operators locally in the Kummer-étale topology of X by taking charts to products of tori and closed affinoid discs. The fact that $\theta_{\mathcal{F}}$ does not depend on the charts and that it is functorial with respect to the sheaf and the space will be proved in Section 3.3; this globalization of Sen operators should not be surprising thanks to the isomorphism $R^1\nu_*\widehat{\mathcal{O}}_X = \Omega^1_X(\log)(-1)$.

An overview of the paper

In Section 2.1 we briefly review some basic facts of the theory of solid locally analytic representations of the author and Rodrigues Jacinto [18, 19]. In Sections 2.2–2.5, we extend the abstract formalism of Sen theory of Berger–Colmez [1] that will be used in the main applications of the paper.

In Section 3.1 we extend the computation of $R^1\nu_*\widehat{\mathcal{O}}_X(1) = \Omega^1_X$ of [25] from smooth to log smooth adic spaces; this follows formally from Scholze's proof using the formalism of log adic spaces. In Section 3.2 we construct the geometric Sen operator in local coordinates, proving local versions of Theorems 1.0.4 and 1.0.5; then, in Section 3.3 we prove the global versions of the theorems. In Section 3.4 we study locally analytic vectors of torsors of p-adic Lie groups. We finish in Section 3.5 by discussing the relation of geometric Sen theory to the p-adic Simpson correspondence of [7, 15, 33].

2. Abstract Sen theory after Berger–Colmez

Sen theory has shown to be a powerful tool in the Galois theory of p-adic fields. For example, it is used to compute Galois cohomology over period rings:

Proposition 2.0.1 ([32, Proposition 8]). *Let $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and let $G_{\mathbb{Q}_p}$ be the absolute Galois group. For $i \in \mathbb{Z}$ we let $\mathbb{C}_p(i)$ denote the i-th Tate twist. Then*

$$H^k(G_{\mathbb{Q}_p}, \mathbb{C}_p(i)) = \begin{cases} 0 & \text{if } i \neq 0, \\ \mathbb{Q}_p & \text{if } i = 0 \text{ and } k = 0, \\ \mathbb{Q}_p \log \chi_{\mathrm{cyc}} & \text{if } i = 0 \text{ and } k = 1, \end{cases}$$

where $\chi_{\mathrm{cyc}}: G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ *is the cyclotomic character, and* $\log: \mathbb{Z}_p^\times \to \mathbb{Z}_p$ *is the logarithm map defined by* $\log(a) = \log(a/[a])$ *with* $[a] \in \mu_{p-1}(\mathbb{Z}_p)$ *the Teichmüller lift of* $\overline{a} \in \mathbb{F}_p^\times$.[5]

In [1], Berger–Colmez defined an axiomatic framework where Sen theory can be applied. Using this formalism, different constructions attached to finite-dimensional Galois representations become formally the same: the Sen module (relative to $\mathbb{Q}_p^{\mathrm{cyc}}$), the overconvergent (φ,Γ)-module (relative to $\widetilde{\mathbb{B}}^\dagger(\mathbb{Q}_p^{\mathrm{cyc}})$), the module $\mathbf{D}_{\mathrm{diff}}$ of differential equations (relative to $\mathbb{B}_{\mathrm{dR}}(\mathbb{Q}_p^{\mathrm{cyc}})$). Moreover, using Sen theory Berger–Colmez [2] describe the locally analytic vectors of completed Galois extensions of $\mathbb{Q}_p$ with Galois group isomorphic to a p-adic Lie group.

The work of Lue Pan [17] is an excellent application of this tool. Inspired by the work of Berger–Colmez, Pan describes the p-adic Simpson correspondence of the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant local systems of modular curves in terms of sheaves over the flag variety. Furthermore, based on the strategy of [2], he manages to use the axiomatic Sen theory to describe the Hodge–Tate structure of locally analytic “interpolations” of finite rank local systems.

The main goal of this section is to provide a more conceptual understanding of the work of Pan and Berger–Colmez via the theory of (solid) locally analytic representations. More precisely, we will prove that the construction of the Sen module holds not only for finite rank representations, but for a larger class of orthonormalizable (also denoted ON in this paper) Banach locally analytic representations.

2.1. Recollections on locally analytic representations

In this section we recall some tools from the theory of locally analytic representations of compact p-adic Lie groups. We will only be concerned with continuous representations on (colimits of) Banach spaces, so everything can be stated in the classical language of Schneider–Teitelbaum [21, 22] and Emerton [8]. However, we will use some technology from the theory of solid locally analytic representations, as developed in [18, 19], which will facilitate some technical arguments. For example, we will consistently use the general comparison theorem between continuous and locally analytic cohomology.

Let G be a compact p-adic Lie group and let $C^{\mathrm{la}}(G,\mathbb{Q}_p)$ be the LB space of locally analytic functions on G; it admits a description as a colimit $C^{\mathrm{la}}(G,\mathbb{Q}_p) = \varinjlim_h C^h(G,\mathbb{Q}_p)$ of $|p^h|$-analytic (or h-analytic for simplicity) functions on G, which depends on the choice of a uniform pro-p-subgroup $G_0 \subset G$ [19, Section 2]. Throughout this paper we will take an arbitrary choice of G_0, and we will freely change G_0 to a smaller subgroup when necessary.

We let $C^{\mathrm{la}}(G,\mathbb{Q}_p)_{\star_1}$ denote $C^{\mathrm{la}}(G,\mathbb{Q}_p)$ with the left regular action (use similarly $\star_2$ for the right regular action). Let $\mathbb{Q}_p[[G]] = \mathbb{Z}_p[[G]][\frac{1}{p}]$ denote the rational Iwasawa algebra of G, seen as a solid $\mathbb{Q}_p$-algebra. Let $\mathrm{Solid}(\mathbb{Q}_p[[G]])$ be the abelian cat-

[5] If $p = 2$ one needs to modify the definition since $\mathbb{Z}_2^\times = \{\pm 1\} \times 1 + 4\mathbb{Z}_2$, namely, under this presentation one projects $\mathbb{Z}_2^\times \to 1 + 4\mathbb{Z}_2$ and then takes the logarithm $\log: 1 + 4\mathbb{Z}_2 \to \mathbb{Z}_2$.

egory of solid $\mathbb{Q}_p[[G]]$-representations, or equivalently the category of $\mathbb{Q}_p$-linear solid G-representations [18, Definition 4.20]. We let $\mathcal{D}_{\blacksquare}(\mathbb{Q}_p[[G]])$ be the derived category of $\mathrm{Solid}(\mathbb{Q}_p[[G]])$.

Definition 2.1.1. Let $V \in \mathcal{D}_{\blacksquare}(\mathbb{Q}_p[[G]])$.

(1) The *solid group cohomology* of V is given by

$$R\underline{\Gamma}_{\blacksquare}(G, V) := R\underline{\mathrm{Hom}}_{\mathbb{Q}_p[[G]]}(\mathbb{Q}_p, V),$$

where $\underline{\mathrm{Hom}}_{\mathbb{Q}_p[[G]]}$ is the solid Hom space of $\mathbb{Q}_p[[G]]$-modules.[6] We denote by $R\Gamma_{\blacksquare}(G, V) := R\underline{\Gamma}_{\blacksquare}(G, V)(*)$ the underlying $\mathbb{Q}_p$-vector space. We also write $\underline{H}^i(G, V)$ for the i-th cohomology group of $R\underline{\Gamma}_{\blacksquare}(G, V)$, and $H^i(G, V) = \underline{H}^i(G, V)(*)$ for the underlying solid $\mathbb{Q}_p$-vector space.

(2) The *derived locally analytic vectors* of V is the solid $\mathbb{Q}_p[[G]]$-module defined by

$$V^{RG\text{-la}} := R\underline{\Gamma}_{\blacksquare}(G, (V \otimes^L_{\mathbb{Q}_p,\blacksquare} C^{\mathrm{la}}(G, \mathbb{Q}_p))_{\star_{1,3}}),$$

where $\otimes^L_{\mathbb{Q}_p,\blacksquare}$ is the solid tensor product, and the $\star_{1,3}$ is the diagonal action on V and $C^{\mathrm{la}}(G, \mathbb{Q}_p)_{\star_1}$. The G-action on $V^{RG\text{-la}}$ arises from the right regular action on $C^{\mathrm{la}}(G, \mathbb{Q}_p)$. If $V \in \mathrm{Solid}(\mathbb{Q}_p[[G]])$ we denote $V^{G\text{-la}} := H^0(V^{RG\text{-la}})$.

(3) We say that a solid G-representation V is *locally analytic* if the natural map $V^{RG\text{-la}} \to V$ is a quasi-isomorphism of solid $\mathbb{Q}_p[[G]]$-modules (equivalently of solid $\mathbb{Q}_p$-vector spaces).

The following theorem summarizes the main features of the category of solid locally analytic representations.

Theorem 2.1.2. *Let $\mathrm{Rep}^{LC}_{\mathbb{Q}_p}(G)$ be the category of $\mathbb{Q}_p$-linear compactly generated complete locally convex continuous representations of G. The following hold:*

(1) *There is a fully faithful inclusion*

$$\underline{(-)} : \mathrm{Rep}^{LC}_{\mathbb{Q}_p}(G) \hookrightarrow \mathrm{Solid}(\mathbb{Q}_p[[G]]).$$

(2) *Under the inclusion of* (1)*, there is a natural quasi-isomorphism of $\mathbb{Q}_p$-vector spaces*

$$R\Gamma_{\blacksquare}(G, \underline{V}) = R\Gamma(G, V)$$

between solid and continuous cohomology.

(3) *The functor $(-)^{RG\text{-la}}$ is idempotent. Let $\mathcal{D}_{\blacksquare}(\mathbb{Q}_p)^{G^{\mathrm{la}}} \subset \mathcal{D}_{\blacksquare}(\mathbb{Q}_p[[G]])$ be the full subcategory of locally analytic representations. Then $\mathcal{D}_{\blacksquare}(\mathbb{Q}_p)^{G^{\mathrm{la}}}$ is stable under colimits and tensor products. Moreover, the inclusion admits a right adjoint given by the functor of derived locally analytic vectors $(-)^{RG\text{-la}}$.*

[6] The object $\mathbb{Q}_p[[G]]$ is an algebra in solid $\mathbb{Q}_p$-vector spaces; this makes $\mathrm{Solid}(\mathbb{Q}_p[[G]])$ naturally enriched on solid $\mathbb{Q}_p$-vector spaces.

(4) *Let* $V \in \mathcal{D}_{\blacksquare}(\mathbb{Q}_p[[G]])$. *Then there is a natural equivalence*

$$R\underline{\Gamma}_{\blacksquare}(G, V) = R\underline{\Gamma}\big(G^{\mathrm{sm}}, R\underline{\Gamma}(\mathrm{Lie}\, G, V^{RG\text{-la}})\big),$$

where $R\underline{\Gamma}(\mathrm{Lie}\, G, -)$ *is* (*the solid enriched*) *Lie algebra cohomology, and* $R\underline{\Gamma}(G^{\mathrm{sm}}, -)$ *is* (*the solid enriched*) *smooth group cohomology.*

Proof. Parts (1) and (2) follow from [18, Proposition 3.7 and Lemma 5.2] respectively. Part (3) follows from [19, Propositions 3.2.10 and 3.3.3]. Finally, part (4) follows from [18, Theorems 5.3 and 5.5], or [19, Theorem 6.3.4]. ■

Remark 2.1.3. In Theorem 2.1.2 (4) smooth G-cohomology and Lie algebra cohomology are as in [19, Definition 1.2.4], namely, the right adjoints of the natural trivial representation map from solid $\mathbb{Q}_p$-vector spaces to the derived categories of smooth G-representations or $U(\mathrm{Lie}\, G)$-modules respectively. For objects in the heart of the t-structure of G-smooth representations, [19, Proposition 6.3.3] shows that smooth group cohomology can be computed via smooth cochains. By the standard Koszul resolution of $\mathbb{Q}_p$ as a $U(\mathrm{Lie}\, G)$-module, Lie algebra cohomology can also be computed via the explicit Chevalley–Eilenberg complex.

An immediate consequence of the theorem is the following corollary:

Corollary 2.1.4. *Let* $V \in \mathrm{Solid}(\mathbb{Q}_p[[G]])$ *be a solid* G*-representation. Suppose that* $V^{RG\text{-la}} = V^{G\text{-la}}$, *i.e. the higher derived locally analytic vectors vanish. Then there are isomorphisms*

$$\underline{H}^i_{\blacksquare}(G, V) = \underline{H}^i(\mathrm{Lie}\, G, V^{G\text{-la}})^G$$

of solid $\mathbb{Q}_p$*-vector spaces. In particular, if* V *arises from a compactly generated locally convex* $\mathbb{Q}_p$*-vector space and has no higher locally analytic vectors, then we have isomorphisms*

$$H^i(G, V) = H^i(\mathrm{Lie}\, G, V^{G\text{-la}})^G$$

of $\mathbb{Q}_p$*-vector spaces.*

Proof. This follows from parts (2) and (4) of Theorem 2.1.2. ■

The property of being a locally analytic representation roughly means that the operators $[g] - [1]$ for $g \in G$ have norm < 1. An example of this phenomenon appears in the following lemma.

Lemma 2.1.5. *Let* V *be a* $\mathbb{Q}_p$*-Banach representation of* G. *The following hold:*

(1) *If* V *is locally analytic then for any* $\mathbb{Z}_p$*-lattice* $V_0 \subset V$ *there is an open compact subgroup* G_0 *stabilizing* V_0 *such that* G_0 *acts trivially on* V_0/p.

(2) *Conversely, suppose that there is a finite extension* $K/\mathbb{Q}_p$ *with uniformizer* ϖ *such that there is an* $\mathcal{O}_K$*-lattice* W_0 *of* $V \otimes_{\mathbb{Q}_p} K$ *and an open compact subgroup* G_0 *stabilizing* W_0, *with* G_0 *acting trivially on* W_0/ϖ. *Then* V *is locally analytic.*

Moreover, if V is locally analytic, then it is analytic in the following sense: there is a uniform pro-p-subgroup $G_0 \subset G$ (cf. [22, Section 4]) with coordinates, $G_0 \cong \mathbb{Z}_p^d$, such that the orbit map $\mathcal{O}_V : V \to C(G_0, V)$ factors through $C^{\mathrm{an}}(G_0, V)$ where $C^{\mathrm{an}}(G_0, V) = C^{\mathrm{an}}(G_0, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} V$ and $C^{\mathrm{an}}(G_0, \mathbb{Q}_p) \cong \mathbb{Q}_p\langle T_1, \dots, T_d\rangle$ is the space of analytic functions on G_0 with respect to the coordinates $(T_1, \dots, T_d)$ of $\mathbb{Z}_p^d$.

Proof. Let us prove part (1). Let V be a locally analytic representation of G. By shrinking G we can assume without loss of generality that G is a uniform pro-p-group and that we have an isomorphism $G \cong \mathbb{Z}_p^d$ of p-adic manifolds. The powers G^{p^n} are then uniform pro-p-groups and we can write $C^{\mathrm{la}}(G, \mathbb{Q}_p) = \varinjlim_{h\to\infty} C^h(G, \mathbb{Q}_p)$ where $C^h(G, \mathbb{Q}_p)$ is the space of analytic functions in the coset decomposition $G = \bigsqcup_{g\in G/G^{p^h}} gG^{p^h}$. Therefore, since V is locally analytic, its orbit map is a map of solid $\mathbb{Q}_p$-vector spaces

$$\mathcal{O}_V^{\mathrm{la}} : V \to \varinjlim_h V \widehat{\otimes}_{\mathbb{Q}_p} C^h(G, \mathbb{Q}_p).$$

By [18, Lemma 3.32] there is some h such that $\mathcal{O}_V$ factors through

$$\mathcal{O}_V^h : V \to V \widehat{\otimes}_{\mathbb{Q}_p} C^h(G, \mathbb{Q}_p).$$

The map $\mathcal{O}_V^h$ is G-equivariant for the natural action on V and for the right regular action on the tensor. One can find a suitable $\mathbb{Z}_p$-lattice $V_0 \subset V$ such that $\mathcal{O}_V^h$ sends V_0 to $V_0 \widehat{\otimes}_{\mathbb{Z}_p} C^h(G, \mathbb{Z}_p)$ (with $C^h(G, \mathbb{Z}_p)$ the space of $\mathbb{Z}_p$-valued h-analytic functions). But then it is easy to see that for any $k \geq 1$ there is $h' > h$ such that the action of $G^{p^{h'}}$ on $C^h(G, \mathbb{Z}_p)/p^k$ is trivial (e.g. since $C^h(G, \mathbb{Z}_p)/p^k$ is a $\mathbb{Z}/p^k$-algebra of finite type and G acts via algebra morphisms, so that it suffices to trivialize the action on algebra generators). Pick k such that $\mathcal{O}_V^h(V_0) \cap p^k(V_0 \widehat{\otimes}_{\mathbb{Z}_p} C^h(G, \mathbb{Z}_p)) \subset \mathcal{O}_V^h(pV_0)$ (which is guaranteed by continuity of the orbit map). Then V_0/p is a subquotient of $V_0 \widehat{\otimes}_{\mathbb{Z}_p} C^h(G, \mathbb{Z}_p)/p^k$ and so it is a trivial representation of $G^{p^{h'}}$ as desired.

Part (2) follows from [19, Proposition 3.4.3] but let us give the details of how the proposition is used. First, since the category of locally analytic representations is stable under colimits on $\mathbb{Q}_p$-solid G-modules (Theorem 2.1.2 (3)), it suffices to show that $W = V \otimes_{\mathbb{Q}_p} K$ is locally analytic (as V is a retract of W). Then the same proof of [19, Proposition 3.3.3] applies for p replaced by the pseudo-uniformizer ϖ and the ϖ-adically complete representation W_0.

The last claim about the analyticity of a suitable uniform compact open subgroup $G_0 \subset G$ follows from the proof of (1). ∎

Finally, we will need a projection formula for the functor of locally analytic vectors.[7]

Lemma 2.1.6. *Let A be a Banach $\mathbb{Q}_p$-algebra endowed with a locally analytic action of G. Let X and M be G-equivariant Banach A-modules such that M has an ON basis*

[7] This projection formula holds for general G-equivariant A-semilinear solid representations; we keep the assumptions of the lemma for simplicity of exposition.

over A and the action of G on M is locally analytic. Then there is a natural equivalence of $\mathbb{Q}_p$-solid G-representations

$$\underline{X}^{RG\text{-la}} \otimes^L_{\underline{A},\blacksquare} \underline{M} \xrightarrow{\sim} (\underline{X \widehat{\otimes}_A M})^{RG\text{-la}}. \tag{2.1}$$

Proof. By the projection formula for locally analytic vectors [19, Corollary 3.2.14 (3)] we know that, when $A = \mathbb{Q}_p$, the natural map (2.1) is an equivalence. We want to reduce the general case to this situation by applying the bar construction of the relative tensor product; cf. [34, discussion before Proposition 8.6.3] or [16, Construction 4.4.2.7].

First, since $M \cong \widehat{\bigoplus}_I A$, by [18, Lemma 3.13] we know that $\underline{M} = \underline{A} \otimes^L_{\mathbb{Q}_p,\blacksquare} P$ with $P = \widehat{\bigoplus}_I \mathbb{Q}_p$. Then *loc. cit.* implies that

$$\underline{X \widehat{\otimes}_A M} \cong \underline{X \widehat{\otimes}_{\mathbb{Q}_p} P} = \underline{X} \otimes^L_{\mathbb{Q}_p,\blacksquare} \underline{P} \cong \underline{X} \otimes^L_{\underline{A},\blacksquare} \underline{M},$$

in particular $\underline{X} \otimes^L_{\underline{A},\blacksquare} \underline{M} = \underline{X} \otimes_{\underline{A},\blacksquare} \underline{M}$ sits in degree 0. Then, since Banach spaces are flat solid $\mathbb{Q}_p$-vector spaces [18, Lemma 3.21], the bar resolution for the tensor product gives rise to a G-equivariant long exact sequence of solid $\mathbb{Q}_p$-vector spaces

$$\cdots \to \underline{X} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{A}^{\otimes n} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{M} \to \cdots \to \underline{X} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{M} \to \underline{X} \otimes_{\underline{A},\blacksquare} \underline{M} \to 0.$$

Hence, by the projection formula of [19, Corollary 3.2.14 (3)],

$$(\underline{X} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{A}^{\otimes n} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{M})^{RG\text{-la}} = \underline{X}^{RG\text{-la}} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{A}^{\otimes n} \otimes_{\mathbb{Q}_p,\blacksquare} \underline{M}.$$

Since $(-)^{RG\text{-la}}$ has finite cohomological dimension [19, Corollary 3.2.14 (1)], one deduces that

$$\underline{X}^{RG\text{-la}} \otimes^L_{\underline{A},\blacksquare} \underline{M} \xrightarrow{\sim} (\underline{X} \otimes^L_{\underline{A},\blacksquare} \underline{M})^{RG\text{-la}}$$

is an equivalence as desired. ■

Remark 2.1.7. The key input in the proof of Lemma 2.1.6 is the projection formula for locally analytic vectors of [19, Corollary 3.2.14 (3)]. An analogous statement holds for h-analytic vectors for a suitable notion of h-analyticity. Indeed, consider $C^{h^-}(G,\mathbb{Q}_p) = \varinjlim_{h'<h} C^{h'}(G,\mathbb{Q}_p)$, the space of locally analytic functions of G with radius of analyticity strictly greater than p^{-h}. By the proof of [19, Proposition 3.2.10] the functor of h^--analytic vectors defined to be the group cohomology of $C^{h^-}(G,\mathbb{Q}_p) \otimes^L_{\mathbb{Q}_p,\blacksquare} -$ is idempotent and so there is a well defined full subcategory of h^--analytic representations of $\mathcal{D}_\blacksquare(\mathbb{Q}[[G]])$. Then the proof of [19, Corollary 3.2.14] applies, which in particular yields a projection formula for h^--representations. As a consequence, Lemma 2.1.6 also holds for h^--analytic representations.

2.2. *Colmez–Sen–Tate axioms*

In this section we introduce the abstract framework of Sen theory that will be needed in this paper. Our discussion follows [1].

Let $F/\mathbb{Q}_p$ be an algebraic extension with p-adic completion $\widehat{F}$, a perfectoid field. Let A be a sous-perfectoid $\widehat{F}$-algebra endowed with the spectral norm $|-|$ (see [27, Section 6.3] for the definition of sous-perfectoid ring). We let $A^0 \subset A$ be the p-complete open $\mathbb{Z}_p$-algebra of elements of norm ≤ 1, equivalently, the open subalgebra of power bounded elements. For a closed Banach vector subspace $W \subset A$ we let $W^0 = W \cap A^0$ be the open subspace of W of elements of norm ≤ 1. Let $I \subset \mathbb{R}_{\geq 0}$ be the positive additive valuation monoid of $\overline{F}$ normalized so that $|p| = p^{-1}$. Given $\varepsilon \in I \setminus \{0\}$ let $\varpi^{\varepsilon} \in F$ be an algebraic pseudo-uniformizer with absolute value $|\varpi^{\varepsilon}| = |p|^{\varepsilon}$. The additive submonoid $I \subset \mathbb{R}_{\geq 0}$ is dense as $\widehat{F}$ is a perfectoid field, and from now on we always take $\varepsilon \in I$. In the following we shall consider almost mathematics with respect to the maximal ideal $\mathfrak{m}_{\widehat{F}} \subset \mathcal{O}_{\widehat{F}}$.

2.2.1. One-dimensional Sen theory. Let $\Gamma \cong \mathbb{Z}_p$ be a one-parameter group with generator γ. Suppose that Γ acts continuously on A, in particular it acts by isometries with respect to the spectral norm. For $k \in \mathbb{N}_{\geq 0}$, let $\Gamma^{(k)} := \Gamma^{p^k} \subset \Gamma$.

Definition 2.2.1 (Colmez–Sen–Tate axioms). We consider the following axioms on the pair (A, Γ):

(CST1) *Sen traces* (*or Tate's normalized traces*). We are given the following data:

(a) For $n \gg 0$ we have a closed $\mathbb{Q}_p$-Banach subalgebra $A_n \subset A$ stable under the action of Γ.

(b) For $n \gg 0$ we have A_n-linear and Γ-equivariant projection maps $R_n : A \to A_n$, i.e. maps such that the composite $A_n \to A \xrightarrow{R_n} A_n$ is the identity morphism.[8] We write $X_n := \ker R_n$.

The data are subject to the following conditions:

(1) We have $F \subset \bigcup_n A_n$. Let $F_n := \widehat{F} \cap A_n$ and let $I_n \subset I$ be its additive valuation submonoid. For $\varepsilon \in I_n$ we choose $\varpi^{\varepsilon} \in F_n$.

(2) For all $\varepsilon > 0$ there is $n_1(\varepsilon) \gg 0$ such that for $n \geq n_1(\varepsilon)$ and $x \in A^0$ we have $R_n(x) \in \varpi^{-\varepsilon} A_n^0$. In other words, $\|R_n(x)\| \leq |\varpi^{-\varepsilon}|\,|x|$ for $x \in A$.

(3) Given $x \in A$ the sequence $(R_n(x))_{n \in \mathbb{N}}$ converges to x in A.

(4) The action of Γ on the Banach algebra A_n is locally analytic. Equivalently, by Lemma 2.1.5, for any $\varepsilon \in I_n$ there is some $k \in \mathbb{N}$ (depending on n and ε) such that the action of $\Gamma^{(k)} \subset \Gamma$ on $A_n^0 / \varpi^{\varepsilon}$ is trivial.

(CST2) *Bounds for the vanishing of cohomology.* For all $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $n_2(\varepsilon, k) \geq n_1(\varepsilon)$ such that if $n \geq n_2(\varepsilon, k)$ then the map $1 - \gamma^{p^k} : X_n \to X_n$ is invertible with inverse satisfying $\|(1 - \gamma^{p^k})^{-1}\| \leq |\varpi|^{-\varepsilon}$.

We say that the tuple $(A, \Gamma, (R_n)_{n \in \mathbb{N}})$ is a *one-dimensional Sen theory* if the axioms (CST1) and (CST2) hold.

[8] In particular, all the algebras A_n are sous-perfectoid and $A_n^0 = A^0 \cap A_n$ is the subring of power bounded elements. Hence, our notation $W^0 := W \cap A^0$ for closed Banach subspaces $W \subset A$ is consistent with that of Banach rings.

The following lemma will be used later in the paper.

Lemma 2.2.2. *Let $(A, \Gamma, (R_n))$ be a one-dimensional Sen theory. Then the map $\varinjlim_n A_n^0 \to A_\infty^0$ induces an almost isomorphism after p-adic completions. In particular, A is the uniform completion of the colimit $\varinjlim_n A_n$ of Banach rings.*

Proof. Let $B = \varinjlim_n A_n$ and $B^0 = \varinjlim_n A_n^0$. By Nakayama's lemma it suffices to see that the map

$$B^0/\varpi \to A^0/\varpi \tag{2.2}$$

is an almost equivalence. Let us first show that it is almost surjective. Let $a \in A^0$ and $\varepsilon \in I$; we want to show that there is $b \in B^0$ such that $b - \varpi^\varepsilon a \in \varpi A^0$. By (CST1) (3) there is some $n \gg 0$ such that $b := R_n(\varpi^\varepsilon a)$ satisfies $|b - \varpi^\varepsilon a| \leq |\varpi|$. By (CST1) (2) we can even pick n such that $b \in A_n^0$, proving what we wanted.

Now we need to show that the map (2.2) is almost injective; we will even show that it is injective. For that, it suffices to see that $A_n^0 \cap \varpi A^0 = \varpi A_n^0$. But this is clear as $A_n^0 = A_n \cap A^0$. ∎

Example 2.2.3. (1) The most naive example of a Sen theory is a *trivial Sen theory*. Namely, we let A be a Banach ring endowed with a locally analytic action of Γ. Then the Sen traces R_n are just the identity maps on A, and it is straightforward to check that all the axioms of Definition 2.2.1 hold. One may think that this example is strange, but it actually shows up in practice: see Example 2.2.7 (5).

(2) The first non-trivial example of a Sen theory is the classical one presented by Tate [32] and Sen [30], given by an abelian totally wildly ramified extension K_∞ of a discretely valued field K with perfect residue field with $\mathrm{Gal}(K_\infty/K)$ a one-dimensional p-adic Lie group. In this case one takes $A = \widehat{K}_\infty$ to be the completion of the abelian extension, and the Sen traces R_n are constructed from normalized traces [1, Proposition 4.1.1]. The prototypical example is given by the cyclotomic extension $\mathbb{Q}_p(\zeta_{p^\infty})$.

(3) The next example relevant for us arises from geometry. Let $C/\mathbb{Q}_p$ be a perfectoid field containing $\mathbb{Q}_p^{\mathrm{cyc}}$. Let $A = C\langle T^{\pm 1/p^\infty}\rangle$ be the ring of functions on the perfectoid torus and $A_n = C\langle T^{\pm 1/p^n}\rangle$. Let $\Gamma = \mathbb{Z}_p(1)$ be the Tate group of p-power roots of unity, and let us fix a compatible sequence $\gamma = (\zeta_{p^n})_n$ of p-power roots of 1. We have a natural action of Γ on A given by

$$\gamma^a \cdot T^{1/p^k} = \zeta_{p^k}^a T^{1/p^k} \tag{2.3}$$

for all $a \in \mathbb{Z}_p$. The ring A has a standard Banach C-basis given by the elements T^r for $r \in \mathbb{Z}[1/p]$. We have maps $R_n : A \to A_n$ given by taking the projection on the elements $\{T^r\}_r$ in the basis with $|r|_p \leq |p^{-n}|$. It is straightforward to check that the axioms of Definition 2.2.1 hold in this situation. Indeed, in this case one even has $\|R_n\| \leq 1$ as operators, i.e. that they send power bounded elements in A to power bounded elements in A_n. Letting $X_n = \ker R_n$, one sees that X_n is the Banach direct sum of the elements T^r with $|r| > |p^{-n}|$. By (2.3), for $k \leq n$ one sees that

$$(1 - \gamma^{p^k})T^r = (1 - \zeta_{p^{s-k}}^a)T^r$$

where $r = a/p^s$ with $(a, p) = 1$. This immediately shows that $1 - \gamma^{p^k}$ is invertible on X_n. Since $|1 - \zeta_{p^m}^b|_p = p^{-\frac{1}{(p-1)p^{m-1}}}$ for $(b, m) = 1$, we find that the inverse operator $(1 - \gamma^{p^k})^{-1}$ has norm $\leq |(1 - \zeta_{p^{n+1-k}})^{-1}|_p = p^{\frac{1}{(p-1)p^{n-k}}}$.

(4) We will also need a variant of the previous example. In this case we take $A = C\langle S^{1/p^\infty}\rangle$, the ring of functions on a perfectoid affinoid disc of radius 1. We take $\Gamma = \mathbb{Z}_p(1)$ as before acting in the obvious way, the rings $A_n = C\langle S^{1/p^n}\rangle$ and the maps $R_n : A \to A_n$ given by projecting to the basis elements $\{S^r\}_r$ with $|r| \leq |p^{-n}|$.

Remark 2.2.4. Strictly speaking, in order to have a Sen theory in Example 2.2.3 (2) in the sense of Definition 2.2.1, one would need to take the restriction to an open subgroup of the Galois group. Since Sen theory is of infinitesimal nature, i.e. it only depends on open neighbourhoods of the identity of the group, we will omit this detail now and in the future.

2.2.2. Products of Sen theories. In the previous section we defined a Sen theory for one-dimensional p-adic Lie groups. In the following we extend the previous axioms to the case when $\Gamma \cong \mathbb{Z}_p^d$ as a p-adic Lie group. We fix a basis $\gamma_i \in \Gamma$ with $i = 1, \dots, d$ and let $\Gamma_i := \gamma_i^{\mathbb{Z}_p} \subset \Gamma$ be the one-parameter subgroup induced by γ_i. Finally, consider a continuous action of Γ on a sous-perfectoid ring A.

Definition 2.2.5. A *d-dimensional Sen theory* $(A, \Gamma, (R_n^i)_{i=1}^d)$ for (A, Γ) is the datum of Sen theories $(A, \Gamma_i, (R_n^i)_{n\in\mathbb{N}})$ in the sense of Definition 2.2.1 (with closed subrings A_n^i) for all $i = 1, \dots, d$, satisfying the following properties:

(a) For all $i, j \in \{1, \dots, d\}$ the Sen traces commute, i.e. $R_n^i R_m^j = R_m^j R_n^i$ for all $n, m \geq 0$.

(b) The subrings A_n^i are Γ-stable and the Sen traces R_n^i are Γ-equivariant.

For all $J \subset \{1, \dots, d\}$ and all tuples $\underline{\mathbf{n}} = (n_j)_{j\in J} \in \mathbb{N}^J$ we let $A_{\underline{\mathbf{n}}}^J \subset A$ be the closed subring consisting of the image of the operator $\prod_{j\in J} R_{n_j}^j : A \to A$. For $n \gg 0$ we will also simply write A_n for the image of $\prod_{i=1}^d R_n^i$.

Remark 2.2.6. Keep the notation of Definition 2.2.5 and let $J \subset \{1, \dots, d\}$. By an inductive argument, Lemma 2.2.2 implies that A is the uniform completion of the colimit $\varinjlim_{\underline{\mathbf{n}}\in\mathbb{N}^J} A_{\underline{\mathbf{n}}}^J$ of Banach rings, namely, by induction one can show that the map $\varinjlim_{\underline{\mathbf{n}}\in\mathbb{N}^J} A_{\underline{\mathbf{n}}}^{J,0} \to A^0$ is an almost isomorphism after p-completion.

Example 2.2.7. (1) A first example of this kind of multivariable Sen theory appears in the work of Brinon [3] who introduces generalizations of Sen theories in the case of non-archimedean fields with imperfect residue characteristic. Examples of such fields arise from completions of fields on rigid varieties at Zariski generic points in their Berkovich or adic spectrum. Since this theory will not be necessary for us, we will not attempt to precisely construct a d-dimensional Sen theory as in Definition 2.2.5 in this situation.

(2) The main example for this paper will arise from products of tori. For later convenience, we will introduce the relevant geometric notation.

Let $C/\mathbb{Q}_p$ be a perfectoid field containing $\mathbb{Q}_p^{\mathrm{cyc}}$ and $C^+ \subset C$ an open bounded integrally closed subring. Let $\mathbb{T}_C := \mathrm{Spa}(C\langle T^{\pm 1}\rangle, C^+\langle T^{\pm 1}\rangle)$ be the torus of dimension 1 with coordinate T over $\mathrm{Spa}(C, C^+)$. For integers $k, n \geq 0$ we let $\mathbb{T}^k_{C,n}$ be the k-dimensional torus with coordinates $T_1^{1/p^n}, \ldots, T_k^{1/p^n}$. For $n = \infty$ we let $\mathbb{T}^k_{C,\infty} = \varprojlim_n \mathbb{T}^k_{C,n}$ be the k-dimensional perfectoid torus with variables $T_1^{1/p^\infty}, \ldots, T_k^{1/p^\infty}$. Let A be the ring of functions on $\mathbb{T}^k_{C,n}$ and let $\Gamma = \mathbb{Z}_p(1)^k$ act on A in the natural way. By taking completed tensor products over C, the maps R^i_n on the one-dimensional torus with coordinate T_i of Example 2.2.3 (3) give rise to a Sen theory $(A, \Gamma, (R^i_n)_n)$ in the sense of Definition 2.2.5. Note that in this case, the map $\mathbb{T}^k_{C,\infty} \to \mathbb{T}^k_C$ is a pro-finite-étale $\mathbb{Z}_p(1)^k$-torsor.

(3) A variant of the previous example that will be needed in this paper is obtained by discs instead of tori; we introduce the relevant geometric notation. Let $\mathbb{D}_C := \mathrm{Spa}(C\langle S\rangle, C^+\langle S\rangle)$ be the one-dimensional disc over C with coordinate S. We endow $\mathbb{D}_C$ with the log structure arising from $S = 0$ (cf. [6, Example 2.3.17]). For integers $k, n \geq 0$ let $\mathbb{D}^k_{C,n}$ be the k-dimensional polydisc with coordinates $S_1^{1/p^n}, \ldots, S_k^{1/p^n}$. Let $\mathbb{D}^k_{C,\infty} = \varprojlim_n \mathbb{D}^k_{C,n}$ be the perfectoid polydisc of dimension k and coordinates $S_1^{1/p^\infty}, \ldots, S_k^{1/p^\infty}$. By letting A be the ring of functions on $\mathbb{D}^k_{C,\infty}$ and $\Gamma = \mathbb{Z}_p(1)^k$, the one-dimensional Sen theories of Example 2.2.3 (4) will combine together to a Sen theory $(A, \Gamma, (R^i_n)_{n,i})$ in the sense of Definition 2.2.5. Note that in this case the map $\mathbb{D}^k_{C,\infty} \to \mathbb{D}^k_C$ is a pro-Kummer-étale $\mathbb{Z}_p(1)^k$-torsor.

(4) By mixing the previous two examples of Sen theories, we can construct a Sen theory for the product $\mathbb{S}_C^{(e,d-e),\infty} := \mathbb{T}^e_{C,\infty} \times \mathbb{D}^{d-e}_{C,\infty}$ of tori and polydiscs with coordinates $T_1, \ldots, T_e, S_{e+1}, \ldots, S_d$. We shall write $\mathbb{S}^{(e,d-e)}_{C,n} := \mathbb{T}^d_{C,n} \times \mathbb{D}^{d-e}_{C,n}$ for $n \in \mathbb{N} \cup \{\infty\}$. By letting A be the ring of functions on $\mathbb{S}^{(e,d-e)}_{C,\infty}$ and $\Gamma = \mathbb{Z}_p(1)^d$, one has a natural action of Γ on A and the Sen theories of (2) and (4) combine to a Sen theory on (A, Γ).

(5) We finally discuss an example that combines the mixed case of (4) and the trivial Sen theory of Example 2.2.3 (1). Let $\mathbb{S}^{(e,d-e)}_C$ be as in (4). We see $\mathbb{S}^{(e,d-e)}_C$ as a log adic space endowed with log structure arising from the normal crossing divisor $S_{e+1} \cdots S_d = 0$. For $i \in \{e+1, \ldots, d\}$ let $D_i \subset \mathbb{S}^{(e,d-e)}_C$ be the divisor defined by $S_i = 0$, and given $J \subset \{e+1, \ldots, d\}$ we let $D_J = \bigcap_{i \in J} D_i$. We endow D_J with the log structure $\mathcal{M}$ given by pullback along $D_J \to \mathbb{S}^{(e,d-e)}_C$. For $n \in \mathbb{N} \cup \{\infty\}$ let $D_{J,n} = \mathbb{S}^{(e,d-e)}_{C,n} \times_{\mathbb{S}^{(e,d-e)}_C} D_J$ where the fibre product is an fs log adic space [6, Proposition 2.3.27]. Without loss of generality assume that $J = \{i+1, \ldots, d\}$ for some $i \geq e$. The underlying adic space of $D_{J,\infty}$ is the perfectoid space

$$D_J = \mathrm{Spa}(C\langle \underline{T}^{\pm \frac{1}{p^\infty}}, S_{e+1}^{\frac{1}{p^\infty}}, \ldots, S_i^{\frac{1}{p^\infty}}\rangle, C^+\langle \underline{T}^{\pm \frac{1}{p^\infty}}, S_{e+1}^{\frac{1}{p^\infty}}, \ldots, S_i^{\frac{1}{p^\infty}}\rangle) = \mathbb{S}^{(e,i-e)}_C.$$

On the other hand, the log structure $\mathcal{M}$ of D_∞ is induced by the map of monoids

$$\delta : \left(\mathbb{N}\left[\frac{1}{p}\right]\right)^d \to A_J := C\langle \underline{T}^{\pm \frac{1}{p^\infty}}, S_{e+1}^{\frac{1}{p^\infty}}, \ldots, S_i^{\frac{1}{p^\infty}}\rangle$$

mapping the j-th component to S_j if $j \leq i$, and to 0 if $j > i$. In particular, $(D_{J,\infty}, \mathcal{M}_\infty) \to (D_J, \mathcal{M})$ is a $\Gamma = \mathbb{Z}_p(1)^d$-torsor in the pro-Kummer-étale site of $(D_J, \mathcal{M})$. Note, however, that only the first $e+i$ components of Γ act non-trivially on A_J. The natural normalized traces attached to the coordinates $T_1, \ldots, T_d, S_{e+1}, \ldots, S_i$ lead to Sen traces on A_J and they give rise to a d-dimensional Sen theory for the pair (A_J, Γ), where the last $d-i$ components act trivially on A_J.

2.2.3. Base change and retracts of Sen theories. In this subsection we prove some stability properties of Sen theories that will be necessary for our applications. First, we need stability under retracts in a suitable sense:

Lemma 2.2.8. *Let $\Gamma = \mathbb{Z}_p$ and consider a one-dimensional Sen theory $(A, \Gamma, (R_n))$, and let $B \subset A$ be a closed Banach $\widehat{F}$-subalgebra stable under Γ and $\pi : A \to B$ a Γ-equivariant B-linear projection. Suppose that $\pi R_n = R_n \pi$ for all $n \gg 0$ so that the maps R_n restrict to $R'_n : B \to B$. Then the tuple $(B, \Gamma, (R'_n)_n)$ is a one-dimensional Sen theory.*

Proof. Since B is a retract of A via the map π, B is sous-perfectoid and we see that $B^0 = B \cap A^0$. Thus, the restriction of the spectral norm of A to B agrees with the spectral norm of B. The condition $\pi R_n = R_n \pi$ implies that the Sen traces R_n leave B stable, and so their images define closed Banach subalgebras $B_n = B \cap A_n$. The axiom (CST1) of Definition 2.2.1 holds for (B, Γ, R'_n). It is left to see that (CST2) holds, so let $\varepsilon \in I$ and let $n \gg k$ be such that $1-\gamma^{p^k}$ is invertible on X_n with inverse $(1-\gamma^{p^k})^{-1}$ bounded by $\varpi^{-\varepsilon}$. Let X'_n be the kernel of R'_n. We have $X'_n = X_n \cap B$ and so $1-\gamma^{p^k}$ is also invertible on X'_n (since the projection π must commute with $(1-\gamma^{p^k})^{-1}$ in X_n) and its inverse is bounded by $\varpi^{-\varepsilon}$. This proves the lemma. ∎

To extend Sen theories along étale maps we will need the following approximation of étale maps for affinoid perfectoid rings.

Lemma 2.2.9. *Let $\{A_i\}_{i \in I}$ be a filtered system of sous-perfectoid $\widehat{F}$-algebras with uniform completed colimit $A = (\varinjlim_i A_i^0)^{\wedge\text{-}p}[\frac{1}{p}]$ given by a perfectoid ring (where the subscript $(-)^{\wedge\text{-}p}$ denotes p-adic completion). Let $A \to A'$ be an étale extension of A that factors as a composite of finite étale extensions and rational localizations.*

1. *There is some index i and an étale extension $A_i \to A'_i$ that factors as a composite of finite étale maps and rational localizations such that $A' = A \widehat{\otimes}_{A_i} A'_i$.*
2. *Let $i \in I$ and let $A_i \to A'_i$ be an étale map that factors as a composite of rational localizations and finite étale maps. For $j \geq i$ let $A'_j := A_j \otimes_{A_i} A'_i$. Then for any $\varepsilon > 0$ there is some $j \geq i$ such that the map*
$$A'^0_j \,\widehat{\otimes}_{A_j^0} A^0 \to A'^0$$
has cokernel killed by ϖ^{ε}.

Proof. Part (1) follows from [23, Lemma 6.13 (ii)] for rational localizations and from [11, Proposition 5.4.53] for finite étale maps. The general case follows by an inductive argument.

Part (2) is [24, Lemma 4.5 (iii)]. Note that, in the notation of *loc. cit.* the fact that the R_i are Tate algebras of finite type is never used in the proof, only that they are uniform Tate algebras and that the étale extension $R_i \to S_i$ remains uniform. This holds in our situation since the algebras A_i are sous-perfectoid. ■

The following lemma extends actions of profinite groups along étale maps.

Lemma 2.2.10. *Let Π be a profinite group acting continuously on a sous-perfectoid ring A over $\mathbb{Q}_p$. Let $A \to A'$ be an étale map that factors as a composite of finite étale maps and rational localizations. Then there is an open compact subgroup $\Pi' \subset \Pi$ and a continuous action ρ of Π' on A extending that on A. Given two such extensions ρ_1 to Π'_1 and ρ_2 to Π'_2, there is a compact open subgroup $\Pi'_3 \subset \Pi'_1 \cap \Pi'_2$ such that ρ_1 and ρ_2 agree on Π'_3. Furthermore, if Π is a p-adic Lie group and the action on A is locally analytic, then the action of Π' on A' is also locally analytic.*

Proof. One can separate the problem into one for open immersions and one for finite étale maps. For open immersions $U = \mathrm{Spa}(A', A'^{+}) \subset \mathrm{Spa}(A, A^0)$ it suffices to take the stabilizer G_U of U in G. The case of finite étale maps is [26, Lemma] (except that there it is stated only for strongly noetherian Tate rings but the argument works in general).

By shrinking Π we can assume that it acts on both A and A'. It is left to show that the action of Π on A' is locally analytic if it is on A. Since the map $A \to A'$ is topologically of finite type (as it factors as a composite of finite étale maps and rational localizations, and each of these maps is topologically of finite type), there is a ring of definition $A_0 \subset A$ and elements $f_1, \ldots, f_k \in A'^0$ such that the p-complete algebra $A'_0 := A_0\langle f_1, \ldots, f_k\rangle \subset A'$ generated by the elements f_i is a ring of definition of A'. By Lemma 2.1.5 it suffices to show that the action of Π on A'_0/p factors through a finite quotient. Since the action on A is locally analytic, the same lemma implies that the action on A_0/p factors through a finite quotient, and by trivializing the action of the elements f_i modulo p (which is possible since the action of Π on A'_0/p is smooth) one gets the desired factorization. ■

Lemma 2.2.11. *Let $(A, \Gamma, (R_n)_n)$ be a one-dimensional Sen theory. Let $n \gg 0$ be an integer such that there is a Banach basis $\{e_i\}_{i\in I}$ of A^0 as an A^0_n-module (contained in A^0) such that*

- *$I = \bigcup_{k\geq n} I_k$ with $\{I_k \subset I\}_k$ a sequential system of finite subsets,*
- *for $m \geq n$,*
$$A^0_m = \bigoplus_{i\in I_m} e_i A^0_n,$$
- *for $m \geq n$ the Sen traces $R_m : A \to A_m$ are given by the projection along the basis $\{e_i\}_{i\in I_m}$.*

Then the following hold:

(1) *The solid base change along $A_n \to A$ is flat.*

(2) *Let M be a Banach A_n-module and consider the base change $M_A = M \widehat{\otimes}_{A_n} A$ as well as $M_m = M \widehat{\otimes}_{A_n} A_m$. Then $M_m \subset M_A$ is a closed Banach A-module, the base*

change of $R_m : A \to A_m$ gives rise to a retraction $R_m : M_A \to M_m$, and for $x \in M_A$ the sequence $(R_m(x))_m$ of elements in M_A converges to x.

Proof. By hypothesis, A is a Banach direct sum of copies of A_m for $m \geq n$ giving rise to an isomorphism $A \cong A_n \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\bigoplus}_I \mathbb{Q}_p$ of A_n-modules. Since Banach $\mathbb{Q}_p$-vector spaces are flat for the solid tensor product [18, Lemma 3.21], by base change the solid tensor product along $A_m \to A$ is flat for all $m \geq n$, and then so is the p-complete tensor product of Banach A_m-modules. The fact that $M_m \subset M_A$ is a closed Banach A_m-module and that we have the retraction $M_A \to M_m$ is clear. This proves (1).

For part (2), by assumption on A we can write

$$M_A = \widehat{\bigoplus_{i \in I}} e_i M$$

and the projection maps $M_A \to M_n$ are given by projecting onto a suitable subbasis of $\{e_i\}_{i \in I}$. The convergence of the sequence $(R_m(x))_x$ to $x \in M_A$ is then clear. ∎

Finally, the following lemma extends Sen theories along étale maps under certain hypotheses that will hold in practice.

Lemma 2.2.12. *Let $\Gamma = \mathbb{Z}_p$ and consider a Sen theory $(A, \Gamma, (R_n)_n)$ as in Definition 2.2.1 with A perfectoid. Let $A \to A'$ be an étale map obtained as a composite of finite étale maps and rational localizations.*

(1) *There is some $k \gg 0$ and an étale map $A_k \to A'_k$ that factors as a composite of rational localizations and finite étale maps such that $A' \cong A \otimes_{A_k} A'_k$. Furthermore, after taking a smaller open subgroup if necessary, the action of Γ on A_k extends to A'_k and so to A'. For $n \geq k$ let $R'_n : A' \to A'_n$ be the base change of the map $R_n : A \to A_n$ along $A_k \to A'_k$.*

(2) *Suppose that there is an integer $n \gg 0$ such that there is an A_n^0-Banach basis $\{e_i\}_{i \in I}$ as in Lemma 2.2.11. Then the tuple $(A', \Gamma, (R'_n)_n)$ is a Sen theory in the sense of Definition 2.2.1.*

Proof. Part (1) follows from Lemmas 2.2.9 (1) and 2.2.10. Indeed, A is the uniform completion of $\varinjlim_n A_n$ by Lemma 2.2.2.

Let us now suppose that $A_k \to A'_k$ is an étale map that factors as a composite of rational localizations and finite étale maps, and suppose that the action of Γ on A_k extends to A'_k. For $n \geq k$ let $A'_n := A'_k \widehat{\otimes}_{A_k} A_n$ and let $R'_n : A' \to A'_n$ be the A'_k-linear extension of the map $R_n : A \to A_n$. We want to see that the axioms (CST1) and (CST2) of Definition 2.2.1 hold for $(A', \Gamma, (R'_n)_n)$ provided they do for $(A, \Gamma, (R_n)_n)$.

First, (CST1) (1) is obvious, (CST1) (3) follows from Lemma 2.2.11 when $M = A'_k$, and (CST1) (4) follows from Lemma 2.2.10. We need to show that the Sen traces R'_n satisfy the desired bound condition of (CST1) (2). Let $\varepsilon > 0$. By Lemma 2.2.9 there is some $N \gg 0$ such that for $m \gg N$ the map

$$A'^0_m \widehat{\otimes}_{A^0_m} A^0 \to A'^0$$

has cokernel killed by ϖ^{ε}. Note that the map must be injective since A^0 has a Banach basis as an A^0_m-module. In particular, $A'^0_m \widehat{\otimes}_{A^0_m} A^0$ is a ring of definition of A'. Similarly, by (CST1) (2) for $(A, \Gamma, (R_n)_n)$ we can take $N \gg 0$ such that for all $m \gg N$ the trace R_m induces a map

$$R_m : A^0 \to \varpi^{-\varepsilon} A^0_m.$$

This implies that the base change $R'_m : A' \to A'_m$ induces a map

$$R'_m : A'^0_m \widehat{\otimes}_{A^0_m} A^0 \to \varpi^{-\varepsilon} A'^0_m.$$

Hence, R'_m induces a map $R'_m : A'_0 \to \varpi^{-2\varepsilon} A'^0_m$. Since ε was arbitrary, this proves the axiom.

To show that (CST2) holds, let $X_m = \ker R_m$ and let X'_m be its base change along $A_k \to A'_k$. By hypothesis, we can write $A^0 = X^0_m \oplus A^0_m$, and $R_m|_{A^0}$ is given by the projection onto A^0_m under this decomposition. In particular, the maps $X^0_m \widehat{\otimes}_{A^0_m} A'^0_m \to X^0_m \widehat{\otimes}_{A^0_m} A'^0 \to X'^0_m := X'_m \cap A'^0$ are injective.

Let $\varepsilon > 0$. By taking m large enough we can assume that the injective map $X^0_m \widehat{\otimes}_{A^0_m} A'^0_m \to X'^0_m$ has cokernel killed by ϖ^{ε}. Therefore, $\widetilde{X}'^0_m := X^0_m \widehat{\otimes}_{A^0_m} A'^0_m$ is open in X'^0 and we can prove (CST2) using the subspace $\widetilde{X}'^0_m$ instead of X'^0_m. Let $\gamma \in \Gamma$ be a generator. By (CST2) for $(A, \Gamma, (R_n)_n)$, for $\varepsilon > 0$ and $s \in \mathbb{N}$ there is $n_2(\varepsilon, s) \geq n_1(\varepsilon)$ such that if $n \geq n_2(\varepsilon, s)$ then the map $1 - \gamma^s : X_n \to X_n$ has an inverse $(1 - \gamma^{p^s})^{-1}$ of norm $\leq |\varpi|^{-\varepsilon}$. This means that the map $1 - \gamma^{p^s} : X^0_n \to X^0_n$ has an inverse up to an ϖ^{ε}-factor

$$(1 - \gamma^{p^s})^{-1} : X^0_n \to \varpi^{-\varepsilon} X^0_n.$$

On the other hand, by taking $s \gg 0$, since the action of Γ on A_k is locally analytic, by Lemma 2.1.5 we can assume that γ^{p^s} acts trivially on $A^0_k / \varpi^{2\varepsilon}$ and $A'^0_k / \varpi^{2\varepsilon}$. Consider the image $Y^0_n = (1 - \gamma^{p^s}) X^0_n$. The space Y^0_n is contained in X^0_n, contains $\varpi^{\varepsilon} X^0_n$, and is stable under multiplication by A^0_k.

By assumption (2) the A^0_k-module X^0_n has an ON basis, say $\{f_j\}_{j \in J}$. Since the elements ϖ^{ε} are algebraic over $\mathbb{Q}_p$, we can choose $s \gg 0$ such that γ^{p^s} is ϖ^{ε}-linear. Since $1 - \gamma^{p^s}$ is an isomorphism on X^0_n after inverting p, the above shows that the image of $\{f_j\}_{j \in J}$ by $1 - \gamma^{p^s}$ gives rise to an ON basis of Y^0_n as an A^0_k-module. Indeed, one has an A^0_k-linear map of torsion free ϖ-complete A^0_k-modules

$$\widehat{\bigoplus_{j \in J}} A^0_k (1 - \gamma^{p^s}) f_j \to Y^0_{\varpi}$$

which is an equivalence modulo $\varpi^{2\varepsilon}$, so by Nakayama's lemma it must be an isomorphism.

We deduce that the base change $Y'^0_n = Y^0_n \otimes_{A^0_k} A'^0_k$ is an ON Banach A'^0_k-module and that it gives rise to an open ball in $Y'^0_n[\frac{1}{p}] = X'^0_n$ such that

$$\varpi^{\varepsilon} \widetilde{X}'^0_n \subset Y'^0_n \subset \widetilde{X}'^0_n.$$

Finally, one sees that the induced map $1-\gamma^{p^s}: X'_n \to X'_n$ sends $\widetilde{X}_n'^0$ to $Y_n'^0$. By reducing modulo $\varpi^{2\varepsilon}$ this map is $A_k'^0/\varpi^{2\varepsilon}$-linear and is the base change along $A_k^0 \to A_k'^0$ of the isomorphism

$$1-\gamma^{p^s}: X_n^0/\varpi^{2\varepsilon} \xrightarrow{\sim} Y_n^0/\varpi^{2\varepsilon}.$$

By Nakayama's lemma this implies that $1-\gamma^{p^s}: \widetilde{X}_n'^0 \xrightarrow{\sim} Y_n'^0$ is also an isomorphism. One deduces that $1-\gamma^{p^s}: X'_n \to X'_n$ has an inverse $(1-\gamma^{p^s})^{-1}$ with norm $\leq |\varpi^{-\varepsilon}|$ with respect to the Banach ball $\widetilde{X}_n'^0$. Since $\widetilde{X}_n'^0 \subset X_n'^0$ has cokernel killed by ϖ^{ε}, one deduces that the norm of $(1-\gamma^{p^s})^{-1}$ with respect to the ball $X_n'^0$ is $\leq |\varpi^{-2\varepsilon}|$. As ε was arbitrary, this shows (CST2) and finishes the proof of the lemma. ∎

Remark 2.2.13. Thanks to Examples 2.2.3 (2)–(4), Lemma 2.2.12 applies for the perfectoid torus, the perfectoid disc and the cyclotomic tower. This will cover all the relevant applications in this paper.

We conclude with the construction of Sen theories for smooth affinoid spaces. We keep the notation of Example 2.2.7.

Proposition 2.2.14. *Let $(X, \mathcal{M}_X)$ be a log affinoid adic space over a perfectoid field (C, C^+) containing $\mathbb{Q}_p^{\mathrm{cyc}}$ with $X = \mathrm{Spa}(A, A^+)$. Suppose that there is an étale chart*

$$\psi : X \to \mathbb{S}_C^{(e,d-e)}$$

written as a composite of finite étale maps and rational localizations, where the log structure of X arises from the log structure of the polydisc term on the right-hand side. Let $\mathbb{S}_{K,\infty}^{(e,d-e)}$ be the perfectoid product of tori and discs endowed with the action of $\Gamma = \mathbb{Z}_p(1)^d$, and let $X_\infty = X \times_{\mathbb{S}_C^{(e,d-e)}} \mathbb{S}_{C,\infty}^{(e,d-e)}$ be its base change to a pro-Kummer-étale Γ-torsor of X. Let A_∞ be the ring of functions of X_∞, and let $R_n'^i$ be the base change of the Sen traces of $\mathbb{S}_{C,\infty}^{(e,d-e)}$ to A_∞. Then the tuple $(A_\infty, \Gamma, (R_n'^i)_n)$ is a d-dimensional Sen theory. A similar statement holds for the rings of functions $A_{\infty,J}$ obtained from boundary divisors for some subset $J \subset \{e+1, \dots, d\}$ as in Example 2.2.7 (5).

Proof. The one-dimensional case follows from Lemma 2.2.12, and the higher-dimensional case follows by induction on the number of coordinates. ∎

2.3. *Relative locally analytic representations*

Let A be a sous-perfectoid ring and let $A^0 \subset A$ be the subring of power bounded elements. Let Π be a profinite group acting continuously on A.

The goal of this section is to introduce a relative analogue of ON Banach locally analytic representations over A for the semilinear action of Π. The motivation is provided by Lemma 2.1.5, saying that a continuous action of a compact p-adic Lie group G on a Banach space V is locally analytic if and only if there is a G-stable lattice $V^0 \subset V$ such that G acts through a finite quotient on V^0/p.

Moreover, in order to adapt the decompletions of [1] in Section 2.4 for relative locally analytic representations, we shall need to study infinite-dimensional analogues of 1-cocycles. This requires a topological understanding of the automorphism group of an infinite-dimensional ON Banach space:

Lemma 2.3.1. *For I and J index sets, we have isomorphisms of A- and A^0-modules*

$$\begin{aligned} \mathrm{Hom}_A\Big(\widehat{\bigoplus_I} A, \widehat{\bigoplus_J} A\Big) &= \Big(\prod_I \widehat{\bigoplus_J} A^0\Big)\Big[\frac{1}{p}\Big], \\ \mathrm{Hom}_{A^0}\Big(\widehat{\bigoplus_I} A^0, \widehat{\bigoplus_J} A^0\Big) &= \prod_I \widehat{\bigoplus_J} A^0, \end{aligned} \tag{2.4}$$

where the Hom *spaces consist of continuous homomorphisms of A- or A^0-modules. We endow the* Hom_{A^0} *space with its natural product topology, and the* Hom_A *space with the locally convex topology making the* Hom_{A^0} *subspace bounded. Equivalently, we can endow the* Hom_{A^0} *and* Hom_A *spaces with the compact-open topology* (*these two topologies agree thanks to* [5, Proposition 4.2]).

Proof. We have $\mathrm{Hom}_A(\widehat{\bigoplus}_I A, \widehat{\bigoplus}_J A) = \mathrm{Hom}_{A^0}(\widehat{\bigoplus}_I A^0, \widehat{\bigoplus}_J A^0)[\frac{1}{p}]$, so it suffices to prove the second equality. One has

$$\begin{aligned} \mathrm{Hom}_{A^0}\Big(\widehat{\bigoplus_I} A^0, \widehat{\bigoplus_J} A^0\Big) &= \varprojlim_n \mathrm{Hom}_{A^0/p^n}\Big(\bigoplus_I A^0/p^n, \bigoplus_J A^0/p^n\Big) \\ &= \varprojlim_n \prod_I \bigoplus_J A^0/p^n = \prod_I \widehat{\bigoplus_J} A^0. \end{aligned}$$

■

Remark 2.3.2. (1) Let V be an ON Banach A-module with a fixed ON lattice V^0 over A^0. Classically, $\mathrm{End}_A(V)$ is endowed with two topologies. First, we have the strong (or operator norm) topology, in which case it is a Banach space. Concretely, after fixing the norm in V arising from V^0, the elements of norm ≤ 1 in $\mathrm{End}_A(V)$ correspond to $\mathrm{End}_{A^0}(V^0)$.

On the other hand, the topology used in Lemma 2.3.1 is the compact-open topology, which naturally agrees with the condensed structure and with the weak topology. One has a natural continuous map

$$\mathrm{End}'_A(V) \to \mathrm{End}_A(V)$$

where the left-hand side is endowed with the strong topology while the right-hand side is endowed with the weak topology. This map is an isomorphism of the underlying vector spaces but it is not an isomorphism of condensed A-modules. However, the existence of this map shows that a sequence of endomorphisms of V that converges for the operator norm topology will also converge for the compact-open topology.

(2) We can reinterpret Lemma 2.1.5 by saying that a representation of a p-adic Lie group G on a Banach space V is locally analytic if and only if the action is continuous with respect to the norm topology; we thank the referee for this observation.

Definition 2.3.3. Let V be an ON Banach A-module. We define the topological group $\mathrm{Aut}_A(V)$ to be the pullback

$$\begin{array}{ccc} \mathrm{Aut}_A(V) & \longrightarrow & \mathrm{End}_A(V)\times \mathrm{End}_A(V) \\ \downarrow & & \downarrow \phi \\ \{\mathrm{id}_V\}\times\{\mathrm{id}_V\} & \longrightarrow & \mathrm{End}_A(V)\times \mathrm{End}_A(V) \end{array}$$

where $\phi(f,g)=(f\circ g, g\circ f)$. Equivalently, it is the closed subspace of $\mathrm{End}_A(V)\times \mathrm{End}_A(V)$ consisting on those pairs (f,g) such that $f\circ g=\mathrm{id}_V=g\circ f$. If V^0 is an ON A^0-lattice on V, we define $\mathrm{Aut}_{A^0}(V^0)$ in a similar way.

The following lemma will be used to construct invertible elements in $\mathrm{Aut}_A(V)$.

Lemma 2.3.4. *Let V be an ON Banach A-module with a fixed ON lattice V^0 over A^0. Let $M\in \mathrm{End}_A(V)$ be an endomorphism whose operator norm with respect to V^0 satisfies $\|M\|\leq |\varpi^{\varepsilon}|$ for some $\varepsilon>0$. Then $1-M\in \mathrm{Aut}_A(V)$ and its inverse is given by the convergent series $(1-M)^{-1}=\sum_{n=0}^{\infty} M^n$.*

Proof. Write $V^0=\widehat{\bigoplus}_I A^0$ so that $\mathrm{End}_A(V)\cong (\prod_I \widehat{\bigoplus}_I A^0)[\frac{1}{p}]$. It is enough to show that $\sum_{n=0}^{\infty} M^n$ converges in $\mathrm{End}_A(V)$, and that the sequence $((1-M)\sum_{n=0}^{m} M^n)_{m\in\mathbb{N}}$ converges to id_V. By hypothesis, $M'=\frac{1}{\varpi^{\varepsilon}}M$ is an operator on V^0, thus $\sum_{n=0}^{\infty}M^n=\sum_{n=0}^{\infty}\varpi^{\varepsilon n}M'^n$ converges since $\mathrm{End}_{A^0}(V^0)\cong\prod_I\widehat{\bigoplus}_I A^0$ is p-adically complete, and both ϖ^{ε} and p are topologically nilpotent units of A. One shows in a similar way that the sequence $((1-M)\sum_{n=0}^{m}M^n)_{m\in\mathbb{N}}$ converges to id_V, finishing the proof. ∎

Let Π be a profinite group acting continuously on A. Given an index set I, we denote by e_i the standard basis of $\widehat{\bigoplus}_I A$, and write $\mathrm{GL}_I(A)=\mathrm{Aut}_A(\widehat{\bigoplus}_I A)$ (and similarly for $\mathrm{GL}_I(A^0)$). The groups $\mathrm{GL}_I(A)$ and $\mathrm{GL}_I(A^0)$ have a natural continuous action of Π on the coefficients. Then ON Banach A-semilinear representations of "rank I" with fixed ON basis are equivalent to continuous 1-cocycles of Π on $\mathrm{GL}_I(A)$.

More precisely, consider an A-semilinear representation $\rho:\Pi\times V\to V$ on an ON A-Banach module and let $\Upsilon:\widehat{\bigoplus}_{i\in I}Ae_i\xrightarrow{\sim}V$ be an A-linear isomorphism. Let σ be the A-semilinear action on $\widehat{\bigoplus}_{i\in I}Ae_i$ where Π acts trivially on the e_i's. One defines the 1-cocycle attached to ρ as

$$U_\rho:\Pi\to \mathrm{GL}_I(A) \tag{2.5}$$

via the formula

$$U_\rho(g)=\Upsilon^{-1}\circ\rho(g)\circ\Upsilon\circ\sigma(g)^{-1}.$$

Conversely, given $U:\Pi\to\mathrm{GL}_I(A)$ a continuous 1-cocycle, one has the attached representation ρ_U on V via the formula

$$\rho_U(g)=\Upsilon\circ U(g)\circ\sigma(g)\circ\Upsilon^{-1}.$$

One easily verifies that this produces a bijection between A-linear representations of Π on V (with a fixed ON basis) and continuous 1-cocycles of Π on $\mathrm{GL}_I(A)$.

We give the following definition of relative locally analytic ON Banach representation.

Definition 2.3.5. An ON Banach A-semilinear representation $\rho : \Pi \times V \to V$ is said to be *relative locally analytic* if there exists a basis $\{v_i\}_{i\in I}$ generating an ON lattice V^0 over A^0 such that

- there is an open subgroup $\Pi' \subset \Pi$ stabilizing V^0 and $\varepsilon > 0$ such that the action of Π' on $\{v_i \bmod \varpi^{\varepsilon}\}_{i\in I}$ is trivial.

We say that $\{v_i\}_{i\in I}$ is a *relative locally analytic basis* of V.

Remark 2.3.6. The previous definition can be rewritten in terms of 1-cocycles. Indeed, let ρ be a continuous A-semilinear representation of Π on an ON A-Banach module V. Let $\{v_i\}_{i\in I}$ be a fixed ON basis generating an ON A^0-lattice V^0 stable under the action of Π. By the construction of (2.5), attached to ρ we have a 1-cocycle U_ρ in $\mathrm{GL}_I(A)$. Since Π leaves V^0 stable, it even lives in $\mathrm{GL}_I(A^0)$. Then $\{v_i\}_{i\in I}$ is a relative locally analytic basis if there is some ϖ^{ε} such that the composite $U_\rho : \mathrm{GL}_I(A^0) \to \mathrm{GL}_I(A^0/\varpi^{\varepsilon})$ factors through a finite quotient Π/Π', that is, the 1-cocycle U_ρ is trivial on $\Pi' \subset \Pi$ modulo ϖ^{ε}.

The following lemma says that composition with matrices in $\mathrm{Aut}_A(V)$ that are close enough to id_V preserves the relative locally analytic bases.

Lemma 2.3.7. *Let V be a relative locally analytic representation of Π, $\mathbf{v} = \{v_i\}_{i\in I}$ a relative locally analytic basis, and V^0 the A^0-lattice spanned by $\{v_i\}_{i\in I}$. Let $\psi \in \mathrm{End}_A(V)$ be an operator such that $\|1-\psi\| \leq |\varpi^{\varepsilon}|$ for the operator norm induced by V^0 and some $\varepsilon > 0$. Then $\psi(\mathbf{v}) = \{\psi(v_i)\}_{i\in I}$ is a relative locally analytic basis of V.*

Proof. Let $\Pi' \subset \Pi$ be an open subgroup stabilizing V^0, and let $\varepsilon' > 0$ be such that the action of Π' on $\{v_i \bmod \varpi^{\varepsilon'}\}_{i\in I}$ is trivial. Let $\varepsilon'' = \min\{\varepsilon, \varepsilon'\}$. Then $\psi(v_i) \equiv v_i \bmod \varpi^{\varepsilon''}$ and Π' acts on $\{\psi(v_i) \bmod \varpi^{\varepsilon''}\}_{i\in I}$ trivially. This proves the lemma. ∎

Example 2.3.8. (1) Let $\Pi = G$ be a compact p-adic Lie group and W be a Banach locally analytic representation over $\mathbb{Q}_p$. Then, by Lemma 2.1.5, $W \widehat{\otimes}_{\mathbb{Q}_p} A$ is a relative locally analytic representation of Π.

(2) Slightly more generally, suppose that Π admits a compact p-adic Lie group G as a quotient. Let W be a Banach locally analytic representation of G over $\mathbb{Q}_p$. Then $W \widehat{\otimes}_{\mathbb{Q}_p} A$ is a relative locally analytic representation of Π. This is the situation we will face in the applications to rigid spaces.

(3) Let V be an A-finite free semilinear representation of Π. Then V is relative locally analytic. Indeed, let $V^0 \subset V$ be an A^0-lattice with basis $v_1, \ldots, v_k$ and $\Pi^0 \subset \Pi$ a subgroup leaving V^0 stable. Then, after shrinking Π^0 if necessary, it acts trivially on all the v_i modulo pV^0.

2.4. The Sen functor

Let A be a sous-perfectoid ring and $\Gamma \cong \mathbb{Z}_p^d$ a torsion free abelian p-adic Lie group of dimension d. Let $(A, \Gamma, (R_n^i)_{i=1}^d)$ be a d-dimensional Sen theory. In this section we define the *Sen functor* for relative locally analytic ON A-Banach representations of Γ; this is nothing but the decompletion by taking (derived) locally analytic vectors.

Definition 2.4.1 (The Sen functor). Let V be a relative locally analytic ON A-Banach representation of Γ. We define the *Sen module of* V to be the subspace of locally analytic vectors for the action of Γ:

$$S(V) := V^{\Gamma\text{-la}}.$$

We also define the *derived Sen module of* V to be the solid complex of derived Γ-locally analytic vectors:

$$RS(V) := V^{R\Gamma\text{-la}}.$$

Remark 2.4.2. In Definition 2.4.1 (2) we see V as a solid $\mathbb{Q}_p$-vector space. Note also that the derived Sen module can be constructed for any complex of solid A-semilinear Γ-representations.

Before stating the main theorem, let us study the behaviour of passing through analytic vectors for the action of Γ on A. Write $C^{\mathrm{la}}(\Gamma,\mathbb{Q}_p) = \varinjlim_{h\to\infty} C^h(\Gamma,\mathbb{Q}_p)$ as a colimit of h-analytic functions. The following lemma says that the h-analytic vectors of A are sandwiched between the images of the Sen traces.

Lemma 2.4.3. *Let (A,Γ,R_n) be a one-dimensional Sen theory and let $\gamma\in\Gamma$ be a generator. The following hold:*

(1) *Given $h>0$ there is $n(h)$ such that for $k\geq n(h)$ one has $X_k^{Rh\text{-an}}=0$. In particular, $A^{Rh_n\text{-an}} = A_{n(h)}^{Rh\text{-an}}$, and the natural map $A^{Rh\text{-an}}\to A$ induces an inclusion of subalgebras of A,*

$$A^{h\text{-an}} \subset A_{n(h)}.$$

(2) *Conversely, given $n\in\mathbb{N}$ there is $h(n)>0$ such that A_n is $h(n)$-analytic. In particular, we have an inclusion of subalgebras of A,*

$$A_n \subset A^{h(n)\text{-an}}.$$

(3) *Taking colimits as $n\to\infty$ we get*

$$A^{R\Gamma\text{-la}} = A^{\Gamma\text{-la}} = \varinjlim_n A_n.$$

Proof. The second statement is just (CST1) (4). For the first statement, we need to see that given $h>0$ there is $n(h)$ such that for $k\geq n(h)$ the space $X_k=\ker(R_k : A\to A_k)$ has no h-analytic vectors.

By Lemma 2.1.5 given $h>0$ and $\varepsilon>0$ there is some $s\in\mathbb{N}$ such that the action of $1-\gamma^{p^s}$ satisfies $\|(1-\gamma^{p^s})\|\leq|\varpi^{2\varepsilon}|$ as an operator on $C^h(\Gamma,\mathbb{Q}_p)$. By (CST2) given $\varepsilon>0$ and $s\in\mathbb{N}$ there is $n(\varepsilon,s)$ such that for $k\geq n(\varepsilon,s)$ the operator $1-\gamma^{p^s}$ is invertible on X_k with inverse satisfying $\|(1-\gamma^{p^s})^{-1}\|\leq|\varpi|^{-\varepsilon}$. Then the operator $1-\gamma^{p^s}$ is invertible on $X_k\widehat{\otimes}_{\mathbb{Q}_p} C^h(\Gamma,\mathbb{Q}_p)$ with inverse given by the power series

$$(1-\gamma^{p^s})^{-1} = -\sum_{i=0}^{\infty}\gamma^{-p^s}(\gamma^{-p^s}-1)^{-(i+1)}\otimes(\gamma^{p^s}-1)^i.$$

This implies that

$$X_k^{Rh\text{-an}} = R\Gamma(\Gamma, X_k \widehat{\otimes}_{\mathbb{Q}_p} C^h(\Gamma, \mathbb{Q}_p)) = 0.$$

Taking $n(h) = n(s, \varepsilon)$ for a fixed ε we get the first statement.

For the third statement, it suffices to write $A = X_k \oplus A_k$ and to note that for some h (which can be large as long as k is even larger) one has $X_k^{Rh\text{-an}} = 0$. Hence

$$A^{Rh\text{-an}} = A_k^{Rh\text{-an}}.$$

Taking colimits as $h, k \to \infty$ one then gets

$$A^{R\Gamma\text{-la}} = \varinjlim_{k,h} A_k^{Rh\text{-an}} = \varinjlim_{k} A_k$$

as desired. ■

Let $(A, \Gamma, (R_n^i)_{i=1}^d)$ be a d-dimensional Sen theory. For a tuple $\underline{\mathbf{n}} \in \mathbb{N}^d$ we let $A_{\underline{\mathbf{n}}} = \prod_{i=1}^d R_{n_i}^i(A)$ be the image of the composition of Sen traces. For $n \in \mathbb{N}$ we will simply write $A_n = \prod_{i=1}^d R_n^i(A)$. We state the main theorem of this section regarding the nice behaviour of the Sen functor for relative locally analytic ON A-Banach representations (cf. [1, Proposition 3.3.1]).

Theorem 2.4.4. *Let V be a relative locally analytic ON A-Banach representation of Γ with a fixed ON A^0-lattice V^0 and $\mathbf{v} = \{v_i\}_{i\in I}$ a relative locally analytic A^0-basis of V^0. The following hold:*

(1) *For all $\varepsilon > 0$ close enough to zero and $n \gg 0$ the A-module V contains a closed ON Banach A_n-submodule $S_n(V)$ with ON A_n^0-lattice $S_n(V)^0 = S_n(V) \cap V^0$ such that:*

  (a) *$S_n(V)^0$ admits an ON basis $\mathbf{v}' = \{v_i'\}_{i\in I}$ with $v_i' \equiv v_i \bmod \varpi^\varepsilon V^0$. In particular, $A \widehat{\otimes}_{A_n} S_n(V) = V$.*

  (b) *The A_n-module $S_n(V)$ is stable under the action of Γ. Moreover, $S_n(V)$ is a locally analytic representation.*

  (c) *For $m \geq n$ we have $S_m(A) = A_m \widehat{\otimes}_{A_n} S_n(A)$.*

  We call $S_m(V)$ the m-th Sen module of V, or the Sen module of V over A_m.

(2) *Given $h \gg 0$ depending on V, there is $n(h) \in \mathbb{N}$ with $n(h) \to \infty$ as $h \to \infty$ such that:*

  (a) *The Banach $A^{h\text{-an}}$-module $V^{h\text{-an}}$ admits an ON basis.*

  (b) *We have a map $A^{h\text{-an}} \to A_{n(h)}$ as in Lemma 2.4.3 and*

$$S_{n(h)}(A) = A_{n(h)} \widehat{\otimes}_{A^{h\text{-an}}} V^{h\text{-an}}.$$

  In particular, for $n \gg 0$ the A_n-module $S_n(A)$ of (1) is unique and equal to the (base change of the) h-analytic vectors of V for a suitable h.

(3) *We have*

$$RS(V) = S(V) = \varinjlim_n S_n(V).$$

In other words, the derived Sen functor is concentrated in degree 0 *and equal to the colimit along* n *of the decompletions* $S_n(V)$.

The proof of Theorem 2.4.4 consists of two steps. First, we generalize the decompletion of [1] from finite rank A-modules to ON Banach A-modules; this will show part (1) of the theorem. Second, we use the decompletion of part (1) and Lemma 2.4.3 to deduce parts (2) and (3).

2.4.1. Reduction to the one-dimensional case. As a first technical step, let us reduce Theorem 2.4.4 to the one-dimensional case by a simple inductive argument.

Proposition 2.4.5. *Suppose that Theorem 2.4.4 holds for one-dimensional Sen theories. Then it holds for Sen theories of arbitrary dimensions.*

Proof. Let $\Gamma \cong \mathbb{Z}_p^d$ and let $(A, \Gamma, (R_n^i)_{i=1}^d)$ be a d-dimensional Sen theory. Suppose by induction that Theorem 2.4.4 holds for $d-1$-dimensional Sen theories. Write $\Gamma = \Gamma_1 \times \Gamma_{[2,d]}$ as a product of the first coordinate and the last $d-1$ coordinates. Let $R_n^1 : A \to A_n^1$ be the Sen traces attached to the first coordinate. Then (A, Γ_1, R_n^1) is a one-dimensional Sen theory by definition, and by assumption Theorem 2.4.4 holds for it. Thus, for all $\varepsilon > 0$ close enough to 0 and $n \gg 0$ we have a Sen module $S_n^1(V) \subset V$ over A_n^1 such that:

(i) There is an ON basis $\{v_i'\}_{i\in I}$ of $S_n^1(V)^0 = S_n^1(V) \cap V^0$ such that $v_i' \equiv v_i \mod \varpi^{\varepsilon} V^0$.

(ii) For $n \gg 0$ we can recover $S_n^1(V)$ by the formula

$$S_n^1(V) = A_n^1 \otimes_{A^{\Gamma_1, h\text{-an}}} V^{\Gamma_1, h\text{-an}}$$

where $W^{\Gamma_1, h\text{-an}}$ are the h-analytic vectors of W for the action of Γ_1 (for a suitable choice of h). In particular, the action of Γ leaves $S_n^1(V)$ stable.

(iii) We have $RS^1(V) = S^1(V) = \varinjlim_n S_n^1(V)$, i.e. the higher locally analytic vectors for the action of Γ_1 vanish.

Let us now fix $n_1 \gg 0$ such that the previous properties hold for all $m \geq n_1$. Then (i) implies that $\{v_i'\}_{i\in I}$ is a relative locally analytic basis of $S_n^1(V)$ as an A_n^1-module for the action of $\Gamma_{[2,d]}$. By Lemma 2.2.8 the tuple $(A_n^1, \Gamma_{[2,d]}, (R_m^i)_{i=2}^d)$ is a $(d-1)$-dimensional Sen theory. By the inductive hypothesis, for all $m \gg n_1$ we have a Sen module $S_m(V)$ over $A_m = \prod_{i=2}^d R_m^i(A_m^1)$ satisfying the analogous properties (i)–(iii) for the group $\Gamma_{[2,d]}$.

We now deduce that (i)–(iii) hold for the whole group Γ (after increasing m if necessary). Property (i) does not depend on the group. For (ii), since the action of Γ_1 was already locally analytic on $S_m^1(V)$ and hence on $S_m(V)$, then by [19, Proposition 3.4.2] the action of Γ on $S_m^1(V)$ is also locally analytic. Since $S_m^1(V)$ is a Banach space, Lemma

2.1.5 implies that it is h-analytic as a Γ-representation for some h. By increasing m we can then find h such that

$$S_m(V) = A_m \otimes_{A^{h\text{-an}}} V^{h\text{-an}},$$

obtaining (ii). Note that, by taking h and m such that $A_m \subset A^{h\text{-an}}$ and $S_m(V)$ is h-analytic, one has

$$V^{h\text{-an}} = (A \widehat{\otimes}_{A_m} S_m(V))^{h\text{-an}} = A^{h\text{-an}} \widehat{\otimes}_{A_m} S_m(V), \tag{2.6}$$

in particular $V^{h\text{-an}}$ is also an ON $A^{h\text{-an}}$-Banach representation of Γ.

Let us briefly justify the projection formula for h-analytic vectors of (2.6): by using the argument of Lemma 2.1.6 one reduces to the projection formula $(V \widehat{\otimes}_{\mathbb{Q}_p} W)^{Rh\text{-an}} = V^{Rh\text{-an}} \otimes^L_{\mathbb{Q}_p,\blacksquare} W$ for W, V Banach representations of Γ with V being h-analytic. The derived locally analytic vectors of the tensor $V \widehat{\otimes}_{\mathbb{Q}_p} W$ are the group cohomology of the representation $C^h(\Gamma, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} W \widehat{\otimes}_{\mathbb{Q}_p} V$, where Γ acts on W and V as usual, and on the h-analytic functions via the left regular action. Since the action of V is h-analytic, there is a Γ-equivariant isomorphism

$$\psi : C^h(\Gamma, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} V \cong C^h(\Gamma, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} V_0$$

where the action of Γ on the right-hand side is trivial on V_0 and the left regular action on the h-analytic functions; this map is given by sending a function $f : \Gamma \to V$ to the function $\widetilde{f}(g) = g^{-1} f(g)$. Thus, we have

$$\begin{aligned}(V \widehat{\otimes}_{\mathbb{Q}_p} W)^{Rh\text{-an}} &= R\Gamma(G, C^h(\Gamma, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} W \widehat{\otimes}_{\mathbb{Q}_p} V) \\ &\cong R\Gamma(G, C^h(\Gamma, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} W \widehat{\otimes}_{\mathbb{Q}_p} V_0) \\ &= R\Gamma(G, C^h(\Gamma, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} W) \otimes^L_{\mathbb{Q}_p,\blacksquare} V \\ &= W^{Rh\text{-an}} \otimes^L_{\mathbb{Q}_p,\blacksquare} V\end{aligned}$$

where in the second equivalence we use the isomorphism ψ, in the third equivalence we use the projection formula for group cohomology, and the last equivalence is the definition of h-analytic vectors.

For (iii), fix m such that the Sen module $S_m(V)$ is defined. Then $V = A \widehat{\otimes}_{A_m} S_m(V)$. We pick h such that A_m and $S_m(V)$ are h-analytic. Then by Lemma 2.1.6 one has

$$V^{R\Gamma\text{-la}} = (A \widehat{\otimes}_{A_m} S_m(V))^{R\Gamma\text{-la}} = A^{R\Gamma\text{-la}} \otimes^L_{A,\blacksquare} S_m(V),$$

where $\otimes^L_{A,\blacksquare}$ is the derived tensor product of A-modules in solid $\mathbb{Q}_p$-vector spaces. For $k \gg m$ we can write $A = X^1_k \oplus A^1_m$. Lemma 2.4.3 implies that $\varinjlim_{k\to\infty} (X^1_k)^{R\Gamma_1\text{-la}} = 0$, and so $A^{R\Gamma_1\text{-la}} = \varinjlim_n A^1_n$. By an inductive argument one deduces $A^{R\Gamma\text{-la}} = \varinjlim_n A_n$, which yields

$$RS(V) = V^{R\Gamma\text{-la}} = A^{R\Gamma\text{-la}} \otimes^L_{A,\blacksquare} S_m(V) = \varinjlim_{k\geq m} A_k \otimes^L_{A,\blacksquare} S_m(V) = \varinjlim_{k\geq m} S_k(V) = S(V)$$

sits in degree 0 and is the colimit of the Sen modules $S_m(V)$ over A_m. This finishes the proof of (iii). ■

The following lemma will reduce the proof of Theorem 2.4.4 to the construction of the Sen modules $S_n(V)$.

Lemma 2.4.6. *Keep the notation of Theorem 2.4.4 and suppose that Γ is a one-dimensional p-adic Lie group. Suppose that we can find Sen modules $S_m(V)$ over A_m for $m \gg 0$ satisfying parts* (1) *and* (2) *of Theorem 2.4.4. Then part* (3) *of the theorem holds, namely,*

$$RS(V) = S(V) = \varinjlim_{m} S_m(V).$$

Proof. Let us take $h > 0$ and $m \gg 0$ such that $A^{h\text{-an}} \subset A_m$, $V^{h\text{-an}}$ is an ON-Banach module over $A^{h\text{-an}}$ and $S_m(V) = A_m \widehat{\otimes}_{A^{h\text{-an}}} V^{h\text{-an}}$. Then $S_m(V)$ is a locally analytic representation of Γ and by Lemma 2.1.6 we have

$$RS(V) = V^{R\Gamma\text{-la}} = (A \widehat{\otimes}_{A_m} S_m(V))^{R\Gamma\text{-la}} = A^{R\Gamma\text{-la}} \otimes^L_{A_m,\blacksquare} S_m(V).$$

By Lemma 2.4.3 we get $A^{R\Gamma\text{-la}} = \varinjlim_k A_k$, which implies what we want. ∎

2.4.2. Decompletion after [1]. Let A be a sous-perfectoid ring. Let (A, Γ, R_n) be a one-dimensional Sen theory with $\gamma \in \Gamma$ a generator of Γ. Given $n \gg 0$ let $c_1, c_2 > 0$ be such that $\|R_n\| \leq |\varpi|^{-c_1}$ as a map $A \to A_n$ and $\|(1-\gamma)^{-1}\| \leq |\varpi|^{-c_2}$ on $X_n = \ker R_n$. Recall that we can take $c_1, c_2 \to 0$ as $n \to \infty$. We let V be an ON A-Banach representation of Γ with A^0-lattice V^0. Suppose that V is relative locally analytic and that we are given a relative locally analytic basis $\{v_i\}_{i\in I}$ of V^0 as in Definition 2.3.5. Let $U_V : \Gamma \to \mathrm{GL}_I(A)$ be the 1-cocycle constructed in (2.5) depending on the basis $\{v_i\}_{i\in I}$; we will simply write $U = U_V(\gamma)$.

Lemma 2.4.7 ([1, Lemme 3.2.3]). *Let $\delta, a, b \in \mathbb{R}_{>0}$ be such that $a \geq c_1 + c_2 + \delta$ and $b \geq \sup\{a + c_1, 2c_1 + 2c_2 + \delta\}$. Suppose that*

- *$U = 1 + U_1 + U_2$ where $U_1 \in \prod_I \widehat{\bigoplus}_I A_n^0$ and $U_2 \in \prod_I \widehat{\bigoplus}_I A^0$,*
- *$U_1 \equiv 0 \bmod \varpi^a$ and $U_2 \equiv 0 \bmod \varpi^b$.*

Then there exists $M \in \mathrm{GL}_I(A^0)$ with $M \equiv 1 \bmod \varpi^{b-c_2-c_3}$ such that

- *$M^{-1}U\gamma(M) = 1 + V_1 + V_2$ with $V_1 \in \prod_I \widehat{\bigoplus}_I A_n^0$ and $V_2 \in \prod_I \widehat{\bigoplus}_I A^0$,*
- *$V_1 \equiv 0 \bmod \varpi^a$ and $V_2 \equiv 0 \bmod \varpi^{b+\delta}$.*

Proof. The proof is exactly the same as the one of [1, Lemme 3.2.3]; we give the details for completeness. Let $R_n : A \to A_n$ be the Sen trace and let X_n be its kernel. Since we have the decomposition $A = A_n \oplus X_n$, the following space decomposes via R_n:

$$\Big(\prod_I \widehat{\bigoplus_I} A^0\Big)\Big[\frac{1}{p}\Big] = \Big(\prod_I \widehat{\bigoplus_I} A_n^0\Big)\Big[\frac{1}{p}\Big] \oplus \Big(\prod_I \widehat{\bigoplus_I} X_n\Big)\Big[\frac{1}{p}\Big].$$

Then, using the bound of (CST1) (2), we can write $U_2 = R_n(U_2) + (1 - R_n)(U_2)$ with

- $R_n(U_2) \in \prod_I \widehat{\bigoplus}_I A_n^0$ and $R_n(U_2) \equiv 0 \bmod \varpi^{b-c_2}$,
- $(1 - R_n)(U_2) \in \prod_I \widehat{\bigoplus}_I X_n^0$ and $(1 - R_n)(U_2) \equiv 0 \bmod \varpi^{b-c_2}$.

Using (CST2) we can write $(1-R_n)(U_2) = (1-\gamma)V$ with $V \in \prod_I \widehat{\bigoplus}_I X_n^0$ such that $V \equiv 0 \mod \varpi^{b-c_2-c_1}$. We now modify the cocycle U by the matrix $M = 1+V$, which amounts to computing

$$(1+V)^{-1}U\gamma(1+V).$$

By the conditions on a, b and δ we deduce that V^2, $U_1\gamma(V)$, $U_2\gamma(V)$, VU_1 and VU_2 are equivalent to 0 mod $\varpi^{b+\delta}$. Reducing modulo $\varpi^{b+\delta}$ we find that

$$\begin{aligned}(1+V)^{-1}U\gamma(1+V) &\equiv (1-V)U(1+\gamma(V)) \mod \varpi^{b+\delta}\\ &\equiv 1 - V + U_1 + U_2 + \gamma(V) \mod \varpi^{b+\delta}\\ &\equiv 1 + U_1 + R_n(U_2) \mod \varpi^{b+\delta}.\end{aligned}$$

This shows that $M^{-1}U\gamma(M) = 1 + V_1 + V_2$ with $V_1 = U_1 + R_n(U_2)$ and satisfying the requirements of the lemma. ■

Corollary 2.4.8. *Let $\delta > 0$ and $b \geq 2c_2 + 2c_3 + \delta$. Let $U \in \mathrm{GL}_I(A^0)$ be a matrix such that $U \equiv 1 \mod \varpi^b$. Then there exists $M \in \mathrm{GL}_I(A^0)$ with $M \equiv 1 \mod \varpi^{b-c_3-c_2}$ such that $M^{-1}U\gamma(M) \in \mathrm{GL}_I(A_n^0)$.*

Proof. By Lemma 2.4.7 there exists $M^{(1)} \in \mathrm{GL}_I(A^0)$ with $M^{(1)} \equiv 1 \mod \varpi^{b-c_2-c_3}$ such that

$$U^{(1)} := M^{(1),-1}U\gamma(M^{(1)}) \in \mathrm{GL}_I(A_n^0) \mod \varpi^{b+\delta}.$$

Let $k \in \mathbb{N}_{\geq 1}$. By induction we can find matrices $M^{(k)} \in \mathrm{GL}_I(A^0)$ with

$$M^{(k)} \equiv 1 \mod \varpi^{b+\delta(k-1)-c_2-c_3}$$

with

$$U^{(k)} := M^{(k),-1}U^{(k-1)}\gamma(M^{(k)}) \in \mathrm{GL}_I(A_n^0) \mod \varpi^{b+\delta k}.$$

Letting $k \to \infty$ and $M := M^{(1)}M^{(2)}\cdots$ one sees that the matrix $U' = M^{-1}U\gamma(M)$ takes values in $\mathrm{GL}_I(A_n^0)$, and $U' \equiv 1 \mod \varpi^{b-c_2-c_3}$. ■

Lemma 2.4.9 ([1, Lemme 3.2.5]). *Let $B \in \mathrm{GL}_I(A)$. Suppose that we are given $V_1, V_2 \in \mathrm{GL}_I(A_n^0)$ such that $V_1 \equiv V_2 \equiv 1 \mod \varpi^{c_2+\varepsilon}$ for some $\varepsilon > 0$, and $\gamma(B) = V_1BV_2$. Then $B \in \mathrm{GL}_I(A_n)$.*

Proof. Consider $C = B - R_n(B)$. Then $\gamma(C) = V_1CV_2$. We have

$$\gamma(C) - C = (V_1-1)CV_2 + V_1C(V_2-1) - (V_1-1)C(V_2-1).$$

Then $C \in \varpi^r \prod_I \widehat{\bigoplus}_I X_n^0$ implies $\gamma(C) - C \in \varpi^{r+c_2+\varepsilon} \prod_I \widehat{\bigoplus}_I X_n^0$. On the other hand, (CST2) provides an isomorphism $1-\gamma : X_n \cong X_n$ whose inverse has norm $\leq |\varpi^{-c_2}|$. We deduce that $C \in \varpi^{r+\varepsilon} \prod_I \widehat{\bigoplus}_I X_n^0$. Since r was arbitrary, one gets $C \in \varpi^{r+s\varepsilon} \prod_I \widehat{\bigoplus}_I X_n^0$ for all $s \geq 1$ and so $C = 0$, or equivalently $B = R_n(B)$, proving what we wanted. ■

We finally prove parts (1) and (2) of Theorem 2.4.4.

Proposition 2.4.10. *Let* (A, Γ, R_n) *be a one-dimensional Sen theory. Then parts* (1) *and* (2) *of Theorem* 2.4.4 *hold.*

Proof. Let V be an ON A-Banach relative locally analytic representation of Γ with ON A^0-Banach lattice $V^0 \subset V$. Let $\{v_i\}_{i\in I}$ be a relative locally analytic basis of V^0 and let $U_V : \Gamma \to \mathrm{GL}_I(A^0)$ be the 1-cocycle attached to V and the fixed basis. Given $\gamma \in \Gamma$ we will write U_γ for $U_V(\gamma)$.

Let us write $\Gamma_k := \Gamma^{p^k}$ for a basis of open neighbourhoods of 1 in Γ, and let $\gamma_k \in \Gamma_n$ be a generator. Since V is relative locally analytic, there are some $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $U_\gamma \equiv 1 \mod \varpi^\varepsilon$ in $\mathrm{End}_A(V^0) \cong \prod_I \widehat{\bigoplus}_I A^0$ for all $\gamma \in \Gamma_n$. Let us fix a $k \in \mathbb{N}$ satisfying this condition.

Now, by (CST1), for $n \gg k$ the trace map $R_n : A \to A_n$ has norm $\leq |\varpi^{-c_1}|$ with $c_1 \to 0$ as $n \to \infty$. By (CST2), for k fixed, we can find $m_0 \gg k$ and a constant $c_2 > 0$ such that for all $m \geq m_0$ the operator $1 - \gamma_k$ is invertible in $X_m = \ker R_m$ with inverse of norm bounded by $|\varpi^{-c_2}|$. We can take $c_2 \to 0$ as $m \to \infty$. Taking the constants c_1 and c_2 such that $\varepsilon > 2c_1 + 2c_2$, we fix $n = m$ satisfying the previous conditions. Thus, we found some $n \gg k$ such that $\|R_n\| \leq |\varpi^{-c_1}|$ and $1 - \gamma_k$ is invertible on X_n with inverse satisfying $\|(1-\gamma_k)^{-1}\| \leq |\varpi^{-c_2}|$.

Then $U(\gamma_k) \in \mathrm{GL}_I(A^0)$ is a matrix satisfying $U \equiv 1 \mod \varpi^\varepsilon$ and by taking $\delta > 0$ such that $\varepsilon > 2c_1 + 2c_2 + \delta$, Corollary 2.4.9 says that there is a matrix $M \in \mathrm{GL}_I(A^0)$ with $M \equiv 1 \mod \varpi^{\varepsilon - c_1 + c_2}$ such that $M^{-1}U(\gamma_k)\gamma_k(M) \in \mathrm{GL}_I(A_n^0)$. Let $U' : \Gamma \to \mathrm{GL}_I(A^0)$ be the cocycle $U'(\gamma) := M^{-1}U(\gamma)\gamma(M)$. Then U' takes values in $\mathrm{GL}_I(A_n^0)$ for $\gamma \in \Gamma_k$ by construction. We claim that U' takes values in $\mathrm{GL}_I(A_n)$ for $\gamma \in \Gamma$. Indeed, let $\gamma \in \Gamma$. Then, as Γ is commutative, we have the relation $\gamma\gamma_k = \gamma_k\gamma$, which yields the identity at the level of cocycles

$$U'(\gamma)\gamma(U'(\gamma_k)) = U'(\gamma\gamma_k) = U'(\gamma_k\gamma) = U'(\gamma_k)\gamma_k(U'(\gamma)).$$

Therefore,

$$\gamma_k(U'(\gamma)) = U'(\gamma_k)^{-1}U(\gamma)\gamma U'(\gamma_k).$$

Since $U'(\gamma_k) \equiv 1 \mod \varpi^\varepsilon$ and $\varepsilon > c_2$, Lemma 2.4.9 shows that $U'(\gamma) \in \mathrm{GL}_I(A_n)$ for $\gamma \in \Gamma$, proving the claim.

Translating from cocycles to representations, U' gives rise to an ON A_n-Banach representation $S_n(V)$ of Γ such that $A \widehat{\otimes}_{A_n} S_n(V) \cong V$ as a A-semilinear Γ-representation. For $m \geq n$ we simply define $S_m(V) := A_m \widehat{\otimes}_{A_n} S_n(V)$. Since the action of Γ is locally analytic on A_n, and the action of Γ_k is trivial on the modified basis $v_i' = Mv_i$ of $S_n(V)$ modulo some power of ϖ, Lemma 2.1.5 implies that the action of Γ on $S_n(V)$ is locally analytic (and so h-analytic for some radius p^{-h}). It is straightforward to check that the representation $S_n(V)$ satisfies condition (1) of the theorem.

Let us now show that part (2) follows formally from part (1). Suppose that $S_n(V)$ and V_n are h-analytic for some $h > 0$. Then by (2.6) we have

$$V^{h\text{-an}} = (A \widehat{\otimes}_{A_n} S_n(V))^{h\text{-an}} = A^{h\text{-an}} \widehat{\otimes}_{A_n} S_n(V)$$

and so $V^{h\text{-an}}$ is an ON A^h-Banach representation of Γ. Moreover, taking $m \gg n$ such that

$A^h \subset A_m$ we get

$$A_m \widehat{\otimes}_{A^{h\text{-an}}} V^{h\text{-an}} = A_m \widehat{\otimes}_{A_n} S_n(V) = S_m(V),$$

proving part (2). ■

2.5. *Group cohomology via Sen theory*

We finish the general discussion of Sen theory with some applications to the computation of group cohomology. Let A be a sous-perfectoid ring, $\Gamma \cong \mathbb{Z}_p^d$ a torsion free abelian compact p-adic Lie group of dimension d and $(A, \Gamma, (R_n^i)_{i=1}^d)$ a d-dimensional Sen theory on A. For $n \gg 0$ let A_n denote the image of the composition $\prod_{i=1}^d R_n$. In the rest of the section we make the following additional assumption that will always hold in practice, and that already appeared in Lemmas 2.2.11 and 2.2.12.

Hypothesis 2.5.1. The maps $A_n \to A_m$ are finite flat and A has an ON basis over A_n for all $n \gg 0$.

A first corollary is the computation of group cohomology in terms of Lie algebra and smooth cohomology.

Corollary 2.5.2. *Let V be a relative locally analytic ON A-Banach representation of* Γ. *Then*

$$R\Gamma(\Gamma, V) = R\Gamma\big(\Gamma^{\mathrm{sm}}, R\Gamma(\mathrm{Lie}\,\Gamma, S(V))\big).$$

In particular,

$$H^i(\Gamma, V) = H^i(\mathrm{Lie}\,\Gamma, S(V))^{\Gamma}.$$

Proof. By Theorem 2.4.4 we know that $V^{R\Gamma\text{-la}} = S(V) = \varinjlim_n S_n(V)$ sits in degree 0 and is the colimit of the decompletion by Sen traces. Then Theorem 2.1.2 (4) yields the desired result. ■

We will consider a last list of axioms that holds in the main geometric application of this paper. In the arithmetic case over $\mathbb{Q}_p$, these axioms are key in the proof of the Ax–Sen–Tate theorem.

(AST) We define the following axioms, inductively on the dimension of the Sen theory.

(1) A one-dimensional Sen theory $(A, \Gamma, (R_n))$ satisfies the *Ax–Sen–Tate property of dimension* 1 if the following conditions hold:

- (i) Hypothesis 2.5.1 holds for $(A, \Gamma, (R_n))$.
- (ii) $A_n = A^{\Gamma_n}$ where $\Gamma_n = \Gamma^{p^n}$.
- (iii) The Sen traces $R_n : A \to A_n$ are constructed from the normalized traces

$$R_n^m : A_m \to A_n, \quad x \mapsto \frac{1}{p^{m-n}} \sum_{g \in \Gamma_m/\Gamma_n} g(x).$$

More precisely, the colimit as $m \to \infty$ of the operators R_n^m defines an operator $R_n : \varinjlim_m A_m \to A_n$ that completes to an operator on the uniform completion, $R_n : A = (\varinjlim_m A_m)^u \to A_n$, which agrees with the Sen trace (cf. Lemma 2.2.2).

(2) Let $(A, \Gamma, (R_n^i)_{i=1}^d)$ be a d-dimensional Sen theory. Write $\Gamma = \Gamma_1 \times \Gamma_{[2,d]} \cong \mathbb{Z}_p \times \mathbb{Z}_p^{d-1}$. We say that $(A, \Gamma, (R_n^i)_{i=1}^d)$ satisfies the *Ax–Sen–Tate property of dimension d* if $(A, \Gamma_1, (R_n^1))$ satisfies the Ax–Sen–Tate property of (1), and for all $n \gg 0$ the $d-1$-dimensional Sen theories $(A_n^1, \Gamma_{[2,d]}, (R_n^i)_{i=2}^d)$ satisfy Hypothesis 2.5.1, and the Ax–Sen–Tate property of dimension $d-1$.

Example 2.5.3. (1) Let K be a p-adic discretely valued field over $\mathbb{Q}_p$ with perfect residue field. Then the cyclotomic tower $K \to K^{\text{cyc}}$ satisfies the axiom (AST).

(2) Let (C, C^+) be a perfectoid field containing $\mathbb{Q}_p^{\text{cyc}}$ and let X be an affinoid fs log smooth adic space over (C, C^+) admitting a chart $\psi : X \to \mathbb{S}_C^{(e,d-e)}$ that factors as a composite of finite étale maps and rational localizations. Let $\mathbb{S}_{C,\infty}^{(e,d-e)}$ be the perfectoid product of tori and discs obtained by taking p-power roots in the coordinates and let $X_\infty = \mathrm{Spa}(A_\infty, A_\infty^+)$ be the pullback over X. Let Γ be the Galois group of $X_\infty \to X$. Then A_∞ is a perfectoid ring and by Proposition 2.2.14 the Sen traces R_n^i of Example 2.2.7 (4) extend to a Sen theory $(A_\infty, \Gamma, (R_n^i)_n)$ satisfying the (AST) axiom. The same holds for the rings of functions on the boundary divisor $D_{J,\infty} \subset X_\infty$ defined by a subset $J \subset \{e+1, \dots, d\}$.

A Sen theory satisfying the Ax–Sen–Tate axiom can be endowed with a Sen operator as follows.

Definition 2.5.4. Let $(A, \Gamma, (R_n^i)_{i=1}^d)$ be a d-dimensional Sen theory satisfying (AST). Let V be a relative locally analytic ON A-Banach representation of Γ. The *Sen operator* of V is the A-linear map

$$\theta_V : V \to V \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\,\Gamma)^\vee$$

given by the A-extension of scalars of the connection

$$S(V) \to S(V) \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\,\Gamma)^\vee.$$

Equivalently, we define the *Sen operator*

$$\mathrm{Sen}_V : \mathrm{Lie}\,\Gamma \otimes V \to V$$

to be the extension of scalars of the derivation

$$\mathrm{Lie}\,\Gamma \otimes S(V) \to S(V).$$

The Higgs cohomology $R\Gamma(\theta_V, V)$ of V is defined to be the complex

$$0 \to V \xrightarrow{\theta_V} V \otimes (\mathrm{Lie}\,\Gamma_{H'})^\vee \to \cdots \to V \otimes \bigwedge^d (\mathrm{Lie}\,\Gamma_{H'})^\vee \to 0,$$

which is nothing other than the A-extension of scalars of the Chevalley–Eilenberg complex of $S(V)$ computing Lie Γ-cohomology. We also denote $H^i(\theta_V, V) := H^i(R\Gamma(\theta_V, V))$.

In the rest of the section we suppose that the d-dimensional Sen theory $(A, \Gamma, (R_n^i)_{i=1}^d)$ satisfies the Ax–Sen–Tate axiom. The next result describes some cohomological properties of A-semilinear Γ-representations in terms of their Sen operators.

Proposition 2.5.5. *Let V and W be relative locally analytic ON A-Banach representations of Γ, and $W \to V$ an A-linear Γ-equivariant map. Consider the exact sequence in solid $\mathbb{Q}_p$-vector spaces*

$$0 \to K \to W \to V \to Q \to 0.$$

Then the derived Sen modules $RS(-)$ of K and Q (cf. Remark 2.4.2) are in degree 0, and we have an exact sequence of solid $\mathbb{Q}_p$-vector spaces

$$0 \to S(K) \to S(W) \to S(V) \to S(Q) \to 0.$$

In particular, we have isomorphisms of solid group cohomology

$$\underline{H}^i(\Gamma, V) = \underline{H}^i(\theta_V, V)^{\Gamma}.$$

Remark 2.5.6. One of the reasons we need to see W and V as solid $\mathbb{Q}_p$-vector spaces in the previous proposition is for the Sen module of Q to be well defined. Otherwise Q could be a non-Hausdorff topological space and there would not be a good theory of locally analytic vectors.

We thank Lue Pan for the explanation of the following argument.

Proof of Proposition 2.5.5. By an inductive argument on the dimension of the Sen theory we can assume without loss of generality that $\Gamma \cong \mathbb{Z}_p$. Consider the map $f : W \to V$. By taking h-analytic vectors and extending scalars to some A_n such that $A^{h\text{-an}} \subset A_n$, Theorem 2.4.4 (2) gives a decompletion of f to a map $f_n : S_n(W) \to S_n(V)$. By Hypothesis 2.5.1, A has an ON basis over A_n and so it is a flat solid A_n-module, being isomorphic to $V \otimes_{\mathbb{Q}_p,\blacksquare} A_n$ with V a Banach $\mathbb{Q}_p$-vector space (cf. [18, Lemma 3.21]). Let $S_n(K) := \ker f_n$ and $S_n(Q) := \operatorname{coker} f_n$; these are locally analytic representations of Γ. The flatness of $A_n \to A$ implies that

$$K = A \otimes^L_{A_n,\square} S_n(K) \quad \text{and} \quad Q = A \otimes^L_{A_n,\square} S_n(Q).$$

Passing to locally analytic vectors, we see by the projection formula (Lemma 2.1.6) and by Lemma 2.4.3 that

$$RS(K) = K^{R\Gamma\text{-la}} = (A)^{R\Gamma\text{-la}} \otimes^L_{A_n,\square} S_n(K) = \varinjlim_m A_m \otimes^L_{A_n} S_n(K) = S(K),$$

and similarly for Q, where in the last equivalence we have used the fact that the maps $A_n \to A_m$ are finite flat. We deduce that K and Q have no higher locally analytic vectors and thus we have an exact sequence of locally analytic representations

$$0 \to S(K) \to S(W) \to S(V) \to S(Q) \to 0.$$

Let us now show the last statement about Γ-cohomology on V. By Corollary 2.5.2 we know that $R\Gamma(\Gamma, V) = R\Gamma(\Gamma^{\mathrm{sm}}, R\Gamma(\mathrm{Lie}\,\Gamma, S(V)))$ and so

$$H^i(\Gamma, V) = H^i(\mathrm{Lie}\,\Gamma, S(V))^{\Gamma}.$$

Fix a basis of $\mathrm{Lie}\,\Gamma$ and consider the exact sequence

$$0 \to K \to V \xrightarrow{\theta_V} V \to Q \to 0$$

given by the Sen operator of V. Then the previous point shows that we have an exact sequence

$$0 \to S(K) \to S(V) \xrightarrow{\theta_V} S(V) \to S(Q) \to 0.$$

We deduce that

$$H^i(\theta_V, V)^{R\Gamma\text{-la}} = H^i(\mathrm{Lie}\,\Gamma, S(V)),$$

and by taking invariants,

$$H^i(\theta_V, V)^{\Gamma} = H^i(\mathrm{Lie}\,\Gamma, S(V))^{\Gamma} = H^i(\Gamma, V),$$

proving what we wanted. ■

Corollary 2.5.7. *Keep the notation of Proposition 2.5.5. Suppose that $\theta_V = 0$. Then there is an equivalence*

$$R\Gamma(\Gamma, V) = \bigoplus_{i=0}^{d} V^{\Gamma} \otimes \bigwedge^{i} \mathrm{Lie}\,\Gamma[-i].$$

Moreover, for n large enough we have $S_n(V) = V^{\Gamma_{p^n}}$ and so

$$V = A \mathbin{\widehat{\otimes}}_{A_n} V^{\Gamma_{p^n}}.$$

Proof. The first claim of the corollary follows from Proposition 2.5.5 since the Higgs complex of V is split.

For the second claim, by hypothesis we have $\theta_V = 0$. This means that the action of $\mathrm{Lie}\,\Gamma$ on $S(V) = (V)^{\Gamma\text{-la}}$ is zero, so that $S(V)$ is a smooth representation of Γ. Since $A^{\Gamma_{p^n}\text{-an}} = A^{\Gamma_{p^n}} = A_n$ by the (AST) axiom, by Theorem 2.4.4 (2) there is $n \gg 0$ such that $S_n(V) = V^{\Gamma_{p^n}\text{-an}} = V^{\Gamma_{p^n}}$.

■

Remark 2.5.8. The splitting of the cohomology of Corollary 2.5.7 depends on the Lie algebra $\mathrm{Lie}\,\Gamma$ and so on the group Γ. In applications we will allow Γ to vary, and hence to guarantee that the splitting is independent of Γ we shall need some additional structure (e.g. Hodge–Tate weights arising from an arithmetic Galois action).

3. Geometric Sen theory

Let $\mathbb{Q}_p^{\mathrm{cyc}}$ be the completed cyclotomic extension of $\mathbb{Q}_p$, (C, C^+) a perfectoid field over $\mathbb{Q}_p^{\mathrm{cyc}}$, and let X be an fs log smooth adic space over (C, C^+) with log structure given by reduced normal crossing divisors. For ? ∈ {an, ét, két, proét, prokét} we let $X_?$ denote the corresponding site over X (see [6, Example 2.3.17 and Sections 4 and 5]). We let $\widehat{\mathcal{O}}_X^{(+)}$ denote the (bounded) complete structural sheaf over $X_{\text{prokét}}$, and for ? ∈ {an, ét, két} we let $\mathcal{O}_X^{(+)}$ be the (bounded) structural sheaf on $X_?$. We also let $\nu_X : X_{\text{prokét}} \to X_{\text{két}}$ and $\eta_X : X_{\text{prokét}} \to X_{\text{an}}$ be the projections of sites; if X is clear from the context we write ν and η instead. Suppose that (C, C^+) is the completion of an algebraic extension of a discretely valued field with perfect residue field (K, K^+), and that X has a form X' over (K, K^+), we shall write $\mathcal{O}\mathbb{B}_{\text{dR,log}}^{(+)}$ and $\mathcal{O}\mathbb{C}_{\log} := \mathrm{gr}^0(\mathcal{O}\mathbb{B}_{\text{dR,log}})$ for the log de Rham and Hodge–Tate period sheaves over $X'_{\text{prokét}}$ (see [7]).

The main goal of this section is to use the abstract Sen theory formalism of Section 2 to study the Hodge–Tate cohomology of X, obtaining Theorems 1.0.4 and 1.0.5 of the introduction. In Section 3.1 we prove that $R\nu_*^1\widehat{\mathcal{O}}_X(1) \cong \Omega_X^1(\log)$ where $\nu : X_{\text{prokét}} \to X_{\text{két}}$ is the projection of sites, following an argument of Scholze. In Section 3.2 we construct the geometric Sen operator of X locally on toric coordinates, for which we are essentially reduced to the Sen theory of a product of tori and discs as in Example 2.2.7 (4). Then, in Section 3.3, we show that these local constructions of the Sen operator glue, following an argument suggested by Lue Pan using the isomorphism $R^1\nu_*\widehat{\mathcal{O}}_X \cong \Omega_X^1(\log)(-1)$. Finally, in Section 3.4, we apply the previous results to study the locally analytic vectors of the completed structural sheaf of pro-Kummer-étale torsors of p-adic Lie groups. We finish by explaining the relation of geometric Sen theory to [7, 15, 33].

All the fibre products considered in the next sections are as fs log adic spaces in the sense of [6, Proposition 2.3.27], in particular they might differ from the fibre products of usual adic spaces (but both agree for trivial log structures). We shall consider almost mathematics with respect to the ideal $\mathfrak{m}_C \subset C^+$ of topologically nilpotent elements of C.

3.1. Log-Kummer exact sequence

Let X be an fs log smooth adic space over (C, C^+) with log structure given by normal crossing divisors. Equivalently, locally in the étale topology, X admits an étale map towards $\mathbb{S}_C^{(e,d-e)} := \mathbb{T}_C^e \times \mathbb{D}_C^{d-e}$ with

$$\mathbb{T}_C^e := \mathrm{Spa}(C\langle T_1^{\pm 1}, \dots, T_e^{\pm 1}\rangle, C^+\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$$

and

$$\mathbb{D}_C^{d-e} := \mathrm{Spa}(C\langle S_{e+1}, \dots, S_d\rangle, C^+\langle S_{e+1}, \dots, S_d\rangle),$$

such that the log structure of X is the pullback of the log structure on $\mathbb{S}_C^{(e,d-e)}$ defined by $S_{e+1}\cdots S_d = 0$. An étale map $\psi : X \to \mathbb{S}_C^{(e,d-e)}$ factoring as a composite of finite étale maps and rational localizations is called a *toric chart* of X (also called a frame in the literature).

The log-Kummer exact sequence is constructed as follows.

Lemma 3.1.1. *Let $\mathcal{M}_X$ be the Kummer-étale sheaf of monoids defining the log structure of X, and let $\mathcal{M}_X^{\text{gp}}$ be its group completion or group of fractions. We have a short exact* *sequence of pro-Kummer-étale sheaves over X,*

$$0 \to \widehat{\mathbb{Z}}_p(1) \to \varprojlim_p \mathcal{M}_X^{\mathrm{gp}} \to \mathcal{M}_X^{\mathrm{gp}} \to 0,$$

where the limit is given by multiplication by p.

Proof. Consider the usual Kummer short exact sequence

$$0 \to \widehat{\mathbb{Z}}_p(1) \to \varprojlim_p \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0$$

where $\mathcal{O}_X$ is the uncompleted structural sheaf. To prove the lemma it suffices to see that the quotient

$$\bar{\mathcal{M}}_X^{\mathrm{gp}} = \mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^\times$$

is a $\mathbb{Z}[\frac{1}{p}]$-module. This property can be checked at the level of geometric points $\bar{x}$ of $X_{\text{két}}$. If $\bar{x}$ is disjoint from the divisor D defining the log structure then $\bar{\mathcal{M}}_{X,\bar{x}}^{\mathrm{gp}} = 0$ and we are done. Otherwise, we can assume that X has a toric chart $X \to \mathbb{S}_C^{(e,d-e)}$, which implies that

$$\bar{\mathcal{M}}_{X,\bar{x}} \cong \mathbb{Q}_{\geq 0}^k \tag{3.1}$$

with $0 \leq k \leq d-e$. Indeed, étale locally the log structure of X around x is modelled by the monoid $\mathbb{N}^k$ for $0 \leq k \leq d-e$, and (3.1) follows from [6, Construction 4.4.3]. Then the group of fractions of $\bar{\mathcal{M}}_{X,\bar{x}}$ is a $\mathbb{Q}$-vector space proving the claim. ∎

Proposition 3.1.2 ([25, Proposition 3.23] and [13, Proposition 2.25]). *Let X be as before, and let $\nu : X_{\text{prokét}} \to X_{\text{két}}$ be the projection of sites. Let $\delta : \mathcal{M}_X \to \Omega_X^1(\log)$ be the map of log differentials of the log structure* [6, Construction 3.3.2]. *Then there is a natural isomorphism*

$$R^1\nu_*\widehat{\mathcal{O}}_X(1) \cong \Omega_X^1(\log)$$

making the following diagram commute:

$$\begin{array}{ccc} \mathcal{M}_X^{\mathrm{gp}} & \longrightarrow & R^1\nu_*\widehat{\mathbb{Z}}_p(1) \\ \big\downarrow{\scriptstyle\delta} & & \big\downarrow \\ \Omega_X^1(\log) & \xrightarrow{\sim} & R^1\nu_*\widehat{\mathcal{O}}_X(1) \end{array}$$

obtained by the log-Kummer-étale sequence and the log differential map δ.

Remark 3.1.3. We prefer to keep the Tate twist even if C contains $\mathbb{Q}_p^{\mathrm{cyc}}$; when X has a form X' over (K, K^+), the isomorphism of Proposition 3.1.2 is Galois equivariant. Later in Section 3.4 we will show that, if X admits the form X', then

$$R\nu_*\widehat{\mathcal{O}}_X = \bigoplus_{i=1}^{d} \Omega_X^i(\log)(-i)[-i],$$

i.e. the pro-Kummer-étale cohomology of $\widehat{\mathcal{O}}_X$ naturally splits thanks to the Galois action.

Proof of Proposition 3.1.2. If X has trivial log structure this is [25, Proposition 3.23]; let us show that the same argument holds in the log smooth situation. First, the image of the map δ generates $\Omega^1_X(\log)$ as an $\mathcal{O}_X$-module, so if such an equivalence exists it must be unique. We can then argue locally in the étale topology of X and assume it has toric coordinates $X \to \mathbb{S}^{(e,d-e)}_C$. Then, using the approximation argument of [25, Lemma 3.24] (which only requires that the map $X \to \mathbb{S}^{(e,d-e)}_C$ factors as a composite of finite étale maps and rational localizations), we can assume that $X \to \mathbb{S}^{(e,d-e)}_C$ arises from base change of an étale map $X' \to \mathbb{S}^{(e,d-e)}_Y = Y \times_{\mathrm{Spa}\,\mathbb{Q}_p} \mathbb{S}^{(e,d-e)}_{\mathbb{Q}_p}$ via a map $\mathrm{Spa}(C, C^+) \to Y$ where Y is smooth of finite type over $\mathbb{Q}_p$. We endow X' with the log structure arising from the normal crossing divisors of $\mathbb{S}^{(e,d-e)}_{\mathbb{Q}_p}$. Finally, the same argument of *loc. cit.* holds by using instead the log-Faltings extension of X':

$$0 \to \widehat{\mathcal{O}}_{X'}(1) \to \mathrm{gr}^1\, \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log,X'} \to \Omega^1_{X'}(\log) \otimes \widehat{\mathcal{O}}_{X'} \to 0;$$

we leave the details to the reader. ■

Corollary 3.1.4. *Let $f : Y \to X$ be a map of fs log smooth adic spaces over (C, C^+) with normal crossing divisors. The following diagram is commutative:*

$$\begin{array}{ccc} f^*\Omega^1_X(\log) & \xrightarrow{\sim} & f^*R^1\nu_{X,*}\widehat{\mathcal{O}}_X(1) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega^1_Y(\log) & \xrightarrow{\sim} & f^*R^1\nu_{Y,*}\widehat{\mathcal{O}}_Y(1) \end{array}$$

Proof. This follows from Proposition 3.1.2, the commutative diagram

$$\begin{array}{ccc} f^*\mathcal{M}^\times_X & \xrightarrow{\delta} & f^*\Omega^1_X(\log) \\ \downarrow & & \downarrow \\ \mathcal{M}^\times_Y & \xrightarrow{\delta} & \Omega^1_Y(\log) \end{array}$$

and the fact that the image of δ generates the sheaf of differentials as vector bundles. ■

We give a different construction of the isomorphism $R^1\nu_*\widehat{\mathcal{O}}_X(1) \cong \Omega^1_X(\log)$ suggested by the referee. We first need a lemma:

Lemma 3.1.5. *Let X be an fs log adic space over (C, C^+) and let $V \in X_{\text{prokét}}$ be an object in the pro-Kummer-étale site of X. Let $V_\infty \to V$ be a pro-Kummer-étale torsor with Galois group Π, a profinite group with trivial pro-p-Sylow subgroup, and with V_∞ a log affinoid perfectoid space* [6, Definition 5.3.1].

Let $\mathcal{F}$ be a p-torsion free, p-adically complete $\widehat{\mathcal{O}}^+_X$-module, pro-Kummer-étale sheaf on $X_{\text{prokét}}$ which is almost acyclic on log affinoid perfectoid spaces and such that $\mathcal{F}/p$ arises from the fully-faithful map $\widetilde{X}_{\text{két}} \hookrightarrow \widetilde{X}_{\text{prokét}}$ of topoi [6, Proposition 5.1.7]. *Then there is an almost quasi-isomorphism*

$$R\Gamma_{\text{prokét}}(V, \mathcal{F}) =^{\text{ae}} \mathcal{F}(V).$$

Moreover, for any $\varepsilon > 0$ one has

$$R\Gamma_{\text{prokét}}(V, \mathcal{F}/p^{\varepsilon}) =^{\text{ae}} \mathcal{F}(V)/p^{\varepsilon}.$$

Proof. Since $\mathcal{F}$ is p-complete and p-torsion free, we have $\mathcal{F} = R\varprojlim_n \mathcal{F}/^{\mathbb{L}} p^n$ as sheaves in $X_{\text{prokét}}$ (where $M/^{\mathbb{L}} p^n$ is the cofiber of the multiplication by p^n map $\text{cofib}(M \xrightarrow{p^n} M)$). Moreover, $\mathcal{F}/^{\mathbb{L}} p^n = \mathcal{F}/p^n$ sits in degree 0. One formally has

$$R\Gamma_{\text{prokét}}(V, \mathcal{F}) = R\varprojlim_n R\Gamma_{\text{prokét}}(V, \mathcal{F}/p^n) = R\varprojlim_n R\Gamma_{\text{két}}(V, \mathcal{F}/p^n)$$

where in the last equivalence we use [6, Proposition 5.1.7]. Since V_∞ is log affinoid perfectoid, $\mathcal{F}$ is almost acyclic on V_∞, which implies that $\mathcal{F}/p^{\varepsilon}$ is almost acyclic on V_∞ and $(\mathcal{F}/p^{\varepsilon})(V_\infty) =^{\text{ae}} \mathcal{F}(V_\infty)/p^{\varepsilon}$ for all $\varepsilon > 0$. Namely, from the short exact sequence

$$0 \to \mathcal{F} \xrightarrow{p^{\varepsilon}} \mathcal{F} \to \mathcal{F}/p^{\varepsilon} \to 0 \tag{3.2}$$

we get an almost short exact sequence

$$0 \to \mathcal{F}(V_\infty) \xrightarrow{p^{\varepsilon}} \mathcal{F}(V_\infty) \to (\mathcal{F}/p^{\varepsilon})(V_\infty) \to 0.$$

Since $V_\infty \to V$ is a Π-torsor, one deduces that

$$R\Gamma_{\text{prokét}}(V, \mathcal{F}) =^{\text{ae}} R\varprojlim_n R\Gamma(\Pi, \mathcal{F}(V_\infty)/p^n),$$
$$R\Gamma_{\text{két}}(V, \mathcal{F}/p^{\varepsilon}) =^{\text{ae}} R\Gamma(\Pi, \mathcal{F}(V_\infty)/p^{\varepsilon}).$$

Notice that the action of Π on $\mathcal{F}(V_\infty)/p^{\varepsilon}$ is smooth as $\mathcal{F}/p^{\varepsilon}$ arises from the Kummer-étale site. Now, since Π has no pro-p-Sylow subgroup, it admits a Haar measure modulo p^{ε} for all $\varepsilon > 0$, and Π-cohomology is exact on smooth representations modulo p^{ε}. This implies that

$$R\Gamma_{\text{prokét}}(V, \mathcal{F}/p^{\varepsilon}) =^{\text{ae}} (\mathcal{F}/p^{\varepsilon})(V).$$

On the other hand, for $m \geq n$ the map $(\mathcal{F}/p^m)(V) \to (\mathcal{F}/p^n)(V)$ is almost surjective and one gets

$$R\Gamma_{\text{prokét}}(V, \mathcal{F}) =^{\text{ae}} R\varprojlim_n (\mathcal{F}/p^n)(V) = \lim_n (\mathcal{F}/p^n)(V) = \mathcal{F}(V),$$

proving the first claim. We deduce the second claim from the almost acyclicity of $\mathcal{F}$ in V and the short exact sequence (3.2). ∎

Proposition 3.1.6. *Let X be an fs log adic space over (C, C^+) and consider the short exact sequence of pro-Kummer-étale sheaves*

$$0 \to \widehat{\mathcal{O}}_X(1) \to \mathbb{B}_{\text{dR}}^+/t^2 \to \widehat{\mathcal{O}}_X \to 0$$

where t is a generator of the kernel of $\mathbb{B}_{\text{dR}}^+ \to \widehat{\mathcal{O}}_X$. Then the connecting map

$$d : \mathcal{O}_X = \nu_* \widehat{\mathcal{O}}_X \to R^1\nu_* \widehat{\mathcal{O}}_X(1)$$

is a C-linear continuous derivation inducing a natural map of $\mathcal{O}_X$-modules

$$\beta : \Omega^1_X \to R^1\nu_*\widehat{\mathcal{O}}_X(1).$$

Moreover, the map β extends uniquely to an isomorphism

$$\Omega^1_X(\log) \xrightarrow{\sim} R^1\nu_*\widehat{\mathcal{O}}_X(1).$$

Proof. All statements are étale local on X, so we can assume without loss of generality that $X = \mathrm{Spa}(A, A^+)$ is affinoid and admits a chart $\psi : X \to \mathbb{S}^{(e,d-e)}_C$. Let $T_1, \ldots, T_e$ be the torus coordinates and $S_{e+1}, \ldots, S_d$ the disc coordinates of $\mathbb{S}^{(e,d-e)}_C$. Let $\mathbb{S}^{(e,d-e)}_{C,\infty}$ be the pro-Kummer-étale cover over $\mathbb{S}^{(e,d-e)}_C$ obtained by all n-th roots of the variables T_i and S_j for all $n \in \mathbb{N}$. By [6, Definition 5.3.1], $\mathbb{S}^{(e,d-e)}_{C,\infty}$ is a log affinoid perfectoid space and then so is its pulback $X_\infty = \mathrm{Spa}(A_\infty, A^+_\infty)$ over X. Note that $X_\infty \to X$ is a pro-Kummer-étale torsor for the group $\widehat{\mathbb{Z}}(1)^d$ obtained as the Tate module of roots of unity.

Let $\mathbb{S}^{(e,d-e)}_{C,p^\infty}$ be the pro-Kummer-étale $\Gamma = \mathbb{Z}_p(1)^d$-torsor over $\mathbb{S}^{(e,d-e)}_C$ obtained by taking only p-power roots of the T_i and S_i, let $X_{p^\infty} = \mathrm{Spa}(A_{p^\infty}, A^+_{p^\infty})$ be its pullback to X. The map $X_\infty \to X_{p^\infty}$ is a $\widehat{\mathbb{Z}}^{(p),k}(1)$-torsor, where $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \widehat{\mathbb{Z}}_\ell$ has trivial pro-p-Sylow subgroups. Since $\mathcal{O}^+_X$ is almost acyclic on log affinoid perfectoids (equivalently the topologically nilpotent elements $\mathcal{O}^{\circ\circ}_X \subset \mathcal{O}^+_X$ are acyclic, cf. [6, Theorem 5.4.3]), Lemma 3.1.5 implies that

$$R\Gamma_{\text{prokét}}(X_{p^\infty}, \mathcal{F}) = \mathcal{F}(X_{p^\infty})$$

for $\mathcal{F} = \widehat{\mathcal{O}}_X$ and $\mathbb{B}^+_{\mathrm{dR}}/t^2$.

Then, since $X_{p^\infty} \to X$ is a pro-Kummer-étale Γ-torsor, one has an isomorphism of cohomologies

$$R\Gamma_{\text{prokét}}(X, \mathcal{F}) = R\Gamma(\Gamma, \mathcal{F}(X_{p^\infty}))$$

where the right-hand side is the continuous cohomology of Banach representations. By Proposition 2.2.14 there are Sen traces $R^i_n : A_{p^\infty} \to A^i_{p^\infty,n}$ arising from the normalized traces of the product of tori and polydiscs $\mathbb{S}^{(e,d-e)}_C$, and the triple $(A_{p^\infty}, \Gamma, (R^i_n))$ is a d-dimensional Sen theory. Note that as A-semilinear Γ-representations, A_{p^∞} and $A_{p^\infty}(1)$ are trivial, so they have trivial Sen operator and by Corollary 2.5.7 we have

$$R\Gamma(\Gamma, A_{p^\infty}) \cong \bigoplus_{i=0}^{d} A \otimes \bigwedge^i (\mathrm{Lie}\,\Gamma)^\vee[-i].$$

In particular, $H^1(X, \widehat{\mathcal{O}}_X(1))$ is a free A-module of rank d. Let us now see that the connecting map

$$d : A \to H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1)) \cong A^d$$

is a derivation. For this, we see $H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1))$ as isomorphic to the group cohomology $H^1(\Gamma, A_\infty(1)) = H^1(\Gamma, A(1)) \cong A^d$ which can be computed via 1-cocycles of Γ. The map d is constructed as follows. Let $\gamma_1, \ldots, \gamma_d \in \Gamma$ be the standard basis obtained by fixing a sequence $\varepsilon = (\zeta_{p^n})_n$ of p-power roots of unity. Let $a, b \in A \subset A_{p^\infty}$ and let $\widetilde{a}, \widetilde{b} \in \mathbb{B}_{\mathrm{dR}}^+/t^2(A_{p^\infty})$ be lifts of a and b respectively. The map d sends the element a to the 1-cocycle of Γ on A given by the tuple $d(a) = ((1-\gamma_1)(\widetilde{a}), \ldots, (1-\gamma_d)(\widetilde{a})) \in A^d$. One has

$$(1-\gamma_i)(\widetilde{a}\widetilde{b}) = (1-\gamma_i)(\widetilde{a})\gamma_i(\widetilde{b}) + \widetilde{a}(1-\gamma_i)(\widetilde{b}) = d(a)\gamma_i(b) + ad(b) = d(a)b + ad(b)$$

as elements in A, showing that d satisfies the Leibniz rule. The map d is clearly C-linear as Γ acts trivially on $\mathbb{B}_{\mathrm{dR}}^+(C)$. This proves that d is a C-linear derivation and so it induces a natural map

$$\beta : \Omega_A^1 \to H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1)).$$

We now want to show that β induces an isomorphism $\Omega_A^1(\log) \xrightarrow{\sim} H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1))$; note that if this isomorphism exists it must be unique since $\Omega_X^1 \to \Omega_X^1(\log)$ is an inclusion of A-vector bundles of the same rank and $H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1))$ is a vector bundle itself.

By construction, the space of log differentials of X has basis given by $\frac{dT_1}{T_1}, \ldots, \frac{dT_e}{T_e}$, $\frac{dS_{e+1}}{S_{e+1}}, \ldots, \frac{dS_d}{S_d}$. Thus, in order to show that β induces the desired isomorphism it suffices to compute the map d on the coordinates T_i and S_j. For this, let $[T_i^\flat] \in \mathbb{B}_{\mathrm{dR}}^+(A_{p^\infty})$ be the Teichmüller lift of the sequence of p-power roots of T_i (and similarly $[S_j^\flat]$ for S_j). Then

$$d(T_i) = \big((1-\gamma_1)([T_i^\flat]), \ldots, (1-\gamma_d)([T_i^\flat])\big),$$

which vanishes in all but the i-th entry which is equal to the class of

$$(1-[\varepsilon])[T_i^\flat] = \frac{1-[\varepsilon]}{1-[\varepsilon]^{1/p}}(1-[\varepsilon]^{1/p})[T_i^\flat] = t(1-[\varepsilon]^{1/p})[T_i^\flat]$$

in $A(1) \cong A$ which is nothing but $\theta((1-[\varepsilon]^{1/p})[T_i^\flat]) = (1-\zeta_p)T_i$ (after trivializing the Tate twist with t), with $\theta : \mathbb{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X$ being Fontaine's map. Similarly, one finds that $d(S_j) = (1-\zeta_p)S_j$. Therefore, if $\{v_i\}_{i=1}^d$ is the standard basis of A^d, the map β sends $\frac{dT_i}{T_i}$ to the vector $(1-\zeta_p)v_i$ and dS_j to the vector $(1-\zeta_p)S_j v_j$. This implies that β extends uniquely to an isomorphism

$$\Omega_X^1(\log) \xrightarrow{\sim} H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1)),$$

proving what we wanted. ■

Remark 3.1.7. One can easily show that the two isomorphisms $\Omega_X^1(\log) \cong R^1\nu_*\widehat{\mathcal{O}}_X$ of Propositions 3.1.2 and 3.1.6 are the same. For this it suffices to see that one has a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \varprojlim_p \mathcal{M}_X & \longrightarrow & \mathcal{M}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle [\alpha(-)]} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & \widehat{\mathcal{O}}_X(1) & \longrightarrow & \mathbb{B}_{\mathrm{dR}}^+/t^2 & \longrightarrow & \widehat{\mathcal{O}}_X & \longrightarrow & 0
\end{array}$$

where the middle map sends a sequence $(a^{1/p^n})_n$ in $\mathcal{M}_X$ to the Teichmüller lift $[(\alpha(a^{1/p^n}))_n]$ where $\alpha : \mathcal{M}_X \to \mathcal{O}_X \to \widehat{\mathcal{O}}_X$ is the monoid map of the log structure.

3.2. The geometric Sen operator: Local computation

In this section we prove a local version of Theorems 1.0.4 and 1.0.5 depending on toric charts. Let (C, C^+) be a perfectoid field containing $\mathbb{Q}_p^{\mathrm{cyc}}$. Let X be an fs log adic space over (C, C^+) with log structure arising from reduced normal crossing divisors. We define the relevant pro-Kummer-étale sheaves that will admit decompletions via locally analytic vectors.

Definition 3.2.1. A pro-Kummer-étale $\widehat{\mathcal{O}}_X$-module $\mathcal{F}$ over X is a *relative locally analytic ON Banach $\widehat{\mathcal{O}}_X$-sheaf*[9] if there is a Kummer-étale cover $\{U_i\}_{i\in I}$ of X such that

(i) for all i, the restriction $\mathcal{F}|_{U_i}$ admits a p-adically complete $\widehat{\mathcal{O}}_X^+$-lattice $\mathcal{F}_i^0$,

(ii) there is $\varepsilon > 0$ (depending on i) such that $\mathcal{F}_i^0/p^\varepsilon \cong^{\mathrm{ae}} \bigoplus_J \mathcal{O}_X^+/p^\varepsilon$ as almost $\mathcal{O}_X^+/p^\varepsilon$-modules[10] for some index set J.

3.2.1. The set-up. For the rest of the section we will assume that $X = \mathrm{Spa}(A, A^+)$ is affinoid and has a toric chart $\psi : X \to \mathbb{S}_C^{(e,d-e)}$ where $\mathbb{S}_C^{(e,d-e)} = \mathbb{T}_C^e \times \mathbb{D}_C^{d-e}$, and that ψ factors as a composite of finite étale maps and rational localizations. We highlight that, by definition of X, the charts ψ exist étale locally on X [6, Example 2.3.17]. We let T_i for $i = 1, \dots, e$ and S_j for $j = e+1, \dots, d$ denote the coordinates of the torus and disc components of $\mathbb{S}_C^{(e,d-e)}$ respectively. Let $\mathbb{S}_{C,\infty}^{(e,d-e)}$ be the pro-Kummer-étale torsor over $\mathbb{S}_C^{(e,d-e)}$ obtained by taking p-power roots of the coordinates T_i and S_j, and let $X_\infty = \mathrm{Spa}(A_\infty, A_\infty^+)$ be its pullback along ψ. Let $\Gamma = \mathbb{Z}_p(1)^d$ denote the Galois group of $X_\infty \to X$, and let $\gamma_1, \dots, \gamma_d$ denote the coordinates of Γ obtained after fixing $\varepsilon = (\zeta_{p^n})_n$, a compatible sequence of p-power roots of unity.

Remark 3.2.2. Note that the underlying adic space of X_∞ is an affinoid perfectoid space, but as an object in $X_{\text{prokét}}$ it is not a log affinoid perfectoid space in the sense of [6, Definition 5.3.1], namely, the sheaf of monoids of X_∞ is not modelled on an n-divisible monoid for $n \neq p$. However, thanks to Lemma 3.1.5 the sheaves $\mathcal{F}$ of Definition 3.2.1 are acyclic on X_∞ (after passing to a Kummer-étale cover such that $\mathcal{F}$ has a lattice $\mathcal{F}^0$ as in the definition). This yields a quasi-isomorphism

$$R\Gamma_{\text{prokét}}(X, \mathcal{F}) \cong R\Gamma(\Gamma, \mathcal{F}(X_\infty))$$

between the pro-Kummer-étale cohomology of $\mathcal{F}$ and the continuous Γ-cohomology of its X_∞-points, whenever $\mathcal{F}$ admits such a lattice $\mathcal{F}^0$. Furthermore, the lemma also implies that $\mathcal{F}^0(X_\infty)/p^\varepsilon \cong^{\mathrm{ae}} \bigoplus_{j\in J} A_\infty^+/p^\varepsilon$ as Γ-representations. Then, after modifying the lattice $\mathcal{F}^0(X_\infty)$ if necessary, the Γ-representation $\mathcal{F}(X_\infty)$ is a relative locally analytic ON A_∞-Banach representation of Γ as in Definition 2.3.5. Furthermore, the natural map

$$\mathcal{F}(X_\infty) \widehat{\otimes}_{A_\infty} \widehat{\mathcal{O}}_{X_\infty} \xrightarrow{\sim} \mathcal{F}|_{X_\infty}$$

is an equivalence of Γ-equivariant pro-Kummer-étale sheaves on X_∞.

[9] The abbreviation *ON* comes from orthonormalizable, meaning that locally we have a Banach basis.

[10] Recall that $\mathcal{O}_X^+/p^\varepsilon = \widehat{\mathcal{O}}_X^+/p^\varepsilon$ as $\widehat{\mathcal{O}}_X^+$ is the p-completion of $\mathcal{O}_X^+$.

3.2.2. Local version of Theorem 1.0.4. Proposition 2.2.14 implies that the ring A_∞ has Sen traces $R_n^i : A_\infty \to A_n^i$ for $i = 1, \dots, d$ such that the triple $(A_\infty, \Gamma, (R_n^i)_n)$ is a d-dimensional Sen theory as in Definition 2.2.5. We deduce the following proposition:

Proposition 3.2.3. *Let $\mathcal{F}$ be a relative locally analytic ON $\widehat{\mathcal{O}}_X$-module over X admitting a lattice $\mathcal{F}^0$ such that $\mathcal{F}^0/p^\varepsilon \cong^{\mathrm{ae}} \bigoplus \mathcal{O}_X^+/p^\varepsilon$ for some $\varepsilon > 0$. Then there is an $\widehat{\mathcal{O}}_X$-linear local geometric Sen operator functorial in $\mathcal{F}$ (but a priori depending on the chart ψ)*

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1(\log)(-1)$$

such that:

(1) *$\theta_{\mathcal{F}}$ is a Higgs field, i.e. $\theta_{\mathcal{F}} \wedge \theta_{\mathcal{F}} = 0$.*

(2) *Higgs cohomology computes pro-Kummer-étale cohomology: if $\nu : X_{\text{prokét}} \to X_{\text{két}}$ and $\eta : X_{\text{prokét}} \to X_{\text{an}}$ are the projections of sites, then*

$$R^i\nu_*\mathcal{F} = \nu_* H^i(\theta_{\mathcal{F}}, \mathcal{F}) \quad \textit{and} \quad R^i\eta_*\mathcal{F} = \eta_* H^i(\theta_{\mathcal{F}}, \mathcal{F})$$

where $H^i(\theta_{\mathcal{F}}, \mathcal{F})$ is the cohomology of the Higgs complex

$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1(\log)(-1) \to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^d(\log)(-d) \to 0.$$

(3) *Suppose that $\theta_{\mathcal{F}} = 0$. Then there are natural equivalences*

$$R\nu_*\mathcal{F} \cong \bigoplus_{i=0}^d \nu_*\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^i(\log)(-i)[-i],$$
$$R\eta_*\mathcal{F} \cong \bigoplus_{i=0}^d \eta_*\mathcal{F} \otimes_{\mathbb{Q}_p} \otimes_{\mathcal{O}_X} \Omega_X^i(\log)(-i)[-i],$$

depending on the toric chart ψ. Moreover, $\nu_*\mathcal{F}$ is an ON $\mathcal{O}_X$-Banach sheaf locally in the finite-Kummer-étale site of X, and we have

$$\mathcal{F} = \widehat{\mathcal{O}}_X \widehat{\otimes}_{\mathcal{O}_X} \nu_*\mathcal{F}.$$

Conversely, if $\mathcal{G}$ is a locally ON Banach $\mathcal{O}_X$-module in the Kummer-étale topology, then the geometric Sen operator of $\widehat{\mathcal{O}}_X \widehat{\otimes}_{\mathcal{O}_X} \mathcal{G}$ vanishes.

We write $\mathrm{Sen}_{\mathcal{F}} : \Omega_X^1(\log)^\vee(1) \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ *for the adjoint of $\theta_{\mathcal{F}}$.*

Proof. By Remark 3.2.2 the sheaf $\mathcal{F}$ is acyclic on X_∞ and $\mathcal{F}(X_\infty)$ is a relative locally analytic ON A_∞-Banach representation of Γ. Since (A_∞, Γ) admits Sen traces R_n^i and $(A_\infty, \Gamma, (R_n^i)_n)$ is a Sen theory by Proposition 2.2.14, we have an A_∞-linear Γ-equivariant Sen operator

$$\theta_{\mathcal{F}} : \mathcal{F}(X_\infty) \to \mathcal{F}(X_\infty) \otimes (\mathrm{Lie}\,\Gamma)^\vee. \tag{3.3}$$

By Proposition 2.5.7 we can identify (depending on the chart ψ)

$$H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X) \cong A \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\,\Gamma)^\vee.$$

From Propositions 3.1.2 or 3.1.6 we can also naturally identify (independently of the chart ψ)

$$H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_X(1)) = \Omega^1_A(\log).$$

Combining these two isomorphisms the map (3.3) becomes

$$\theta_{\mathcal{F}} : \mathcal{F}(X_\infty) \to \mathcal{F}(X_\infty) \otimes_A \Omega^1_A(\log)(-1).$$

Taking the completed base change to $\widehat{\mathcal{O}}_{X_\infty}$ and keeping track of the Γ-equivariance we have constructed an $\widehat{\mathcal{O}}_X$-linear map of pro-Kummer-étale sheaves on $X_{\text{prokét}}$,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}(X_\infty) \otimes_A \Omega^1_A(\log)(-1).$$

The map $\theta_{\mathcal{F}}$ is clearly functorial in $\mathcal{F}$ (though a priori it depends on ψ) being constructed by taking the derivation of $\operatorname{Lie}\Gamma$ on the locally analytic vectors of $\mathcal{F}(X_\infty)$. From the construction it is also clear that $\theta_{\mathcal{F}}$ is a Higgs field, proving (1). The comparison between invariants of Higgs cohomology and pro-étale cohomology is a consequence of Proposition 2.5.5; this gives (2). The cohomology computation of (3) follows from Proposition 2.5.7, and similarly for the statements about $\nu_*\mathcal{F}$ being ON $\mathcal{O}_X$-Banach module locally finite Kummer-étale and $\widehat{\mathcal{O}}_X \widehat{\otimes}_{\mathcal{O}_X} \nu_*\mathcal{F} = \mathcal{F}$. ∎

3.2.3. Local version of Theorem 1.0.5. Next, we construct the geometric Sen operator for a pro-Kummer-étale torsor with Galois group given by a p-adic Lie group. We let G denote a compact p-adic Lie group and let $\widetilde{X} \to X$ be a pro-Kummer-étale torsor over X with Galois group G.

Proposition 3.2.4. *Let V be a locally analytic Banach representation of G, and let $V_{\text{két}}$ be the pro-Kummer-étale sheaf over X constructed by V via the G-torsor $\widetilde{X} \to X$. Then there is a geometric Sen operator for the torsor $\widetilde{X} \to X$ (a priori depending on the chart ψ)*

$$\theta_{\widetilde{X}} : \widehat{\mathcal{O}}_X \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee_{\text{két}} \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1), \tag{3.4}$$

or dually a map

$$\mathrm{Sen}_{\widetilde{X}} : \Omega^1_X(\log)^\vee(1) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X \to (\operatorname{Lie} G)_{\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X,$$

where $\operatorname{Lie} G$ *is endowed with the adjoint action of G, and such that we have a commutative diagram of pro-Kummer-étale sheaves*

$$\begin{array}{ccc} V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X & \xrightarrow{d_V \otimes \mathrm{id}_{\widehat{\mathcal{O}}_X}} & (V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee_{\text{két}} \\ & \searrow^{\theta_V} & \downarrow{\scriptstyle \mathrm{id}_V \otimes \theta_{\widetilde{X}}} \\ & & (V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1) \end{array}$$

such that $d_V : V \to V \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee$ is induced by the derivation, and θ_V is the geometric Sen operator of $V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ of Proposition 3.2.3.

Proof. First, we highlight that by Lemma 2.1.5, $V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ is a relative locally analytic ON $\widehat{\mathcal{O}}_X$-sheaf on $X_{\text{prokét}}$, and so by Proposition 3.2.3 it admits a geometric Sen operator θ_V. We want to show that θ_V actually only depends on a universal map as in (3.4). Let $W = C^h(G, \mathbb{Q}_p)_{\star_2}$ be the space of h-analytic functions of G (for a fixed local chart $G_0 \cong \mathbb{Z}_p^k$ and radius of analyticity p^{-h}) endowed with the right regular action of G. Let $\widetilde{X}_\infty = \widetilde{X} \times_X X_\infty$ be the pullback in the pro-Kummer-étale site of X with ring of functions $\widetilde{A}_\infty$. The map $\widetilde{X}_\infty \to X$ is a pro-Kummer-étale $G \times \Gamma$-torsor.

By construction, the action of the Sen operators of $\operatorname{Lie}\Gamma$ on the algebra $W \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}_\infty$ is by left G-invariant $\widetilde{A}_\infty$-linear derivations. Indeed, set $\mathcal{F} = W_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$. The sheaf $\mathcal{F}$ is an $\widehat{\mathcal{O}}_X$-algebra, so its evaluation at X_∞ is an A_∞-algebra. By construction, the Sen operators are given by the derivations of $\operatorname{Lie}\Gamma$ on the locally analytic vectors $S(\mathcal{F}(X_\infty)) = \mathcal{F}(X_\infty)^{\Gamma\text{-la}}$. Since $\mathcal{F}(X_\infty)^{\Gamma\text{-la}}$ is an algebra, this action by derivations satisfies the Leibniz rule. In addition, $\operatorname{Lie}\Gamma$ acts $A_\infty^{\Gamma\text{-la}} = A_\infty^{\Gamma\text{-sm}}$-linearly on $\mathcal{F}(X_\infty)^{\Gamma\text{-la}}$. Since $\mathcal{F}$ is nothing other than the base change of $S(\mathcal{F}(X_\infty))$ from $A_\infty^{\Gamma\text{-sm}}$ to $\widehat{\mathcal{O}}_X$, one deduces that $\operatorname{Lie}\Gamma$ acts on $\mathcal{F}$ by $\widehat{\mathcal{O}}_X$-linear derivations. Furthermore, the action of $\operatorname{Lie}\Gamma$ is G-equivariant for the left regular action on $\mathcal{F}$, hence the Sen operators act by left G-invariant derivations.

The previous shows that the action of $\operatorname{Lie}\Gamma$ on $W \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}_\infty$ must factor through an $\widetilde{A}_\infty$-linear $G \times \Gamma$-equivariant map

$$\operatorname{Sen}_{\widetilde{X}} : \operatorname{Lie}\Gamma \otimes_{\mathbb{Q}_p} \widetilde{A}_\infty \to \operatorname{Lie} G \otimes_{\mathbb{Q}_p} \widetilde{A}_\infty$$

where $\operatorname{Lie} G$ acts on W via right derivations (i.e. left G-invariant derivations). Dually, we have a map

$$\theta_{\widetilde{X}} : \widetilde{A}_\infty \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee \to \widetilde{A}_\infty \otimes_{\mathbb{Q}_p} (\operatorname{Lie}\Gamma)^\vee \cong \widetilde{A}_\infty \otimes_A \Omega_A^1(\log)(-1).$$

In particular, we have a $G \times \Gamma$-equivariant commutative diagram

$$\begin{array}{ccc} W \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}_\infty & \xrightarrow{d_W \otimes \mathrm{id}} & W \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}_\infty \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee \\ & \searrow^{\theta_V} & \big\downarrow {\scriptstyle \mathrm{id}_V \otimes \theta_{\widetilde{X}}} \\ & & W \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}_\infty \otimes_A \Omega_A^1(\log)(-1) \end{array}$$

Taking base change to $\widehat{\mathcal{O}}_{\widetilde{X}_\infty}$ and keeping track of $G \times \Gamma$-equivariance, we have a map of pro-Kummer-étale sheaves over $X_{\text{prokét}}$ as in (3.4) which is compatible with the geometric Sen operator of W. Let now V be a general Banach locally analytic representation of G. Then by using the orbit map we get a G-equivariant inclusion

$$\mathcal{O}_V : V \hookrightarrow C^h(G, \mathbb{Q}_p)_{\star_2} \widehat{\otimes}_{\mathbb{Q}_p} V_0$$

for some $h > 0$ where V_0 has the trivial action of G. Passing to pro-Kummer-étale sheaves and since the formation of θ_V is functorial in V, we deduce that $\theta_{\widetilde{X}}$ also computes the geometric Sen operator of V, finishing the proof of the proposition. ∎

Remark 3.2.5. Proposition 3.2.4 shows that in order to compute the geometric Sen operator of a torsor it suffices to compute the geometric Sen operator of a faithful representation of $\mathrm{Lie}\, G$.

A direct consequence of the previous proposition is the vanishing of the action of the geometric Sen operators at infinite level. Let $\widetilde{A} = \widehat{\mathcal{O}}_X(\widetilde{X})$ be the algebra of completed functions on $\widetilde{X}$.

Corollary 3.2.6. *Let $V = C^{\mathrm{la}}(G, \mathbb{Q}_p)_{\star_1}$ be the left regular locally analytic representation of G and let $V_{\mathrm{k\acute{e}t}}$ be the pro-Kummer-étale sheaf over X obtained from V via the G-torsor $\widetilde{X} \to X$. The following hold:*

(1) $H^0_{\mathrm{prok\acute{e}t}}(X, V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = \widetilde{A}^{G\text{-la}}$.

(2) *Consider the action of $\widetilde{A}^{G\text{-la}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$ on $\widetilde{A}^{G\text{-la}}$ obtained by the $\widetilde{A}^{G\text{-la}}$-linear extension of the derivation action of* $\mathrm{Lie}\, G$. *Then the restriction of this action to the geometric Sen operators*

$$\mathrm{Sen}_{\widetilde{X}} : \widetilde{A}^{G\text{-la}} \otimes_A \Omega^1_A(\log)(-1) \to \widetilde{A}^{G\text{-la}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$$

vanishes.

Proof. For the first claim note that

$$H^0_{\mathrm{prok\acute{e}t}}(X, V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = H^0(G, C^{\mathrm{la}}(G, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}) = \widetilde{A}^{G\text{-la}}.$$

For the second statement, by Proposition 3.2.3 we have

$$\begin{aligned} H^0_{\mathrm{prok\acute{e}t}}(X, V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) &= H^0_{\mathrm{prok\acute{e}t}}(X, (V_{\mathrm{k\acute{e}t}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X)^{\theta_V = 0}) \\ &= H^0(G, (C^{\mathrm{la}}(G, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A})^{\theta_{V,\star_1} = 0}). \end{aligned}$$

In other words, if $\mathrm{Orb} : \widetilde{A}^{G\text{-la}} \to C^{\mathrm{la}}(G, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}^{G\text{-la}}$ denotes the orbit map $a \mapsto (g \mapsto ga)$, it factors through the subspace $(C^{\mathrm{la}}(G, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{A}^{G\text{-la}})^{\theta_{V,\star_1} = 0}$ where θ_V acts by zero via $\widetilde{A}^{G\text{-la}}$-linear left derivations.

Let $g \in G$ and $f : G \to \widetilde{A}^{G\text{-la}}$ a locally analytic function. Let us write $\star_2$ for the right regular action. We have

$$(g \star_1 f)(1) = f(g^{-1}) = (g^{-1} \star_2 f)(1).$$

This implies that for $\mathfrak{X} \in \widetilde{A}^{G\text{-la}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$ and f as before,

$$(\mathfrak{X} \star_1 f)(1) = -(\mathfrak{X} \star_2 f)(1).$$

On the other hand, a retract of the orbit map Orb is given by evaluation at 1, and additionally the orbit map is G-equivariant for the right regular action on the locally analytic functions. Thus, for $\mathfrak{X} \in \mathrm{Im}(\mathrm{Sen}_{\widetilde{X}}) \subset \widetilde{A}^{G\text{-la}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$ and $a \in \widetilde{A}^{G\text{-la}}$ we get

$$\mathfrak{X} \cdot a = (\mathfrak{X} \star_2 \mathrm{Orb}(a))(1) = -(\mathfrak{X} \star_1 \mathrm{Orb}(a))(1) = 0,$$

proving the vanishing of the Sen operators on $\widetilde{A}^{G\text{-la}}$ as desired. ∎

3.2.4. Vanishing of higher locally analytic vectors. As a first application of Proposition 3.2.4 let us prove local vanishing of higher locally analytic vectors at infinite level. Let $\psi : X \to \mathbb{S}_C^{(e,d-e)}$ be as before. Let G be a compact p-adic Lie group and let $\widetilde{X} \to X$ be a pro-Kummer-étale G-torsor. Let us write $X = \operatorname{Spa}(A, A^+)$ and let $\widetilde{A} := \widehat{\mathcal{O}}_X(\widetilde{X})$ be the completed functions on $\widetilde{X}$.

Proposition 3.2.7. *Suppose that the geometric Sen operator*

$$\theta_{\widetilde{X}} : \widehat{\mathcal{O}}_X \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^{\vee}_{\text{két}} \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1)$$

is surjective. Let $V = C^{\text{la}}(G, \mathbb{Q}_p)_{\star_1}$ be the space of locally analytic functions of G endowed with the left regular action, and let $V_{\text{két}}$ be the pro-Kummer-étale sheaf over X attached to V via the G-torsor $\widetilde{X} \to X$. Then the natural map

$$\widetilde{A}^{G\text{-la}} \xrightarrow{\sim} R\Gamma_{\text{prokét}}(X, V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X)$$

is a quasi-isomorphism. In particular, the right-hand side is concentrated in degree 0.

Proof. By Corollary 3.2.6 we know that $H^0_{\text{proként}}(X, V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = \widetilde{A}^{G\text{-la}}$. Therefore, it suffices to prove the vanishing of higher cohomology groups.

Write $V = \varinjlim_h V_h$ with $V_h = C^h(G, \mathbb{Q}_p)$ the spaces of h-analytic functions for $h \to \infty$. Since X is qcqs, we have

$$R\Gamma_{\text{proként}}(X, V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = \varinjlim_h R\Gamma_{\text{proként}}(X, V_{h,\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X).$$

By Proposition 3.2.4, the Sen operator of V_h arises from the left derivation of $\operatorname{Lie} G$ and the dual of the map $\theta_{\widetilde{X}}$. On the other hand, by Proposition 3.2.3 (2) we have

$$H^i_{\text{proként}}(X, V_h \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = H^0_{\text{proként}}(X, H^i(\theta_{V_h}, V_h \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X)).$$

Therefore, in order to prove the vanishing of higher cohomology groups it suffices to show that

$$H^i(\theta_V, V \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) := \varinjlim_h H^i(\theta_{V_h}, V_h \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = 0 \tag{3.5}$$

for $i \geq 1$.

The vanishing will follow essentially from the Poincaré–Birkhoff–Witt theorem by using a complementary basis of the image of the Sen operators in $\operatorname{Lie} G$ to construct local coordinates of the group G. To make this idea precise, it is more convenient to take colimits along compact-open subgroups $G_0 \subset G$. For $G_0 \subset G$ normal compact-open let $X_{G_0} \to X$ be the finite Kummer-étale extension obtained by the quotient $\widetilde{X}/G_0$ in the pro-Kummer-étale site. One has

$$C^{\text{la}}(G, \mathbb{Q}_p) = \operatorname{Ind}_{G_0}^G(C^{\text{la}}(G_0, \mathbb{Q}_p)),$$

and by Shapiro's lemma we have

$$R\Gamma_{\text{proként}}(X, (C^{\text{la}}(G, \mathbb{Q}_p))_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) = R\Gamma_{\text{proként}}(X_{G_0}, (C^{\text{la}}(G_0, \mathbb{Q}_p))_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X).$$

Therefore, in order to show that (3.5) vanishes it will suffice to work locally on G, and even to work as $G_0 \to 1$. We fix a basis $\mathfrak{X}_1, \ldots, \mathfrak{X}_g$ of the Lie algebra $\operatorname{Lie} G$ and for $h \gg 0$ we let $\mathring{\mathbb{G}}_h$ be the Stein analytic group induced by the exponential of the basis $p^h\mathfrak{X}_1, \ldots, p^h\mathfrak{X}_g$. Then $\mathring{\mathbb{G}}_h$ is isomorphic to the open polydisc $\mathring{\mathbb{D}}^g_{\mathbb{Q}_p}(p^{-h})$ of radius p^{-h} and dimension g. Given $h \gg 0$ we let $G_h = \mathring{\mathbb{G}}_h(\mathbb{Q}_p)$; it is an open compact subgroup of G. We can write the space of locally analytic functions as the colimit

$$C^{\mathrm{la}}(G, \mathbb{Q}_p) = \varinjlim_{h\to\infty} \operatorname{Ind}_{G_h}^{G} \mathcal{O}(\mathring{\mathbb{G}}_h).$$

Therefore, to prove the vanishing of (3.5) it suffices to show that

$$\varinjlim_{h} H^0(X_{G_h}, H^i(\theta_V, \mathcal{O}(\mathring{\mathbb{G}}_h)_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X)) = 0,$$

where $\mathcal{O}(\mathring{\mathbb{G}}_h)_{\text{két}}$ is the pro-Kummer-étale sheaf over X_{G_h} associated to $\mathcal{O}(\mathring{\mathbb{G}}_h)$. Let us take an arbitrary map $U = \operatorname{Spa}(R, R^+) \to \widetilde{X}$ with U log affinoid perfectoid. It suffices to see that

$$\varinjlim_{h} H^i(\theta_V, \mathcal{O}(\mathring{\mathbb{G}}_h) \widehat{\otimes}_{\mathbb{Q}_p} R) = 0.$$

By hypothesis, we have a direct sum decomposition

$$\operatorname{Lie} G \otimes_{\mathbb{Q}_p} R \cong \Omega^1_X(\log)(1) \otimes_A R \oplus W.$$

Fix R^0-lattices $W^0 \subset W$ and $\mathcal{L} \subset \Omega^1_X(\log)(1) \otimes_A R$. For $r \gg 0$ the R^0-lattice $p^r W^0 \oplus p^r\mathcal{L}$ of $\operatorname{Lie} G \otimes_{\mathbb{Q}_p} R$ is stable under the Lie bracket and it gives rise to Stein analytic groups $\mathring{\mathbb{H}}_r$ over $\operatorname{Spa}(R, R^+)$. Given h there is some r such that $p^r W^0 \oplus p^r\mathcal{L}$ contains the lattice $R^0 p^h\mathfrak{X}_1 \oplus \cdots \oplus R^0 p^h \mathfrak{X}_g$ and conversely. We deduce that

$$\varinjlim_{h} \mathcal{O}(\mathring{\mathbb{G}}_h) \widehat{\otimes}_{\mathbb{Q}_p} R = \varinjlim_{r} \mathcal{O}(\mathring{\mathbb{H}}_r) \tag{3.6}$$

as modules over the R-linear Lie algebra $R \otimes \operatorname{Lie} G$. Finally, we can write

$$\mathcal{O}(\mathring{\mathbb{H}}_r) = \mathcal{O}(\exp(\mathring{p^r\mathcal{L}})) \widehat{\otimes}_R \mathcal{O}(\exp(\mathring{p^r W^0}))$$

as modules over the Lie algebra $R \otimes_{\mathbb{Q}_p} \operatorname{Lie} \Gamma$ (acting by left derivations on the left term), where the exponentials of the lattices are simply given by open polydiscs over $\operatorname{Spa}(R, R^+)$ after picking a basis. Finally, the action of the image of the Sen operators in $\operatorname{Lie} \Gamma$ is induced by the left regular action, and so it only acts on the term $\mathcal{O}(\exp(\mathring{p^r\mathcal{L}}))$ of the tensor product. Since $\operatorname{Lie} \Gamma$ is abelian, this action is induced by the linear action of $p^r\mathcal{L}$ on the adic space, and by picking coordinates the cohomology

$$R\Gamma(\operatorname{Lie} \Gamma, \mathcal{O}(\exp(\mathring{p^r\mathcal{L}})))$$

is identified with the de Rham cohomology of $\mathcal{O}(\exp(\mathring{p^r\mathcal{L}}))$. One has

$$R\Gamma(\operatorname{Lie} \Gamma, \mathcal{O}(\exp(\mathring{p^r\mathcal{L}}))) = R$$

by the Poincaré lemma for open polydiscs [31, Lemma 26]. We deduce that

$$R\Gamma(\theta_V, \mathcal{O}(\mathring{\mathbb{H}}_r)) \cong \mathcal{O}(\exp(\mathring{p}^r W^0))$$

is concentrated in degree 0. This implies that the θ_V-cohomology of (3.6) is in degree 0 and therefore (3.5) holds. This finishes the proof of the proposition. ∎

Remark 3.2.8. Let G be a compact p-adic Lie group and let $\widetilde{X} \to X$ be a pro-Kummer-étale G-torsor. Let $X_\infty \to X$ be the Γ-torsor that arises from the coordinates. Then the $G \times \Gamma$-torsor $\widetilde{X}_\infty = \widetilde{X} \times_X X_\infty \to X$ satisfies the hypothesis of Proposition 3.2.7.

3.3. *The geometric Sen operator: Globalization*

In Propositions 3.2.3 and 3.2.4 we proved local versions of Theorems 1.0.4 and 1.0.5 respectively. In order to obtain the global version we need to show that the Sen operators glue in toric charts ψ according to $\Omega^1_X(\log)$.

3.3.1. Key case. Let X and Y be affinoid fs log smooth adic spaces over (C, C^+), and suppose we have two toric charts $\psi_X : X \to \mathbb{S}_C^{(e,d-e)}$ and $\psi_Y : Y \to \mathbb{S}_C^{(g,h-g)}$. Let X_∞ and Y_∞ be the pro-Kummer-étale torsors over X and Y obtained by taking p-power roots of the coordinates of the charts, and let Γ_X and Γ_Y denote the Galois groups of $X_\infty \to X$ and $Y_\infty \to Y$ respectively. Let $f : Y \to X$ be a morphism over (C, C^+), and let $f^*X_\infty = Y \times_X X_\infty$, seen as an object in $X_{\text{prokét}}$. The following is a direct generalization of [17, Lemma 3.4.3]; we thank Lue Pan for the simplifications of a previous proof.

Proposition 3.3.1. *Let $\nu_X : X_{\text{prokét}} \to X_{\text{két}}$ and $\nu_Y : Y_{\text{prokét}} \to Y_{\text{két}}$ be the projections of sites. We have a commutative diagram of $\widehat{\mathcal{O}}_Y$-linear pro-Kummer-étale sheaves over Y,*

$$\begin{array}{ccc}
(\operatorname{Lie}\Gamma_X)^\vee \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_Y & \xrightarrow{\theta_{f^*X_\infty}} & \Omega^1_Y(\log)(-1) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y \\
\tilde{\beta}\uparrow & \nearrow_{f^*} & \\
f^*\Omega^1_X(\log)(-1) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y & &
\end{array}$$

where the upper horizontal map is the Sen operator of the Γ_X-torsor $f^*X_\infty \to Y$ of Proposition 3.2.4, the diagonal map is the pullback of differentials, and the left vertical map is the composite

$$\Omega^1_X(-1) \xrightarrow{\beta} R^1\nu_{X,*}\widehat{\mathcal{O}}_X \cong (\operatorname{Lie}\Gamma_X)^\vee \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$

with the first map being the natural isomorphism of Propositions 3.1.2 and 3.1.6, and the second map obtained from Corollary 2.5.7 by computing $R^1\nu_{X,*}\widehat{\mathcal{O}}_X$ as Γ_X-cohomology.

Proof. It suffices to compute the geometric Sen operators over Y for a faithful representation of Γ_X. Let V be the unique unipotent algebraic representation of Γ_X fitting in a short exact sequence

$$0 \to \mathbb{Q}_p e \to V \to (\operatorname{Lie}\Gamma_X)^\vee \to 0 \tag{3.7}$$

such that

- the action of $\operatorname{Lie}\Gamma_X$ is trivial on the left and right terms of (3.7),
- if $\mathfrak{Y}^\vee \in (\operatorname{Lie}\Gamma_X)^\vee$ and $\mathfrak{X} \in \operatorname{Lie}\Gamma_X$, then for any lift $\widetilde{\mathfrak{Y}}^\vee \in V$ of $\mathfrak{Y}$ we have $\mathfrak{X}\cdot\widetilde{\mathfrak{Y}}^\vee = \mathfrak{Y}^\vee(\mathfrak{X})\cdot e$.

It is clear that V is a faithful representation of Γ_X. Let $V_{Y,\text{két}}$ be the pro-Kummer-étale sheaf over Y that arises from V via the Γ_X-torsor $f^*X_\infty \to Y$, and let

$$\operatorname{Sen}_{V,Y} : (\Omega^1_Y(\log))^\vee(1) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y \otimes_{\mathbb{Q}_p} V_{Y,\text{két}} \to V_{Y,\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_Y$$

be the local Sen operator of $V_{\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_Y$ of Proposition 3.2.3. Similarly, we let $V_{X,\text{két}}$ be the pro-Kummer-étale sheaf over X associated to V via the Γ_X-torsor $X_\infty \to X$. We also have a geometric Sen operator over X,

$$\operatorname{Sen}_{V,X} : (\Omega^1_X(\log))^\vee(1) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X \otimes_{\mathbb{Q}_p} V_{X,\text{két}} \to V_{X,\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X.$$

Note that $f^*V_{X,\text{proket}} = V_{Y,\text{proket}}$ as pro-Kummer-étale sheaves. We want to show that the following diagram is commutative:

$$\begin{array}{ccc}
(\Omega^1_Y(\log))^\vee(1) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y \otimes_{\mathbb{Q}_p} V_{Y,\text{két}} & & \\
\downarrow f_* & \overset{\operatorname{Sen}_{V,Y}}{\searrow} & \\
f^*(\Omega^1_X(\log))^\vee(1) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y \otimes_{\mathbb{Q}_p} V_{Y,\text{két}} & \xrightarrow[f^*\operatorname{Sen}_{V,X}]{} & V_{Y,\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_Y
\end{array} \tag{3.8}$$

where $f_* : (\Omega^1_Y(\log))^\vee \to f^*(\Omega^1_X(\log))^\vee$ is dual to the pullback map of log differentials.

For this we need to do a computation. Write $X = \operatorname{Spa}(A, A^+)$ and $Y = \operatorname{Spa}(B, B^+)$ as affinoid spaces. Let $X_\infty = \operatorname{Spa}(A_\infty, A^+_\infty)$ and $Y_\infty = \operatorname{Spa}(B_\infty, B^+_\infty)$ be the perfectoid torsors arising from the coordinates of X and Y respectively, and write $f^*X_\infty \times_X Y_\infty = \operatorname{Spa}(B_{\infty,\infty}, B^+_{\infty,\infty})$ for the $\Gamma_X \times \Gamma_Y$-torsor over Y. We have maps of rings

$$\begin{array}{ccccc}
B & \longrightarrow & B_\infty & \longrightarrow & B_{\infty,\infty} \\
\uparrow & & & \nearrow & \\
A & \longrightarrow & A_\infty & &
\end{array}$$

By Remark 3.2.8 we find that $B^{R(\Gamma_X\times\Gamma_Y)\text{-la}}_{\infty,\infty} = B^{\Gamma_X\times\Gamma_Y\text{-la}}_{\infty,\infty}$ sits in degree 0. By Lemma 3.1.5 we can compute

$$H^1_{\text{proket}}(Y, \widehat{\mathcal{O}}_Y) \cong H^1(\Gamma_X \times \Gamma_Y, B_{\infty,\infty}).$$

By Theorem 2.1.2 (4) and the vanishing of higher locally analytic vectors of $B_{\infty,\infty}$ we have

$$H^1(\Gamma_X \times \Gamma_Y, B_{\infty,\infty}) = H^1(\operatorname{Lie}\Gamma_X \times \operatorname{Lie}\Gamma_Y, B^{\Gamma_X\times\Gamma_Y\text{-la}}_{\infty,\infty})^{\Gamma_X\times\Gamma_Y}.$$

The map β induces an isomorphism

$$\Omega^1_B(\log)(-1) = H^1_{\text{proet}}(Y, \widehat{\mathcal{O}}_Y) \cong H^1(\operatorname{Lie}\Gamma_X \times \operatorname{Lie}\Gamma_Y, B^{\Gamma_X\times\Gamma_Y\text{-la}}_{\infty,\infty})^{\Gamma_X\times\Gamma_Y}. \tag{3.9}$$

Let $(\gamma_{i,X})_{i=1}^d$ and $(\gamma_{j,Y})_{j=1}^h$ be the standard bases of the groups $\Gamma_X = \mathbb{Z}_p(1)^d$ and $\Gamma_Y = \mathbb{Z}_p(1)^h$ obtained after fixing a compatible sequence $\varepsilon = (\zeta_{p^n})_n$ of p-power roots of unity. Let $(\log \gamma_{i,X})$ and $(\log \gamma_{j,Y})$ be the corresponding bases of the Lie algebras.

By Corollary 3.1.4 we have a commutative diagram

$$\begin{array}{ccccc} \Omega^1_B(\log)(-1) & \xrightarrow{\beta_Y} & H^1_{\text{proét}}(Y, \widehat{\mathcal{O}}_Y) & \xrightarrow{\sim} & H^1(\Gamma_Y, B) \cong (\operatorname{Lie}\Gamma_Y)^\vee \otimes_{\mathbb{Q}_p} B \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle A} \\ \Omega^1_A(\log)(-1) & \xrightarrow{\beta_X} & H^1_{\text{proét}}(Y, \widehat{\mathcal{O}}_Y) & \xrightarrow{\sim} & H^1(\Gamma_X, A) \cong (\operatorname{Lie}\Gamma_X)^\vee \otimes_{\mathbb{Q}_p} A \end{array}$$

where we have identified group cohomology with Lie algebra cohomology of the trivial representation, and where $A = (a_{i,j})$ is the induced matrix from the basis $((\log \gamma_{i,X})^\vee)_i$ to $((\log \gamma_{j,Y})^\vee)_j$.

Thanks to (3.9), and the interpretation of Lie algebra cohomology as cocycles, we can find locally analytic functions $(z_j)_{j=1}^h$ in $B_{\infty,\infty}^{\Gamma_X\times\Gamma_Y\text{-la}}$ such that

$$\log \gamma_{l,X} \cdot z_k = \delta_{k,l} \quad \text{and} \quad \log \gamma_{l,Y} \cdot z_k = -a_{k,l}. \tag{3.10}$$

Indeed, the 1-cocycle $(\log \gamma_{k,X})^\vee - \sum_j a_{k,j}(\log \gamma_{j,Y})^\vee$ in the Koszul complex of $B^{\Gamma_X\times\Gamma_Y\text{-la}}$ for $\operatorname{Lie}\Gamma_X \times \operatorname{Lie}\Gamma_Y$ is cohomologically trivial and so there exists an element $z_k \in B^{\Gamma_X\times\Gamma_Y\text{-la}}$ such that

$$\begin{aligned} \sum_i (\log \gamma_{i,X} \cdot z_k)(\log \gamma_{i,X})^\vee + \sum_j (\log \gamma_{j,Y} \cdot z_k)(\log \gamma_{j,Y})^\vee \\ = (\log \gamma_{k,X})^\vee - \sum_j a_{k,j}(\log \gamma_{j,Y})^\vee, \end{aligned}$$

which yields (3.10).

Consider a $\mathbb{Q}_p$-vector space splitting $V \cong \mathbb{Q}_p e \oplus (\operatorname{Lie}\Gamma_X)^\vee$ and write $\widetilde{(\log \gamma_{i,X})}^\vee$ for the lift of $(\log \gamma_{i,X})^\vee$. Using the elements v_k, and after shrinking Γ_X and Γ_Y if necessary for them to act analytically on the z_k, we find a $\Gamma_X \times \Gamma_Y$-equivariant isomorphism

$$(V \otimes_{\mathbb{Q}_p} B_{\infty,\infty}) \cong B_{\infty,\infty} e \oplus \bigoplus_{k=1}^h v_k B_{\infty,\infty}$$

such that $v_k = \widetilde{(\log \gamma_{k,X})}^\vee - z_k \cdot e$ has a trivial action of Γ_X. Taking invariants under Γ_X we get

$$(V \otimes_{\mathbb{Q}_p} B_{\infty,\infty})^{\Gamma_X} \cong B_\infty e \oplus \bigoplus_{k=1}^h v_k B_\infty$$

as B_∞-semilinear Γ_Y-representations. The basis $(e, v_1, \ldots, v_h)$ is Γ_Y-locally analytic, e is already Γ_Y-invariant, and the other vectors satisfy

$$(\log \gamma_{j,Y}) \cdot v_k = -((\log \gamma_{j,Y}) \cdot z_k)e = a_{k,j} e.$$

This shows the commutativity of (3.8), proving the proposition. ■

Theorem 3.3.2. *Let $\mathcal{F}$ be a relative locally analytic ON Banach $\widehat{\mathcal{O}}_X$-sheaf over X. Then the local geometric Sen operators of Proposition 3.2.3 glue to an $\widehat{\mathcal{O}}_X$-linear global geometric Sen operator*

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1).$$

Furthermore, the following properties hold:

(1) *The formation of $\theta_{\mathcal{F}}$ is functorial in $\mathcal{F}$ and $\theta_{\mathcal{F}} \wedge \theta_{\mathcal{F}} = 0$.*

(2) *Let $\nu : X_{\text{prokét}} \to X_{\text{két}}$ be the projection from the pro-Kummer-étale site to the Kummer-étale site. Then there is a natural equivalence*

$$R^i \nu_* \mathcal{F} = \nu_* H^i(\theta_{\mathcal{F}}, \mathcal{F}),$$

where $H^i(\theta_{\mathcal{F}}, \mathcal{F})$ is the cohomology of the Higgs complex

$$0 \to \mathcal{F} \xrightarrow{\theta_{\mathcal{F}}} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1) \to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^d_X(\log)(-d) \to 0.$$

(3) *Suppose that $\theta_{\mathcal{F}} = 0$. Then $\nu_*\mathcal{F}$ is locally in the Kummer-étale topology of X an ON $\mathcal{O}_X$-Banach module and $\mathcal{F} = \widehat{\mathcal{O}}_X \widehat{\otimes}_{\mathcal{O}_X} \nu_*\mathcal{F}$. Conversely, for any locally ON $\mathcal{O}_X$-Banach module $\mathcal{G}$ the pullback $\widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \mathcal{G}$ has trivial Sen operator.*

(4) *If X has a form X' over a discretely valued field with perfect residue field (K, K^+) and $\mathcal{F}$ is defined over X', then $\theta_{\mathcal{F}}$ is Galois equivariant. In particular, we recover the natural splitting*

$$R\nu_*\widehat{\mathcal{O}}_X = \bigoplus_{i=0}^{d} \Omega^i_X(\log)(-i)[-i].$$

(5) *Let Y be another fs log smooth adic space over (C, C^+), and let $f : Y \to X$ be a morphism. Then there is a commutative diagram of geometric Sen operators*

$$\begin{array}{ccc} f^*\mathcal{F} & \xrightarrow{f^*\theta_{\mathcal{F}}} & f^*\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\Omega^1_X(\log)(-1) \\ & \searrow^{\theta_{f^*\mathcal{F}}} & \downarrow{\scriptstyle \mathrm{id}\otimes f^*} \\ & & f^*\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_Y(\log)(-1) \end{array}$$

Proof. The gluing of the local Sen operators of Proposition 3.2.3 follows from Proposition 3.3.1. Indeed, we can suppose that X is affinoid and has two charts $\psi_1, \psi_2 : X \to \mathbb{S}_C^{(e,d-e)}$ (these exist by definition of X having log structure arising from reduced normal crossing divisors; see [6, Example 2.3.17]). Let $X_{\infty,0}$ and $X_{0,\infty}$ be the pro-Kummer-étale Γ-torsors obtained from the perfectoid toric charts ψ_1 and ψ_2 respectively, and let $X_{\infty,\infty} = X_{\infty,0} \times_X X_{0,\infty}$. Let us write Γ_1 and Γ_2 for the Galois groups of $X_{\infty,0}$ and $X_{0,\infty}$ respectively. Then, by Proposition 2.2.14 and Theorem 2.4.4 there is $n \gg 0$ such that we have a $\Gamma_1 \times \Gamma_2$-equivariant isomorphism

$$\widehat{\mathcal{O}}_X(X_{\infty,\infty}) \widehat{\otimes}_{\mathcal{O}_X(X_{\infty,0})^{p^n\Gamma_1}} \mathcal{F}(X_{\infty,0})^{p^n\Gamma_1\text{-an}} = \mathcal{F}(X_{\infty,\infty}) = \widehat{\mathcal{O}}_X(X_{\infty,\infty}) \widehat{\otimes}_{\mathcal{O}_X(X_{0,\infty})^{p^n\Gamma_2}} \mathcal{F}(X_{0,\infty})^{p^n\Gamma_2\text{-an}}.$$

Therefore, by Proposition 3.2.4, the action of the Sen operators

$$\operatorname{Lie}\Gamma_1 \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X \cong (\Omega^1_X(\log))^\vee(1) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X$$

on $\mathcal{F}$ via the chart ψ_1 can be computed either as the extension of the scalars via the natural derivations on $\mathcal{F}(X_{\infty,0})^{p^n\Gamma_1\text{-an}}$, or as the extension of the scalars of the action of $\operatorname{Lie}\Gamma_2$ on $\mathcal{F}(X_{0,\infty})^{p^n\Gamma_2\text{-an}}$ by precomposing with the Sen map

$$\mathrm{Sen} : (\operatorname{Lie}\Gamma_1) \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X \to (\operatorname{Lie}\Gamma_2) \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X \cong (\Omega^1_X(\log))^\vee(1) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X.$$

These two actions agree thanks to Proposition 3.3.1.

Now, part (1) follows from the gluing and the functoriality of Proposition 3.2.3. Parts (2) and (3) are consequences of parts (2) and (3) of *loc. cit.* respectively. For part (4), note that the Galois action of Gal_K on the Lie algebra $\operatorname{Lie}\Gamma$ induced by some local toric coordinate $\psi : X \to \mathbb{S}^{(e,d-e)}_C$ is given by the cyclotomic character χ. Finally, part (5) follows from Proposition 3.3.1 by using an analogous argument to the one at the beginning of the proof. ∎

Under further assumptions on the space X and the sheaf $\mathcal{F}$ we can even compute the projection to the analytic site:

Corollary 3.3.3. *Let $\eta : X_{\text{prokét}} \to X_{\text{an}}$ be the projection of sites. Suppose that $\mathcal{F}$ admits a lattice $\mathcal{F}^+$ such that $\mathcal{F}^+/p^\varepsilon \cong^{\mathrm{ae}} \bigoplus_I \mathcal{O}^+_X/p^\varepsilon$ for some $\varepsilon > 0$ and some index set I, and X admits toric charts locally in the analytic topology. Then there are natural isomorphisms*

$$R^i\eta_*\mathcal{F} = \eta_* H^i(\theta_{\mathcal{F}}, \mathcal{F}).$$

Proof. This follows from Theorems 2.4.4, 3.3.2 and Corollary 2.5.7 by applying Sen theory to toric coordinates arising locally in the analytic topology of X. ∎

3.3.2. Gluing for general G. We have made all the preparations to prove Theorem 1.0.5.

Theorem 3.3.4. *Let X be an fs log smooth adic space over $\mathrm{Spa}(C, C^+)$ with log structure given by normal crossing divisors. Let G a p-adic Lie group and $\widetilde{X} \to X$ a pro-Kummer-étale G-torsor. Then the geometric Sen operators of Proposition 3.2.4 given by local charts of X glue to a morphism of $\widehat{\mathcal{O}}_X$-vector bundles over $X_{\text{prokét}}$,*

$$\theta_{\widetilde{X}} : \widehat{\mathcal{O}}_X \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee_{\text{két}} \to \widehat{\mathcal{O}}_X(-1) \otimes_{\mathcal{O}_X} \Omega^1_X(\log),$$

such that $\theta_{\widetilde{X}} \wedge \theta_{\widetilde{X}} = 0$. In particular, for any locally analytic Banach representation V of G, we have a commutative diagram

$$\begin{array}{ccc} V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X & \xrightarrow{d_V \otimes \mathrm{id}_{\widehat{\mathcal{O}}}} & (V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X) \otimes_{\mathbb{Q}_p} (\operatorname{Lie} G)^\vee_{\text{két}} \\ & \searrow^{\theta_V} & \downarrow{\scriptstyle \mathrm{id}_V \otimes \theta_{\widetilde{X}}} \\ & & (V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X(-1)) \otimes_{\mathcal{O}_X} \Omega^1_X(\log) \end{array}$$

such that $d_V : V \to V \otimes (\mathrm{Lie}\, G)^\vee$ is induced by derivations, and θ_V is the geometric Sen operator of $V_{\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$.

Moreover, let $H \to G$ be a morphism of p-adic Lie groups, let Y an fs log smooth adic space over (C, C^+) and let $\widetilde{Y} \to Y'$ be an H-torsor. Suppose we are given a commutative diagram compatible with the group actions

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{3.11}$$

Then the following square is commutative:

$$\begin{array}{ccc} f^*(\mathrm{Lie}\, G)^\vee_{\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_Y & \xrightarrow{f^*\theta_{\widetilde{X}}} & f^*\Omega^1_X(\log) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y(-1) \\ \downarrow & & \downarrow \\ (\mathrm{Lie}\, H)^\vee_{\text{két}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_Y & \xrightarrow{\theta_{\widetilde{Y}}} & \Omega^1_Y(\log) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y(-1) \end{array} \tag{3.12}$$

Proof. We first construct the geometric Sen operator $\theta_{\widetilde{X}}$ for the G-torsor $\widetilde{X} \to X$. Let V be a locally analytic representation of G on a Banach space, and consider the inclusion of the orbit map

$$V \hookrightarrow V \widehat{\otimes}_{\mathbb{Q}_p} C^{\mathrm{la}}(G, \mathbb{Q}_p)_{\star_2},$$

where G acts via the right regular action on the right-hand side. By writing $C^{\mathrm{la}}(G, \mathbb{Q}_p) = \varinjlim_h C^h(G, \mathbb{Q}_p)$ as the colimit of h-analytic functions on G (for some fixed local coordinates), we can just assume without loss of generality that $V = C^h(G, \mathbb{Q}_p)_{\star_2}$ endowed with the right regular action. Let $\mathcal{F} = C^h(G, \mathbb{Q}_p)_{\star_2, \text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ be the relative locally analytic ON $\widehat{\mathcal{O}}_X$-Banach module over X defined by V, and let $\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1)$ be the geometric Sen operator on $\mathcal{F}$ of Theorem 3.3.2. Consider the adjoint map

$$\mathrm{Sen}_{\mathcal{F}} : \Omega^{1,\vee}_X(\log)(1) \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}.$$

By Proposition 3.2.4, locally in the Kummer-étale topology of X, $\mathrm{Sen}_{\mathcal{F}}$ acts by $\widehat{\mathcal{O}}_X$-linear derivations on $\mathcal{F}$ that are in addition left G-equivariant. Then the local maps of Proposition 3.2.4 glue to an $\widehat{\mathcal{O}}_X$-linear morphism

$$\mathrm{Sen}_{\widetilde{X}} : \Omega^{1,\vee}_X(\log) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(1) \to (\mathrm{Lie}\, G)_{\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X,$$

or equivalently, by taking adjoints, to a map

$$\theta_{\widetilde{X}} : (\mathrm{Lie}\, G)^\vee_{\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X \to \Omega^1_X(\log) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(-1)$$

satisfying the conclusion of the theorem.

Next, we prove functoriality for the commutative square (3.11). Consider the space $C^{\mathrm{la}}(G, \mathbb{Q}_p) = \varinjlim_h C^h(G, \mathbb{Q}_p)$ of locally analytic functions of G, written as the colimit of h-analytic functions. By restriction, we can also see $C^h(G, \mathbb{Q}_p)$ as an H-representation via the map $H \to G$. Let

$$\mathcal{F} = C^h(G, \mathbb{Q}_p)_{\star_2, \text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$$

be the pro-Kummer-étale sheaf over X associated to $C^h(G, \mathbb{Q}_p)$. The sheaf $f^*\mathcal{F}$ over Y is also the $\widehat{\mathcal{O}}_Y$-extension of scalars of the pro-Kummer-étale sheaf associated to the H-representation $C^h(G, \mathbb{Q}_p)|_H$. Let $\theta_{\mathcal{F}}$ and $\theta_{f^*\mathcal{F}}$ be the geometric Sen operators on $\mathcal{F}$ and $f^*\mathcal{F}$ respectively. By Theorem 3.3.2 (5) we have a commutative diagram

$$\begin{array}{ccc} f^*\mathcal{F} & \xrightarrow{f^*\theta_{\mathcal{F}}} & f^*\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\Omega^1_X(\log)(-1) \\ & \searrow^{\theta_{f^*\mathcal{F}}} & \downarrow{\scriptstyle \mathrm{id} \otimes f^*} \\ & & f^*\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_Y(\log)(-1) \end{array}$$

The action of $f^*\theta_{\mathcal{F}}$ and $\theta_{f^*\mathcal{F}}$ is also given by left G-invariant $\widehat{\mathcal{O}}_Y$-linear derivations. This implies that they must arise from the Lie algebra action of $\operatorname{Lie} G$ on the square (3.12), proving that it commutes. This finishes the proof of the theorem. ∎

The Sen morphism should encode the directions of perfectoidness of X. We have the following conjecture, which is a generalization of a theorem of Sen saying that a p-adic Galois representation of a p-adic field has vanishing Sen operator if and only if it is potentially unramified [10, Corollary 3.32].

Conjecture 3.3.5. *Let X be an fs log smooth adic space over (C, C^+) with normal crossing divisors, let G be a compact p-adic Lie group and $\widetilde{X} \to X$ a pro-Kummer-étale G-torsor. Then the geometric Sen operator $\theta_{\widetilde{X}} : \widehat{\mathcal{O}}_X \otimes_{\widehat{\mathbb{Q}}_p} (\operatorname{Lie} G)^\vee_{\text{két}} \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \Omega^1_X(\log)^\vee(-1)$ is surjective if and only if $\widetilde{X}$ is a perfectoid space.*

Remark 3.3.6. In [20], we show that the pro(-Kummer-)étale torsors defining the infinite level Shimura varieties satisfy the hypothesis of Conjecture 3.3.5. The proof of this fact never uses the perfectoidness of the Shimura variety, only the p-adic Riemann–Hilbert correspondence of [7]. Moreover, in [20] we use the explicit construction of the geometric Sen operators for Shimura varieties to prove the vanishing part of the rational Calegari–Emerton conjectures [4].

Remark 3.3.7. In his original work [28, 29], Sen shows that an algebraic extension K of $\mathbb{Q}_p$ with Galois group a p-adic Lie group G is deeply ramified (equivalently, its completion is perfectoid) if and only if the Sen operator seen as an object $\theta \in \operatorname{Lie} G \otimes_{\mathbb{Q}_p} \widehat{K}$ is non-zero. Recently in [12], He proved that Conjecture 3.3.5 holds for residue fields of rigid varieties. This pointwise perfectoidness criterion is sufficient for proving the vanishing of the Calegari–Emerton conjectures integrally.

3.4. Locally analytic vectors of pro-Kummer-étale towers

We keep the previous notations, i.e. (C, C^+) is a perfectoid field containing $\mathbb{Q}_p^{\text{cyc}}$, X is an fs log smooth adic space over (C, C^+) with log structure given by normal crossing divisors, G a compact p-adic Lie group, and $\widetilde{X}$ a pro-Kummer-étale G-torsor over X. In this last section we apply Theorem 3.3.2 to study the locally analytic vectors of $\widehat{\mathcal{O}}_{\widetilde{X}}$ for the action of G. We also extend the cohomology computations of the theorem to log adic spaces arising from the boundary of X.

3.4.1. Pro-Kummer-étale cohomology of the boundary. Let X and $\widetilde{X} \to X$ be as before. Let $D \subset X$ be the boundary divisor. By definition, étale locally on X, D can be written as a disjoint union of irreducible components, $D = \bigcup_{a \in I} D_a$, where the finite intersections of the D_a's are smooth (cf. [6, Examples 2.3.17 and 2.3.18]). For the rest of the section we work with a boundary divisor D decomposed as $D = \bigcup_{a \in I} D_a$ as before. Given a finite subset $J \subset I$ we set $D_J = \bigcap_{a \in J} D_a$ endowed with the log structure pulled back from X, and write $\iota_J : D_J \subset X$ for the inclusion map. We denote by $\widehat{\mathcal{O}}_{D_J}^{(+)}$ the sheaf $\iota_{J,*}\widehat{\mathcal{O}}_{D_J}^{(+)}$ over $X_{\text{prokét}}$.

In this section we write $\mathcal{O}_{X,\text{két}}$ and $\mathcal{O}_{X,\text{an}}$ for the structural sheaves on the Kummer-étale site and the analytic site respectively. We also denote by $\mathcal{O}_{D_J,\text{két}}$ and $\mathcal{O}_{D_J,\text{an}}$ the sheaves defined by the boundary divisor D_J.

We start with a partial extension of Theorem 3.3.2 to the boundary.

Proposition 3.4.1. *Let $\mathcal{F}$ be a relative locally analytic $\widehat{\mathcal{O}}_X$-module over $X_{\text{prokét}}$ and let $\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1)$ be the geometric Sen operator of $\mathcal{F}$. Let $\nu_* : X_{\text{prokét}} \to X_{\text{két}}$ be the projection of sites, and let $J \subset I$ be a finite subset. Then there is a natural isomorphism*

$$R^i \nu_* \iota_{J,*}\iota_J^* \mathcal{F} = \nu_*(H^i(\theta_{\mathcal{F}}, \iota_* \iota_J^* \mathcal{F})). \tag{3.13}$$

Moreover, let $\eta : X_{\text{prokét}} \to X_{\text{an}}$ be the projection of sites and suppose that $\mathcal{F}$ has a lattice $\mathcal{F}^+$ such that $\mathcal{F}^+/p^{\varepsilon} =^{\text{ae}} \bigoplus_I \mathcal{O}_X^+/p^{\varepsilon}$ as Kummer-étale $\mathcal{O}_X^+/p^{\varepsilon}$-modules for some $\varepsilon > 0$. Then

$$R^i \eta_* \iota_{J,*}\iota_J^* \mathcal{F} = \eta_*(H^i(\theta_{\mathcal{F}}, \iota_* \iota_J^* \mathcal{F})). \tag{3.14}$$

Proof. All the statements are local on X for the Kummer-étale or analytic topology, so we can assume that we have a toric chart $\psi : X \to \mathbb{S}_C^{(e,d-e)}$, and take $D = D_J$ to be the vanishing locus of $S_{e+1} = \cdots = S_d = 0$. We also write $\iota : D \to X$ for the inclusion map. We can also assume that $\mathcal{F}$ admits an almost $\widehat{\mathcal{O}}_X^+$-lattice $\mathcal{F}^+$ as in Definition 3.2.1. In particular, notice that

$$\iota_* \iota^*(\mathcal{F}^+/p^{\varepsilon}) \cong^{\text{ae}} \bigoplus_I \mathcal{O}_D^+/p^{\varepsilon}.$$

Let X_∞ be the $\Gamma \cong \mathbb{Z}_p(1)^d$-torsor over X obtained via ψ by taking p-power roots of the coordinates T_i and S_j, set $D_\infty = X_\infty \times_X D$. Let $A_\infty = \widehat{\mathcal{O}}_X(X_\infty)$ be the sheaf of functions of X_∞ and $A_{\infty,D} = \widehat{\mathcal{O}}_D(D_\infty)$. By Example 2.2.7 (5), the boundary of $\mathbb{S}_\infty^{(e,d-e)} = \text{Spa}(C\langle T_i^{\frac{1}{p^\infty}}, S_j^{\frac{1}{p^\infty}}\rangle, C^+\langle T_i^{\frac{1}{p^\infty}}, S_j^{\frac{1}{p^\infty}}\rangle)$ admits Sen traces for the action of Γ, and by Proposition 2.2.14 these Sen traces extend to a d-dimensional Sen theory $(A_{D,\infty}, \Gamma, (R_n^i)_{n,i})$ to the boundary of X_∞.

By Lemma 3.1.5 applied to $\iota_*\iota^*\mathcal{F}$, we have

$$R\Gamma(X_\infty, \iota_*\iota^*\mathcal{F}) = \mathcal{F}(D_\infty).$$

The condition on $\mathcal{F}^+/p^\varepsilon$ together with the derived Nakayama lemma yields

$$\mathcal{F}(D_\infty) = A_{D,\infty} \widehat{\otimes}^L_{A_\infty} \mathcal{F}(X_\infty)$$

as Γ-equivariant A_∞-modules.

The Sen operators on $\mathcal{F}(D_\infty)$ are constructed from the action of $\operatorname{Lie}\Gamma$ on its locally analytic vectors. By the projection formula for locally analytic vectors of Lemma 2.1.6 we get

$$\begin{aligned}\mathcal{F}(D_\infty)^{\Gamma\text{-la}} &= (A_{D,\infty} \widehat{\otimes}^L_{A_\infty} \mathcal{F}(X_\infty))^{\Gamma\text{-la}} \\ &= (A_{D,\infty} \widehat{\otimes}^L_{A_\infty^{\Gamma\text{-la}}} \mathcal{F}(X_\infty)^{\Gamma\text{-la}})^{\Gamma\text{-la}} \\ &= A_{D,\infty}^{\Gamma\text{-la}} \widehat{\otimes}^L_{A_\infty^{\Gamma\text{-la}}} \mathcal{F}(X_\infty)^{\Gamma\text{-la}}.\end{aligned}$$

where in the second equivalence we have used Theorem 2.4.4 to decomplete $\mathcal{F}(X_\infty)$ via locally analytic vectors.

This shows that the geometric Sen operators on $\iota_*\iota^*\mathcal{F}$ are just the base change along $\widehat{\mathcal{O}}_X \to \widehat{\mathcal{O}}_D$ of the geometric Sen operators on $\mathcal{F}$. Since $(A_{D,\infty}, \Gamma, (R^i_n)_{n,i})$ is a Sen theory satisfying (AST), formulas (3.13) and (3.13) follow from Proposition 2.5.5 as the left-hand sides are computed via Γ-group cohomology. ∎

3.4.2. Locally analytic vectors of $\widehat{\mathcal{O}}_X$. Let (C, C^+) be a perfectoid field containing $\mathbb{Q}_p^{\mathrm{cyc}}$, X an fs log smooth adic space over (C, C^+) with normal crossing divisors, and $\widetilde{X} \to X$ a pro-Kummer-étale torsor for a compact p-adic Lie group G. For an open subgroup $G_0 \subset G$ we let X_{G_0} be the finite Kummer-étale extension over X given by the quotient $\widetilde{X}/G_0$ in $X_{\text{prokét}}$. The space $\widetilde{X}$ has an underlying topological space given by $|\widetilde{X}| = \varprojlim_{G_0 \subset G} |X_{G_0}|$, where $|X_{G_0}|$ is the underlying adic space of X_{G_0}. The analytic site of $\widetilde{X}$ is the site of disjoint unions of open subspaces of $|\widetilde{X}|$.

Definition 3.4.2. Let $\widehat{\mathcal{O}}_{\widetilde{X}}$ be the restriction of the completed structural sheaf of $X_{\text{prokét}}$ to the analytic site of $\widetilde{X}$. We let $\mathcal{O}^{\mathrm{la}}_{\widetilde{X}} \subset \widehat{\mathcal{O}}_{\widetilde{X}}$ be the presheaf mapping a qcqs open subspace $\widetilde{U} \subset \widetilde{X}$ to the ind-Banach space

$$\mathcal{O}^{\mathrm{la}}_{\widetilde{X}}(\widetilde{U}) := \widehat{\mathcal{O}}_{\widetilde{X}}(\widetilde{U})^{G_{\widetilde{U}}\text{-la}}$$

of $G_{\widetilde{U}}$-locally analytic sections of $\widehat{\mathcal{O}}_{\widetilde{X}}(\widetilde{U})$, where $G_{\widetilde{U}}$ is the stabilizer of $\widetilde{U}$.

Lemma 3.4.3. *The presheaf $\mathcal{O}^{G\text{-la}}_{\widetilde{X}}$ is a sheaf for the analytic topology of $\widetilde{X}$.*

Proof. Note that for any qcqs open subspace $\widetilde{U} \subset \widetilde{X}$, the ring $\widehat{\mathcal{O}}_{\widetilde{X}}(\widetilde{U})$ is a $\mathbb{Q}_p$-Banach space. Let $\{\widetilde{V}_i\}_{i=1}^s$ be an open cover of $\widetilde{U}$ by qcqs open subspaces. We have a short exact sequence

$$0 \to \widehat{\mathcal{O}}_{\widetilde{X}}(\widetilde{U}) \to \prod_i \widehat{\mathcal{O}}_{\widetilde{X}}(\widetilde{V}_i) \to \prod_{i,j} \widehat{\mathcal{O}}_{\widetilde{X}}(\widetilde{V}_i \cap \widetilde{V}_j). \tag{3.15}$$

Let $G_0 \subset G$ be an open compact subgroup stabilizing all the $\widetilde{V}_i$'s. Then G_0 stabilizes $\widetilde{U}$ and the intersections $\widetilde{V}_i \cap \widetilde{V}_j$, and (3.15) is a G_0-equivariant exact sequence of Banach spaces. Then, tensoring with $C^{\mathrm{la}}(G_0,\mathbb{Q}_p)_{\star_1}$ and taking G_0-invariant vectors we get an exact sequence

$$0\to \mathcal{O}^{\mathrm{la}}_{\widetilde{X}}(\widetilde{U})\to \prod_i \mathcal{O}^{\mathrm{la}}_{\widetilde{X}}(\widetilde{V}_i)\to \prod_{i,j}\mathcal{O}^{\mathrm{la}}_{\widetilde{X}}(\widetilde{V}_i\cap \widetilde{V}_j),$$

proving what we wanted. ∎

The following definition is useful to construct sheaves over $\widetilde{X}$.

Definition 3.4.4. Let $G_0\subset G$ be an open compact subgroup, denote by $\eta_{G_0}: X_{G_0,\text{prokét}}\to X_{G_0,\text{an}}$ the projection of sites. Let $\mathcal{F}:=(\mathcal{F}_{G_0})_{G_0\subset G}$ be a compatible sequence of pro-Kummer-étale sheaves on X_{G_0} in the sense that if $G_0'\subset G_0$ we have a map $\psi_{G_0}^{G_0'}:\mathcal{F}_{G_0}|_{X_{G_0'}}\to \mathcal{F}_{G_0'}$, and for $G_0''\subset G_0'\subset G_0$ we have $\psi_{G_0'}^{G_0''}\circ\psi_{G_0}^{G_0'}=\psi_{G_0}^{G_0''}$. We define

$$R\eta_{\infty,*}\mathcal{F}:=\varinjlim_{G_0} R\eta_{G_0,*}\mathcal{F}_{G_0}$$

seen as a sheaf over $|\widetilde{X}|$.

Let $D\subset X$ be the boundary divisor and suppose that $D=\bigcup_{i\in I}D_i$, a disjoint union of irreducible divisors with smooth finite intersections. Given a finite subset $J\subset I$ as we let $D_J=\bigcap_{i\in J}D_i$ we declare $D_\emptyset=X$. Let $\widehat{\mathcal{O}}_{D_J}$ be the sheaf of completed functions on $D_{J,\text{prokét}}$ seen as a sheaf over $X_{\text{prokét}}$ after pushforward along the inclusion.

Theorem 3.4.5. *Let $\eta: X_{\text{prokét}}\to X_{\text{an}}$ be the projection of sites, and let*

$$C^{\mathrm{la}}(\operatorname{Lie}G,\mathbb{Q}_p)_{\star_1,\text{két}}:=(C^{\mathrm{la}}(G_0,\mathbb{Q}_p)_{\star_1,\text{két}})_{G_0\subset G}$$

be the ind-sequence of pro-Kummer-étale sheaves obtained from left regular representations over the tower $(X_{G_0})_{G_0\subset G}$ as in Definition 3.4.4. Suppose that the following axiom holds:

(BUN) *The geometric Sen operator of $\widetilde{X}$*

$$\theta_{\widetilde{X}}:\widehat{\mathcal{O}}_X\otimes_{\mathbb{Q}_p}(\operatorname{Lie}G)^\vee_{\text{két}}\to \widehat{\mathcal{O}}_X\otimes_{\mathcal{O}_X}\Omega^1_X(\log)(-1)$$

is surjective.

Then, for any finite subset $J\subset I$, we have

$$R\eta_{\infty,*}(C^{\mathrm{la}}(\operatorname{Lie}G,\mathbb{Q}_p)_{\star_1,\text{két}}\widehat{\otimes}_{\mathbb{Q}_p}\widehat{\mathcal{O}}_{D_J})=\mathcal{O}^{\mathrm{la}}_{D_J},$$

where $\mathcal{O}^{\mathrm{la}}_{D_J}$ is the sheaf of locally analytic sections of $\widehat{\mathcal{O}}_{D_J}$ restricted to $|\widetilde{X}|$. Furthermore, we have an exact sequence of sheaves in $|\widetilde{X}|$,

$$\mathcal{O}^{\mathrm{la}}_{\widetilde{X}}\to\bigoplus_{a\in I}\mathcal{O}^{\mathrm{la}}_{D_a}\to\bigoplus_{\substack{J\subset I\\|J|=2}}\mathcal{O}^{\mathrm{la}}_{D_J}\to\cdots\to\mathcal{O}^{\mathrm{la}}_{D_I}\to 0, \tag{3.16}$$

induced by the boundary divisors.

Proof. This follows formally from Propositions 3.2.7 and 3.4.1. Indeed, we can work locally in the analytic topology of X, and even after a finite Kummer-étale cover of the form X_{G_0}. Thus, we can assume without loss of generality that $X = \mathrm{Spa}(A, A^+)$ is affinoid and that it admits a chart to $\mathbb{S}_C^{(e,d-e)}$ with coordinates T_i, S_j such that $D = D_J$ is the vanishing locus of $S_{e+1} = \cdots = S_d = 0$. Let us write $V_{G_0} = C^{\mathrm{la}}(G_0, \mathbb{Q}_p)_{\star_1}$ for the left regular representation of G_0. We want to see that

$$\varinjlim_{G_0} R\Gamma(X_{G_0,\text{prokét}}, V_{G_0,\text{két}} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_D) = \mathcal{O}_D^{\mathrm{la}}.$$

The vanishing of higher cohomology groups follows from Proposition 3.2.7. The identification of the degree 0 cohomology is also clear.

It is left to show that (3.16) is an exact sequence. By Theorem 3.3.2 (2) we know that

$$\mathcal{O}^{\mathrm{la}}(\widetilde{X}) = \varinjlim_{G_0} H^0_{\text{prokét}}(X_{G_0}, (V_{G_0} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X)^{\theta_V=0}).$$

By the proof of Proposition 3.2.7, the filtered system $(V_{G_0} \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X)^{\theta_V=0}$ of pro-étale sheaves on the tower X_{G_0} is ind-equivalent to a system $(\mathcal{F}_{G_0})_{G_0}$ of pro-étale sheaves where each $\mathcal{F}_{G_0}$ is a relative locally analytic sheaf on X_{G_0} admitting a lattice $\mathcal{F}^+_{G_0}$ as in Definition 3.2.1. Indeed, in the notation of the proposition, these are the sheaves corresponding to

$$\mathcal{O}(\exp(p^r W^0)) = \mathcal{O}(\mathbb{H}_r)^{\mathrm{Lie}\,\Gamma=0}.$$

Hence, we can write

$$\mathcal{O}^{\mathrm{la}}(\widetilde{X}) = \varinjlim_{G_0} \mathcal{F}_{G_0}(X_{G_0}).$$

Let X_∞ be the Γ-torsor over X obtained by adding p-power roots of the coordinates S_i and T_j, set $\widetilde{X}_\infty = \widetilde{X}_\infty \times_X X_\infty$. For $G_0 \subset G$ an open compact subgroup denote $X_{G_0,\infty} = X_{G_0} \times_X X_\infty$. Then we can write

$$H^0(X_{G_0}, \mathcal{F}_{G_0}) = H^0(G_0 \times \Gamma, \mathcal{F}_{G_0}(\widetilde{X}_\infty)) = H^0(\Gamma, \mathcal{F}(X_{G_0,\infty})).$$

By Proposition 2.2.14 we have a Sen theory on $\mathcal{O}(X_{G_0,\infty})$ for the action of Γ. Since the Sen operators on $\mathcal{F}_{G_0}$ are trivial, by Corollary 2.5.7 there is an open subgroup $\Gamma_{G_0} \subset \Gamma$ (depending on G_0) such that

$$\mathcal{F}_{G_0} = \mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}} \widehat{\otimes}_{\mathcal{O}(X_{G_0,\infty})^{\Gamma_{G_0}}} \widehat{\mathcal{O}}_{X_{G_0}}$$

with $\mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}}$ an ON $\mathcal{O}(X_{G_0,\infty})^{\Gamma_{G_0}}$-Banach module.

We have an exact sequence of pro-Kummer-étale $\widehat{\mathcal{O}}_X$-modules on $X_{\text{prokét}}$,

$$\widehat{\mathcal{O}}_X \to \bigoplus_{a \in I} \widehat{\mathcal{O}}_{D_a} \to \bigoplus_{\substack{J \subset I \\ |J|=2}} \widehat{\mathcal{O}}_{D_J} \to \cdots \to \widehat{\mathcal{O}}_{D_I} \to 0.$$

Tensoring with $\mathcal{F}_{G_0}$, evaluating at $X_{G_0,\infty}$ and taking Γ_{G_0}-invariant vectors, we get an exact sequence

$$\mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}} \to \bigoplus_{a\in I} \mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}} \widehat{\otimes}_{\mathcal{O}(X_{G_0,\infty})^{\Gamma_{G_0}}} \mathcal{O}_{D_a}(X_{G_0,\infty})^{\Gamma_{G_0}} \to \cdots \to \mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}} \widehat{\otimes}_{\mathcal{O}(X_{G_0,\infty})^{\Gamma_{G_0}}} \mathcal{O}_{D_I}(X_{G_0,\infty})^{\Gamma_{G_0}} \to 0$$

(this is the same as tensoring $\mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}}$ with the exact sequence induced by divisors of the rigid space $X_{G_0,\Gamma_0} := X_{G_0,\infty}/\Gamma_{G_0}$; it remains exact since $\mathcal{F}(X_{G_0,\infty})^{\Gamma_{G_0}}$ admits an ON basis). Taking group cohomology for the finite group action Γ/Γ_{G_0}, we obtain an exact sequence

$$\mathcal{F}_{G_0}(X_{G_0}) \to \bigoplus_{a\in I} (\mathcal{F}_{G_0} \widehat{\otimes}_{\widehat{\mathcal{O}}_X} \widehat{\mathcal{O}}_{D_a})(X_{G_0}) \to \cdots \to (\mathcal{F}_{G_0} \widehat{\otimes}_{\widehat{\mathcal{O}}_X} \widehat{\mathcal{O}}_{D_I})(X_{G_0}) \to 0.$$

Taking colimits as $G_0 \to 1$ one obtains the exact sequence (3.16) as desired. ■

3.5. Relation to the p-adic Simpson correspondence

We finish this section with the relationship between the geometric Sen operator and the p-adic Simpson correspondence of [7, 15, 33]. We shall write (K, K^+) for a complete discretely valued non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field, and let (C, C^+) be the completion of an algebraic closure of K. We let $t \in \mathbb{B}_{\mathrm{dR}}^+(C, C^+)$ be a generator of the kernel of the map of Fontaine $\theta : \mathbb{B}_{\mathrm{dR}}^+(C, C^+) \to C$, e.g. $t = \log([\varepsilon])$ where $\varepsilon = (\zeta_{p^n})_n$ is a sequence of compatible p-power roots of unity.

Let us recall the following result.

Theorem 3.5.1 ([15, Theorem 2.1] and [7, Theorem 3.2.4]). *Let X be an fs log smooth adic space over* $\mathrm{Spa}(K, K^+)$ *and let $\mathbb{L}^0$ be a pro-Kummer-étale $\mathbb{Z}_p$-local system, and write $\mathbb{L} = \mathbb{L}^0[\frac{1}{p}]$. Let $\mathcal{O}\mathbb{C}_{\log} = \mathrm{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ be the Hodge–Tate period sheaf, and $\nu : X_{C,\text{prokét}} \to X_{C,\text{két}}$ be the projection of sites. Then*

$$\mathcal{H}(\mathbb{L}) := R\nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{C}_{\log})$$

is a Gal_K*-equivariant* log *Higgs bundle concentrated in degree* 0*. Let θ denote the Higgs field of $\mathcal{H}(\mathbb{L})$. Then*

$$R\nu_*\mathbb{L} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X = R\Gamma(\theta, \mathcal{H}(\mathbb{L})). \tag{3.17}$$

To re-prove this theorem in our theory we first need to compute the geometric Sen operator of $\mathcal{O}\mathbb{C}_{\log}$.

Proposition 3.5.2. *The geometric Sen operator of*

$$\mathcal{O}\mathbb{C}_{\log} = \mathrm{gr}^0(\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}) = \varinjlim_n \mathrm{Sym}^n(\mathrm{gr}^1\, \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}) \cdot t^{-n}$$

is given by

$$\mathcal{O}\mathbb{C}_{\log} \xrightarrow{-\overline{\nabla}} \mathcal{O}\mathbb{C}_{\log}(-1) \otimes_{\mathcal{O}_X} \Omega^1(\log),$$

where $\overline{\nabla}$ is the residual connection of $\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \otimes_{\mathcal{O}_X} \Omega^1(\log)$.

Proof. It suffices to identify $\theta_{\mathcal{O}\mathbb{C}_{\log}}$ with $-\nabla$ locally on X, we can then assume that X has toric coordinates $\psi : X \to \mathbb{S}_K^{(e,d-e)}$. Let $X_{C,\infty} = X \times_{\mathbb{S}_C^{(e,d-e)}} \mathbb{S}_{C,\infty}^{(e,d-e)}$ the Γ-torsor over X. Then by [7, Proposition 2.3.15] we have a presentation

$$\mathcal{O}\mathbb{C}_{\log}|_{X_{C,\infty}} = \widehat{\mathcal{O}}_{X_{C,\infty}}\left[\frac{\log(\underline{T}^{-1}[\underline{T}]^{b})}{t}, \frac{\log(\underline{S}^{-1}[\underline{S}^{\flat}])}{t}\right],$$

where $T^{\flat} = (T^{1/p^n})_{n\in\mathbb{N}}$, $S^{\flat} = (S^{1/p^n})_{n\in\mathbb{N}}$, and $t = \log([\varepsilon])$. We see from this presentation that

$$\overline{\nabla}(\log(\underline{T}^{-1}[T^{\flat}])) = -\frac{d\underline{T}}{\underline{T}} \quad \text{and} \quad \overline{\nabla}(\log(\underline{S}^{-1}[\underline{S}^{\flat}])) = -\frac{d\underline{S}}{\underline{S}}.$$

Note that the products of the variables $\frac{\log(T_i^{-1}[T_i]^{\flat})}{t}$ and $\frac{\log(S_j^{-1}[S_j]^{\flat})}{t}$ form a locally analytic basis of the $\widehat{\mathcal{O}}_X$-module $\mathcal{O}\mathbb{C}_{\log}$ over $X_{C,\infty}$. Hence, to compute the Sen operator of $\mathcal{O}\mathbb{C}_{\log}$ it suffices to know the derivative of the action of Γ on them. Let γ_i be the standard basis of Γ. Since

$$\gamma_i^a(\log(T_k^{-1}[T_k]^{\flat})) = \log(T_k^{-1}[T_k]^{\flat}) + \delta_{i,k}at$$

(and similarly for the S_j), we deduce that

$$\theta_{\mathcal{O}\mathbb{C}_{\log}}(\log \underline{T}^{-1}[T^{\flat}]) = \frac{d\underline{T}}{\underline{T}} \quad \text{and} \quad \theta_{\mathcal{O}\mathbb{C}_{\log}}(\log \underline{S}^{-1}[\underline{S}^{\flat}]) = \frac{d\underline{S}}{\underline{S}},$$

proving that $\theta_{\mathcal{O}\mathbb{C}_{\log}} = -\overline{\nabla}$ as desired. ■

Proof of Theorem 3.5.1. We first need to make a construction. Let us suppose without loss of generality that $\mathbb{L}^0$ is of rank n. Define the $\mathrm{GL}_n(\mathbb{Z}_p)$-torsor

$$\widetilde{X} := \mathrm{Isom}(\mathbb{Z}_p^n, \mathbb{L}^0).$$

Thus, $\mathbb{L}$ is constructed from the standard representation of GL_n via the torsor $\widetilde{X}$. In particular, by Theorem 3.3.4, $\mathbb{L} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X$ has a Sen operator $\theta_{\mathbb{L}}$ arising from a map of pro-Kummer-étale sheaves

$$\theta_{\widetilde{X}} : (\mathfrak{gl}_n)^{\vee}_{\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X \to \Omega_X^1(\log) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(-1),$$

or equivalently a map

$$\mathrm{Sen}_{\widetilde{X}} : \Omega_X^1(\log)^{\vee} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(1) \to \mathfrak{gl}_{n,\text{két}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_X.$$

On the other hand, by Proposition 3.5.2, the Sen operator on $\mathcal{O}\mathbb{C}_{\log}$ is $-\overline{\nabla}$. Thus, by Theorem 3.3.2 (2) one gets

$$R^i\nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{C}_{\log}) = \nu_* H^i(-\theta_{\mathbb{L}\otimes\mathcal{O}\mathbb{C}_{\log}}, \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{C}_{\log}),$$

where $\theta_{\mathbb{L}\otimes\mathcal{O}\mathbb{C}_{\log}} = \theta_{\mathbb{L}} \otimes \mathrm{id}_{\mathcal{O}\mathbb{C}_{\log}} - \mathrm{id}_{\mathbb{L}} \otimes_{\mathbb{Q}_p} \overline{\nabla}$. We want to show that $R^i\nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{C}_{\log}) = 0$ for $i > 0$ and that it is a vector bundle of rank n for $i = 0$. We need the following lemma, which is the Lie algebra incarnation of [15, Lemma 2.15].

Lemma 3.5.3. *The image of* $\mathrm{Sen}_{\widetilde{X}}:\Omega^1_X(\log)^\vee\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X(1)\to\mathfrak{gl}_{n,\text{két}}\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_X$ *is contained in a nilpotent subalgebra.*

Proof. Since $\mathbb{L}$ is obtained by the standard representation of $\mathfrak{gl}_n$, it is enough to prove that the action of $\mathrm{Sen}_{\widetilde{X}}$ on $\mathbb{L}\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_X$ is nilpotent. The coefficients of the characteristic polynomial of $\mathrm{Sen}_{\widetilde{X}}$ are given by Galois equivariant maps

$$\sigma_i:\Omega^1_X(\log)^\vee\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X(1)\to\mathrm{End}_{\widehat{\mathcal{O}}_X}\Big({\bigwedge}^i\mathbb{L}\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_X\Big)\xrightarrow{\mathrm{Tr}}\widehat{\mathcal{O}}_X.$$

Since we are working over a discrete valuation field, Tate's theorem on the vanishing of Galois cohomology [32, Proposition 8] forces $\sigma_i=0$ for all $i=1,\dots,n$, proving that $\mathrm{Sen}_{\widetilde{X}}$ is nilpotent. ∎

We have proven that the Sen operator $\theta_{\mathbb{L}}$ of $\mathbb{L}\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_X$ is nilpotent. We claim that $\mathcal{H}(\mathbb{L}):=\nu_*(\mathbb{L}\otimes_{\mathbb{Q}_p}\mathcal{O}\mathbb{C}_{\log})$ is a vector bundle of rank n and that the natural map

$$\mathcal{H}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{C}_{\log}\xrightarrow{\sim}\mathbb{L}\otimes_{\mathbb{Q}_p}\mathcal{O}\mathbb{C}_{\log}$$

is an isomorphism. If this holds, then

$$R\nu_*(\mathbb{L}\otimes_{\mathbb{Q}_p}\mathcal{O}\mathbb{C}_{\log})=R\nu_*(\mathcal{H}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{C}_{\log})=\mathcal{H}(\mathbb{L})\otimes_{\mathcal{O}_X}R\nu_*\mathcal{O}\mathbb{C}_{\log}=\mathcal{H}(\mathbb{L}),$$

where the vanishing of $R\nu_*\mathcal{O}\mathbb{C}_{\log}$ follows from the computation of its Sen operator in Proposition 3.5.2 and Theorem 3.3.2 (2). The quasi-isomorphism (3.17) then follows by taking cohomology for the Higgs field of $\mathcal{O}\mathbb{C}_{\log}$ and $\mathcal{H}(\mathbb{L})$.

We are left to show that $\mathcal{H}(\mathbb{L})$ is an $\mathcal{O}_X$-vector bundle in the Kummer-étale site with the aforementioned properties. This property is local in the Kummer-étale site, so we can assume that X is affinoid and has toric coordinates $\psi:X\to\mathbb{S}^{(e,d-e)}_K$ with coordinates T_i,S_j. Let $X_{\infty,C}$ be the Γ-torsor over X_C obtained by taking p-power roots of T_i and S_j, let $A_\infty=\mathcal{O}(X_{\infty,C})$ be the perfectoid algebra of functions of $X_{\infty,C}$. Write $\mathcal{F}=\mathbb{L}\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_X$.

By shrinking X if necessary, we can assume that $\mathcal{F}(X_{\infty,C})$ is a free module over A_∞ of rank n. Let $v\in\mathcal{F}(X_{\infty,C})$. Let us write $\Gamma\cong\mathbb{Z}_p^d$ with coordinates γ_i, and let θ_i be the Sen operator arising from the derivation along the direction of γ_i. By Lemma 3.5.3 the Sen operators θ_i are nilpotent on $\mathcal{F}$. On the other hand, we have an explicit description as a polynomial algebra

$$\mathcal{O}\mathbb{C}_{\log}(X_{\infty,C})=A_\infty[Z_1,\dots,Z_d]$$

where the Z_i satisfy $\theta_i(Z_j)=\delta_{i,j}$; see the proof of Proposition 3.5.2. Then, in the tensor product $(\mathcal{F}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}_{\log})(X_{\infty,C})\cong\mathcal{F}(X_{\infty,C})[\underline{Z}]$ we can consider the element

$$v^{(1)}=\sum_{k=0}^{n-1}(-1)^k\theta_1^k(v)\frac{Z_1^k}{k!}.$$

One has $\theta_1(v^{(1)})=0$. By repeating this construction for all the operators θ_i, we find an element $\widetilde{v}\in\mathcal{F}(X_{\infty,C})[\underline{Z}]$ with constant term given by v such that $\theta_i(\widetilde{v})=0$ for all i.

By picking a locally analytic basis $\{v_1, \ldots, v_n\}$ of $\mathcal{F}(X_{\infty,C})$, the previous algorithm constructs locally analytic elements $\widetilde{v}_1, \ldots, \widetilde{v}_n \in \mathcal{F}(X_{\infty,C})[\underline{Z}]$ with $\theta_j(\widetilde{v}_i) = 0$ for all i, j. Thus, the elements $\widetilde{v}_1$ are fixed by the action of an open compact subgroup $\Gamma_0 \subset \Gamma$ and, after localizing in the Kummer-étale topology, they give rise to a vector bundle $\mathcal{H}(\mathbb{L})$ over X_C such that

$$\mathcal{H}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{C}_{\log} = \mathcal{F} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_{\log}.$$

Thus, we must have

$$\mathcal{H}(\mathbb{L}) = \nu_*(\mathcal{F} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_{\log}),$$

proving what we wanted. ■

Remark 3.5.4. The previous argument can be extended to any $\widehat{\mathcal{O}}_X$-vector bundle $\mathcal{F}$. Indeed, we can always find locally Kummer-étale on X a lattice $\mathcal{F}^+ \subset \mathcal{F}$ such that $\mathcal{F}^+/p =^{\mathrm{ae}} \bigoplus_{i=1}^n \mathcal{O}_X^+/p$ as pro-Kummer-étale sheaves. By Theorem 3.3.2, the sheaf $\mathcal{F}$ admits a geometric Sen operator $\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}(-1) \otimes \Omega_X^1(\log)$ computing its cohomology. Then, since X is defined over a discretely valued field, the proof of Lemma 3.5.3 shows that $\theta_{\mathcal{F}}$ is actually a nilpotent operator. This would prove in particular that Theorem 3.5.1 holds not just for local systems but for any $\widehat{\mathcal{O}}_X$-vector bundle when X is defined over a discretely valued field. We thank Ben Heuer for pointing this out.

Finally, let us mention the relation to the work of Wang [33]. Let X be a rigid analytic space over $\mathbb{C}_p$ admitting a liftable good reduction $\mathfrak{X}$ over $\mathcal{O}_{\mathbb{C}_p}$ (this means that $\mathfrak{X}$ admits a lifting over $A_{\inf}/\xi^2$ where $\xi = ([\varepsilon]-1)/([\varepsilon^{1/p}]-1)$). Recall the following theorem.

Theorem 3.5.5 ([33, Theorem 5.3]). *Let $\mathcal{O}\mathbb{C}_{\log}^{\dagger}$ denote the overconvergent Hodge–Tate period sheaf of Wang. Let $a \geq 1/(p-1)$ and $\nu : X_{\mathrm{prok\acute{e}t}} \to X_{\mathrm{k\acute{e}t}}$ be the projection of sites. Then the functor*

$$\mathcal{H}(\mathcal{L}) := \nu_*(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_{\log}^{\dagger})$$

induces an equivalence from the category of a-small generalized representations to the category of a-small Higgs bundles.

We do not pretend to give a new proof of this statement, instead let us translate some of the main players into the language used in this paper. An a-small generalized representation of rank l is a locally free $\widehat{\mathcal{O}}_X$-module $\mathcal{L}$ admitting a lattice $\mathcal{L}^0$ such that there is $b > a + \mathrm{val}(\rho_k)$ with $\mathcal{L}^0/p^b =^{\mathrm{ae}} (\mathcal{O}_X^+/p^b)^l$ (ρ_K being an element in $\mathfrak{m}_{\mathbb{C}_p}$ depending on the ramification of a discretely valued subfield). In particular, this is a relative locally analytic $\widehat{\mathcal{O}}_X$-sheaf as in Definition 3.2.1. Wang constructs the sheaf $\mathcal{O}\mathbb{C}_{\log}^{\dagger}$ by considering a particular lattice of the Faltings extension provided by the lifting of $\mathfrak{X}$ to A_2 (cf. [33, Corollary 2.19]). Locally in coordinates, the ring $\mathcal{O}\mathbb{C}_{\log}^{\dagger}$ is nothing other than the algebra of functions of an overconvergent polydisc of radius $|\rho_k|$ (cf. [33, Theorem 2.27]). The a-smallness condition is a finite rank version of the relative locally analytic condition, where one imposes a fixed radius of analyticity. Finally, the decompletion used by Wang [33, Section 3.1] is the integral version of the decompletion provided by Berger–Colmez axiomatic Sen theory [1].

Acknowledgements. First and foremost, I would like to thank my advisor Vincent Pilloni who guided me throughout the development of this project. I want to thank George Boxer and Joaquín Rodrigues Jacinto for their patience and my enlightenment after visits to their office several times to discuss different stages of the work. I am grateful to the participants of the study group in Lyon in the Spring of 2021 on Pan's work on locally analytic completed cohomology for the modular curve. I thank Laurent Berger for his comments on an earlier version of the Sen theory formalism. I also want to thank Fabrizio Andreatta, Gabriel Dospinescu, Andrew Graham, Valentin Hernandez, Sean Howe, Ben Heuer, Damien Junger and Arthur-César Le Bras for fruitful exchanges. Special thanks to Lue Pan for many conversations that enhanced the content of the document, in particular his suggestion to improve a cohomological computation that leads to Proposition 2.5.5. I want to thank the Max Planck Institute for Mathematics for the excellent mathematical exchanges that helped me to give a better presentation of this paper. Finally, I want to thank the anonymous referee for the many corrections and suggestions that notably improved and simplified the exposition of the document. This work was done during the PhD studies of the author at the ENS de Lyon.

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