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Rigidities of isoperimetric inequality under nonnegative Ricci curvature

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Abstract. The sharp isoperimetric inequality for non-compact Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth has been obtained in increasing generality with different approaches in a number of contributions culminated by Balogh and Kristály (2023) also covering metric-measure spaces satisfying the nonnegative Ricci curvature condition in the synthetic sense of Lott, Sturm and Villani. In sharp contrast with the compact case of positive Ricci curvature, for a large class of spaces including weighted Riemannian manifolds, no complete characterization of the equality cases is present in the literature. The scope of this paper is to settle this problem by proving, in the same generality as Balogh and Kristály (2023), that the equality in the isoperimetric inequality can be attained only by metric balls. Whenever this happens the space is forced, in a measure theoretic sense, to be a cone. Our result applies to different frameworks yielding as corollaries new rigidity results: it extends the rigidity results of Brendle (2023) for weighted Riemannian manifolds and the rigidity results of Antonelli et al. (2023) for general RCD spaces. It also applies to the Euclidean setting by proving that optimizers in the anisotropic and weighted isoperimetric inequality for Euclidean cones are necessarily the Wulff shapes.

Keywords: isoperimetric inequality, metric geometry, CD condition, Ricci curvature, localization, anisotropic perimeter.

1. Introduction

The Levy–Gromov isoperimetric inequality [36, Appendix C] asserts that if E is a (sufficiently regular) subset of a Riemannian manifold (M^n, g) with dimension n and $\mathrm{Ric}_g \geq K > 0$, then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S|}, \tag{1.1}$$

where B is a spherical cap in the model sphere, i.e., the n-dimensional sphere with constant Ricci curvature equal to K, and $|M|$, $|S|$, $|\partial E|$, $|\partial B|$ denote the corresponding n- or

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$(n-1)$-dimensional volumes, and where B is chosen so that $\frac{|E|}{|M|} = \frac{|B|}{|S|}$. If there exists a set $E \subset M$ with smooth boundary attaining the equality in (1.1), then M^n is isometric to the model space, i.e., the n-dimensional sphere of the same Ricci curvature of M^n, and E is a metric ball.

If (M^n, g) is a Riemannian manifold, it is natural to consider more general measures other than the Vol_g. Then the relevant object to control is the N-Ricci tensor introduced in [11]: if $h \in C^2(M)$ with $h > 0$, the generalized N-Ricci tensor, with $N \geq n$, is defined by

$$\mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g - (N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}}.$$

The weighted manifold $(M^n, g, h\,\mathrm{Vol}_g)$ is said to satisfy the Bakry–Émery curvature-dimension condition $\mathsf{CD}(K, N)$ [12] if $\mathrm{Ric}_{g,h,N} \geq Kg$. The $\mathsf{CD}(K, N)$ condition incorporates information on curvature and dimension from both the geometry of (M^n, g) and the measure $h\,\mathrm{Vol}_g$. In its most general form, the sharp (with respect to all parameters) extension of (1.1) to weighted manifolds (with also bounded diameter) verifies the Bakry–Émery $\mathsf{CD}(K, N)$ due to [45]; we refer as well to [45] for the long list of previous contributions that are too many to list them all.

In their seminal works, Lott–Villani [43] and Sturm [50, 51] introduced a synthetic definition of $\mathsf{CD}(K, N)$ for complete and separable metric spaces (X, d) endowed with a (locally-finite Borel) reference measure $\mathfrak{m}$ ("metric-measure space" or m.m.s.). The synthetic $\mathsf{CD}(K, N)$ is formulated in terms of optimal transport (see Section 2 for its definition) and it was shown to coincide with the Bakry–Émery one in the smooth Riemannian setting (and in particular in the classical non-weighted one), that it is stable under measured Gromov–Hausdorff convergence of metric-measure spaces, and that Finsler manifolds and Alexandrov spaces satisfy it.

In [24], the Levy–Gromov isoperimetric inequality has been generalized to the metric-measure spaces satisfying the synthetic $\mathsf{CD}(K, N)$ by showing that the same sharp lower bounds obtained in [45] applies to metric setting. The approach of [24] is based on the localization paradigm, a powerful dimensional reduction tool from convex geometry extended to weighted Riemannian manifolds by means of an L^1 optimal transport approach by Klartag [41], and then obtained for $\mathsf{CD}(K, N)$ spaces in [24].

In [24], in the case $K > 0$, the rigidity of (1.1) has been generalized as well. The equality in the isoperimetric inequality implies that $(X, \mathsf{d}, \mathfrak{m})$ has maximal diameter. If in addition $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{RCD}(0, N)$ condition (see Section 2), then X is isomorphic as m.m.s. to a spherical suspensions [24, Theorem 1.4]. As a consequence, the optimal sets are characterized as well (are metric balls centered on the tips of the spherical suspensions) producing a rather clear picture of the isoperimetric inequality in the setting $K > 0$.

On the other hand, it is well known that without an additional condition on the geometry of the space no isoperimetric inequality holds true in general spaces in the regime of $K = 0$, i.e., those with nonnegative Ricci curvature.

However, the classical Euclidean isoperimetric inequality asserts that any Borel set $E \subset \mathbb{R}^n$ with smooth boundary satisfies

$$|\partial E| \geq n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}}.$$

Thus, a way to reconcile Levy–Gromov with the previous inequality is to impose a growth condition on the space to match the Euclidean one. Letting

$$B_r(x) = \{y \in X : \mathsf{d}(x, y) < r\}$$

denoting the metric ball with center $x \in X$ and radius $r > 0$, by Bishop–Gromov volume growth inequality, see [50, Theorem 2.3], the map $r \mapsto \frac{\mathfrak{m}(B_r(x))}{r^N}$ is non-increasing over $(0, \infty)$ for any $x \in X$. The *asymptotic volume ratio* is then naturally defined by

$$\mathsf{AVR}_{(X,\mathsf{d},\mathfrak{m})} = \lim_{r\to\infty} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}.$$

It is easy to see that it is indeed independent of the choice of $x \in X$; the constant ω_N is the volume of the Euclidean unit ball in $\mathbb{R}^N$ whenever $N \in \mathbb{N}$, and it is classically extended to real values of N via the gamma function. When $\mathsf{AVR}_{(X,\mathsf{d},\mathfrak{m})} > 0$, we say that $(X, \mathsf{d}, \mathfrak{m})$ has *Euclidean volume growth*. Whenever no ambiguity is possible, we will prefer the shorter notation AVR_X. In particular, if (M, g) is a non-compact, complete n-dimensional Riemannian manifold having nonnegative Ricci curvature, the asymptotic volume ratio of (M, g) is given by $\mathsf{AVR}_g := \mathsf{AVR}_{(M,\mathsf{d}_g,\mathrm{Vol}_g)}$. By the Bishop–Gromov theorem, one has that $\mathsf{AVR}_g \leq 1$ with $\mathsf{AVR}_g = 1$ if and only if (M, g) is isometric to the usual Euclidean space $\mathbb{R}^n$ endowed with the Euclidean metric g_0.

The sharp isoperimetric inequality for Riemannian manifolds with Euclidean volume growth has been obtained in increasing generality with different approaches in a number of contributions [1, 15, 35, 39]. The most general version (including as subclasses the previous contributions) is the one valid for m.m.s.'s satisfying the $\mathsf{CD}(0, N)$ condition; it has been obtained by Balogh and Kristály in [13] and follows from a refined application of the Brunn–Minkowski inequality given by optimal transport.

Theorem 1.1 ([13, Theorem 1.1]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an m.m.s. satisfying the* $\mathsf{CD}(0, N)$ *condition for some* $N > 1$*, and having Euclidean volume growth. Then for every bounded Borel subset* $E \subset X$ *it holds*

$$\mathfrak{m}^+(E) \geq N\omega_N^{\frac{1}{N}} \mathsf{AVR}_X^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{1.2}$$

Moreover, inequality (1.2) *is sharp.*

More challenging to prove are the rigidity properties of (1.2). So far it has been obtained only under special assumptions on the space without matching the generality of Theorem 1.1.

To the best of our knowledge, the following two are the most general results in the literature. The first one is for the smooth setting and is due to Brendle [15].

Theorem 1.2 ([15, Theorem 1.2]). *Inequality* (1.2) *is valid for any non-compact, complete n-dimensional Riemannian manifold (M, g) with nonnegative Ricci curvature and Euclidean volume growth. The equality holds in* (1.2) *for some $E \subset M$ with C^1 smooth regular boundary and smooth manifold M if and only if $\mathsf{AVR}_g = 1$ and E is isometric to a ball $B \subset \mathbb{R}^n$.*

Antonelli, Pasqualetto, Pozzetta and Semola [9] generalize [15, Theorem 1.2] to the non-smooth setting by considering $\mathsf{RCD}(0, N)$ spaces and removes the regularity assumptions on the boundary of E.

Theorem 1.3 ([9, Theorem 1.3]). *Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathsf{RCD}(0, N)$ m.m.s. having Euclidean volume growth. Then equality* (1.2) *holds for some $E \subset X$ with $\mathcal{H}^N(E) \in (0, \infty)$ if and only if X is isometric to a Euclidean metric-measure cone over an $\mathsf{RCD}(N-2, N-1)$ space and E is isometric to a ball centered at one of the tips of X.*

Both theorems deal with the unweighted case, i.e., $\mathfrak{m} = \mathrm{Vol}_g$ and $\mathfrak{m} = \mathcal{H}^N$, respectively.

The scope of the present paper is to improve the generality of these rigidity results. We will be able to characterize all the sets attaining identity (1.2) within the same generality of Theorem 1.1; the rigidity of the space will follow as well.

1.1. The result

The following is the main result of the paper.

Theorem 1.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. satisfying the $\mathsf{CD}(0, N)$ condition for some $N > 1$, and having Euclidean volume growth. Let $E \subset X$ be a bounded Borel set that saturates* (1.2). *Then there exists (a unique) $o \in X$ such that, up to a negligible set, $E = B_\rho(o)$, with $\rho = (\frac{\mathfrak{m}(E)}{\mathsf{AVR}_X \omega_N})^{\frac{1}{N}}$. Moreover, considering the disintegration of $\mathfrak{m}$ with respect to $\mathsf{d}(\cdot, o)$, the measure $\mathfrak{m}$ has the following representation:*

$$\mathfrak{m} = \int_{\partial B_\rho(o)} \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q} \in \mathcal{P}(\partial B_\rho(o)), \; \mathfrak{m}_\alpha \in \mathcal{M}_+(X), \tag{1.3}$$

where $\mathfrak{m}_\alpha$ is concentrated on the geodesic ray from o through α and identified (via the unitary speed parametrization of the ray) with $N\omega_N \mathsf{AVR}_X t^{N-1} \mathcal{L}^1 \llcorner_{[0,\infty)}$.

The essentially non-branching assumption in Theorem 1.4 is necessary to prevent pathological situation within the synthetic CD theory (see, for instance, the local-to-global property [22]) and it is verified both by the class of weighted manifolds and the $\mathsf{RCD}(0, N)$ spaces. The uniqueness of the point o has to be meant in the sense that ball E has a unique center, i.e., if $B_\rho(o) = E = B_\rho(o')$, then $o = o'$. The minimizer can be non-unique: for instance, in the $\mathsf{CD}(0, 2)$ space $(\mathbb{R} \times [0, \infty), |\cdot|, \mathcal{L}^2)$, all balls centered in $(x, 0)$, $x \in \mathbb{R}$ are isoperimetric sets.

It also has to be noticed that it is usually rare to obtain rigidities in our generality where the failure of the linearity of Laplace operator prevents the use of several known rigidities results. Nonetheless the isoperimetric inequality seems to make an exception (see also [20] where optimal sets have been obtained in the same generality in the regime $K > 0$ by a quantitative analysis).

In the more regular setting of $\mathsf{RCD}(0, N)$ spaces one can invoke [27, Theorem 1.1], the so-called "volume cone implies metric cone", to improve the measure rigidity of Theorem 1.4 valid in the $\mathsf{CD}(0, N)$ setting to the stronger metric rigidity.

Theorem 1.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an m.m.s. satisfying the $\mathsf{RCD}(0, N)$ condition for some $N > 1$, and having Euclidean volume growth. Then equality (1.2) holds for some bounded set $E \subset X$ if and only if X is isometric to an Euclidean metric-measure cone over an $\mathsf{RCD}(N-2, N-1)$ space and E is isometric to the ball centered at one of the tips of X.*

Thus, Theorem 1.5 recovers and extends both Theorems 1.2 and 1.3 by allowing more general measures (other than the volume measure or the Hausdorff measure) and spaces with not necessarily infinitesimally linear structures. In the case $\mathfrak{m} = \mathcal{H}^N$, the hypothesis on the boundedness of E can be dropped. Indeed, it was proven [8, Theorem 1.3] that minimizers of the perimeter are bounded: apply the cited theorem to our setting with $G = 0$; the Bishop–Gromov inequality ensures $\mathcal{H}^N(B_1(x)) \geq \omega_N \mathsf{AVR}_X > 0$.

Remark 1.6. After this paper was submitted for publication, it has been proved [10] that the minimizers of the perimeter are bounded for $\mathsf{RCD}(0, N)$ spaces whose reference measure is possibly not the Hausdorff measure $\mathcal{H}^N$. Therefore, the boundedness hypothesis in Theorem 1.5 can be dropped.

Theorem 1.4 finds applications and covers new cases also in the Euclidean setting. We postpone this discussion to the final part of the introduction and we now proceed presenting the proof strategy and the structure of the paper.

The classical approach to rigidity results goes by inspecting known proofs of inequalities to extract extra information whenever the equality happens. Our approach goes indeed in this direction by starting from the proof of the isoperimetric inequality for non-compact MCP spaces obtained in [21]. In [21], the argument uses the localization given by the optimal transport problem between the given set E and its complement inside a large ball of radius R containing E. This produces a disjoint family of one-dimensional transport rays and a corresponding disintegration of the reference measure $\mathfrak{m}$ restricted to the metric ball B_R. Then one can apply the one-dimensional weighted Levy–Gromov isoperimetric inequality to the traces of E along the transport rays and conclude the proof of (1.2) by taking $R \to \infty$. For the reader's convenience, we have included this proof also here, see Theorem 4.3.

In order to capture the equality case following this proof, it is therefore necessary to deal with this limit procedure. The intuition suggests that whenever a region E attains the equality in (1.2), then for large values of R the one-dimensional traces have to be almost optimal. The almost optimality has to be intended in many respects: along each geodesic ray, the diameter has to be almost optimal, the one-dimensional conditional measures have to be almost t^{N-1} and the set has to be almost an interval starting from the starting point of the ray. The main difficulty here is to perform a quantitative analysis of the right order that will not vanish when $R \to \infty$. This is done in Section 5 that culminates with Theorem 5.11, where we summarize the crucial stability estimates for the one-dimensional densities and the geometry of the traces of E.

Then the natural prosecution is to take the limit as $R \to \infty$ and hopefully obtain a disintegration of $\mathfrak{m}$ on the whole space having conditional measures verifying the limit estimates. Disintegration formulas are however typically hard to treat under a limit procedure. For instance, the maximality of the transport rays is likely not preserved preventing any chances to get limit estimates. Nonetheless, the almost maximality of E and all the almost optimal information deduced from it in Section 5 permit to bypass this intricate issue and obtain a well-behaved limit disintegration. The limit is analyzed in Section 6 and summarized by Corollary 6.14.

The final part of Section 6 is then dedicated to the proof that the optimal set E is a ball and that the disintegration formula (1.3) holds true (see Theorems 7.5 and 7.12).

1.2. Applications in the Euclidean setting

Theorem 1.4 implies new results also in the Euclidean setting, namely the characterization of optimal regions for the anisotropic isoperimetric inequality for weighted cones.

The setting is the following: let $\Sigma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin, and let $H : \mathbb{R}^n \to [0, \infty)$ be a *gauge*, that is a nonnegative, convex and positively homogeneous function of degree one. Moreover, w is weight that is supposed to be continuous on $\bar{\Sigma}$ and positive and locally Lipschitz in Σ.

For a smooth set $E \subset \mathbb{R}^n$, the *weighted anisotropic perimeter* relative to the cone Σ is given by

$$\mathsf{P}_{w,H}(E; \Sigma) = \int_{\partial E} H(\nu(x)) w(x)\, dS,$$

where $\nu(x)$ is the unit outward normal at $x \in \partial E$, and dS is the surface measure. The main result of [16] is the sharp isoperimetric inequality for the weighted anisotropic perimeter: if in addition w is positively homogeneous of degree $\alpha > 0$ and $w^{\frac{1}{\alpha}}$ is concave in Σ, then

$$\frac{\mathsf{P}_{w,H}(E; \Sigma)}{w(E \cap \Sigma)^{\frac{N-1}{N}}} \geq \frac{\mathsf{P}_{w,H}(W; \Sigma)}{w(W \cap \Sigma)^{\frac{N-1}{N}}}, \tag{1.4}$$

where $N = n + \alpha$ and W is the Wulff shape associated to H, for the details see [16, Theorem 1.3]. The expression $w(A)$ with $A \subset \mathbb{R}^n$ is a shorthand notation for the integral of w over A in dx.

Inequality (1.4), taking $w = 1$, $\Sigma = \mathbb{R}^n$, and $H = \|\cdot\|_2$, recovers the classical sharp isoperimetric inequality. Taking $w = 1$ and $H = \|\cdot\|_2$, (1.4) gives back the isoperimetric inequality in convex cones originally obtained by Lions and Pacella [42]. Finally, if $w = 1$, $\Sigma = \mathbb{R}^n$ and H is some other gauge, then (1.4) is the Wulff inequality.

As observed in [16], Wulff balls W centered at the origin intersected with Σ are always minimizers (1.4). However, in [16] a characterization of the equality case (or a proof of uniqueness of those minimizers), is not carried over (see also [38] for a different approach). Despite the many recent contributions (and an announcement in [16] of the forthcoming work solving the problem), this characterization, in its full generality, seems to be still not present in the literature.

We now briefly recall the known results. The characterization of the optimal sets has been obtained in the unweighted and isotropic case ($w = 1$ and $H = \|\cdot\|_2$) for smooth cones in [42] and for general cones in [34] via a quantitative analysis. The same approach of [34] has been recently used in [28] to characterize optimal sets in the unweighted and anisotropic case with the gauge H assumed additionally to be a norm with strictly convex unitary ball. Finally, [26] extended [28] to the case of H being a positive gauge (i.e., a not necessarily reversible norm) still uniformly elliptic.

The characterization in weighted setting has been solved in [25], in the isotropic case ($H = \|\cdot\|_2$); in [46], the anisotropic case is treated, but the minimizer is assumed to be convex.

It has been already observed in [16] that the assumption that $w^{\frac{1}{\alpha}}$ is concave has a natural interpretation as the $\mathsf{CD}(0, N)$ condition, where $N = n + \alpha$, and reported as well in [13] that (1.4) can be obtained as a particular case of (1.2) when H is a norm. To be precise, if H is a norm, then its dual function H_0 is a norm as well, and one can associate to the triple Σ, H and w the metric-measure space $(\Sigma, \mathsf{d}_{H_0}, w\mathscr{L}^n)$, where $\mathsf{d}_{H_0}(x, y) := H_0(x - y)$. The perimeter associated to these metric-measure spaces (see Section 2.2) will indeed coincide with $\mathsf{P}_{w,H}$.

Moreover, it is well known that for any norm $\|\cdot\|$, the associated metric-measure space $(\mathbb{R}^n, \|\cdot\|, \mathscr{L}^n)$ satisfies the synthetic $\mathsf{CD}(0, n)$, see, for instance, [53]. From [31, Proposition 3.3], one deduces that $(\mathbb{R}^n, \|\cdot\|, w\mathscr{L}^n)$ satisfies $\mathsf{CD}(0, n + \alpha)$, provided $w^{\frac{1}{\alpha}}$ is concave. Moreover, by the homogeneity properties of H and w, one can check that

$$\mathsf{AVR}_{(\Sigma, \mathsf{d}_{H_0}, w\mathscr{L}^n)} = \lim_{R\to\infty} \frac{\int_{B_{d_{H_0}}(R)\cap\Sigma} w\, d\mathscr{L}^n}{\omega_N R^N} = \frac{\int_{B_{d_{H_0}}(1)\cap\Sigma} w\, d\mathscr{L}^n}{\omega_N} > 0.$$

Indeed, recall that the Wulff shape W of H is the unitary ball of the dual norm H_0, hence the measure scales with power $N = n + \alpha$. Conversely, the perimeter of the rescaled Wulff shape is the derivative with respect to the scaling factor of the measure, hence the perimeter of the Wulff shape is N times its measure, thus (1.4) can be seen as a particular case of (1.2).

We can therefore apply Theorem 1.4 to $(\Sigma, \mathsf{d}_{H_0}, w\mathscr{L}^n)$: the differentiability of H implies that the unitary ball of H_0 is strictly convex, and therefore the distance d_{H_0} is non-branching.

Theorem 1.7. *Let $\Sigma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin, and let $H_0 \colon \mathbb{R}^n \to [0, \infty)$ be a norm with strictly convex unitary ball. Consider moreover the α-homogeneous weight $w \colon \overline{\Sigma} \to [0, \infty)$ such that $w^{\frac{1}{\alpha}}$ is concave. Then the equality in (1.4) is attained if and only if $E = W \cap \Sigma$, where W is a rescaled Wulff shape.*

To conclude, we stress that assumption on w being α-homogeneous can actually be removed and obtained as a consequence of the measure rigidity part of Theorem 1.4 if we consider the modified version of (1.4) with the asymptotic volume ratio, i.e., we assume the right-hand side of (1.4) to be equal to $(\omega_{n+\alpha}\mathsf{AVR}_{(\Sigma, \mathsf{d}_{H_0}, w\mathscr{L}^n)})^{\frac{1}{n+\alpha}} > 0$.

In this case, Theorem 1.4 applies (the strict convexity of H and the concavity of $w^{\frac{1}{\alpha}}$ imply the non-branching hypothesis and the $\mathsf{CD}(0,N)$ condition, respectively).

The first part of Theorem 1.4 says that the isoperimetric set is a ball in the dual norm of H, i.e., it is a rescaled Wulff shape.

The second part of Theorem 1.4, regarding the disintegration of the measures along the rays, provides an integration formula in polar coordinates, where the Jacobian determinant grows with exponent $N-1=n+\alpha-1$. Since the density of Lebesgue measure in polar coordinates is $(n-1)$-homogeneous, we deduce that w is α-homogeneous.

2. Preliminaries

In this section, we recall the main constructions needed in the paper. The reader familiar with curvature-dimension conditions and metric-measure spaces will just need to check Sections 2.3 and 2.4 for the decomposition of X into transport rays (localization) which is going to be used throughout the paper. In Section 2.1, we review geodesics in the Wasserstein distance and the curvature-dimension conditions; in Section 2.2, we recall the perimeter and BV functions in the metric setting.

2.1. Wasserstein distance and the curvature-dimension condition

A metric-measure space (m.m.s.) $(X,\mathsf{d},\mathfrak{m})$ is a triple with (X,d) being a complete and separable metric space and $\mathfrak{m}$ being a Borel nonnegative measure over X. By $\mathcal{M}^+(X)$, $\mathcal{P}(X)$, and $\mathcal{P}_2(X)$ we denote the space of nonnegative Borel measures on X, the space of probability measures, and the space of probability measures with finite second moment, respectively. On the space $\mathcal{P}_2(X)$, we define the L^2-Wasserstein distance W_2, by setting, for $\mu_0,\mu_1\in\mathcal{P}_2(X)$,

$$W_2(\mu_0,\mu_1)^2=\inf_{\pi}\int_{X\times X}\mathsf{d}^2(x,y)\,\pi(dxdy), \tag{2.1}$$

making $(\mathcal{P}_2(X),W_2)$ complete. In W_2, the infimum is taken over all $\pi\in\mathcal{P}(X\times X)$ with μ_0 and μ_1 as the first and the second marginals, i.e., $(P_1)_\sharp\pi=\mu_0$, $(P_2)_\sharp\pi=\mu_1$. Of course, P_1 (resp. P_2) is the projection on the first (resp. second) factor, and $(P_i)_\sharp$ denotes the corresponding push-forward map on measures.

Denote the space of geodesics on (X,d) by

$$\mathrm{Geo}(X):=\{\gamma\in C([0,1],X):\mathsf{d}(\gamma_s,\gamma_t)=|s-t|\mathsf{d}(\gamma_0,\gamma_1)\text{ for every } s,t\in[0,1]\}.$$

Any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}_2(X),W_2)$ can be lifted to a measure $\nu\in\mathcal{P}(\mathrm{Geo}(X))$, so that $(e_t)_\sharp\nu=\mu_t$ for all $t\in[0,1]$, where, for each $t\in[0,1]$, e_t is the evaluation map,

$$e_t\colon \mathrm{Geo}(X)\to X,\quad e_t(\gamma):=\gamma_t.$$

Given $\mu_0,\mu_1\in\mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0,\mu_1)$ the space of all $\nu\in\mathcal{P}(\mathrm{Geo}(X))$ for which $(e_0\otimes e_1)_\sharp\nu$ realizes the minimum in (2.1). If (X,d) is geodesic, then the set $\mathrm{OptGeo}(\mu_0,\mu_1)$ is non-empty for any $\mu_0,\mu_1\in\mathcal{P}_2(X)$.

A set $F \subset \mathrm{Geo}(X)$ is a set of non-branching geodesics if and only if for any $\gamma^1, \gamma^2 \in F$, it holds

$$\exists\, \bar{t} \in (0,1) \text{ such that } \forall t \in [0,\bar{t}]\ \gamma_t^1 = \gamma_t^2 \ \Rightarrow\ \gamma_s^1 = \gamma_s^2 \ \ \forall s \in [0,1].$$

With this terminology, we recall from [49] the following definition.

Definition 2.1. A metric-measure space $(X, \mathsf{d}, \mathfrak{m})$ is *essentially non-branching* if and only if for any $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, with μ_0, μ_1 absolutely continuous with respect to $\mathfrak{m}$, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

The $\mathsf{CD}(K, N)$ condition for m.m.s.'s has been introduced in the seminal works of Sturm [50, 51] and Lott–Villani [43]; here we briefly recall only the basics in the case $K = 0$, $1 < N < \infty$ (the setting of the present paper) and its form under the additional assumptions on the space to be essentially non-branching. For the general definition of $\mathsf{CD}(K, N)$, see [43, 50, 51]. The equivalence between the two formulations follows from [23] (see also [50, Proposition 4.2]).

Definition 2.2 ($\mathsf{CD}(0, N)$ for essentially non-branching spaces). An essentially non-branching m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}(0, N)$ if and only if for all $\mu_0, \mu_1 \in \mathscr{P}_2(X, \mathsf{d}, \mathfrak{m})$, there exists a unique $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ induced by a map (i.e., $\nu = S_\sharp(\mu_0)$ for some map $S\colon X \to \mathrm{Geo}(X)$), $\mu_t := (e_t)_\# \nu \ll \mathfrak{m}$ for all $t \in [0, 1]$, and writing $\mu_t = \rho_t \mathfrak{m}$, we have for all $t \in [0, 1]$,

$$\rho_t^{-\frac{1}{N}}(\gamma_t) \geq (1-t)\, \rho_0^{-\frac{1}{N}}(\gamma_0) + t\, \rho_1^{-\frac{1}{N}}(\gamma_1) \quad \text{for } \nu\text{-a.e. } \gamma \in \mathrm{Geo}(X).$$

If $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}(0, N)$ condition, then the same is valid for $(\mathrm{supp}\,\mathfrak{m}, \mathsf{d}, \mathfrak{m})$; hence we directly assume $X = \mathrm{supp}\,\mathfrak{m}$.

If (M, g) is a Riemannian manifold of dimension n and $h \in C^2(M)$, with $h > 0$, then the m.m.s. $(M, \mathsf{d}_g, h\,\mathrm{Vol}_g)$ satisfies $\mathsf{CD}(0, N)$ with $N \geq n$ if and only if (see [50, Theorem 1.7])

$$\mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g - (N-n) \frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}} \geq 0,$$

in other words, if and only if the weighted Riemannian manifold $(M, g, h\,\mathrm{Vol}_g)$ has nonnegative generalized N-Ricci tensor. If $N = n$, the generalized N-Ricci tensor $\mathrm{Ric}_{g,h,N} = \mathrm{Ric}_g$ requires h to be constant.

Unless otherwise stated, we shall always assume that the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is essentially non-branching and satisfies $\mathsf{CD}(0, N)$ for some $N > 2$ with $\mathrm{supp}(\mathfrak{m}) = X$. This implies directly that (X, d) is a geodesic, complete, and locally compact metric space.

2.2. *Perimeter and BV functions in metric-measure spaces*

Given $u \in \mathrm{Lip}(X)$, the space of real-valued Lipschitz functions over X, its slope $|Du|(x)$ at $x \in X$ is defined by

$$|Du|(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{\mathsf{d}(x, y)}.$$

Following [2,3,47] and the more recent [5], given a Borel subset $E \subset X$ and Ω open, the perimeter of E relative to Ω is denoted by $\mathsf{P}(E;\Omega)$ and is defined as follows:

$$\mathsf{P}(E;\Omega) := \inf\Big\{\liminf_{n\to\infty} \int_\Omega |Du_n|\, d\mathfrak{m} : u_n \in \mathrm{Lip}(\Omega),\ u_n \to \mathbf{1}_E \text{ in } L^1(\Omega,\mathfrak{m})\Big\}.$$

We say that $E \subset X$ has finite perimeter in X if $\mathsf{P}(E;X) < \infty$. We recall also a few properties of the perimeter functions:

(a) $\mathsf{P}(E;\Omega) = \mathsf{P}(F;\Omega)$, whenever $\mathfrak{m}((E\Delta F)\cap\Omega) = 0$ (locality);

(b) the map $E \mapsto \mathsf{P}(E;\Omega)$ is lower-semicontinuous (l.s.c.) with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence;

(c) $\mathsf{P}(E;\Omega) = \mathsf{P}(X\setminus E;\Omega)$ (complementation).

Moreover, if E is a set of finite perimeter, then the set function $\Omega \to \mathsf{P}(E;A)$ is the restriction to open sets of a finite Borel measure $\mathsf{P}(E;\cdot)$ in X (see [5, Lemma 5.2]), defined by

$$\mathsf{P}(E;A) := \inf\{\mathsf{P}(E;\Omega) : \Omega \supset A,\ \Omega \text{ open}\}.$$

In order to simplify the notation, we will write $\mathsf{P}(E)$ instead of $\mathsf{P}(E;X)$. Finally, we recall that the perimeter can be seen [6] as the l.s.c. envelope of the Minkowski content

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon\to 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon},$$

where $E^\varepsilon = \{x \in X : \mathrm{dist}(x,E) < \varepsilon\}$.

The *isoperimetric profile function* of $(X,\mathsf{d},\mathfrak{m})$, denoted by $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}$, is defined as the pointwise maximal function such that $\mathsf{P}(A) \geq \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mathfrak{m}(A))$ for every Borel set $A \subset X$, that is,

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) := \inf\{\mathsf{P}(A) : A \subset X \text{ Borel},\ \mathfrak{m}(A) = v\}.$$

Milman [45] gave an explicit isoperimetric profile $\mathcal{I}_{K,N,D}$ function such that if a Riemannian manifold (M,g) with smooth density h has diameter at most $D > 0$, generalized N-Ricci tensor $\mathrm{Ric}_{g,h,N} \geq K \in \mathbb{R}$, then the isoperimetric profile function of $(M,\mathsf{d}_g, h\,\mathrm{Vol}_g)$ is bounded below by $\mathcal{I}_{K,N,D}$. In this paper, we will make extensive use of $\mathcal{I}_{K,N,D}$ in the case $K = 0$. In this case, the isoperimetric profile computed by Milman [45, Corollary 1.4, Case 4] is indeed given by the formula

$$\mathcal{I}_{0,N,D}(v) := \frac{N}{D}\inf_{\xi\geq 0} \frac{\big(v\wedge(1-v)(\xi+1)^N + v\vee(1-v)\xi^N\big)^{\frac{N-1}{N}}}{(\xi+1)^N - \xi^N},$$

and it is obtained by optimizing among a family of one-dimensional spaces; we will expand this analysis in Section 4. In order to keep the notation short, we will write $\mathcal{I}_{N,D}$ in place of $\mathcal{I}_{0,N,D}$.

The classical theory of the perimeter in $\mathbb{R}^n$ makes extensive use of BV function. The notion of BV functions in the setting of m.m.s. has been first introduced in [47] and then more deeply studied in [5]. In particular, three different definitions of BV functions has been proven to be equivalent.

One of these three notions is given by relaxation of the energy functional. We say that a function $f \in L^1(X)$ is in $BV_*((X, \mathsf{d}, \mathfrak{m}))$ if there exists a sequence $f_n \in \mathrm{Lip}(X) \cap L^1(X)$ converging to f in L^1 such that $\sup_n \int_X |\nabla f_n| \, d\mathfrak{m} < \infty$. In this case, one can define the relaxed total variation

$$|Df|_*(\Omega) := \inf\left\{\liminf_{n\to\infty} \int_X |Df_n| \, d\mathfrak{m} : f_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ f_n \to f \text{ in } L^1(\Omega)\right\},$$

where $\Omega \subset X$ is an open set. It has been shown in [47] that the total variation extends uniquely to a finite Borel measure.

Another definition of BV functions is given using test plans. We say that a probability measure $\pi \in \mathscr{P}(C([0,1]; X))$ is an ∞-test plan if

(1) π is concentrated on Lipschitz-continuous curves;

(2) there exists a constant $C = C(\pi) > 0$ (named *compression* of the test plan) such that $(e_t)_\#\pi \le C\mathfrak{m}$.

A Borel subset $\Gamma \subset C([0,1]; X)$ is said to be 1-negligible if $\pi(\Gamma) = 0$, for every ∞-test plan π. We say that a function $f \in L^1(X)$ is of weak-bounded variation ($f \in$ w-$BV((X, \mathsf{d}, \mathfrak{m}))$) if the following two conditions hold:

(1) there exists a 1-negligible subset Γ such that $f \circ \gamma \in BV((0,1))$ $\forall \gamma \in C([0,1]; X) \setminus \Gamma$, and

$$|f(\gamma_0) - f(\gamma_1)| \le |D(f \circ \gamma)|((0,1));$$

(2) there exists a measure $\mu \in \mathscr{M}^+(X)$ such that for every ∞-test plan π, for every Borel set $B \subset X$, we have that

$$\int \gamma_\# |D(f \circ \gamma)|(B) \, \pi(d\gamma) \le C(\pi) \| \sup_{t\in[0,1]} |\dot\gamma_t| \|_{L^\infty(\pi)} \mu(B). \tag{2.2}$$

Moreover, one can prove that there exists the least measure satisfying (2.2). Such measure is named weak-total variation and it is denoted by $|Df|_w$.

Theorem 2.3 ([5, Theorem 1.1]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric-measure space, with $\mathfrak{m}$ a locally finite Borel measure (i.e., for all $x \in X$, there exists $r > 0$ such that $\mathfrak{m}(B_r(x)) < \infty$). Then the spaces $BV_*((X, \mathsf{d}, \mathfrak{m}))$ and w-$BV((X, \mathsf{d}, \mathfrak{m}))$ coincide and for every function*

$$f \in BV_*((X, \mathsf{d}, \mathfrak{m})) = w\text{-}BV((X, \mathsf{d}, \mathfrak{m})),$$

it holds

$$|Df|_*(B) = |Df|_w(B) \quad \textit{for every Borel set } B.$$

It is clear that a set $E \subset X$ has finite perimeter whenever $\mathbf{1}_E \in BV_*((X, \mathsf{d}, \mathfrak{m}))$, and in this case it holds

$$\mathsf{P}(E; A) = |D\mathbf{1}_E|_*(\Omega) = |D\mathbf{1}_E|_w(\Omega) \quad \forall \Omega \subset X \text{ open}.$$

2.3. Localization

The localization method reduces the task of establishing various analytic and geometric inequalities on a full-dimensional space to the one-dimensional setting.

In the Euclidean setting, it goes back to Payne and Weinberger [48], and it has been developed and popularized by Gromov–Milman [37], Lovász–Simonovits [44], and Kannan–Lovász–Simonovits [40]. In 2015, Klartag [41] reinterpreted the localization method as a measure disintegration adapted to L^1-optimal transport, and extended it to weighted Riemannian manifolds satisfying $\mathsf{CD}(K,N)$. The first author and Mondino [24] have succeeded to generalize this technique to essentially non-branching m.m.s.'s satisfying the $\mathsf{CD}(K,N)$ condition with $N \in (1,\infty)$. Here we only report the case $K = 0$.

Theorem 2.4 (Localization on $\mathsf{CD}(0,N)$ spaces [24, Teorem 3.28]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. with* $\mathrm{supp}(\mathfrak{m}) = X$ *satisfying* $\mathsf{CD}(0,N)$ *for some $N \in (1,\infty)$. Let $f : X \to R$ be $\mathfrak{m}$-integrable with $\int_X f\,\mathfrak{m} = 0$ and*

$$\int_X |f(x)|\,\mathsf{d}(x,x_0)\mathfrak{m}(dx) < \infty$$

for some (hence for all) $x_0 \in X$. Then there exist an $\mathfrak{m}$-measurable subset $\mathcal{T} \subset X$ (named transport set) and a family $\{X_\alpha\}_{\alpha\in Q}$ of subsets of X, such that there exists a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}}$ on $\{X_\alpha\}_{\alpha\in Q}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha),$$

and for $\mathfrak{q}$-a.e. $\alpha \in Q$,

(1) *X_α is a closed geodesic in (X, d).*

(2) *$\mathfrak{m}_\alpha$ is a Radon measure supported on X_α with $\mathfrak{m}_\alpha \ll \mathcal{H}^1\llcorner_{X_\alpha}$.*

(3) *$(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies* $\mathsf{CD}(0,N)$.

(4) *$\int f\,d\mathfrak{m}_\alpha = 0$, and $f = 0$ $\mathfrak{m}$-a.e. on $X \setminus \mathcal{T}$.*

Moreover, the X_α are called transport rays, and two distinct transport rays can only meet at their extremal points (having measure zero for $\mathfrak{m}_\alpha$).

A few comments are in order.

By $\mathcal{H}^1$ we denote the one-dimensional Hausdorff measure on the underlying metric space.

Given $\{X_\alpha\}_{\alpha\in Q}$ a partition of X, a disintegration of $\mathfrak{m}$ on $\{X_\alpha\}_{\alpha\in Q}$ is a measure space structure $(Q, \mathcal{Q}, \mathfrak{q})$ and a map

$$Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}(X, \mathcal{X})$$

such that

(1) For $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on X_α.

(2) For all $B \in \mathcal{X}$, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable.

(3) For all $B \in \mathcal{X}$, $\mathfrak{m}(B) = \int_Q \mathfrak{m}_\alpha(B)\mathfrak{q}(d\alpha)$; this is abbreviated by $\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha)$.

We point out that the disintegration is unique for fixed $\mathfrak{q}$. That means that if there is a family $(\widetilde{\mathfrak{m}}_\alpha)_\alpha$ satisfying the conditions above, then for $\mathfrak{q}$-a.e. α, $\mathfrak{m}_\alpha = \widetilde{\mathfrak{m}}_\alpha$. If we replace $\mathfrak{q}$ by a different measure $\hat{\mathfrak{q}}$ such that $\hat{\mathfrak{q}} = \rho\mathfrak{q}$, then the map $\alpha \mapsto \rho(\alpha)\mathfrak{m}_\alpha$ still satisfies the conditions above, with $\hat{\mathfrak{q}}$ in place of $\mathfrak{q}$.

Concerning the fact that $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(0, N)$, since (X_α, d) is a geodesic, it is isometric to a real interval, and therefore the $\mathsf{CD}(0, N)$ condition is equivalent to having $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ and $h_\alpha^{\frac{1}{N-1}}$ being concave (here we identify X_α with a real interval).

2.4. L^1-optimal transportation

In this section, we recall only some facts from the theory of L^1-optimal transportation which are of some interest for this paper; we refer to [4, 7, 14, 19, 22, 32, 33, 41, 52] and references therein for more details on the theory of L^1-optimal transportation.

Theorem 2.4 has been proven by studying the optimal transportation problem between $\mu_0 := f^+\mathfrak{m}$ and $\mu_1 := f^-\mathfrak{m}$, where $f^\pm$ denote the positive and the negative part of f, with the distance as cost function.

By the summability properties of f (see the hypothesis of Theorem 2.4) one deduces the existence of an L^1-Kantorovich potential φ, solution of the dual problem. Using φ, we can construct the set

$$\Gamma := \{(x, y) \in X \times X : \varphi(x) - \varphi(y) = \mathsf{d}(x, y)\},$$

inducing a partial order relation whose maximal chains produce a partition made of one-dimensional sets of a certain subset of the space, provided that the ambient space X satisfies some mild regularity properties.

This procedure has been already presented and used in several contributions [7, 14, 33, 41, 52] when the ambient space is the Euclidean space, a manifold or a non-branching metric space (see [14, 17] for extended metric spaces). The analysis in our framework started with [19] and has been refined and extended in [22]; we will follow the notation of [22] to which we refer for more details.

The *transport relation* $\mathcal{R}^e$ and the *transport set with end-points* $\mathcal{T}^e$ are defined as

$$\mathcal{R}^e := \Gamma \cup \Gamma^{-1} = \{|\varphi(x) - \varphi(y)| = \mathsf{d}(x, y)\},$$
$$\mathcal{T}^e := P_1(\mathcal{R}^e \setminus \{x = y\}),$$

where $\{x = y\}$ denotes the diagonal $\{(x, y) \in X^2 : x = y\}$ and $\Gamma^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma\}$. Since φ is 1-Lipschitz, Γ, Γ^{-1} and $\mathcal{R}^e$ are closed sets and therefore, from the local compactness of (X, d), σ-compact; consequently, $\mathcal{T}^e$ is σ-compact.

We restrict $\mathcal{T}^e$ to a smaller set where $\mathcal{R}^e$ is an equivalent relation. To exclude possible branching, we need to consider the following sets, introduced in [19]:

$$A^+ := \{x \in \mathcal{T}^e : \exists z, w \in \Gamma(x),\ (z, w) \notin \mathcal{R}^e\},$$
$$A^- := \{x \in \mathcal{T}^e : \exists z, w \in \Gamma^{-1}(x),\ (z, w) \notin \mathcal{R}^e\},$$

where $\Gamma(x) = \{y \in X : (x, y) \in \Gamma\}$ denotes the section of Γ through x in the first coordinate; $\Gamma^{-1}(x)$ and $\mathcal{R}^e(x)$ are defined in the same way. The sets $A^{\pm}$ are called the sets of forward and backward branching points, respectively. Note that both $A^{\pm}$ are σ-compact sets. Then the non-branched transport set has been defined as

$$\mathcal{T} := \mathcal{T}^e \setminus (A^+ \cup A^-),$$

and it is a Borel set; in the same way, define the non-branched relation as $\mathcal{R} = \mathcal{R}^e \cap (\mathcal{T} \times \mathcal{T})$. It was shown in [19] (cf. [14]) that $\mathcal{R}$ is an equivalence relation over $\mathcal{T}$ and that for any $x \in \mathcal{T}$, $\mathcal{R}(x) \subset (X, \mathsf{d})$ is isometric to a closed interval in $(\mathbb{R}, |\cdot|)$.

A priori, the non-branched transport set $\mathcal{T}$ can be much smaller than $\mathcal{T}^e$. However, under fairly general assumptions one can prove that the sets $A^{\pm}$ of forward and backward branching are both $\mathfrak{m}$-negligible. In [19, Proposition 4.5], this was shown for an m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfying $\mathsf{RCD}^*(K, N)$ and $\text{supp}(\mathfrak{m}) = X$. The same proof works for an essentially non-branching m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfying $\mathsf{CD}(0, N)$ and $\text{supp}(\mathfrak{m}) = X$ (see [23]).

One can choose $Q \subset \mathcal{T}$ a Borel section of the equivalence relation $\mathcal{R}$ (this choice is possible as it was shown in [14, Proposition 4.4]). Define the quotient map $\mathfrak{Q}: \mathcal{T} \to Q$ as $\mathfrak{Q}(x) = \alpha$, where α is the unique element of $\mathcal{R}(x) \cap Q$. Given a finite measure $\mathfrak{q} \in \mathcal{M}^+(Q)$ such that $\mathfrak{q} \ll \mathfrak{Q}_\#(\mathfrak{m}_{\llcorner \mathcal{T}})$, the disintegration theorem applied to $(\mathcal{T}, \mathcal{B}(\mathcal{T}), \mathfrak{m}_{\llcorner \mathcal{T}})$ gives an essentially unique disintegration of $\mathfrak{m}_{\llcorner \mathcal{T}}$ consistent with the partition of $\mathcal{T}$ given by the equivalence classes $\{\mathcal{R}(\alpha)\}_{\alpha \in Q}$ of $\mathcal{R}$:

$$\mathfrak{m}_{\llcorner \mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha).$$

In the sequel, we will also use the notation X_α to denote the equivalence class $\mathcal{R}(\alpha)$. Note that such measure $\mathfrak{q}$ can always be build by taking the push-forward via $\mathfrak{Q}$ of a suitable finite measure absolutely continuous with respect to $\mathfrak{m}_{\mathcal{T}}$.

The existence of a measurable section also permits a construction of a measurable parametrization of the transport rays. First, define the (possibly infinite) length of a transport ray $|X_\alpha| := \sup_{x, y \in X_\alpha} \mathsf{d}(x, y)$. Then, we can define

$$g\colon \text{Dom}(g) \subset Q \times [0, +\infty) \to \mathcal{T}$$

that associates to (α, t) the unique $x \in \mathcal{R}(\alpha)$ in such a way that $\varphi(g(\alpha, t)) - \varphi(g(\alpha, s)) = s - t$, provided $t, s \in (0, |X_\alpha|)$. In other words, $g(\alpha, \cdot)$ is the unit-speed, maximal parametrization of X_α such that $\frac{d}{dt}\varphi(g(\alpha, t)) = -1$. We specify that this parametrization ensures that $f(g(\alpha, 0)) \geq 0$. By continuity of g with respect to the variable t, we extend g in order to also map the end-points of the rays X_α; the restriction of g to the set $\{(\alpha, t) : t \in (0, |X_\alpha|)\}$ is injective.

Finally, to prove that the disintegration is $\mathsf{CD}(0, N)$, i.e., that for $\mathfrak{q}$-a.e. $\alpha \in Q$, the space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is $\mathsf{CD}(0, N)$, one uses the presence of the L^2-Wasserstein geodesics inside the transport set $\mathcal{T}$ (see [18, Lemma 4.6]). We refer to [24, Theorem 4.2] for all the details.

The measure $\mathfrak{m}_\alpha$ will be absolutely continuous with respect to $\mathcal{H}^1\llcorner_{X_\alpha}$ as a consequence of the $\mathsf{CD}(0,N)$ condition in one-dimensional spaces: there exists a map h_α: $(0,|X_\alpha|) \to \mathbb{R}$ such that $\mathfrak{m}_\alpha = (g(\alpha,\cdot))_\#(h_\alpha \mathcal{L}^1\llcorner_{(0,|X_\alpha|)})$. The construction does not depend on the function f but only on the L^1-Kantorovich potential φ.

Theorem 2.5. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching m.m.s. with* $\mathrm{supp}(\mathfrak{m}) = X$ *satisfying* $\mathsf{CD}(0,N)$ *for some $N \in (1,\infty)$. Assume that $\varphi: X \to R$ is a 1-Lipschitz function, and let $\mathcal{T}$ and $(X_\alpha)_{\alpha\in Q}$ be respectively the transport set and the transport rays as they were defined in the previous paragraphs. Let Q and $\mathfrak{Q}: \mathcal{T} \to Q$ be the quotient set and the quotient map, respectively, and assume that there exists a measure $\mathfrak{q} \ll \mathfrak{Q}_\#(\mathfrak{m}_\mathcal{T})$. Then there exists a disintegration of $\mathfrak{m}\llcorner_\mathcal{T}$ on $\{X_\alpha\}_{\alpha\in Q}$,*

$$\mathfrak{m}\llcorner_\mathcal{T} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha),$$

and for $\mathfrak{q}$-a.e. $\alpha \in Q$,

(1) *X_α is a closed geodesic in (X,d).*

(2) *$\mathfrak{m}_\alpha$ is a Radon measure supported on X_α with $\mathfrak{m}_\alpha \ll \mathcal{H}^1\llcorner_{X_\alpha}$.*

(3) *The metric-measure space $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha)$ satisfies* $\mathsf{CD}(0,N)$.

Theorem 2.4 follows from the previous theorem, provided that we are able to localize constraint $\int_X f \, d\mathfrak{m} = 0$. The localization is a consequence of the properties of the L^1-optimal transport problem (see [24, Theorem 5.1]).

3. Localization of the measure and the perimeter

To prove Theorem 1.4, we will need to consider the isoperimetric problem inside a family of large subsets of X with diameter approaching ∞. In order to apply the classical dimension reduction argument provided by localization theorem (see Sections 2.3 and 2.4), one needs in principle these subsets to also be convex. As the existence of an increasing family of convex subsets recovering at the limit the whole space X is in general false, we will overcome this issue in the following way.

Given any bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then consider $R > 0$ such that $E \subset B_R$ (hereinafter we will adopt the notation $B_R := B_R(x_0)$). Consider then the following family of zero mean functions:

$$f_R(x) = \Big(\chi_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\Big)\chi_{B_R}.$$

Clearly, f_R satisfies the hypothesis of Theorem 2.4, so we obtain an $\mathfrak{m}$-measurable subset $\mathcal{T}_R \subset X$ and a family $\{X_{\alpha,R}\}_{\alpha\in Q_R}$ of transport rays, such that there exists a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_R}$ on $\{X_{\alpha,R}\}_{\alpha\in Q_R}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}_R} = \int_{Q_R} \mathfrak{m}_{\alpha,R} \, \mathfrak{q}_R(d\alpha), \quad \mathfrak{q}_R(Q_R) = \mathfrak{m}(\mathcal{T}_R), \tag{3.1}$$

with the probability measures $\mathfrak{m}_{\alpha,R}$ having a $\mathsf{CD}(0, N)$ density with respect to $\mathcal{H}^1 \llcorner_{X_{\alpha,R}}$. The localization of the zero mean implies that

$$\mathfrak{m}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \mathfrak{m}_{\alpha,R}(B_R), \quad \mathfrak{q}_R\text{-a.e. } \alpha \in Q_R. \tag{3.2}$$

We denote by $g_R(\alpha, \cdot)\colon [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ the unit speed parametrization of the geodesic $X_{\alpha,R}$. For this reason, it holds

$$\mathfrak{m}_{\alpha,R} = (g_R(\alpha, \cdot))_\# (h_{\alpha,R} \mathcal{L}^1 \llcorner_{[0,|X_{\alpha,R}|]})$$

for some $\mathsf{CD}(0, N)$ density $h_{\alpha,R}$.

Also, we specify that the direction of the parametrization of $X_{\alpha,R}$ is chosen such that $g_R(\alpha, 0) \in E$. Equivalently, if φ_R denotes a Kantorovich potential associated to the localization of g_R, then the parametrization is chosen in such a way that φ_R is decreasing along $X_{\alpha,R}$ with slope -1.

We then define $T_{\alpha,R}$ to be the unique element of $[0, |X_{\alpha,R}|]$ such that

$$\mathfrak{m}_{\alpha,R}(g_R(\alpha, [0, T_{\alpha,R}])) = \mathfrak{m}_{\alpha,R}(B_R).$$

Since $\mathfrak{m}_{\alpha,R}$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner_{X_{\alpha,R}}$, the existence of a unique $T_{\alpha,R}$ follows. Moreover, from the measurability in α of $\mathfrak{m}_{\alpha,R}$, we deduce the same measurability of $T_{\alpha,R}$.

Notice that $\operatorname{diam}(B_R \cap X_{\alpha,R}) \le R + \operatorname{diam}(E)$: since $g_R(\alpha, \cdot)$ is a unit speed parametrization of $X_{\alpha,R}$, then $\mathsf{d}(g_R(\alpha, 0), g_R(\alpha, t)) \le \mathsf{d}(g_R(\alpha, 0), x_0) + \mathsf{d}(g_R(\alpha, t), x_0) \le \operatorname{diam}(E) + R$, provided $g_R(\alpha, t) \in B_R \cap X_{\alpha,R}$. Hence, the same upper bound is valid for $T_{\alpha,R}$, i.e., $T_{\alpha,R} \le R + \operatorname{diam}(E)$.

We restrict $\mathfrak{m}_{\alpha,R}$ to $\hat{X}_{\alpha,R} := g_R(\alpha, [0, T_{\alpha,R}])$ to have the following disintegration:

$$\begin{aligned} &\mathfrak{m}\llcorner_{\hat{\mathcal{T}}_R} = \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}\, \hat{\mathfrak{q}}_R(d\alpha), \quad \hat{\mathfrak{m}}_{\alpha,R} := \frac{\mathfrak{m}_{\alpha,R}\llcorner_{\hat{X}_{\alpha,R}}}{\mathfrak{m}_{\alpha,R}(B_R)} \in \mathcal{P}(X), \\ &\hat{\mathfrak{q}}_R = \mathfrak{m}_{\cdot,R}(B_R)\mathfrak{q}_R, \end{aligned} \tag{3.3}$$

where $\hat{\mathcal{T}}_R := \bigcup_{\alpha \in Q_R} \hat{X}_{\alpha,R}$; in particular, $\hat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R)$, using (3.1) and the fact that $B_R \subset \mathcal{T}_R$.

Disintegration (3.3) will be a localization like (3.2) only if $(E \cap X_{\alpha,R}) \subset \hat{X}_{\alpha,R}$, implying that

$$\hat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \quad \hat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R.$$

To prove this inclusion, we will impose that $E \subset B_{\frac{R}{4}}$. Since $g_R(\alpha, \cdot)\colon [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ has unit speed, we notice that

$$\mathsf{d}(g_R(\alpha, t), x_0) \le \mathsf{d}(g_R(\alpha, 0), x_0) + t \le \operatorname{diam}(E) + t \le \frac{R}{2} + t,$$

where in the second inequality we have used that each starting point of the transport ray has to be inside E, being precisely where $f_R > 0$. Hence, $g_R(\alpha, t) \in B_R$ for all $t < \frac{R}{2}$.

This implies that $(g_R(\alpha,\cdot))^{-1}(B_R) \supset [0, \min\{\frac{R}{2}, |X_{\alpha,R}|\}]$, hence there are "no holes" inside $(g_R(\alpha,\cdot))^{-1}(B_R)$ before $\min\{\frac{R}{2}, |X_{\alpha,R}|\}$, implying that $|\widehat{X}_{\alpha,R}| \geq \min\{\frac{R}{2}, |X_{\alpha,R}|\}$. Since $\mathrm{diam}(E) \leq \frac{R}{2}$, we deduce that $(g_R(\alpha,\cdot))^{-1}(E) \subset [0, \min\{\frac{R}{2}, |X_{\alpha,R}|\}]$ implying that $(E \cap X_{\alpha,R}) \subset \widehat{X}_{\alpha,R}$.

We can give an explicit description of the measure $\hat{\mathfrak{q}}_R$ in term of a push-forward via the quotient map $\mathfrak{Q}_R$ of the measure $\mathfrak{m}\llcorner_E$,

$$\begin{aligned}\hat{\mathfrak{q}}_R(A) &= \int_{Q_R} \mathbf{1}_A(\alpha)\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\hat{\mathfrak{m}}_{\alpha,R}(E)\hat{\mathfrak{q}}_R(d\alpha)\\ &= \int_{Q_R} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\hat{\mathfrak{m}}_{\alpha,R}(E\cap \mathfrak{Q}_R^{-1}(A))\hat{\mathfrak{q}}_R(d\alpha)\\ &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\mathfrak{m}(E\cap \mathfrak{Q}_R^{-1}(A)),\end{aligned}$$

hence $\hat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}(\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)$.

We need to study the relation between the perimeter and the disintegration of measure (3.3). Fix an open set $\Omega \subset X$ and consider the relative perimeter $\mathsf{P}(E;\Omega)$. Let $u_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a sequence such that $u_n \to \mathbf{1}_E$ in $L^1_{\mathrm{loc}}(\Omega)$ and $\lim_{n\to\infty}\int_\Omega |Du_n|\,d\mathfrak{m} = \mathsf{P}(E;\Omega)$. Using the Fatou lemma, we can compute

$$\begin{aligned}\mathsf{P}(E;\Omega) &= \lim_{n\to\infty}\int_\Omega |Du_n|\,d\mathfrak{m} \geq \liminf_{n\to\infty}\int_{\Omega\cap\widehat{\mathcal{T}}_R}|Du_n|\,d\mathfrak{m}\\ &= \liminf_{n\to\infty}\int_{Q_R}\int_\Omega |Du_n|\,\hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &\geq \int_{Q_R}\liminf_{n\to\infty}\int_\Omega |Du_n|\,\hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &\geq \int_{Q_R}\liminf_{n\to\infty}\int_{X_{\alpha,R}\cap\Omega} |u_n'|\,\hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &\geq \int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\Omega)\,\hat{\mathfrak{q}}_R(d\alpha),\end{aligned}$$

where u_n' denotes the derivative along the curve $g_R(\alpha,\cdot)$ and $\mathsf{P}_{\widehat{X}_{\alpha,R}}$ the perimeter of m.m.s. $(\widehat{X}_{\alpha,R}, \mathsf{d}, \hat{\mathfrak{m}}_{\alpha,R})$.

By arbitrariness of Ω, we deduce the following disintegration inequality:

$$\mathsf{P}(E;\cdot) \geq \int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\cdot)\,\hat{\mathfrak{q}}_R(d\alpha).$$

Moreover, the fact that the geodesic

$$g_R(\alpha,\cdot)\colon [0, |\widehat{X}_{\alpha,R}|] \to \widehat{X}_{\alpha,R}$$

has unit speed implies that

$$\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\cdot) = (g_R(\alpha,\cdot))_\#(\mathsf{P}_{h_{\alpha,R}}((g_R(\alpha,\cdot))^{-1}(E);\cdot)).$$

We summarize this construction in the following assertion.

Proposition 3.1. *Given any bounded $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then fix $R > 0$ such that $E \subset B_{\frac{R}{4}}(x_0)$. Then there exists a Borel set $\widehat{\mathcal{T}}_R \subset X$, with $E \subset \widehat{\mathcal{T}}_R$ and a disintegration formula*

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}\, \hat{\mathfrak{q}}_R(d\alpha), \quad \hat{\mathfrak{m}}_{\alpha,R}(\widehat{X}_{\alpha,R}) = 1, \quad \hat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R), \tag{3.4}$$

such that

$$\hat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \quad \text{for } \hat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R, \quad \hat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E), \tag{3.5}$$

and the one-dimensional m.m.s. $(\widehat{X}_{\alpha,R}, \mathsf{d}, \hat{\mathfrak{m}}_{\alpha,R})$ satisfies the $\mathsf{CD}(0,N)$ condition and has diameter bounded by $R + \operatorname{diam}(E)$. Furthermore, the following formula holds true:

$$\mathsf{P}(E;\cdot) \geq \int_{Q_R} \mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\cdot)\hat{\mathfrak{q}}_R(d\alpha). \tag{3.6}$$

The rescaling introduced in Proposition 3.1 will be crucially used to obtain non-trivial limit estimates as $R \to \infty$.

4. One-dimensional analysis

Proposition 3.1 is the first step to obtain an almost optimality of $E \cap \widehat{X}_{\alpha,R}$ from the optimality of a bounded set E. We now have to analyze in detail the one-dimensional isoperimetric profile function.

We fix a few notations and conventions.

We will be considering the m.m.s. $(I, |\cdot|, h\mathcal{L}^1)$ with an interval $I \subset \mathbb{R}$ satisfying the $\mathsf{CD}(0,N)$ condition; when the interval has finite diameter, we will always assume that $I = [0,D]$. We will assume also that $\int_0^D h\,dx = 1$, unless otherwise specified. For consistency with the conditional measures from disintegration theorem, we will use the notation $\mathfrak{m}_h = h\mathcal{L}^1$.

We also introduce the functions

$$v_h\colon [0,D] \to [0,1] \quad \text{and} \quad r_h\colon [0,1] \to [0,D]$$

as

$$v_h(r) := \int_0^r h(s)\,ds, \quad r_h(v) := (v_h)^{-1}(v);$$

notice that from the $\mathsf{CD}(0,N)$ condition, $h > 0$ over I makes v_h invertible and in turn the definition of r_h well-posed.

We will denote by P_h the perimeter in the space $([0,D], |\cdot|, h\mathcal{L}^1_{[0,D]})$. If $E \subset [0,D]$ is a set of finite perimeter, then it can be decomposed (up to a negligible set) into a family of disjoint intervals

$$E = \bigcup_i (a_i, b_i),$$

and the union is at most countable. In this case, we have that the perimeter is given by

$$\mathsf{P}_h(E) = \sum_{i:a_i \neq 0} h(a_i) + \sum_{i:b_i \neq D} h(b_i).$$

We shall denote by $\mathcal{I}_h$ the isoperimetric profile $\mathcal{I}_h(v) := \inf_{E:\mathfrak{m}_h(E)=v} \mathsf{P}_h(E)$.

4.1. *Properties of the isoperimetric profile function*

For our purpose, we consider the model spaces $([0, D], |\cdot|, h_{N,D}(\xi, \cdot)\mathcal{L}^1 \llcorner_{[0,D]})$, for $N > 1$, $D > 0$, and, $\xi \geq 0$, where

$$h_{N,D}(\xi, x) := \frac{N}{D^N} \frac{(x + \xi D)^{N-1}}{(\xi + 1)^N - \xi^N}.$$

For the model spaces, we can easily compute the functions $v_{N,D}(\xi, \cdot) := v_{h_{N,D}(\xi,\cdot)}$ and $r_{N,D}(\xi, \cdot) := r_{h_{N,D}(\xi,\cdot)}$:

$$\begin{aligned} v_{N,D}(\xi, r) &= \frac{(r + \xi D)^N - (\xi D)^N}{D^N((1 + \xi)^N - \xi^N)}, \\ r_{N,D}(\xi, v) &= D((v(1 + \xi)^N + (1 - v)\xi^N)^{\frac{1}{N}} - \xi). \end{aligned} \tag{4.1}$$

We can easily deduce that if E is an isoperimetric set of measure $v \in (0, 1)$ for a model space, then (up to a negligible set)

$$E = \begin{cases} [0, r_{N,D}(\xi, v)] & \text{if } v \leq \dfrac{1}{2}, \\ [r_{N,D}(\xi, 1 - v), D] & \text{if } v \geq \dfrac{1}{2}, \end{cases}$$

with the convention that if $v = \frac{1}{2}$, both cases are possible. Indeed, if $v \leq \frac{1}{2}$, we can "push" all the mass to the left obtaining a new set $E' = [0, r_{N,D}(\xi, v)]$; the monotonicity of $h_{N,D}(\xi, \cdot)$ ensures that E' has smaller perimeter than E'. If, on the contrary, $v \geq \frac{1}{2}$, then we have that the complementary $[0, D]\backslash E$ is an isoperimetric set, then $[0, D]\backslash E = [0, r_{N,D}(1 - v)]$. This allows us to explicitly compute the isoperimetric profile of the model spaces

$$\begin{aligned} \mathcal{I}_{N,D}(\xi, v) &= h_{N,D}(\xi, r_{N,D}(\min\{v, 1 - v\})) \\ &= \frac{N}{D} \frac{(\min\{v, 1 - v\}(\xi + 1)^N + \max\{v, 1 - v\}\xi^N)^{\frac{N-1}{N}}}{(\xi + 1)^N - \xi^N}. \end{aligned}$$

We also define an auxiliary function $\mathcal{G}_N$ as

$$\mathcal{G}_N(\xi, v) := \frac{((\xi + 1)^N + (\frac{1}{v} - 1)\xi^N)^{\frac{N-1}{N}}}{(\xi + 1)^N - \xi^N}. \tag{4.2}$$

Notice that, if $v \leq \frac{1}{2}$, then

$$\mathcal{G}_N(\xi, v) = \frac{D}{N} \frac{\mathcal{I}_{N,D}(\xi, v)}{v^{1-\frac{1}{N}}}.$$

One advantage of this function is that it does not depend on D. This is indeed quite natural, as the isoperimetric profile scales with D.

For the family of one-dimensional $\mathsf{CD}(0, N)$ spaces with diameter not larger than D, an explicit and sharp lower bound for the isoperimetric profile function has been established in [45] (see, for instance, [45, Corollary 1.4] and [24, Section 6.1] for the non-smooth analog). Defining

$$\begin{aligned}\mathcal{I}_{N,D}(v) &:= \frac{N}{D}\inf_{\xi\geq 0}\frac{(\min\{v,1-v\}(\xi+1)^N+\max\{v,1-v\}\xi^N)^{\frac{N-1}{N}}}{(\xi+1)^N-\xi^N}\\ &= \inf_{\xi\geq 0}\mathcal{I}_{N,D}(\xi,v),\end{aligned}$$

then one obtains that $\mathcal{I}_h(v) \geq \mathcal{I}_{N,D}(v)$ for every $h\colon [0, D'] \to \mathbb{R}$ satisfying the $\mathsf{CD}(0, N)$ condition, with $D' \in (0, D]$.

We obtain the following lower bound.

Lemma 4.1. *Fix $N > 1$. Then, we have the following estimate for $\mathcal{I}_{N,D}$:*

$$\mathcal{I}_{N,D}(w) \geq \frac{N}{D}w^{1-\frac{1}{N}}(1-O(w^{\frac{1}{N}})) = \frac{N}{D}(w^{1-\frac{1}{N}}-O(w)) \quad \textit{as } w\to 0.$$

Proof. Recalling the definition of $\mathcal{G}_N$, what we have to prove becomes

$$\inf_{\xi\geq 0}\mathcal{G}_N(\xi,w)\geq 1-O(w^{\frac{1}{N}}) \quad \text{as } w\to 0.$$

The minimum in the infimum in the expression above is attained, at least for all w small enough. Indeed, we have that

$$\mathcal{G}_N(\xi,v) = \frac{((1+\xi^{-1})^N+(\frac{1}{w}-1))^{\frac{N-1}{N}}}{\xi((1+\xi^{-1})^N-1)} = \frac{((1+\xi^{-1})^N+(\frac{1}{w}-1))^{\frac{N-1}{N}}}{\xi(1+N\xi^{-1}-o(\xi^{-1})-1)}, \tag{4.3}$$

thus the limit

$$\lim_{\xi\to\infty}\mathcal{G}_N(\xi,w) = \frac{(\frac{1}{w}-1)^{\frac{N-1}{N}}}{N}\geq 1 = \mathcal{G}_N(0,w)$$

implies the coerciveness of $\xi\mapsto\mathcal{G}_N(\xi,w)$. Define $\xi_w\in\arg\min_{\xi\in[0,\infty]}\mathcal{G}_N(\xi,w)$; we trivially have that $\mathcal{G}_N(\xi_w,w)\leq 1$.

First, we prove that $\limsup_{w\to 0}\xi_w<\infty$ (we soon will improve this estimate). Suppose the contrary, i.e., there exists some sequence $w_n\to 0$ such that $\xi_{w_n}\to\infty$. Then we have

$$\begin{aligned}1 &\geq \limsup_{n\to\infty}\mathcal{G}_N(\xi_{w_n},w_n)\geq\limsup_{n\to\infty}\frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}\xi_{w_n}^{N-1}}{(\xi_{w_n}+1)^N-\xi_{w_n}^N}\\ &= \limsup_{n\to\infty}\frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}}{\xi_{w_N}(1+N\xi_{w_n}^{-1}+o(\xi_{w_n}^{-1})-1)} = \infty,\end{aligned} \tag{4.4}$$

which is a contradiction. Since $\limsup_{w\to 0}\xi_w<\infty$, we have $(\xi_w+1)^{N-1}-\xi_w^{N-1}\leq C$, for all w small enough, for some constant $C>0$.

We improve the estimate above

$$1 \geq \limsup_{w \to 0} \mathcal{G}_N(\xi_w, w) \geq \limsup_{w \to 0} \frac{((\frac{1}{w} - 1)\xi_w^N)^{\frac{N-1}{N}}}{(\xi_w + 1)^{N-1} - \xi_w^N} \geq \limsup_{w \to 0} \frac{((\frac{1}{w} - 1)\xi_w^N)^{\frac{N-1}{N}}}{C}, \tag{4.5}$$

which implies $\limsup_{w \to 0} \xi_w \leq 0$, i.e., $\xi_w \to 0$ as $w \to 0$. We can improve the estimate again

$$1 \geq \limsup_{w \to 0} \mathcal{G}_N(\xi_w, w) = \limsup_{w \to 0} \frac{((1 + \xi_w)^N + \frac{\xi_w^N}{w} - \xi_w^N)^{\frac{N-1}{N}}}{(\xi_w + 1)^{N-1} - \xi_w^N} = (1 + \limsup_{w \to 0} \frac{\xi_w^N}{w})^{\frac{N-1}{N}}, \tag{4.6}$$

yielding $\limsup_{w \to 0} \frac{\xi_w}{w^{\frac{1}{N}}} \leq 0$, i.e., $\xi_w = o(w^{\frac{1}{N}})$ as $w \to 0$. Finally, we can conclude noticing that

$$\inf_{\xi \geq 0} \mathcal{G}_N(\xi, w) = \mathcal{G}_N(\xi_w, w) = \frac{((\xi_w + 1)^N + (\frac{1}{w} - 1)\xi_w^N)^{\frac{N-1}{N}}}{(\xi_w + 1)^N - \xi_w^N} = \frac{(1 + o(1))^{\frac{N-1}{N}}}{1 + O(\xi_w)} = 1 - O(w^{\frac{1}{N}}). \qquad \blacksquare$$

Corollary 4.2. *Fix $N > 1$. Then for all $D \geq D' > 0$ and for all $h \colon [0, D'] \to \mathbb{R}$ satisfying the $\mathsf{CD}(0, N)$ condition, it holds that*

$$\mathsf{P}_h(E) \geq \mathcal{I}_h(\mathfrak{m}_h(E)) \geq \frac{N}{D'} \mathfrak{m}_h(E)^{1-\frac{1}{N}} (1 - O(\mathfrak{m}_h(E)^{\frac{1}{N}})) \geq \frac{N}{D} \mathfrak{m}_h(E)^{1-\frac{1}{N}} (1 - O(\mathfrak{m}_h(E)^{\frac{1}{N}})),$$

for any Borel set $E \subset [0, D']$.

4.2. *Sharp isoperimetric inequalities in $\mathsf{CD}(0, N)$ spaces with Euclidean volume growth*

We re-obtain Theorem 1.2 via localization.

Theorem 4.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching $\mathsf{CD}(0, N)$ space having $\mathsf{AVR}_X > 0$. Let $E \subset X$ be any bounded Borel set, then*

$$\mathsf{P}(E) \geq N \omega_N^{\frac{1}{N}} \mathsf{AVR}_X^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{4.7}$$

Proof. Let $x_0 \in E$ be any point. We then consider $R > 0$ such that $E \subset B_{R(x)}$. For shortness, we will write $B_R = B_R(x_0)$. We use Proposition 3.1 and in particular (3.6), obtaining

$$\mathsf{P}(E) \geq \int_{Q_R} \mathsf{P}_{\hat{X}_{\alpha,R}}(E) \, \hat{\mathfrak{q}}_R(d\alpha). \tag{4.8}$$

Using Corollary 4.2 and the fact that each ray $\hat{X}_{\alpha,R}$ has length at most diam $E + R$, we deduce

$$\begin{aligned}\mathsf{P}(E) &\geq \int_{Q_R} \mathcal{I}_{N,\operatorname{diam} E+R}(\hat{\mathfrak{m}}_{\alpha,R}(E))\,\hat{\mathfrak{q}}_R(d\alpha) \geq \mathfrak{m}(B_R)\,\mathcal{I}_{N,\operatorname{diam} E+R}\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\Big)\\ &\geq \mathfrak{m}(B_R)\,\frac{N}{\operatorname{diam} E + R}\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\Big)^{1-\frac{1}{N}}\Big(1 - O\Big(\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\Big)^{\frac{1}{N}}\Big)\Big)\\ &= N\Big(\frac{\mathfrak{m}(B_R)}{R^N}\Big)^{\frac{1}{N}}\mathfrak{m}(E)^{1-\frac{1}{N}} - \frac{O(1)}{\operatorname{diam} E + R}.\end{aligned}$$

We conclude by taking the limit as $R \to \infty$ in the equation above. ■

4.3. One-dimensional reduction for the optimal region

Assuming that $E \subset X$ turns inequality (4.7) into an identity and following the proof of Theorem 4.3, a natural guess is that the right-hand side of (4.8) converges to the left-hand side as $R \to \infty$. The measure

$$\hat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R)$$

is converging to infinity with order $O(R^N)$, so the integrand should converge to 0 with order $O(R^{-N})$. We now confirm this heuristic.

Definition 4.4. Let $D \geq D' > 0$ and let $h\colon [0, D'] \to \mathbb{R}$ be a $\mathsf{CD}(0, N)$ density. If $E \subset [0, D']$ is Borel subset, we define the D-residual of E as

$$\operatorname{Res}_h^D(E) := \frac{D\mathsf{P}_h(E)}{N(\mathfrak{m}_h(E))^{1-\frac{1}{N}}} - 1. \tag{4.9}$$

If $v \in (0, \frac{1}{2})$, we define the D-residual of v as

$$\operatorname{Res}_h^D(v) := \operatorname{Res}_h^D([0, r_h(v)]) = \frac{Dh(r_h(v))}{Nv^{1-\frac{1}{N}}} - 1.$$

Corollary 4.2 can be restated as

$$\operatorname{Res}_h^D(E) \geq -O(\mathfrak{m}_h(E)^{\frac{1}{N}}). \tag{4.10}$$

We now apply the definition of residual to the disintegration rays.

In order to simplify the notation, we denote by $\mathsf{P}_{\alpha,R}$ the perimeter measure of the one-dimensional m.m.s. $(\hat{X}_{\alpha,R}, \mathsf{d}, \hat{\mathfrak{m}}_{\alpha,R})$. The measure $\hat{\mathfrak{m}}_{\alpha,R}$ will be identified with the ray map g to $h_{\alpha,R}\mathcal{L}^1$. Then

$$\begin{aligned}\operatorname{Res}_{\alpha,R} &:= \operatorname{Res}_{h_{\alpha,R}}^{R+\operatorname{diam}(E)}(g(\alpha,\cdot)^{-1}(E \cap \hat{X}_{\alpha,R})) \quad \text{for } \alpha \in Q_R,\\ \operatorname{Res}_{x,R} &:= \operatorname{Res}_{\mathfrak{Q}_R(x),R} \quad \text{for } x \in E.\end{aligned}$$

The good rays are those rays having small residual. We quantify their abundance.

Proposition 4.5. *Assume* $(X, \mathsf{d}, \mathfrak{m})$ *is an essentially non-branching* $\mathsf{CD}(0, N)$ *space such that* $\mathsf{AVR}_X > 0$. *If* $E \subset X$ *is a bounded set attaining equality in* (4.7)*, then*

$$\lim_{R \to \infty} \frac{\|\mathrm{Res}_{\alpha,R}\|_{L^1(Q_R)}}{\mathfrak{m}(B_R)} = 0, \tag{4.11}$$

where the reference measure for the Lebesgue space $L^1(Q_R)$ *is* $\mathfrak{q}_R$.

Proof. We first check that the function $\alpha \to \mathrm{Res}_{\alpha,R}$ is integrable. To this extent, it is enough to check that $(\mathrm{Res}_{\alpha,R})^-$ is integrable; indeed, this last fact derives from the isoperimetric inequality $\mathrm{Res}_{\alpha,R} \geq -O((\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)})^{\frac{1}{N}})$, as stated in (4.10). We can now compute the integral in (4.11)

$$\begin{aligned}\frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} |\mathrm{Res}_{\alpha,R}|\, \hat{\mathfrak{q}}_R(d\alpha) &= \frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} (2(\mathrm{Res}_{\alpha,R})^- + \mathrm{Res}_{\alpha,R}) \hat{\mathfrak{q}}_R(d\alpha) \\ &\leq O\Big(\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\Big)^{\frac{1}{N}}\Big) + \frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} \mathrm{Res}_{\alpha,R}\, \hat{\mathfrak{q}}_R(d\alpha).\end{aligned}$$

The first term is infinitesimal, so we focus on the second one

$$\begin{aligned}\int_{Q_R} \mathrm{Res}_{\alpha,R}\, \hat{\mathfrak{q}}_R(d\alpha) &= \int_{Q_R} \Big(\frac{(R + \mathrm{diam}(E))\mathsf{P}_{\alpha,R}(E)}{N} \Big(\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\Big)^{1-\frac{1}{N}} - 1\Big) \hat{\mathfrak{q}}_R(d\alpha) \\ &= \frac{R + \mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}-1}\, N \mathfrak{m}(E)^{1-\frac{1}{N}}} \int_{Q_R} \mathsf{P}_{\alpha,R}(E)\, \hat{\mathfrak{q}}_R(d\alpha) - \mathfrak{m}(B_R) \\ &\leq \frac{R + \mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}-1}\, N \mathfrak{m}(E)^{1-\frac{1}{N}}} \mathsf{P}(E) - \mathfrak{m}(B_R) \\ &\leq \mathfrak{m}(B_R) \frac{R + \mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}}} (\mathsf{AVR}_X \omega_N)^{\frac{1}{N}} - \mathfrak{m}(B_R),\end{aligned}$$

yielding

$$\frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} \mathrm{Res}_{\alpha,R}\, \mathfrak{q}_R(d\alpha) \leq \frac{R + \mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}}} (\mathsf{AVR}_X \omega_N)^{\frac{1}{N}} - 1,$$

and the right-hand side goes to 0 as $R \to \infty$. ■

Corollary 4.6. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching* $\mathsf{CD}(0, N)$ *space having* $\mathsf{AVR}_X > 0$. *Let* $E \subset X$ *be a set saturating the isoperimetric inequality* (4.7)*, then it holds true*

$$\lim_{R \to \infty} \|\mathrm{Res}_{\mathfrak{Q}_R(x),R}\|_{L^1(E)} = 0.$$

Proof. A direct computation gives

$$\begin{aligned}\|\mathrm{Res}_{\mathfrak{Q}_R(x),R}\|_{L^1(E)} &= \int_{Q_R} \int_E |\mathrm{Res}_{\mathfrak{Q}_R(x),R}| \hat{\mathfrak{m}}_{\alpha,R}(dx) \hat{\mathfrak{q}}_R(d\alpha) \\ &= \int_{Q_R} |\mathrm{Res}_{\alpha,R}| \hat{\mathfrak{m}}_{\alpha,R}(E) \hat{\mathfrak{q}}_R(d\alpha) \\ &= \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \|\mathrm{Res}_{\alpha,R}\|_{L^1(Q_R)} \to 0.\end{aligned}$$

■

5. Analysis along the good rays

We now use the residual to control how distant the density $h\colon [0, D'] \to \mathbb{R}$ is from the model density $x \in [0, D] \mapsto \frac{Nx^{N-1}}{D}$ as well as the one-dimensional traces of E from the optimal ones.

The results in this section go in the direction of proving that, given $D \geq D' > 0$, $h\colon [0, D'] \to \mathbb{R}$ a $\mathsf{CD}(0, N)$ density, a subset $E \subset [0, D']$, if the measure $\mathfrak{m}_h(E)$ and the residual $\mathrm{Res}_h^D(E)$ are small, then the set E is closed to the interval $[0, D\mathfrak{m}_h(E)^{\frac{1}{N}}]$ and the density h is closed to the model density $\frac{Nx^{N-1}}{D}$.

Remark 5.1. We will make an extensive use of Landau's "big-O" and "small-o" notation. If we are in a situation where several variables appear, but only a few of them are converging, either the "big-O" or "small-o" could depend on the non-converging variables.

In our setting, the converging variables will be $w \to 0$ and $\delta \to 0$. The free variables will be the following:

(1) D, a bound from above on the diameter of the space;

(2) $D' \in (0, D]$, the diameter of the space;

(3) $h\colon [0, D'] \to \mathbb{R}$, a $\mathsf{CD}(0, N)$ density;

(4) $E \subset [0, D']$, a set with measure $\mathfrak{m}_h(E) = w$ and residual $\mathrm{Res}_h^D(E) \leq \delta$.

The following estimates are infinitesimal expansions as $w \to 0$ and $\delta \to 0$ and whenever a "big-O" or "small-O" appears, it has to be understood that this expression can be substituted with a function going to 0 with the same order uniformly with respect to the other free variables.

Remark 5.2. Another point to remark is the fact that we focus only on the case when E is on the left. We will sometimes assume that E is of the form $[0, r] \subset [0, D']$ and sometimes that $E \subset [0, L]$, with the tacit understanding that $r \ll D'$ or $L \ll D'$. This is possible because the rays come from the L^1-optimal transport problem from the measure $\frac{\mathfrak{m}\llcorner_E}{\mathfrak{m}(E)}$ to the measure $\frac{\mathfrak{m}\llcorner_{B_R}}{\mathfrak{m}(B_R)}$, where E is our original set. Hence, the rays are lines starting from E and going away, thus the intersection of E with any ray lies at the beginning of the ray.

5.1. Almost rigidity of the diameter

We start our analysis by focusing on the diameter of the space: the inequality $D \geq D'$ tends to attain equality if $\mathfrak{m}_h(E) = w \to 0$ and $\mathrm{Res}_h^D(E) \leq \delta \to 0$. It follows from the fact that the isoperimetric profile $\mathcal{I}_{N,D}$ scales according to D.

Proposition 5.3. *Fix $N > 1$. The following estimate holds for $w \to 0$ and $\delta \to 0$:*

$$D' \geq D(1 - o(1)), \tag{5.1}$$

where $D \geq D' > 0$ and $h\colon [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density such that $E \subset [0, D']$ is a subset satisfying $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \leq \delta$.

Proof. The definition of residual (4.9) gives

$$\frac{N}{D'}w^{1-\frac{1}{N}}(1+\mathrm{Res}_h^{D'}(E)) = \mathsf{P}_h(E) = \frac{N}{D}w^{1-\frac{1}{N}}(1+\mathrm{Res}_h^{D}(E)). \tag{5.2}$$

Since

$$\mathrm{Res}_h^D(E) \geq O(w^{\frac{1}{N}})$$

by (4.10), if w is small enough, we can multiply equation (5.2) by the factor $\frac{D' w^{\frac{1}{N}-1}}{N(1+\mathrm{Res}_h^D(E))}$, obtaining

$$\frac{D'}{D} = \frac{1+\mathrm{Res}_h^{D'}(E)}{1+\mathrm{Res}_h^{D}(E)} \geq \frac{1-O(w^{\frac{1}{N}})}{1+\mathrm{Res}_h^{D}(E)} \geq \frac{1-O(w^{\frac{1}{N}})}{1+\delta} = 1-o(1). \qquad \blacksquare$$

5.2. *Almost rigidity of the set E: The convex case*

We now prove that the set E has to be close to $[0, D\mathfrak{m}_h(E)^{\frac{1}{N}}]$. We start by considering the special case when the set E is of the form $E = [0, r]$.

Proposition 5.4. *Fix $N > 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$:*

$$r_h(w) \leq D(w^{\frac{1}{N}}(1+o(1))), \tag{5.3}$$

$$r_h(w) \geq D(w^{\frac{1}{N}}(1+o(1))), \tag{5.4}$$

where $D \geq D' > 0$ and $h\colon [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density such that $\mathrm{Res}_h^D(w) = \mathrm{Res}_h^D([0, r_h(w)]) \leq \delta$.

Proof. In order to simplify the notation, we write $r = r_h(w)$.

Part 1. *Inequality* (5.3). By the $\mathsf{CD}(0, N)$ of the function h, we have $h(x) \leq \frac{h(r)}{r^{N-1}}x^{N-1}$ for $r \leq x \leq D'$. If we integrate in $[r, D']$, we obtain

$$1-w \leq \int_r^{D'} \frac{h(r)}{r^{N-1}}x^{N-1}dx = \frac{h(r)(D'^N - r^N)}{Nr^{N-1}} \leq \frac{h(r)D'^N}{Nr^{N-1}} \leq \frac{h(r)D^N}{Nr^{N-1}},$$

yielding to

$$r^{N-1} \leq \frac{D^N}{N(1-w)}h(r) = \frac{D^N}{N(1-w)}\frac{N}{D}w^{1-\frac{1}{N}}(1+\mathrm{Res}_h^D(w)) \leq (Dw^{\frac{1}{N}})^{N-1}\frac{1+\delta}{1-w}.$$

Part 2. *Inequality* (5.4). This second part is a bit more difficult. The first step is to show that we can lead ourselves back to the case of model spaces, namely that we can assume $h = h_{N,D'}(\xi, \cdot)$ for some $\xi \geq 0$ (cf. (4.1)). That is, we want to show that given h, we find ξ such that

$$\mathrm{Res}_{h_{N,D'}(\xi,\cdot)}^D(w) \leq \mathrm{Res}_h^D(w) \leq \delta \quad \text{and} \quad r_{h_{N,D'}(\xi,\cdot)}(w) \leq r.$$

To this extent, consider the function $s\colon [0, \infty) \to \mathbb{R}$ given by

$$s(a) := \int_r^{D'} (h(r)^{\frac{1}{N-1}} + a(x-r))^{N-1}dx.$$

Clearly, this function is strictly increasing and it holds

$$s\Big(\frac{h(r)^{\frac{1}{N-1}}}{r}\Big) = \int_r^{D'} \frac{h(r)}{r^{N-1}} x^{N-1} dx \geq \int_r^{D'} h(x)\,dx = 1 - w,$$
$$s(0) = (D'-r)h(r) = (D'-r)\frac{N}{D} w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(w)) \leq 2N w_N^{1-\frac{1}{N}} < 1 - w_N \leq 1 - w,$$

where we assumed that $\mathrm{Res}_h^D(w) \leq \delta \leq 1$ and $w \leq w_N$ (for some $w_N > 0$ depending only on N), which is possible since $w \to 0$ and $\delta \to 0$. From the two inequalities above, it follows that there exists a unique $a \in (0, \frac{h(r)^{\frac{1}{N-1}}}{r}]$ such that $s(a) = 1 - w$. We can define the $\mathsf{CD}(0, N)$ density $\overline{h}(x) := (h(r)^{\frac{1}{N-1}} + a(x-r))^{N-1}$, which satisfies

$$\int_r^{D'} \overline{h}(x)\,dx = \int_r^{D'} h(x)\,dx = 1 - w.$$

By mean value theorem, there exists $y \in (r, D')$ such that $h(y) = \overline{h}(y)$, thus, by convexity of $h^{\frac{1}{N-1}}$, $\overline{h}(x) \geq h(x)$ for all $x \in [0, r]$. This implies that

$$\int_0^r \overline{h}(x)\,dx \geq \int_0^r h(x)\,dx = w.$$

Define

$$V := \int_0^{D'} \overline{h}(x)\,dx = \frac{(h(r)^{\frac{1}{N-1}} + a(D'-r))^N - (h(r)^{\frac{1}{N-1}} - ar)^N}{Na}$$
$$= \int_r^{D'} \overline{h}(x)\,dx + \int_0^r \overline{h}(x)\,dx \geq 1 - w + \int_0^r h(x)\,dx = 1,$$
$$\overline{r} := r_{\overline{h}}(wV) \leq r_{\overline{h}}(V - (1-w)) = r_{\overline{h}}\Big(\int_0^r \overline{h}(x)\,dx\Big) = r,$$

where $wV \leq V - (1-w)$ follows from $1 - w \in [0,1]$ and $V \geq 1$. Finally, we renormalize $\overline{h}$, defining

$$\hat{h}(x) := \frac{\overline{h}(x)}{V} = Na \frac{(h(r)^{\frac{1}{N-1}} + a(x-r))^{N-1}}{(h(r)^{\frac{1}{N-1}} + a(D'-r))^N - (h(r)^{\frac{1}{N-1}} - ar)^N}.$$

If we set $\xi = \frac{h(r)^{\frac{1}{N-1}} - ar}{aD'} \geq 0$, then it turns out that (cf. (4.1))

$$\hat{h}(x) = h_{N,D'}(\xi, x) = \frac{N(x + D'\xi)^{N-1}}{D'^N((1+\xi)^N - \xi^N)}.$$

This function satisfies

$$r_{N,D'}(\xi, w) = \overline{r} \leq r_h(w) \quad \text{and} \quad h_{N,D'}(\xi, \overline{r}) \leq \overline{h}(\overline{r}) \leq \overline{h}(r) = h(r)$$

(the inequality $\overline{h}(\overline{r}) \leq \overline{h}(r)$ follows from the fact that $a \geq 0$, hence $\overline{h}$ is non-increasing). This latter inequality can be restated as

$$\mathrm{Res}^{D}_{h_{N,D'}(\xi,\cdot)}(w) \leq \mathrm{Res}^{D}_{h}(w) \leq \delta. \tag{5.5}$$

For this reason, we can assume that h is of the type $h_{N,D'}(\cdot,\xi)$ for some $\xi \geq 0$.

Recalling equation (4.1), we notice that

$$r_{N,D'}(\xi,w) = D'((w(1+\xi)^N + (1-w)\xi^N)^{\frac{1}{N}} - \xi) \geq D'(w^{\frac{1}{N}} - \xi). \tag{5.6}$$

What we are going to prove is that ξ is "small" in a sense that we will soon specify. Using the definition of residual, inequality (5.5) can be restated as (we already defined $\mathcal{G}_N$ in equation (4.2))

$$\mathcal{G}_N(\xi,w) = \frac{((1+\xi)^N + (\frac{1}{w}-1)\xi^N)^{\frac{N-1}{N}}}{(1+\xi)^N - \xi^N} \leq \frac{D'}{D}(1+\delta) \leq 1+\delta.$$

Define the set

$$L_\delta(w) := \{\xi : \mathcal{G}_N(\eta,w) > 1+\delta \ \forall \eta > \xi\}.$$

We have already proved in (4.3) that

$$\lim_{\xi\to\infty} \mathcal{G}_N(\xi,w) = N^{-1} w^{\frac{1-N}{N}},$$

hence the set $L_\delta(w)$ is non-empty.

At this point, define the function $\xi_\delta(w) := \inf L_\delta(w)$. By the definition of ξ_δ and the continuity of $\mathcal{G}_N$, it clearly holds that

$$\mathcal{G}_N(\xi,w) \leq 1+\delta \ \Rightarrow \ \xi \leq \xi_\delta(w), \quad \mathcal{G}_N(\xi_\delta(w),w) = 1+\delta.$$

Now we follow the line of the proof of Proposition 4.1. First, like in (4.4), we can see that $\xi_\delta(w)$ is bounded as $w \to 0$ and $\delta \to 0$. Indeed, suppose the contrary, i.e., that there exist two sequences $w_n \to 0$ and $\delta_n \to 0$ such that $\xi_{\delta_n}(w_n) \to \infty$. Then we have

$$\begin{aligned} 1 &\geq \limsup_{n\to\infty} \mathcal{G}_N(\xi_{\delta_n}(w_n),w_n) \geq \limsup_{n\to\infty} \frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}} \xi_{\delta_n}(w_n)^{N-1}}{(\xi_{\delta_n}(w_n)+1)^N - \xi_{\delta_n}(w_n)^N} \\ &\geq \limsup_{n\to\infty} \frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}}{\xi_{\delta_n}(w_n)((\frac{1}{\xi_{\delta_n}(w_n)}+1)^N - 1)} = \infty, \end{aligned}$$

which is a contradiction. Like in (4.5), we can prove that $\xi_\delta(w) \to 0$ as $w \to 0$ and $\delta \to 0$:

$$\begin{aligned} 1 &\geq \limsup_{\substack{w\to 0\\ \delta\to 0}} \mathcal{G}_N(\xi_\delta(w),w) \geq \limsup_{\substack{w\to 0\\ \delta\to 0}} \frac{((\frac{1}{w}-1)\xi_\delta(w)^N)^{\frac{N-1}{N}}}{(\xi_\delta(w)+1)^N - \xi_\delta(w)^N} \\ &\geq \limsup_{\substack{w\to 0\\ \delta\to 0}} \frac{((\frac{1}{w}-1)\xi_\delta(w)^N)^{\frac{N-1}{N}}}{C}. \end{aligned}$$

Finally, like in (4.6), we have that

$$\begin{aligned} 1 &= \lim_{\substack{w\to 0\\ \delta\to 0}} \mathcal{G}_N(\xi_\delta,(w),w) = \lim_{\substack{w\to 0\\ \delta\to 0}} \frac{((1+\xi_\delta(w))^N + (\frac{1}{w}-1)\xi_\delta(w)^N)^{\frac{N-1}{N}}}{(1+\xi_\delta(w))^N - \xi_\delta(w)^N} \\ &= \lim_{\substack{w\to 0\\ \delta\to 0}} \left(1+\frac{\xi_\delta(w)^N}{w}\right)^{\frac{N-1}{N}}, \end{aligned}$$

yielding

$$\lim_{\substack{w\to 0\\ \delta\to 0}} \frac{\xi_\delta(w)^N}{w} = 1.$$

Using Landau's notation, the above becomes $\xi_\delta(w) = o(w^{\frac{1}{N}})$ as $w \to 0$ and $\delta \to 0$.

At this point, we can recall (5.6), obtaining

$$r_{N,D'}(\xi_\delta(w),w) \geq D'(w^{\frac{1}{N}} - \xi_\delta(w)) \geq D'(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})).$$

If we use estimate (5.1), we can continue the chain on inequalities and conclude

$$\begin{aligned} \frac{r_{N,D'}(\xi_\delta(w),w)}{D} &\geq \frac{D'}{D}(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})) \geq (1-o(1))(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})) \\ &= w^{\frac{1}{N}}(1-o(1)). \end{aligned}$$

■

5.3. *Almost rigidity of the set E: The general case*

We now drop the assumption $E = [0,r]$. Up to a negligible set, $E = \bigcup_{i\in\mathbb{N}}(a_i,b_i)$, where the intervals (a_i,b_i) are far away from each other (i.e., $b_i < a_j$ or $b_j < a_i$ for $i \neq j$). By boundedness of the original set of our isoperimetric problem, we can also assume that E is included in the interval $[0,L]$ for some $L > 0$. Define $b(E) := \operatorname{ess\,sup} E \leq L$.

In the next proposition, we exclude the existence of a sequence such that (a_{i_n}, b_{i_n}) goes to $b(E)$.

Lemma 5.5. *Fix $N > 1$ and $L > 0$. Then there exist two constants $\bar{w} > 0$ and $\bar{\delta} > 0$ (depending only on N and L) such that the following happens. For all $D \geq D' > 0$ with $D \geq 3L$, for all $h\colon [0,D'] \to \mathbb{R}$ satisfying the $\mathsf{CD}(0,N)$ condition, and for all $E \subset [0,L]$, such that $\mathfrak{m}_h(E) \leq \bar{w}$ and $\mathrm{Res}_h^D(E) \leq \bar{\delta}$, there exist $a \in [0,b(E))$ and an at-most-countable family of intervals $((a_i,b_i))_i$ such that, up to a negligible set,*

$$E = \bigcup_i (a_i,b_i) \cup (a,b(E)),$$

with $a_i, b_i < a$ $\forall i$. Moreover, h is strictly increasing on $[0,b(E)]$.

Proof. By Proposition 5.3, we have that if $\mathfrak{m}_h(E)$ and $\mathrm{Res}_h^D(E)$ are small enough, then D' is close to $D \geq 3L$ and in particular $D' \geq 2L$. We already know that the set E is of the form $E = \bigcup_i (a_i,b_i)$ (up to a negligible set); our aim is to prove that there exists j

such that $a_i, b_i < a_j$ for all $i \neq j$. In this case, $a = a_j$. Suppose the contrary, i.e., $\forall j$ $\exists i \neq j$ such that $a_i > a_j$. With this assumption, we can build a sequence $(j_n)_n$, so that $(a_{j_n})_n$ is increasing, thus converging to some $y \in (0, L]$. By continuity of h, we have that $h(a_{j_n}) \to h(y) > 0$. We can compute the perimeter

$$\infty = \sum_{n \in \mathbb{N}} h(a_{j_n}) \leq \mathsf{P}_h(E) = \frac{N}{D}(\mathfrak{m}_h(E))^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(E)) < \infty,$$

which is a contradiction.

It remains to prove that h is increasing on $[0, b(E)]$. In order to simplify the notation, let $b := b(E)$. Denote by

$$t := \lim_{z \searrow 0} \frac{h(b+z)^{\frac{1}{N-1}} - h(b)^{\frac{1}{N-1}}}{z}$$

the right derivative of $h^{\frac{1}{N-1}}$ in b, which must exists because $h^{\frac{1}{N-1}}$ is concave. We want to prove that $t > 0$; from this and the fact that $h^{\frac{1}{N-1}}$ is concave, it will follow that h is strictly increasing in $[0, b]$. Suppose the contrary, i.e., $t \leq 0$. Then, by concavity of $h^{\frac{1}{N-1}}$, we have that

$$h(x) \leq h(b)\Big(\frac{D'-x}{D'-b}\Big)^{N-1} \quad \forall x \in [0, b], \quad \text{and} \quad h(x) \leq h(b) \quad \forall x \in [b, D'].$$

If we integrate, we obtain

$$\begin{aligned}
1 &\leq \int_0^b h(b)\Big(\frac{D'-x}{D'-b}\Big)^{N-1} dx + \int_b^{D'} h(b)\, dx \\
&= \frac{h(b)}{N}\Big(\frac{D'^N - (D'-b)^N}{(D'-b)^{N-1}} + N(D'-b)\Big) \\
&\leq \frac{\mathsf{P}_h(E)}{N}\Big(\frac{D'^N}{(D'-b)^{N-1}} + ND'\Big) = \frac{\mathsf{P}_h(E)D'}{N}\Big(\Big(1 - \frac{b}{D'}\Big)^{1-N} + N\Big) \\
&= \frac{\mathsf{P}_h(E)D'}{N}\Big(1 + (N-1)\frac{b}{D'} + o\Big(\frac{b}{D'}\Big) + N\Big).
\end{aligned} \tag{5.7}$$

Consider the two factors in the right-hand side of the estimate above. The former is controlled by using the definition of residual

$$\frac{\mathsf{P}_h(E)D'}{N} \leq \frac{\mathsf{P}_h(E)D}{N} = \mathfrak{m}_h(E)^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(E)),$$

and if $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}_h^D(E)$ is bounded, then the term above goes to 0. Regarding the latter factor, we just need to prove that $\frac{b}{D'}$ is bounded,

$$\frac{b}{D'} \leq \frac{L}{D'} \leq \frac{L}{2L} = \frac{1}{2}.$$

If we put together these last two estimates, we obtain that the right-hand side of (5.7) converges to 0 as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}_h^D(E) \to 0$, whereas the left-hand side is equal to 1, obtaining a contradiction. ∎

What we have just proven is that there exists a connected component of E, which is extremal on the right; this component is precisely the interval $(a, b(E))$. We will denote by $a(E)$ the number a given by the proposition we just have proved. Since our estimates are infinitesimal expansions in the limit as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}_h^D(E) \to 0$, we will always assume that $\mathfrak{m}_h(E) \leq \bar{w}$ and $\mathrm{Res}_h^D(E) \leq \bar{\delta}$, so that the expression $a(E)$ makes sense. We will make an extensive use of the fact that h is increasing in the interval $[0, b(E)]$: since, again, our estimates are in the limit as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}_h^D(E) \to 0$, the fact that h is increasing in $[0, b(E)]$ will be taken into account, without explicitly referring to the previous lemma.

We now prove that this component $(a(E), b(E))$ tends to fill the set E and that $b(E)$ converges as expected to $D\mathfrak{m}_h(E)^{\frac{1}{N}}$.

Proposition 5.6. *Fix $N > 1$ and $L > 0$. The following estimates hold for $w \to 0$ and $\delta \to 0$:*

$$b(E) \leq Dw^{\frac{1}{N}} + Do(w^{\frac{1}{N}}), \tag{5.8}$$

$$b(E) \geq Dw^{\frac{1}{N}} - Do(w^{\frac{1}{N}}), \tag{5.9}$$

$$a(E) \leq Do(w^{\frac{1}{N}}), \tag{5.10}$$

where $D \geq 3L$, $D' \in (0, D]$, $h\colon [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \leq \delta$.

Proof. Part 1. *Inequality* (5.9). Since the density h is strictly increasing on $[0, b(E)]$ and $E \subset [0, b(E)]$ (up to a null measure set), we have that $r_h(w) \leq b(E)$ and

$$\mathrm{Res}_h^D(w) = \frac{Dh(r_h(w))}{Nw^{1-\frac{1}{N}}} - 1 \leq \frac{Dh(b(E))}{Nw^{1-\frac{1}{N}}} - 1 \leq \frac{D\mathsf{P}_h(E)}{Nw^{1-\frac{1}{N}}} - 1 = \mathrm{Res}_h^D(E) \leq \delta.$$

We now exploit Proposition 5.4 (in particular, estimate (5.4)), yielding

$$D(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})) \leq r_h(w) \leq b(E),$$

and we have concluded the proof of (5.9).

Part 2. *Inequality* (5.10). First we prove that $a(E) < r_h(w)$ for w and δ small enough. Suppose the contrary, i.e., that $a(E) \geq r_h(w)$. This implies that $h(a(E)) \geq h(r_h(w))$, hence

$$\mathsf{P}_h(E) \geq 2h(r_h(w)).$$

We deduce that

$$\begin{aligned}-O(w^{\frac{1}{N}}) \leq \mathrm{Res}_h^D(w) &= \frac{Dh(r_h(w))}{Nw^{1-\frac{1}{N}}} - 1 \leq \frac{D\mathsf{P}_h(E)}{2Nw^{1-\frac{1}{N}}} - 1 \\ &= \frac{1}{2}(\mathrm{Res}_h^D(E) - 1) \leq \frac{\delta - 1}{2}.\end{aligned}$$

If we take the limit as $w \to 0$ and $\delta \to 0$, we obtain a contradiction.

We exploit the Bishop–Gromov inequality and the isoperimetric inequality (respectively) to obtain

$$h(a(E)) \geq h(r_h(w))\Big(\frac{a(E)}{r_h(w)}\Big)^{N-1},$$
$$h(b(E)) \geq h(r_h(w)) \geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}})).$$

Putting together the inequalities above and using the definition of residual, we obtain

$$\begin{aligned}\frac{N}{D} w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(E)) = \mathsf{P}_h(E) &\geq h(b(E)) + h(a(E)) \geq h(r_h(w)) + h(a(E))\\ &\geq h(r_h(w))\Big(1 + \Big(\frac{a(E)}{r_h(w)}\Big)^{N-1}\Big)\\ &\geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}}))\Big(1 + \Big(\frac{a(E)}{r_h(w)}\Big)^{N-1}\Big),\end{aligned}$$

yielding

$$\begin{aligned}a(E) &\leq r_h(w)\Big(\frac{1 + \mathrm{Res}_h^D(E)}{1 + O(w^{\frac{1}{N}})} - 1\Big)^{\frac{1}{N-1}} \leq r_h(w)((1+\delta)(1 - O(w^{\frac{1}{N}})) - 1)^{\frac{1}{N-1}}\\ &\leq r_h(w) o(1) \leq D w^{\frac{1}{N}}(1 + o(1)) o(1) = D o(w^{\frac{1}{N}}),\end{aligned}$$

where estimate (5.3) was taken into account. This concludes the proof of (5.10).

Part 3. *Inequality* (5.8). To simplify the notation, in the following integrals, we omit writing the differentials: the integrals are performed with the respect to the Lebesgue measure. Since

$$\int_E h = \int_0^{r_h(w)} h,$$

we can deduce (together with the fact that $a(E) \leq r_h(w) \leq b(E)$)

$$\begin{aligned}\int_{E \cap [0, r_h(w)]} h + \int_{r_h(w)}^{b(E)} h &= \int_{E \cap [0, r_h(w)]} h + \int_{[0, r_h(w)] \setminus E} h\\ &= \int_{E \cap [0, r_h(w)]} h + \int_{[0, a(E)] \setminus E} h,\end{aligned}$$

hence

$$(b(E) - r_h(w)) h(r_h(w)) \leq \int_{r_h(w)}^{b(E)} h = \int_{[0, a(E)] \setminus E} h \leq \int_0^{a(E)} h \leq a(E) h(a(E)),$$

yielding

$$b(E) - r_h(w) \leq a(E) \frac{h(a(E))}{h(r_h(w))} \leq a(E).$$

We conclude by combining the inequality above with estimate (5.9) and estimate (5.3) from Proposition 5.4. ∎

5.4. Almost rigidity of the density h

We now prove that the density h converges to the density of the model space $\frac{Nx^{N-1}}{D^N}$. Relying on the Bishop–Gromov inequality, we obtain an estimate of h from below.

Proposition 5.7. *Fix $N > 1$ and $L > 0$. The following estimate holds for $w \to 0$ and $\delta \to 0$:*

$$h(x) \geq \frac{N}{D^N} x^{N-1}(1 - o(1)), \quad \textit{uniformly w.r.t. } x \in [0, b(E)], \tag{5.11}$$

where $D \geq 3L$, $D' \in (0, D]$, $h\colon [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \leq \delta$.

Proof. Fix $x \in [0, b(E)]$. We can compute, using the Bishop–Gromov inequality,

$$h(x) \geq h(b(E)) \frac{x^{N-1}}{b(E)^{N-1}} \geq h(r_h(w)) \frac{x^{N-1}}{b(E)^{N-1}}.$$

The first factor is controlled by using the isoperimetric inequality

$$h(r_h(w)) \geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}})) = \frac{N}{D} w^{1-\frac{1}{N}}(1 - o(1)).$$

For the term $b(E)$, we use estimate (5.8)

$$b(E) \leq D w^{\frac{1}{N}}(1 + o(1)).$$

The thesis follows from the combination of these last two inequalities. ∎

Before going on, we prove the following, purely technical lemma.

Lemma 5.8. *Fix $N > 1$ and consider the function $f\colon [0, 1) \times [0, \infty] \to \mathbb{R}$ given by*

$$f(t, \eta) = \frac{1 + \eta - t^N}{1 - t}.$$

Define the function g by

$$g(\eta) = \sup\{t - s : f(t, 0) \leq f(s, \eta)\}.$$

Then $\lim_{\eta \to 0} g(\eta) = 0$.

Proof. The proof is by contradiction. Suppose that there exist $\varepsilon > 0$ and three sequences $(\eta_n)_n$, $(t_n)_n$ and $(s_n)_n$ such that $\eta_n \to 0$, $f(t_n, 0) \leq f(s_n, \eta_n)$ and $t_n - s_n > \varepsilon$. Up to taking a subsequence, we can assume that $t_n \to t$ and $s_n \to s$, hence $1 \geq t \geq s + \varepsilon$. The functions $f(\cdot, \eta_n)$ converge to $f(\cdot, 0)$, uniformly in the interval $[0, 1 - \frac{\varepsilon}{2}]$. This implies $f(s_n, \eta_n) \to f(s, 0)$, yielding $f(t, 0) \leq f(s, 0)$. Since $t \mapsto f(t, 0)$ is strictly increasing, we obtain $t \leq s \leq t - \varepsilon$, which is a contradiction. ∎

We now obtain an estimate of h from above in the interval $[a(E), b(E)]$ going in the opposite direction of the Bishop–Gromov inequality.

Proposition 5.9. *Fix $N > 1$ and $L > 0$. The following estimate holds for $w \to 0$ and $\delta \to 0$:*

$$h(x) \le h(b(E))\Big(\frac{x}{b(E)} + o(1)\Big)^{N-1}, \quad \text{uniformly w.r.t. } x \in [a(E), b(E)], \tag{5.12}$$

where $D \ge 3L$, $D' \in (0, D]$, $h\colon [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \le \delta$.

Proof. Fix $x \in [a(E), b(E)]$ and, in order to simplify the notation, define

$$a := a(E), \quad b := b(E), \quad k := h(x)^{\frac{1}{N-1}}, \quad l := h(b(E))^{\frac{1}{N-1}}.$$

By concavity of $h^{\frac{1}{N-1}}$, it holds true that

$$\begin{aligned} h(y) &\ge \Big(\frac{y}{x}\Big)^{N-1} k^{N-1} \quad \forall y \in [a, x],\\ h(y) &\ge \Big(l + (k-l)\frac{b-y}{b-x}\Big)^{N-1} \quad \forall y \in [x, b]. \end{aligned}$$

We can integrate these two inequalities, obtaining

$$\begin{aligned} w &\ge \int_a^x \frac{y^{N-1}}{x^{N-1}} k^{N-1} dy + \int_x^b \Big(l + (k-l)\frac{b-y}{b-x}\Big)^{N-1} dy\\ &= \frac{k^{N-1}\,(x^N - a^N)}{N x^{N-1}} + \frac{b-x}{N}\frac{l^N - k^N}{l-k}, \end{aligned}$$

yielding

$$\begin{aligned} \frac{1-(\frac{k}{l})^N}{1-\frac{k}{l}} &\le \frac{Nw - \frac{k^{N-1}(x^N-a^N)}{x^{N-1}}}{l^{N-1}(b-x)} = \frac{\frac{Nw}{bl^{N-1}} - \frac{k^{N-1}(x^N-a^N)}{b(lx)^{N-1}}}{1-\frac{x}{b}}\\ &\le \frac{\frac{Nw}{bl^{N-1}} - \frac{x^N-a^N}{b^N}}{1-\frac{x}{b}} = \frac{\frac{Nw}{bl^{N-1}} + \frac{a^N}{b^N} - \frac{x^N}{b^N}}{1-\frac{x}{b}}, \end{aligned}$$

where in the last inequality we used the Bishop–Gromov inequality written in the form $\frac{k^{N-1}}{l^{N-1}} \ge \frac{x^{N-1}}{b^{N-1}}$. At this point, we estimate the terms $\frac{Nw}{bl^{N-1}}$ and $\frac{a^N}{b^N}$. Regarding the former, taking into account (5.9) and the isoperimetric inequality, we notice

$$\begin{aligned} \frac{Nw}{bl^{N-1}} &= \frac{Nw}{b(E)\,h(b(E))} \le \frac{Nw}{b(E)\,h(r_h(w))}\\ &\le \frac{Nw}{Dw^{\frac{1}{N}}(1-o(1))\frac{N}{D}w^{1-\frac{1}{N}}(1-O(w^{\frac{1}{N}}))} = 1 + o(1). \end{aligned}$$

The latter term is even more simple (recall (5.8) and (5.10)),

$$\frac{a^N}{b^N} = \frac{a(E)^N}{b(E)^N} \le \frac{D^N o(w)}{D^N w(1-o(1))^N} = o(1).$$

We can put all the pieces together obtaining

$$f\left(\frac{k}{l},0\right)=\frac{1-(\frac{k}{l})^N}{1-\frac{k}{l}}\le\frac{\frac{Nw}{bl^{N-1}}+\frac{a^N}{b^N}-\frac{x^N}{b^N}}{1-\frac{x}{b}}\le\frac{1+o(1)-\frac{x^N}{b^N}}{1-\frac{x}{b}}=f\left(\frac{x}{b},o(1)\right),$$

where f is the function of Lemma 5.8. We can apply said lemma (and in particular (5.8)), and we get

$$\frac{k}{l}-\frac{x}{b}\le g(o(1))=o(1).$$

If we explicit the definitions of k, l, and b, it turns out that the inequality above is precisely the thesis. ■

5.5. *Rescaling the diameter and renormalizing the measure*

We now obtain a first limit estimate of the density h. The presence of factor $\frac{1}{D^N}$ in estimate (5.11) suggests the need of a rescaling to get a non-trivial limit estimate. We will rescale by $\frac{1}{b(E)}$ and renormalize the measure by $\mathfrak{m}_h(E)$.

Fix $k>0$ and define the rescaling transformation $S_k(x)=\frac{x}{k}$. If $h\colon[0,D']\to\mathbb{R}$ is a density and $E\subset[0,L]$, we can define

$$\nu_{h,E}=(S_{b(E)})_\#\left(\frac{\mathfrak{m}_h\llcorner E}{\mathfrak{m}_h(E)}\right)\in\mathscr{P}([0,1]).$$

The probability measure $\nu_{h,E}$ is absolutely continuous with respect to $\mathscr{L}^1$. Denote by $\tilde{h}_E\colon[0,1]\to\mathbb{R}$ the Radon–Nikodym derivative $\frac{d\nu_{h,e}}{d\mathscr{L}^1}$. The density $\tilde{h}_E$ can be computed explicitly

$$\tilde{h}_E(t)=\mathbf{1}_E(b(E)t)\frac{b(E)}{\mathfrak{m}_h(E)}h(b(E)t). \tag{5.13}$$

Since E could be disconnected, the indicator function in (5.13) prevents $\tilde{h}_E^{\frac{1}{N-1}}$ from being concave and therefore $([0,1],|\cdot|,\nu_{h,E})$ from satisfying the $\mathsf{CD}(0,N)$ condition.

Proposition 5.10. *Fix $N>1$ and $L>0$. The following estimate holds for $w\to0$ and $\delta\to0$:*

$$\|\tilde{h}_E-Nt^{N-1}\|_{L^\infty(0,1)}\le o(1),$$

where $D\ge3L$, $D'\in(0,D]$, $h\colon[0,D']\to\mathbb{R}$ is a $\mathsf{CD}(0,N)$ density, and the set $E\subset[0,L]$ satisfies $\mathfrak{m}_h(E)=w$ and $\mathrm{Res}_h^D(E)\le\delta$.

Proof. Fix $t\in[0,1]$. The proof is divided into four parts.

Part 1. *Estimate from below and $t>\frac{a(E)}{b(E)}$.* Since $t>\frac{a(E)}{b(E)}$, then $tb(E)\in E$ (for a.e. t). By a direct computation, we have

$$\begin{aligned}\tilde{h}_E(t)&=\frac{b(E)}{w}h(tb(E))\ge\frac{Nb(E)^N}{D^Nw}t^{N-1}(1-o(1))\\&\ge\frac{ND^Nw(1+o(1))^N}{D^Nw}t^{N-1}(1-o(1))=Nt^{N-1}-Nt^{N-1}o(1),\end{aligned}$$

where we have used estimate (5.11), with $x = tb(E)$, in the first inequality and (5.9) in the first and second inequalities, respectively. Since $t \in [0, 1]$, then $-t^{N-1}o(1) \geq o(1)$ and we conclude this first part.

Part 2. *Estimate from below and* $t \leq \frac{a(E)}{b(E)}$. In this case, it may happen that $tb(E) \notin E$, so the best we can say about $\widetilde{h}_E$ is that it is nonnegative in t. The point here is to exploit the fact that the interval $[0, \frac{a(E)}{b(E)}]$ is "short" and that $t \leq \frac{a(E)}{b(E)}$. By a direct computation (we recall (5.9) and (5.10)), we have

$$\begin{aligned}\widetilde{h}_E(t) &\geq 0 \geq Nt^{N-1} - Nt^{N-1} \geq Nt^{N-1} - N\frac{a(E)^{N-1}}{b(E)^{N-1}}\\ &\geq Nt^{N-1} - N\frac{D^{N-1}o(w^{1-\frac{1}{N}})}{D^{N-1}w^{1-\frac{1}{N}}(1+o(1))^{N-1}} \geq Nt^{N-1} - o(1).\end{aligned}$$

Part 3. *Estimate from above and* $t > \frac{a(E)}{b(E)}$. We take into account estimate (5.12) with $x = tb(E)$ and compute

$$\begin{aligned}\widetilde{h}_E(t) &= \frac{b(E)}{w}h(tb(E)) \leq \frac{b(E)}{w}h(b(E))(t+o(1))^{N-1}\\ &\leq \frac{b(E)}{w}h(b(E))(t^{N-1}+o(1)) \leq \frac{Dw^{\frac{1}{N}}(1+o(1))}{w}\mathsf{P}_h(E)(t^{N-1}+o(1))\\ &= \frac{Dw^{\frac{1}{N}}(1+o(1))}{w}\frac{N}{D}w^{1-\frac{1}{N}}(1+\mathrm{Res}_h^D(E))(t^{N-1}+o(1))\\ &\leq N(1+o(1))(1+\delta)(t^{N-1}+o(1)) = Nt^{N-1}+o(1)\end{aligned}$$

(in the second inequality, we exploited the uniform continuity of $t \in [0, 1] \mapsto t^{N-1}$; in the third one, estimate (5.8)).

Part 4. *Estimate from above and* $t \leq \frac{a(E)}{b(E)}$. Fix $\varepsilon > 0$ and compute

$$\begin{aligned}\widetilde{h}_E(t) &= b(E)\frac{\mathbf{1}_E(tb(E))}{\mathfrak{m}_h(E)}h(b(E)t) \leq \frac{b(E)}{\mathfrak{m}_h(E)}h(b(E)t)\\ &\leq \frac{b(E)}{\mathfrak{m}_h(E)}h\Big(b(E)\Big(\frac{a(E)}{b(E)}+\varepsilon\Big)\Big) = \widetilde{h}_E\Big(\frac{a(E)}{b(E)}+\varepsilon\Big),\end{aligned}$$

and the last equality holds true for a.e. ε small enough. At this point, we can take into account the previous part and continue

$$\widetilde{h}_E(t) \leq \widetilde{h}_E\Big(\frac{a(E)}{b(E)}+\varepsilon\Big) \leq N\Big(\frac{a(E)}{b(E)}+\varepsilon\Big)^{N-1} + o(1).$$

If we take the limit as $\varepsilon \to 0$, we can conclude

$$\widetilde{h}_E(t) \leq N\Big(\frac{a(E)}{b(E)}\Big)^{N-1} + o(1) \leq o(1) \leq Nt^{N-1} + o(1). \qquad \blacksquare$$

The following theorem summarizes the content of this section.

Theorem 5.11. *Fix $N > 1$ and $L > 0$. Then there exists a function $\omega\colon \mathrm{Dom}(\omega) \subset (0,\infty) \times \mathbb{R} \to \mathbb{R}$, infinitesimal in 0, such that the following holds. For all $D \geq 3L$, $D' \in (0, D)$, for all $h\colon [0, D'] \to \mathbb{R}$ a $\mathsf{CD}(0, N)$ density, and for all $E \subset [0, L]$, it holds*

$$|b(E) - D\mathfrak{m}_h(E)^{\frac{1}{N}}| \leq D\mathfrak{m}_h(E)^{\frac{1}{N}}\omega(\mathfrak{m}_h(E), \mathrm{Res}_h^D(E)), \tag{5.14}$$

$$\|\tilde{h}_E - Nt^{N-1}\|_{L^\infty} \leq \omega(\mathfrak{m}_h(E), \mathrm{Res}_h^D(E)), \tag{5.15}$$

where $b(E) = \operatorname{ess\,sup} E$ and

$$\tilde{h}_E = \frac{1}{\mathfrak{m}_h}\frac{d(S_{b(E)})_\#\mathfrak{m}_h\llcorner_E}{d\mathcal{L}^1} \quad \text{with } S_{b(E)}(x) = \frac{x}{b(E)}.$$

6. Passage to the limit as $R \to \infty$

We now go back to studying the identity case of the isoperimetric inequality, where E is a bounded Borel set such that

$$\mathsf{P}(E) = N(\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}\mathfrak{m}(E)^{1-\frac{1}{N}},$$

where $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching $\mathsf{CD}(0, N)$ space having $\mathsf{AVR}_X > 0$.

We make use of the notation of Section 3; denote by φ_R the Kantorovich potential associated to f_R and (3.1). Since the construction does not change if we add a constant to φ_R, we can assume that φ_R is equibounded on every bounded set. Using the Ascoli–Arzelà theorem and a diagonal argument, we deduce that, up to subsequences, φ_R converges to a certain 1-Lipschitz function φ_∞, uniformly on every bounded set.

We recall the disintegration given by Proposition 3.1,

$$\mathfrak{m}\llcorner_{\hat{\mathcal{T}}_R} = \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}\,\hat{\mathfrak{q}}_R(d\alpha) \quad \text{and} \quad \mathsf{P}(E;\cdot) \geq \int_{Q_R} \mathsf{P}_{\hat{X}_{\alpha,R}}(E;\cdot)\hat{\mathfrak{q}}_R(d\alpha). \tag{6.1}$$

We would like to take the limit in the disintegration formula (6.1). To the knowledge of the authors, there is no easy way to take such a limit. For this reason, the effort of this section goes in the direction to understand how the properties of the disintegration behave at the limit.

6.1. Passage to the limit of the radius

We start by defining the *radius* function $r_R\colon \bar{E} \to [0, \operatorname{diam} E]$. Fix $x \in E \cap \hat{\mathcal{T}}_R$, and let $E_{x,R} := (g_R(\mathfrak{Q}_R(x), \cdot))^{-1}(E) \subset [0, |\hat{X}_{\mathfrak{Q}_R(x),R}|]$. Define

$$r_R(x) := \begin{cases} \operatorname{ess\,sup} E_{x,R} & \text{if } x \in E \cap \hat{\mathcal{T}}_R, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Notice that $r_R(x) = b(E_{x,E})$, where the notation $b(E)$ was introduced in Section 5.3.

The function r_R is defined on $\bar{E}$ for two motivations: we require a common domain to not depend on R and the domain must be a compact metric space.

Remark 6.1. The set $E \cap \widehat{\mathcal{T}}_R$ has full $\mathfrak{m}\llcorner_E$-measure in $\bar{E}$. This means that it does not really matter how r_R is defined outside $E \cap \widehat{\mathcal{T}}_R$. This fact is particularly relevant because we will only take limits in the $\mathfrak{m}\llcorner_E$-a.e. sense or in senses which are weaker than the pointwise convergence.

The next proposition ensures that, in the limit as $R \to \infty$, the function r_R converges to the constant $(\frac{\mathfrak{m}_h(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$, which is precisely the radius that we expect.

Proposition 6.2. *Up to subsequences, it holds true*

$$\lim_{R\to\infty} r_R = \Big(\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\Big)^{\frac{1}{N}}, \quad \mathfrak{m}\llcorner_E\text{-}a.e.$$

Proof. By Corollary 4.6, we have that $\|\mathrm{Res}_{R,\mathfrak{Q}_R(x)}\|_{L^1(\bar{E};\mathfrak{m}\llcorner_E)} \to 0$, as $R \to \infty$, hence there exist a negligible subset $N \subset E$ and a sequence $R_n \to \infty$, such that

$$\lim_{n\to\infty} \mathrm{Res}_{x,R_n} = 0 \quad \text{for all } x \in E\backslash N.$$

Define $G := \bigcap_n \widehat{\mathcal{T}}_{R_n} \backslash N$ and notice that $\mathfrak{m}(E\backslash G) = 0$. Now fix $n \in \mathbb{N}$ and $x \in G$, and let $\alpha := \mathfrak{Q}_{R_n}(x) \in Q_{R_n}$. By triangular inequality, it holds

$$\Big| r_{R_n}(x) - \Big(\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\Big)^{\frac{1}{N}} \Big| \le \Big| r_{R_n}(x) - (R_n + \mathrm{diam}\, E)\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\Big)^{\frac{1}{N}} \Big| + \Big| (R_n + \mathrm{diam}\, E)\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\Big)^{\frac{1}{N}} - \Big(\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\Big)^{\frac{1}{N}} \Big|,$$

and the second term goes to 0 by definition of AVR.

Let us focus on the first term. Consider the ray $(\widehat{X}_{\alpha,R_n}, \mathsf{d}, \hat{\mathfrak{m}}_{\alpha,R_n})$. By definition, we have that

$$\mathrm{Res}_{h_{\alpha,R_n}}^{R_n+\mathrm{diam}\, E}(E_{x,R_n}) = \mathrm{Res}_{\alpha,R_n}.$$

We are in position to use Theorem 5.11 and, in particular, estimate (5.14) implies

$$\begin{aligned}
&\Big| r_{R_n}(x) - (R_n + \mathrm{diam}\, E)\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\Big)^{\frac{1}{N}} \Big| \\
&\quad = \Big| r_{R_n}(x) - (R_n + \mathrm{diam}\, E)(\mathfrak{m}_{h_{\alpha,R_n}}(E_{x,R_n}))^{\frac{1}{N}} \Big| \\
&\quad \le (R_n + \mathrm{diam}\, E)\mathfrak{m}_{h_{\alpha,R_n}}(E)^{\frac{1}{N}} \omega(\mathfrak{m}_{h_{\alpha,R_n}}(E), \mathrm{Res}_{h_{\alpha,R_n}}^{R_n+\mathrm{diam}\, E}(E_{x,R_n})) \\
&\quad = (R_n + \mathrm{diam}\, E)\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\Big)^{\frac{1}{N}} \omega\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}, \mathrm{Res}_{\mathfrak{Q}_R(x),R_n}\Big).
\end{aligned}$$

Since the right-hand side in the inequality above is infinitesimal, we can take the limit as $n \to \infty$ and conclude. ∎

Hence, in the limit the length of the rays converges to a well-defined constant; this will turn out to be the radius of E. From now on, we will write $\rho := (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$.

6.2. Passage to the limit of the rays

Consider now a constant-speed parametrization of the rays inside E:

$$\gamma_s^{x,R} := \begin{cases} g_R(\mathfrak{Q}_R(x), s\, r_R(x)) & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ x & \text{otherwise,} \end{cases}$$

where $x \in \bar{E}$ and $s \in [0,1]$. Remark 6.1 also applies to the map $x \mapsto \gamma^{x,R}$. A direct consequence of the definition of $\gamma^{x,R}$ is

$$\mathsf{d}(\gamma_t^{x,R}, \gamma_s^{x,R}) = \varphi_R(\gamma_t^{x,R}) - \varphi_R(\gamma_s^{x,R}) \quad \forall 0 \le t \le s \le 1, \text{ for } \mathfrak{m}\text{-a.e. } x \in E, \tag{6.3}$$

$$\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R}) = r_R(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E, \tag{6.4}$$

$$x \in \gamma^{x,R} \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E. \tag{6.5}$$

We stress out the order of the quantifiers in (6.3): the said equation has to be understood in the sense that $\exists N \subset E$ such that $\mathfrak{m}(N) = 0$ and $\forall t \le s$, $\forall x \in E \setminus N$, (6.3) holds true. Regarding (6.5), we point out that the expression $x \in \gamma^{x,R}$ means that $\exists t \in [0,1]$ such that $x = \gamma_t^{x,R}$, or, equivalently, $\min_{t\in[0,1]} \mathsf{d}(x, \gamma_t^{x,R}) = 0$.

In order to capture the limit behavior of $\gamma^{x,R}$ as $R \to \infty$, we proceed as follows. First define $K := \{\gamma \in \mathrm{Geo}(X) : \gamma_0, \gamma_1 \in \bar{E}\}$. Since a $\mathsf{CD}(K,N)$ space is locally compact and E is bounded, $\bar{E}$ is compact and so is K. Then define the measure

$$\tau_R := (\mathrm{Id} \times \gamma^{\cdot,R})_\# \mathfrak{m}\llcorner_E \in \mathcal{M}(\bar{E} \times K).$$

The measure τ_R has mass $\mathfrak{m}(E)$ and enjoys the following immediate properties:

$$(P_1)_\# \tau_R = \mathfrak{m}\llcorner_E \quad \text{and} \quad \gamma = \gamma^{x,R} \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \bar{E} \times K.$$

We can restate properties (6.3)–(6.5) using a more measure-theoretic language

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_R(e_t(\gamma)) + \varphi_R(e_s(\gamma)) = 0 \quad \forall\, 0 \le t \le s \le 1, \tag{6.6}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x) = 0, \tag{6.7}$$

$$x \in \gamma, \tag{6.8}$$

for τ_R-a.e. $(x,\gamma) \in \bar{E} \times K$. Since the measures τ_R have the same mass and $\bar{E} \times K$ is compact, the family of measures $(\tau_R)_{R>0}$ is tight, thus we can extract a subsequence (which we do not relabel) such that $\tau_R \rightharpoonup \tau$ weakly, i.e., $\int_{\bar{E}\times K} \psi\, d\tau_R \to \int_{\bar{E}\times K} \psi\, d\tau$, for all $\psi \in C_b(\bar{E} \times K)$.

The next proposition affirms that properties (6.6)–(6.8) pass to the limit as $R \to \infty$.

Proposition 6.3. *For τ-a.e. $(x,\gamma) \in \bar{E} \times K$, it holds that*

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)) \quad \forall\, 0 \le t \le s \le 1, \tag{6.9}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{6.10}$$

$$x \in \gamma. \tag{6.11}$$

Proof. Fix $t \le s$ and integrate (6.6) in $\bar{E} \times K$, obtaining

$$0 = \int_{\bar{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_R(e_t(\gamma)) + \varphi_R(e_s(\gamma)))\tau_R(dx\, d\gamma) = \int_{\bar{E}\times K} L^{t,s}_{\varphi_R}(\gamma)\tau_R(dx\, d\gamma),$$

where we have set

$$L^{t,s}_{\psi}(\gamma) := \mathsf{d}(e_t(\gamma), e_s(\gamma)) - \psi(e_t(\gamma)) + \psi(e_s(\gamma)).$$

The map $L^{t,s}_{\varphi_R}\colon K \to \mathbb{R}$ is clearly continuous and converges uniformly (recall that $\varphi_R \to \varphi_\infty$ uniformly on every compact) to $L^{t,s}_{\varphi_\infty}$. For this reason, we can take the limit in the equation above obtaining

$$0 = \int_{\bar{E}\times K} L^{t,s}_{\varphi_\infty}(\gamma)\tau(dx\, d\gamma) = \int_{\bar{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_\infty(e_t(\gamma)) + \varphi_\infty(e_s(\gamma)))\tau(dx\, d\gamma).$$

The 1-lipschitzianity of φ_∞ yields $L^{t,s}_{\varphi_\infty}(\gamma) \ge 0\ \forall \gamma \in K$, hence

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)) \quad \text{for } \tau\text{-a.e. } (x,\gamma) \in \bar{E} \times K.$$

In order to conclude, fix a countable dense subset $P \subset [0,1]$, and find a τ-negligible set $N \subset \bar{E} \times K$ such that

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)) \quad \forall t,s \in P, \text{ with } t \le s,\ \forall (x,\gamma) \in (\bar{E} \times K)\setminus N.$$

If we have $0 \le t \le s \le 1$, we approximate t and s with two sequences in P, and we can pass to the limit in the equation above concluding the proof of (6.9).

Now we prove (6.10). The idea is similar, but in this case we need to be more careful, because the function r_R fails to be continuous. Like before, we can integrate equation (6.7) obtaining

$$0 = \int_{\bar{E}\times X} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\tau_R(dx\, d\gamma).$$

If the functions r_R were continuous and converged uniformly to ρ, then we could pass to the limit and conclude. Unfortunately, Proposition 6.2 provides a limit only in the a.e. sense. We overcome this issue using Lusin's and Egorov's theorems. Fix $\varepsilon > 0$ and find a compact set $L \subset E$ such that

(1) the restrictions $r_R|_L$ are continuous;

(2) the restricted maps $r_R|_L$ converge uniformly to ρ;

(3) $\mathfrak{m}(E\setminus L) \le \varepsilon$.

We can now compute the limit

$$
\begin{aligned}
0 &= \lim_{R\to\infty} \int_{\bar{E}\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\tau_R(dx\,d\gamma) \\
&\geq \liminf_{R\to\infty} \int_{L\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\tau_R(dx\,d\gamma) \\
&\geq \int_{L\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - \rho|\tau(dx\,d\gamma) \geq 0,
\end{aligned}
$$

hence

$$
\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho \quad \text{for } \tau\text{-a.e. } (x,\gamma) \in L \times K.
$$

This means that the equation above holds true except for a set of measure at most ε. By arbitrariness of ε, we conclude the proof of (6.10). Finally, we prove (6.11). Consider the continuous, nonnegative function

$$
L(x,\gamma) := \inf_{t\in[0,1]} \mathsf{d}(x, e_t(\gamma)).
$$

Equation (6.8) implies

$$
0 = \int_{\bar{E}\times K} L(x,\gamma)\,\tau_R(dx\,d\gamma).
$$

The equation above passes to the limit as $R \to \infty$, hence we deduce $L(x,\gamma) = 0$ for τ-a.e. $(x,\gamma) \in \bar{E} \times K$, which is precisely (6.8). ∎

6.3. Disintegration of the measure and the perimeter

Recalling the disintegration formula (6.1), we define the map $\bar{E} \ni x \mapsto \mu_{x,R} \in \mathscr{P}(\bar{E})$ as

$$
\mu_{x,R} := \begin{cases} \dfrac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (\hat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R})\llcorner_E & \text{if } x \in E \cap \hat{\mathcal{T}}_R, \\ \delta_x & \text{otherwise.} \end{cases}
$$

This new family of measures satisfies the disintegration formula

$$
\mathfrak{m}\llcorner_E = \int_{\bar{E}} \mu_{x,R}\mathfrak{m}\llcorner_E(dx). \tag{6.12}
$$

Indeed, by a direct computation (recall (3.4)–(3.5))

$$
\begin{aligned}
\mathfrak{m}(A\cap E) &= \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}(A\cap E)\hat{\mathfrak{q}}_R(d\alpha) \\
&= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}(A\cap E)(\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)(d\alpha) \\
&= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_X \hat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R}(A\cap E)\mathfrak{m}\llcorner_E(dx) \\
&= \int_X \mu_{x,R}(A)\mathfrak{m}\llcorner_E(dx).
\end{aligned}
$$

Remark 6.4. We give a few details regarding the measurability of the integrand function in equation (6.12). The said equation should be interpreted in the following sense: the map $x \mapsto \mu_{x,R}(A)$ is measurable and formula (6.12) holds. Indeed, the map $x \mapsto \mu_{x,R}(A)$ is (up to excluding the negligible set $\bar{E}\backslash(E \cap \hat{\mathcal{T}}_R)$) the composition of

$$Q_R \ni \alpha \mapsto \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \hat{\mathfrak{m}}_{\alpha,R}(A \cap E)$$

and the projection $\mathfrak{Q}_R$. The former map is $\hat{\mathfrak{q}}_R$-measurable, while the map $\mathfrak{Q}_R$ is $\mathfrak{m}$-measurable, with respect to the σ-algebra of Q_R, thus the composition is measurable.

Since $\hat{\mathfrak{m}}_{\alpha,R} = (g_R(\alpha,\cdot))_\#(h_{\alpha,R}\mathcal{L}^1\llcorner_{[0,|\hat{X}_{\alpha,R}|]})$, we can explicitly compute the measure $\mu_{x,R}$ (recall that by (6.2) $r_R(x) = \operatorname{ess\,sup} E_{x,R}$ for $\mathfrak{m}\llcorner_E$-a.e. x)

$$\begin{aligned}
\mu_{x,R} &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (g_R(\mathfrak{Q}_R(x),\cdot))_\#((g_R(\mathfrak{Q}_R(x),\cdot))^{-1}(E) h_{\mathfrak{Q}_R(x),R}\mathcal{L}^1\llcorner_{[0,r_R(r)]}) \\
&= (g_R(\mathfrak{Q}_R(x),\cdot))_\#\Big(\mathbf{1}_{E_{x,R}} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} h_{\mathfrak{Q}_R(x),R}\mathcal{L}^1\llcorner_{[0,r_R(x)]}\Big) \\
&= (\gamma^{x,R})_\#(\tilde{h}_E^{x,R}\mathcal{L}^1\llcorner_{[0,1]}) \quad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \bar{E},
\end{aligned}$$

where

$$\tilde{h}_E^{x,R}(t) = \mathbf{1}_{E_{x,R}}(r_R(x)t) r_R(x) \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} h_{\mathfrak{Q}_R(x),R}(r_R(x)t).$$

Thanks to (3.6), we can perform a similar operation for the perimeter. Having in mind that $h_{R,\mathfrak{Q}_R(x)}(r_R(x))\delta_{r_R(x)} \le \mathsf{P}_{h_{R,\mathfrak{Q}_R(x)}}(E_{x,R};\cdot)$, we define the map

$$p_{x,R} := \begin{cases} \min\Big\{\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} h_{R,\mathfrak{Q}_R(x)}(r_R(x)), \frac{N}{\rho}\Big\}\delta_{g_R(\mathfrak{Q}_R(x),r_R(x))} & \text{if } x \in E \cap \hat{\mathcal{T}}_R, \\ \frac{N}{\rho}\delta_x & \text{if } x \in \bar{E}\backslash(E \cap \mathcal{T}_R). \end{cases}$$

Using the maps $\gamma^{x,R}$ and $\tilde{h}_{x,R}$, we can rewrite $p_{x,R}$ as

$$p_{x,R} = \begin{cases} \min\Big\{\frac{\tilde{h}_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R},\gamma_1^{x,R})}, \frac{N}{\rho}\Big\}\delta_{\gamma_1^{x,R}} & \text{if } x \in E \cap \hat{\mathcal{T}}_R, \\ \frac{N}{\rho}\delta_x & \text{if } x \in \bar{E}\backslash(E \cap \mathcal{T}_R). \end{cases}$$

The definition of $p_{x,R}$ immediately yields

$$p_{x,R} \le \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;\cdot) \quad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \bar{E},$$

hence we deduce the following "disintegration" formula (equations (3.6) and (3.5) are taken into account)

$$\begin{aligned}
\mathsf{P}(E;A) &\ge \int_{Q_R} \mathsf{P}_{X_{\alpha,R}}(E;A)\,\hat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_{\bar{E}} \mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;A)\,\mathfrak{m}\llcorner_E(dx) \\
&\ge \int_{\bar{E}} p_{x,R}(A)\mathfrak{m}(dx) \quad \forall A \subset \bar{E} \text{ Borel}.
\end{aligned} \tag{6.13}$$

Let $F := e_{(0,1)}(K) = \{\gamma_t : \gamma \in K, t \in [0,1]\}$, and let $S \subset \mathcal{M}^+(F)$ be the subset of the nonnegative measures on F with mass at most $\frac{N}{\rho}$. We endow the sets $\mathcal{P}(F)$ and S with the weak topology of measures. Since K and F are compact Hausdorff spaces, by Riesz–Markov representation theorem, the weak topology on $\mathcal{P}(F)$ and S coincides with the weak* topology induced by the duality against continuous functions $C(F)$. It is well known that the weak* convergence can be metrized on bounded sets if the primal space is separable. For instance, a possible suitable metric is given by

$$\mathsf{d}(\mu,\nu) = \sum_{k=1}^{\infty} \frac{1}{2^k \| f_k \|_\infty} \Big| \int_X f_k \, d\mu - \int_X f_k \, d\nu \Big|, \tag{6.14}$$

where $\{f_k\}_k$ is dense set in $C(X)$. We endow the spaces $\mathcal{P}(F)$ and S with the distance defined in (6.14).

Define the map $G_R \colon \bar{E} \times K \to \mathcal{P}(F) \times S$ as

$$G_R(x,\gamma) := \Big(\gamma_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]}), \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho}\Big\} \delta_{e_1(\gamma)}\Big).$$

The function G_R is measurable with respect to the variable x and continuous with respect to the variable γ. At this point, we can define the measure

$$\sigma_R := (\mathrm{Id} \times G_R)_\# \tau_R \in \mathcal{M}^+(\bar{E} \times K \times \mathcal{P}(F) \times S).$$

Notice that the mass of σ_R is $\mathfrak{m}(E)$ for all $R > 0$. In order to simplify the notation, set $Z = \bar{E} \times K \times \mathcal{P}(F) \times S$.

Proposition 6.5. *The measure σ_R satisfies the following properties:*

$$\begin{aligned} \int_E \psi \, d\mathfrak{m} &= \int_Z \int_E \psi(y)\mu(dy)\sigma_R(dx\, d\gamma\, d\mu\, dp) \ \ \forall \psi \in C_b^0(\bar{E}), \\ \int_{\bar{E}} \psi(y)\mathsf{P}(E,dy) &\geq \int_Z \int_{\bar{E}} \psi(y)p(dy)\sigma_R(dx\, d\gamma\, d\mu\, dp) \ \ \forall \psi \in C_b^0(\bar{E}),\ \psi \geq 0. \end{aligned} \tag{6.15}$$

Proof. Fix a test function $\psi \in C_b^0(\bar{E})$. First we notice that for σ_R-a.e. $(x,\gamma,\mu,p) \in Z$, we have that $\mu = \mu_{x,R}$. Indeed, it holds that

$$\mu = \gamma_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]}) = (\gamma^{x,R})_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]}) = \mu_{x,R} \quad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p) \in Z,$$

and we used the fact that $\gamma = \gamma_{x,R}$ for τ_R-a.e. $(x,\gamma) \in \bar{E} \times K$. We conclude this first part by a direct computation,

$$\begin{aligned} \int_E \psi \, d\mathfrak{m} &= \int_E \int_E \psi(y)\mu_{x,R}\mathfrak{m}(dx) = \int_Z \int_E \psi(y)\mu_{x,R}(dy)\sigma_R(dx\, d\gamma\, d\mu\, dp) \\ &= \int_Z \int_E \psi(y)\mu(dy)\sigma_R(dx\, d\gamma\, d\mu\, dp) \end{aligned}$$

that gives the proof of inequality (6.15).

Now fix an open set $\Omega \subset X$ and compute using (6.13)

$$\begin{aligned}\mathsf{P}(E;\Omega) &\geq \int_E \min\Big\{\frac{\widetilde{h}^E_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R},\gamma_1^{x,R})},\frac{N}{\rho}\Big\}\delta_{\gamma_1^{x,R}}(\Omega)\,d\,\mathfrak{m}(dx)\\ &= \int_Z \min\Big\{\frac{\widetilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma^{x,R}),e_1(\gamma^{x,R}))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma^{x,R})}(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp)\\ &= \int_Z \min\Big\{\frac{\widetilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma)}(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp).\end{aligned}$$

If we use the fact that

$$p = \min\Big\{\frac{\widetilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N(\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}}{\mathfrak{m}(E)^{\frac{1}{N}}}\Big\}\delta_{e_1(\gamma)}(\Omega)\quad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p)\in Z,$$

we continue the chain of inequalities obtaining

$$\begin{aligned}\mathsf{P}(E;\Omega) &\geq \int_Z \min\Big\{\frac{\widetilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma)}(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp)\\ &= \int_Z p(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp).\end{aligned}$$

Since the perimeter is outer-regular, i.e., $\mathsf{P}(E;A) = \inf\{\mathsf{P}(E;\Omega) : \Omega \supset A \text{ is open}\}$, we can conclude. ■

At this point, we are in position to take the limit as $R \to \infty$, as the properties we have proven pass to the limit, but before proceeding we prove the following technical lemma.

Lemma 6.6. *Let X be a complete and separable metric space, let Y, Z be two compact metric spaces, and let $\mathfrak{m}$ be a finite Radon measure on X. Consider a sequence of functions $f_n \colon X \times Y \to Z$ and $f \colon X \times Y \to Z$, such that f_n and f are Borel-measurable in the first variable and continuous in the second. Assume that for $\mathfrak{m}$-a.e. $x \in X$, the sequence $f_n(x,\cdot)$ converges uniformly to $f(x,\cdot)$. Consider a sequence of measures $\mu_n \in \mathcal{M}^+(X \times Y)$ such that $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{M}^+(X \times Y)$ and $(\pi_X)_\#\mu_n = \mathfrak{m}$. Then it holds*

$$(\mathrm{Id} \times f_n)_\#\mu_n \rightharpoonup (\mathrm{Id} \times f)_\#\mu, \quad \textit{weakly in } \mathcal{M}(X \times Y \times Z).$$

Proof. In order to simplify the notation, set $\nu_n = (\mathrm{Id} \times f_n)_\#\mu_n$ and $\nu = (\mathrm{Id} \times f)_\#\mu$. Fix $\varepsilon > 0$.

We would like to use an extension of Egorov's and Lusin's theorems for functions taking values in separable metric spaces. The reader can find a proof these theorems in [30, Theorem 7.5.1] (for Egorov's theorem) and [29, Appendix D] (for Lusin's theorem). In this setting, we deal with maps taking value in $C(Y, Z)$, the space of continuous functions between the compact spaces Y and Z, which turns out to be separable.

Using said theorems, we can find a compact $K \subset X$ such that

(1) the maps $x \in K \mapsto f_n(x,\cdot) \in C(Y,Z)$ are continuous (and the same holds for f in place of f_n);

(2) the restricted maps $x \in K \mapsto f_n(x,\cdot)$ converge to $x \in K \mapsto f(x,\cdot)$, uniformly in the space $C(K, C(Y,Z))$;

(3) $\mathfrak{m}(X \backslash K) \le \varepsilon$.

Regarding point (2), this immediately implies that the restriction $f_n|_{K\times Y} \to f|_{K\times Y}$ converges uniformly in $K \times Y$.

We check the convergence of ν_n using the test function $\varphi \in C_b^0(X \times Y \times Z)$,

$$\begin{aligned}\Big|\int_{X\times Y\times Z} \varphi\, d\nu_n - \int_{X\times Y\times Z}\varphi\, d\nu\Big| &\le \|\varphi\|_{C^0}(\nu_n((X\backslash K)\times Y\times Z) + \nu((X\backslash K)\times Y\times Z)) \\ &\quad + \Big|\int_{K\times Y\times Z}\varphi\, d\nu_n - \int_{K\times Y\times Z}\varphi\, d\nu\Big| \\ &= \|\varphi\|_{C^0}(\mathfrak{m}(X\backslash K)+\mathfrak{m}(X\backslash K)) + \Big|\int_{K\times Y\times Z}\varphi\, d\nu_n - \int_{K\times Y\times Z}\varphi\, d\nu\Big| \\ &\le 2\varepsilon\|\varphi\|_{C^0} + \Big|\int_{K\times Y\times Z}\varphi\, d\nu_n - \int_{K\times Y\times Z}\varphi\, d\nu\Big|.\end{aligned}$$

We focus on the second term and compute the integral

$$\int_{K\times Y\times Z}\varphi\, d\nu_n = \int_{K\times Y}\varphi(x,y,f_n(x,y))\,\mu_n(dx\,dy).$$

The function $\varphi|_{K\times Y\times Z}$ is uniformly continuous (because it is continuous and defined on a compact space), hence $\varphi(x,y,f_n(x,y))$ converges to $\varphi(x,y,f(x,y))$ uniformly in $K\times Y$. For this reason, together with the fact that $\mu_n \rightharpoonup \mu$ weakly, we can take the limit in the equation above obtaining

$$\begin{aligned}\lim_{n\to\infty}\int_{K\times Y}\varphi(x,y,f_n(x,y))\mu_n(dx\,dy) &= \int_{K\times Y}\varphi(x,y,f(x,y))\mu(dx\,dy) \\ &= \int_{K\times Y\times Z}\varphi\, d\nu,\end{aligned}$$

and this concludes the proof. ∎

Corollary 6.7. *Consider the function* $G\colon \overline{E}\times K \to \mathcal{P}(F)\times S$ *defined as*

$$G(x,\gamma) = \Big(\gamma_\#(Nt^{N-1}\mathcal{L}^1\llcorner_{[0,1]}), \max\Big\{\frac{N}{\mathsf{d}(e_0(\gamma),e_1(\gamma))}, \frac{N}{\rho}\Big\}\delta_{e_1(\gamma)}\Big),$$

and let $\sigma := (\mathrm{Id}\times G)_\#\tau$. *Then we have that* $\sigma_R \rightharpoonup \sigma$ *in the weak topology of measures.*

Proof. We just need to check the hypotheses of the previous lemma. The set $\overline{E}$ is compact, hence complete and separable. The set K is compact and so is $\mathcal{P}(F)\times S$ (with respect to the weak topology). As we have already pointed out, the maps G_R are measurable in the first variable and continuous in the second variable. Finally, we need to see that for a.e. x, the limit $G_R(x,\gamma) \to G(x,\gamma)$ holds uniformly with respect to γ. Fix x and γ and pick a test function $\psi \in C_b(F)$. Compute

$$\begin{aligned}\Big|\int_F \psi(y)\gamma_\#(\widetilde{h}_E^{x,R}\mathcal{L}^1\llcorner_{[0,1]})(dy) - \int_F \psi(y)\gamma_\#(Nt^{N-1}\mathcal{L}^1\llcorner_{[0,1]})(dy)\Big| \\ = \Big|\int_0^1 \psi(\gamma_t)(\widetilde{h}_E^{x,R} - Nt^{N-1})\,dt\Big| \le \|\psi\|_{C(F)}\|\widetilde{h}_E^{x,R} - Nt^{N-1}\|_{L^\infty}.\end{aligned}$$

The right-hand side of the inequality above does not depend on γ (but only on x and ψ) and converges to 0 by Theorem 5.11, in particular (5.15). This means that the first component of $G_R(x,\gamma)$ converges (in the weak topology of $\mathcal{P}(F)$), uniformly with respect to γ (see (6.14)). For the other component, the proof is analogous, so we omit it. ■

The next proposition reports all the relevant properties of the limit measure σ.

Proposition 6.8. *The measure σ satisfies the following disintegration formulas:*

$$\int_E \psi(y)\mathfrak{m}(dy) = \int_Z \int_0^1 \psi(e_t(\gamma))Nt^{N-1}\,dt\,\sigma(dx\,d\gamma\,d\mu\,dp) \tag{6.16}$$

for all $\psi \in L^1(E;\mathfrak{m}\llcorner_E)$, and

$$\int_{\bar{E}} \psi(y)\mathsf{P}(E;dy) = \frac{N}{\rho}\int_Z \psi(e_1(\gamma))\psi\,\sigma(dx\,d\gamma\,d\mu\,dp) \tag{6.17}$$

for all $\psi \in L^1(\bar{E};\mathsf{P}(E;\cdot))$. Furthermore, for σ-a.e. $(x,\gamma,\mu,p) \in Z$, it holds

$$\mathsf{d}(e_t(\gamma),e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)) \quad \forall 0 \le t \le s \le 1, \tag{6.18}$$

$$\mathsf{d}(e_0(\gamma),e_1(\gamma)) = \rho, \tag{6.19}$$

$$x \in \gamma, \tag{6.20}$$

$$\mu = \gamma_\#(Nt^{N-1}\mathcal{L}^1\llcorner_{[0,1]}), \tag{6.21}$$

$$p = \frac{N}{\rho}\delta_{e_1(\gamma)}. \tag{6.22}$$

Proof. Equations (6.18)–(6.20) are just a restatement of (6.9)–(6.11), respectively. Equation (6.21) is an immediate consequence of the definition of G. Similarly, taking into account (6.19), we can deduce (6.22),

$$p = \min\Big\{\frac{N}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma)} = \frac{N}{\rho}\delta_{e_1(\gamma)}.$$

In order to prove (6.16), fix $\psi \in C_b^0(F) = C_b^0(e_{(0,1)}(K))$ and define the function $L_\psi\colon \mathcal{P}(F) \to \mathbb{R}$ as

$$L_\psi(\mu) = \int_F \psi\,d\mu.$$

This function is bounded and continuous with respect to the weak topology of $\mathcal{P}(F)$. Hence, we take into account the definition of weak convergence of measures and compute the limit using (6.15) and (6.21),

$$\begin{aligned}\int_E \psi \, d\mathfrak{m} &= \lim_{R\to\infty} \int_Z \int_F \psi(y)\, \mu(dy)\sigma_R(dx\, d\gamma\, d\mu\, dp)\\ &= \lim_{R\to\infty} \int_Z L_\psi(\mu)\sigma_R(dx\, d\gamma\, d\mu\, dp)\\ &= \int_Z \int_F \psi(y)\mu(dy)\sigma(dx\, d\gamma\, d\mu\, dp)\\ &= \int_Z \int_0^1 \psi(e_t(\gamma))Nt^{N-1}\, dt\sigma(dx\, d\gamma\, d\mu\, dp).\end{aligned}$$

Using standard approximation arguments, we see that the equation above holds true also for any $\psi \in L^1(E; \mathfrak{m}\llcorner_E)$.

Regarding (6.17), using the same argument we can deduce that

$$\int_{\bar{E}} \psi(y)\mathsf{P}(E; dy) \geq \frac{N}{\rho} \int_Z \psi(e_1(\gamma))\sigma(dx\, d\gamma\, d\mu\, dp) \quad \forall \psi \in L^1(\bar{E}; \mathsf{P}(E; \cdot)),\ \psi \geq 0.$$

If we test the inequality above with $\psi = 1$, the inequality is satisfied meaning that the two measures have the same mass, so the inequality improves to an equality. ■

6.4. Back to the classical localization notation

We are now in position to re-obtain the "classical" disintegration formulas for both the measure $\mathfrak{m}$ and the perimeter of E.

We recall the definitions of some of the objects that were introduced in Section 2.4. For instance, let $\Gamma_\infty = \{(x, y) : \varphi_\infty(x) - \varphi_\infty(y) = \mathsf{d}(x, y)\}$ and $\mathcal{R}^e_\infty = \Gamma_\infty \cup \Gamma^{-1}_\infty$ be the transport relation. The transport set with endpoints is $\mathcal{T}^e_\infty := P_1(\mathcal{R}^e_\infty \setminus \{x = y\})$; clearly, $E \subset \mathcal{T}^e_\infty$, up to a negligible set. The sets of forward and backward branching points are defined as

$$\begin{aligned}A^+_\infty &:= \{x \in \mathcal{T}^e_\infty : \exists z, w \in \Gamma_\infty(x), (z, w) \notin \mathcal{R}^e_\infty\},\\ A^-_\infty &:= \{x \in \mathcal{T}^e_\infty : \exists z, w \in \Gamma^{-1}_\infty(x), (z, w) \notin \mathcal{R}^e_\infty\}.\end{aligned}$$

The transport set is defined as $\mathcal{T}_\infty := \mathcal{T}^e_\infty \setminus (A^+_\infty \cup A^-_\infty)$; since the sets A^+_∞ and A^-_∞ are negligible, then $\mathcal{T}_\infty$ has full measure in $\mathcal{T}^e_\infty$. Let Q_∞ be the quotient set and let $\mathfrak{Q}_\infty \colon \mathcal{T}_\infty \to Q_\infty$ be the quotient map; denote by $X_{\alpha,\infty} := \mathfrak{Q}^{-1}(\alpha)$ the disintegration rays and let $g_\infty \colon \mathrm{Dom}(g_\infty) \subset \mathbb{R} \times Q_\infty \to X$ be the parametrization of the rays such that $\frac{d}{dt}\varphi_\infty(g_\infty(t, \alpha)) = -1$. For every $\alpha \in Q_\infty$, let $t_\alpha \colon \overline{X_{\alpha,\infty}} \to [0, \infty)$ be the function $t_\alpha(x) := (g(\alpha, \cdot))^{-1} = \mathsf{d}(x, g_\infty(\mathfrak{Q}_\infty(x), 0))$; the function t_α measures how much points are translated from the starting point of the ray $X_{\alpha,\infty}$.

The following proposition guarantees that the geodesics, on which the measure σ is supported, lay on the transport set $\mathcal{T}_\infty$.

Proposition 6.9. *For σ-a.e. $(x, \gamma, \mu, p) \in Z$, it holds that $e_t(\gamma) \notin A^+_\infty \cup A^-_\infty$ for all $t \in (0, 1)$.*

Proof. Fix $\varepsilon > 0$ and let

$$P := \{(x, \gamma, \mu, p) \in Z : e_\varepsilon(\gamma) \in A^+_\infty \text{ and conditions (6.16)–(6.22) hold}\}.$$

Notice that by definition of A^+_∞, if $(x, \gamma, \mu, p) \in P$, then $\gamma_t \in A^+_\infty$, for all $t \in [0, \varepsilon]$, thus we can compute

$$\begin{aligned} 0 = \mathfrak{m}(A^+_\infty) &= \int_Z \int_0^1 \mathbf{1}_{A^+_\infty}(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \\ &\geq \int_P \int_0^\varepsilon \mathbf{1}_{A^+_\infty}(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \geq \varepsilon^N \sigma(P), \end{aligned}$$

so P is negligible. Fix now $(x, \gamma, \mu, p) \notin P$. By definitions of A^+_∞ and P, we have that $\gamma_t \notin A^+_\infty$, for all $t \in [\varepsilon, 1]$. By arbitrariness of ε, we deduce that for σ-a.e. $(x, \gamma, \mu, p) \in Z$, it holds that $e_t(\gamma) \notin A^+_\infty$, for all $t \in (0, 1]$. The proof for the set A^-_∞ is analogous. ∎

Corollary 6.10. *For σ-a.e. $(x, \gamma, \mu, p) \in Z$, it holds that $e_t(\gamma) \in \overline{X_{\mathfrak{Q}(x),\infty}}$ and*

$$e_t(\gamma) = g(\mathfrak{Q}(x), t_{\mathfrak{Q}(x)}(e_0(\gamma)) + \rho t). \tag{6.23}$$

Define $\hat{\mathfrak{q}} := \frac{1}{\mathfrak{m}(E)}(\mathfrak{Q}_\infty)_\#(\mathfrak{m}\llcorner_E) \ll (\mathfrak{Q}_\infty)_\#\mathfrak{m}\llcorner_{\mathcal{T}_\infty}$, and let $\tilde{\mathfrak{q}}$ be a probability measure such that $(\mathfrak{Q}_\infty)_\#\mathfrak{m}\llcorner_{\mathcal{T}_\infty} \ll \tilde{\mathfrak{q}}$. The disintegration theorem gives the following two formulas:

$$\mathfrak{m}\llcorner_E = \int_{Q_\infty} \hat{\mathfrak{m}}_{\alpha,\infty} \hat{\mathfrak{q}}(d\alpha) \quad \text{and} \quad \mathfrak{m}\llcorner_{\mathcal{T}_\infty} = \int_{Q_\infty} \tilde{\mathfrak{m}}_{\alpha,\infty} \tilde{\mathfrak{q}}(d\alpha), \tag{6.24}$$

where the measures $\hat{\mathfrak{m}}_{\alpha,\infty}$ and $\tilde{\mathfrak{m}}_{\alpha,\infty}$ are supported on $X_{\alpha,\infty}$. By comparing the two expressions above, it turns out that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha)\hat{\mathfrak{m}}_{\alpha,\infty} = \mathbf{1}_E \tilde{\mathfrak{m}}_{\alpha,\infty}$. Theorem 2.5 ensures that the space $(X_{\alpha,\infty}, \mathsf{d}, \tilde{\mathfrak{m}}_{\alpha,\infty})$ satisfies the $\mathsf{CD}(0, N)$ condition. Note that the disintegration $\alpha \mapsto \hat{\mathfrak{m}}_{\alpha,\infty}$ does not fall under the hypothesis of Theorem 2.5: indeed, in this case we are disintegrating a measure concentrated on E and not on the transport set $\mathcal{T}_\infty$. Define the functions $\hat{h}_\alpha$ and $\tilde{h}_\alpha$ such that

$$\hat{\mathfrak{m}}_{\alpha,\infty} = (g(\alpha, \cdot))_\#(\hat{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}) \quad \text{and} \quad \tilde{\mathfrak{m}}_{\alpha,\infty} = (g(\alpha, \cdot))_\#(\tilde{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}).$$

Clearly, it holds that

$$\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha)\hat{h}_\alpha(t) = \mathbf{1}_E(g(\alpha, t))\tilde{h}_\alpha(t),$$

thus we can derive a somehow weaker concavity condition for the function $\hat{h}_\alpha^{\frac{1}{N-1}}$: for all $x_0, x_1 \in (0, |X_{\alpha,\infty}|)$ and for all $t \in [0, 1]$, it holds that

$$\hat{h}_\alpha((1-t)x_0 + t x_1)^{\frac{1}{N-1}} \geq (1-t)\hat{h}_\alpha(x_0)^{\frac{1}{N-1}} + t\hat{h}_\alpha(x_1)^{\frac{1}{N-1}},$$

if $\hat{h}_\alpha((1-t)x_0 + t x_1) > 0$.

The inequality above implies that

$$\text{the map } r \mapsto \frac{\hat{h}_\alpha(r)}{r^{N-1}} \text{ is decreasing on the set } \{r \in (0, |X_{\alpha,\infty}|) : \hat{h}_\alpha(r) > 0\}. \tag{6.25}$$

Define the set $\hat{Z} \subset Z$ as

$$\hat{Z} := \big\{(x,\gamma,\mu,p) \in Z : x \in E \cap \mathcal{T}_\infty, \text{ and the properties given by equations (6.16)–(6.17) and (6.23) hold}\big\}.$$

Clearly, $\hat{Z}$ has full σ-measure in Z. We give a partition for $\hat{Z}$

$$\hat{Z}_\alpha := \{(x,\gamma,\mu,p) \in \hat{Z} : \mathfrak{Q}_\infty(x) = \alpha\},$$

and we disintegrate the measure σ according to the partition $(\hat{Z}_\alpha)_{\alpha \in Q_\infty}$

$$\sigma = \int_{Q_\infty} \sigma_\alpha \mathfrak{q}(d\alpha), \tag{6.26}$$

where the measures σ_α are supported on $\hat{Z}_\alpha$. Moreover, let $\nu_\alpha \in \mathcal{P}([0,\infty))$ be the measure given by

$$\nu_\alpha := \frac{1}{\mathfrak{m}(E)} (t_\alpha \circ e_0 \circ \pi_K)_\#(\sigma_\alpha)$$

(we recall that $t_\alpha = (g(\alpha,\cdot))^{-1}$ and $\pi_K(x,\gamma,\mu,p) = \gamma$).

The following proposition states that the density $\hat{h}_\alpha$ is given by the convolution of the model density and the measure ν_α.

Proposition 6.11. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\hat{h}_\alpha(r) = N\omega_N \mathsf{AVR}_X \int_{[0,\infty)} (r-t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r) \nu_\alpha(dt) \quad \forall r \in (0, |X_{\alpha,\infty}|).$$

Proof. Fix $\psi \in L^1(\mathfrak{m}\llcorner_E)$ and compute its integral using equations (6.16) and (6.26)

$$\begin{aligned}\int_E \psi(x)\mathfrak{m}(dx) &= \int_{\hat{Z}} \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \\ &= \int_{Q_\infty} \int_{\hat{Z}_\alpha} \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \mathfrak{q}(d\alpha).\end{aligned}$$

Fix now $\alpha \in Q_\infty$ and compute (recall (6.23) and the definition of $\hat{Z}$)

$$\begin{aligned}&\int_{\hat{Z}_\alpha} \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\ &\quad = \int_{\hat{Z}_\alpha} \int_0^\rho \psi(e_{\frac{s}{\rho}}(\gamma)) N \frac{s^{N-1}}{\rho^N}\, ds\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\ &\quad = \int_{\hat{Z}_\alpha} \int_0^\rho \psi(g_\infty(\mathfrak{Q}(x), t(\alpha,\gamma_0) + s)) N \frac{s^{N-1}}{\rho^N}\, ds\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\widehat{Z}_\alpha} \int_0^{|X_{\alpha,\infty}|} \psi(g_\infty(\alpha,r)) N \frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N} \mathbf{1}_{(t(\alpha,\gamma_0),t(\alpha,\gamma_0)+\rho)}(r) \\
&\qquad \times dr\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\
&= \int_0^{|X_{\alpha,\infty}|} \psi(g_\infty(\alpha,r)) \int_{\widehat{Z}_\alpha} N \frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N} \mathbf{1}_{(t(\alpha,\gamma_0),t(\alpha,\gamma_0)+\rho)}(r) \\
&\qquad \times \sigma_\alpha(dx\, d\gamma\, d\mu\, dp)\, dr,
\end{aligned}
$$

hence, by the uniqueness of the disintegration, we deduce that

$$
\begin{aligned}
\widehat{h}_\alpha(r) &= \int_{\widehat{Z}_\alpha} N \frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N} \mathbf{1}_{(t(\alpha,\gamma_0),t(\alpha,\gamma_0)+\rho)}(r) \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\
&= N\omega_N \mathsf{AVR}_X \int_{[0,\infty)} (r-t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r) \nu_\alpha(dt).
\end{aligned}
$$

∎

Proposition 6.12. *For $\widehat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that $\nu_\alpha = \delta_0$.*

Proof. Let $T := \inf \operatorname{supp} \nu_\alpha$. If we set $r \in (T, T+\rho)$, we can compute

$$
\begin{aligned}
\frac{\widehat{h}_{\alpha,\infty}(r)}{N\omega_N \mathsf{AVR}_X} &= \int_{[0,\infty)} (r-t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r) \nu_\alpha(dt) = \int_{[T,r)} (r-t)^{N-1} \nu_\alpha(dt) \\
&\geq \int_{[T,r)} \Big(\frac{r-T}{2} \mathbf{1}_{[T,\frac{r+T}{2}]}(t) \Big)^{N-1} \nu_\alpha(dt) \\
&= \frac{(r-T)^{N-1}}{2^{N-1}} \nu_\alpha\Big(\Big[T, \frac{r+T}{2}\Big]\Big).
\end{aligned}
\tag{6.27}
$$

By definition of T, we have that $\nu_\alpha([T, \frac{r+T}{2}]) > 0$, hence

$$
\widehat{h}_\alpha(r) > 0 \quad \text{for all } r \in (T, T+\rho).
$$

On the other hand,

$$
\begin{aligned}
\widehat{h}_{\alpha,\infty}(r) &= N\omega_N \mathsf{AVR}_X \int_{[T,r)} (r-t)^{N-1} \nu_\alpha(dt) \\
&\leq N\omega_N \mathsf{AVR}_X (r-T)^{N-1} \nu_\alpha([T,r)) \to 0 \quad \text{as } r \to T^+.
\end{aligned}
\tag{6.28}
$$

We claim that $T = 0$. Indeed, if $T > 0$, then $\lim_{r\to T^+} \frac{\widehat{h}_\alpha(r)}{r^{N-1}} = 0$ contradicting (6.25).

We now derive the non-increasing function

$$
(0,\rho) \ni r \mapsto \frac{\widehat{h}_\alpha(r)}{r^{N-1}} = \frac{N\omega_N \mathsf{AVR}_X}{r^{N-1}} \int_{[0,r)} \nu_\alpha(dt),
$$

obtaining

$$
0 \geq N\omega_N \mathsf{AVR}_X \Big(\frac{1-N}{r^N} \int_{[0,r)} (r-t)^{N-1} \nu_\alpha(dt) + \frac{1}{r^{N-1}} \frac{d}{dr} \int_{[0,r)} (r-t)^{N-1} \nu_\alpha(dt) \Big).
$$

The second term can be computed as

$$
\begin{aligned}
\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\nu_\alpha(dt) &= \lim_{h\to 0}\int_{[r,r+h)}\frac{(r+h-t)^{N-1}}{h}\nu_\alpha(dt)\\
&\quad+\lim_{h\to 0}\int_{[0,r)}\frac{(r+h-t)^{N-1}-(r-t)^{N-1}}{h}\nu_\alpha(dt)\\
&\geq 0+\int_{[0,r)}\lim_{h\to 0}\frac{(r+h-t)^{N-1}-(r-t)^{N-1}}{h}\nu_\alpha(dt)\\
&=(N-1)\int_{[0,r)}(r-t)^{N-2}\nu_\alpha(dt),
\end{aligned}
$$

yielding

$$
\begin{aligned}
0 &\geq (1-N)\int_{[0,r)}(r-t)^{N-1}\nu_\alpha(dt)+r\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\nu_\alpha(dt)\\
&\geq (N-1)\int_{[0,r)}(r(r-t)^{N-2}-(r-t)^{N-1})\nu_\alpha(dt)\\
&=(N-1)\int_{[0,r)}t(r-t)^{N-2}\nu_\alpha(dt).
\end{aligned}
$$

The inequality above implies that $\nu_\alpha((0,r))=0$ for all $r\in(0,\rho)$, hence $\nu_\alpha(0,\rho)=0$. We deduce that

$$
\begin{aligned}
\hat{h}_\alpha(r) &= N\omega_N \mathsf{AVR}_X\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\
&= N\omega_N \mathsf{AVR}_X\, r^{N-1}\nu_\alpha(\{0\}) \quad \forall r\in(0,\rho).
\end{aligned}
$$

If $\nu_\alpha([\rho,\infty))=0$, then $\nu_\alpha=\delta_0$ (because ν_α has mass 1) completing the proof. Assume on the contrary that $\nu_\alpha([\rho,\infty))>0$, and let $S:=\inf\operatorname{supp}(\nu_\alpha\llcorner_{[\rho,\infty)})\geq\rho$. In this case, we follow computations (6.27) and (6.28), with S in place of T, deducing $\lim_{r\to S^+}\hat{h}_\alpha(r)=0$, contradicting (6.25). ∎

Corollary 6.13. *For $\hat{\mathfrak{q}}$-a.e. $\alpha\in Q_\infty$, for σ_α-a.e. $(x,\gamma,\mu,p)\in Z_\alpha$, it holds that $e_t(\gamma)=g(\alpha,\rho t)$, $\forall t\in[0,1]$.*

Proof. The fact that $\nu_\alpha=\delta_0$, implies $t_\alpha(\gamma_0)=0$ for σ_α-a.e. $(x,\gamma,\mu,p)\in\hat{Z}_\alpha$, hence, recalling (6.23) and the definition of $\hat{Z}$, we have that

$$
e_t(\gamma)=g(\alpha,t_\alpha(e_0)+\rho t)=g(\alpha,\rho t). \qquad ∎
$$

The next corollary concludes the discussion of the limiting procedures of the localization.

Corollary 6.14. *For $\hat{\mathfrak{q}}$-a.e. $\alpha\in Q_\infty$, it holds that*

$$
\hat{h}_\alpha(r)=N\omega_N\mathsf{AVR}_X\mathbf{1}_{(0,\rho)}(r)r^{N-1}.
$$

Moreover, the following disintegration formulas hold:

$$\mathfrak{m} = N\omega_N \mathsf{AVR}_X \int_{Q_\infty} (g(\alpha,\cdot))_\#(r^{N-1}\mathcal{L}^1\llcorner_{(0,\rho)})\hat{\mathfrak{q}}(d\alpha), \tag{6.29}$$

$$\mathsf{P}(E;\cdot) = \mathsf{P}(E)\int_{Q_\infty} \delta_{g(\alpha,\rho)}\hat{\mathfrak{q}}(d\alpha). \tag{6.30}$$

Proof. The only non-trivial part is equation (6.30). Using (6.17) and Corollary 6.13, we can deduce that $\forall\psi \in L^1(\bar{E};\mathsf{P}(E;\cdot))$

$$\begin{aligned}\int_{\bar{E}} \psi(x)\mathsf{P}(E;dx) &= \frac{N}{\rho}\int_{\hat{Z}} \psi(e_1(\gamma))\psi\,\sigma(dx\,d\gamma\,d\mu\,dp)\\ &= \frac{N}{\rho}\int_{Q_\infty}\int_{\hat{Z}_\alpha} \psi(e_1(\gamma))\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\hat{\mathfrak{q}}(d\alpha)\\ &= \frac{N}{\rho}\int_{Q_\infty} \psi(g(\alpha,\rho))\int_{\hat{Z}_\alpha} \sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\hat{\mathfrak{q}}(d\alpha). \end{aligned}$$ ∎

7. E is a ball

The aim of this section is to prove that E coincides with a ball of radius ρ. Before starting the proof, we give a few technical lemmas. The first lemma states that a BV function with null differential on an open connected set is constant. This fact is already known for Sobolev functions and it follows from either the Sobolev-to-Lipschitz property or the local Poincaré inequality.

Lemma 7.1. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}(K,N)$ space with $X = \operatorname{supp}\mathfrak{m}$, and let $\Omega \subset X$ be an open connected set. If $v \in$ w-BV$((\Omega,\mathsf{d},\mathfrak{m}))$ and $|Du| = 0$, then u is constant in Ω (i.e., there exists $C \in \mathbb{R}$ such that $v(x) = C$ for $\mathfrak{m}$-a.e. $x \in \Omega$).*

Proof. The proof in given only for the case $K = 0$. We refer to Section 2.2 for the notation. Fix $x \in \Omega$ and let $r > 0$ be such that $B_{3r}(x) \subset \Omega$. Assume by contradiction that there are two constants $a < b$ such that the sets

$$A := \{y \in B_r(x) : v(y) \le a\} \quad \text{and} \quad B := \{y \in B_r(x) : v(y) \ge b\}$$

have strictly positive measures. Let us consider the probability measures $\mu_0 = \frac{\mathfrak{m}\llcorner A}{\mathfrak{m}(A)}$ and $\mu_1 = \frac{\mathfrak{m}\llcorner B}{\mathfrak{m}(B)}$. Let $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ and $\mu_t = (e_t)_\#\pi$. The $\mathsf{CD}(K,N)$ condition (as stated in Definition 2.2) is the following:

$$\begin{aligned}\rho_t(\gamma_t) &\le ((1-t)\rho_0^{-\frac{1}{N}}(\gamma_0) + t\rho_1^{-\frac{1}{N}}(\gamma_1))^{-N} \le (1-t)\rho_0(\gamma_0) + t\rho_1(\gamma_1)\\ &= \mathfrak{m}(A)^{-1} + \mathfrak{m}(B)^{-1} \quad \text{for } \pi\text{-a.e. } \gamma,\end{aligned}$$

and this proves that there exists a constant $C > 0$ such that $(e_t)_\#\pi \le C\mathfrak{m}$. What we have proven and the fact that $\mathrm{Lip}(\gamma) = \mathsf{d}(\gamma_0,\gamma_1) \le 2r$, for π-a.e. γ, implies that π is an ∞-test plan. For π-a.e. γ, we have that $|D(v\circ\gamma)|([0,1]) \geq b-a$, because γ is a curve from A to B, thus

$$b-a \leq \int \gamma_\# |D(v\circ\gamma)|(X)\,\pi(d\gamma) \leq C\,\|\mathrm{Lip}(\gamma)\|_{L^\infty(\pi)}\mu(X) \leq 2rC\mu(X),$$

where μ is any weak upper gradient for v. Since we can choose the null measure as weak upper gradient, we obtain a contradiction. Thus, there exists a constant c_x such that $v=c_x$ a.e. in $B_r(x)$. Taking into account the connectedness of Ω, we deduce that v is globally constant. ■

The following lemma is topological. It can be seen as a weak formulation of the following statement: let Ω be an open connected subset of a topological space X, and let $E\subset X$ be any set; if $\Omega\cap E\neq\emptyset$ and $\Omega\backslash E\neq\emptyset$, then $\partial E\cap\Omega\neq\emptyset$.

Lemma 7.2. *Assume that* $(X,\mathsf{d},\mathfrak{m})$ *is an essentially non-branching* $\mathsf{CD}(K,N)$ *space with* $X=\operatorname{supp}\mathfrak{m}$. *Let* $E\subset X$ *be a Borel set, and let* $\Omega\subset X$ *be an open connected set. If* $\mathfrak{m}(E\cap\Omega)>0$ *and* $\mathfrak{m}(\Omega\backslash E)>0$, *then* $\mathsf{P}(E;\Omega)>0$.

Proof. Assume on the contrary that $\mathsf{P}(E;\Omega)=0$. In this case, there exists a sequence $u_n\in\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $u_n\to\mathbf{1}_E$ in L^1_{loc} and $\int_\Omega|\operatorname{lip}u_n|\,d\mathfrak{m}\to 0$. This immediately implies that $u_n\to v$ in the space $\mathrm{BV}_*((\Omega,\mathsf{d},\mathfrak{m}))$ for some $v\in\mathrm{BV}_*((\Omega,\mathsf{d},\mathfrak{m}))$ such that $|Dv|=0$. By uniqueness of the limit, $\mathbf{1}_E=v$ a.e. in Ω, whereas Lemma 7.1 implies that v is constant, which is a contradiction. ■

The next lemma ensures that if two balls coincide, then they must share their center.

Lemma 7.3. *Assume that* $(X,\mathsf{d},\mathfrak{m})$ *is an essentially non-branching,* $\mathsf{CD}(K,N)$ *space with* $X=\operatorname{supp}\mathfrak{m}$ *and let* $x,y\in X$ *and* $r>0$. *If* $B_r(x)=B_r(y)$ *and* $\mathfrak{m}(X\backslash B_r(x))>0$, *then* $x=y$.

Proof. Assume by contradiction $x\neq y$. Since (X,d) is a geodesic space, then $(B_r(x))^t=B_{r+t}(x)=B_{r+t}(y)$, hence if $z\in X$ is such that $\mathsf{d}(z,x)=r+t$, then $z\in B_{r+t+\varepsilon}(x)=B_{r+t+\varepsilon}(y)$ for all $\varepsilon>0$, thus $\mathsf{d}(z,y)\leq r+t=\mathsf{d}(z,x)$. We deduce $\mathsf{d}(z,y)=\mathsf{d}(z,x)$ for all $z\in X\backslash B_r(x)$. Consider now two disjoint sets $A,B\subset X\backslash B_r(x)$, such that $\mathfrak{m}(A)=\mathfrak{m}(B)$. Consider the maps

$$T(z)=\begin{cases} x & \text{if } z\in A,\\ y & \text{if } z\in B,\end{cases}\qquad S(z)=\begin{cases} y & \text{if } z\in A,\\ x & \text{if } z\in B.\end{cases}$$

Since $\mathsf{d}(S(z),x)=\mathsf{d}(S(z),y)=\mathsf{d}(T(z),x)=\mathsf{d}(T,x)\ \forall z\in A\cup B$, these maps are two different solutions of the Monge problem $\inf_R\int_{A\cup B}\mathsf{d}^2(z,R(z))\mathfrak{m}(dz)$, among all possible maps $R\colon X\to X$ such that $R_\#(\mathfrak{m}\llcorner_{A\cup B})=\mathfrak{m}(A)(\delta_x+\delta_y)$. Since said problem admits a unique solution [23, Theorem 5.1], we have found a contradiction. ■

Proposition 7.4. *For* $\hat{\mathfrak{q}}$*-a.e.* $\alpha\in Q_\infty$, *it holds that*

$$\varphi_\infty(g_\infty(\alpha,0))\leq\operatorname*{ess\,sup}_E\varphi_\infty\quad\text{and}\quad\varphi_\infty(g_\infty(\alpha,\rho))\geq\operatorname*{ess\,inf}_E\varphi_\infty.$$

Proof. We prove only the former inequality since the latter has the same proof. In order to simplify the notation, define $M := \operatorname{ess\,sup}_E \varphi_\infty$. Let $H := \{\alpha \in Q_\infty : \varphi_\infty(g_\infty(\alpha, 0)) \geq M + 2\varepsilon\}$. Define the following measure on E:

$$\mathfrak{n}(T) = N\omega_N \mathsf{AVR}_X \int_H \int_0^\varepsilon \mathbf{1}_T(g_\infty(\alpha, r)) r^{N-1}\, dr\, \hat{\mathfrak{q}}(d\alpha) \quad \forall T \subset E \text{ Borel.}$$

Clearly, $\mathfrak{n} \ll \mathfrak{m}$ (compare with (6.29)), so $\varphi_\infty(x) \leq M$ for $\mathfrak{n}$-a.e. $x \in E$. We can compute the integral

$$\begin{aligned} 0 &\geq \int_E (\varphi_\infty(x) - M)\mathfrak{n}(dx) \\ &= N\omega_N \mathsf{AVR}_X \int_H \int_0^\varepsilon (\varphi_\infty(g_\infty(\alpha, t)) - M) t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= N\omega_N \mathsf{AVR}_X \int_H \int_0^\varepsilon (\varphi_\infty(g_\infty(\alpha, 0)) - t - M) t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &\geq N\omega_N \mathsf{AVR}_X \int_H \int_0^\varepsilon \varepsilon t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) = \varepsilon^N \hat{\mathfrak{q}}(H). \end{aligned}$$

We deduce that $\hat{\mathfrak{q}}(H) = 0$ and, by arbitrariness of ε, we can conclude. ■

Theorem 7.5. *There exists a unique point $o \in X$, such that, up to a negligible set, $E = B_\rho(o)$, where $\rho = (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$. Moreover, it holds that*

$$\varphi_\infty(o) = \operatorname*{ess\,sup}_E \varphi_\infty = \max_{B_\rho(o)} \varphi_\infty. \tag{7.1}$$

Proof. Define $\tilde{E} := \operatorname{supp} \mathbf{1}_E$. Recall that by definition of support, $\tilde{E} = \bigcup_C C$, where the intersection is taken among all closed sets C such that $\mathfrak{m}(E \setminus C) = 0$; in particular, $\mathfrak{m}(E \setminus \tilde{E}) = 0$. Let $o \in \arg\max_{\tilde{E}} \varphi_\infty$. The uniqueness will follow from Lemma 7.3.

First, we prove the first equality of (7.1). Let $N := \{x \in E : \varphi_\infty(x) > \varphi_\infty(o) = \max_{\tilde{E}} \varphi_\infty\}$. By definition of maximum, $N \cap \tilde{E} = \emptyset$, so $N \subset E \setminus \tilde{E}$, hence $\mathfrak{m}(N) = 0$, thus

$$M = \operatorname*{ess\,sup}_E \varphi_\infty \leq \varphi_\infty(o).$$

On the other side, consider the open set $P := \{x : \varphi(x) > M\}$. By definition of essential supremum, we have that $\mathfrak{m}(E \cap P) = 0$, hence $\tilde{E} \subset X \setminus P$, thus $\varphi_\infty(o) \leq M$. The other equality in (7.1) will follow from the fact $E = B_\rho(o)$ (up to a negligible set).

It is sufficient to prove only that $B_\rho(o) \subset E$, for the other inclusion is a consequence. Indeed, the Bishop–Gromov inequality, together with the definition of asymptotic volume ratio yields

$$\mathfrak{m}(E) \geq \mathfrak{m}(B_\rho(o)) \geq \omega_N \mathsf{AVR}_X \rho^N = \mathfrak{m}(E),$$

and the equality of measures improves to an equality of sets.

Fix now $\varepsilon > 0$ and define $A = B_{\rho-\varepsilon}(o)$. If $\mathfrak{m}(E \setminus A) = 0$, then we deduce that $B_{\rho-\varepsilon}(o) \subset E$ and, by arbitrariness of ε, we can conclude.

Suppose the contrary, i.e., that $\mathfrak{m}(E\setminus A) > 0$. Clearly, the set A is connected and $\mathfrak{m}(A \cap E) > 0$ (otherwise $o \notin \tilde{E}$), so we exploit Lemma 7.2 obtaining $\mathsf{P}(E; A) > 0$. Define $H = \{\alpha \in Q_\infty : g_\infty(\alpha, \rho) \in A\}$. A simple computation shows that the set H is non-negligible (recall (6.30)),

$$0 < \frac{\mathsf{P}(E; A)}{\mathsf{P}(E)} = \int_{Q_\infty} \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \int_H \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \hat{\mathfrak{q}}(H).$$

The Lipschitz-continuity of φ_∞ yields

$$\varphi_\infty(x) \geq \varphi_\infty(o) - \rho + \varepsilon \geq M - \rho + \varepsilon \quad \forall x \in A = B_{\rho-\varepsilon}(o),$$

hence

$$\varphi_\infty(g_\infty(\alpha, \rho)) \geq M - \rho + \varepsilon \quad \forall \alpha \in H.$$

We continue the chain of inequalities, obtaining

$$\varphi_\infty(g_\infty(\alpha, 0)) \geq \varphi_\infty(g_\infty(\alpha, \rho)) + \rho \geq M + \varepsilon \quad \forall \alpha \in H.$$

The line above, together with the fact that $\hat{\mathfrak{q}}(H) > 0$, contradicts Proposition 7.4. ■

7.1. $\varphi_\infty(x)$ coincides with $-\mathsf{d}(x, o)$

The present section is devoted to proving that $\varphi_\infty(x) = -\mathsf{d}(x, o) + \varphi_\infty(o)$.

Proposition 7.6. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\mathsf{d}(o, g(\alpha, t)) = t \quad \forall t \in [0, \rho]. \tag{7.2}$$

Proof. The 1-lipschitzianity of φ_∞ and the fact that $E = B_\rho(o)$ (up to a negligible set) imply that $\varphi_\infty(x) \geq \varphi_\infty(o) - \rho$ for $\mathfrak{m}$-a.e. $x \in E$. Thus, we deduce, using Proposition 7.4 and equation (7.1), that

$$\varphi_\infty(g_\infty(\alpha, 0)) \leq \varphi_\infty(o) \quad \text{and} \quad \varphi_\infty(g_\infty(\alpha, \rho)) \geq \varphi_\infty(o) - \rho.$$

Since $\frac{d}{dt}\varphi_\infty(g_\infty(\alpha, t)) = -1$, $t \in (o, \rho)$, the inequalities above are satisfied and

$$\varphi_\infty(g_\infty(\alpha, t)) = \varphi_\infty(o) - t \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty.$$

The 1-lipschitzianity of φ_∞, together with the equation above, yields

$$\mathsf{d}(o, g_\infty(\alpha, t)) \geq \varphi_\infty(o) - \varphi_\infty(g_\infty(\alpha, t)) = t \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty. \tag{7.3}$$

Fix $\varepsilon > 0$ and let $C = \{\alpha \in Q_\infty : \mathsf{d}(o, g_\infty(\alpha, 0)) > 2\varepsilon\}$. The function

$$f(t) := \inf\{\mathsf{d}(o, g_\infty(\alpha, t)) : \alpha \in C\}$$

is 1-Lipschitz and satisfies $f(0) \geq 2\varepsilon$, hence $f(t) \geq 2\varepsilon - t$, yielding (cf. (7.3))

$$f(t) \geq \max\{(2\varepsilon - t), t\} \geq \varepsilon.$$

The inequality above implies that $g_\infty(\alpha, t) \notin B_\varepsilon(o)$ for all $t \in [0,1]$ for all $\alpha \in C$. We compute the measure of $B_\varepsilon(o)$ using the disintegration formula (6.29)

$$\begin{aligned}\frac{\mathfrak{m}(B_\varepsilon(o))}{N\omega_N \mathsf{AVR}_X} &= \int_{Q_\infty}\int_0^\rho \mathbf{1}_{B_\varepsilon(o)}(g_\infty(\alpha,t))t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha)\\ &= \int_{Q_\infty\setminus C}\int_0^\rho \mathbf{1}_{B_\varepsilon(o)}(g_\infty(\alpha,t))t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha).\end{aligned}$$

If $\mathbf{1}_{B_\varepsilon(o)}(g_\infty(\alpha,t)) = 1$, then inequality (7.3) yields $t \le \varepsilon$, so we continue the computation

$$\begin{aligned}\frac{\mathfrak{m}(B_\varepsilon(o))}{N\omega_N \mathsf{AVR}_X} &= \int_{Q_\infty\setminus C}\int_0^\rho \mathbf{1}_{B_\varepsilon(o)}(g_\infty(\alpha,t))t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha)\\ &= \int_{Q_\infty\setminus C}\int_0^\varepsilon \mathbf{1}_{B_\varepsilon(o)}(g_\infty(\alpha,t))t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha)\\ &\le \int_{Q_\infty\setminus C}\int_0^\varepsilon t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha) = (\hat{\mathfrak{q}}(Q_\infty) - \hat{\mathfrak{q}}(C))\frac{\varepsilon^N}{N}.\end{aligned}$$

On the other hand, the Bishop–Gromov inequality states that

$$\mathfrak{m}(B_\varepsilon(o)) \ge \frac{\varepsilon^N}{\rho^N}\mathfrak{m}(B_\rho(o)) = \frac{\varepsilon^N}{\rho^N}\mathfrak{m}(E) = \varepsilon^N\omega_N \mathsf{AVR}_X,$$

thus, comparing with the previous inequality, we obtain $\hat{\mathfrak{q}}(C) = 0$. By arbitrariness of ε, we deduce that $g_\infty(\alpha, 0) = o$ for $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$.

Finally, using again (7.3), we can conclude

$$t \le \mathsf{d}(o, g_\infty(\alpha,t)) \le \mathsf{d}(o, g_\infty(\alpha,0)) + \mathsf{d}(g_\infty(\alpha,0), g_\infty(\alpha,t)) = t$$

for all $t \in [0,\rho]$, for $\hat{\mathfrak{q}}$-a.e $\alpha \in Q_\infty$. ∎

Corollary 7.7. *It holds that for all $x \in B_\rho(o)$, $\varphi_\infty(x) = \varphi_\infty(o) = -\mathsf{d}(x,o)$.*

Proof. If $x \in E \cap \mathcal{T}_\infty$, then $x = g(\alpha,t)$ for some t, with $\alpha = \mathfrak{Q}_\infty(x)$. By the previous proposition, we may assume that $g_\infty(\alpha,0) = o$, hence we have that

$$\varphi_\infty(x) - \varphi_\infty(o) = \varphi_\infty(g_\infty(\alpha,t)) - \varphi_\infty(g_\infty(\alpha,0)) = -\mathsf{d}(g_\infty(\alpha,t), g_\infty(\alpha,0)) = -\mathsf{d}(x,o).$$

Since $\mathcal{T}_\infty \cap E$ has full measure in $B_\rho(o)$ and $\operatorname{supp}\mathfrak{m} = X$, we conclude. ∎

7.2. *Localization of the whole space*

At this point, we are in position to extend the localization given in Section 6.4 to the whole space X. Since we do not know the behavior of φ_∞ outside of $B_\rho(o)$, we take as reference function $-\mathsf{d}(o,\cdot)$, which coincides with φ_∞ on $B_\rho(o)$.

In this section, we will use some of the concepts introduced in Section 2.4. In particular, we will refer to the transport relation $\mathcal{R}^e$; the transport set $\mathcal{T}$ turns out to have full $\mathfrak{m}$-measure. We will denote by Q the quotient set, and let $\mathfrak{Q}\colon \mathcal{T} \to Q$ be the quotient map; let $X_\alpha := \mathfrak{Q}^{-1}(\alpha)$ be the disintegration rays, and let $g\colon \mathrm{Dom}(g) \subset \mathbb{R} \times Q \to X$ be the standard parametrization. Define $\mathfrak{q} := \frac{1}{\mathfrak{m}(E)} \mathfrak{Q}_\#(\mathfrak{m}\llcorner_E)$ (note that for the moment we still do not know if $\mathfrak{Q}_\#(\mathfrak{m}\llcorner_E) \ll \mathfrak{q}$).

Proposition 7.8. *For $\mathfrak{q}$-a.e. $\alpha \in Q$, it holds that $\mathsf{d}(o, g(\alpha, t)) = t$ for all $t \in [0, |X_\alpha|]$.*

Proof. Let $\tilde{\mathfrak{q}} \in \mathcal{P}(Q)$ be a measure such that $\tilde{\mathfrak{q}} \ll \mathfrak{Q}_\#(\mathfrak{m}) \ll \tilde{\mathfrak{q}}$. The maximality of the rays (see [22, Theorem 7.10]) guarantees that $\mathring{\mathcal{R}}^e(\alpha) \subset X_\alpha$, for $\tilde{\mathfrak{q}}$-a.e. $\alpha \in Q$, where $\mathring{\mathcal{R}}^e(\alpha)$ denotes the relative interior of $\mathcal{R}^e(\alpha)$. By definition of distance, $o \in \mathcal{R}^b(\alpha)$ for all $\alpha \in Q$, thus $g(\alpha, 0) = o$ for $\tilde{\mathfrak{q}}$-a.e. $\alpha \in Q$. Since $\mathfrak{q} \ll \mathfrak{Q}_\#\mathfrak{m} \ll \tilde{\mathfrak{q}}$, the thesis follows. ■

Proposition 7.9. *It holds true that $\mathfrak{Q}_\#\mathfrak{m} \ll \mathfrak{q}$.*

Proof. Let $\tilde{\mathfrak{q}} \in \mathcal{P}(Q)$ be a measure such that $\mathfrak{Q}_\#(\mathfrak{m}) \ll \tilde{\mathfrak{q}}$. Using the localization theorem, we get that $\mathfrak{m} = \int_Q \tilde{\mathfrak{m}}_\alpha \tilde{\mathfrak{q}}(d\alpha)$, where the measures $\tilde{\mathfrak{m}}_\alpha$ are supported on X_α and satisfy the $\mathsf{CD}(0, N)$ condition. Let $A \subset Q$ be a set such that $\mathfrak{q}(A) = 0$, that is,

$$0 = \mathfrak{m}(B_\rho(0) \cap \mathfrak{Q}^{-1}(A)) = \int_A \mathfrak{m}(B_\rho(o))\, \tilde{\mathfrak{q}}(d\alpha),$$

thus $\tilde{\mathfrak{m}}_\alpha(B_\rho) = 0$, for $\tilde{\mathfrak{q}}$-a.e. $\alpha \in A$. Since $\mathsf{d}(o, g(\alpha, t)) = t$, for $\tilde{\mathfrak{q}}$-a.e. $\alpha \in A$ (compare with the previous proof), the $\mathsf{CD}(0, N)$ condition applied to every $\tilde{\mathfrak{m}}_\alpha$ yields $\tilde{\mathfrak{m}}_\alpha = 0$ for $\tilde{\mathfrak{q}}$-a.e. $\alpha \in Q$. It follows that $\mathfrak{m}(\mathfrak{Q}^{-1}(A)) = 0$. ■

The previous proposition allows us to use Theorem 2.5, hence there exists a unique disintegration for the measure $\mathfrak{m}$,

$$\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha) \tag{7.4}$$

such that

(1) the measures $\mathfrak{m}_\alpha$ are supported on X_α;

(2) the space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies the $\mathsf{CD}(0, N)$ condition.

We denote by $h_\alpha\colon (0, |X_\alpha|) \to \mathbb{R}$ the density function such that

$$\mathfrak{m}_\alpha = (g(\alpha, \cdot))_\#(h_\alpha \mathcal{L}\llcorner_{(0,|X_\alpha|)}).$$

The next two propositions bound together the localization obtained in Section 6.4 (in particular Corollary (6.14)) with the localization using $-\mathsf{d}(o, \cdot)$ as 1-Lipschitz reference function.

Proposition 7.10. *There exists a unique measurable map*

$$L\colon \mathrm{Dom}(L) \subset Q_\infty \to Q$$

such that the domain of L has full $\hat{\mathfrak{q}}$ in Q_∞ and it holds

$$L(\mathfrak{Q}_\infty(x)) = \mathfrak{Q}(x) \quad \forall x \in B_\rho(o) \cap \mathcal{T}_\infty \cap \mathcal{T}, \quad \textit{and} \quad \mathfrak{q} = L_\#\hat{\mathfrak{q}}.$$

Proof. Since $\varphi_\infty = \varphi_\infty(o) - \mathsf{d}(o,\cdot)$ on $B_\rho(o)$, the partitions $(X_{\alpha,\infty})_{\alpha\in Q_\infty}$ and $(X_\alpha)_{\alpha\in Q}$ agree on the set $B_\rho(o)\cap\mathcal{T}_\infty\cap\mathcal{T}$, that is, given $x, y \in B_\rho(o)\cap\mathcal{T}_\infty\cap\mathcal{T}$, we have that $(x, y) \in \mathcal{R}_\infty$ if and only if $(x, y) \in \mathcal{R}$. Consider the set

$$H := \{(x,\alpha,\beta) \in (B_\rho(o)\cap\mathcal{T}_\infty\cap\mathcal{T}) \times Q_\infty \times Q : \mathfrak{Q}_\infty(x) = \alpha \text{ and } \mathfrak{Q}(x) = \beta\},$$

and let $G := \pi_{Q_\infty\times Q}(H)$ be the projection of H on the second and third variable. For what we have said, G is the graph of a map $L\colon \mathrm{Dom}(L) \subset Q_\infty \to Q$. The other properties easily follow. ■

Proposition 7.11. *For q-a.e. $\alpha \in Q$, it holds that $|X_\alpha| \ge \rho$ and*

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1} \quad \forall r \in [0,\rho].$$

Proof. Comparing equation (7.2) with Proposition 7.8, we deduce that for $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that

$$g_\infty(\alpha,t) = g_\infty(L(\alpha),t) \quad \forall t \in (0,\min\{\rho,|X_\alpha|\}).$$

Comparing the disintegration formulas (6.24) and (7.4), we deduce

$$\mathfrak{m}\llcorner_E = \int_Q \hat{\mathfrak{m}}_{\alpha,\infty}\hat{\mathfrak{q}}(d\alpha) = \int_Q \mathfrak{m}_\alpha\llcorner_E\,\mathfrak{q}(d\alpha) = \int_{Q_\infty} (\mathfrak{m}_{L(\alpha)})\llcorner_E\,\hat{\mathfrak{q}}(d\alpha),$$

hence $\hat{\mathfrak{m}}_{\alpha,\infty} = (\mathfrak{m}_{L(\alpha)})\llcorner_E$, thus, recalling (6.29), we deduce that

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1} \quad \forall r \in (0,\min\{\rho,|X_\alpha|\}).$$

The fact that $|X_\alpha| \ge \rho$ follows from the expression above. ■

Theorem 7.12. *For $\mathfrak{q}$-a.e. $\alpha \in Q$, it holds that $|X_\alpha| = \infty$ and*

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1} \quad \forall r > 0.$$

Proof. Fix $\varepsilon > 0$ and let

$$C := \Big\{\alpha \in Q : \lim_{R\to\infty}\int_0^R \frac{h_\alpha}{R^N} < \omega_N \mathsf{AVR}_X(1-\varepsilon)\Big\},$$

with the convention that the limit above is 0 if $|X_\alpha| < \infty$ (notice that the limit always exists and it is not larger than $\omega_N\mathsf{AVR}_X$ by the Bishop–Gromov inequality applied to each density h_α). We compute the asymptotic volume ratio using the disintegration

$$\begin{aligned}
\mathsf{AVR}_X\omega_N &= \lim_{R\to\infty}\frac{\mathfrak{m}(B_R)}{R^N} = \lim_{R\to\infty}\int_Q\int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha)\\
&= \int_Q \lim_{R\to\infty}\int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha)\\
&= \int_C \lim_{R\to\infty}\int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha) + \int_{Q\setminus C}\lim_{R\to\infty}\int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha)\\
&\le \int_C \omega_N\mathsf{AVR}_X(1-\varepsilon)\mathfrak{q}(d\alpha) + \int_{Q\setminus C}\omega_N\mathsf{AVR}_X\,\mathfrak{q}(d\alpha)\\
&= \omega_N\mathsf{AVR}_X(1-\varepsilon\mathfrak{q}(C)),
\end{aligned}$$

thus $\mathfrak{q}(C) = 0$. By arbitrariness of ε we deduce that $\lim_{R\to\infty} \int_0^R \frac{h_\alpha}{R^N} = \omega_N \mathsf{AVR}_X$, hence $h_\alpha(t) = N\omega_N \mathsf{AVR}_X t^{N-1}$ for $\mathfrak{q}$-a.e. $\alpha \in \tilde{Q}$. ∎

The proof of Theorem 1.4 is therefore concluded. As described in the introduction, Theorems 1.5 and 1.7 are immediate consequences.

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