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Graded Lie algebras, compactified Jacobians and arithmetic statistics

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Abstract. A simply laced Dynkin diagram gives rise to a family of curves over $\mathbb{Q}$ and a coregular representation, using deformations of simple singularities and Vinberg theory, respectively. Thorne conjectured and partially proved a strong link between the arithmetic of these curves and the rational orbits of these representations. In this paper, we complete Thorne's picture and show that 2-Selmer elements of the Jacobians of the smooth curves in each family can be parametrised by integral orbits of the corresponding representation. Using geometry-of-numbers techniques, we deduce statistical results on the arithmetic of these curves. We prove these results in a uniform manner. This recovers and generalises results of Bhargava, Gross, Ho, Shankar, Shankar and Wang. The main innovations are an analysis of torsors on affine spaces using results of Colliot-Thélène and the Grothendieck–Serre conjecture, a study of geometric properties of compactified Jacobians using the Białynicki-Birula decomposition, and a general construction of integral orbit representatives.

Keywords: arithmetic statistics, Selmer groups, Vinberg theory, compactified Jacobians.

Contents

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1. Introduction

1.1. Context

This paper is a contribution to arithmetic statistics of algebraic curves: given a family of curves $\mathcal{F}$ over $\mathbb{Q}$, what can be said about the rational points of C (or related objects) as C varies in $\mathcal{F}$? Over the last twenty years, Bhargava and his collaborators have made spectacular progress in this direction. One of their key ideas is that many arithmetic objects can be parametrised by rational or integral orbits of a representation (G, V). When the representation is coregular, meaning that the ring of invariants $\mathbb{Q}[V]^G$ is a polynomial ring, they have developed powerful geometry-of-numbers techniques to count integral orbits of V. Combining orbit parametrisations with these counting techniques has led to many striking results; see [7,9,12–15] for some highlights and [5,38] for surveys of these results.

This raises the question: how does one find such orbit parametrisations? Typically they arise from classical algebro-geometric constructions. For example, elements of $\mathrm{Sel}_2 E$ for an elliptic curve $E/\mathbb{Q}$ correspond to locally soluble genus-1 curves C that are double covers of $\mathbb{P}^1_{\mathbb{Q}}$ [29, §1.3], so give rise to $\mathrm{PGL}_2(\mathbb{Q})$-orbits of binary quartic forms [14, Theorem 3.5]. This example goes back to Birch and Swinnerton-Dyer [17] (building on ideas of Mordell) and has been used by Bhargava and Shankar to compute the average size of the 2-Selmer group of elliptic curves [14]. See [10] for an exhaustive list of orbit parametrisations of genus-1 curves which are obtained using similar (but more difficult) algebro-geometric constructions. See also [7] for an orbit parametrisation of 2-Selmer groups of odd hyperelliptic curves using the geometry of pencils of quadrics [82]. Even though these considerations have been hugely successful, Wei Ho writes that ‘Finding appropriate groups G and vector spaces V related to the Selmer elements is still a relatively ad hoc process’ [38, p. 45].

Gross [35] observed that most coregular representations employed in arithmetic statistics arise from *Vinberg theory*, that is the theory of graded Lie algebras. This suggests the possibility to take Vinberg theory as a starting point, and to attempt to naturally construct families of curves in this setting. This is exactly the perspective taken in Thorne’s Ph.D. thesis [77] in the case of 2-Selmer groups. Given a simply laced Dynkin diagram of type A, D, E, he *canonically* constructs a family of curves and a coregular representation whose rational orbits should be related to the arithmetic of the curves in the family. This canonical construction unifies many orbit parametrisations in the literature and has already produced new results in arithmetic statistics; see [65,78,80]. However, to obtain all the expected consequences it remained to be shown that all elements of the 2-Selmer group give rise to rational orbits [77, Conjecture 4.16] and that such rational orbits admit integral representatives.

The main goal of this paper is to resolve both these questions, and to do so in a uniform manner for all the ADE-families considered. By using geometry-of-numbers techniques developed by Bhargava and his collaborators, we obtain an upper bound on the average size of the 2-Selmer group of the Jacobians of the smooth curves in each family. This has

Type	**Equation**	***m***
A_{2g}	$y^2 = x^{2g+1} + p_2x^{2g-1} + \cdots + p_{2g+1}$	1
A_{2g+1}	$y^2 = x^{2g+2} + p_2x^{2g} + \cdots + p_{2g+2}$	2
D_{2g+2} $(g \geq 1)$	$y(xy + p_{2g+2}) = x^{2g+1} + p_2x^{2g} + p_4x^{2g-1} + \cdots + p_{4g+2}$	3
D_{2g+1} $(g \geq 2)$	$y(xy + p_{2g+1}) = x^{2g} + p_2x^{2g-1} + p_4x^{2g-2} + \cdots + p_{4g}$	2
E_6	$y^3 = x^4 + (p_2x^2 + p_5x + p_8)y + (p_6x^2 + p_9x + p_{12})$	1
E_7	$y^3 = x^3y + p_{10}x^2 + x(p_2y^2 + p_8y + p_{14}) + p_6y^2 + p_{12}y + p_{18}$	2
E_8	$y^3 = x^5 + (p_2x^3 + p_8x^2 + p_{14}x + p_{20})y + (p_{12}x^3 + p_{18}x^2 + p_{24}x + p_{30})$	1

Tab. 1. Families of curves.

consequences for the ranks of the Jacobians and the rational points of the curves in these families.

1.2. Statement of results

Let D be a Dynkin diagram of type A_n, D_n or E_n and let $C \to B$ be the family of projective curves over $\mathbb{Q}$ with affine equation given by Table 1. For example, if $\mathsf{D} = A_{2g}$, then $B = \operatorname{Spec} \mathbb{Q}[p_2, \ldots, p_{2g+1}]$ and $C \to B$ is the family of all monic odd hyperelliptic curves of genus g. If $\mathsf{D} = E_7$, then $C \to B$ is the family of all plane quartic curves with a marked rational flex point. The family $C \to B$ is a semi-universal deformation of its central fibre (by setting all coefficients p_i equal to zero), which is a simple singularity of type D. (See Proposition 3.13.) We exclude the case $\mathsf{D} = A_1$.

Write $B^{\mathrm{rs}} \subset B$ for the locus above which $C \to B$ is smooth, the complement of a discriminant hypersurface. For every field $k/\mathbb{Q}$ and $b \in B^{\mathrm{rs}}(k)$, write J_b for the Jacobian of the smooth projective curve C_b, an abelian variety over k of dimension equal to the genus of C_b. Our first main theorem is an orbit parametrisation for elements of $J_b(k)/2J_b(k)$.

To each diagram D, one may canonically associate a representation V of a reductive group $G/\mathbb{Q}$. This construction, due to Thorne [77], is recalled in Section 3.1 and is based on Vinberg's theory of graded Lie algebras. See Section 3.2 for an explicit description of G and V, although we will almost never use this description. The geometric quotient $V /\!\!/ G = \operatorname{Spec} \mathbb{Q}[V]^G$ (parametrising G-invariant polynomials of V) turns out to be isomorphic to B. For every field $k/\mathbb{Q}$ and $b \in B(k)$, write V_b for the subset of elements of V which map to b under the map $V \to V /\!\!/ G \simeq B$.

Theorem 1.1 (Theorem 6.6). *For every field $k/\mathbb{Q}$ and element $b \in B^{\mathrm{rs}}(k)$, there exists an injection $\eta_b\colon J_b(k)/2J_b(k) \hookrightarrow G(k)\backslash V_b(k)$ compatible with base change.*

See Theorem 6.6 for a more precise formulation and an explicit construction of this injection. Using a local-global principle for G, one can also embed the 2-Selmer group of J_b inside the $G(\mathbb{Q})$-orbits of $V(\mathbb{Q})$. Recall that the 2-Selmer group of an abelian variety $A/\mathbb{Q}$ is a finite-dimensional $\mathbb{F}_2$-vector space $\operatorname{Sel}_2 A$ defined by local conditions and fitting inside an exact sequence

$$0 \to A(\mathbb{Q})/2A(\mathbb{Q}) \to \operatorname{Sel}_2 A \to \text{Ш}(A/\mathbb{Q})[2] \to 0.$$

Theorem 1.2 (Corollary 6.9). *For every $b \in B^{\mathrm{rs}}(\mathbb{Q})$, the injection η_b extends to an injection* $\operatorname{Sel}_2 J_b \hookrightarrow G(\mathbb{Q})\backslash V_b(\mathbb{Q})$.

If D is of type A_2, $C \to B$ is the family of elliptic curves in short Weierstrass form, and we essentially recover the orbit-parametrisation of Birch–Swinnerton-Dyer [17] used by Bhargava–Shankar [14]. If D is of type A_{2g}, we recover the orbit parametrisation of Bhargava and Gross [7].

Crucially, we additionally show that the $G(\mathbb{Q})$-orbits corresponding to $\operatorname{Sel}_2 J_b$ using Theorem 1.2 have integral representatives away from small primes, see Corollary 7.21. Using geometry-of-numbers techniques to count integral orbits of V, Theorem 1.2 may thus be used to give an upper bound on the average size of the 2-Selmer group of $\operatorname{Sel}_2 J_b$. Using the identification $B = \operatorname{Spec} \mathbb{Q}[p_{d_1}, \dots, p_{d_r}]$ from Table 1, let $\mathcal{F}$ be the subset of elements $b = (p_{d_1}(b), \dots, p_{d_r}(b)) \in \mathbb{Z}^r$ with $b \in B^{\mathrm{rs}}(\mathbb{Q})$. We define the height of $b \in \mathcal{F}$ by the formula

$$\operatorname{ht}(b) := \max(|p_{d_1}(b)|^{1/d_1}, \dots, |p_{d_r}(b)|^{1/d_r}).$$

Note that for every $X \in \mathbb{R}_{>0}$, the set $\{b \in \mathcal{F} \mid \operatorname{ht}(b) < X\}$ is finite. To state the next theorem, note that each curve C_b has points at infinity not lying in the affine patch of Table 1, and we call those points the *marked points*. Their cardinality is displayed in Table 1.

Theorem 1.3 (Theorem 9.1). *Let m be the number of marked points. Then when ordered by height, the average size of the 2-Selmer group of J_b for $b \in \mathcal{F}$ is bounded above by $3 \cdot 2^{m-1}$. More precisely, we have*

$$\limsup_{X \to +\infty} \frac{\sum_{b \in \mathcal{F}, \operatorname{ht}(b) < X} \# \operatorname{Sel}_2 J_b}{\#\{b \in \mathcal{F} \mid \operatorname{ht}(b) < X\}} \le 3 \cdot 2^{m-1}.$$

The same result holds true even if we impose finitely many congruence conditions on $\mathcal{F}$. Assuming a certain plausible uniformity estimate (Conjecture 8.19), we show that the limit exists and the bound $3 \cdot 2^{m-1}$ is sharp, see Section 9.2. The constant $3 \cdot 2^{m-1}$ is consistent with the heuristics of Poonen and Rains [58], once we incorporate the fact that the m marked points give rise to a 'trivial' subgroup of $\operatorname{Sel}_2 J_b$ which has size 2^{m-1} most of the time (Proposition 9.4). See Section 1.3 for a comparison of this theorem with previously obtained results.

Just like in previous cases, Theorem 1.3 implies a bound on the average of the Mordell–Weil rank $\operatorname{rk}(J_b)$ of J_b, the rank of the finitely generated abelian group $J_b(\mathbb{Q})$. Let $\operatorname{Sel}_2^{\mathrm{triv}} J_b \subset \operatorname{Sel}_2 J_b$ be the subgroup generated by the differences of the marked points. By the chain of inequalities

$$\operatorname{rk} J_b \le \dim_{\mathbb{F}_2}(\operatorname{Sel}_2 J_b / \operatorname{Sel}_2^{\mathrm{triv}} J_b) + (m-1) \le \frac{1}{2}(\#(\operatorname{Sel}_2 J_b / \operatorname{Sel}_2^{\mathrm{triv}} J_b)) + (m-1)$$

combined with Theorem 1.3 and the fact that $\mathrm{Sel}_2^{\mathrm{triv}} J_b$ is of size 2^{m-1} for 100% of curves, we obtain the following assertion.

Corollary 1.4. *Let m be the number of marked points of the family $C \to B$. Then when ordered by height, the average rank $\mathrm{rk}(J_b)$, where $b \in \mathcal{F}$, is bounded above by $m + 1/2$.*

1.3. Relation to other works

Theorem 1.3 has been previously obtained for many D:

- A_2: Bhargava–Shankar [14], who prove in this case that the average is exactly 3.
- A_{2g}: Bhargava–Gross [7].
- A_{2g+1}, $g \geq 2$: Shankar–Wang [69].
- D_{2g+1}, $g \geq 2$: Shankar [70].
- A_3, D_4: Bhargava–Ho [11, Theorem 1.1 (c), (g)].
- E_6: Laga [43].

All these works combine geometry-of-numbers techniques with the orbit parametrisation of Theorem 1.2 to obtain Theorem 1.3 in their specific case, just as we do here. However, their construction of orbits (in other words, the proof of Theorem 1.2) and analysis of the representation (G, V) require specific arguments in each case. One of the main points of this paper is that we are able to prove Theorem 1.3 in a uniform way. Inspecting the above list, we see that the only cases not previously considered in the literature are $\mathsf{D} = D_{2g+2}$ with $g \geq 2$, E_7 and E_8. Concretely, this concerns the universal family of hyperelliptic genus g curves with two non-conjugate non-Weierstrass points (D_{2g+2}), the universal family of plane quartic curves with a marked flex point (E_7) and the universal family of trigonal genus 4 curves with a marked triple ramification point (E_8).

Theorem 1.3 has a number of interesting consequences for the rational points of C_b, typically using various forms of the Chabauty–Coleman method. See, for example, [7, Corollary 1.4] and [59] for such results in the case $\mathsf{D} = A_{2g}$, and [43, Corollary 1.3 and Theorem 1.4] in the case $\mathsf{D} = E_6$. Theorem 1.3 should give similar consequences for $\mathsf{D} = D_{2g+2}$, E_7 and E_8, but we have not pursued this in this paper.

We describe what is new in this paper compared with Thorne's work. He has shown the analogue of Theorem 1.1 for the subset of $J_b(k)/2J_b(k)$ lying in the image of the Abel–Jacobi map $C_b(k) \to J_b(k)/2J_b(k)$ with respect to a fixed marked point [77, Theorem 4.15]. This allowed him and Romano to deduce arithmetic statistical results on the 2-*Selmer set* of the curve C_b (a pointed subset of $\mathrm{Sel}_2 J_b$) and the integral points of the affine curve C_b°, see [65, 78]. The first main innovation of this work is the construction of orbits associated with *all* elements of $J_b(k)/2J_b(k)$, as was conjectured in [77, Conjecture 4.16]. The second main innovation is an integral study of the representations (G, V). In particular, we show that orbits arising from Theorem 1.2 admit integral representatives away from small primes (Theorem 7.6). This technical result is essential for applying orbit-counting methods and allows us to obtain new results on the arithmetic of the curves C_b.

1.4. Method of proof

We briefly describe the proof of Theorem 1.1, which is the first main novelty of this paper. Thorne has shown that the stabiliser $Z_G(v)$ of an arbitrary element $v \in V_b(k)$ is canonically isomorphic to $J_b[2]$, the 2-torsion subgroup of the Jacobian of C_b. In fact, there always exists a distinguished orbit $\kappa_b \in G(k)\backslash V_b(k)$, and a well-known lemma in arithmetic invariant theory (Lemma 2.13) shows that by twisting κ_b the set $G(k)\backslash V_b(k)$ can be identified with the pointed kernel of the map on Galois cohomology $\mathrm{H}^1(k, Z_G(\kappa_b)) \to \mathrm{H}^1(k, G)$.

To prove Theorem 1.1, it therefore suffices to prove that the composition

$$J_b(k)/2J_b(k) \xrightarrow{\delta} \mathrm{H}^1(k, J_b[2]) \simeq \mathrm{H}^1(k, Z_G(\kappa_b)) \to \mathrm{H}^1(k, G), \tag{1.1}$$

where δ is the 2-descent map of J_b, is trivial. We solve this problem by considering it universally. More precisely, a 'categorified' version of (1.1) associates to every element $P \in J_b(k)$ a G-torsor $T_P \to \operatorname{Spec} k$ such that its isomorphism class $[T_P] \in \mathrm{H}^1(k, G)$ equals the image of P under (1.1). This process can be carried out in a relative setting: let J^{rs} be the relative Jacobian of the family of smooth curves $C|_{B^{\mathrm{rs}}} \to B^{\mathrm{rs}}$. Then we may construct a G-torsor $T \to J^{\mathrm{rs}}$ whose pullback along a point $P\colon \operatorname{Spec} k \to J^{\mathrm{rs}}$ is isomorphic to T_P. The crucial observation is that the geometry of the total space J^{rs} is very simple, despite the fibres of $J^{\mathrm{rs}} \to B^{\mathrm{rs}}$ being abelian varieties so arguably not so simple. For example, J^{rs} is a rational variety. This fact (or rather a similar, more precise statement), together with an analysis of G-torsors on affine spaces and progress on the Grothendieck–Serre conjecture, allows us to prove J^{rs} admits a Zariski open cover above which T is trivial. This implies that each T_P is trivial, proving the theorem.

To analyse the geometry of J^{rs}, we introduce a compactification of J^{rs} over the whole of B: there exists a projective scheme $\bar{J} \to B$ restricting to J^{rs} over B^{rs} called the *compactified Jacobian* of $C \to B$. The scheme $\bar{J}$ parametrises rank-1 torsion-free sheaves following Altman–Kleiman [1], with the caveat that in the reducible fibres of $C \to B$ we have to impose a stability condition in the sense of Esteves [33] to obtain a well-behaved moduli problem. The main selling point of this paper can be summarised as follows: geometric properties of $\bar{J}$ are very useful in the construction of orbits associated with elements of J^{rs}. For example, we show that even though the fibres of $\bar{J} \to B$ might be highly singular, $\bar{J}$ is a smooth and geometrically integral variety. Moreover, the Białynicki-Birula decomposition from geometric representation theory shows that $\bar{J}$ has a decomposition into affine cells, so has a very transparent geometry. The consequences for the geometry of J^{rs} are strong enough to carry out the strategy of the previous paragraph and consequently prove Theorem 1.1. The smoothness of $\bar{J}$ is essential and follows from the fact that $C \to B$ is a semi-universal deformation of its central fibre.

The second main innovation of this paper is a uniform construction of integral representatives, and we again exploit the geometry of the compactified Jacobian. We achieve this by deforming to the case of square-free discriminant and using a general result on extending reductive group schemes over open dense subschemes of regular arithmetic surfaces (Lemma 7.13). We are able to deform to this case using Bertini theorems over $\mathbb{Q}_p$

and $\mathbb{F}_p$ and (again!) the smoothness of $\bar{J}$. Many of the ideas were already present in our earlier work [43] treating the E_6 case, but we use stack-theoretic language to streamline it significantly. Constructing integral orbits has often been a subtle point in the past, and we expect that our methods will have applications to settings different to the one considered here.

To deduce Theorem 1.3 from Theorem 1.2, we use the robust geometry-of-numbers methods developed by Bhargava and his collaborators to count integral orbits coregular representations, about which we make two remarks. Firstly, the only step in the counting argument that cannot be carried out uniformly is controlling orbits lying in the cuspidal region of the fundamental domain, the so-called 'cutting of the cusp'. For every ADE diagram, this relies on combinatorial calculations in the associated root system. These calculations have appeared in the literature (on which we rely) except for the D_{2g+2} case, which we handle explicitly in Appendix A. This is the only part of the paper where we rely on the previous works listed at the beginning of Section 1.3. It would be very interesting to find a less computational or even uniform proof for these calculations. This remark extends to other representations employed in arithmetic statistics, for example, the ones used in [12, 13]. See [13, Table 2] for an example of the intricacies involved. Secondly, the reason that we only obtain an upper bound in Theorem 1.3 is our failure to prove the uniformity estimate of Conjecture 8.19, which is the crucial ingredient for the square-free sieve needed to obtain the lower bound. For the representations considered in this paper, this conjecture has only been solved in the A_2 case [14, Theorem 2.13]. Establishing more cases of this conjecture would be very interesting but seems difficult at present.

1.5. Other coregular representations

There are at least two ways in which coregular representations can arise from Vinberg theory that are not treated in this paper. In both cases, we expect that our methods go a long way towards proving analogous results to Theorem 1.3.

Firstly, one may try to incorporate gradings on nonsimply laced Lie algebras (so of type B, C, F, G) into the picture. Again, there will be families of curves, but the relevant Selmer groups may arise from a general isogeny, not just multiplication by an integer. Such gradings have already appeared in the literature, implicitly and explicitly: a $\mathbb{Z}/2\mathbb{Z}$-grading on G_2 has been used to study 2-Selmer groups of elliptic curves with a marked 3-torsion point [11, Theorem 1.1 (f)]; a $\mathbb{Z}/3\mathbb{Z}$-grading on G_2 has been used to study 3-isogeny Selmer groups of the curves $y^2 = x^3 + k$ [6]; a $\mathbb{Z}/2\mathbb{Z}$-grading on F_4 has been used to study 2-Selmer groups of a family of Prym surfaces [44].

Secondly, although their occurrence is more sporadic, there are also interesting $\mathbb{Z}/m\mathbb{Z}$-gradings on simple Lie algebras for $m \geq 3$; see, for example, [66], where the authors calculate the average size of the 3-Selmer group of the family of odd genus-2 curves using a $\mathbb{Z}/3\mathbb{Z}$-grading on E_8. The representations used by Bhargava–Shankar for 3-, 4- and 5-Selmer groups of elliptic curves [12, 13, 15] can also be interpreted this way.

To the best of our knowledge, every coregular representation appearing in the literature on arithmetic statistics of algebraic curves arises from Vinberg theory (that is, from a $\mathbb{Z}/m\mathbb{Z}$-grading on a semisimple Lie algebra), *except* for one family of notable examples: the representation of SL_n acting on pairs of symmetric matrices $\mathrm{Sym}^2(n) \oplus \mathrm{Sym}^2(n)$. This representation is used in [9] to show that a positive proportion of locally soluble hyperelliptic curves over $\mathbb{Q}$ of fixed genus have no points over any odd degree extension. One feature that distinguishes their setting from ours is that their representation lacks a 'Kostant section', which is related to the fact that the curves they study do not come with specified marked points. We wonder if one can still interpret this representation in terms of Lie theory and study its arithmetic from this perspective.

1.6. Organisation

We now summarise the sections of this paper. In Section 2, we recall some background results in Vinberg theory and arithmetic invariant theory. In Section 3, we recall the constructions and main results of Thorne's thesis and introduce the Vinberg representation (G, V) and family of curves $C \to B$. In Section 4, we extend the results of Thorne's thesis from the smooth fibres of $C \to B$ to those fibres admitting at most one singular nodal point. In Section 5, we introduce and study compactified Jacobians of the family $C \to B$. In Section 6, we analyse torsors on affine spaces and use this and our results from the previous sections to prove Theorems 1.1 and 1.2 on the construction of orbits. In Section 7, we prove that such orbits admit integral representatives away from small primes. In Section 8, we employ Bhargava's orbit-counting techniques and count integral orbits of the representation (G, V). We combine all the results from the previous sections in Section 9 to obtain Theorem 1.3. In Appendix A, we perform some combinatorial calculations in the root system of type D_{2n} to complete the proof of Proposition 8.12 (cutting off the cusp) in this case. We note that the main novel contributions of this paper lie in Sections 4, 5, 6 and 7.

1.7. Notation

Table 2 summarises the main objects used throughout the paper.

1.7.1. General. For a field k, we write k^s for a fixed separable closure and

$$\Gamma_k = \mathrm{Gal}(k^s/k)$$

for its absolute Galois group.

If X is a scheme over S and $T \to S$ is a morphism, we write X_T for the base change of X to T. If $T = \operatorname{Spec} A$ is an affine scheme, we also write X_A for X_T.

If G is a smooth group scheme over S, then we write $\mathrm{H}^1(S, G)$ for the set of isomorphism classes of étale sheaf torsors under G over S, which is a pointed set coming from nonabelian Čech cohomology. If $S = \operatorname{Spec} R$, we write $\mathrm{H}^1(R, G)$ for the same

Symbol	Definition	Reference in paper
H	Split adjoint group of type ADE	Section 3.1
$(T, P, \{X_\alpha\})$	Pinning of H	Section 3.1
θ	Split stable involution of H	Section 3.1
G	Fixed points of θ on H	Section 3.1
V	(-1)-part of action of θ on $\mathfrak{h}$	Section 3.1
B	GIT quotient $V /\!\!/ G$	Section 3.1
$\pi: V \to B$	Invariant map	Section 3.1
$\kappa_E: B \to V$	Kostant section associated to E	Section 3.6
$C^\circ \to B$	Family of affine curves	Section 3.7
$C \to B$	Family of projective curves	Section 3.7
$\infty_1, \dots, \infty_m$	Marked points of $C \to B$	Section 3.7
$p_{d_1}, \dots, p_{d_r}$	Invariant polynomials of G-action on V	Section 3.7
$\kappa: B \to V$	Fixed choice of Kostant section	Section 3.8
$A, Z \to B^{\mathrm{rs}}$	Centraliser of $\kappa\|_{B^{\mathrm{rs}}}$ in H and G	Section 3.8
$\Lambda \to B^{\mathrm{rs}}$	Character group scheme of the torus $A \to B^{\mathrm{rs}}$	Section 3.8
N_Λ	$\mathrm{image}(\Lambda/2\Lambda \to \Lambda^\vee/2\Lambda^\vee)$	Section 3.8
$J^{\mathrm{rs}} \to B^{\mathrm{rs}}$	Jacobian variety of $C^{\mathrm{rs}} \to B^{\mathrm{rs}}$	Section 3.7
D	Zero locus of discriminant $\Delta \in \mathbb{Q}[B]$	Section 4.1
B^1	Complement of singular locus of D in B	Section 4.1
$\bar{J} \to B$	Compactified Jacobian	Section 5.2
$\underline{H}, \underline{G}, \underline{V}$	Extensions of above objects over $\mathbb{Z}$	Section 7.1
N	Sufficiently large integer	Section 7.2
S	$\mathbb{Z}[1/N]$	Section 7.2

Tab. 2. Notation used throughout the paper.

object. If k is a field, then $\mathrm{H}^1(k, G)$ coincides with the first nonabelian Galois cohomology set of $G(k^s)$.

If $G \to S$ is a group scheme acting on $X \to S$ and $x \in X(T)$ is a T-valued point, we write $Z_G(x) \to T$ for the centraliser of x in G. It is defined by the following pullback square:

$$\begin{tikzcd}
Z_G(x) \arrow[r] \arrow[d] & T \arrow[d] \\
G \times_S X \arrow[r] & X \times_S X.
\end{tikzcd}$$

Here $G \times_S X \to X \times_S X$ denotes the map $(g, x) \mapsto (g \cdot x, x)$, and $T \to X \times_S X$ denotes the composition of x with the diagonal $X \to X \times_S X$.

If V is a vector space over a field k, we write $k[V]$ for the graded algebra $\mathrm{Sym}(V^\vee)$. Then V is naturally identified with the k-points of the scheme $\operatorname{Spec} k[V]$, and we call this latter scheme V as well. If G is a group scheme over k acting on V, we write $V /\!\!/ G := \operatorname{Spec} k[V]^G$ for the *GIT quotient* of V by G.

1.7.2. Root lattices. We define a *lattice* to be a finitely generated free $\mathbb{Z}$-module Λ together with a symmetric and positive-definite bilinear form $(\cdot,\cdot)\colon \Lambda\times\Lambda\to\mathbb{Z}$. We write $\Lambda^\vee := \{\lambda\in\Lambda\otimes\mathbb{Q}\mid(\lambda,\Lambda)\subset\mathbb{Z}\}$ for the *dual lattice* of Λ, which is naturally identified with $\mathrm{Hom}(\Lambda,\mathbb{Z})$. We say Λ is a *root lattice* if (λ,λ) is an even integer for all $\lambda\in\Lambda$ and the set

$$\{\alpha\in\Lambda\mid(\alpha,\alpha)=2\}$$

generates Λ. If $\Phi\subset\mathbb{R}^n$ is a simply laced root system, then $\Lambda=\mathbb{Z}\Phi$ is a root lattice. In that case, we define the type of Λ to be the Dynkin type of Φ.

If S is a scheme, an *étale sheaf of root lattices* Λ over S is defined as a locally constant étale sheaf of finite free $\mathbb{Z}$-modules together with a bilinear pairing $\Lambda\times\Lambda\to\mathbb{Z}$ (where $\mathbb{Z}$ denotes the constant étale sheaf on S) such that for every geometric point $\overline{s}$ of S, the stalk $\Lambda_{\overline{s}}$ is a root lattice. In that case, $\mathrm{Aut}(\Lambda)$ is a finite étale S-group.

1.7.3. Reductive groups and Lie algebras. A reductive group scheme over S is a smooth S-affine group scheme $G\to S$ whose geometric fibres are connected reductive groups. See [72] for the basics of reductive groups over a field and [27] for reductive group schemes over a general base. A reductive group is assumed to be connected.

If $G, H, \dots$ are algebraic groups, then we will use Gothic letters $\mathfrak{g},\mathfrak{h},\dots$ to denote their Lie algebras. If G is a reductive group with split maximal torus $T\subset G$, we shall write $\Phi_{\mathfrak{t}}\subset X^*(T)$ for the set of roots of T in $\mathfrak{g}$, and $\Phi_{\mathfrak{t}}^\vee\subset X_*(T)$ for its set of coroots. The map $\alpha\in\Phi_{\mathfrak{t}}\mapsto d\alpha\in\mathrm{Hom}(\mathfrak{t},k)$ identifies $\Phi_{\mathfrak{t}}$ with the set of roots of $\mathfrak{t}$ in $\mathfrak{g}$, and we will use this identification without further comment.

If x is an element of a Lie algebra $\mathfrak{g}$, then we write $\mathfrak{z}_{\mathfrak{g}}(x)$ for the centraliser of x in $\mathfrak{g}$, a subalgebra of $\mathfrak{g}$. We note that if G is an algebraic group over a field k and $x\in\mathfrak{g}$ any element, then the inclusion $\mathrm{Lie}\, Z_G(x)\subset\mathfrak{z}_{\mathfrak{g}}(x)$ is an equality if the characteristic of k is zero or if x is semisimple [39, Proposition 1.10].

2. Background

2.1. *The adjoint quotient of a Lie algebra*

To motivate the results in Vinberg theory, we first recall some classical results in the invariant theory of Lie algebras.

Let H be a connected reductive group over a field k of characteristic zero with Lie algebra $\mathfrak{h}$. The group H acts on $\mathfrak{h}$ via the adjoint representation. Let $p\colon\mathfrak{h}\to\mathfrak{h}/\!\!/H=\mathrm{Spec}\,k[\mathfrak{h}]^H$ be the so-called *adjoint quotient* induced by the inclusion $k[\mathfrak{h}]^H\subset k[\mathfrak{h}]$. We interpret $\mathfrak{h}/\!\!/H$ as the space of invariants of the H-action on $\mathfrak{h}$ and p as the morphism of taking invariants. Recall that an element $x\in\mathfrak{h}$ is said to be *regular* if $\dim\mathfrak{z}_{\mathfrak{h}}(x)$ is minimal among elements of $\mathfrak{h}$; this minimal value equals the rank of H. The subset of regular elements defines an open subscheme $\mathfrak{h}^{\mathrm{reg}}\subset\mathfrak{h}$. The following classical proposition summarises the invariant theory of $\mathfrak{h}$.

Proposition 2.1. (1) *Every semisimple element of $\mathfrak{h}$ is contained in a Cartan subalgebra, and if k is algebraically closed, every two Cartan subalgebras are $H(k)$-conjugate.*

(2) *Let $\mathfrak{c} \subset \mathfrak{h}$ be a Cartan subalgebra and let $W = N_H(\mathfrak{c})/Z_H(\mathfrak{c})$. Then the inclusion $\mathfrak{c} \subset \mathfrak{h}$ induces an isomorphism (the Chevalley isomorphism)*

$$\mathfrak{c} /\!\!/ W \simeq \mathfrak{h} /\!\!/ H.$$

Since W is a finite reflection group, this quotient is isomorphic to affine space.

(3) *If k is algebraically closed and $b \in (\mathfrak{h} /\!\!/ H)(k)$, the fibre $p^{-1}(b)$ contains a unique open $H(k)$-orbit (consisting of the regular elements with invariants b) and a unique closed $H(k)$-orbit (consisting of the semisimple elements with invariants b).*

We will often use induction arguments to reduce a statement for $\mathfrak{h}$ to a reductive Lie algebra of smaller rank. To this end, the following lemma will be helpful. We suppose for this lemma that H is split and $T \subset H$ is a split maximal torus. This determines a root datum $(X^*(T), \Phi_{\mathfrak{t}}, X_*(T), \Phi_{\mathfrak{t}}^{\vee})$ (in the sense of [72, §7.4]) and a Weyl group $W = N_G(T)/T$.

Lemma 2.2. *Let $x \in \mathfrak{t}$ be a semisimple element. Then the centraliser $Z_H(x)$ is a (connected) reductive group. Moreover, let*

$$\Phi_{\mathfrak{t}}(x) = \{\alpha \in \Phi_{\mathfrak{t}} \mid \alpha(x) = 0\} \quad \text{and} \quad \Phi_{\mathfrak{t}}^{\vee}(x) = \{\alpha^{\vee} \in \Phi_{\mathfrak{t}}^{\vee} \mid \alpha \in \Phi_{\mathfrak{t}}(x)\}.$$

Let $W_x = Z_W(x)$. Then the root datum of $Z_H(x)$ is $(X^*(T), \Phi_{\mathfrak{t}}(x), X_*(T), \Phi_{\mathfrak{t}}^{\vee}(x))$, and the Weyl group of $Z_H(x)$ with respect to T is isomorphic to W_x.

Proof. The centraliser $Z_H(x)$ is connected by [74, Theorem 3.14]. The fact that it is reductive and has the above root datum follows from [74, Lemma 3.7]. The claim about the Weyl group of $Z_H(x)$ follows from [74, Lemma 3.7 (c)], again using the fact that $Z_H(x)$ is connected. ■

Let $\mathfrak{c} \subset \mathfrak{h}$ be a Cartan subalgebra. The *discriminant polynomial* $\Delta \in k[\mathfrak{h}]^H$ is the image of the product of all the roots $\prod \alpha \in k[\mathfrak{c}]^W$ with respect to $\mathfrak{c}$ under the Chevalley isomorphism $k[\mathfrak{c}]^W \xrightarrow{\sim} k[\mathfrak{h}]^H$; it is independent of the choice of $\mathfrak{c}$. For $x \in \mathfrak{h}$, we have $\Delta(x) \neq 0$ if and only if x is regular semisimple. The *discriminant locus* (or discriminant divisor) $D \subset \mathfrak{h} /\!\!/ H$ is the zero locus of Δ. This subscheme will play a fundamental role later in this paper (in particular, in Section 4).

The next lemma says that the étale local structure of $\mathfrak{h} /\!\!/ H$ and D near a point is determined by the centraliser of a semisimple lift of that point.

Lemma 2.3. *Let $x \in \mathfrak{h}$ be a semisimple element with centraliser $\mathfrak{z}_{\mathfrak{h}}(x)$. Let $\mathfrak{c} \subset \mathfrak{h}$ be a Cartan subalgebra containing x. Let W and W_x be the respective Weyl groups of $\mathfrak{h}$ and $\mathfrak{z}_{\mathfrak{h}}(x)$ with respect to $\mathfrak{c}$. Consider the diagram*

$$\begin{tikzcd}
\mathfrak{c} \arrow[d, "\phi_x"'] \arrow[dr, "\phi"] & \\
\mathfrak{c} /\!\!/ W_x \arrow[r, "\psi"] & \mathfrak{c} /\!\!/ W.
\end{tikzcd}$$

Then ψ is étale at $\phi_x(x)$. Moreover, if D and D_x denote the discriminant divisors of $\mathfrak{h}$ and $\mathfrak{z}_{\mathfrak{h}}(x)$, respectively, then $\psi^* D = D_x + R$, where R is a divisor of $\mathfrak{c} /\!\!/ W_x$ not containing $\phi_x(x)$ in its support.

Proof. We may assume that k is algebraically closed and hence that $\mathfrak{h}$ is split. Since ϕ and ϕ_x are finite and faithfully flat (they are even Galois with Galois groups W and W_x), the map ψ is finite and faithfully flat. The fact that ψ is étale at $\phi_x(x)$ follows from the fact that the stabiliser of the W-action on x is precisely W_x (Lemma 2.2).

To prove the claim about the discriminant divisors, let $\Delta = \prod_{\alpha \in \Phi_{\mathfrak{c}}} \alpha \in k[\mathfrak{c}]^W$ and $\Delta_x = \prod_{\alpha \in \Phi_{\mathfrak{c}}(x)} \alpha \in k[\mathfrak{c}]^{W_x}$ denote the respective discriminant polynomials. By definition of $\Phi_{\mathfrak{c}}(x)$, $\Delta = \Delta_x \cdot \mathcal{R}$ as elements of $k[\mathfrak{c}]^{W_x}$, where $\mathcal{R} \in k[\mathfrak{c}]^{W_x}$ is a polynomial that does not vanish at x. Since D and D_x are the zero loci of Δ and Δ_x, respectively, this proves the claim. ■

2.2. *Vinberg theory*

We keep the notations from Section 2.1. Let $m \geq 1$ be an integer. A *$\mathbb{Z}/m\mathbb{Z}$-grading* on $\mathfrak{h}$ is, by definition, a direct sum decomposition

$$\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{h}(i)$$

into linear subspaces satisfying $[\mathfrak{h}(i), \mathfrak{h}(j)] \subset \mathfrak{h}(i+j)$. Given a $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{h}$, let $\mathfrak{g} := \mathfrak{h}(0)$ and $V := \mathfrak{h}(1)$. Then $\mathfrak{g}$ is a subalgebra of $\mathfrak{h}$, and the restriction of the adjoint representation of $\mathfrak{h}$ induces an action of $\mathfrak{g}$ on V. If $\zeta \in k$ is a primitive m-th root of unity, giving a $\mathbb{Z}/m\mathbb{Z}$-grading amounts to giving, by considering ζ^i-eigenspaces, an automorphism θ of $\mathfrak{h}$ of order dividing m. In general, when no such ζ exists or is fixed, giving a $\mathbb{Z}/m\mathbb{Z}$-grading amounts to giving a homomorphism $\mu_m \to \mathrm{Aut}(\mathfrak{h})$ of group schemes over k.

Suppose that $\mu_m \to \mathrm{Aut}(H)$ is a morphism of group schemes. The composition $\mu_m \to \mathrm{Aut}(H) \to \mathrm{Aut}(\mathfrak{h})$ determines a $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{h}$. If G is the identity component of the centraliser of μ_m in H, then G has Lie algebra $\mathfrak{g}$ and acts on $V = \mathfrak{h}(1)$ by restriction of the adjoint action. The pair (G, V) is called a *Vinberg representation*, and its study is dubbed *Vinberg theory* [81]. If $\mathfrak{h}$ is a semisimple $\mathbb{Z}/m\mathbb{Z}$-graded Lie algebra, a natural choice for H is the adjoint group $\mathrm{Aut}(\mathfrak{h})^\circ$ of $\mathfrak{h}$: this is the unique (up to non-unique isomorphism) connected semisimple group with trivial centre and Lie algebra $\mathfrak{h}$.

We now summarise some of the highlights of Vinberg theory, referring to [47, 55] for proofs. We call an element $x \in V$ semisimple, nilpotent or regular if it is so when considered as an element of $\mathfrak{h}$. We call a subspace $\mathfrak{c} \subset V$ that consists of semisimple elements, that satisfies $[\mathfrak{c}, \mathfrak{c}] = 0$, and is maximal with these properties (among subspaces of V) a *Cartan subspace*.

Lemma 2.4. *If $x \in V$ has Jordan decomposition $x = x_s + x_n$, where x_s, x_n are commuting elements that are semisimple and nilpotent respectively, then $x_s, x_n \in V$.*

Proposition 2.5. *Every semisimple $x \in V$ is contained in a Cartan subspace of V. Every two Cartan subspaces are $G(k^s)$-conjugate.*

We call a triple (e, h, f) an $\mathfrak{sl}_2$*-triple* of $\mathfrak{h}$ if e, h, f are nonzero elements of $\mathfrak{h}$ satisfying the following relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The classical Jacobson–Morozov lemma states that every nilpotent element in $\mathfrak{h}$ can be completed to an $\mathfrak{sl}_2$-triple. If $\mathfrak{h}$ is $\mathbb{Z}/m\mathbb{Z}$-graded, we say an $\mathfrak{sl}_2$-triple is *normal* if $e \in \mathfrak{h}(1)$, $h \in \mathfrak{h}(0)$ and $f \in \mathfrak{h}(-1)$.

Lemma 2.6 (Graded Jacobson–Morozov). *Every nilpotent $e \in \mathfrak{h}(1)$ is contained in a normal $\mathfrak{sl}_2$-triple (e, h, f).*

Proof. See [42, Proposition 4], which only treats the case $m = 2$ but whose proof works for any m. (We will only need the $m = 2$ case in this paper.) ∎

The next proposition describes the basic geometric invariant theory of the representation (G, V). Let $\pi\colon V \to V /\!\!/ G = \operatorname{Spec} k[V]^G$ be the graded analogue of the adjoint quotient from Section 2.1.

Proposition 2.7. (1) *Let $\mathfrak{c} \subset V$ be a Cartan subspace and $W(\mathfrak{c}) := N_G(\mathfrak{c})/Z_G(\mathfrak{c})$. Then the inclusion $\mathfrak{c} \subset V$ induces an isomorphism*

$$\mathfrak{c} /\!\!/ W(\mathfrak{c}) \simeq V /\!\!/ G.$$

The group $W(\mathfrak{c})$ is a finite pseudo-reflection group, so the quotient is isomorphic to affine space.

(2) *If k is algebraically closed and $b \in (V /\!\!/ G)(k)$, then the fibre $\pi^{-1}(b)$ contains a unique closed $G(k)$-orbit, the set of semisimple elements with invariants b.*

Remark 2.8. In contrast to the $m = 1$ case, it is not true in general that two regular elements $x, y \in V(k)$ with the same invariants (that is, with the same images in $V /\!\!/ G$) are $G(k^s)$-conjugate. Indeed, it follows from Proposition 3.8 below that the $\mathbb{Z}/2\mathbb{Z}$-gradings introduced in Section 3.1 can have multiple $G(k^s)$-orbits of regular nilpotent elements.

2.3. Stable gradings

We keep the notations of Sections 2.1 and 2.2. Of particular interest to us are the stable gradings.

Definition 2.9. Suppose that k is algebraically closed. We say a vector $v \in V$ is *stable* (in the sense of geometric invariant theory) if the G-orbit of v is closed and its stabiliser $Z_G(v)$ is finite. We say (G, V) is *stable* if V contains stable vectors. If k is not necessarily algebraically closed, we say (G, V) is stable if (G_{k^s}, V_{k^s}) is.

Stable gradings of simple Lie algebras over an algebraically closed field of characteristic zero have been classified [62, §§7.1 and 7.2] in terms of regular elliptic conjugacy classes of (twisted) Weyl groups. In the case of involutions (in other words, $\mathbb{Z}/2\mathbb{Z}$-gradings), this classification takes the simple form of Lemma 2.10, see [77, Lemma 2.6] for a proof. We say two involutions $\theta, \theta' \colon H \to H$ are $H(k)$-conjugate if there exists an $h \in H(k)$ such that $\theta' = \mathrm{Ad}(h) \circ \theta \circ \mathrm{Ad}(h)^{-1}$.

Lemma 2.10. *Suppose that k is algebraically closed. Then there exists a unique $H(k)$-conjugacy class of stable involutions.*

For example, if H is a torus, then the only stable involution is given by the inversion map $h \mapsto h^{-1}$.

One of the main advantages of stable gradings is that they have a particularly good invariant theory. The next proposition describes this more precisely in the case of $\mathbb{Z}/2\mathbb{Z}$-gradings. In particular, it shows that regular semisimple orbits over algebraically closed fields are well understood. We refer to [77, §2] for precise references.

Proposition 2.11. *Suppose that $\theta \colon H \to H$ is a stable involution, with associated Vinberg representation (G, V). Then the following properties are satisfied:*

(1) *Let $\mathfrak{c} \subset V$ be a Cartan subspace and $W(\mathfrak{c}) := N_G(\mathfrak{c})/Z_G(\mathfrak{c})$. Then $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{h}$ and the map $N_G(\mathfrak{c}) \to W_{\mathfrak{c}} := N_H(\mathfrak{c})/Z_H(\mathfrak{c})$ is surjective. Consequently, the inclusions $\mathfrak{c} \subset V \subset \mathfrak{h}$ induce isomorphisms*

$$\mathfrak{c} /\!\!/ W_{\mathfrak{c}} \simeq V /\!\!/ G \simeq \mathfrak{h} /\!\!/ H.$$

In particular, the quotient is isomorphic to affine space.

(2) *Let k be algebraically closed and let $x, y \in V(k)$ be regular semisimple elements. Then x is $G(k)$-conjugate to y if and only if x, y have the same image in $V /\!\!/ G$.*

(3) *Let $\Delta \in \mathbb{Q}[V]^G$ be the restriction of the Lie algebra discriminant of $\mathfrak{h}$ to the subspace V and let k be algebraically closed. Then for all $x \in V(k)$, x is regular semisimple if and only if $\Delta(x) \neq 0$, if and only if x is stable in the sense of Definition 2.9.*

2.4. *Arithmetic invariant theory*

Let k be a field with separable closure k^s. Let G/k be a smooth algebraic group acting on a k-vector space V. In general, a fixed $G(k^s)$-orbit in $V(k^s)$ might break up into multiple $G(k)$-orbits, and the study of this phenomenon is referred to as arithmetic invariant theory [8]. We recall its relation to Galois cohomology, which lies at the basis of the orbit parametrisations in this paper.

Lemma 2.12. *Suppose that G acts on a k-scheme X. Suppose that the k-point $e \in X(k)$ has smooth stabiliser $Z_G(e)$ and that the action of $G(k^s)$ on $X(k^s)$ is transitive. Then there is a natural bijection*

$$G(k)\backslash X(k) \overset{1:1}{\longleftrightarrow} \ker(\mathrm{H}^1(k, Z_G(e)) \to \mathrm{H}^1(k, G)).$$

Proof. This is [8, Proposition 1]. The bijection is explicitly constructed as follows: if $x \in X(k)$, transitivity ensures that there exists an element $g \in G(k^s)$ with $x = g \cdot e$. For every element $\sigma \in \Gamma_k = \mathrm{Gal}(k^s/k)$, we again have $x = \sigma(g) \cdot e$, so the map $\sigma \mapsto g^{-1}\sigma(g)$ defines a 1-cocycle with values in $Z_G(e)$ which is trivial in $\mathrm{H}^1(k, G)$. ■

In fact, we will need a relative version of Lemma 2.12 which is valid over any base scheme.

Lemma 2.13. *Let $G \to S$ be a smooth affine group scheme acting on an S-scheme X. Let $e \in X(S)$ be an S-point, and suppose that the action map $m\colon G \to X$, $g \mapsto g \cdot e$ is smooth and surjective. Then the assignment $x \mapsto$ 'isomorphism class of the $Z_G(e)$-torsor $m^{-1}(x)$' induces a bijection between the set of $G(S)$-orbits of $X(S)$ and the kernel of the map of pointed sets* $\mathrm{H}^1(S, Z_G(e)) \to \mathrm{H}^1(S, G)$.

Proof. This is [27, Exercise 2.4.11]: the conditions imply that $X \simeq G/Z_G(e)$ and since G and $Z_G(e)$ (the fibre above e of the smooth map m) are S-smooth, we can replace fppf cohomology by étale cohomology. ■

2.5. *The Grothendieck–Serre conjecture*

In this subsection, we discuss some general results concerning principal bundles over reductive group schemes which will be useful in Section 6. Recall from [27, Definition 3.1.1] that a reductive group scheme over S is a smooth S-affine group scheme $G \to S$ whose geometric fibres are connected reductive groups.

Definition 2.14. Let R be a regular local ring with fraction field K and let $G \to \mathrm{Spec}\, R$ be a reductive group scheme. We say that the *Grothendieck–Serre conjecture holds for R and G* if the restriction map

$$\mathrm{H}^1(R, G) \to \mathrm{H}^1(K, G)$$

is injective.

Note that the injectivity $\mathrm{H}^1(R, G) \to \mathrm{H}^1(K, G)$ is stronger than requiring that this map has a trivial kernel, since this is merely a map of pointed sets. It is conjectured that the Grothendieck–Serre conjecture holds for every reductive group scheme over every regular local ring; see [53] for a survey and [23, §1.4] for a short summary of known results. Below we will single out the known cases that we will need.

Lemma 2.15. *Let X be a regular integral scheme with function field K. Let G be a reductive X-group scheme. Suppose that the Grothendieck–Serre conjecture holds for all local rings of X and G. Then every two G-torsors over X that are generically isomorphic (that is, isomorphic over K) are Zariski locally isomorphic (that is, isomorphic after restricting to a Zariski open cover).*

Proof. Let T, T' be two G-torsors over X which are generically isomorphic and let $x \in X$. We need to prove that x has an open neighbourhood over which T and T' are isomorphic. Since the Grothendieck–Serre conjecture holds for the local ring $\mathcal{O}_{X,x}$, the torsors T and T' are isomorphic when restricted to $\operatorname{Spec} \mathcal{O}_{X,x}$. The result follows from spreading out this isomorphism. ■

Proposition 2.16. *Let R be a regular local ring and G a reductive R-group. Suppose that at least one of the following is satisfied:*

- *R is a discrete valuation ring;*
- *R contains an infinite field.*

Then the Grothendieck–Serre conjecture holds for R and G.

Proof. The case of a discrete valuation ring was proved by Nisnevich [51], with corrections by Guo [37]. The case where R contains an infinite field was proved by Fedorov and Panin [34]. ■

The conjecture is known in many other cases; see [23] for a recent general result when R is of mixed characteristic and [54] for the case where R contains a finite field.

Corollary 2.17. *Let X be a regular integral scheme and G a reductive X-group. Suppose that at least one of the following conditions is satisfied:*

- *X is a Dedekind scheme;*
- *X has a map to the spectrum of an infinite field.*

Then every two G-torsors over X that are generically isomorphic are Zariski locally isomorphic.

Proof. Combine Lemma 2.15 and Proposition 2.16. ■

3. Around Thorne's thesis

In the remainder of this paper, we will focus on a particular Vinberg representation. Given a Dynkin diagram of type A, D, E, we will canonically construct a stable $\mathbb{Z}/2\mathbb{Z}$-grading on the Lie algebra of the corresponding type following Thorne's thesis [77, §2]. We then recall and extend some of its basic properties in Sections 3.2–3.6. In Section 3.7, we introduce the corresponding family of curves $C \to B$, and in Section 3.8 we recall the relation between stabilisers in V and the 2-torsion in the Jacobians of smooth fibres of $C \to B$. We do not claim any originality in this section, except maybe for some of the calculations in Section 3.9.

3.1. *A split stable $\mathbb{Z}/2\mathbb{Z}$-grading*

Let H be a split adjoint simple group of type A, D, E over $\mathbb{Q}$ with Dynkin diagram D. We have an exact sequence

$$1 \to H \to \operatorname{Aut}(H) \to \operatorname{Aut}(\mathsf{D}) \to 1. \tag{3.1}$$

Assume that H is equipped with a pinning $(T, P, \{X_\alpha\})$. By definition, this means that $T \subset H$ is a split maximal torus (which determines a root system $\Phi_H := \Phi_{\mathfrak{t}}$), $P \subset H$ is a Borel subgroup containing T (which determines a root basis $S_H \subset \Phi_H$), and X_α is a generator for each root space $\mathfrak{h}_\alpha$ for $\alpha \in S_H$. The subgroup $\mathrm{Aut}((H, T, P, \{X_\alpha\})) \subset \mathrm{Aut}(H)$ of elements preserving the pinning determines a splitting of sequence (3.1).

On the other hand, if $W = N_H(T)/T$ denotes the Weyl group of Φ_H, we have an exact sequence

$$1 \to W \to \mathrm{Aut}(\Phi_H) \to \mathrm{Aut}(\mathsf{D}) \to 1. \tag{3.2}$$

We define $\vartheta \in \mathrm{Aut}(H)$ as the unique element of $\mathrm{Aut}((H, T, P, \{X_\alpha\}))$ whose image in $\mathrm{Aut}(\mathsf{D})$ under (3.1) coincides with the image of $-1 \in \mathrm{Aut}(\Phi_H)$ in $\mathrm{Aut}(\mathsf{D})$ under (3.2). Note that $\vartheta = 1$ if and only if $-1 \in W$.

Write $\check{\rho} \in X_*(T)$ for the sum of the fundamental coweights with respect to S_H, characterised by the property that $(\alpha \circ \check{\rho})(t) = t$ for all $\alpha \in S_H$. Let

$$\theta := \vartheta \circ \mathrm{Ad}(\check{\rho}(-1)) = \mathrm{Ad}(\check{\rho}(-1)) \circ \vartheta.$$

Then θ defines an involution of $\mathfrak{h}$, and thus by considering (± 1)-eigenspaces it determines a $\mathbb{Z}/2\mathbb{Z}$-grading

$$\mathfrak{h} = \mathfrak{h}(0) \oplus \mathfrak{h}(1).$$

Let $G := (H^\theta)^\circ$ be the identity component of the centraliser of θ in H and let $V := \mathfrak{h}(1)$. The space V defines a representation of G by restricting the adjoint representation. If we write $\mathfrak{g}$ for the Lie algebra of G, then V is a Lie algebra representation of $\mathfrak{g} = \mathfrak{h}(0)$. The Vinberg representation (G, V) is a central object of study of this paper. Its relevance can be summarised in the following proposition, proved in [79, Proposition 1.9].

Proposition 3.1. *Up to $H(\mathbb{Q})$-conjugacy, θ is the unique involution of H with the property that θ is stable (Definition 2.9) and the reductive group G is split over $\mathbb{Q}$.*

The first property of Proposition 3.1 is geometric: it characterises the $H(\bar{\mathbb{Q}})$-conjugacy class of θ. The second property is arithmetic, and it is equivalent to requiring the existence of a regular nilpotent in $V(\mathbb{Q})$. (For the last claim, see [77, Corollary 2.15].) Note that in our construction of θ the element $E = \sum_{\alpha \in S_H} X_\alpha$ is a regular nilpotent in $V(\mathbb{Q})$.

Write $B := V /\!\!/ G = \mathrm{Spec}\, \mathbb{Q}[V]^G$ and $\pi\colon V \to B$ for the natural quotient map. We have a $\mathbb{G}_m$-action on V given by $\lambda \cdot v = \lambda v$, and there is a unique $\mathbb{G}_m$-action on B such that π is $\mathbb{G}_m$-equivariant. The invariant theory of the pair (G, V) is summarised in Proposition 2.11. We additionally record the following fact concerning the smooth locus of π, which follows from the proof of [77, Proposition 3.10].

Lemma 3.2. *Let $x \in V$ and let $d\pi_x$ be the induced map on tangent spaces $T_x V \to T_{\pi(x)} B$. Then $d\pi_x$ is surjective if and only if x is regular. Consequently, the smooth locus of π coincides with $V^{\mathrm{reg}} := V \cap \mathfrak{h}^{\mathrm{reg}}$.*

Type	G	V	$\pi_0(H^\theta)$
A_{2r}	SO_{2r+1}.	$\mathrm{Sym}_0^2(2r+1)$	1
A_{2r+1}	PSO_{2r+2}	$\mathrm{Sym}_0^2(2r+2)$	$\mathbb{Z}/2\mathbb{Z}$
D_{2r}	$(\mathrm{SO}_{2r} \times \mathrm{SO}_{2r})/\Delta(\mu_2)$	$(2r) \boxtimes (2r)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
D_{2r+1}	$\mathrm{SO}_{2r+1} \times \mathrm{SO}_{2r+1}$	$(2r+1) \boxtimes (2r+1)$	$\mathbb{Z}/2\mathbb{Z}$
E_6	PSp_8	$\wedge_0^4(8)$	1
E_7	SL_8/μ_4	$\wedge^4(8)$	$\mathbb{Z}/2\mathbb{Z}$
E_8	Spin_{16}/μ_2	half spin	1

Tab. 3. Short description of each representation.

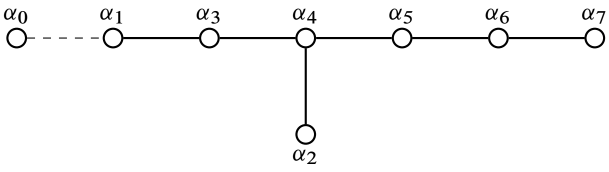
3.2. *Explicit determination of* (G, V)

Using the results of [61] applied to the Kac diagram of θ [62, §§7.1 and 7.2] (or an explicit description of θ in the case of classical groups), one may calculate the isomorphism class of the split group G and the representation V explicitly. These results are summarised in Table 3, where we have used the following notation:

- If G is defined as a subgroup of GL_n, then (n) denotes the representation of G corresponding to this embedding.
- In the case A_n, $\mathrm{Sym}_0^2(n)$ denotes the unique codimension one G-subrepresentation of $\mathrm{Sym}^2(n)$. Equivalently, $\mathrm{Sym}_0^2(n)$ can be viewed as the set of self-adjoint linear maps $(n) \to (n)$ of trace zero.
- In the case D_{2r}, $\Delta(\mu_2)$ denotes the image of μ_2 diagonally embedded in the centre $\mu_2 \times \mu_2$ of $\mathrm{SO}_{2r} \times \mathrm{SO}_{2r}$.
- In the case E_6, $\wedge_0^4(8)$ denotes the unique 42-dimensional subrepresentation of the PSp_8-representation $\wedge^4(8)$.
- In the case E_8, Spin_{16}/μ_2 denotes a μ_2-quotient of Spin_{16} that is not isomorphic to SO_{16}; it does not seem to have a more succinct name.

We will only need these explicit identifications in the proof of Proposition 8.12. Moreover, we will calculate the component group of H^θ and the centre of G more uniformly in Sections 3.3 and 3.4.

We treat the E_7 case as an example. The extended Dynkin diagram is given by



The normalised Kac coordinates of θ (given in [62, §7.1, Table 4]) are everywhere zero, except at the bottom node α_2, which has coordinate 1. We may now apply the results of [61, §2.4]. Since the Kac coordinates are invariant under the automorphism of the extended diagram, the component group of H^θ is of order 2. Since the highest root has coordinate 2 at α_2, the centre of G is of order 2. If we delete the node α_2, we obtain a diagram of type A_7, so G semisimple is of type A_7. Since G is split, it follows that $G \simeq \mathrm{SL}_8/\mu_4$. Moreover, the representation V has highest weight the fundamental weight corresponding to α_4, so is isomorphic to $\wedge^4(8)$, where (8) denotes the defining representation of SL_8.

3.3. The component group of H^θ

The group H^θ is typically disconnected, and we have a tautological exact sequence

$$1 \to G \to H^\theta \to \pi_0(H^\theta) \to 1.$$

The component group $\pi_0(H^\theta)$ is a finite étale group scheme over $\mathbb{Q}$. We will show that $\pi_0(H^\theta)$ is split and describe it in two different ways, which will be useful in the proof of Proposition 6.10.

Firstly, we use Weyl groups. Recall that $W_H = N_H(T)/T$ denotes the Weyl group of H. We know that $T^\theta = T^\vartheta$ is a maximal torus of G, and moreover the centraliser $Z_H(T^\theta)$ of T^θ equals T; these claims can be verified explicitly or follow from [63, Lemmas 5.1 and 5.3]. It follows that $N_H(T^\theta) \subset N_H(T)$, so $W_{H^\theta} := N_{H^\theta}(T^\theta)/T^\theta$ is naturally a subgroup of W_H. Let $W_G := N_G(T^\theta)/T^\theta$ be the Weyl group of G, a normal subgroup of W_{H^θ}. We have inclusions $W_G \subset W_{H^\theta} \subset W_H$.

Lemma 3.3. *The inclusion $N_{H^\theta}(T^\theta) \subset H^\theta$ induces an isomorphism*

$$W_{H^\theta}/W_G \simeq \pi_0(H^\theta).$$

Proof. This is implicit in the proof of [61, Lemma 3.9]; we sketch the details. It suffices to prove that $H^\theta = G \cdot N_{H^\theta}(T^\theta)$. This can be checked on geometric points, so let $k/\mathbb{Q}$ be an algebraically closed field and let $h \in H^\theta(k)$. The conjugate subgroup $\mathrm{Ad}(h) \cdot T^\theta$ is a maximal torus of G_k. Since G_k is reductive, $G(k)$ acts transitively on its maximal tori, so $\mathrm{Ad}(h) \cdot T^\theta = \mathrm{Ad}(g) \cdot T^\theta$ for some $g \in G$. We see that $g^{-1}h \in N_{H^\theta}(T^\theta)$, as claimed. ■

Corollary 3.4. *The finite étale $\mathbb{Q}$-group $\pi_0(H^\theta)$ is constant (in other words, has trivial Galois action) and the map $H^\theta(\mathbb{Q}) \to \pi_0(H^\theta)$ is surjective.*

Proof. It suffices to prove the latter claim. Since T is a maximal torus of H, W_H is a constant group scheme, so its subgroup W_{H^θ} is constant too. By Lemma 3.3, it suffices to show that $N_{H^\theta}(T^\theta)(\mathbb{Q}) \to W_{H^\theta}$ is surjective. This follows from Hilbert's Theorem 90 since the torus T^θ is $\mathbb{Q}$-split. ■

For the second description, choose a Cartan subspace $\mathfrak{c} \subset V$ and let $C \subset H$ be the maximal torus with Lie algebra $\mathfrak{c}$. Since θ acts as -1 on $\mathfrak{c}$, it acts via inversion on C hence $C[2] \subset H^\theta$. The next lemma is [42, Proposition 1].

Lemma 3.5. *We have* $H^\theta = G \cdot C[2]$. *In other words, the inclusion* $C[2] \subset H^\theta$ *induces a surjection* $C[2] \twoheadrightarrow \pi_0(H^\theta)$.

Lemma 3.5 allows us to give an explicit description of $\pi_0(H^\theta)$.

Corollary 3.6. *Let* $H_{\mathrm{sc}} \to H$ *be the simply connected cover of* H *and let* $\pi_1(H)$ *denote the centre of* H_{sc}. *Then there is an isomorphism* $\pi_0(H^\theta) \simeq \pi_1(H)/2\pi_1(H)$.

Proof. Let $C \subset H$ be a maximal torus whose Lie algebra is a Cartan subspace of V. (Such a torus certainly exists: take the centraliser of a regular semisimple element of V.) Let $C_{\mathrm{sc}} \subset H_{\mathrm{sc}}$ be its preimage in H_{sc}. We have an exact sequence

$$1 \to \pi_1(H) \to C_{\mathrm{sc}} \to C \to 1. \tag{3.3}$$

Examining the long exact sequence associated with the 2-torsion of (3.3) shows that

$$\frac{C[2]}{\mathrm{image}(C_{\mathrm{sc}}[2] \to C[2])} \simeq \pi_1(H)/2\pi_1(H).$$

We claim that the left-hand side is isomorphic to $\pi_0(H^\theta)$. Indeed, the involution θ: $H \to H$ uniquely extends to an involution of H_{sc}, still denoted by θ, and a theorem of Steinberg [73, Theorem 8.1] shows that H_{sc}^θ is connected. It follows that the induced map $H_{\mathrm{sc}}^\theta \to H^\theta$ surjects onto G. Therefore, the kernel of the natural map $C[2] \to \pi_0(H^\theta)$ (which is surjective by Lemma 3.5) agrees with the image of the map $C_{\mathrm{sc}}[2] \to C[2]$, as claimed. ■

3.4. *The fundamental group of G*

Proposition 3.7. *The group G is semisimple and its fundamental group has order equals* $2\#\pi_0(H^\theta)$.

Proof. Let $H_{\mathrm{sc}} \to H$ be the simply connected cover of H and let $\pi_1(H)$ denote the centre of H_{sc}. By the previously invoked theorem of Steinberg [73, Theorem 8.1], H_{sc}^θ is connected. Therefore, the induced map $H_{\mathrm{sc}}^\theta \to G$ is surjective with kernel $\pi_1(H)[2]$. Moreover, $\pi_1(H)[2]$ has cardinality $\#\pi_0(H^\theta)$ by Corollary 3.6. Hence it suffices to prove that H_{sc}^θ is semisimple and its fundamental group is of order 2. This is a result of Kaletha, see [79, Proposition A.1]. ■

3.5. *Regular nilpotent elements in V*

The next proposition describes the set of regular nilpotent elements in V [77, Lemma 2.14].

Proposition 3.8. *For every field* $k/\mathbb{Q}$, *the group* $H^\theta(k)$ *acts simply transitively on the set of regular nilpotent elements of* $V(k)$.

Corollary 3.9. *Let $k/\mathbb{Q}$ be a field and $E \in V(k)$ a regular nilpotent. (For example, $E = \sum_{\alpha \in S_H} X_\alpha$.) Then the map $h \mapsto h \cdot E$ induces a bijection between $\pi_0(H^\theta)$ and the set of $G(k)$-orbits of regular nilpotent elements in $V(k)$.*

Proof. Follows from Proposition 3.8 and the fact that $H^\theta(k) \to \pi_0(H^\theta)$ is surjective (Corollary 3.4). ■

We see in particular that if H^θ is disconnected, then there are multiple G-orbits of regular nilpotent elements in V. To state the next result, recall from Section 2.2 the notion of a normal $\mathfrak{sl}_2$-triple.

Corollary 3.10. *Let $k/\mathbb{Q}$ be a field and $E \in V(k)$ a regular nilpotent element. Then E is contained in a unique normal $\mathfrak{sl}_2$-triple.*

Proof. Proposition 3.8 shows that the stabiliser $Z_G(E)$ is trivial. Therefore, the corollary follows from [77, Lemma 2.17]. ■

3.6. Kostant sections

We describe sections of the GIT quotient $\pi \colon V \to B$ whose remarkable construction is originally due to Kostant. Let $E \in V(\mathbb{Q})$ be a regular nilpotent element and let (E, X, F) be the unique normal $\mathfrak{sl}_2$-triple containing E using Corollary 3.10. We define the affine linear subspace

$$\kappa_E := (E + \mathfrak{z}_{\mathfrak{h}}(F)) \cap V \subset V.$$

We call κ_E the *Kostant section* associated with E, or simply a Kostant section.

Proposition 3.11. (1) *The composition $\kappa_E \hookrightarrow V \to B$ is an isomorphism.*

(2) *κ_E is contained in the open subscheme of regular elements of V.*

(3) *The morphism $G \times \kappa_E \to V$, $(g, v) \mapsto g \cdot v$ is étale.*

Proof. Parts (1) and (2) are [77, Lemma 3.5]; the last part is [77, Proposition 3.4], together with the fact that $G \times \kappa_E$ and V have the same dimension (apply [77, Lemma 2.21] to $x = 0$). ■

Every Kostant section κ_E determines a morphism $B \to V$ that is a section of the quotient map $\pi \colon V \to B$, and we denote this section by κ_E too. For any $b \in B(k)$, we write $\kappa_{E,b}$ for the fibre of κ_E over b.

Definition 3.12. Let $k/\mathbb{Q}$ be a field and $v \in V(k)$. We say v is *k-reducible* if v is not regular semisimple or v is $G(k)$-conjugate to $\kappa_{E,b}$ for some Kostant section κ_E and where $b = \pi(v)$. Otherwise, we call v *k-irreducible*.

If k is algebraically closed, then every element of $V(k)$ is k-reducible by Proposition 2.11.

3.7. *A family of curves*

If $k/\mathbb{Q}$ is a field, an element $v \in \mathfrak{h}(k)$ is called *subregular* if $\dim \mathfrak{z}_{\mathfrak{h}}(x) = r + 2$, where r is the rank of $\mathfrak{h}$. By [77, Proposition 2.27], the vector space V contains subregular nilpotent elements; let $e \in V(\mathbb{Q})$ be such an element, and fix (e, x, f) a normal $\mathfrak{sl}_2$-triple extending it, using Lemma 2.6.

Slodowy [71] has shown that the restriction of the invariant map $(e + \mathfrak{z}_{\mathfrak{h}}(f)) \to B$ is a family of surfaces. Moreover, he has shown that this family is a semi-universal deformation of its central fibre, which is a simple surface singularity of the type corresponding to that of H. Proposition 3.13 is a $\mathbb{Z}/2\mathbb{Z}$-graded analogue of Slodowy's result, due to Thorne. Define $C^\circ := (e + \mathfrak{z}_{\mathfrak{h}}(f)) \cap V$. Restricting the invariant map $\pi\colon V \to B$ to C° defines a morphism $\varphi\colon C^\circ \to B$.

Proposition 3.13. (1) *The geometric fibres of φ are reduced connected curves.*

(2) *The central fibre $C_0^\circ = \varphi^{-1}(0)$ has a unique singular point which is a simple singularity of type A_r, D_r, E_r, corresponding to that of H.*

(3) *We can choose coordinates $p_{d_1}, \dots, p_{d_r}$ on B, where p_{d_i} is homogeneous of degree d_i, and coordinates $(x, y, p_{d_1}, \dots, p_{d_{r-1}})$ on C° such that $C^\circ \to B$ is given by the affine equation of Table 1.*

(4) *The formal completion of $C^\circ \to B$ along its central fibre defines a morphism of formal schemes $\widehat{C^\circ} \to \widehat{B}$ which is a semi-universal deformation of its central fibre.*

(5) *The morphism φ is faithfully flat. It is smooth at $x \in C^\circ$ if and only if x is a regular element of V.*

(6) *The action map $G \times C^\circ \to V$, $(g, x) \mapsto g \cdot x$ is smooth.*

Proof. This is proved in Thorne's thesis. The first three parts are [77, Theorem 3.8]; for the definition of a simple curve singularity, see [76, end of §2]. The fourth part follows from the fact that the semi-universal deformation of an isolated hypersurface singularity can be explicitly computed [71, §2.4] and agrees with the equations given in Table 1. The last two parts are contained in [77, Propositions 3.4 and 3.10]. ∎

The next lemma describes the singularities of the fibres of $C^\circ \to B$ very precisely; see [77, Corollary 3.16] for its proof.

Lemma 3.14. *Let $k/\mathbb{Q}$ be a field, let $b \in B(k)$ and let $v \in V_b(k)$ be a semisimple element. Then there is a bijection between the connected components of the Dynkin diagram of $Z_H(v)$ and the singularities of C_b°, which takes each (connected, simply laced) Dynkin diagram to a singularity of the corresponding type.*

We compactify the flat affine family of curves $C^\circ \to B$ to a flat projective family of curves $C \to B$ as described in [77, Lemma 4.9]. That lemma implies that the complement $C \setminus C^\circ$ is a disjoint union of sections $\infty_1, \dots, \infty_m\colon B \to C$ and $C \to B$ is smooth in a Zariski open neighbourhood of these sections. For every field $k/\mathbb{Q}$ and $b \in B(k)$, the curve C_b has k-rational points $\infty_{1,b}, \dots, \infty_{m,b} \in C_b(k)$; we call these the *marked points of C_b*.

Lemma 3.15. *There are natural bijections between*

(1) *the sections $\infty_1, \dots, \infty_m$ of $C \to B$;*

(2) *irreducible components of C_0;*

(3) *G-orbits of regular nilpotent elements of V whose closure contains e.*

The bijections are given as follows: given a section ∞_i, map it to the irreducible component containing $\infty_{i,0} \in C_0$; given an irreducible component of C_0, map it to the G-orbit of any point on its smooth locus.

Proof. See [77, Lemma 4.14] and its proof. ■

For the remainder of this paper, we fix a section $\infty_1 = \infty$ of $C \to B$ and a regular nilpotent element $E \in V(\mathbb{Q})$ whose G-orbit corresponds to ∞ under Lemma 3.15. Moreover, we fix a choice of polynomials $p_{d_1}, \dots, p_{d_r} \in \mathbb{Q}[V]^G$ and coordinates x, y of C° satisfying the conclusions of Proposition 3.13. Recall that we have defined a $\mathbb{G}_m$-action on B which satisfies $\lambda \cdot p_{d_i} = \lambda^{d_i} p_{d_i}$. There exist unique positive integers a, b such that $\lambda \cdot (x, y, p_{d_1}, \dots, p_{d_{r-1}}) := (\lambda^a x, \lambda^b y, \lambda^{2d_1} p_{d_1}, \dots, \lambda^{2d_{r-1}} p_{d_{r-1}})$ defines a $\mathbb{G}_m$-action on C and such that the morphism $C \to B$ is $\mathbb{G}_m$-equivariant with respect to the square of the usual $\mathbb{G}_m$-action on B. (The integers (a, b) are given by (w_r, w_{r+1}) in the table of [77, Proposition 3.6]. These weights can also be defined Lie theoretically, but we will not need this fact in what follows.)

3.8. *Universal centralisers*

Recall from the last paragraph of Section 3.7 that we have fixed a regular nilpotent $E \in V(\mathbb{Q})$; let $\kappa\colon B \to V$ be the Kostant section corresponding to E constructed in Section 3.6. Recall from our conventions in Section 1.7 that if $v\colon S \to V$ is an S-point of V, then $Z_G(v) \to S$ denotes the centraliser of v in G.

Definition 3.16. Let $Z \to B$ be the centraliser $Z_G(\kappa)$ of the Kostant section $\kappa\colon B \to V$ with respect to the G-action on V. Similarly, let $A \to B$ be the centraliser $Z_H(\kappa)$ of $\kappa\colon B \to \mathfrak{h}$ with respect to the H-action on $\mathfrak{h}$.

For every field $k/\mathbb{Q}$ and $b \in B(k)$, the group scheme Z_b (resp. A_b) is the centraliser $Z_G(\kappa_b) \subset G$ of κ_b in G (resp. $Z_H(\kappa_b) \subset H$). We have $Z = A \cap (G \times B)$. Since κ lands in the regular locus of V, A and Z are commutative group schemes. To state the next lemma, recall that $V^{\mathrm{reg}} \subset V$ denotes the open subscheme of regular elements and that $\pi\colon V \to B$ and $p\colon \mathfrak{h} \to B$ denote the morphisms of taking invariants.

Lemma 3.17. *Let $v\colon S \to V^{\mathrm{reg}}$ be a morphism with $b = \pi(v) \in B(S)$. Then there is a canonical isomorphism $Z_G(v) \simeq Z_b$. Similarly, if $v\colon S \to \mathfrak{h}^{\mathrm{reg}}$ is a morphism with invariants $b = p(v) \in B(S)$, then there is a canonical isomorphism $Z_H(v) \simeq A_b$.*

Proof. The isomorphism $Z_G(v) \simeq Z_b$ follows from [77, Proposition 4.1] and a very similar proof works for A; we briefly sketch it. The morphism $H \times B \to \mathfrak{h}^{\mathrm{reg}}$, $(h, b) \mapsto h \cdot \kappa_b$

is smooth and surjective [64, Lemma 3.3.1], so has sections étale locally. It follows that v is H-conjugate to κ_b étale locally on S. Conjugating defines isomorphisms $Z_H(v) \simeq A_b$, again étale locally on S. Since A_b is commutative, these isomorphisms do not depend on the choice of element by which we conjugate v to κ_b. Using étale descent, these isomorphisms glue to give an isomorphism of group schemes $Z_H(v) \simeq A_b$. ■

The next lemma gives a useful description of the fibres of $Z \to B$.

Lemma 3.18. *Let $k/\mathbb{Q}$ be a field and $x \in V(k)$ a regular element, with Jordan decomposition $x = x_s + x_n$. Let $\mathfrak{c} \subset V$ be a Cartan subspace containing x_s and let $C \subset H$ denote the maximal torus with Lie algebra $\mathfrak{c}$. Let $H_{\mathrm{sc}} \to H$ be the simply connected cover of H and $C_{\mathrm{sc}} \to C$ its restriction to C. Then there is a canonical isomorphism*

$$\operatorname{Hom}(Z_G(x), \mathbb{F}_2) \simeq \operatorname{image}\left(\frac{X^*(C)}{2X^*(C) + \mathbb{Z}\Phi_{\mathfrak{c}}(x)} \to \frac{X^*(C_{\mathrm{sc}})}{2X^*(C_{\mathrm{sc}}) + \mathbb{Z}\Phi_{\mathfrak{c}}(x)}\right).$$

Proof. A theorem of Steinberg [73, Theorem 8.1] shows that $(H_{\mathrm{sc}})^\theta$ is connected. Therefore, $Z_G(x) = \operatorname{image}(Z_{(H_{\mathrm{sc}})^\theta}(x) \to Z_{H^\theta}(x))$. Now use [77, Corollary 2.9]. ■

Let B^{rs} denote the image of the subscheme regular semisimple elements in V under $\pi\colon V \to B$. Then B^{rs} is also the complement of the discriminant locus $(\Delta = 0)$ in B, by part (3) of Proposition 2.11. For a B-scheme X, we denote its restriction to B^{rs} by X^{rs}. The group scheme $Z^{\mathrm{rs}} \to B^{\mathrm{rs}}$ is finite étale and $A^{\mathrm{rs}} \to B^{\mathrm{rs}}$ is a family of maximal tori.

Definition 3.19. Let $\Lambda \to B^{\mathrm{rs}}$ be the character group of A^{rs}.

In other words, Λ is the Cartier dual $\operatorname{Hom}(A^{\mathrm{rs}}, \mathbb{G}_m)$ of A^{rs}. The B^{rs}-scheme Λ is an étale sheaf of root lattices in the sense of Section 1.7.2. In particular, it comes equipped with a pairing $\langle\cdot,\cdot\rangle\colon \Lambda \times \Lambda \to \mathbb{Z}$. This pairing induces an alternating pairing $(\cdot,\cdot)\colon \Lambda/2\Lambda \times \Lambda/2\Lambda \to \mathbb{F}_2$ which might be degenerate. Setting

$$N_\Lambda := \operatorname{image}(\Lambda/2\Lambda \to \Lambda^\vee/2\Lambda^\vee),$$

we see [77, Lemma 2.11] that $(\cdot,\cdot)$ descends to a nondegenerate pairing on N_Λ. Lemma 3.18 implies the following.

Lemma 3.20. *There exists a canonical isomorphism $Z^{\mathrm{rs}} \simeq N_\Lambda$.*

We use the isomorphism of Lemma 3.20 to transport the pairing from N_Λ to Z^{rs}: we thus obtain a nondegenerate pairing $Z^{\mathrm{rs}} \times Z^{\mathrm{rs}} \to \mathbb{F}_2$.

It follows from Lemma 3.14 that the restriction $C^{\mathrm{rs}} \to B^{\mathrm{rs}}$ is a family of smooth projective curves; write $J^{\mathrm{rs}} \to B^{\mathrm{rs}}$ for the relative Jacobian of the family of smooth projective curves $C^{\mathrm{rs}} \to B^{\mathrm{rs}}$ [21, §9.3; Theorem 1]. The next result is one of the main results of Thorne's thesis and a first step towards relating the curves $C^{\mathrm{rs}} \to B^{\mathrm{rs}}$ to the representation (G, V).

Proposition 3.21. *There exists a canonical isomorphism $J^{\mathrm{rs}}[2] \simeq Z^{\mathrm{rs}}$ of finite étale group schemes that sends the Weil pairing on $J^{\mathrm{rs}}[2]$ to the pairing on Z^{rs} defined above.*

Proof. Since both group schemes are finite étale and B^{rs} is normal, it suffices to prove the statement above for the generic point of B^{rs} by [75, Tag 0BQM]. In that case, the statement follows from [77, Corollary 4.12]. ∎

Part (2) of Proposition 3.23 implies that the isomorphism $J^{\mathrm{rs}}[2] \simeq Z^{\mathrm{rs}}$ is unique.

3.9. *Monodromy of* $J^{\mathrm{rs}}[2]$

We give some additional properties of the group scheme $J^{\mathrm{rs}}[2] \to B^{\mathrm{rs}}$, which by Lemma 3.20 and Proposition 3.21 we may identify with $N_\Lambda \to B^{\mathrm{rs}}$. Before we state them, we recall some definitions and set up notation.

Recall from Section 3.1 that T is a split maximal torus of H with Lie algebra $\mathfrak{t}$ and Weyl group W. Let $L := X^*(T)$ be its character group and $N_L := \mathrm{image}(L/2L \to L^\vee/2L^\vee)$. Consider the composition $\mathfrak{t} \to \mathfrak{t} /\!\!/ W \xrightarrow{\sim} \mathfrak{h} /\!\!/ H \xrightarrow{\sim} V /\!\!/ G = B$, where $\mathfrak{t} \to \mathfrak{t} /\!\!/ W$ is the natural projection, $\mathfrak{t} /\!\!/ W \xrightarrow{\sim} \mathfrak{h} /\!\!/ H$ is the Chevalley restriction isomorphism (Proposition 2.1), and $\mathfrak{h} /\!\!/ H \xrightarrow{\sim} V /\!\!/ G$ is the isomorphism induced from the inclusion $V \subset \mathfrak{h}$ (Proposition 2.11). Restricting to regular semisimple elements defines a finite étale cover $f : \mathfrak{t}^{\mathrm{rs}} \to B^{\mathrm{rs}}$ with Galois group W.

Proposition 3.22. *The finite étale group scheme $J^{\mathrm{rs}}[2] \to B^{\mathrm{rs}}$ becomes trivial after the base change $f : \mathfrak{t}^{\mathrm{rs}} \to B^{\mathrm{rs}}$, where it becomes isomorphic to the constant group scheme N_L. The monodromy action is induced by the natural action of W on L.*

Proof. Since $J^{\mathrm{rs}}[2]$ is isomorphic to N_Λ, it suffices to prove that the torus $A \to B^{\mathrm{rs}}$ is isomorphic to the constant torus $T \times \mathfrak{t}^{\mathrm{rs}} \to \mathfrak{t}^{\mathrm{rs}}$ after pulling back along f, with monodromy given by the action of W on T.

To prove this, note that by Lemma 3.17, if $x : S \to \mathfrak{h}^{\mathrm{rs}}$ is an S-point with invariants $b = p(x) \in B^{\mathrm{rs}}(S)$, then $Z_H(x) \simeq A_b$ as group schemes over S. (Here $\mathfrak{h}^{\mathrm{rs}} \subset \mathfrak{h}$ denotes the subset of regular semisimple elements.) In particular, we can apply this to the $\mathfrak{t}^{\mathrm{rs}}$-point $i : \mathfrak{t}^{\mathrm{rs}} \to \mathfrak{h}^{\mathrm{rs}}$ (where i is the inclusion map), giving an isomorphism $T \times \mathfrak{t}^{\mathrm{rs}} \simeq A_{\mathfrak{t}^{\mathrm{rs}}}$. Since this isomorphism is induced by étale locally conjugating i to κ by elements of H, the monodromy action is indeed given by the natural action of W on T. ∎

Proposition 3.22 shows that it suffices to understand the W-action on N_L if we wish to understand the group scheme $J^{\mathrm{rs}}[2]$. To this end, we perform some root system calculations in the following proposition.

Proposition 3.23. *Suppose that H is not of type A_1.*

1. *$L^\vee/2L^\vee$ has no nonzero W-invariant elements.*
2. *The $\mathbb{F}_2[W]$-representation N_L is absolutely irreducible, so every W-equivariant automorphism $N_L \to N_L$ is the identity.*
3. *There exists an element $w \in W$ that has no nonzero fixed points on N_L.*

Proof. (1) Note that $L^\vee/2L^\vee = \mathrm{Hom}(L/2L, \mathbb{F}_2)$, so let $f : L/2L \to \mathbb{F}_2$ be a nonzero W-invariant functional. If f vanishes on a root of L, then f vanishes on all of them since they form a single W-orbit. Since the roots of L generate $L/2L$, it follows that $f(\alpha) = 1$ for every root. Since we have assumed that L is not of type A_1, there exist roots α, β such that $\alpha + \beta$ is also a root. But then we would have $1 = f(\alpha + \beta) = f(\alpha) + f(\beta) = 1 + 1 = 0$. This is a contradiction, hence no such nonzero f exists.

(2) Let $S \subset N_L \otimes \overline{\mathbb{F}}_2$ be a W-stable subspace, and assume that $v \in S$ is a nonzero element. Since the pairing $(\cdot, \cdot)$ on N_L is nondegenerate, there exists a root $\alpha \in L$ such that $(v, \alpha) \neq 0$ in $\overline{\mathbb{F}}_2$. If $w_\alpha \in W$ denotes the reflection associated with α, then $w_a(v) = v - (v, \alpha)\alpha$ also lies in S. It follows that $w_a(v) - v = (v, \alpha)\alpha$ lies in S, hence α also lies in S. Since W acts transitively on the roots, every root is contained in S. Since the roots generate L, it follows that $S = N_L \otimes \overline{\mathbb{F}}_2$, as claimed. The second claim follows from Schur's lemma. (We thank Beth Romano for helping us with the proof of this fact.)

(3) We first consider the case that the pairing on $L/2L$ is nondegenerate, which is equivalent to the projection map $L/2L \to N_L$ being injective. Since $L/2L$ and $L^\vee/2L^\vee$ have the same order, the latter statement is also equivalent to the fact that $L/2L \to L^\vee/2L^\vee$ is an isomorphism. We show that in this case it suffices to take a Coxeter element w_{cox} of W. Indeed, let $H_{\mathrm{sc}} \to H$ be the simply connected cover of H, let π_1 be the centre of H_{sc} and let T_{sc} be the preimage of T in H_{sc}. It is a classical fact that the inclusion $\pi_1 \subset T_{\mathrm{sc}}$ restricts to an equality $\pi_1 = T_{\mathrm{sc}}^{w_{\mathrm{cox}}}$, see [32, Theorem 1.6].[1] Taking 2-torsion implies that $\pi_1[2] = T_{\mathrm{sc}}[2]^{w_{\mathrm{cox}}}$. Since the map $L/2L \to L^\vee/2L^\vee$ is an isomorphism, the same is true for the map $T_{\mathrm{sc}}[2] \to T[2]$ which has kernel $\pi_1[2]$, hence $T_{\mathrm{sc}}[2]^{w_{\mathrm{cox}}} = \pi_1[2] = \{1\}$. Since $T_{\mathrm{sc}}[2] \simeq L/2L$, we have shown that $(L/2L)^{w_{\mathrm{cox}}} = N_L^{w_{\mathrm{cox}}} = 0$, as claimed.

We now consider the general case. Let S be a root basis of L, which is an $\mathbb{F}_2$-basis of the vector space $L/2L$. Since $L/2L \to N_L$ is surjective, there exists a subset $S_M \subset S$ projecting onto a basis of N_L. Let M be the $\mathbb{F}_2$-span of S_M. Then M is a (possibly reducible) root lattice associated with the sub-root system generated by S_M, and the composition $M/2M \hookrightarrow L/2L \twoheadrightarrow N_L$ is an isomorphism. The pairing $(\cdot, \cdot)$ on $L/2L$ restricts to a pairing on $M/2M$, and the previous sentence shows that this pairing on $M/2M$ is nondegenerate. It follows that M is a direct sum of irreducible root lattices of the form considered in the first case of this proof. Let w be a Coxeter element with respect to S_M, i.e., a product of the simple reflections in S_M. Then $(M/2M)^w = 0$ by the first case of the proof, so $N_L^w = 0$ too. ■

4. The mildly singular locus

We keep the notations from Section 3. Recall from Section 3.8 that $B^{\mathrm{rs}} \subset B$ denotes the locus where the discriminant polynomial Δ is nonzero, and that the family of curves $C \to B$ is smooth exactly above B^{rs} (Lemma 3.14). In this section, we introduce an

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[1] The reference assumes that T is a maximal torus in a compact Lie group, but this implies the corresponding result for a maximal torus in a semisimple group over a field of characteristic zero.

open subset $B^1 \subset B$ strictly containing B^{rs} where we allow the fibres of $C \to B$ to have one nodal singular point. We therefore call $B^1 \setminus B^{\mathrm{rs}}$ the 'mildly singular locus' of B. We then extend some results concerning the representation V and the family of curves from B^{rs} to B^1 in Sections 4.2 and 4.3, and generalise Proposition 3.21 in Section 4.4. This will be useful for the construction of orbits in Section 6 and for the analysis of integral orbits of square-free discriminant in Section 7.4. To avoid stating the same assumption repeatedly, we will make the following assumption throughout the rest of this paper.

Convention 4.1. The group H is not of type A_1.

4.1. The discriminant locus

Recall that we have fixed a maximal torus $T \subset H$ in Section 3.1. Recall from Section 2.1 that the discriminant polynomial $\Delta \in \mathbb{Q}[\mathfrak{h}]^H$ is the image of $\prod_{\alpha \in \Phi_{\mathfrak{t}}} \alpha \in \mathbb{Q}[\mathfrak{t}]^W$ under the isomorphism $\mathbb{Q}[\mathfrak{t}]^W \xrightarrow{\sim} \mathbb{Q}[\mathfrak{h}]^H$ of the Chevalley restriction theorem. Using the isomorphism $\mathbb{Q}[\mathfrak{h}]^H \xrightarrow{\sim} \mathbb{Q}[V]^G = \mathbb{Q}[B]$ from Proposition 2.11, we view Δ as an element of $\mathbb{Q}[B]$.

Lemma 4.2. *For every field $k/\mathbb{Q}$, Δ is irreducible in $k[B]$.*

Proof. It suffices to prove that we cannot partition $\Phi_{\mathfrak{t}}$ into two nonempty W-invariant subsets. Equivalently, we need to prove that W acts transitively on $\Phi_{\mathfrak{t}}$. This is true since $\Phi_{\mathfrak{t}}$ is irreducible and simply laced. ■

We write D for the subscheme of B cut out by Δ. Lemma 4.2 implies the following.

Corollary 4.3. *The scheme D is geometrically integral.*

Write D_{sing} for the singular locus of D, a closed subscheme of D.

Definition 4.4. We define B^1 as the complement of D_{sing} in B. We define D^1 as the complement of D_{sing} in D; we call D^1 the *mildly singular locus.*

The subscheme $B^1 \subset B$ is open, and we have inclusions $B^{\mathrm{rs}} \subsetneq B^1 \subsetneq B$. Since D is geometrically integral, the complement of B^1 in B has codimension ≥ 2. (In fact, Lemma 4.5 shows that it has codimension exactly 2.) As a general piece of notation, if X is a B-scheme, we write X^1 for its restriction to B^1.

4.2. Representation theory over B^1

If b is a point of B, we write $\mathfrak{h}_b$ for the fibre of the adjoint quotient $\mathfrak{h} \xrightarrow{p} \mathfrak{h} /\!\!/ H = B$ along this point.

Lemma 4.5. *Suppose that $k/\mathbb{Q}$ is an algebraically closed field and $b \in B(k)$. Then $b \in D^1(k)$ if and only if some (equivalently, every) semisimple $x \in \mathfrak{h}_b(k)$ has the property that the derived subgroup of $Z_H(x)$ is of type A_1.*

Proof. As every two semisimple elements in $\mathfrak{h}_b(k)$ are $H(k)$-conjugate (Proposition 2.1), requiring the last claim for some semisimple element of $\mathfrak{h}_b(k)$ is equivalent to requiring it for all of them. Let $T \subset H$ be the fixed maximal torus of Section 3.1, with root system $\Phi_{\mathfrak{t}}$ and Weyl group W. Let x be an element of $\mathfrak{t}$ with invariants b. By Lemma 2.2, $Z_H(x)$ is a reductive group with root system $\Phi_{\mathfrak{t}}(x) = \{\alpha \in \Phi_{\mathfrak{t}} \mid \alpha(x) = 0\}$, and its Weyl group W_x is the subgroup of W generated by the reflections through $\Phi_{\mathfrak{t}}(x)$.

To prove the lemma, we need to prove that $b \in D^1(k)$ if and only if $\Phi_{\mathfrak{t}}(x)$ is of type A_1. Let b_x be the image of x in $\mathfrak{t} /\!\!/ W_x$ and let $D_x \subset \mathfrak{t} /\!\!/ W_x$ be the discriminant locus of $Z_H(x)$, with smooth locus D_x^1. By Lemma 2.3, $D_x \to D$ is étale at b_x, hence $b \in D^1(k)$ if and only if $b_x \in D_x^1(k)$. So it suffices to prove that $b_x \in D_x^1(k)$ if and only if $\Phi_{\mathfrak{t}}(x)$ is of type A_1.

Firstly, suppose that $\Phi_{\mathfrak{t}}(x) = \{\alpha, -\alpha\}$ is of type A_1, so $W_x = \{1, w_\alpha\}$ is generated by the reflection through α. Then one can compute D_x explicitly: $\mathfrak{t} /\!\!/ W_x$ is given, up to taking a product with an affine space, by the quotient of $\operatorname{Spec} k[X]$ by the $\mathbb{Z}/2\mathbb{Z}$-action $X \mapsto -X$. This quotient is $\operatorname{Spec} k[X^2]$ and D_x is the vanishing locus of X^2, hence D_x is smooth. Therefore, $b_x \in D_x^1(k)$, which proves one direction.

Conversely, suppose that $\Phi_{\mathfrak{t}}(x)$ is not of type A_1. If $\Phi_{\mathfrak{t}}(x)$ were empty, then $b \in B^{\mathrm{rs}}(k)$, so $\Phi_{\mathfrak{t}}(x)$ is nonempty and of rank ≥ 2. We need to prove that D is singular at b. Since the singular locus of D is closed and x is the specialisation of a point y for which the rank of $\Phi_{\mathfrak{t}}(y)$ is exactly 2, we may assume that $\Phi_{\mathfrak{t}}(x)$ is either of type A_2 or $A_1 \times A_1$. In both cases, one can compute explicitly that D_x is not smooth at b_x, as required. ∎

Recall from Proposition 3.21 that if $k/\mathbb{Q}$ is a field and $b \in B^{\mathrm{rs}}(k)$, we have an isomorphism $Z_b \simeq J_b[2]$ of finite étale k-groups. So if g denotes the common arithmetic genus of the curves C_b, the group scheme Z_b has order 2^{2g}.

Lemma 4.6. *If $b \in D^1(k)$, the group scheme Z_b has order 2^{2g-1}.*

Proof. The group scheme Z_b is the centraliser of the element κ_b, which is regular by Proposition 3.11. By Lemma 3.18, it suffices to prove that if L is a root lattice of the same type as H, $N_L = \operatorname{image}(L/2L \to L^\vee/2L^\vee)$ and $\alpha \in L$ is a root, then α is nonzero in N_L. Since L is not of type A_1 (Convention 4.1), there exists a root β with $(\alpha, \beta) = -1$. Therefore, $\alpha \notin 2L^\vee$, so α is nonzero in N_L, as claimed. ∎

Before we state the last result of this section, we record a useful lemma.

Lemma 4.7. *Let $k/\mathbb{Q}$ be a field and $x \in \mathfrak{h}(k)$ a semisimple element with centraliser $L := Z_H(x)$. Then the centre of L is connected.*

Proof. We may assume that k is algebraically closed and that x lies in $\mathfrak{t}(k)$, the Lie algebra of the maximal torus $T \subset H$ fixed in Section 3.1. It suffices to prove that the character group of the centre of L is torsion-free. By Lemma 2.2, this group can be identified with $X^*(T)/\mathbb{Z}\Phi_{\mathfrak{t}}(x)$. Since H is adjoint, $X^*(T) = \mathbb{Z}\Phi_{\mathfrak{t}}$. The definition of the root system $\Phi_{\mathfrak{t}}(x)$ shows that it is $\mathbb{Q}$-closed in the sense of [71, §3.5]. By [71, Proposition 3.5], every root basis of $\Phi_{\mathfrak{t}}(x)$ can be extended to a root basis of $\Phi_{\mathfrak{t}}$. This implies that $\mathbb{Z}\Phi_{\mathfrak{t}}(x)$ is a direct summand of $\mathbb{Z}\Phi_{\mathfrak{t}}$ so the quotient $\mathbb{Z}\Phi_{\mathfrak{t}}/\mathbb{Z}\Phi_{\mathfrak{t}}(x)$ is indeed torsion-free. ■

To state the next proposition, recall that $V^{\mathrm{reg}} \subset V$ denotes the open subscheme of regular elements and that we have fixed a Kostant section κ in Section 3.8.

Proposition 4.8. *The action map $G \times B^1 \to V^{\mathrm{reg}}|_{B^1}$, $(g, b) \mapsto g \cdot \kappa_b$ is surjective.*

Proof. Part (2) of Proposition 2.11 implies that this map is surjective when restricted to B^{rs}. Therefore, it suffices to prove that if $k/\mathbb{Q}$ is algebraically closed and $b \in D^1(k)$, then every two elements $x, y \in V_b^{\mathrm{reg}}(k)$ are $G(k)$-conjugate. Let $x = x_s + x_n$ and $y = y_s + y_n$ be the Jordan decompositions of x and y. Since the semisimple parts x_s and y_s are $G(k)$-conjugate by Proposition 2.7, we may assume that $x_s = y_s$. The centraliser $L := Z_H(x_s)$ is a reductive group with derived subgroup of type A_1 (Lemma 4.5); write $\mathfrak{l}$ for its Lie algebra. The involution θ restricts to a stable involution $\theta|_L$ on L [77, Lemma 2.5]. Since x and y are regular, x_n, y_n are regular nilpotent elements of $\mathfrak{l}^{\theta=-1}$. Therefore, to prove the lemma, it suffices to prove that $L^\theta \cap G$ acts transitively on the regular nilpotents in $\mathfrak{l}^{\theta=-1}$. (Note that $L^\theta \subset H^\theta$ but we do not have $L^\theta \subset G$ in general.)

We first claim that L^θ acts transitively on the regular nilpotents in $\mathfrak{l}^{\theta=-1}$. To this end, let $Z(L)$ denote the centre of L and consider the exact sequence

$$0 \to Z(L) \to L \to \mathrm{PGL}_2 \to 0. \tag{4.1}$$

The involution θ preserves $Z(L)$ (acting via inversion by [77, Lemma 2.7 (3)]), and by Lemma 2.10 we may choose the isomorphism $L/Z(L) \simeq \mathrm{PGL}_2$ so that θ corresponds to the standard stable involution $\xi = \mathrm{Ad}(\mathrm{diag}(1, -1))$ of PGL_2 from Section 3.1. An elementary computation in $\mathfrak{sl}_2$ (or Proposition 3.8) shows that PGL_2^ξ acts transitively on the regular nilpotents in $\mathfrak{l}^{\xi=-1}$. To prove the claim, we only need to show that $L^\theta \to \mathrm{PGL}_2^\xi$ is surjective. Since $Z(L)$ is connected (Lemma 4.7), $Z(L)/(1-\theta)Z(L)$ is trivial and therefore taking θ-invariants of (4.1) shows that indeed $L^\theta \to \mathrm{PGL}_2^\xi$ is surjective, proving the claim.

To prove that $L^\theta \cap G$ acts transitively on the regular nilpotents in $\mathfrak{l}^{\theta=-1}$, it suffices to prove by the previous paragraph that $L^\theta \cap G$ surjects onto PGL_2^ξ. We first claim that there exists a semisimple element $t \in V$ with centraliser $M = Z_H(t)$ such that $L \subset M$ and such that the derived subgroup of M is of type A_2. Indeed, take a Cartan subspace $\mathfrak{c} \subset V$ containing x_s; then $\Phi_{\mathfrak{c}}(x_s) = \{\pm\alpha\}$ for some root α. Since $\Phi_{\mathfrak{c}}$ is not of type A_1, there exists a root β such that $\{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\} \subset \Phi_{\mathfrak{c}}$. Taking t to be an element of $\mathfrak{c}$ that vanishes exactly on those roots satisfies the requirements.

Again, θ restricts to a stable involution on M and the isomorphism $M/Z(M) \simeq \mathrm{PGL}_3$ can be chosen so that θ agrees with the standard stable involution ψ of PGL_3 from Section 3.1. Again, $Z(M)$ is connected by Lemma 4.7 and taking θ-invariants of the analogue of sequence (4.1) for M gives an exact sequence

$$1 \to Z(M)^\theta \to M^\theta \to \mathrm{PGL}_3^\psi \to 1.$$

A component group calculation (Corollary 3.6) shows that PGL_3^{ψ} is connected. Therefore, the identity component $(M^{\theta})^{\circ}$ maps surjectively onto PGL_3^{ψ}. Since $(M^{\theta})^{\circ} \subset M^{\theta} \cap G$, this implies that $M^{\theta} = (M^{\theta} \cap G) \cdot Z(M)^{\theta}$. It follows that $L^{\theta} = (L^{\theta} \cap G) \cdot Z(L)^{\theta}$. Indeed, if $l \in L^{\theta}$, there exists an element $z \in Z(M)^{\theta}$ such that $lz \in M^{\theta} \cap G$. Since $Z(M)^{\theta} \subset Z(L)^{\theta}$, we have $lz \in L^{\theta} \cap G$. We have established the equality $L^{\theta} = (L^{\theta} \cap G) \cdot Z(L)$, and it implies that $L^{\theta} \cap G$ surjects onto PGL_2^{ξ}. ∎

Remark 4.9. Proposition 4.8 is false when H is of type A_1. Indeed, in that case $0 \in B^1 = B$, $G = \left\{ \left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix} \right) \mid xy = 0 \right\} \simeq \mathbb{G}_m$ and there exist two G-orbits on $V_0^{\mathrm{reg}} = \left\{ \left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix} \right) \mid xy = 0 \right\}$. The problem is that, in the notation of the proof, $L^{\theta} \cap G = G$ does not surject onto the disconnected group PGL_2^{ξ}.

4.3. Geometry over B^1

Recall from Section 3.7 that we have introduced a family of projective curves $C \to B$ which is smooth exactly above B^{rs}.

Lemma 4.10. *Let $k/\mathbb{Q}$ be a field and let $b \in B(k)$. Then $b \in D^1(k)$ if and only if the curve C_b has a unique nodal singularity.*

Proof. A node is a simple singularity of type A_1. Therefore, the lemma follows from Lemmas 3.14 and 4.5. ∎

The fibres of the morphism $C \to B$ may be reducible. However, this does not happen over B^1.

Lemma 4.11. *The fibres of $C^1 \to B^1$ are geometrically integral.*

Proof. The geometric fibres of $C \to B$ are reduced, connected, and over B^{rs} these fibres are smooth. Therefore, it suffices to prove that C_b is irreducible if $k/\mathbb{Q}$ is algebraically closed and $b \in D^1(k)$. This can be verified by computation in each case, or by the following uniform argument.

Let $Z \subset D^1$ be the locus above which the fibres fail to be geometrically integral. Then Z is closed by [36, Théorème 12.2.1 (x)]. We claim that Z is also open. To prove this, it suffices to prove that Z is closed under generalisation [75, Tag 0903]. By [75, Tag 054F], this amounts to showing that for every complete discrete valuation ring R and morphism $\operatorname{Spec} R \to D^1$, the generic point of $\operatorname{Spec} R$ lands in Z if the closed point of $\operatorname{Spec} R$ does. If $b \in D^1(k)$, then C_b has a unique nodal singularity by Lemma 4.10, so either C_b is irreducible with one node or a union of two irreducible components intersecting transversally. Therefore, if $b \in D^1(k)$, then C_b is reducible (in other words, $b \in Z$) if and only if the generalised Jacobian $\mathrm{Pic}^0_{C_b/k}$ is an abelian variety [21, §9.2, Example 8]. On the other hand, since the locus of $\operatorname{Spec} R$ where the relative Jacobian $\mathrm{Pic}^0_{C_R/R} \to \operatorname{Spec} R$ (which exists and is a semi-abelian scheme by [21, §9.3, Theorem 7]) is an abelian variety is open (this follows from looking at torsion points), we conclude that the generic fibre of $\mathrm{Pic}^0_{C_R/R}$ is an abelian variety if its special fibre is. It follows that Z is closed under generalisation, proving the claim. Since D^1 is irreducible and Z is open and closed, it follows that Z is empty or equal to D^1; we will exclude the latter case.

To this end, it suffices to prove that C_η is geometrically integral, where η is the generic point of D. Assume by contradiction that this is not the case. As observed in the previous paragraph, this implies that $\mathrm{Pic}^0_{C_\eta/\eta}$ is an abelian variety. Therefore, the finite étale group scheme $J^{\mathrm{rs}}[2] \to B^{\mathrm{rs}}$ is unramified along D. By Zariski–Nagata purity for finite étale covers [75, Tag 0BJE], this implies that $J^{\mathrm{rs}}[2]$ extends to a finite étale cover over B. Since B is isomorphic to affine space over k, this cover must be trivial. However, a monodromy calculation (Propositions 3.22 and 3.23) shows that $J^{\mathrm{rs}}[2]$ is nontrivial. (Recall that we have excluded that H is of type A_1.) This is a contradiction, completing the proof of the lemma. ■

Since $C^1 \to B^1$ has geometrically integral fibres by Lemma 4.11, the group scheme $\mathrm{Pic}^0_{C^1/B^1}$ is well defined, and we denote it by $J^1 \to B^1$ [21, §9.3, Theorem 1]. It is a semi-abelian scheme. The 2-torsion subgroup $J^1[2] \to B^1$ is a quasi-finite étale group scheme; we may therefore view it as a sheaf on the étale site of B^1.

Lemma 4.12. *Let $j\colon B^{\mathrm{rs}} \hookrightarrow B^1$ be the open inclusion. Then $j_*J^{\mathrm{rs}}[2] = J^1[2]$ as étale sheaves on B^1.*

Proof. Consider the natural morphism

$$\phi\colon J^1[2] \to j_*j^*J^1[2] = j_*J^{\mathrm{rs}}[2]$$

obtained by adjunction. Since $J^1 \to B^1$ is separated, $J^1[2] \to B^1$ is separated as well, so ϕ is injective. To prove that ϕ is an isomorphism, it suffices to check this at geometric points of B^1. Combining the last two sentences, it suffices to prove that $(J^1[2])_{\bar b}$ and $(j_*J^{\mathrm{rs}}[2])_{\bar b}$ have the same cardinality for all geometric points $\bar b$ of B^1, or even that the cardinality of the latter is bounded above by the cardinality of the first. This is obvious if $\bar b$ lands in B^{rs}, so assume that $\bar b$ lands in D^1.

By Lemma 4.11, $C_{\bar b}$ is integral and has a unique singularity, which is a node. It follows that $J^1_{\bar b}$ has order 2^{2g-1}, where g is the arithmetic genus of C_b. On the other hand, the order of $(j_*J^{\mathrm{rs}}[2])_x$ for $x \in D^1$ can only go down under specialisation. It therefore suffices to prove that if η denotes the generic point of D, then $(j_*J^{\mathrm{rs}}[2])_\eta$ has order 2^{2g-1}. In fact, we claim that ϕ_η is an isomorphism.

To this end, let K be the fraction field of the discrete valuation ring $\mathcal{O}_{B,\eta}$ and let j_η be the inclusion $\mathrm{Spec}\, K \hookrightarrow \mathrm{Spec}\, \mathcal{O}_{B,\eta}$. Then the pullback of $j_*J^{\mathrm{rs}}[2]$ along $\mathrm{Spec}\, \mathcal{O}_{B,\eta} \to B$ equals $(j_\eta)_*J^{\mathrm{rs}}_K[2]$, where J^{rs}_K denotes the pullback J^{rs} along the generic point of B. The curve $C_{\mathcal{O}_{B,\eta}}$ is regular since $\mathrm{Spec}\, \mathcal{O}_{B,\eta} \to B$, hence $C_{\mathcal{O}_{B,\eta}} \to C$ is formally smooth and the total space C is smooth. Therefore, a result of Raynaud [21, §9.5, Theorem 1] shows that the identity component of the Picard functor of $C_{\mathcal{O}_{B,\eta}}$, which equals $J^1_{\mathcal{O}_{B,\eta}}$ by definition, is isomorphic to the Néron model of J_K. By the Néron mapping property, this shows that $J^1_\eta[2] = (j_*J^{\mathrm{rs}}[2])_\eta$. This completes the proof of the claim, hence that of the lemma. ■

4.4. *Summary of properties of B^1*

We summarise the properties of D^1 in the next theorem.

Theorem 4.13. *Let $k/\mathbb{Q}$ be an algebraically closed field and let $b \in B(k)$. Then the following are equivalent:*

(1) *$b \in D^1(k)$;*

(2) *for every semisimple $v \in V_b(k)$, the derived subgroup of $Z_H(v)$ is of type A_1;*

(3) *C_b is irreducible and has a unique singular point, which is a node.*

Proof. Combine Lemmas 4.5 and 4.10. ■

Theorem 4.14. *The isomorphism $J^{\mathrm{rs}}[2] \simeq Z^{\mathrm{rs}}$ from Proposition 3.21 uniquely extends to an isomorphism $J^1[2] \simeq Z^1$ of separated étale group schemes over B^1.*

Proof. Since $J^{\mathrm{rs}}[2]$ and Z^{rs} are dense in $J^1[2]$ and Z^1, respectively, uniqueness is clear. For the existence, denote the open immersion $B^{\mathrm{rs}} \hookrightarrow B^1$ by j. Consider the composition

$$\psi \colon Z^1 \to j_* Z^{\mathrm{rs}} \xrightarrow{\sim} j_* J^{\mathrm{rs}}[2] \xrightarrow{\sim} J^1[2]$$

of the adjunction morphism $Z^1 \to j_* Z^{\mathrm{rs}}$, the pushforward of the isomorphism $Z^{\mathrm{rs}} \xrightarrow{\sim} J^{\mathrm{rs}}[2]$ along j and the isomorphism $j_* J^{\mathrm{rs}}[2] \xrightarrow{\sim} J^1[2]$ of Lemma 4.12. Since $Z^1 \to B^1$ is separated, ψ is injective. It therefore suffices to prove that $Z^1_{\bar{b}}$ and $J^1_{\bar{b}}$ have the same cardinality for every geometric point $\bar{b}$ of D^1. If g denotes the common arithmetic genus of the fibres of $C \to B$, then $Z^1_{\bar{b}}$ has cardinality 2^{2g-1} by Lemma 4.6. On the other hand, $J^1_{\bar{b}}$ also has cardinality 2^{2g-1} by Lemmas 4.10 and 4.11. ■

Although it is not necessary for this paper, we expect that an isomorphism similar to the one of Theorem 4.14 holds over the whole of B, but we have not been able to prove it.

5. The compactified Jacobian

We keep the notations from Section 3. Recall from Section 3.8 that the family of smooth projective curves $C^{\mathrm{rs}} \to B^{\mathrm{rs}}$ has Jacobian variety $J^{\mathrm{rs}} \to B^{\mathrm{rs}}$ which is itself a smooth and projective morphism. The goal of this section is to extend the B^{rs}-scheme J^{rs} to a proper B-scheme $\bar{J}$ with good geometric properties. We achieve this using the theory of the compactified Jacobian of Altman–Kleiman [1], extended by Esteves [33] to incorporate reducible curves. Its construction is given in Section 5.2 and its basic properties are summarised in Theorem 5.14. Note that the occurrence of reducible fibres is the reason why the definition of $\bar{J}$ is more involved here than in our previous work [43, §4.3], which treats the E_6 case and where only irreducible fibres are present.

The results of this section will be useful for the construction of orbits in Section 6 (specifically Section 6.2) and the construction of integral representatives in Section 7.5.

5.1. Generalities on sheaves

The following material is largely taken from [33, 48]. Let k be an algebraically closed field. By a *curve* we mean a reduced projective scheme of pure dimension 1 over k.

Definition 5.1. A coherent sheaf I on a connected curve X is said to be

(1) *rank*-1 if $I_\eta \simeq \mathcal{O}_{X,\eta}$ as $\mathcal{O}_{X,\eta}$-modules for every generic point $\eta \in X$;

(2) *torsion-free* if the associated points of I are precisely the generic points of X;

(3) *simple* if $\operatorname{End}_k(I) = k$.

We remark that the first two conditions imply the third if X is irreducible and that every torsion-free rank-1 sheaf on a smooth curve is invertible.

To obtain a well-behaved moduli problem of torsion-free rank-1 sheaves on a reducible curve, we use stability conditions introduced by Esteves [33]. A *subcurve* Z of a curve X is a closed k-subscheme that is reduced and of pure dimension 1. If I is a torsion-free sheaf on X, its restriction to a subcurve $I|_Z$ is not necessarily torsion-free; it contains a biggest torsion subsheaf and the quotient of $I|_Z$ by this subsheaf is denoted by I_Z. The sheaf I_Z is the unique torsion-free quotient of I whose support is equal to Z.

Definition 5.2. Let E be a vector bundle on a connected curve X of rank $r \geq 1$ and degree $-rd$. Let I be a torsion-free rank-1 sheaf on X with Euler characteristic $\chi(I) = d$. We say that I is *E-semistable* if for every nonempty proper subcurve $Y \subsetneq X$, we have that

$$\chi(I_Y) \geq -\frac{\deg(E|_Y)}{r}. \tag{5.1}$$

We say that I is *E-stable* if for every nonempty proper subcurve, inequality (5.1) is strict.

Given a vector bundle E on X, we may define its *multislope* $\underline{q}^E = \{q^E_{C_i}\}$ as follows. It is a tuple of rational numbers, one for each irreducible component C_i of X, defined by setting

$$q^E_{C_i} := -\frac{\deg(E|_{C_i})}{\operatorname{rank} E}.$$

If $Y \subset X$ is a subcurve, write $q^E_Y := \sum_{C_i \subset Y} q^E_{C_i}$, where the sum is taken over those irreducible components C_i that are contained in Y. If E is of rank r and degree $-rd$, then $q^E_X = d$. When the vector bundle E is clear from the context, we omit the superscript from the notation $\underline{q}^E$.

Definition 5.3. Let X be a curve and E a vector bundle on X of rank r and degree $-rd$ with multislope $\underline{q}$. We say that E is *general* if $q_Y \notin \mathbb{Z}$ for any nonempty proper subcurve $Y \subsetneq X$.

If I is torsion-free rank-1 on X, then I is E-semistable if and only if $\chi(I_Y) \geq q_Y$ for every nonempty proper subcurve $Y \subset X$, and E-stable if every such inequality is strict. Therefore, if E is general, a torsion-free rank-1 sheaf on X is E-semistable if and only if it is E-stable.

The next lemma shows that a family of simple torsion-free rank-1 sheaves has no unexpected endomorphisms. For a quasi-coherent sheaf $\mathscr{F}$ on a scheme X, we write $\mathscr{E}\mathrm{nd}(\mathscr{F})$ for the sheaf of $\mathscr{O}_X$-module endomorphisms of $\mathscr{F}$, which is again a quasi-coherent sheaf on X.

Lemma 5.4. *Let $p\colon \mathcal{X} \to T$ be a flat family of projective curves whose geometric fibres are reduced and connected. Let I be a locally finitely presented $\mathscr{O}_{\mathcal{X}}$-module, flat over T, whose geometric fibres above T are simple torsion-free rank-1. Then $p_*\mathscr{E}\mathrm{nd}(I) = \mathscr{O}_T$.*

Proof. Use [1, Corollary 5.3] and the assumption that I is simple in each geometric fibre. ∎

5.2. *The definition*

Recall from Section 3.7 that $C \to B$ is a flat projective morphism whose geometric fibres are reduced connected curves, and that this morphism has sections $\infty_1, \dots, \infty_m\colon B \to C$ landing in the smooth locus.

Lemma 5.5. *For every field $k/\mathbb{Q}$ and $b \in B(k)$, the irreducible components of C_b are geometrically irreducible. Moreover, every such irreducible component contains $\infty_{i,b}$ in its smooth locus for some i.*

Proof. The first claim follows from the second one. For the second one, we may assume that k is algebraically closed. Consider the line bundle $\mathscr{L} = \mathscr{O}_C(\infty_1 + \cdots + \infty_m)$ on C associated with the divisors ∞_i of C. For every $b \in B(k)$, $\mathscr{L}_b$ is ample if and only if every irreducible component of C_b contains $\infty_{i,b}$ for some i. Moreover, the locus of elements $b \in B$ for which $\mathscr{L}_b$ is ample is open [36, Corollaire 9.6.4], $\mathbb{G}_m$-invariant (with respect to the $\mathbb{G}_m$-action on $C \to B$ introduced in Section 3.7) and contains the central point by Lemma 3.15. These three facts imply that it must be the whole of B. ∎

In order to define a compactified Jacobian of $C \to B$, we first construct a vector bundle E on C using properties of the central fibre C_0. Recall from Lemma 3.15 that each of the m irreducible components of C_0 contains a unique marked point $\infty_{i,0}$. Let $\underline{q} = \{q_1, \dots, q_m\}$ be a tuple of rational numbers such that $\sum_{i=1}^m q_i = \chi(\mathscr{O}_{C_0}) = 1 - p_a(C_0)$ and $\sum_{i\in I} q_i \notin \mathbb{Z}$ for every nonempty proper subset $I \subset \{1, \dots, m\}$; it is easy to see that such a tuple exists. Write $q_i = e_i/r$ for some $e_i \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 1}$. By further multiplying e_i and r, we may assume that $r \geq m$. Let E be the following vector bundle on C:

$$E = \mathscr{O}_C(-e_1 \cdot \infty_1) \oplus \cdots \oplus \mathscr{O}_C(-e_m \cdot \infty_m) \oplus \mathscr{O}_C^{\oplus r-m}.$$

Since the image of $\infty_i\colon B \to C$ is a divisor of C, the line bundles $\mathscr{O}_C(-e_i \cdot \infty_i)$ are well defined. Note that the vector bundle $E|_{C_0}$ has multislope $\underline{q}$ by construction. For every geometric point b of B, $E|_{C_b}$ is a vector bundle of rank r and degree $-r(1 - p_a(C_0))$ on the curve C_b.

Lemma 5.6. *For every geometric point b of B, the vector bundle $E|_{C_b}$ is general in the sense of Definition 5.3.*

Proof. Follows from Lemma 5.5 and the construction of E. ■

We are now ready to define the compactified Jacobian associated with E. We assume we have made a choice of $\underline{q}$ and E as above. Consider the functor

$$\overline{\mathbb{J}}_E \colon \{B\text{-schemes}\} \to \{\text{sets}\}$$

sending a B-scheme T to the set of equivalence classes of pairs (I, ϕ), where

- I is a locally finitely presented $\mathcal{O}_{C_T}$-module, flat over T, with the property that for every geometric point t of T, I_t is simple torsion-free rank-1, $\chi(I_t) = \chi(\mathcal{O}_{C_0})$ and I_t is E_t-stable;
- ϕ is an isomorphism $\infty_{1,T}^{*} I \simeq \mathcal{O}_T$ of $\mathcal{O}_T$-modules.

We say two pairs (I, ϕ) and (I', ϕ') are equivalent if there is an isomorphism $I \simeq I'$ mapping ϕ to ϕ'. We have the following basic representability result [33, Theorem B].

Proposition 5.7 (Esteves). *The functor $\overline{\mathbb{J}}_E$ is representable by a B-scheme $\overline{J}_E$.*

Proof. Let F be the functor from B-schemes to sets, sending a B-scheme T to the set of equivalence classes of locally finitely presented $\mathcal{O}_{C_T}$-modules I, flat over T, with the property that for every geometric point t of T, I_t is simple torsion-free rank-1, $\chi(I_t) = \chi(\mathcal{O}_{C_0})$ and I_t is E_t-stable. (In contrast to $\overline{\mathbb{J}}_E$, we omit the rigidification ϕ.) Here we say I and I' are equivalent if there exists an invertible sheaf $\mathscr{L}$ on T such that $I' \simeq \mathscr{L}_{C_T} \otimes I$. Let F^{et} denote the étale sheafification of F. By [33, Proposition 34], the functor F^{et} is representable by an open subspace of the algebraic space parametrising simple torsion-free rank-1 sheaves with no Euler characteristic or stability condition. By Lemma 5.5 and [33, Theorem B], the latter algebraic space is in fact a scheme, so F^{et} is representable by a scheme as well.

On the other hand, the forgetful morphism $\overline{\mathbb{J}}_E \to F$, $(T, \phi) \mapsto T$ is an isomorphism of functors, since every $I \in F(T)$ is equivalent to another element $I' \in F(T)$ admitting a rigidification. Since elements of $\overline{\mathbb{J}}_E$ have no nontrivial automorphisms by Lemma 5.4, étale descent of quasi-coherent sheaves implies that $\overline{\mathbb{J}}_E$ is an étale sheaf, so we have natural identifications $\overline{\mathbb{J}}_E = F = F^{\text{et}}$. Since F^{et} is representable by a scheme by the previous paragraph, the same is true for $\overline{\mathbb{J}}_E$. ■

Definition 5.8. We call $\overline{J}_E$ a compactified Jacobian of $C \to B$ associated with E.

If $C \to B$ has reducible fibres, different choices of $\underline{q}$ may give rise to different compactified Jacobians. For our purposes, these differences will be harmless and for the remainder of this paper we fix a choice of $\underline{q}$ and E as above and we simply write $\overline{J} = \overline{J}_E$.

Lemma 5.9. *Let k be a field and $b \in B(k)$ such that the curve C_b is integral. Then $\overline{J}_b$ parametrises torsion-free rank-1 sheaves on C_b with degree zero, i.e., Euler characteristic $1 - p_a(C_0)$.*

Proof. If C_b is integral, the E_b-stability condition and the simplicity of the sheaves are automatic. ■

5.3. Basic properties of $\overline{J}$

Lemma 5.10. *The morphism $\overline{J} \to B$ is projective.*[2]

Proof. Since the vector bundle E is chosen to be general (Lemma 5.6), the notions of E-stable and E-semistable agree. Therefore, a theorem of Esteves [33, Theorems C.1 and C.4] shows that $\overline{J}$ is quasi-projective. Moreover, [33, Theorem A.1] shows that $\overline{J} \to B$ is universally closed. We conclude that $\overline{J}$ is projective over B. ■

Recall from Section 3.7 that we have defined a $\mathbb{G}_m$-action on C such that $C \to B$ is $\mathbb{G}_m$-equivariant with respect to the square of the usual $\mathbb{G}_m$-action on B. By functoriality, this induces a $\mathbb{G}_m$-action on $\overline{J}$ too such that $\overline{J} \to B$ is $\mathbb{G}_m$-equivariant (again with respect to the square of the usual $\mathbb{G}_m$-action on B). The following argument will be used in the next two lemmas: if $U \subset \overline{J}$ is an open $\mathbb{G}_m$-invariant subset containing the central fibre $\overline{J}_0$, then $U = \overline{J}$. Indeed, by the properness of $\overline{J} \to B$, the complement of U in $\overline{J}$ projects to a closed $\mathbb{G}_m$-invariant subset of B that does not contain the central point $0 \in B$, so must be empty.

Lemma 5.11. *The variety $\overline{J}$ is smooth.*

Proof. The family $C \to B$ is a semi-universal deformation of the plane curve singularity C_0 (Proposition 3.13). Therefore, [48, Fact 4.2 (ii)] implies that $\overline{J}$ is smooth in a neighbourhood of $\overline{J}_0$. Since the smooth locus of $\overline{J}$ is open, $\mathbb{G}_m$-invariant and contains $\overline{J}_0$, it must be the whole of $\overline{J}$. ■

We emphasise that the fibres of $\overline{J} \to B$ might be singular above points that do not lie in B^{rs}. In fact, we have a precise description of the smooth locus.

Lemma 5.12. *The morphism $\overline{J} \to B$ is flat of relative dimension $p_a(C_0)$. The smooth locus of $\overline{J} \to B$ coincides with the locus of invertible sheaves.*

Proof. By [48, Theorem 5.5 (ii)], the morphism $\overline{J} \to B$ is flat in a neighbourhood of $\overline{J}_0$. Since the flat locus is open and $\mathbb{G}_m$-invariant, it follows that it must equal the whole of $\overline{J}$. The claim about the smooth locus is [48, Theorem 5.5 (iii)]. ■

Lemma 5.13. *The geometric fibres of $\overline{J} \to B$ are reduced and connected. Consequently, $\overline{J}$ is geometrically integral. Moreover, if $k/\mathbb{Q}$ is an algebraically closed field and $b \in B(k)$ is such that C_b is integral, then $\overline{J}_b$ is integral.*

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[2] There are several nonequivalent definitions of a projective morphism but in this case they all agree, see [75, Tag 0B45].

Proof. Since all the fibres of $C \to B$ have planar singularities, [48, Theorem A (i)–(iii)] shows that $\bar{J} \to B$ has geometrically reduced and connected fibres. Since $\bar{J}$ is B-flat, this implies that $\bar{J}$ is geometrically connected. Since $\bar{J}$ is smooth, it follows that it is geometrically irreducible. To establish the last claim, [48, Corollary 5.14] shows that the number of irreducible components of $\bar{J}_b$ can be calculated in terms of the intersections between the irreducible components of C_b. This number is always 1 when C_b is irreducible, as can be seen from [48, Definition 5.12]. ∎

For future reference, we summarise the above properties in the following theorem. Write $\bar{J}^1$ for the restriction of $\bar{J}$ to B^1. Recall from Section 4.3 that $J^1 \to B^1$ is the relative generalised Jacobian of the family of integral curves $C^1 \to B^1$. Note that by Lemma 5.9 and the definition of J^1 we have an open embedding $J^1 \to \bar{J}^1$.

Theorem 5.14. *Let $\bar{J} \to B$ be a compactified Jacobian associated with some choice of E as in Definition 5.8. Then the morphism $\bar{J} \to B$ is flat, projective and restricts to J^{rs} over B^{rs}. Its geometric fibres are reduced and connected. The scheme $\bar{J}$ is geometrically integral and smooth over $\mathbb{Q}$. The smooth locus of $\bar{J}^1 \to B^1$ is isomorphic to $J^1 \to B^1$. The complement of J^1 in $\bar{J}$ has codimension ≥ 2.*

Proof. Only the last two sentences remain to be established. The claim about the smooth locus of $\bar{J}^1$ follows from Lemmas 5.9 and 5.12 and the definition of J^1. For the claim about the codimension, let Z be the complement of J^1 in $\bar{J}$. Then Z is supported above the discriminant locus D of B. Moreover, the fibres of the map $\bar{J}|_{D^1} \to D^1$ are geometrically integral by Lemmas 4.11 and 5.13, so the fibres of the map $Z|_{D^1} \to D^1$ have dimension strictly less than those of $\bar{J}|_{D^1} \to D^1$. Combining the last two sentences proves the claim. ∎

5.4. *The Białynicki-Birula decomposition of $\bar{J}$*

We recall the Białynicki-Birula decomposition [16] from geometric representation theory. If k is a field and X is a scheme of finite type of k, we define a *decomposition* of X to be a collection of locally closed subschemes $X_1, \dots, X_n$ of X such that the underlying topological space of X is a disjoint union of the underlying topological spaces of the X_i. If in addition X is separated over k and endowed with a $\mathbb{G}_m$-action and if $x \in X$, we say that $\lim_{\lambda \to 0} \lambda \cdot x$ exists if the action map

$$\mathbb{G}_m \to X, \quad \lambda \mapsto \lambda \cdot x$$

extends (necessarily uniquely) to a morphism $\mathbb{A}^1_k \to X$.

Proposition 5.15. *Suppose that X is a smooth and separated scheme of finite type over a field k, endowed with a $\mathbb{G}_m$-action. Then the closed subscheme of fixed points $X^{\mathbb{G}_m}$ is smooth; let $F_1, \dots, F_n$ denote its connected components. Suppose in addition that $\lim_{\lambda \to 0} \lambda \cdot x$ exists for every $x \in X$. Then there exists a decomposition of X into locally closed subschemes X_i and morphisms $X_i \to F_i$ which are affine space fibrations in the Zariski topology.*

Proof. See [40, Theorem 1.5] for a modern proof, which treats the generality in which we have stated it. We may informally describe X_i as those points $x \in X$ whose limit $\lim_{\lambda \to 0} \lambda \cdot x$ lies in F_i, and the map $X_i \to F_i$ as taking the limit $x \mapsto \lim_{\lambda \to 0} \lambda \cdot x$. ■

Corollary 5.16. *In the setting of Proposition 5.15, assume furthermore that X is geometrically integral and $X^{\mathbb{G}_m}$ is finite. Then there exists an open subset of X isomorphic to affine space $\mathbb{A}_k^{\dim X}$.*

Proof. Let $F_1, \dots, F_n$ denote the connected components of $X^{\mathbb{G}_m}$; since $X^{\mathbb{G}_m}$ is smooth and finite, each F_i is the spectrum of a separable field extension k_i of finite degree over k. Let $X_1, \dots, X_n$ be the decomposition of X of Proposition 5.15. Then each X_i is isomorphic to $\mathbb{A}_{k_i}^{n_i}$ for some integer $n_i \geq 0$. There exists an X_i, say X_1, which is of maximal dimension $\dim X$. Since X_1 is locally closed, it is an open subset of its closure $\bar{X}_1 = X$, so X_1 is an open subset of X. Since X is geometrically irreducible, the same is true for X_1. This implies that $X_1 \times_k k_1$ is irreducible, so $k_1 = k$. Therefore, X_1 is isomorphic to $\mathbb{A}_k^{\dim X}$. ■

Remark 5.17. The proof shows that under the assumptions of Corollary 5.16, X is even decomposed into affine cells.

We will apply Corollary 5.16 to the compactified Jacobian $\bar{J} \to B$ constructed in Section 5.2. Recall from Section 5.3 that $\bar{J}$ inherits a $\mathbb{G}_m$-action from C. We denote the central fibre of $\bar{J}$ by $\bar{J}_0$.

Lemma 5.18. *The set of $\mathbb{G}_m$-fixed points $\bar{J}_0^{\mathbb{G}_m}$ is finite.*

Proof. This follows from calculations of Beauville [3, §4.1] if C_0 is integral. It seems likely that one can extend his analysis to reducible curves, but we will proceed differently. We will assume that all schemes are base changed to a fixed algebraic closure k of $\mathbb{Q}$. Let $J(C_0)$ be the generalised Jacobian of C_0 parametrising line bundles having multidegree zero, i.e., degree zero on each irreducible component of C_0. Then $J(C_0)$ is an algebraic group acting on $\bar{J}_0$, compatibly with the $\mathbb{G}_m$-actions on $J(C_0)$ and $\bar{J}_0$.

Firstly, we claim that the closure of every $\mathbb{G}_m$-orbit of a point in $J(C_0)$ contains the identity. Indeed, every point of $[L] \in J(C_0)(k)$ is represented by a Cartier divisor $D_1 - \deg(D_1)\infty_{1,0} + \cdots + D_m - \deg(D_m)\infty_{m,0}$, where D_i is a Cartier divisor supported on the smooth affine part of the irreducible component $C_{0,i}$ of C_0 containing $\infty_{i,0}$. Since every smooth point P of $C_{0,i}$ satisfies $\lim_{\lambda \to \infty} \lambda \cdot P = \infty_{i,0}$ (as can be seen from the definition of the $\mathbb{G}_m$-action on C_0 in Section 3.7 and Table 1), we see that $\lambda \cdot [L] \to 0$ as $\lambda \to \infty$, proving the claim.

Secondly, we claim that the action of $J(C_0)$ on $\bar{J}_0$ has finitely many orbits. Indeed, let $p \in C_0$ be the unique singular point. Since p is an ADE-singularity, there are only finitely many isomorphism classes of torsion-free rank-1 modules over the completed local ring $\widehat{\mathcal{O}}_{C_0,p}$ (see [41], in fact this property can be used to characterise ADE-singularities amongst Gorenstein singularities). It therefore suffices to prove that if $[\mathscr{F}], [\mathscr{G}] \in \bar{J}_0(k)$ are two sheaves whose completed stalks at p are isomorphic, then $\mathscr{F} \simeq \mathscr{G} \otimes \mathscr{L}$, where $\mathscr{L}$ is a line bundle on C_0 whose multidegree can only take finitely many values (independently of $\mathscr{F}$ and $\mathscr{G}$). To prove this, consider the Hom-sheaf $\mathscr{H} = \mathscr{H}\mathrm{om}(\mathscr{F}, \mathscr{G})$ and the endomorphism sheaf $\mathscr{E} = \mathscr{E}\mathrm{nd}(\mathscr{F})$. Since $\mathscr{F}|_{C_0 \setminus \{p\}}$ is a line bundle, $\mathscr{E}$ is a coherent commutative $\mathscr{O}_{C_0}$-algebra which is generically isomorphic to $\mathscr{O}_{C_0}$, and $\mathscr{H}$ is a coherent $\mathscr{E}$-module. Since the formation of $\mathscr{H}$ and $\mathscr{E}$ commutes with flat base change [75, Tag 0C6I], the completed stalk $\mathscr{H}_p \otimes \widehat{\mathscr{O}}_{C_0,p}$ is free of rank 1 over $\mathscr{E}_p \otimes \widehat{\mathscr{O}}_{C_0,p}$. It follows that $\mathscr{H}_p$ is free of rank 1 over $\mathscr{E}_p$, so the stalks $\mathscr{F}_p$, $\mathscr{G}_p$ are isomorphic $\mathscr{O}_{C_0,p}$-modules. (An isomorphism is given by choosing an $\mathscr{E}_p$-generator of $\mathscr{H}_p$.) By spreading out such an isomorphism, we may find an open subset $U \subset C_0$ containing p and an isomorphism $\phi_U \colon \mathscr{F}|_U \xrightarrow{\sim} \mathscr{G}|_U$. The restrictions of $\mathscr{F}, \mathscr{G}$ to $C_0 \setminus \{p\}$ are line bundles. Since C_0 is connected and p is its unique singular point, the complement $C_0 \setminus U$ is a union of finitely many points. We may therefore find an open subset $V \subset C_0 \setminus \{p\}$ containing those points and an isomorphism $\phi_V \colon \mathscr{F}|_V \xrightarrow{\sim} \mathscr{G}|_V$. The transition map $(\phi_V)|_{U\cap V}^{-1} \circ (\phi_U)|_{U \cap V} \colon \mathscr{F}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}|_{U \cap V}$ defines an element $f \in \mathrm{H}^0(U \cap V, \mathscr{O}_{C_0}^{\times})$. Let $\mathscr{L}$ be the line bundle on C_0 obtained by glueing $\mathscr{O}_U$ and $\mathscr{O}_V$ along the automorphism f. One can then explicitly check that the maps

$$\mathscr{F}_U \otimes \mathscr{O}_U \to \mathscr{G}_U \colon s \otimes 1 \mapsto \phi_U(s),$$
$$\mathscr{F}_V \otimes \mathscr{O}_V \to \mathscr{G}_V \colon s \otimes 1 \mapsto \phi_V(s),$$

glue to an isomorphism $\mathscr{F} \otimes \mathscr{L} \simeq \mathscr{G}$. The multidegree of $\mathscr{L}$ can only take finitely many values (when we vary $\mathscr{F}$ and $\mathscr{G}$ in $\bar{J}_0$) because of the E-stability condition imposed on sheaves in $\bar{J}_0$. This completes the proof of the claim. (We thank Jesse Leo Kass for his help with the proof of this claim.)

We now use the last two paragraphs to show that $\bar{J}_0^{\mathbb{G}_m}$ is finite. Indeed, by the second claim, it suffices to prove that every $J(C_0)$-orbit contains at most one $\mathbb{G}_m$-fixed point. If $x \in \bar{J}_0^{\mathbb{G}_m}$ and $g \in J(C_0)$ are such that $g \cdot x \in \bar{J}_0^{\mathbb{G}_m}$, then $g^{-1}(\lambda \cdot g)$ lies in the stabiliser of x in $J(C_0)$ for all $\lambda \in \mathbb{G}_m$. Since this stabiliser is closed, the first claim implies that it contains $\lim_{\lambda \to \infty} g^{-1} \lambda \cdot g = g^{-1}$. Therefore, g lies in the stabiliser of x, that is $g \cdot x = x$. We conclude that x is the only $\mathbb{G}_m$-fixed point in the $J(C_0)$-orbit of x. ∎

Theorem 5.19. *The variety* $\bar{J}$ *has a dense open subset isomorphic to affine space* $\mathbb{A}^d_{\mathbb{Q}}$ *for some* $d \geq 1$.

Proof. The compactified Jacobian is a smooth, geometrically integral and quasi-projective scheme over $\mathbb{Q}$ (Theorem 5.14). Since $\bar{J} \to B$ is proper and $\lim_{\lambda \to 0} \lambda \cdot b$ exists for every $b \in B$, $\lim_{\lambda \to 0} \lambda \cdot x$ exists for every $x \in \bar{J}$. We wish to apply Corollary 5.16, so it suffices to prove that the fixed point locus $\bar{J}^{\mathbb{G}_m}$ is finite. Since $B^{\mathbb{G}_m}$ consists of the central point 0, it suffices to prove that $\bar{J}_0^{\mathbb{G}_m}$ is finite, which is exactly Lemma 5.18. ∎

Remark 5.20. The Białynicki-Birula decomposition gives a canonical decomposition of $\bar{J}$ into locally closed subschemes isomorphic to affine space. This can be made very explicit when H is of type A_2. In that case, $C \to B$ is the universal family of elliptic curves with short Weierstrass form $y^2 = x^3 + p_2 x + p_3$ (see Table 1), and it turns out [48, Proposition 7.3] that there is an isomorphism of B-schemes $\bar{J} \simeq C$ respecting the $\mathbb{G}_m$-actions. The BB decomposition of C has the form

$$C = C^\circ \sqcup \infty(B),$$

where C° is the affine part and $\infty(B) \simeq B$ is the image of the marked section at infinity $\infty\colon B \to C$. The fact that these strata are isomorphic to affine space can be seen directly: $\infty(B) \simeq B = \mathbb{A}^2_{\mathbb{Q}}$ and C° is isomorphic to the closed subscheme $\{(x, y, p_2, p_3) \mid y^2 = x^3 + p_2 x + p_6\} \subset \mathbb{A}^4_{\mathbb{Q}}$, which via projection to the first three coordinates is isomorphic to $\mathbb{A}^3_{\mathbb{Q}}$.

In the case A_{2g}, it is true (but we do not show) that the BB decomposition stratifies the smooth locus of $\bar{J} \to B$ according to the Mumford representation of a divisor on a hyperelliptic curve [50, Chapter IIIa, Proposition 1.2]. It would be interesting to obtain a similarly concrete interpretation of this decomposition for other families of curves studied here.

6. Constructing orbits

We keep the notations from Section 3. The goal of this section is to construct for every $b \in B^{\mathrm{rs}}(\mathbb{Q})$ and every element of $\mathrm{Sel}_2\, J_b$ a $G(\mathbb{Q})$-orbit of $V_b(\mathbb{Q})$, see Corollary 6.9. The technical input is the Zariski triviality of a certain universal torsor on J^{rs} in Section 6.2, see Theorem 6.4. This will be achieved using generalities concerning torsors on open subsets of affine spaces developed in Section 6.1.

6.1. *Torsors on open subsets of affine space*

The purpose of this subsection is to prove the following theorem, which will be useful in the proof of Theorem 6.4.

Theorem 6.1. *Let k be a field of characteristic zero and X an open subset of $\mathbb{A}^n_k$ whose complement has codimension ≥ 2. Let G be a reductive group over k and let $T \to X$ be a G-torsor. Suppose that X contains a k-rational point over which T is trivial. Then T is Zariski locally trivial.*

Example 6.2. We illustrate Theorem 6.1 in the concrete case $G = \mathrm{PGL}_2$. If k and X are as in the theorem, then a PGL_2-torsor can alternatively be viewed as a Severi–Brauer curve $\mathcal{C} \to X$. In other words, $\mathcal{C} \to X$ is a smooth projective family of genus zero curves (i.e., conics). Suppose that X contains a point $x \in X(k)$ such that the conic $\mathcal{C}_x$ has a k-rational point. Then Theorem 6.1 says that $\mathcal{C}$ is a projective bundle over X. This implies that all the other fibres of $\mathcal{C} \to X$ also contain a k-rational point.

Theorem 6.1 might be unsurprising to experts, although we have not been able to locate it explicitly in the literature. It follows from a slight variant of the formalism developed in [25, §1] using pointed sets rather than abelian groups. We take a different route and give a short proof which was suggested to us by Colliot-Thélène. We thank him for letting us to include the argument here.

The crucial input is the following result of Raghunathan–Ramanathan, see [60, Theorem 1.1] and the remark immediately thereafter.

Proposition 6.3. *Let k be a perfect field and G a reductive group over k. Then every G-torsor on $\mathbb{A}^1_k$ is isomorphic to the pullback of a G-torsor on* $\operatorname{Spec} k$ *along the map* $\mathbb{A}^1_k \to \operatorname{Spec} k$.

Proof of Theorem 6.1. Let $p\colon \mathbb{A}^n_k \to \mathbb{A}^{n-1}_k$ be the projection onto the first $n-1$ coordinates and let $p_X\colon X \to \mathbb{A}^{n-1}_k$ be its restriction to X. If K denotes the function field of $\mathbb{A}^{n-1}_k$, then the assumptions on X imply that the generic fibre of p_X is isomorphic to $\mathbb{A}^1_K$. By Proposition 6.3, the restriction of T to this generic fibre is induced from a torsor T_0 on $\operatorname{Spec} K$ along the map $\mathbb{A}^1_K \to \operatorname{Spec} K$.

Since k is infinite, we may choose a section $s\colon \mathbb{A}^{n-1}_k \to \mathbb{A}^n_k$ of p such that its image does not contain any generic point of the complement of X in $\mathbb{A}^n_k$. Suppose that $X_1 = s^{-1}(X \cap s(\mathbb{A}^{n-1}_k))$ and let T_1 be the pullback of T along $s|_{X_1}\colon X_1 \to X$. By our choice of s, X_1 is an open subset of $\mathbb{A}^{n-1}_k$ whose complement has codimension ≥ 2. The generic fibre of T_1 is a torsor on $\operatorname{Spec} K$ isomorphic to T_0.

Replacing T by T_1 and iterating this process, we obtain a torsor T_n on $\operatorname{Spec} k$ whose pullback along $X \to \operatorname{Spec} k$ is generically isomorphic to T. By known cases of the Grothendieck–Serre conjecture (Corollary 2.17), T and $T_n \times_k X$ are Zariski locally isomorphic. Since X has a k-point above which T is trivial, the same is true for $T_n \times_k X$, hence T_n is trivial itself. We conclude that T is Zariski locally trivial, as desired. ■

6.2. *A universal torsor*

Recall from Section 3.8 that $J^{\mathrm{rs}} \to B^{\mathrm{rs}}$ denotes the relative Jacobian of the family of smooth curves $C^{\mathrm{rs}} \to B^{\mathrm{rs}}$, that $Z \to B$ denotes the universal stabiliser of the Kostant section κ, and that there is an isomorphism of finite étale group schemes $J^{\mathrm{rs}}[2] \simeq Z^{\mathrm{rs}}$ over B^{rs}.

Since $J^{\mathrm{rs}} \to B^{\mathrm{rs}}$ is an abelian scheme, the multiplication-by-2 map $J^{\mathrm{rs}} \xrightarrow{\times 2} J^{\mathrm{rs}}$ is a $J^{\mathrm{rs}}[2]$-torsor. Pushing out this torsor along the maps $J^{\mathrm{rs}}[2] \xrightarrow{\sim} Z^{\mathrm{rs}} \hookrightarrow G$ defines a G-torsor $T^{\mathrm{rs}} \to J^{\mathrm{rs}}$. (This procedure is also called ‘changing the structure group’.) The following theorem is one of the main technical results of this paper, and is the essential input for constructing orbits associated with elements of $J_b(\mathbb{Q})$ (Theorem 6.6).

Theorem 6.4. *The torsor T^{rs} is Zariski locally trivial. That is, for every $x \in J^{\mathrm{rs}}$ there exists an open subset $U \subset J^{\mathrm{rs}}$ containing x such that $T^{\mathrm{rs}}|_U$ is trivial.*

To briefly explain why this is relevant for constructing orbits, note that if $b \in B^{\mathrm{rs}}(\mathbb{Q})$ and $P \in J_b(\mathbb{Q})$, the image of P under the composition

$$J_b(\mathbb{Q})/2J_b(\mathbb{Q}) \to \mathrm{H}^1(\mathbb{Q}, J_b[2]) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{Q}, Z_b) \to \mathrm{H}^1(\mathbb{Q}, G)$$

coincides with the isomorphism class of the pullback of T^{rs} along $P\colon \operatorname{Spec} \mathbb{Q} \to J^{\mathrm{rs}}$. Theorem 6.4 implies that this pullback defines the trivial class in $\mathrm{H}^1(\mathbb{Q}, G)$, which implies that it corresponds to a $G(\mathbb{Q})$-orbit of $V_b(\mathbb{Q})$; see Theorem 6.6 for full details.

Proof of Theorem 6.4. Recall from Section 4 that $B^1 \subset B$ is an open subset containing B^{rs} and that the family of curves $J^1 \to B^1$ is the relative (generalised) Jacobian of the family of curves $C^1 \to B^1$. By Theorem 4.14, the isomorphism $J^{\mathrm{rs}}[2] \simeq Z^{\mathrm{rs}}$ extends to an isomorphism $J^1[2] \simeq Z^1$ of quasi-finite étale group schemes over B^1. The multiplication-by-two map $J^1 \xrightarrow{\times 2} J^1$ is a $J^1[2]$-torsor, and pushing out this torsor along the composition $J^1[2] \xrightarrow{\sim} Z^1 \to G$ defines a G-torsor $T^1 \to J^1$. By construction, the restriction of T^1 to J^{rs} is isomorphic to T^{rs}.

To prove the theorem, it suffices to prove that T^1 is Zariski locally trivial. Using known cases of the Grothendieck–Serre conjecture (Corollary 2.17), it even suffices to prove that T^1 is Zariski locally trivial when restricted to a nonempty open subset of J^1.

Recall from Section 5 that we have constructed a scheme $\bar{J} \to B$ containing J^1 as an open subscheme. By Theorem 5.14, the complement of J^1 in $\bar{J}$ has codimension ≥ 2; by Theorem 5.19, $\bar{J}$ contains an open dense subscheme U isomorphic to affine $\mathbb{Q}$-space. This implies that the complement of $U^1 := U \cap J^1$ in U has codimension ≥ 2.

We claim that $T^1|_{U^1}$ is Zariski locally trivial. By Theorem 6.1, it suffices to prove that T^1_x is trivial for some $x \in U^1(\mathbb{Q})$. In fact, we will show the stronger statement that $\{x \in J^1(\mathbb{Q}) \mid T^1_x \text{ is trivial}\}$ is Zariski dense in J^1. Indeed, since J^1 is a rational variety (it contains U^1 as a dense open subscheme), the set $J^1(\mathbb{Q})$ is dense in J^1. Since the multiplication-by-two map $J^1 \xrightarrow{\times 2} J^1$ is dominant, the subset $2J^1(\mathbb{Q}) \subset J^1(\mathbb{Q})$ is still dense in J^1. By construction of T^1, the pullback of T^1 along a point $x \in 2J^1(\mathbb{Q})$ is trivial. This completes the proof of the claim, hence the proof of the theorem. ∎

6.3. *Constructing orbits for* 2*-descent elements*

We start by applying a well-known lemma from arithmetic invariant theory recalled in Section 2.4 to give a cohomological description of the G-orbits of V. For every $\mathbb{Q}$-algebra R and $b \in B^{\mathrm{rs}}(R)$, we write $J_b := J^{\mathrm{rs}}_b$ for the Jacobian of C_b.

Corollary 6.5. *Let R be a $\mathbb{Q}$-algebra and let $b \in B^{\mathrm{rs}}(R)$. Then the association $v \mapsto \{g \in G \mid g \cdot v = \kappa_b\}$ induces an injection*

$$\gamma_b \colon G(R)\backslash V_b(R) \hookrightarrow \mathrm{H}^1(R, J_b[2]).$$

Its image coincides with the pointed kernel of the map $\mathrm{H}^1(R, J_b[2]) \xrightarrow{\sim} \mathrm{H}^1(R, Z_b) \to \mathrm{H}^1(R, G)$.

Proof. We apply Lemma 2.13 to the action of $G_{B^{\mathrm{rs}}}$ on V^{rs}. Indeed, the action map

$$G \times B^{\mathrm{rs}} \to V^{\mathrm{rs}}, \quad (g, b) \mapsto g \cdot \kappa_b$$

is étale (Proposition 3.11) and it is surjective by Proposition 2.11. Pulling back along $b \colon \operatorname{Spec} R \to B^{\mathrm{rs}}$ and using the isomorphism $J_b[2] \simeq Z_b$ from Proposition 3.21 gives the desired bijection. ∎

We now piece all the ingredients obtained so far together to deduce our first main theorem.

Theorem 6.6. *Let R be a local $\mathbb{Q}$-algebra (for example, a field of characteristic zero) and let $b \in B^{\mathrm{rs}}(R)$. Then the image of the 2-descent map $J_b(R)/2J_b(R) \to \mathrm{H}^1(R, J_b[2])$ lies in the image of γ_b of Corollary 6.5. Consequently, there is a canonical injection*

$$\eta_b \colon J_b(R)/2J_b(R) \hookrightarrow G(R)\backslash V_b(R)$$

compatible with base change.

Proof. By Corollary 6.5, it suffices to prove that the composition $J_b(R)/2J_b(R) \to \mathrm{H}^1(R, J_b[2]) \simeq \mathrm{H}^1(R, Z_b) \to \mathrm{H}^1(R, G)$ is trivial. Recall that in Section 6.2, we have constructed a G-torsor $T^{\mathrm{rs}} \to J^{\mathrm{rs}}$ such that its pullback along a point $P \colon \operatorname{Spec} R \to J^{\mathrm{rs}}$ defines a G-torsor $T_P^{\mathrm{rs}} \to \operatorname{Spec} R$ whose isomorphism class equals the image of P under the above composite map. Since T^{rs} is Zariski locally trivial by Theorem 6.4, T_P^{rs} is Zariski locally trivial. Since R is a local ring, it follows that T_P^{rs} is trivial. This completes the proof. ■

Remark 6.7. In the proof of Theorem 6.4, we have shown the stronger statement that the torsor $T^1 \to J^1$ (a natural extension of T^{rs} to J^1) is Zariski locally trivial. A straightforward adaption of the proof of Theorem 6.6 then shows that if R is a local ring and $b \in B^1(R)$ (instead of $b \in B^{\mathrm{rs}}(R)$), then there exists an injection $J_b^1(R)/2J_b^1(R) \hookrightarrow G(R)\backslash V_b^{\mathrm{reg}}(R)$. We do not know if this observation is useful.

6.4. *Constructing orbits for 2-Selmer elements*

The next proposition might be well known to experts – see, for example, [56, Remark after Theorem 6.22] – but we believe it deserves to be stated explicitly. We slightly deviate from our standing notation and allow G to be an arbitrary split semisimple group in this proposition.

Proposition 6.8. *Let G be a split semisimple group over a number field k. Then the kernel of $\mathrm{H}^1(k, G) \to \prod_v \mathrm{H}^1(k_v, G)$ (where v runs over all places) is trivial.*

We emphasise that $\mathrm{H}^1(k, G) \to \prod_v \mathrm{H}^1(k_v, G)$ is merely a map of pointed sets, and that it need not be injective. (Take G to be a special orthogonal group.)

Proof of Proposition 6.8. We have an exact sequence

$$1 \to \mu \to G_{\mathrm{sc}} \to G \to 1,$$

where G_{sc} is simply connected and μ is a finite subgroup of a split torus (i.e., a product of μ_n's). This sequence induces a long exact sequence in nonabelian cohomology. Let $\alpha \in \mathrm{H}^1(k, G)$ be a class with $\alpha_v = 1$ for all places v of k. Since $\mathrm{H}^2(k, \mu) \to \prod_v \mathrm{H}^2(k_v, \mu)$ is injective by the Hasse principle for the Brauer group, we see that α lifts to a class $\beta \in \mathrm{H}^1(k, G_{\mathrm{sc}})$. Since μ is a central subgroup of G_{sc}, any other lift of α is given by $\lambda\beta$, where $\lambda \in \mathrm{H}^1(k, \mu)$ is a cocycle. We will show that we can choose λ so that $\lambda\beta$ is trivial. By the Hasse principle for simply connected groups [56, Theorem 6.6], the map

$\mathrm{H}^1(k, G_{\mathrm{sc}}) \to \prod_v \mathrm{H}^1(k_v, G_{\mathrm{sc}})$ is injective. (This map is even bijective.) If v is a finite or complex place, then $\mathrm{H}^1(k_v, G_{\mathrm{sc}})$ is trivial [57, Theorem 5.12.24 (b)]. If v is real, then $\beta_v \in \mathrm{H}^1(k_v, G_{\mathrm{sc}})$ has trivial image in $\mathrm{H}^1(k_v, G)$ so comes from an element of $\mathrm{H}^1(k_v, \mu)$. Since $\mathrm{H}^1(k, \mu) \to \prod_{v \text{ real}} \mathrm{H}^1(k_v, \mu)$ is surjective (this follows from the case $\mu = \mu_n$), we may choose $\lambda \in \mathrm{H}^1(k, \mu)$ such that $\lambda_v \beta_v = 1$ for every real place v. This implies that $\lambda\beta$ is trivial, as required. ■

Corollary 6.9. *Let k be a number field and let $b \in B^{\mathrm{rs}}(k)$. Let $\mathrm{Sel}_2 J_b$ be the 2-Selmer group of the abelian variety J_b/k. Then $\mathrm{Sel}_2 J_b \subset \mathrm{H}^1(k, J_b[2])$ is contained in the image of γ_b. Consequently, the injection η_b from Theorem 6.6 extends to an injection*

$$\mathrm{Sel}_2 J_b \hookrightarrow G(k) \backslash V_b(k).$$

Proof. We have a commutative diagram for every place v,

$$\begin{tikzcd}
J_b(k)/2J_b(k) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^1(k, J_b[2]) \arrow[r] \arrow[d] & \mathrm{H}^1(k, G) \arrow[d] \\
J_b(k_v)/2J_b(k_v) \arrow[r, "\delta_v"] & \mathrm{H}^1(k_v, J_b[2]) \arrow[r] & \mathrm{H}^1(k_v, G).
\end{tikzcd}$$

By Corollary 6.5, it suffices to prove that 2-Selmer elements in $\mathrm{H}^1(k, J_b[2])$ are killed under the composition $\mathrm{H}^1(k, J_b[2]) \xrightarrow{\sim} \mathrm{H}^1(k, Z_G(\kappa_b)) \to \mathrm{H}^1(k, G)$. By definition, an element of $\mathrm{Sel}_2 J_b$ consists of a class in $\mathrm{H}^1(k, J_b[2])$ whose restriction to $\mathrm{H}^1(k_v, J_b[2])$ lies in the image of δ_v for every place v. So by Theorem 6.6, the image of such an element in $\mathrm{H}^1(k_v, G)$ is trivial for every v. Proposition 6.8 completes the proof. ■

6.5. *Reducible orbits and marked points*

Recall from Definition 3.12 that an element of $V^{\mathrm{rs}}(k)$ is k-reducible if it is $G(k)$-conjugate to a Kostant section. Recall from Section 3.7 that $\infty_1, \ldots, \infty_m$ denote the set of marked points of $C \to B$.

Proposition 6.10. *Let $k/\mathbb{Q}$ be a field and suppose that $b \in B^{\mathrm{rs}}(k)$. Then the image under $\eta_b\colon J_b(k)/2J_b(k) \hookrightarrow G(k)\backslash V_b(k)$ of the subgroup of $J_b(k)/2J_b(k)$ generated by $\{\infty_2 - \infty_1, \ldots, \infty_m - \infty_1\}$ coincides with the set of k-reducible $G(k)$-orbits of $V_b(k)$. Moreover, the set of k-reducible $G(k)$-orbits has the maximal size 2^{m-1} if and only if the inclusion $Z_G(\kappa_b) \subset Z_{H^\theta}(\kappa_b)$ is surjective on k-points.*

Proof. The proof is very similar to [65, Lemma 2.11]. For a scheme X/k, we write $\mathrm{H}_1(X, \mathbb{F}_2) := \mathrm{Hom}(\mathrm{H}^1_{\mathrm{et}}(X_{k^s}, \mathbb{F}_2), \mathbb{F}_2)$, where $\mathrm{H}^1_{\mathrm{et}}$ denotes étale cohomology. We have an exact sequence of étale homology groups

$$1 \to \mu_2^m/\Delta(\mu_2) \to \mathrm{H}_1(C_b^\circ, \mathbb{F}_2) \to \mathrm{H}_1(C_b, \mathbb{F}_2) \to 1. \tag{6.1}$$

Let $H_{\mathrm{sc}} \to H$ be the simply connected cover of H and let $C_{H_{\mathrm{sc}}}$ be the centre of H_{sc}. By [77, Theorem 4.10], sequence (6.1) is isomorphic to

$$1 \to C_{H_{\mathrm{sc}}}[2] \to Z_{H_{\mathrm{sc}}^\theta}(\kappa_b) \to Z_G(\kappa_b) \to 1. \tag{6.2}$$

It follows that the duals of these sequences are also isomorphic. We will calculate these duals and their connecting maps in Galois cohomology.

The dual of (6.1) is isomorphic to

$$1 \to J_b[2] \to \mathrm{H}^1_{\mathrm{et}}(C^{\circ}_{b,k^s}, \mathbb{F}_2) \to (\mu_2^m)_{\Sigma=0} \to 1. \tag{6.3}$$

Here we use the identification $J_b[2] = \mathrm{H}^1_{\mathrm{et}}(C_{b,k^s}, \mathbb{F}_2)$, and $(\mu_2^m)_{\Sigma=0}$ denotes the subset of μ_2^m of elements summing to zero. An explicit calculation shows that the image of the connecting map $(\mu_2^m)_{\Sigma=0}(k) \to \mathrm{H}^1(k, J_b[2])$ coincides with the image of the subgroup of $J_b(k)/2J_b(k)$ generated by $\{\infty_2 - \infty_1, \dots, \infty_m - \infty_1\}$ under the 2-descent map $J_b(k)/2J_b(k) \hookrightarrow \mathrm{H}^1(k, J_b[2])$.

On the other hand, we claim that the dual of (6.2) is isomorphic to

$$1 \to Z_G(\kappa_b) \to Z_{H^\theta}(\kappa_b) \to \pi_0(H^\theta) \to 1. \tag{6.4}$$

Indeed, the identification of the first two terms follows from [77, Corollary 2.9] and the existence of a nondegenerate pairing on $Z_G(\kappa_b)$ [77, Corollary 2.12]. It follows from Lemma 3.5 that we may identify the last term with $\pi_0(H^\theta)$. Next, we claim that the image of the connecting map $\pi_0(H^\theta) \to \mathrm{H}^1(k, Z_G(\kappa_b))$ coincides with the image of the k-reducible orbits in $V_b(k)$ under the map $G(k)\backslash V_b(k) \hookrightarrow \mathrm{H}^1(k, Z_G(\kappa_b))$ from Lemma 2.13. Indeed, consider the commutative diagram

$$\begin{tikzcd}
G(k)\backslash V_b(k) \arrow[r] \arrow[d, hook] & H^\theta(k)\backslash V_b(k) \arrow[d, hook] \\
\mathrm{H}^1(k, Z_G(\kappa_b)) \arrow[r] & \mathrm{H}^1(k, Z_{H^\theta}(\kappa_b)),
\end{tikzcd}$$

where the horizontal maps are induced by the inclusions $G \subset H^\theta$ and $Z_G(\kappa_b) \subset Z_{H^\theta}(\kappa_b)$, and the vertical maps arise from Lemma 2.13. It follows from Corollary 3.4 that the map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, H^\theta)$ has trivial pointed kernel. Moreover, all k-reducible elements in $V_b(k)$ are $H^\theta(k)$-conjugate by Proposition 3.8. Therefore, the set of k-reducible $G(k)$-orbits corresponds to the kernel of $\mathrm{H}^1(k, Z_G(\kappa_b)) \to \mathrm{H}^1(k, Z_{H^\theta}(\kappa_b))$ which, using (6.4), coincides with the image of the map $\pi_0(H^\theta) \to \mathrm{H}^1(k, Z_G(\kappa_b))$. This proves the claim and the first part of the proposition.

To prove the remaining part, note that there are 2^{m-1} k-reducible orbits if and only if the map $(\mu_2^m)_{\Sigma=0}(k) \to \mathrm{H}^1(k, J_b[2])$ is injective. By considering the long exact sequences associated with the isomorphic sequences (6.3) and (6.4), this is equivalent to the surjectivity of $\mathrm{H}^0(k, Z_G(\kappa_b)) \to \mathrm{H}^0(k, Z_{H^\theta}(\kappa_b))$. ■

7. Integral representatives

We keep the notations from Section 3. In this section, we introduce integral structures for G and V and prove that for large primes p, the image of the map from Theorem 6.6 applied to $R = \mathbb{Q}_p$ lands in the orbits which admit a representative in $\mathbb{Z}_p$. See Theorem 7.6

for a precise statement. In Section 7.6, we deduce an integrality result for orbits over $\mathbb{Q}$ (as opposed to orbits over $\mathbb{Q}_p$).

7.1. Integral structures

So far we have considered properties of the pair (G, V) over $\mathbb{Q}$. In this subsection, we define these objects over $\mathbb{Z}$.

Let $\underline{H}$ (resp. $\underline{G}$) be the unique (up to isomorphism) split reductive group over $\mathbb{Z}$ with generic fibre H (resp. G). The automorphism $\theta\colon H \to H$ extends by the same formula to an automorphism $\underline{H} \to \underline{H}$, still denoted by θ.

Lemma 7.1. *The equality* $(H^\theta)^\circ = G$ *extends to an isomorphism* $(\underline{H}^\theta_{\mathbb{Z}[1/2]})^\circ \simeq \underline{G}_{\mathbb{Z}[1/2]}$, *where* $(\underline{H}^\theta_{\mathbb{Z}[1/2]})^\circ$ *is the relative identity component of* $\underline{H}^\theta_{\mathbb{Z}[1/2]}$.

Proof. This follows from the fact that $(\underline{H}^\theta_{\mathbb{Z}[1/2]})^\circ$ is a reductive group scheme of the same type as $\underline{G}_{\mathbb{Z}[1/2]}$, which follows from [27, Remark 3.1.5]. ■

To obtain a $\mathbb{Z}$-structure for V, choose a Chevalley basis of $\mathfrak{g}$ with respect to the maximal torus T^θ, and choose an admissible $\mathbb{Z}$-form $\underline{V}$ of the G-representation V with respect to this basis [20, Proposition 2.4]. Since G acts faithfully on V, the results of [20, §3] imply that $\underline{G}$ is isomorphic to the Zariski closure of G in $\mathrm{GL}(\underline{V})$. We henceforth view $\underline{G}$ as a closed subgroup of $\mathrm{GL}(\underline{V})$, so $\underline{V}$ is a representation of $\underline{G}$.

Recall from Section 3.7 that we have fixed polynomials $p_{d_1}, \dots, p_{d_r} \in \mathbb{Q}[V]^G$ satisfying the conclusions of Proposition 3.13. Note that those conclusions are invariant under the $\mathbb{G}_m$-action on B. By rescaling the polynomials p_{d_i} using this $\mathbb{G}_m$-action, we can assume they lie in $\mathbb{Z}[\underline{V}]^{\underline{G}}$. We may additionally assume that the discriminant Δ from Section 4.1 lies in $\mathbb{Z}[\underline{V}]^{\underline{G}}$. Define

$$\underline{B} := \operatorname{Spec} \mathbb{Z}[p_{d_1}, \dots, p_{d_r}] \quad \text{and} \quad \underline{B}^{\mathrm{rs}} := \operatorname{Spec} \mathbb{Z}[p_{d_1}, \dots, p_{d_r}][\Delta^{-1}].$$

Taking invariants defines a map $\pi\colon \underline{V} \to \underline{B}$.

We extend the family of curves given by the equation in Table 1 to the family $\underline{C} \to \underline{B}$ given by that same equation.

Proposition 7.2. *The group* $\underline{G}$ *has class number* 1: $G(\mathbb{A}^\infty) = G(\mathbb{Q})\underline{G}(\widehat{\mathbb{Z}})$.

Proof. The group $\underline{G}$ is the Zariski closure of G in $\mathrm{GL}(\underline{V})$ and in a suitable basis of $\underline{V}$, G contains a maximal $\mathbb{Q}$-split torus T^θ consisting of diagonal matrices in $\mathrm{GL}(V)$. Therefore, $\underline{G}$ has class number 1 by [56, Theorem 8.11; Corollary 2] and the fact that $\mathbb{Q}$ has class number one. ■

7.2. Spreading out

Our constructions and theorems for (G, V) of the previous sections will continue to be valid over $\mathbb{Z}[1/N]$ for some appropriate choice of integer N, in a sense we will now explain.

Let us call a positive integer N *admissible* if the following properties are satisfied (set $S := \mathbb{Z}[1/N]$):

(1) Each prime dividing the order of the Weyl group of H is a unit in S. (In particular, 2 is a unit in S.)

(2) The zero locus $\underline{D}_S \to \operatorname{Spec} S$ of the discriminant Δ is flat and its smooth locus $\underline{D}^1_S$ coincides with the regular locus of $\underline{D}_S$. Moreover, the nonsmooth locus of $\underline{D}_S \to \operatorname{Spec} S$ is flat over $\operatorname{Spec} S$.

(3) The morphism $\underline{C}_S \to \underline{B}_S$ is smooth exactly above $\underline{B}^{\mathrm{rs}}_S$.

(4) The affine curve $\underline{C}^{\circ}_S$ is a closed subscheme of $\underline{V}_S$ and the action map $\underline{G}_S \times \underline{C}^{\circ}_S \to \underline{V}_S$, $(g, x) \mapsto g \cdot x$ is smooth.

(5) For a field k of characteristic not dividing N, $b \in \underline{D}^1(k)$ if and only if every semisimple lift $v \in \underline{V}_b(k)$ has centraliser $Z_H(v)$ of semisimple rank-1 if and only if the curve $\underline{C}_b$ has a unique nodal singularity. In that case, the curve $\underline{C}_b$ is geometrically integral.

(6) There exist open subschemes $\underline{V}^{\mathrm{rs}} \subset \underline{V}^{\mathrm{reg}} \subset \underline{V}_S$ such that if k is a field of characteristic not dividing N and $v \in \underline{V}(k)$, then v is regular if and only if $v \in \underline{V}^{\mathrm{reg}}(k)$, and v is regular semisimple if and only if $v \in \underline{V}^{\mathrm{rs}}(k)$. Moreover, $\underline{V}^{\mathrm{rs}}$ is the open subscheme defined by the nonvanishing of the discriminant polynomial Δ in $\underline{V}_S$.

(7) The morphism $\pi \colon \underline{V}_S \to \underline{B}_S$ is smooth exactly at $\underline{V}^{\mathrm{reg}}$.

(8) $S[\underline{V}]^{\underline{G}} = S[p_{d_1}, \dots, p_{d_r}]$. The Kostant section κ fixed in Section 3.7 extends to a section $\kappa \colon \underline{B}_S \to \underline{V}^{\mathrm{reg}}$ of π satisfying the following property: for any $b \in \underline{B}(\mathbb{Z}) \subset \underline{B}_S(S)$, we have $\kappa_{N \cdot b} \in \underline{V}(\mathbb{Z})$. Moreover, each $G(\mathbb{Q})$-orbit of Kostant sections has a representative which satisfies the same property.

(9) Let $\underline{B}^1$ be the complement of the nonregular locus of $\underline{D}$ in $\underline{B}$. Then the action map $\underline{G}_S \times \underline{B}_S \to \underline{V}^{\mathrm{reg}}$, $(g, b) \mapsto g \cdot \kappa_b$ is étale and its image contains $\underline{V}^{\mathrm{reg}}|_{\underline{B}^1_S}$.

(10) Let $\underline{J}^1_S \to \underline{B}_S$ denote the relative generalised Jacobian of the family of integral curves $\underline{C}_S|_{\underline{B}^1_S} \to \underline{B}^1_S$ [21, §9.3, Theorem 1] and let $\underline{J}^{\mathrm{rs}}_S \to \underline{B}^{\mathrm{rs}}_S$ denote its restriction to $\underline{B}^{\mathrm{rs}}_S$. Let $\underline{Z}_S \to \underline{B}_S$ be the centraliser of the Kostant section κ in $\underline{G}_S$. Then there is an isomorphism $\underline{J}^{\mathrm{rs}}_S[2] \simeq \underline{Z}^{\mathrm{rs}}_S$ of finite étale group schemes over $\underline{B}^{\mathrm{rs}}_S$ whose restriction to B^{rs} is the isomorphism of Proposition 3.21. It extends to an isomorphism $\underline{J}^1_S[2] \simeq \underline{Z}^1_S$.

(11) The B-scheme $\bar{J}$ constructed in Section 5 extends to a $\underline{B}_S$-scheme $\underline{\bar{J}}_S \to \underline{B}_S$ which is flat, projective, with geometrically integral fibres and whose restriction to $\underline{B}^{\mathrm{rs}}_S$ is isomorphic to $\underline{J}^{\mathrm{rs}}_S$. Moreover, $\underline{\bar{J}}_S \to S$ is smooth with geometrically integral fibres, and the smooth locus of the morphism $\underline{\bar{J}}_S \to \underline{B}_S$ is an open subscheme of $\underline{\bar{J}}_S$ whose complement is S-fibrewise of codimension at least two.

(12) The G-torsor $T \to J^{\mathrm{rs}}$ from Section 6.2 extends using the same definition to a $\underline{G}_S$-torsor $\underline{T}_S \to \underline{J}^{\mathrm{rs}}_S$, and $\underline{T}_S$ is Zariski locally trivial.

It might be possible to construct an explicit admissible integer for every pair (G, V). We will content ourselves with the following.

Proposition 7.3. *There exists an admissible integer N.*

Proof. The proof is very similar to the proof of [43, Proposition 4.1]. It follows from the results of the previous sections and the principle of spreading out [57, §3.2]. We omit the details, but refer in each case to the corresponding property over $\mathbb{Q}$.

Properties (1) and (2) follow from spreading out; property (3) follows from Lemma 3.14; property (4) follows from the definition of C° in Section 3.7 and part (6) of Proposition 3.13; property (5) follows from Theorem 4.13; property (6) follows from an argument similar to Property 5 of [43, Proposition 4.1]; property (7) follows from Lemma 3.2; property (8) follows from an argument similar to Property 4 of [43, Proposition 4.1]; property (9) follows from Propositions 3.11 and 4.8; property (10) follows from Proposition 3.21 and Theorem 4.14; property (11) follows from Theorem 5.14; finally, property (12) follows from Theorem 6.4. ■

For the remainder of the paper, we fix an admissible integer N and continue to write $S = \operatorname{Spec} \mathbb{Z}[1/N]$. Moreover, to simplify notation, we will drop the subscript $\underline{()}_S$ and write $G, V, B, J, C, \dots$ for $\underline{G}_S, \underline{V}_S, \underline{B}_S, \underline{J}_S, \underline{C}_S, \dots$

Using these properties, we can extend our previous results to S-algebras rather than $\mathbb{Q}$-algebras. We mention in particular the following two, which follow from properties (9), (10) and (12). (Recall from Section 1.7 the definition of H^1.)

Proposition 7.4 (Analogue of Corollary 6.5). *Let R be an S-algebra and let $b \in B^{\mathrm{rs}}(R)$. Then we have a natural bijection of pointed sets*

$$G(R)\backslash V_b(R) \simeq \ker(\mathrm{H}^1(R, J_b[2]) \to \mathrm{H}^1(R, G)).$$

Proposition 7.5 (Analogue of Theorem 6.6). *Let R be a local S-algebra and let $b \in B^{\mathrm{rs}}(R)$. Then there is an injective map*

$$\eta_b \colon J_b(R)/2J_b(R) \hookrightarrow G(R)\backslash V_b(R)$$

compatible with base change on R.

We are now ready to state the main theorem of this section. Write $\mathscr{E}_p$ for the set of all $b \in \underline{B}(\mathbb{Z}_p)$ that lie in $B^{\mathrm{rs}}(\mathbb{Q}_p)$. It consists of those elements of $\underline{B}(\mathbb{Z}_p)$ of nonzero discriminant. (This is different to $\underline{B}^{\mathrm{rs}}(\mathbb{Z}_p)$, which consists of those $b \in \underline{B}(\mathbb{Z}_p)$ whose discriminant is a *unit* in $\mathbb{Z}_p$ and for which integral orbits are already constructed in Proposition 7.5.)

Theorem 7.6. *Let p be a prime not dividing N. Then for any $b \in \mathscr{E}_p$, the image of the map*

$$\eta_b \colon J_b(\mathbb{Q}_p)/2J_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)\backslash V_b(\mathbb{Q}_p)$$

from Theorem 6.6 is contained in the image of the map $V(\mathbb{Z}_p) \to G(\mathbb{Q}_p)\backslash V(\mathbb{Q}_p)$.

The proof of Theorem 7.6 will be given at the end of Section 7.5. The essential ingredients are Lemmas 7.13 and 7.16 and the good geometric properties of the compactified Jacobian summarised in Theorem 5.14. We refer to the start of Section 7.5 for a general overview of the proof strategy.

7.3. Some stacks

For technical purposes related to the proof of Theorem 7.6, we need to introduce some stacks relevant to our setup. This can be seen as an attempt to 'geometrise' the set of G-orbits of V, and allows for more flexibility in glueing and descent arguments. Hopefully, we soothe the reader by mentioning that we will not need any serious properties of stacks, and we mainly think of them as collections of groupoids where one can glue objects suitably. All stacks introduced in this paper are considered in the étale topology. Recall from Section 7.2 that we have fixed an admissible integer N and we have set $S = \mathbb{Z}[1/N]$.

Definition 7.7. Let $\mathsf{B}G = [\operatorname{Spec} S/G]$ be the *classifying stack* of G. By definition, for any S-scheme X the groupoid $\mathsf{B}G(X)$ has as objects G-torsors over X. Morphisms are given by isomorphisms of G-torsors.

Definition 7.8. Let $\mathcal{M} = [G\backslash V]$ be the quotient stack of V by the natural G-action on V. By definition, for any S-scheme X an object of $\mathcal{M}(X)$ consists of a G-torsor $T \to X$ together with a G-equivariant morphism $\phi\colon T \to V$. A morphism between two objects (T, ϕ) and (T', ϕ') consists of an isomorphism $\alpha\colon T \to T'$ of G-torsors satisfying $\phi' \circ \alpha = \phi$.

Finally, recall that $Z \to B$ denotes the centraliser of the Kostant section κ, an extension of the group scheme of Definition 3.16 to S. Consider the quotient stack $[B/Z] \to B$, where Z acts trivially on B. For any B-scheme X, an X-point of $[B/Z](X)$ consists of a Z-torsor on X.

These stacks come with a few natural maps between them:

- $\mathcal{M} \to \mathsf{B}G$: sends a pair (T, ϕ) to the G-torsor T.
- $\mathcal{M} \to B$: sends a pair $(T \xrightarrow{\alpha} X, T \xrightarrow{\phi} V)$ to the unique morphism $X \xrightarrow{f} B$ fitting in the commutative diagram

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[d, "\alpha"] & V \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B. \end{tikzcd}$$

  (Here π denotes the invariant map, and the existence and uniqueness of f follow from étale descent.) We will often regard $\mathcal{M}$ as a stack over B. In particular, if $b \in B(X)$ is an X-point, we write $\mathcal{M}_b$ for the pullback of $\mathcal{M}$ along this point; it is isomorphic to $[G\backslash V_b]$.
- $V \to \mathcal{M}$: sends an X-point $X \xrightarrow{v} V$ to $(G \times X, \phi_v)$, where $\phi_v\colon G \times X \to V$ sends (g, x) to $g \cdot v(x)$.
- There is a substack $[B/Z] \hookrightarrow \mathcal{M}$ obtained by 'twisting' the Kostant section. For any B-scheme X, its image consists of those elements of $\mathcal{M}$ that are étale locally conjugate to κ_b (or rather to its image under $V \to \mathcal{M}$.)

If $\mathscr{G}$ is a groupoid, we write $\pi_0\mathscr{G}$ for its set[3] of isomorphism classes.

Lemma 7.9. *Let b be an X-point of B. The map $V_b(X) \to \mathcal{M}_b(X)$ induces a bijection between the $G(X)$-orbits of $V_b(X)$ and elements of $\pi_0(\mathcal{M}_b(X))$ that map to the trivial element in $\pi_0(\mathsf{B}G(X))$.*

Proof. This follows formally from the definitions. Indeed, if $v, v' \in V_b(X)$ give rise to isomorphic elements $(G \times X, \phi_v)$ and $(G \times X, \phi_{v'})$ in $\mathcal{M}_b(X)$, then there exists an isomorphism $G \times X \xrightarrow{\sim} G \times X$ of G-torsors mapping ϕ_v to $\phi_{v'}$. Such an isomorphism is defined by multiplying an element of $G(X)$, so v and v' are $G(X)$-conjugate. The argument can be reversed, so we obtain an injection

$$G(X)\backslash V_b(X) \hookrightarrow \pi_0(\mathcal{M}_b(X)).$$

Since an object (T, ϕ) of $\mathcal{M}_b(X)$ is isomorphic to $(G \times X, \phi_v)$ for some $v \in V_b(X)$ if and only if T is the trivial torsor, we conclude. ∎

Example 7.10. Suppose that $X = \operatorname{Spec} \mathbb{Z}_p$, where p is coprime to N and $b \in B(\mathbb{Z}_p)$. Since every G-torsor over $\operatorname{Spec} \mathbb{Z}_p$ is trivial (by [49, Chapter III, Remark 3.11 (a)] and Lang's theorem), Lemma 7.9 gives a bijection between $G(\mathbb{Z}_p)\backslash V_b(\mathbb{Z}_p)$ and $\pi_0(\mathcal{M}_b(\mathbb{Z}_p))$. Therefore, when we want to show that an orbit $x \in G(\mathbb{Q}_p)\backslash V_b(\mathbb{Q}_p)$ has an integral representative, it suffices to extend it to a $\mathbb{Z}_p$-point of quotient stack $\mathcal{M}_b$. This will be used in the proof of Theorem 7.6.

The next lemma can be interpreted as a categorical version of Corollary 6.5.

Lemma 7.11. *The inclusion $[B/Z] \hookrightarrow \mathcal{M}$ induces an isomorphism of stacks $[B^{\mathrm{rs}}/Z^{\mathrm{rs}}] \simeq \mathcal{M}^{\mathrm{rs}}$ over B^{rs}.*

Proof. It suffices to prove that for any B-scheme X and $b \in B^{\mathrm{rs}}(X)$, every two objects in $\mathcal{M}_b(X)$ are étale locally isomorphic. (Since then every object will be étale locally isomorphic to the Kostant section.) By passing to an étale extension, we may assume that these objects map to the trivial element in $\pi_0(\mathsf{B}G(X))$. It therefore suffices to prove that every two elements of $V_b^{\mathrm{rs}}(X)$ are étale locally $G(X)$-conjugate. This is true since $G \times B^{\mathrm{rs}} \to V^{\mathrm{rs}}$ is smooth and surjective so has sections étale locally; see property (9) of Section 7.2. ∎

Let $\mathcal{M}^{\mathrm{reg}} \subset \mathcal{M}$ be the open substack whose X-points consist of those objects (T, ϕ) of $\mathcal{M}(X)$ such that ϕ lands in the locus of regular elements V^{reg} and all morphisms between them. Note that the map $[B/Z] \to \mathcal{M}$ factors through $\mathcal{M}^{\mathrm{reg}}$ by (a spreading out of) part (2) of Proposition 3.11.

Lemma 7.12. *The inclusion $[B/Z] \hookrightarrow \mathcal{M}^{\mathrm{reg}}$ induces an isomorphism of stacks $[B^1/Z^1] \simeq \mathcal{M}^{\mathrm{reg}}|_{B^1}$ over B^1.*

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[3] Assuming it is a set, which will always be the case in this paper.

Proof. By the same reasoning as the proof of Lemma 7.11, it suffices to prove that $G \times B^1 \to V^{1,\mathrm{reg}}$ is smooth and surjective. This follows from property (9) of Section 7.2, which is a spreading out of Propositions 3.11 and 4.8. ■

The next lemma is a purity result for the stack $\mathcal{M}$, and will be a crucial ingredient in the proof of Theorem 7.6.

Lemma 7.13. *Let X be a regular integral 2-dimensional scheme, let $U \subset X$ be an open subscheme whose complement is finite and let $b \in B(X)$. Then the restriction $\mathcal{M}_b(X) \to \mathcal{M}_{b|_U}(U)$ is an equivalence of categories.*

Proof. We will use the following fact [26, Lemme 2.1 (iii)] repeatedly: if Y is an affine X-scheme of finite type, then restriction of sections $Y(X) \to Y(U)$ is bijective. To prove essential surjectivity, let $(T_U, T_U \xrightarrow{\phi_U} V_b)$ be an object of $\mathcal{M}_{b|_U}(U)$. According to [26, Théoreme 6.13], the G-torsor $T_U \to U$ extends to a G-torsor T on X. By the fact above applied to $Y = V_b$, ϕ_U uniquely extends to a morphism $\phi \colon T \to V_b$. The uniqueness of ϕ guarantees that ϕ is G-equivariant. Since the scheme of isomorphisms $\mathrm{Isom}_{\mathcal{M}}(\mathcal{A}, \mathcal{A}')$ between two objects of $\mathcal{M}_b(X)$ is X-affine, fully faithfulness follows from the fact applied to this isomorphism scheme. ■

7.4. *Orbits of square-free discriminant*

In this subsection, we study orbits of square-free discriminant, which will be useful in the proof of Theorem 7.6 and in Section 9. For the remainder of Section 7.4, we fix a discrete valuation ring R with fraction field K, uniformiser π, residue field k and normalised discrete valuation $\mathrm{ord}_K \colon K^\times \twoheadrightarrow \mathbb{Z}$. We assume that the integer N fixed in Section 7.2 is a unit in R.

Lemma 7.14. *Suppose that $b \in B(R)$ satisfies $\mathrm{ord}_K(\Delta(b)) = 1$. Then $b_k \in D^1(k)$.*

Proof. We may assume that R is complete. Since $\Delta(b)$ reduces to $0 \in k$, we have that $b_k \in D$. Since $\Delta(b)$ is a uniformiser of R, the quotient of the regular ring $\mathrm{H}^0(B_R, \mathcal{O}_{B_R}) = R[p_{d_1}, \dots, p_{d_r}]$ by the maximal ideal $(p_{d_1} - p_{d_1}(b), \dots, p_{d_r} - p_{d_r}(b), \Delta)$ is isomorphic to k. Therefore, the elements $\{p_{d_1} - p_{d_1}(b), \dots, p_{d_r} - p_{d_r}(b), \Delta\}$ form a regular system of parameters at b_k. Hence $D_R = \mathrm{Spec}\, R[\{p_{d_i}\}]/(\Delta)$ is regular at b_k. We conclude that b_k is a regular point of D_R. To prove the lemma, it suffices to prove that the regular locus of D_R coincides with the smooth locus D^1_R of $D_R \to \mathrm{Spec}\, R$.

Indeed, let $Z \subset D$ be the nonsmooth locus of $D \to \mathrm{Spec}\, S$, which coincides with the nonregular locus of D and is S-flat by property (2) of Section 7.2. Let Z' be the nonregular locus of D_R, which is closed by the excellence of R. Since taking the smooth locus commutes with base change, Z_R agrees with the nonsmooth locus of $D_R \to \mathrm{Spec}\, R$. It therefore suffices to show that $Z_R = Z'$. Since every smooth point of $D_R \to \mathrm{Spec}\, R$ is regular, $Z' \subset Z_R$. To prove the opposite inclusion, let L be the prime subfield of K, in other words the residue field of the image of $\mathrm{Spec}\, K \to \mathrm{Spec}\, S$. Since L is perfect, Z_L is the nonregular locus of D_L and every point of $D_L \setminus Z_L$ is geometrically regular.

It follows that Z_K is the nonregular locus of D_K and so $Z_K = Z'_K$. Since Z is S-flat by assumption, Z_R is R-flat, so Z_K is dense in Z_R. Since Z' contains the closure of $Z'_K = Z_K$ which is Z_R, we conclude that $Z_R \subset Z'$. ■

Lemma 7.15. *Suppose that* $b \in B(R)$ *satisfies* $\operatorname{ord}_K(\Delta(b)) = 1$. *Then the scheme* V_b *is regular.*

Proof. The idea of the proof is to reduce the statement to $\mathfrak{sl}_2$; this will be achieved by a sequence of standard but somewhat technical reduction steps. Since regularity can be checked after étale extensions and completion, we may assume that R is complete and k is separably closed. By Lemma 7.14, $b_k \in D^1(k)$. Let $v \in V_b(k)$ be a semisimple element. Then the centraliser $Z_H(v)$ is a reductive group of semisimple rank-1 (property (5) of Section 7.2).

We claim that there exists a lift $\widetilde{v} \in V(R)$ of v such that $\widetilde{v}_K \in V(K) \subset \mathfrak{h}(K)$ is semisimple and such that the group scheme $Z_H(\widetilde{v}) \to \operatorname{Spec} R$ is smooth with connected fibres. Indeed, let $\mathfrak{c} \subset V_k$ be a Cartan subspace containing v (here we use the extension of Vinberg theory to positive characteristic of [47]). Let $x \in V(R)$ be a lift of some regular semisimple element in $\mathfrak{c}$. Then x_K is regular semisimple (this being an open condition) and its centraliser $\widetilde{\mathfrak{c}} := \mathfrak{z}_{\mathfrak{h}}(x) \subset V_R$ is a Cartan subspace lifting $\mathfrak{c}$. Since k is separably closed, $\mathfrak{c}$ is a split Cartan subalgebra; since R is complete the same is true for $\widetilde{\mathfrak{c}}$. We may therefore choose an element $\widetilde{v} \in \widetilde{\mathfrak{c}}$ lifting v that vanishes on the same roots of $\mathfrak{c}$ as v; this $\widetilde{v}$ will satisfy the desired properties. The smoothness of the centraliser $L := Z_H(\widetilde{v}) \to \operatorname{Spec} R$ follows from [28, Theorem 1.1 (1)]. The connectedness of the fibres follows from Lemma 2.2, whose proof continues to hold if the characteristic of k is not a torsion prime for $\mathfrak{h}$, which is weaker than our assumption that the order of the Weyl group is invertible in k (property (2) of Section 7.2).

The involution $\theta\colon \mathfrak{h} \to \mathfrak{h}$ restricts to a stable involution of the Lie algebra $\mathfrak{l}$ of L by [77, Lemma 2.5]. We claim that the morphism of R-schemes $G \times \mathfrak{l}^{\theta=-1} \to V_R$, $(g, x) \mapsto g \cdot x$ is smooth. Since the domain and target are R-flat, it suffices to check this R-fibrewise [31, Chapitre I, §7.4]. This follows from [77, Proposition 4.5] (noting that $X^1 = \mathfrak{l}^{\theta=-1}$ in this case), whose proof continues to hold when the characteristic of k does not divide the order of the Weyl group of H.

Let $\mathfrak{l} \xrightarrow{\pi_L} B_L := \mathfrak{l} /\!\!/ L$ be the GIT quotient, and let $\phi\colon B_L \to B$ be the map induced by the inclusion $\mathfrak{l} \subset \mathfrak{h}$. Since ϕ is étale at $\pi_L(v) \in B_L(k)$ (Lemma 2.3), the R-point $b \in B(R)$ uniquely lifts to an R-point $b_L \in B_L(R)$ satisfying $b_{L,k} = \pi_L(v)$. Since b_L is open in the fibre $\phi^{-1}(b)$, $\mathfrak{l}_{b_L} := \pi_L^{-1}(b_L)$ is an open subscheme of $\mathfrak{l} \cap \mathfrak{h}_b$. Using the previous paragraph, this implies that the action map $m\colon G \times \mathfrak{l}_{b_L}^{\theta=-1} \to V_b$ is smooth. We claim that m is also surjective. By property (9) of Section 7.2, the image of m contains the set V_b^{reg} of regular elements (in the sense of Lie theory). The complement $V_b \setminus V_b^{\mathrm{reg}}$ consists of the semisimple elements of the special fibre $V_{b,k}$. Since all such semisimple elements are $G(\bar{k})$-conjugate and since the image of m contains v, we conclude that m is surjective.

Since regularity is a smooth-local property [75, Tag 036D], the smooth surjective morphism m shows that it suffices to prove that $\mathfrak{l}_{b_L}^{\theta=-1}$ is a regular scheme. We now make $\mathfrak{l}$

more explicit. Since 2 is invertible in R and since L is reductive of semisimple rank 1, $\mathfrak{l} = Z(\mathfrak{l}) \oplus \mathfrak{l}^{\mathrm{der}}$, where $Z(\mathfrak{l})$ and $\mathfrak{l}^{\mathrm{der}}$ are the centre and the derived subalgebra of $\mathfrak{l}$, respectively. Since k is separably closed, $\mathfrak{l}^{\mathrm{der}} \simeq \mathfrak{sl}_{2,R}$. We claim that any two stable involutions on $\mathfrak{l}$ are étale locally conjugate. Indeed, the subscheme of elements of L mapping one such involution to another is smooth, so to prove that it has sections étale locally we merely have to show it surjects on Spec R, which follows from spreading out of Lemma 2.10. (See [44, Proposition 5.6] for the proof of a similar statement.) Therefore, we may assume that in the decomposition $\mathfrak{l} \simeq Z(\mathfrak{l}) \oplus \mathfrak{sl}_{2,R}$, θ corresponds to the standard stable involution $\mathrm{Ad}((1,-1))$ of $\mathfrak{sl}_{2,R}$ and to -1 on $Z(\mathfrak{l})$. Moreover, if Δ_L denotes the discriminant polynomial of $\mathfrak{l}$, then $\Delta_L(b_L)$ equals $\Delta(b)$ up to a unit in R (Lemma 2.3). We may now calculate that $\mathfrak{l}_{b_L}^{\theta=-1}$ is isomorphic to the scheme $(xy = \Delta_L(b_L))$. This scheme is regular since $\Delta(b)$, and hence $\Delta_L(b_L)$ is a uniformiser of R. ■

Lemma 7.16. *Let $b \in B(R)$ with $\mathrm{ord}_K(\Delta(b)) = 1$. Then C_b is regular and its geometric special fibre is integral and has a unique singularity, which is a node. Moreover, the group scheme $J_b^1 \to$ Spec R (where J^1 is introduced in Section 4.3) is the Néron model of its generic fibre.*

Proof. By Lemma 7.15, the scheme V_b is regular. Moreover, since C° is (the spreading out of) a transverse slice of the G-action on V, the map $G \times C_b^\circ \to V_b$ is smooth (property (4) of Section 7.2). Since regularity is smooth-local, it follows that C_b° is a regular scheme. Since C_b is smooth in a Zariski neighbourhood of the marked points $\infty_{i,b}$, the scheme C_b is regular too. We have $b_k \in D^1(k)$ by Lemma 7.14. Therefore, the special fibre C_{b_k} is geometrically integral and has a unique nodal singularity (property (5) of Section 7.2). The claim about J_b^1 follows from the regularity of C_b and a result of Raynaud [21, §9.5, Theorem 1]. ■

The next theorem is not necessary for the proof of Theorem 7.6, but completely determines the integral orbits in the case of square-free discriminant and will be useful in Sections 8.10 and 9.

Theorem 7.17. *Let R be a discrete valuation ring in which N is a unit. Let $K = \mathrm{Frac}\, R$ and let $\mathrm{ord}_K \colon K^\times \twoheadrightarrow \mathbb{Z}$ be the normalised discrete valuation. Let $b \in B(R)$ and suppose that $\mathrm{ord}_K \Delta(b) \leq 1$. Then*

1. *If $x \in V_b(R)$, then $Z_G(x)(K) = Z_G(x)(R)$.*
2. *The natural map $\alpha \colon G(R)\backslash V_b(R) \to G(K)\backslash V_b(K)$ is injective and its image contains $\eta_b(J_b(K)/2J_b(K))$.*
3. *If furthermore R is complete and has finite residue field, then the image of α equals $\eta_b(J_b(K)/2J_b(K))$.*

Proof. If $\mathrm{ord}_K \Delta(b) = 0$, J_b is smooth and proper over R. Since $Z_G(x)$ is finite étale over R, the first part follows. By Proposition 7.4 and Lemma 7.18 below, α is injective. Proposition 7.5 and the equality $J_b(K) = J_b(R)$ imply that $\eta_b \colon J_b(K)/2J_b(K) \to G(K)\backslash V_b(K)$ factors through $G(R)\backslash V_b(R)$, so the second part follows. If R is complete and the residue field k is finite, the pointed sets $\mathrm{H}^1(R, G)$ and $\mathrm{H}^1(R, J_b)$ are trivial by [49, Chapter III, Remark 3.11 (a)] and Lang's theorem. The third part then follows from the fact that the 2-descent map $J_b(R)/2J_b(R) \to \mathrm{H}^1(R, J_b[2])$ is an isomorphism.

We now assume that $\mathrm{ord}_K\, \Delta(b) = 1$. Then $b_k \in D^1(k)$ by Lemma 7.14. According to Lemma 7.16, C_b/R is regular, has geometrically integral fibres and its special fibre has a unique nodal singularity. By the same lemma, the group scheme J_b^1/R introduced in Section 4.3 is the Néron model of its generic fibre. Moreover, we have an isomorphism $Z_b^1 \simeq J_b^1[2]$ of quasi-finite étale group schemes over R by Theorem 4.14 (or rather its spreading out, property (10) of Section 7.2).

By Lemmas 7.15 and 3.2 (and the spreading out of the latter, property (7) of Section 7.2), the scheme V_b is regular and the smooth locus of the morphism $V_b \to \operatorname{Spec} R$ coincides with the locus V_b^{reg} of regular elements of V_b (this time in the sense of Lie theory). Since a section of a morphism between regular schemes lands in the smooth locus [21, §3.1, Proposition 2], we see that $V_b(R) = V_b^{\mathrm{reg}}(R)$. By property (9) of Section 7.2, the morphism $G \times \operatorname{Spec} R \to V_b^{\mathrm{reg}}$, $(g, b) \mapsto g \cdot \kappa_b$ is a torsor under the group scheme Z_b^1 from Section 4.2. By Lemma 2.13, we obtain a bijection of pointed sets

$$G(R)\backslash V_b(R) = G(R)\backslash V_b^{\mathrm{reg}}(R) \simeq \ker(\mathrm{H}^1(R, J_b^1[2]) \to \mathrm{H}^1(R, G)). \tag{7.1}$$

We now prove the first part of the theorem. Since $x \in V_b^{\mathrm{reg}}(R)$ is étale locally G-conjugate to κ_b by the previous paragraph, we may assume that $x = \kappa_b$. But then $Z_G(\kappa_b) = Z_b^1 \simeq J_b^1[2]$ and J_b^1 satisfies the Néron mapping property, so $J_b^1[2](R) = J_b^1[2](K)$.

To prove the remaining parts, note that the map $\mathrm{H}^1(R, J_b^1[2]) \to \mathrm{H}^1(K, J_b^1[2])$ is injective (Lemma 7.18 below), so by (7.1) the map $G(R)\backslash V_b(R) \to G(K)\backslash V_b(K)$ is injective too. To show that the image of $G(R)\backslash V_b(R) \to G(K)\backslash V_b(K)$ contains the subset $\eta_b(J_b(K)/2J_b(K))$, note that we have an exact sequence of smooth group schemes

$$0 \to J_b^1[2] \to J_b^1 \xrightarrow{\times 2} J_b^1 \to 0,$$

since J_b^1 has connected fibres. This implies the existence of a commutative diagram

$$\begin{tikzcd}
J_b^1(R)/2J_b^1(R) \arrow[r, "="] \arrow[d] & J_b(K)/2J_b(K) \arrow[d] \\
\mathrm{H}^1(R, J_b^1[2]) \arrow[r] & \mathrm{H}^1(K, J_b[2]).
\end{tikzcd}$$

It therefore suffices to prove that every element in the image of the map $J_b^1(R)/2J_b^1(R) \to \mathrm{H}^1(R, J_b^1[2])$ has trivial image in $\mathrm{H}^1(R, G)$. This is true, since the pointed kernel of the map $\mathrm{H}^1(R, G) \to \mathrm{H}^1(K, G)$ is trivial (Proposition 2.16).

If R has finite residue field, then [49, Chapter III, Remark 3.11 (a)] and Lang's theorem imply that $\mathrm{H}^1(R, G) = \{1\}$. In this case, the $G(R)$-orbits of $V_b(R)$ are in bijection with $\mathrm{H}^1(R, J_b^1[2])$ by (7.1). The triviality of $\mathrm{H}^1(R, J_b^1)$ (again by Lang's theorem) shows that $\mathrm{H}^1(R, J_b^1[2])$ is in bijection with $J_b^1(R)/2J_b^1(R) = J_b(K)/2J_b(K)$. This proves part (3), completing the proof of the theorem. ■

Lemma 7.18. *Let* Γ *be a quasi-finite étale commutative group scheme over* $\operatorname{Spec} R$*. Suppose that* Γ *is a Néron model of its generic fibre: for every étale extension* $R \to R'$ *of discrete valuation rings, we have* $\Gamma(R') = \Gamma(\operatorname{Frac} R')$*. Then the map of étale cohomology groups* $\mathrm{H}^1(R,\Gamma) \to \mathrm{H}^1(K,\Gamma)$ *is injective.*

Proof. Let $j\colon \operatorname{Spec} K \to \operatorname{Spec} R$ denote the natural inclusion. Then the Néron mapping property translates into the equality of étale sheaves $j_*j^*\Gamma = \Gamma$. The map $\mathrm{H}^1(R,\Gamma) \to \mathrm{H}^1(K,\Gamma)$ is therefore injective because it is the first term in the five-term exact sequence associated with the Leray spectral sequence $\mathrm{H}^p(R, \mathrm{R}^q j_* j^*\Gamma) \Rightarrow \mathrm{H}^{p+q}(K,\Gamma)$. ■

7.5. *Proof of Theorem 7.6*

In this section, we use the results from Sections 7.3 and 7.4 to complete the proof of Theorem 7.6.

We start by summarising the broad strategy. Let p be a prime not dividing N, let $b \in B(\mathbb{Z}_p) \cap B^{\mathrm{rs}}(\mathbb{Q}_p)$, $P \in J_b(\mathbb{Q}_p)$, and let $\eta_b(P) \in G(\mathbb{Q}_p)\backslash V_b(\mathbb{Q}_p)$ be the orbit constructed in Theorem 6.6. Let

$$\lambda\colon \operatorname{Spec} \mathbb{Q}_p \to \mathcal{M}_b = [G\backslash V_b]$$

be the $\mathbb{Q}_p$-point of $\mathcal{M}_b$ corresponding to $\eta_b(P)$ under Lemma 7.9. We wish to show that $\eta_b(P)$ has a representative in $V_b(\mathbb{Z}_p)$. According to Example 7.10, this is equivalent to showing that λ extends to a morphism $\widetilde{\lambda}\colon \operatorname{Spec} \mathbb{Z}_p \to \mathcal{M}_b$. Instead of constructing such a $\widetilde{\lambda}$ directly, we will build a 2-dimensional scheme around $\operatorname{Spec} \mathbb{Z}_p$, construct a map to $\mathcal{M}$ on a large open subscheme of this scheme and use Lemma 7.13 to extend this map to the whole scheme, which can then be specialised to $\operatorname{Spec} \mathbb{Z}_p$ giving the desired $\widetilde{\lambda}$. More precisely,

(1) Using Bertini theorems and properties of the compactified Jacobian, we construct a 2-dimensional regular integral scheme $\mathcal{X}$ together with a morphism $x\colon \operatorname{Spec} \mathbb{Z}_p \to \mathcal{X}$ and a morphism $\widetilde{b}\colon \mathcal{X} \to B$ such that $\widetilde{b}$ extends b in the sense that $\widetilde{b} \circ x = b$, and such that $\Delta(\widetilde{b})$ is square-free in some sense. (See Corollary 7.20 for a precise statement.)

(2) Using the results of Section 7.4 (more specifically Lemma 7.16), we find an open subset $U \subset \mathcal{X}$ such that $x_{\mathbb{Q}_p} \in \mathcal{X}(\mathbb{Q}_p)$ lies in $U(\mathbb{Q}_p)$, $\mathcal{X} \setminus U$ is a finite set of closed points and such that $\lambda\colon \operatorname{Spec} \mathbb{Q}_p \to \mathcal{M}_b$ extends to a morphism $U \to \mathcal{M}_{\widetilde{b}}$.

(3) By the purity Lemma 7.13, the latter morphism extends to a morphism $\mathcal{X} \to \mathcal{M}_{\widetilde{b}}$. Precomposing with $x\colon \operatorname{Spec} \mathbb{Z}_p \to \mathcal{X}$ gives a morphism $\widetilde{\lambda}\colon \operatorname{Spec} \mathbb{Z}_p \to \mathcal{M}_b$ extending λ, as desired.

Before we carry out this proof strategy precisely, we need to state the following Bertini type theorem over $\mathbb{Z}_p$, proved in [43, Proposition 4.22].

Proposition 7.19. *Let* p *be a prime number. Let* $\mathcal{Y} \to \mathbb{Z}_p$ *be a smooth, quasi-projective morphism of relative dimension* $d \geq 1$ *with geometrically integral fibres. Let* $\mathcal{D} \subset \mathcal{Y}$ *be an effective Cartier divisor. Assume that* $\mathcal{Y}_{\mathbb{F}_p}$ *is not contained in* $\mathcal{D}$ *(i.e.,* $\mathcal{D}$ *is horizontal) and that* $\mathcal{D}_{\mathbb{Q}_p}$ *is reduced. Let* $P \in \mathcal{Y}(\mathbb{Z}_p)$ *be a section such that* $P_{\mathbb{Q}_p} \notin \mathcal{D}_{\mathbb{Q}_p}$*. Then there* *exists a closed subscheme* $\mathcal{X} \hookrightarrow \mathcal{Y}$ *containing the image of* P *satisfying the following properties:*

- $\mathcal{X} \to \mathbb{Z}_p$ *is smooth of relative dimension* 1 *with geometrically integral fibres.*
- $\mathcal{X}_{\mathbb{F}_p}$ *is not contained in* $\mathcal{D}$ *and the (scheme-theoretic) intersection* $\mathcal{X}_{\mathbb{Q}_p} \cap \mathcal{D}_{\mathbb{Q}_p}$ *is reduced.*

Recall that $\bar{J}$ denotes the compactified Jacobian introduced in Section 5, which has been spread out in Section 7.2 to a scheme over $\mathbb{Z}[1/N]$.

Corollary 7.20. *Suppose that* p *is a prime not dividing* N. *Let* $b \in B(\mathbb{Z}_p) \cap B^{\mathrm{rs}}(\mathbb{Q}_p)$ *and* $P \in J_b(\mathbb{Q}_p)$. *Then there exists a morphism* $\mathcal{X} \to \mathbb{Z}_p$ *which is of finite type, smooth of relative dimension* 1 *and has geometrically integral fibres, together with a morphism* $\mathcal{X} \to \bar{J}_{\mathbb{Z}_p}$ *satisfying the following properties:*

1. *Let* $\tilde{b}$ *be the composition* $\mathcal{X} \to \bar{J}_{\mathbb{Z}_p} \to B_{\mathbb{Z}_p}$. *Then the discriminant of* $\tilde{b}$, *seen as a map* $\mathcal{X} \to \mathbb{A}^1_{\mathbb{Z}_p}$, *is square-free on the generic fibre of* $\mathcal{X}$ *and not identically zero on the special fibre.*
2. *There exists a section* $x \in \mathcal{X}(\mathbb{Z}_p)$ *such that the composition* $\operatorname{Spec} \mathbb{Q}_p \xrightarrow{x_{\mathbb{Q}_p}} \mathcal{X} \to \bar{J}_{\mathbb{Z}_p}$ *coincides with* P.

Proof. We apply Proposition 7.19 with $\mathcal{Y} = \bar{J}_{\mathbb{Z}_p}$. We define $\mathcal{D}$ to be the pullback of the discriminant locus $\{\Delta = 0\} \subset \underline{B}_{\mathbb{Z}_p}$ under the morphism $\bar{J}_{\mathbb{Z}_p} \to \underline{B}_{\mathbb{Z}_p}$. Since the latter morphism is proper, we can extend $P \in J_b(\mathbb{Q}_p)$ to an element of $\bar{J}_b(\mathbb{Z}_p)$, still denoted by P. We claim that the triple $(\mathcal{Y}, \mathcal{D}, P)$ satisfies the assumptions of Proposition 7.19. This follows from an argument identical to [43, Corollary 4.23], using property (11) of Section 7.2. ■

Proof of Theorem 7.6. Choose a relative curve $\mathcal{X} \to \mathbb{Z}_p$, a map $\mathcal{X} \to \bar{J}_{\mathbb{Z}_p}$ and a section $x \in \mathcal{X}(\mathbb{Z}_p)$ satisfying the conclusions of Corollary 7.20, and let $\tilde{b}$ be the composition $\mathcal{X} \to \bar{J}_{\mathbb{Z}_p} \to B_{\mathbb{Z}_p}$. Recall that J^1 is an open subscheme of $\bar{J}$; let $\mathcal{X}^1$ denote the open subscheme of $\mathcal{X}$ landing in $J^1_{\mathbb{Z}_p}$.

We claim that the complement of $\mathcal{X}^1$ in $\mathcal{X}$ is a union of finitely many closed points. Indeed, by Lemma 7.16 and the fact that the discriminant of $\tilde{b}_{\mathbb{Q}_p}$ is square-free, the group scheme

$$J^1_{\tilde{b}_{\mathbb{Q}_p}} \to \mathcal{X}_{\mathbb{Q}_p}$$

is a Néron model of its generic fibre. By the Néron mapping property, the section

$$\mathcal{X}_{\mathbb{Q}_p} \to \bar{J}^1_{\mathbb{Q}_p}$$

must land in $J^1_{\mathbb{Q}_p}$. Since the discriminant of $\mathcal{X}$ is nonzero on the special fibre, it follows that $\mathcal{X}^1_{\mathbb{F}_p}$ is nonempty. Combining the last two sentences and the fact that $\mathcal{X}_{\mathbb{F}_p}$ is irreducible proves the claim.

To carry out the second step in the proof strategy sketched in the beginning of this section, we construct a morphism $\mathcal{X}^1 \to \mathcal{M}_{\tilde{b}}$ such that precomposing this morphism with

$x_{\mathbb{Q}_p}\colon \operatorname{Spec}\mathbb{Q}_p \to \mathcal{X}$ equals the morphism $\lambda\colon \operatorname{Spec}\mathbb{Q}_p \to \mathcal{M}_b$ that corresponds to the orbit $\eta_b(P)$ under Lemma 7.9. The multiplication-by-2 map $J^1 \xrightarrow{\times 2} J^1$ on the semiabelian scheme $J^1 \to B^1$ is a $J^1[2]$-torsor, and pulling back this torsor along the morphism $\mathcal{X}^1 \to J^1$ defines a $J^1[2]$-torsor $T \to \mathcal{X}^1$. Using the isomorphism $J^1[2] \simeq Z^1$ of Theorem 4.14 (and its spreading out version of property (10) in Section 7.2), we obtain a Z^1-torsor $T' \to \mathcal{X}^1$. In the language of Section 7.3, this torsor determines a morphism $\mathcal{X}^1 \to [B/Z]$. Composing this morphism with the inclusion $[B/Z] \hookrightarrow \mathcal{M}$ (given by twisting the Kostant section and described in Section 7.3), we obtain a morphism $\mathcal{X}^1 \to \mathcal{M}_{\tilde{b}}$. Specializing this morphism at $x_{\mathbb{Q}_p}$ corresponds to the orbit $\eta_b(P)$ under Lemma 7.9, by the explicit construction of η_b in Theorem 6.6.

Finally, by Lemma 7.13 – and this is the key point – the above morphism $\mathcal{X}^1 \to \mathcal{M}_{\tilde{b}}$ extends (uniquely) to a morphism $\mathcal{X} \to \mathcal{M}_{\tilde{b}}$. Precomposing with $x\colon \operatorname{Spec}\mathbb{Z}_p \to \mathcal{X}$ defines a morphism $\tilde{\lambda}\colon \operatorname{Spec}\mathbb{Z}_p \to \mathcal{M}_b$ whose $\mathbb{Q}_p$-fibre corresponds to $\eta_b(P)$ under Lemma 7.9. By Example 7.10, $\tilde{\lambda}$ actually arises from an element of $V_b(\mathbb{Z}_p)$. We conclude that $\eta_b(P)$ has a representative in $V_b(\mathbb{Z}_p)$, completing the proof. ■

7.6. *Orbits over* $\mathbb{Z}$

Recall that $\mathscr{E}_p = \underline{B}(\mathbb{Z}_p) \cap B^{\mathrm{rs}}(\mathbb{Q}_p)$ for all p. Define $\mathscr{E} := \underline{B}(\mathbb{Z}) \cap B^{\mathrm{rs}}(\mathbb{Q})$. We state the following corollary, whose proof is completely analogous to the proof of [66, Corollary 5.8] and uses the fact that $\underline{G}$ has class number 1 (Proposition 7.2).

Corollary 7.21. *Let $b_0 \in \mathscr{E}$. Then for each prime p dividing N, we can find an open compact neighbourhood W_p of b_0 in $\mathscr{E}_p$ and an integer $n_p \geq 0$ with the following property. Let $M = \prod_{p\mid N} p^{n_p}$. Then for all $b \in \mathscr{E} \cap (\prod_{p\mid N} W_p)$ and for all $y \in \mathrm{Sel}_2(J_{M\cdot b})$, the orbit $\eta_{M\cdot b}(y) \in G(\mathbb{Q})\backslash V_{M\cdot b}(\mathbb{Q})$ contains an element of $\underline{V}_{M\cdot b}(\mathbb{Z})$.*

This statement about integral representatives will be strong enough to obtain the main theorems in Section 8.

8. Geometry-of-numbers

In this section, we will apply the counting techniques of Bhargava to provide estimates for the integral orbits of bounded height in the representation $(\underline{G}, \underline{V})$. We keep the notation from the previous sections and continue to assume that H is not of type A_1.

8.1. *Heights*

Recall that $\underline{B} = \operatorname{Spec}\mathbb{Z}[p_{d_1}, \dots, p_{d_r}]$ and that $\pi : \underline{V} \to \underline{B}$ denotes the morphism of taking invariants. For any $b \in B(\mathbb{R})$, we define the *height* of b by the formula

$$\mathrm{ht}(b) := \sup_{1\leq i\leq r} |p_i(b)|^{1/i}.$$

We define $\mathrm{ht}(v) = \mathrm{ht}(\pi(v))$ for any $v \in V(\mathbb{R})$. We have $\mathrm{ht}(\lambda \cdot b) = |\lambda|\mathrm{ht}(b)$ for all $\lambda \in \mathbb{R}$ and $b \in B(\mathbb{R})$. If A is a subset of $V(\mathbb{R})$ or $B(\mathbb{R})$ and $X \in \mathbb{R}_{>0}$, we write $A_{<X} \subset A$ for the subset of elements of height $< X$. For every such X, the set $\underline{B}(\mathbb{Z})_{<X}$ is finite.

The next lemma records a numerological fact, which implies that $\underline{B}(\mathbb{Z})_{<X}$ has an order of magnitude $X^{\dim V}$.

Lemma 8.1. *We have* $d_1 + \cdots + d_r = \dim_{\mathbb{Q}} V$.

Proof. Recall from Section 3.1 that Φ_H denotes a root system of H. We prove the two equalities

$$d_1 + \cdots + d_r = \frac{1}{2}\#\Phi_H + \operatorname{rank} H = \dim V.$$

The first one is classical, see [22, Corollary 10.2.4]; the second one follows from [77, Lemma 2.21] applied to $x = 0$. ■

8.2. *Measures on G*

Let ω_G be a generator for the $\mathbb{Q}$-vector space of left-invariant top differential forms on $\underline{G}$ over $\mathbb{Q}$. It is well defined up to an element of $\mathbb{Q}^\times$ and it determines Haar measures dg on $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ for each prime p.

Recall from Section 3.7 that m denotes the number of marked points of the family of curves $C \to B$.

Proposition 8.2. *The product* $\mathrm{vol}(\underline{G}(\mathbb{Z})\backslash\underline{G}(\mathbb{R})) \cdot \prod_p \mathrm{vol}(\underline{G}(\mathbb{Z}_p))$ *converges absolutely and equals* 2^m*, the Tamagawa number of* G.

Proof. Proposition 7.2 implies that the product equals the Tamagawa number $\tau(G)$ of G. By Proposition 3.7, G is semisimple and its fundamental group has order $2\#\pi_0(H^\theta)$; let $G_{\mathrm{sc}} \to G$ be its simply connected cover. The proof of Proposition 6.10 (more precisely, the isomorphism between (6.3) and (6.4)) shows that $\#\pi_0(H^\theta)$ has order 2^{m-1}. Now use the identities $\tau(G) = 2^m\tau(G_{\mathrm{sc}})$ [52, Theorem 2.1.1] and $\tau(G_{\mathrm{sc}}) = 1$ [46]. ■

We study the measure dg on $G(\mathbb{R})$ using the Iwasawa decomposition, after introducing some notation. Recall from Section 3.1 that we have fixed a maximal torus $T \subset H$ with set of roots Φ_H. Moreover, we have fixed a Borel subgroup P containing T, which determines a root basis $S_H \subset \Phi_H$ and a set of positive roots Φ_H^+. Then T^θ is a maximal torus of G and P^θ is a Borel subgroup of G [63, Lemma 5.1]. Let $\Phi_G = \Phi(G, T^\theta)$ be its set of roots, $S_G = \{b_1, \ldots, b_k\}$ the corresponding root basis and $\Phi_G^\pm$ the subsets of positive/negative roots. Fix, once and for all, a maximal compact subgroup $K \subset G(\mathbb{R})$. If N is the unipotent radical of P^θ, we have a decomposition $P^\theta = T^\theta N \subset G$. Let $\bar{P} = T\bar{N} \subset G$ be the opposite Borel subgroup. Then the natural product maps

$$\bar{N}(\mathbb{R}) \times T^\theta(\mathbb{R})^\circ \times K \to G(\mathbb{R}), \quad T^\theta(\mathbb{R})^\circ \times \bar{N}(\mathbb{R}) \times K \to G(\mathbb{R})$$

are diffeomorphisms. If $t \in T^\theta(\mathbb{R})$, let $\delta_G(t) = \prod_{\beta \in \Phi_G^-} \beta(t) = \det \mathrm{Ad}(t)|_{\mathrm{Lie}\, \bar{N}(\mathbb{R})}$. The following result follows from well-known properties of the Iwasawa decomposition; see [45, Chapter 3, §1].

Lemma 8.3. *Let dt, dn, dk be Haar measures on $T^\theta(\mathbb{R})^\circ$, $\bar{N}(\mathbb{R})$, K, respectively. Then the assignment*

$$\begin{aligned} f \mapsto & \int_{t\in T^\theta(\mathbb{R})^\circ} \int_{n\in \bar{N}(\mathbb{R})} \int_{k\in K} f(tnk)\, dk\, dn\, dt \\ &= \int_{t\in T^\theta(\mathbb{R})^\circ} \int_{n\in \bar{N}(\mathbb{R})} \int_{k\in K} f(ntk)\delta_G(t)^{-1}\, dk\, dn\, dt \end{aligned}$$

defines a Haar measure on $G(\mathbb{R})$.

We now fix Haar measures on the groups $T^\theta(\mathbb{R})^\circ$, K and $\bar{N}(\mathbb{R})$, as follows. We give $T^\theta(\mathbb{R})^\circ$ the measure pulled back from the isomorphism $\prod_{\beta\in S_G}\beta\colon T^\theta(\mathbb{R})^\circ \to \mathbb{R}_{>0}^{\#S_G}$, where $\mathbb{R}_{>0}$ gets its standard Haar measure $d^\times\lambda = d\lambda/\lambda$. We give K the probability Haar measure. Finally, we give $\bar{N}(\mathbb{R})$ the unique Haar measure dn such that the Haar measure on $G(\mathbb{R})$ from Lemma 8.3 coincides with dg.

8.3. Measures on V

Let ω_V be a generator of the free rank-1 $\mathbb{Z}$-module of left-invariant top differential forms on $\underline{V}$. Then ω_V is uniquely determined up to sign and it determines Haar measures dv on $V(\mathbb{R})$ and $V(\mathbb{Q}_p)$ for every prime p. We define the top form $\omega_B = dp_{d_1}\wedge\cdots\wedge dp_{d_r}$ on $\underline{B}$. It defines measures db on $B(\mathbb{R})$ and $B(\mathbb{Q}_p)$ for every prime p.

Lemma 8.4. *There exists a unique rational number $W_0 \in \mathbb{Q}^\times$ with the following property. Let $k/\mathbb{Q}$ be a field extension, let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{h}_k$ contained in V_k, and let $\mu_\mathfrak{c}\colon G_k\times\mathfrak{c}\to V_k$ be the action map. Then $\mu_\mathfrak{c}^*\omega_V = W_0\omega_G\wedge\pi|_\mathfrak{c}^*\omega_B$.*

Proof. The proof is identical to that of [78, Proposition 2.13]. Here we use the fact that the sum of the invariants equals the dimension of the representation: $d_1+\cdots+d_r = \dim_\mathbb{Q} V$ (Lemma 8.1). ■

Lemma 8.5. *Let $W_0\in\mathbb{Q}^\times$ be the constant of Lemma 8.4.*

(1) *Let $\underline{V}(\mathbb{Z}_p)^{\mathrm{rs}} := \underline{V}(\mathbb{Z}_p)\cap V^{\mathrm{rs}}(\mathbb{Q}_p)$ and define a function $m_p\colon \underline{V}(\mathbb{Z}_p)^{\mathrm{rs}}\to\mathbb{R}_{\geq 0}$ by the formula*

$$m_p(v) = \sum_{v'\in \underline{G}(\mathbb{Z}_p)\backslash(G(\mathbb{Q}_p)\cdot v\cap \underline{V}(\mathbb{Z}_p))} \frac{\#Z_{\underline{G}}(v')(\mathbb{Q}_p)}{\#Z_{\underline{G}}(v')(\mathbb{Z}_p)}.$$

Then m_p is locally constant.

(2) *Let $\underline{B}(\mathbb{Z}_p)^{\mathrm{rs}} := \underline{B}(\mathbb{Z}_p)\cap B^{\mathrm{rs}}(\mathbb{Q}_p)$ and let $\psi_p\colon \underline{V}(\mathbb{Z}_p)^{\mathrm{rs}}\to\mathbb{R}_{\geq 0}$ be a bounded, locally constant function which satisfies $\psi_p(v)=\psi_p(v')$ when $v,v'\in\underline{V}(\mathbb{Z}_p)^{\mathrm{rs}}$ are conjugate under the action of $G(\mathbb{Q}_p)$. Then we have the formula*

$$\int_{v\in\underline{V}(\mathbb{Z}_p)^{\mathrm{rs}}}\psi_p(v)\mathrm{d}v = |W_0|_p\,\mathrm{vol}(\underline{G}(\mathbb{Z}_p))\int_{b\in\underline{B}(\mathbb{Z}_p)^{\mathrm{rs}}}\sum_{v\in G(\mathbb{Q}_p)\backslash \underline{V}_b(\mathbb{Z}_p)}\frac{m_p(v)\psi_p(v)}{\#Z_{\underline{G}}(v)(\mathbb{Q}_p)}\mathrm{d}b.$$

Proof. The proof is identical to that of [65, Proposition 3.3], using Lemma 8.4. ■

8.4. Fundamental sets

Let $K \subset G(\mathbb{R})$ be the maximal compact subgroup fixed in Section 8.2. For any $c \in \mathbb{R}_{>0}$, define $T_c := \{t \in T^\theta(\mathbb{R})^\circ \mid \forall \beta \in S_G,\, \beta(t) \leq c\}$. A *Siegel set* is, by definition, any subset

$$\mathfrak{S}_{\omega,c} := \omega \cdot T_c \cdot K,$$

where $\omega \subset \bar{N}(\mathbb{R})$ is a compact subset and $c > 0$.

Proposition 8.6. (1) *For every $\omega \subset \bar{N}(\mathbb{R})$ and $c > 0$, the set*

$$\{\gamma \in \underline{G}(\mathbb{Z}) \mid \gamma \cdot \mathfrak{S}_{\omega,c} \cap \mathfrak{S}_{\omega,c} \neq \emptyset\}$$

is finite.

(2) *We can choose $\omega \subset \bar{N}(\mathbb{R})$ and $c > 0$ such that $\underline{G}(\mathbb{Z}) \cdot \mathfrak{S}_{\omega,c} = G(\mathbb{R})$.*

Proof. The first part follows from the Siegel property [19, Corollaire 15.3]. By [56, Theorem 4.15], the second part is reduced to proving that $G(\mathbb{Q}) = P(\mathbb{Q}) \cdot \underline{G}(\mathbb{Z})$, which follows from [18, §6, Lemma 1(b)]. ■

Fix $\omega \subset \bar{N}(\mathbb{R})$ and $c > 0$ so that $\mathfrak{S}_{\omega,c}$ satisfies the conclusions of Proposition 8.6. By enlarging ω, we may assume that $\mathfrak{S}_{\omega,c}$ is semialgebraic. We drop the subscripts, and for the remainder of Section 8 we write $\mathfrak{S}$ for this fixed Siegel set. The set $\mathfrak{S}$ will serve as a fundamental domain for the action of $\underline{G}(\mathbb{Z})$ on $G(\mathbb{R})$.

A $\underline{G}(\mathbb{Z})$-coset of $G(\mathbb{R})$ may be represented more than once in $\mathfrak{S}$, but by keeping track of the multiplicities this will not cause any problems. The surjective map $\varphi\colon \mathfrak{S} \to \underline{G}(\mathbb{Z})\backslash G(\mathbb{R})$ has finite fibres and if $g \in \mathfrak{S}$, we define $\mu(g) := \#\varphi^{-1}(\varphi(g))$. The function $\mu\colon \mathfrak{S} \to \mathbb{N}$ is uniformly bounded by $\mu_{\max} := \#\{\gamma \in \underline{G}(\mathbb{Z}) \mid \gamma\mathfrak{S} \cap \mathfrak{S} \neq \emptyset\}$ and has semialgebraic fibres. By pushing forward measures via φ, we obtain the formula

$$\int_{g \in \mathfrak{S}} \mu(g)^{-1}\, dg = \operatorname{vol}(\underline{G}(\mathbb{Z})\backslash G(\mathbb{R})). \tag{8.1}$$

We now construct special subsets of $V^{\mathrm{rs}}(\mathbb{R})$ which serve as our fundamental domains for the action of $G(\mathbb{R})$ on $V^{\mathrm{rs}}(\mathbb{R})$. By the same reasoning as in [78, §2.9], we can find open subsets $L_1, \ldots, L_k$ of $\{b \in B^{\mathrm{rs}}(\mathbb{R}) \mid \operatorname{ht}(b) = 1\}$ and sections $s_i\colon L_i \to V(\mathbb{R})$ of the map $\pi\colon V \to B$ satisfying the following properties:

- For each i, L_i is connected and semialgebraic and s_i is a semialgebraic map with bounded image.
- Set $\Lambda = \mathbb{R}_{>0}$. Then we have an equality

$$V^{\mathrm{rs}}(\mathbb{R}) = \bigcup_{i=1}^{k} G(\mathbb{R}) \cdot \Lambda \cdot s_i(L_i).$$

If $v \in s_i(L_i)$, let $r_i = \#Z_G(v)(\mathbb{R})$; this integer is independent of the choice of v. We record the following change-of-measure formula, which follows from Lemma 8.4.

Lemma 8.7. *Let $f\colon V(\mathbb{R}) \to \mathbb{C}$ be a continuous function of compact support and $i \in \{1, \dots, k\}$. Let $G_0 \subset G(\mathbb{R})$ be a measurable subset and let $m_\infty(v)$ be the cardinality of the fibre of the map $G_0 \times \Lambda \times L_i \to V(\mathbb{R})$, $(g, \lambda, l) \mapsto g \cdot \lambda \cdot s_i(l)$ above $v \in V(\mathbb{R})$. Then*

$$\int_{v \in G_0 \cdot \Lambda \cdot s_i(L_i)} f(v) m_\infty(v)\, dv = |W_0| \int_{b \in \Lambda \cdot L_i} \int_{g \in G_0} f(g \cdot s_i(b))\, dg\, db,$$

where $W_0 \in \mathbb{Q}^\times$ is the scalar of Lemma 8.4.

8.5. *Counting integral orbits of V*

For any $\underline{G}(\mathbb{Z})$-invariant subset $A \subset \underline{V}(\mathbb{Z})$, define

$$N(A, X) := \sum_{v \in \underline{G}(\mathbb{Z}) \backslash A_{<X}} \frac{1}{\# Z_{\underline{G}}(v)(\mathbb{Z})}.$$

(Recall that $A_{<X}$ denotes the set of elements of A of height $< X$.) Let k be a field of characteristic not dividing N. We say an element $v \in V(k)$ with $b = \pi(v)$ is

- *k-reducible* if $\Delta(b) = 0$ or if it is $G(k)$-conjugate to a Kostant section, and *k-irreducible* otherwise.
- *k-soluble* if $\Delta(b) \neq 0$ and v lies in the image of the map

  $$\eta_b\colon J_b(k)/2J_b(k) \to G(k) \backslash V_b(k)$$

  of Proposition 7.5.

For any $A \subset \underline{V}(\mathbb{Z})$, write $A^{\mathrm{irr}} \subset A$ for the subset of $\mathbb{Q}$-irreducible elements. Write $V(\mathbb{R})^{\mathrm{sol}} \subset V(\mathbb{R})$ for the subset of $\mathbb{R}$-soluble elements. Write g for the common arithmetic genus of the curves $C \to B$.

Theorem 8.8. *We have*

$$N(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V(\mathbb{R})^{\mathrm{sol}}, X) = \frac{|W_0|}{2^g} \operatorname{vol}(\underline{G}(\mathbb{Z}) \backslash G(\mathbb{R})) \operatorname{vol}(B(\mathbb{R})_{<X}) + o(X^{\dim V}),$$

where $W_0 \in \mathbb{Q}^\times$ is the scalar of Lemma 8.4.

Remark 8.9. It is possible to obtain a power saving error term in Theorem 8.8 using a finer analysis and the methods of [68], but we have not pursued this here.

We first explain how to reduce Theorem 8.8 to Proposition 8.10. Recall that there exist $\mathbb{G}_m$-actions on V and B such that the morphism $\pi\colon V \to B$ is $\mathbb{G}_m$-equivariant and that we write $\Lambda = \mathbb{R}_{>0}$. By an argument identical to [43, Lemma 5.5], the subset $V(\mathbb{R})^{\mathrm{sol}} \subset V^{\mathrm{rs}}(\mathbb{R})$ is open and closed in the Euclidean topology. Therefore, by discarding some of the subsets $L_1, \dots, L_k$ of Section 8.4, we may write $V(\mathbb{R})^{\mathrm{sol}} = \bigcup_{i \in J} G(\mathbb{R}) \cdot \Lambda \cdot s_i(L_i)$ for some $J \subset \{1, \dots, k\}$. Moreover, for every $b \in B^{\mathrm{rs}}(\mathbb{R})$ we have equalities

$$\#(G(\mathbb{R}) \backslash V_b(\mathbb{R})^{\mathrm{sol}}) / \# Z_G(\kappa_b)(\mathbb{R}) = \#(J_b(\mathbb{R})/2J_b(\mathbb{R})) / \# J_b[2](\mathbb{R}) = \frac{1}{2^g},$$

where the first follows from the definition of $\mathbb{R}$-solubility and Proposition 3.21, and the second is a general fact about real abelian varieties. Therefore, by the inclusion-exclusion principle, to prove Theorem 8.8 it suffices to prove the following proposition. (See [43, §5.2] for more details concerning this step.)

For any nonempty subset I of $\{1, \dots, k\}$, write $L_I = \pi(\bigcap_{i \in I} G(\mathbb{R}) \cdot s_i(L_i))$. Fix an $i \in I$, write s_I for the restriction of s_i to L_I and write $r_I = r_i$. (The section s_I may depend on i but the number r_I does not if L_I is nonempty.)

Proposition 8.10. *Suppose I is a nonempty subset of $\{1, \dots, k\}$ and write $(L, s, r) = (L_I, s_I, r_I)$. Then as $X \to +\infty$, we have*

$$\begin{aligned} &N((G(\mathbb{R}) \cdot \Lambda \cdot s(L)) \cap \underline{V}(\mathbb{Z})^{\mathrm{irr}}, X) \\ &\quad = \frac{|W_0|}{r} \operatorname{vol}(\underline{G}(\mathbb{Z}) \backslash G(\mathbb{R})) \operatorname{vol}((\Lambda \cdot L)_{<X}) + o(X^{\dim V}). \end{aligned}$$

So to prove Theorem 8.8, it remains to prove Proposition 8.10. For the latter, we will follow the general orbit-counting techniques established by Bhargava, Shankar and Gross [7, 14] closely. The only notable differences are that we work with a Siegel set instead of a true fundamental domain and that we have to carry out a case-by-case analysis for cutting off the cusp in Section 8.11. For the remainder of Section 8, we fix a triple (L, s, r) as above with $L \neq \emptyset$.

8.6. First reductions

We first reduce Proposition 8.10 to estimating the number of (weighted) lattice points in a region of $V(\mathbb{R})$. Recall that $\mathfrak{S}$ denotes the Siegel set fixed in Section 8.4 which comes with a multiplicity function $\mu \colon \mathfrak{S} \to \mathbb{N}$. Because $\underline{G}(\mathbb{Z}) \cdot \mathfrak{S} = G(\mathbb{R})$, every element of $G(\mathbb{R}) \cdot \Lambda \cdot s(L)$ is $\underline{G}(\mathbb{Z})$-equivalent to an element of $\mathfrak{S} \cdot \Lambda \cdot s(L)$. In fact, we can be more precise about how often a $\underline{G}(\mathbb{Z})$-orbit will be represented in $\mathfrak{S} \cdot \Lambda \cdot s(L)$. Let $\nu \colon \mathfrak{S} \cdot \Lambda \cdot s(L) \to \mathbb{R}_{>0}$ be the 'weight' function defined by

$$x \mapsto \nu(x) := \sum_{\substack{g \in \mathfrak{S} \\ x \in g \cdot \Lambda \cdot s(L)}} \mu(g)^{-1}. \tag{8.2}$$

Then ν takes only finitely many values and has semialgebraic fibres. We now claim that if every element of $\mathfrak{S} \cdot \Lambda \cdot s(L)$ is weighted by ν, then the $\underline{G}(\mathbb{Z})$-orbit of an element $x \in G(\mathbb{R}) \cdot \Lambda \cdot s(L)$ is represented exactly $\# Z_G(x)(\mathbb{R}) / \# Z_{\underline{G}}(x)(\mathbb{Z})$ times. More precisely, for any $x \in G(\mathbb{R}) \cdot \Lambda \cdot s(L)$ we have

$$\sum_{x' \in \underline{G}(\mathbb{Z}) \cdot x \cap \mathfrak{S} \cdot \Lambda \cdot s(L)} \nu(x') = \frac{\# Z_G(x)(\mathbb{R})}{\# Z_{\underline{G}}(x)(\mathbb{Z})}.$$

This follows from an argument similar to [14, p. 202] by additionally keeping track of the multiplicity function μ.

In conclusion, for any $\underline{G}(\mathbb{Z})$-invariant subset $A \subset \underline{V}(\mathbb{Z}) \cap G(\mathbb{R}) \cdot \Lambda \cdot s(L)$ we have

$$N(A, X) = \frac{1}{r} \#[A \cap (\mathfrak{S} \cdot \Lambda \cdot s(L))_{<X}], \tag{8.3}$$

with the caveat that elements on the right-hand side are weighted by ν. (Recall that $r = \# Z_G(v)(\mathbb{R})$ for some $v \in s(L)$.)

8.7. Averaging and counting lattice points

We consider an averaged version of (8.3) and obtain a useful expression for $N(A, X)$ (Lemma 8.11) using a trick due to Bhargava. Then we use this expression to count orbits that lie in the 'main body' of V using geometry-of-numbers techniques, see Proposition 8.15.

Fix a compact, semialgebraic subset $G_0 \subset G(\mathbb{R}) \times \Lambda$ of nonempty interior, which in addition satisfies $K \cdot G_0 = G_0$, $\mathrm{vol}(G_0) = 1$ and the projection of G_0 onto Λ is contained in $[1, K_0]$ for some $K_0 > 1$. Moreover, we suppose that G_0 is of the form $G_0' \times [1, K_0]$, where G_0' is a subset of $G(\mathbb{R})$. Equation (8.3) still holds when L is replaced by hL for any $h \in G(\mathbb{R})$, by the same argument as above. Thus for any $\underline{G}(\mathbb{Z})$-invariant $A \subset \underline{V}(\mathbb{Z}) \cap G(\mathbb{R}) \cdot \Lambda \cdot s(L)$, we obtain

$$N(A, X) = \frac{1}{r} \int_{h \in G_0} \#[A \cap (\mathfrak{S} \cdot \Lambda \cdot hs(L))_{<X}]\, dh, \tag{8.4}$$

again with the caveat that elements on the right are weighted by a function similar to (8.2). We use equation (8.4) to *define* $N(A, X)$ for any subset $A \subset \underline{V}(\mathbb{Z}) \cap G(\mathbb{R}) \cdot \Lambda \cdot s(L)$ which is not necessarily $\underline{G}(\mathbb{Z})$-invariant. We can rewrite this integral using the decomposition $\mathfrak{S} = \omega \cdot T_c \cdot K$, Lemma 8.3 and an argument similar to [14, §2.3], which we omit.

Lemma 8.11. *Given $X \geq 1$, $n \in \bar{N}(\mathbb{R})$, $t \in T^{\theta}(\mathbb{R})$ and $\lambda \in \Lambda$, define $B(n, t, \lambda, X) := (nt\lambda G_0 \cdot s(L))_{<X}$. Then for any subset $A \subset \underline{V}(\mathbb{Z}) \cap (G(\mathbb{R}) \cdot \Lambda \cdot s(L))$, we have*

$$\begin{aligned} N(A, X) = \frac{1}{r} \int_{\lambda = K_0^{-1}}^{X} \int_{t \in T_c} \int_{n \in \omega} &\#[A \cap B(n, t, \lambda, X)] \mu(nt)^{-1} \\ &\times \delta_G(t)^{-1}\, dn\, dt\, d^{\times}\lambda, \end{aligned} \tag{8.5}$$

where an element $v \in A \cap B(n, t, \lambda, X)$ on the right-hand side is counted with weight $\#\{h \in G_0 \mid v \in nt\lambda h \cdot s(L)\}$.

Before estimating the integrand of (8.5) by counting lattice points in the bounded regions $B(n, t, \lambda, X)$, we first need to handle the so-called cuspidal region after introducing some notation.

Let Φ_V be the set of weights of the T^{θ}-action on V. Any $v \in V(\mathbb{Q})$ can be decomposed as $\sum v_a$, where v_a lies in the weight space corresponding to $a \in \Phi_V$. For a subset $M \subset \Phi_V$, let $V(M) \subset V$ be the subspace of elements v with $v_a = 0$ for all $a \in M$. Define $S(M) := V(M)(\mathbb{Q}) \cap \underline{V}(\mathbb{Z})$.

Let $a_0 \in X^*(T^\theta)$ denote the restriction of the highest root $\alpha_0 \in \Phi_H$ to T^θ. It turns out that $a_0 \in \Phi_V$: if H is not of type A_{2n}, this follows from the fact that the Coxeter number of H is even so the root height of α_0 with respect to S_H is odd; if H is of type A_{2n}, this can be checked explicitly.

We define $S(\{a_0\})$ (sometimes written $S(a_0)$) as the *cuspidal region* and define $\underline{V}(\mathbb{Z}) \setminus S(a_0)$ as the *main body* of V. The next proposition, proved in Section 8.11, says that the number of irreducible elements in the cuspidal region is negligible.

Proposition 8.12. *There exists* $\delta > 0$ *such that* $N(S(a_0)^{\mathrm{irr}}, X) = O(X^{\dim V - \delta})$.

Remark 8.13. The method of proof shows that an explicit δ can in principle be obtained in each case. For example, we may take $\delta = 1 + \epsilon$ for every $\epsilon > 0$ when H is of type A_{2n} by [7, Proposition 10.5].

Having dealt with the cuspidal region, we may now count lattice points in the main body using the following proposition [2, Theorem 1.3], which strengthens a well-known result of Davenport [30].

Proposition 8.14. *Let* $m, n \geq 1$ *be integers and let* $Z \subset \mathbb{R}^{m+n}$ *be a semialgebraic subset. For* $T \in \mathbb{R}^m$, *let* $Z_T = \{x \in \mathbb{R}^n \mid (T, x) \in Z\}$, *and suppose that all such subsets* Z_T *are bounded. Then for any unipotent upper-triangular matrix* $u \in \mathrm{GL}_n(\mathbb{R})$, *we have*

$$\#(Z_T \cap u\mathbb{Z}^n) = \mathrm{vol}(Z_T) + O(\max\{1, \mathrm{vol}(Z_{T,j})\}),$$

where $Z_{T,j}$ *runs over all orthogonal projections of* Z_T *to all* j*-dimensional coordinate hyperplanes* $(1 \leq j \leq n-1)$. *Moreover, the implied constant depends only on* Z.

Proposition 8.15. *Let* $A = \underline{V}(\mathbb{Z}) \cap (G(\mathbb{R}) \cdot \Lambda \cdot s(L))$. *Then*

$$N(A \setminus S(a_0), X) = \frac{|W_0|}{r} \mathrm{vol}(\underline{G}(\mathbb{Z})\backslash G(\mathbb{R}))\, \mathrm{vol}((\Lambda \cdot L)_{<X}) + o(X^{\dim V}).$$

Proof. Choose generators for the weight space $\underline{V}_a$ (as a finite free $\mathbb{Z}$-module) for every $a \in \Phi_V$ and let $\|\cdot\|$ denote the supremum norm of $V(\mathbb{R})$ with respect to this choice of basis. Since the set $\omega \cdot G_0 \cdot s(L)$ is bounded, we can choose a constant $J > 0$ such that $\|v\| \leq J$ for all $v \in \omega \cdot G_0 \cdot s(L)$. Let $F(n,t,\lambda,X) = \{v \in B(n,t,\lambda,X) \mid v_{a_0} \neq 0\}$. If $F(n,t,\lambda,X) \cap \underline{V}(\mathbb{Z}) \neq \emptyset$, there exists an element $v \in B(n,t,\lambda,X)$ such that $\|v_{a_0}\| \geq 1$, hence $\lambda a_0(t) \geq 1/J$.

We wish to estimate $\#[(A \setminus S(a_0)) \cap B(n,t,\lambda,X)] = \#[\underline{V}(\mathbb{Z}) \cap F(n,t,\lambda,X)]$ for all $t \in T_c$, $n \in \omega$, $\lambda \geq K_0^{-1}$ and X using Proposition 8.14. An element $v \in F(n,t,\lambda,X)$ has weight $\#\{h \in G_0 \mid v \in nt\lambda h \cdot s(L)\}$, and $F(n,t,\lambda,X)$ is partitioned into finitely many bounded semialgebraic subsets of constant weight. Moreover, we have an equality of (weighted) volumes $\mathrm{vol}(F(n,t,\lambda,X)) = \mathrm{vol}(B(n,t,\lambda,X))$. Since t and λ stretch the elements of $\underline{V}_a$ by factors $a(t)$ and λ, respectively, for any $M \subset \Phi_V$ the volume of the projection of $F(n,t,\lambda,X)$ to $V(M)(\mathbb{R})$ is bounded above by $O(\lambda^{\dim V - \#M} \prod_{a \in \Phi_V \setminus M} a(t))$. Since Φ_V is closed under inversion, we have $\prod_{a \in \Phi_V} a(t) = 1$. Moreover, since a_0 is the highest weight of the representation V, we know that for every $a \in \Phi_V$ we can write

$a = a_0 - \sum_{\beta \in S_G} n_\beta \beta$ for some nonnegative rationals n_β. Since $t \in T_c$ by assumption, we have $a(t) \geq c^{-\sum n_\beta} a_0(t)$. It follows that

$$\lambda^{\dim V - \#M} \prod_{a \in \Phi_V \setminus M} a(t) = \lambda^{\dim V - \#M} \prod_{a \in M} a(t)^{-1} \ll \lambda^{\dim V - \#M} a_0(t)^{-\#M}.$$

Putting the results from the previous paragraph together, we conclude by Proposition 8.14 that the number of weighted elements of $[(A \setminus S(a_0)) \cap B(n, t, \lambda, X)]$ is given by the formula

$$\begin{cases} 0 & \text{if } \lambda a_0(t) < 1/J, \\ \operatorname{vol}(B(n,t,\lambda,X)) + O(\lambda^{\dim V - 1} a_0(t)^{-1}) & \text{otherwise.} \end{cases}$$

By that same proposition, the implied constant in this estimate is independent of $t \in T_c$, $n \in \omega$, $\lambda \geq K_0^{-1}$ and X. Therefore, by Lemma 8.11 $N(A \setminus S(a_0), X)$ equals

$$\begin{aligned} \frac{1}{r} \int_{\lambda = K_0^{-1}}^{X} \int_{t \in T_c, a_0(t) \geq 1/\lambda J} \int_{n \in \omega} & (\operatorname{vol}(B(n,t,\lambda,X)) + O(\lambda^{\dim V - 1} a_0(t)^{-1})) \\ & \times \mu(nt)^{-1} \delta_G(t)^{-1} \, dn \, dt \, d^\times \lambda. \end{aligned} \tag{8.6}$$

We first show that the integral of the second summand is $o(X^{\dim V})$. One easily reduces to showing that

$$\int_{\lambda = K_0^{-1}}^{X} \int_{t \in T_c, a_0(t) \geq 1/\lambda J} \lambda^{\dim V - 1} a_0(t)^{-1} \delta_G(t)^{-1} \, dt \, d^\times \lambda = o(X^{\dim V}).$$

Write $S_G = \{\beta_1, \ldots, \beta_k\}$ and identify $T^\theta(\mathbb{R})^\circ$ with $\mathbb{R}^k_{>0}$ using the isomorphism $t \mapsto (\beta_i(t))$. Write $a_0 = \sum h_i \beta_i$ and $\sum_{\Phi_G^+} \beta = \sum \delta_i \beta_i$ with $h_i, \delta_i \in \mathbb{Q}$. Since the coefficients δ_i are strictly positive, there exists an $\epsilon \in (0,1)$ such that $\delta_i - \epsilon h_i > 0$ for all i. Since $\lambda^{1-\epsilon} a_0(t)^{1-\epsilon} \gg 1$ on $\{t \in T_c \mid a_0(t) \geq 1/\lambda J\}$, it follows that

$$\begin{aligned} & \int_{\lambda = K_0^{-1}}^{X} \int_{t \in T_c, a_0(t) \geq 1/\lambda J} \lambda^{\dim V - 1} a_0(t)^{-1} \delta_G(t)^{-1} \, dt \, d^\times \lambda \\ & \quad \ll \int_{\lambda = K_0^{-1}}^{X} \lambda^{\dim V - \epsilon} \int_{t \in T_c, a_0(t) \geq 1/\lambda J} a_0(t)^{-\epsilon} \delta_G(t)^{-1} \, dt \, d^\times \lambda. \end{aligned} \tag{8.7}$$

Since the exponents of t_i in $a_0(t)^{-\epsilon} \delta_G(t)^{-1}$ are strictly positive, the inner integral of the right-hand side of (8.7) is bounded independently of λ. It follows that the right-hand side of (8.7) is $\ll \int_{\lambda = K_0^{-1}}^{X} \lambda^{\dim V - \epsilon} d^\times \lambda = O(X^{\dim V - \epsilon})$, as claimed.

On the other hand, the integral of the first summand in (8.6) is

$$\frac{1}{r} \int_{g \in \mathfrak{S}} \operatorname{vol}((g \cdot \Lambda \cdot G_0 \cdot s(L))_{<X}) \mu(g)^{-1} \, dg + o(X^{\dim V}),$$

using the fact that $\operatorname{vol}(B(n,t,\lambda,X)) = O(\lambda^{\dim V})$. Lemma 8.7 shows that

$$\operatorname{vol}((g \cdot \Lambda \cdot G_0 \cdot s(L))_{<X}) = |W_0| \operatorname{vol}((\Lambda \cdot L)_{<X}) \operatorname{vol}(G_0) = |W_0| \operatorname{vol}((\Lambda \cdot L)_{<X}).$$

The proposition follows from formula (8.1). ∎

8.8. *End of the proof of Proposition* 8.10

The following proposition is proven in Section 8.10.

Proposition 8.16. *Let* V^{red} *denote the subset of* $\mathbb{Q}$*-reducible elements* $v \in \underline{V}(\mathbb{Z})$ *with* $v \notin S(a_0)$*. Then* $N(V^{\text{red}}, X) = o(X^{\dim V})$*.*

We now finish the proof of Proposition 8.10. Again let $A = \underline{V}(\mathbb{Z}) \cap (G(\mathbb{R}) \cdot \Lambda \cdot s(L))$. Then

$$N(A^{\text{irr}}, X) = N(A^{\text{irr}} \setminus S(a_0), X) + N(S(a_0)^{\text{irr}}, X).$$

The second term on the right-hand side is $o(X^{\dim V})$ by Proposition 8.12, and $N(A^{\text{irr}} \setminus S(a_0), X) = N(A \setminus S(a_0), X) + o(X^{\dim V})$ by Proposition 8.16. Using Proposition 8.15, we obtain

$$N(A^{\text{irr}}, X) = \frac{|W_0|}{r} \operatorname{vol}(\underline{G}(\mathbb{Z}) \backslash G(\mathbb{R})) \operatorname{vol}((\Lambda \cdot L)_{<X}) + o(X^{\dim V}).$$

This completes the proof of Proposition 8.10, hence also that of Theorem 8.8.

8.9. *Congruence conditions*

We now introduce a weighted version of Theorem 8.8. If $w \colon \underline{V}(\mathbb{Z}) \to \mathbb{R}$ is a function and $A \subset \underline{V}(\mathbb{Z})$ is a $\underline{G}(\mathbb{Z})$-invariant subset, we define

$$N_w(A, X) := \sum_{\substack{v \in \underline{G}(\mathbb{Z}) \backslash A \\ \operatorname{ht}(v) < X}} \frac{w(v)}{\# Z_{\underline{G}}(v)(\mathbb{Z})}. \tag{8.8}$$

We say a function w is *defined by finitely many congruence conditions* if w is obtained from pulling back a function $\overline{w} \colon \underline{V}(\mathbb{Z}/M\mathbb{Z}) \to \mathbb{R}$ along the projection $\underline{V}(\mathbb{Z}) \to \underline{V}(\mathbb{Z}/M\mathbb{Z})$ for some $M \geq 1$. For such a function, write μ_w for the average of $\overline{w}$ where we put the uniform measure on $\underline{V}(\mathbb{Z}/M\mathbb{Z})$. The following theorem follows immediately from the proof of Theorem 8.8, compare [14, §2.5].

Theorem 8.17. *Let* $w \colon \underline{V}(\mathbb{Z}) \to \mathbb{R}$ *be defined by finitely many congruence conditions. Then*

$$N_w(\underline{V}(\mathbb{Z})^{\text{irr}} \cap V(\mathbb{R})^{\text{sol}}, X) = \mu_w \frac{|W_0|}{2^g} \operatorname{vol}(\underline{G}(\mathbb{Z}) \backslash G(\mathbb{R})) \operatorname{vol}(B(\mathbb{R})_{<X}) + o(X^{\dim V}),$$

where $W_0 \in \mathbb{Q}^\times$ *is the scalar of Lemma* 8.4*.*

Next we will consider infinitely many congruence conditions. Suppose we are given for each prime p a $\underline{G}(\mathbb{Z}_p)$-invariant function $w_p \colon \underline{V}(\mathbb{Z}_p) \to [0, 1]$ with the following properties:

- The $\underline{G}(\mathbb{Z}_p)$-invariant function w_p is locally constant outside the closed subset $\{v \in \underline{V}(\mathbb{Z}_p) \mid \Delta(v) = 0\} \subset \underline{V}(\mathbb{Z}_p)$.
- For p sufficiently large, we have $w_p(v) = 1$ for all $v \in \underline{V}(\mathbb{Z}_p)$ such that $p^2 \nmid \Delta(v)$.

In this case, we can define a function $w\colon \underline{V}(\mathbb{Z}) \to [0,1]$ by the formula $w(v) = \prod_p w_p(v)$ if $\Delta(v) \neq 0$ and $w(v) = 0$ otherwise. Call a function $w\colon \underline{V}(\mathbb{Z}) \to [0,1]$ defined by this procedure *acceptable*.

Theorem 8.18. *Let* $w\colon \underline{V}(\mathbb{Z}) \to [0,1]$ *be an acceptable function. Then*

$$\begin{aligned}&N_w(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V^{\mathrm{sol}}(\mathbb{R}), X)\\ &\quad\le \frac{|\mathcal{W}_0|}{2^g}\Big(\prod_p \int_{\underline{V}(\mathbb{Z}_p)} w_p(v)\mathrm{d}v\Big) \operatorname{vol}(\underline{G}(\mathbb{Z})\backslash G(\mathbb{R})) \operatorname{vol}(B(\mathbb{R})_{<X}) + o(X^{\dim V}).\end{aligned}$$

Proof. This inequality follows from Theorem 8.17; the proof is identical to the first part of the proof of [14, Theorem 2.21]. ■

To obtain a lower bound in Theorem 8.18 when infinitely many congruence conditions are imposed, one needs a uniformity estimate that bounds the number of irreducible $\underline{G}(\mathbb{Z})$-orbits whose discriminant is divisible by the square of a large prime. The following conjecture is the direct analogue of [14, Theorem 2.13].

Conjecture 8.19. *For a prime* p*, let* $\mathcal{W}_p(V)$ *denote the subset of* $v \in \underline{V}(\mathbb{Z})^{\mathrm{irr}}$ *such that* $p^2 \mid \Delta(v)$*. Then for any* $M > 0$*, we have*

$$\lim_{X\to+\infty} \frac{N(\bigcup_{p>M} \mathcal{W}_p(V), X)}{X^{\dim V}} = O\Big(\frac{1}{\log M}\Big),$$

where the implied constant is independent of M*.*

Conjecture 8.19 is related to computing the density of square-free values of polynomials. See [4] for some remarks about similar questions, for known results and why these uniformity estimates seem difficult in general. By an identical proof to that of [14, Theorem 2.21], we obtain the following assertion.

Proposition 8.20. *Assume that Conjecture* 8.19 *holds for* (G, V)*. Let* $w\colon \underline{V}(\mathbb{Z}) \to [0,1]$ *be an acceptable function. Then*

$$\begin{aligned}&N_w(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V^{\mathrm{sol}}(\mathbb{R}), X)\\ &\quad= \frac{|\mathcal{W}_0|}{2^g}\Big(\prod_p \int_{\underline{V}(\mathbb{Z}_p)} w_p(v)\mathrm{d}v\Big) \operatorname{vol}(\underline{G}(\mathbb{Z})\backslash G(\mathbb{R})) \operatorname{vol}(B(\mathbb{R})_{<X}) + o(X^{\dim V}).\end{aligned}$$

8.10. Estimates on reducibility and stabilisers

In this subsection, we give the proof of Proposition 8.16 and the following proposition, which will be useful in Section 9.

Proposition 8.21. *Let* V^{bigstab} *denote the subset of* $\mathbb{Q}$*-irreducible elements* $v \in \underline{V}(\mathbb{Z})$ *with* $\#Z_G(v)(\mathbb{Q}) > 1$*. Then* $N(V^{\mathrm{bigstab}}, X) = o(X^{\dim V})$*.*

By the same reasoning as [7, §10.7] it will suffice to prove Lemma 8.22 below, after having introduced some notation.

Let N be the integer of Section 7.1 and let p be a prime not dividing N. We define $V_p^{\mathrm{red}} \subset V(\mathbb{Z}_p)$ to be the set of vectors whose reduction mod p is $\mathbb{F}_p$-reducible. We define $V_p^{\mathrm{bigstab}} \subset V(\mathbb{Z}_p)$ to be the set of vectors $v \in V(\mathbb{Z}_p)$ such that $p | \Delta(v)$ or the image of v in $V(\mathbb{F}_p)$ has nontrivial stabiliser in $G(\mathbb{F}_p)$.

Lemma 8.22. *We have*

$$\lim_{Y \to +\infty} \prod_{N < p < Y} \int_{V_p^{\mathrm{red}}} dv = 0,$$

and similarly

$$\lim_{Y \to +\infty} \prod_{N < p < Y} \int_{V_p^{\mathrm{bigstab}}} dv = 0.$$

Proof. The proof is very similar to the proof of [66, Proposition 6.9] using the root lattice calculations of Section 3.9. We first treat the case of V_p^{bigstab}. Let p be a prime not dividing N. We have the formula

$$\int_{V_p^{\mathrm{bigstab}}} dv = \frac{1}{\# V(\mathbb{F}_p)} \#\{v \in V(\mathbb{F}_p) \mid \Delta(v) = 0 \text{ or } Z_G(v)(\mathbb{F}_p) \neq 1\}.$$

Since $\{\Delta = 0\}$ is a hypersurface, we have

$$\frac{1}{\# V(\mathbb{F}_p)} \#\{v \in V(\mathbb{F}_p) \mid \Delta(v) = 0\} = O(p^{-1}). \tag{8.9}$$

If $v \in V^{\mathrm{rs}}(\mathbb{F}_p)$, then $\# Z_G(v)(\mathbb{F}_p)$ depends only on $\pi(v)$ by (the $\mathbb{Z}[1/N]$-analogue of) Lemma 3.17. Therefore, if $b \in B^{\mathrm{rs}}(\mathbb{F}_p)$, Proposition 7.4 and Lang's theorem imply that $\# V_b(\mathbb{F}_p)$ is partitioned into $\#\mathrm{H}^1(\mathbb{F}_p, J_b[2])$ many orbits under $G(\mathbb{F}_p)$, each of size $\# G(\mathbb{F}_p) / \# J_b[2](\mathbb{F}_p)$. Since $\# J_b[2](\mathbb{F}_p) = \#(J_b(\mathbb{F}_p)/2J_b(\mathbb{F}_p)) = \#\mathrm{H}^1(\mathbb{F}_p, J_b[2])$, we have $\# V^{\mathrm{rs}}(\mathbb{F}_p) = \# G(\mathbb{F}_p) \# B^{\mathrm{rs}}(\mathbb{F}_p)$.

So to prove the lemma in the case of V_p^{bigstab}, it suffices to prove that there exists a $\delta \in (0,1)$ such that

$$\frac{1}{\# B^{\mathrm{rs}}(\mathbb{F}_p)} \#\{b \in B^{\mathrm{rs}}(\mathbb{F}_p) \mid J_b[2](\mathbb{F}_p) \neq 1\} \to \delta$$

as $p \to +\infty$. We will achieve this using the results of [67, §9.3]. Recall from Section 3.1 that T is a split maximal torus of H with Lie algebra $\mathfrak{t}$ and Weyl group W. These objects spread out to objects $\underline{T}$, $\underline{H}$, $\underline{\mathfrak{t}}$ over $\mathbb{Z}$. In Section 3.9, we have defined a W-torsor $f \colon \mathfrak{t}^{\mathrm{rs}} \to B^{\mathrm{rs}}$ which extends to a W-torsor $\underline{\mathfrak{t}}^{\mathrm{rs}}_S \to \underline{B}^{\mathrm{rs}}_S$, still denoted by f. The group scheme $J[2] \to \underline{B}^{\mathrm{rs}}_S$ is trivialised along f and the monodromy action is given by the natural action of W on N_L using the same logic and notation as Proposition 3.22. Let $C \subset W$ be the subset of elements of W which fix some nonzero element of N_L. Then [67, Proposition 9.15] implies that

$$\frac{1}{\# B^{\mathrm{rs}}(\mathbb{F}_p)} \#\{b \in B^{\mathrm{rs}}(\mathbb{F}_p) \mid J_b[2](\mathbb{F}_p) \neq 1\} = \frac{\# C}{\# W} + O(p^{-1/2}). \tag{8.10}$$

Since $C \neq W$ by part (3) of Proposition 3.23, we conclude the proof of the lemma in this case.

We now treat the case V_p^{red}. Again by (8.9), it suffices to prove that there exists a non-negative $\delta < 1$ such that

$$\frac{1}{\# V^{\mathrm{rs}}(\mathbb{F}_p)} \#\{v \in V^{\mathrm{rs}}(\mathbb{F}_p) \mid v \text{ is } \mathbb{F}_p\text{-reducible}\} < \delta \tag{8.11}$$

for all sufficiently large p. By the first paragraph of the proof of this lemma, there are exactly $\# J_b[2]$ orbits of $V_b(\mathbb{F}_p)$ for all $b \in B^{\mathrm{rs}}(\mathbb{F}_p)$, each of size $\# G(\mathbb{F}_p)/\# J_b[2](\mathbb{F}_p)$. Therefore, (8.11) equals

$$\frac{1}{\# B^{\mathrm{rs}}(\mathbb{F}_p)} \sum_{b \in B^{\mathrm{rs}}(\mathbb{F}_p)} \frac{\#\{\mathbb{F}_p\text{-reducible orbits in } V_b(\mathbb{F}_p)\}}{\# J_b[2](\mathbb{F}_p)}. \tag{8.12}$$

Each summand in (8.12) belongs to the set $\{1/2^{2g}, 2/2^{2g}, \ldots, (2^{2g}-1)/2^{2g}, 1\}$; let η_p be the proportion of $b \in B^{\mathrm{rs}}(\mathbb{F}_p)$ for which this summand equals 1. The quantity in (8.12) is $\leq \eta_p + (1-\eta_p)(2^{2g}-1)/2^{2g} = 1 + (\eta_p - 1)/2^{2g}$. By (8.10), $\eta_p \to \eta := 1 - \# C/\# W$ as p tends to infinity. Since $1 \in C$, we see that $\eta < 1$ and hence $1 + (\eta - 1)/2^{2g} < 1$, completing the proof of the lemma. ∎

Proof of Proposition 8.16. We first claim that if $v \in \underline{V}(\mathbb{Z})$ with $b = \pi(v)$ is $\mathbb{Q}$-reducible, then for each prime p not dividing N the reduction of v in $V(\mathbb{F}_p)$ is $\mathbb{F}_p$-reducible. Indeed, either $\Delta(b) = 0$ in $\mathbb{F}_p$ (in which case v is $\mathbb{F}_p$-reducible), or $p \nmid \Delta(b)$ and v is $G(\mathbb{Q})$-conjugate to κ'_b for some Kostant section κ'. In the latter case, part (2) of Theorem 7.17 implies that v is $G(\mathbb{Z}_p)$-conjugate to κ'_b, so their reductions are $G(\mathbb{F}_p)$-conjugate, proving the claim. By a congruence version of Proposition 8.15, for every subset $L \subset B(\mathbb{R})$ considered in Proposition 8.10 and for every $Y > 0$ we obtain the estimate

$$N(V^{\mathrm{red}} \cap G(\mathbb{R}) \cdot \Lambda \cdot s(L), X) \leq C\Big(\prod_{N<p<Y} \int_{V_p^{\mathrm{red}}} dv\Big) \cdot X^{\dim V} + o(X^{\dim V}),$$

where $C > 0$ is a constant independent of Y. By Lemma 8.22, the product of the integrals converges to zero as Y tends to infinity, so

$$N(V^{\mathrm{red}} \cap G(\mathbb{R}) \cdot \Lambda \cdot s(L), X) = o(X^{\dim V}).$$

Since this holds for every such subset L, the proof is complete. ∎

Proof of Proposition 8.21. Note that we have not used Theorem 8.8 in the proof of Proposition 8.16, but we may use it now to prove Proposition 8.21. Again, the reduction of an element of V^{bigstab} modulo p lands in V_p^{bigstab} if p does not divide N, by part (1) of Theorem 7.17. Since $\lim_{X \to +\infty} N(V^{\mathrm{bigstab}}, X)/X^{\dim V}$ is $O(\prod_{N<p<Y} \int_{V_p^{\mathrm{red}}} dv)$ by Theorem 8.17 and the product of the integrals converges to zero by Lemma 8.22, the proof is complete. ∎

8.11. *Cutting off the cusp*

In this subsection, we consider the only remaining unproved assertion of this section, namely Proposition 8.12. This is the only substantial part of this paper where we rely on previous papers treating specific cases. The case A_{2g} is treated in [7, Proposition 10.5]; the case A_{2g+1} ($g \geq 1$) is [69, Proposition 21]; the case D_{2g+1} ($g \geq 2$) is [70, Proposition 7.6]; the case E_6 is [78, Proposition 3.6]; cases E_7 and E_8 are [65, Proposition 4.5]. Note that these authors sometimes use a power of the height that we use. It remains to consider the case where H is of type D_{2n} and $n \geq 2$. We first reduce the statement to a combinatorial result, after introducing some notation. This reduction step is valid for any H, and we do not yet assume that H is of type D_{2n}.

Recall that every element $a \in X^*(T^\theta) \otimes \mathbb{Q}$ can be uniquely written as $\sum_{i=1}^{k} n_i(a)\beta_i$ for some rational numbers $n_i(a)$. We define a partial ordering on $X^*(T^\theta) \otimes \mathbb{Q}$ by declaring that $a \geq b$ if $n_i(a-b) \geq 0$ for all $i = 1, \ldots, k$. By restriction, this induces a partial ordering on Φ_V. The restriction of the highest root $a_0 \in \Phi_V$ is the unique maximal element with respect to this partial ordering.

If (M_0, M_1) is a pair of disjoint subsets of Φ_V, we define $S(M_0, M_1) := \{v \in \underline{V}(\mathbb{Z}) \mid \forall a \in M_0, v_a = 0; \forall a \in M_1, v_a \neq 0\}$. Let $\mathcal{C}$ be the collection of nonempty subsets $M_0 \subset \Phi_V$ with the property that if $a \in M_0$ and $b \geq a$, then $b \in M_0$. Given a subset $M_0 \in \mathcal{C}$, we define $\lambda(M_0) := \{a \in \Phi_V \setminus M_0 \mid M_0 \cup \{a\} \in \mathcal{C}\}$, i.e., the set of maximal elements of $\Phi_V \setminus M_0$.

By definition of $\mathcal{C}$ and λ, $S(\{a_0\}) = \bigcup_{M_0 \in \mathcal{C}} S(M_0, \lambda(M_0))$. Therefore, to prove Proposition 8.12, it suffices to prove that for each $M_0 \in \mathcal{C}$, either $S(M_0, \lambda(M_0))^{\mathrm{irr}} = \emptyset$ or $N(S(M_0, \lambda(M_0)), X) = O(X^{\dim V - \epsilon})$ for some $\epsilon > 0$. By the same logic as [78, Proposition 3.6 and §5] (itself based on a trick due to Bhargava), the latter estimate holds if there exists a subset $M_1 \subset \Phi_V \setminus M_0$ and a function $f\colon M_1 \to \mathbb{R}_{\geq 0}$ with $\sum_{a \in M_1} f(a) < \# M_0$ such that

$$\sum_{\beta \in \Phi_G^+} \beta - \sum_{a \in M_0} a + \sum_{a \in M_1} f(a)a$$

has strictly positive coordinates with respect to the basis S_G. It will thus suffice to prove the following combinatorial proposition, which is the analogue of [7, Proposition 29].

Proposition 8.23. *Let $M_0 \in \mathcal{C}$ be a subset such that $V(M_0)(\mathbb{Q})$ contains $\mathbb{Q}$-irreducible elements. Then there exist a subset $M_1 \subset \Phi_V \setminus M_0$ and a function $f\colon M_1 \to \mathbb{R}_{\geq 0}$ satisfying the following conditions:*

- *We have $\sum_{a \in M_1} f(a) < \# M_0$.*
- *For each $i = 1, \ldots, k$, we have*

$$\sum_{\beta \in \Phi_G^+} n_i(\beta) - \sum_{a \in M_0} n_i(a) + \sum_{a \in M_1} f(a) n_i(a) > 0.$$

We will prove Proposition 8.23 in the remaining case D_{2n} in Appendix A.

9. The average size of the 2-Selmer group

9.1. An upper bound

In this section, we prove Theorem 1.3 stated in the introduction. Recall that we write $\mathscr{E}$ for the set of elements $b \in \underline{B}(\mathbb{Z})$ of nonzero discriminant. We recall that we have defined a height function ht for $\mathscr{E}$ in Section 8.1. We say a subset $\mathscr{F} \subset \mathscr{E}$ is defined by *finitely many congruence conditions* if $\mathscr{F}$ is the preimage of a subset of $\underline{B}(\mathbb{Z}/n\mathbb{Z})$ under the reduction map $\mathscr{E} \to \underline{B}(\mathbb{Z}/n\mathbb{Z})$ for some $n \geq 1$.

Theorem 9.1. *Let $\mathscr{F} \subset \mathscr{E}$ be a subset defined by finitely many congruence conditions. Let m be the number of marked points. Then we have*

$$\limsup_{X\to+\infty} \frac{\sum_{b\in\mathscr{F},\,\mathrm{ht}(b)<X} \#\,\mathrm{Sel}_2 J_b}{\#\{b \in \mathscr{F} \mid \mathrm{ht}(b) < X\}} \leq 3\cdot 2^{m-1}.$$

The proof is along the same lines as the discussion in [66, §7].

We first prove a ‘local’ result. Recall that $\mathscr{E}_p$ denotes the set of elements $b \in \underline{B}(\mathbb{Z}_p)$ of nonzero discriminant. Define $\mathscr{F}_p$ as the closure of $\mathscr{F}$ in $\mathscr{E}_p$, equivalently $\mathscr{F}_p$ is the preimage in $\mathscr{E}_p$ of a subset of $\underline{B}(\mathbb{Z}/n\mathbb{Z})$ that defines $\mathscr{F}$.

For every $b \in B^{\mathrm{rs}}(\mathbb{Q})$, consider the subgroup of $J_b(\mathbb{Q})/2J_b(\mathbb{Q})$ generated by differences of the marked points $\{\infty_1 - \infty_2, \dots, \infty_1 - \infty_m\}$. The image of this subgroup under the map $J_b(\mathbb{Q})/2J_b(\mathbb{Q}) \hookrightarrow \mathrm{Sel}_2 J_b$ is by definition the subgroup $\mathrm{Sel}_2^{\mathrm{triv}} J_b$ of ‘marked’ elements. Its complement $\mathrm{Sel}_2^{\cancel{\mathrm{triv}}} J_b$ is the subset of ‘nonmarked’ elements.

Proposition 9.2. *Let $b_0 \in \mathscr{F}$. Then we can find for each prime p dividing N an open compact neighbourhood W_p of b_0 in $\mathscr{E}_p$ such that the following condition holds. Let $\mathscr{F}_W = \mathscr{F} \cap (\prod_{p|N} W_p)$. Then we have*

$$\limsup_{X\to+\infty} \frac{\sum_{b\in\mathscr{F}_W,\,\mathrm{ht}(b)<X} \#\,\mathrm{Sel}_2^{\cancel{\mathrm{triv}}} J_b}{\#\{b \in \mathscr{F}_W \mid \mathrm{ht}(b) < X\}} \leq 2^m.$$

Proof. Choose the sets W_p and integers $n_p \geq 0$ for $p|N$ satisfying the conclusion of Corollary 7.21. If p does not divide N, set $W_p = \mathscr{F}_p$ and $n_p = 0$. Let $M := \prod_p p^{n_p}$.

For $v \in \underline{V}(\mathbb{Z})$ with $\pi(v) = b$, define $w(v) \in \mathbb{Q}_{\geq 0}$ by the following formula:

$$w(v) = \begin{cases} \left(\sum_{v'\in\underline{G}(\mathbb{Z})\backslash(\underline{G}(\mathbb{Q})\cdot v\cap\underline{V}(\mathbb{Z}))} \frac{\#Z_{\underline{G}}(v')(\mathbb{Q})}{\#Z_{\underline{G}}(v')(\mathbb{Z})}\right)^{-1} & \text{if } b \in p^{n_p}\cdot W_p \text{ and} \\ & G(\mathbb{Q}_p)\cdot v \in \eta_b(J_b(\mathbb{Q}_p)/2J_b(\mathbb{Q}_p)) \\ & \text{for all } p, \\ 0 & \text{otherwise.} \end{cases}$$

Define $w'(v)$ by the formula

$$w'(v) = \#Z_{\underline{G}}(v)(\mathbb{Q})w(v).$$

Corollaries 6.9 and 7.21 and Proposition 6.10 imply that if $b \in M \cdot \mathscr{F}_W$, nonmarked elements in the 2-Selmer group of J_b correspond bijectively to $G(\mathbb{Q})$-orbits of $V_b(\mathbb{Q})$ that intersect $\underline{V}(\mathbb{Z})$ nontrivially, that are $\mathbb{Q}$-irreducible and that are soluble at $\mathbb{R}$ and $\mathbb{Q}_p$ for all p. In other words, in the notation of (8.8) we have the formula

$$\sum_{\substack{b \in \mathcal{F}_W \\ \mathrm{ht}(b) < X}} \#\,\mathrm{Sel}_2^{\cancel{\mathrm{triv}}} J_b = \sum_{\substack{b \in M \cdot \mathcal{F}_W \\ \mathrm{ht}(b) < MX}} \#\,\mathrm{Sel}_2^{\cancel{\mathrm{triv}}} J_b = N_{w'}(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V(\mathbb{R})^{\mathrm{sol}}, MX). \tag{9.1}$$

Proposition 8.21 implies that

$$N_{w'}(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V(\mathbb{R})^{\mathrm{sol}}, MX) = N_w(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V(\mathbb{R})^{\mathrm{sol}}, MX) + o(X^{\dim V}). \tag{9.2}$$

It is more convenient to work with $w(v)$ than with $w'(v)$ because $w(v)$ is an acceptable function in the sense of Section 8.9. Indeed, for $v \in \underline{V}(\mathbb{Z}_p)$ with $\pi(v) = b$, define $w_p(v) \in \mathbb{Q}_{\geq 0}$ by the following formula:

$$w_p(v) = \begin{cases} \left(\sum_{v' \in \underline{G}(\mathbb{Z}_p) \backslash (\underline{G}(\mathbb{Q}_p) \cdot v \cap \underline{V}(\mathbb{Z}_p))} \frac{\# Z_{\underline{G}}(v')(\mathbb{Q}_p)}{\# Z_{\underline{G}}(v')(\mathbb{Z}_p)}\right)^{-1} & \text{if } b \in p^{n_p} \cdot W_p \\ & \text{and } G(\mathbb{Q}_p) \cdot v \\ & \in \eta_b(J_b(\mathbb{Q}_p)/2J_b(\mathbb{Q}_p)), \\ 0 & \text{otherwise.} \end{cases}$$

Then an argument identical to [14, Proposition 3.6] (using that $\underline{G}$ has class number 1 by Proposition 7.2) shows that $w(v) = \prod_p w_p(v)$ for all $v \in \underline{V}(\mathbb{Z})$. The remaining properties for $w(v)$ to be acceptable follow from part (1) of Lemma 8.5 and Theorem 7.17. Using Lemma 8.5, we obtain the formula

$$\int_{v \in \underline{V}(\mathbb{Z}_p)} w_p(v) dv = |W_0|_p \operatorname{vol}(\underline{G}(\mathbb{Z}_p)) \int_{b \in p^{n_p} \cdot W_p} \frac{\# J_b(\mathbb{Q}_p)/2J_b(\mathbb{Q}_p)}{\# J_b[2](\mathbb{Q}_p)} db. \tag{9.3}$$

Using the equality $\# J_b(\mathbb{Q}_p)/2J_b(\mathbb{Q}_p) = |1/2^g|_p \# J_b[2](\mathbb{Q}_p)$ for all $b \in \mathcal{E}_p$ (which is a general fact about abelian varieties), we see that the integral on the right-hand side equals $|1/2^g|_p \operatorname{vol}(p^{n_p} \cdot W_p) = |1/2^g|_p p^{-n_p \dim_{\mathbb{Q}} V} \operatorname{vol}(W_p)$. Combining identities (9.1) and (9.2) shows that

$$\limsup_{X \to +\infty} X^{-\dim V} \sum_{\substack{b \in \mathcal{F}_W \\ \mathrm{ht}(b) < X}} \#\,\mathrm{Sel}_2^{\cancel{\mathrm{triv}}} J_b = \limsup_{X \to +\infty} X^{-\dim V} N_w(\underline{V}(\mathbb{Z})^{\mathrm{irr}} \cap V(\mathbb{R})^{\mathrm{sol}}, MX).$$

This in turn by Theorem 8.18 is less than or equal to

$$\frac{|W_0|}{2^g} \left(\prod_p \int_{\underline{V}(\mathbb{Z}_p)} w_p(v) dv \right) \operatorname{vol}(\underline{G}(\mathbb{Z}) \backslash G(\mathbb{R})) 2^{\dim B} M^{\dim V}.$$

Using (9.3) this simplifies to

$$\operatorname{vol}(\underline{G}(\mathbb{Z}) \backslash \underline{G}(\mathbb{R})) \prod_p \operatorname{vol}(\underline{G}(\mathbb{Z}_p)) 2^{\dim B} \prod_p \operatorname{vol}(W_p).$$

On the other hand, an elementary point count shows that

$$\lim_{X \to +\infty} \frac{\#\{b \in \mathcal{F}_W \mid \mathrm{ht}(b) < X\}}{X^{\dim V}} = 2^{\dim B} \prod_p \operatorname{vol}(W_p).$$

We conclude that

$$\limsup_{X\to+\infty} \frac{\sum_{b\in\mathscr{F}_W,\,\mathrm{ht}(b)<X} \#\,\mathrm{Sel}_2^{\mathrm{triv}} J_b}{\#\{b\in\mathscr{F}_W \mid \mathrm{ht}(b)<X\}} \leq \mathrm{vol}(\underline{G}(\mathbb{Z})\backslash\underline{G}(\mathbb{R}))\cdot\prod_p \mathrm{vol}(\underline{G}(\mathbb{Z}_p)).$$

Since the Tamagawa number of $\underline{G}$ is 2^m (Proposition 8.2), the proposition follows. ■

To deduce Theorem 9.1 from Proposition 9.2, choose for each $i\geq 1$ sets $W_{p,i}\subset\mathscr{E}_p$ (for p dividing N) such that if $W_i=\mathscr{E}\cap(\prod_{p\mid N}W_{p,i})$, then W_i satisfies the conclusion of Proposition 9.2 and we have a countable partition $\mathscr{F}=\mathscr{F}_{W_1}\sqcup\mathscr{F}_{W_1}\sqcup\cdots$. By an argument identical to the proof of Theorem 7.1 in [66], we see that for any $\varepsilon>0$, there exists $k\geq 1$ such that

$$\limsup_{X\to+\infty} \frac{\sum_{b\in\bigsqcup_{i\geq k}\mathscr{F}_{W_i},\mathrm{ht}(b)<X} \#\,\mathrm{Sel}_2^{\mathrm{triv}} J_b}{\#\{b\in\mathscr{F}\mid \mathrm{ht}(b)<X\}}<\varepsilon.$$

This implies that

$$\begin{aligned}\limsup_{X\to+\infty} \frac{\sum_{b\in\mathscr{F},\mathrm{ht}(b)<X} \#\,\mathrm{Sel}_2^{\mathrm{triv}} J_b}{\#\{b\in\mathscr{F}\mid \mathrm{ht}(b)<X\}} &\leq 2^m \limsup_{X\to+\infty}\frac{\#\{b\in\bigsqcup_{i<k}\mathscr{F}_{W_i}\mid \mathrm{ht}(b)<X\}}{\#\{b\in\mathscr{F}\mid \mathrm{ht}(b)<X\}}+\varepsilon\\ &\leq 2^m+\varepsilon.\end{aligned}$$

Since the above inequality is true for any $\varepsilon>0$, it is true for $\varepsilon=0$. Since the subgroup $\mathrm{Sel}^{\mathrm{triv}} J_b$ has size at most 2^{m-1}, we conclude the proof of Theorem 9.1.

Remark 9.3. A small modification of the above argument shows that Theorem 9.1 remains valid when $\mathscr{F}\subset\mathscr{E}$ is the subset of so-called 'minimal' elements, namely those elements $b\in\mathcal{E}$ with $N^{-1}\cdot b\notin\underline{B}(\mathbb{Z})$ for all integers $N\geq 1$.

9.2. *A conditional lower bound*

We show that the upper bound in Theorem 9.1 is sharp if we assume Conjecture 8.19. We first need to establish (unconditionally) a lower bound for the subgroup of marked elements $\mathrm{Sel}^{\mathrm{triv}} J_b$.

Proposition 9.4. *Let $\mathscr{F}\subset\mathscr{E}$ be a subset defined by finitely many congruence conditions. Then the limit*

$$\lim_{X\to+\infty}\frac{\#\{b\in\mathscr{F}\mid \mathrm{ht}(b)<X,\,\#\,\mathrm{Sel}_2(J_b)^{\mathrm{triv}}=2^{m-1}\}}{\#\{b\in\mathscr{F}\mid \mathrm{ht}(b)<X\}}$$

exists and equals 1.

Proof. Let $b\in\mathscr{F}$ and consider the maximal torus $Z_H(\kappa_b)$ of H. By Proposition 6.10, $\#\,\mathrm{Sel}_2(J_b)^{\mathrm{triv}}=2^{m-1}$ if and only if the map $Z_G(\kappa_b)\to Z_{H^\theta}(\kappa_b)$ is surjective on $\mathbb{Q}$-points. The Galois action on $Z_H(\kappa_b)$ induces a homomorphism $\mathrm{Gal}(\mathbb{Q}^s\mid\mathbb{Q})\to W$ by Proposition 3.22, where W is the Weyl group of the split torus $T\subset H$ with character group L. If this homomorphism is surjective, then $Z_{H^\theta}(\kappa_b)(\mathbb{Q})=T[2]^W=(L^\vee/2L^\vee)^W=\{0\}$

by part (1) of Proposition 3.23, so $Z_G(\kappa_b) \to Z_{H^\theta}(\kappa_b)$ is automatically surjective on $\mathbb{Q}$-points.

It therefore suffices to prove that the limit

$$\lim_{X\to+\infty} \frac{\#\{b \in \mathcal{F} \mid \mathrm{ht}(b) < X, \mathrm{Gal}(\mathbb{Q}^s \mid \mathbb{Q}) \to W \text{ surjective}\}}{\#\{b \in \mathcal{F} \mid \mathrm{ht}(b) < X\}}$$

exists and equals 1. This follows from a version of Hilbert's irreducibility theorem; see [24, Theorem 2.1], adapted as in [24, §5, Notes (iii)] to account for the fact that the coordinates of B have unequal weights. ∎

Theorem 9.5. *Assume that Conjecture 8.19 holds for* (G, V)*. Let* $\mathcal{F} \subset \mathcal{E}$ *be a subset defined by finitely many congruence conditions. Then the limit*

$$\lim_{X\to+\infty} \frac{\sum_{b\in\mathcal{F},\,\mathrm{ht}(b)<X} \#\,\mathrm{Sel}_2^{\mathrm{triv}} J_b}{\#\{b \in \mathcal{F} \mid \mathrm{ht}(b) < X\}}$$

exists and equals 2^m*. Moreover, the average size of the* 2*-Selmer group* $\mathrm{Sel}_2 J_b$ *exists and equals* $3 \cdot 2^{m-1}$.

Proof. The proof of the first statement is identical to the proof of Theorem 9.1, using Proposition 8.20 instead of Theorem 8.18. The second statement follows from the first and Proposition 9.4. ∎

Appendix A. Cutting off the cusp for D_{2n}

In this appendix, we prove Proposition 8.12 in the case that H is of type D_{2n} for all $n \geq 2$. The methods employed here are fairly standard but somewhat intricate, and are sometimes inspired by [70, §7.2.1]. In Appendix A.1, we recall some results and notation on groups of type D_n. In Appendix A.2, we make the representation (G, V) in the case D_{2n} and some related objects explicit. In Appendix A.3, we establish sufficient conditions for a vector $v \in V(\mathbb{Q})$ to be $\mathbb{Q}$-reducible. In Appendix A.4, we finish the proof of Proposition 8.12.

A.1. Recollections on even orthogonal groups

Let $n \geq 2$ be an integer. Let W be a $2n$-dimensional $\mathbb{Q}$-vector space with basis $\mathcal{B} = \{e_1, \dots, e_n, e_n^*, \dots, e_1^*\}$. Let b be the symmetric bilinear form with the property that $b(e_i, e_j) = b(e_i^*, e_j^*) = 0$ and $b(e_i, e_j^*) = \delta_{ij}$ for all $1 \leq i, j \leq n$. For every linear map $f \colon W \to W$, there is a unique adjoint linear map $f^* \colon W \to W$ satisfying $b(fv, w) = b(v, f^*w)$ for all $v, w \in W$. We define the $\mathbb{Q}$-algebraic group $H := \mathrm{SO}(W, b) = \{g \in \mathrm{SL}(W) \mid gg^* = 1\}$. Then $\mathfrak{h} := \mathrm{Lie}\, H$ can be naturally identified with $\{f \in \mathrm{End}(W) \mid f + f^* = 0\}$. Below we will make various aspects of the semisimple group H explicit.

Using $\mathcal{B}$ to represent an element $f \colon W \to W$ as a $2n \times 2n$ matrix A, we have that f^* corresponds to reflecting A along its antidiagonal. Consider the maximal torus $T =$

$\{\operatorname{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})\} \subset H$. Its character group $X^*(T)$ is freely generated by the characters $(t_1, \dots) \mapsto t_i$ with $1 \leq i \leq n$, and we abusively denote these characters by $t_i \in X^*(T)$ too.

Root system. The roots of $\mathfrak{h}$ with respect to T are given by

$$\Phi_H = \{\pm t_i \pm t_j \mid 1 \leq i \neq j \leq n\} \subset X^*(T).$$

The standard upper triangular Borel subgroup of GL_{2n} (with respect to the basis $\mathcal{B}$ of W) determines a root basis of Φ_H, given by

$$S_H = \{t_1 - t_2, \dots, t_{n-1} - t_n, t_{n-1} + t_n\}. \tag{A.1}$$

We denote the elements of this root basis by $\alpha_1, \dots, \alpha_n$. The highest root of Φ_H with respect to S_H is $t_1 + t_2$, which has height $2n - 3$.

Weyl group. The Weyl group W_H is isomorphic to $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$. Explicitly, elements of W_H correspond to pairs $(\sigma, (\epsilon_i))$, where $\sigma \in S_n$ is a permutation and $\epsilon_i \in \{\pm 1\}$ is a sign for each $1 \leq i \leq n$, with the property that $\prod_i \epsilon_i = 1$. An element $(\sigma, (\epsilon_i))$ acts on $X^*(T)$ via the rule $t_i \mapsto \epsilon_i t_{\sigma(i)}$. In particular, $-1 \in W_H$ if and only if n is even.

Stable involution. To describe the stable involution of H in Appendix A.2 in the case that n is even, we determine the elements $s \in T$ with the property that $\alpha(s) = -1$ for every simple root $\alpha \in S_H$. Such an s is not uniquely determined since H is not adjoint, but it is uniquely determined up to multiplication by the element $(-1, \dots, -1) \in T$ of the centre of H. Using description (A.1), we see that s is of the form

$$\pm(1, -1, 1, -1, \dots, (-1)^n). \tag{A.2}$$

Change of variables. We record the following computation which will be useful in Appendix A.4:

$$\begin{cases} t_i = \alpha_i + \dots + \alpha_{n-2} + \dfrac{1}{2}(\alpha_{n-1} + \alpha_n), & 1 \leq i \leq n-2, \\ t_{n-1} = \dfrac{1}{2}(\alpha_{n-1} + \alpha_n), \\ t_n = \dfrac{1}{2}(-\alpha_{n-1} + \alpha_n). \end{cases} \tag{A.3}$$

Sum of positive roots. The sum of the positive roots of Φ_H with respect to S_H is

$$\begin{aligned} \sum_{\alpha \in \Phi_H^+} \alpha &= 2(n-1)t_1 + 2(n-2)t_2 + \dots + 2t_{n-1} \\ &= \sum_{k=1}^{n-2} k(2n-k-1)\alpha_k + \frac{n(n-1)}{2}(\alpha_{n-1} + \alpha_n). \end{aligned}$$

Discriminant. Let $\mathfrak{t} := \operatorname{Lie} T$ and write an element of $\mathfrak{t}$ as $\operatorname{diag}(t_1, \dots, t_n, -t_n, \dots, -t_1)$. Let $\Delta \in \mathbb{Q}[\mathfrak{h}]^H$ be the discriminant polynomial of H, defined as the image of $\prod_{\alpha \in \Phi_H} \alpha$

under the Chevalley isomorphism $\mathbb{Q}[\mathfrak{t}]^{W_H} \to \mathbb{Q}[\mathfrak{h}]^H$ of Proposition 2.1. Let $\operatorname{Pff} \in \mathbb{Q}[\mathfrak{h}]^H$ be the image of the product $\prod_{i=1}^n t_i \in k[\mathfrak{t}]^{W_H}$ under the Chevalley isomorphism. If we write $t_i := t_{2n+1-i}$ for $n+1 \leq i \leq 2n$, then we may compute that

$$\prod_{1 \leq i < j \leq 2n} (t_i - t_j)^2 = \prod_{\alpha \in \Phi_H} \alpha(t)^2 \prod_{i=1}^n t_i^2.$$

It follows that if we write χ_v for the characteristic polynomial of a square matrix v and $\operatorname{disc}(\chi_v)$ for its discriminant, we have the identity

$$\operatorname{disc}(\chi_v) = \Delta(v)^2 \operatorname{Pff}(v)^2 \tag{A.4}$$

for every $v \in \mathfrak{h}$.

A.2. An explicit model for the split stable involution of D_{2n}

Let $n \geq 2$ be an integer. Let W_1 be the $\mathbb{Q}$-vector space with basis $\{e_1, \dots, e_n, e_n^*, \dots, e_1^*\}$, and let b_1 be the symmetric bilinear form with the property $b_1(e_i, e_j) = b_1(e_i^*, e_j^*) = 0$ and $b_1(e_i, e_j^*) = \delta_{ij}$ for all $1 \leq i, j \leq n$. Let W_2 be the $\mathbb{Q}$-vector space with basis $\{f_1, \dots, f_n, f_n^*, \dots, f_1^*\}$, and let b_2 be the bilinear form of W_2 constructed similarly to b_1 with e_i and e_i^* replaced by f_i and f_i^*. Let $(W, b) := (W_1, b_1) \oplus (W_2, b_2)$. Let $H' := \mathrm{SO}(W, b)$, let H be the quotient of H' by its centre of order 2 and let $\mathfrak{h} := \operatorname{Lie} H = \operatorname{Lie} H'$. With respect to the basis

$$\{e_1, \dots, e_n, e_n^*, \dots, e_1^*, f_1, \dots, f_n, f_n^*, \dots, f_1^*\}, \tag{A.5}$$

the adjoint of a $(4n) \times (4n)$-block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with respect to b is given by $\left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$. Here if X is a $2n \times 2n$ matrix, we write X^* for its reflection around the antidiagonal. It follows that in this basis $\mathfrak{h}$ is given by

$$\left\{ \begin{pmatrix} B & A \\ -A^* & C \end{pmatrix} \mid B^* = -B,\, C^* = -C \right\}.$$

Stable involution. The ordered basis

$$\{e_1, f_1, \dots, e_n, f_n, f_n^*, e_n^*, \dots, f_1^*, e_1^*\} \tag{A.6}$$

of W determines a maximal torus and root basis of H as in Appendix A.1. Let θ be the involution of H constructed using the recipe in Section 3.1 with respect to this root basis. Since -1 is contained in the Weyl group of H (as observed in Appendix A.1), θ is inner. The description of (A.2) shows that with respect to the first basis (A.5) of W, θ is given by conjugating by the element $s' = \operatorname{diag}(1, \dots, 1, -1, \dots, -1)$, where the first $2n$ entries are 1 and the last $2n$ entries are -1. Using this description, it is easy to see that

$$\mathfrak{g} := \mathfrak{h}^\theta = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B^* = -B,\, C^* = -C \right\},$$

$$V := \mathfrak{h}^{\theta=-1} = \left\{ \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \mid A \in \mathrm{Mat}_{2n,2n} \right\}.$$

Moreover, $G := (H^{\theta})^{\circ}$ is isomorphic to $(\mathrm{SO}(W_1) \times \mathrm{SO}(W_2))/\Delta(\mu_2)$, where $\Delta(\mu_2)$ denotes the image of the diagonal inclusion of μ_2 into the centre $\mu_2 \times \mu_2$ of $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$. Using these identifications, we see that the map

$$\begin{pmatrix} 0 & A \\ -A^{*} & 0 \end{pmatrix} \mapsto A$$

establishes a bijection between V and the representation $\mathrm{Hom}(W_2, W_1)$, where $(g, h) \in \mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$ acts on $f\colon W_2 \to W_1$ via $g \circ f \circ h^{-1}$. In terms of matrices, the action is given by $(g, h) \cdot A = gAh^{-1}$. We will typically view an element of $V(\mathbb{Q})$ as a $2n \times 2n$ matrix A or a linear operator $f\colon W_2 \to W_1$.

Roots. Let T' be the maximal torus $\mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}, s_1, \dots, s_n, s_n^{-1}, \dots, s_1^{-1})$ of H' (again using the basis (A.5)), and let T be its image in H. Then T is a maximal torus of H and G. Let Φ_H and Φ_G be the corresponding sets of roots. Let $W_H = N_H(T)/T$ and $W_G = N_G(T)/T$ be the respective Weyl groups. Basis (A.6) determines a set of positive roots Φ_H^{+} of H (as in Appendix A.1) and by restriction a set of positive roots Φ_G^{+} of G. The corresponding simple roots are given by

$$\begin{aligned} S_H &= \{t_1 - s_1, s_1 - t_2, \dots, s_{n-1} - t_n, t_n - s_n, t_n + s_n\}, \\ S_G &= \{t_1 - t_2, \dots, t_{n-1} - t_n, t_{n-1} + t_n\} \cup \{s_1 - s_2, \dots, s_{n-1} - s_n, s_{n-1} + s_n\}. \end{aligned}$$

We label the elements of S_G by $\{\beta_1, \dots, \beta_{n-1}, \beta_n\} \cup \{\gamma_1, \dots, \gamma_{n-1}, \gamma_n\}$. We have $\Phi_H = \Phi_G \sqcup \Phi_V$ and $\Phi_V = \{\pm t_i \pm s_j \mid 1 \le i, j \le n\}$.

Component group. Let s be the image of s' (defined below (A.6)) in $T(\mathbb{Q})$. Lemma 3.3 shows that the inclusion $N_{H^{\theta}}(T) \hookrightarrow H^{\theta}$ induces an isomorphism $Z_{W_H}(s)/W_G \simeq H^{\theta}/G$. In fact, let

$$\Omega := \{w \in W_H \mid w(S_G) = S_G\}.$$

Then using the description of the Weyl group of H and G from Appendix A.1, we see that $Z_{W_H}(s) = W_G \rtimes \Omega$ and $\Omega \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. Explicit generators of Ω are given by ω_1, ω_2, where

$$\omega_1\colon t_i \leftrightarrow s_i, \quad \omega_2\colon \begin{cases} s_i \mapsto s_i, & t_i \mapsto t_i, \quad 1 \le i \le n-1, \\ t_n \mapsto -t_n, & s_n \mapsto -s_n. \end{cases}$$

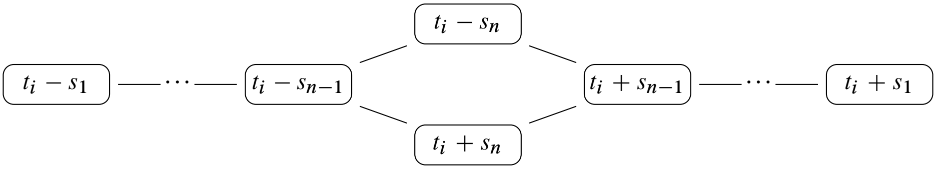
Weights. Using the description of elements of V as $2n \times 2n$ matrices, we organise the weights Φ_V using the position of their eigenspaces:

$$\left(\begin{array}{cc:c|c:cc} t_1 - s_1 & \cdots & t_1 - s_n & t_1 + s_n & \cdots & t_1 + s_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hdashline t_n - s_1 & \cdots & t_n - s_n & t_n + s_n & \cdots & t_n + s_1 \\ \hline -t_n - s_1 & \cdots & -t_n - s_n & -t_n + s_n & \cdots & -t_n + s_1 \\ \hdashline \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -t_1 - s_1 & \cdots & -t_1 - s_n & -t_1 + s_n & \cdots & -t_1 + s_1 \end{array}\right). \tag{A.7}$$

The group Ω acts on the set of weights Φ_V as follows: ω_1 flips the elements of Φ_V along the antidiagonal of (A.7), and ω_2 swaps the two middle rows and the two middle columns.

The partial ordering. Recall from Section 8.11 that we have defined a partial ordering on $X^*(T')$ by declaring that $a \geq b$ if and only if $a - b$ has nonnegative coordinates with respect to the basis S_G. Note that this partial ordering is preserved by the action of Ω on $X^*(T')$.

We describe the induced partial ordering on the subset Φ_V using organisation (A.7). We first consider the restriction of the partial ordering to the rows and columns of (A.7). Let $1 \leq i \leq 2n$ and write $t_i := -t_{2n+1-i}, s_i := -s_{2n+1-i}$ if $i \geq n+1$. The Hasse diagram of the partial ordering restricted to the weights of row i is given by



(In this diagram, $a \leq b$ if and only if b is to the right of a.) The Hasse diagram of column $2n+1-i$ is given by swapping the roles of s_j and t_j in the above diagram for every $1 \leq j \leq 2n$. The partial ordering Φ_V is the one generated by the relations between two elements lying in the same row or column.

For example, $t_1 + s_1$ is the maximal element of Φ_V, and the restriction of the partial ordering to the four $n \times n$ blocks is given by: $a \leq b$ if and only if b is to the top right of a.

Regular nilpotent element. For $\alpha \in \Phi_V$, let X_α be the $2n \times 2n$ matrix with coefficient 1 at the entry corresponding to α using (A.7) and zeroes elsewhere. The element $E := \sum_{\alpha \in S_H} X_\alpha$ is a regular nilpotent element of $V(\mathbb{Q})$ and gives rise to an $\mathfrak{sl}_2$-triple (E, X, F) and Kostant section $\kappa = E + \mathfrak{z}_{\mathfrak{h}}(F)$, see Section 3.6. Since elements of $\mathfrak{z}_{\mathfrak{h}}(F)$ are supported on $\Phi_V \cap \Phi_H^-$ (where $\Phi_H^- = \Phi_H \setminus \Phi_H^+$), every element of κ is of the form

$$\left(\begin{array}{cccc|cccc}
1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
* & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
* & \cdots & * & 1 & 1 & 0 & \cdots & 0 \\
\hline
* & \cdots & \cdots & * & * & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
* & \cdots & \cdots & * & * & \cdots & \cdots & *
\end{array}\right). \tag{A.8}$$

If $\omega \in \Omega$, then $E_\omega := \sum_{\alpha \in \omega(S_H)} X_\alpha$ is again regular nilpotent and gives rise to a Kostant section κ_ω. Then $\{\kappa_\omega \mid \omega \in \Omega\}$ is a full set of representatives of $G(\mathbb{Q})$-orbits of Kostant sections.

A.3. Reducibility conditions

Recall that if A is a $2n \times 2n$ matrix, then A^* denotes its reflection along the antidiagonal.

Proposition A.1. *Let $k/\mathbb{Q}$ be a field and let $A \in V(k)$. The following are equivalent:*

(1) *A is a regular semisimple element of $V(k)$;*

(2) *AA^* is a regular semisimple $2n \times 2n$ matrix (in other words, the characteristic polynomial of AA^* has distinct roots in k^s);*

(3) *A^*A is a regular semisimple $2n \times 2n$ matrix.*

Proof. Let $\Delta \in \mathbb{Q}[V]^G$ be the discriminant polynomial of $\mathfrak{h} \simeq \mathfrak{so}_{4n}$ restricted to V. Let D be the block matrix $\left(\begin{smallmatrix} 0 & A \\ -A^* & 0 \end{smallmatrix}\right)$. If C is a square matrix, write $\chi_C \in k[X]$ for its characteristic polynomial. If f is a polynomial, write $\operatorname{disc}(f)$ for its discriminant in the usual sense. Identity (A.4) implies that $\operatorname{disc}(\chi_D) = \Delta(A)^2 \cdot \operatorname{Pff}(D)^2$. We have $\operatorname{Pff}(D) = \pm \det(A)$ since both square to $\det(D)$, so

$$\operatorname{disc}(\chi_D) = \Delta(A)^2 \cdot \det(A)^2. \tag{A.9}$$

On the other hand, if $f(X) = g(X^2)$ for some polynomial $g \in k[X]$, then it is elementary to check that $\operatorname{disc}(f) = \pm \operatorname{disc}(g)^2 f(0)$. Moreover, by calculating determinants of block matrices we have $\chi_D(X) = \chi_{-AA^*}(X^2)$. Therefore,

$$\operatorname{disc}(\chi_D) = \pm \operatorname{disc}(\chi_{-AA^*})^2 \cdot \det(A)^2. \tag{A.10}$$

Both identities (A.9) and (A.10) hold in $\mathbb{Q}[V][X]$, i.e., they hold when the coefficients of A are interpreted as variables. Since $\det \in \mathbb{Q}[V]$ is not identically zero, it follows that

$$\Delta(A) = \pm \operatorname{disc}(\chi_{-AA^*}) = \pm \operatorname{disc}(\chi_{AA^*}).$$

Since $\chi_{AA^*} = \chi_{A^*A}$, we also have $\Delta(A) = \pm \operatorname{disc}(\chi_{A^*A})$. Since A is a regular semisimple element of $V(k)$ if and only if $\Delta(A) \neq 0$, the proposition follows. ∎

In the next corollary, we organise the set Φ_V using matrix (A.7), and we recall from Section 8.7 that for a subset $M \subset \Phi_V$ we have defined $V(M)$ as the subspace of $v = \sum_{a \in \Phi_V} v_a \in V$ with the property that $v_a = 0$ for all $a \in M$.

Corollary A.2. *Suppose that a subset $M \subset \Phi_V$ satisfies at least one of the following conditions:*

(1) *M contains a top right $i \times (2n + 1 - i)$ block for some $1 \leq i \leq 2n$;*

(2) *M contains the top right $i \times j$ and $j \times i$ blocks for some $i, j \geq 1$ satisfying $i + j = 2n$.*

Then every element of $V(M)(\mathbb{Q})$ is not regular semisimple.

Proof. (1) We may suppose (using the fact that $A \mapsto A^*$ preserves regular semisimplicity) that $i \leq n$. Let $X_1 = \operatorname{span}\{e_1, \dots, e_n\} \subset W_1$. Then $\mathrm{GL}(X_1)$ embeds inside $\mathrm{SO}(W_1)$, using the map $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{smallmatrix}\right)$. Suppose that $A \in V(M)(\mathbb{Q})$. Using the $\mathrm{GL}(X_1)$-action to put the top left $i \times (i - 1)$ block of A in row echelon form, we may suppose that M

contains the top right $1 \times 2n$ block; in other words, we may suppose that $i = 1$. In that case, the matrix AA^* has zeroes on the first row and the last column. This implies that the characteristic polynomial of AA^* is divisible by X^2, which implies that A is not regular semisimple by Proposition A.1.

(2) Assume that $i \leq j$ and let $A \in V(M)(\mathbb{Q})$. Then the matrix $B = AA^*$ is of the following form:

$$\left(\begin{array}{c|c|c} B_1 & 0 & 0 \\ \hline * & B_2 & 0 \\ \hline * & * & B_3 \end{array}\right).$$

Here B_1, B_3 are $i \times i$ matrices and B_2 is a $(j-i) \times (j-i)$ matrix (it is possible that $i = j$). Recall that χ_C denotes the characteristic polynomial of a square matrix C. Then we have

$$\chi_{AA^*} = \chi_{B_1} \chi_{B_2} \chi_{B_3}.$$

Since $B_3 = B_1^*$, the polynomial $\chi_{AA^*} = \chi_{B_1}^2 \chi_{B_2}$ has repeated roots. By Proposition A.1, this shows that A is not regular semisimple. ∎

Recall from Appendix A.2 that we may interpret an element $A \in V(\mathbb{Q})$ as a linear map $W_2 \to W_1$. Using the perfect pairings b_i on W_i, we may thus interpret A^* as a linear map $W_1 \to W_2$, and AA^* as a linear map $W_1 \to W_1$.

Proposition A.3. *Let $k/\mathbb{Q}$ be a field and $A \in V(k)$. Assume that there exists an $(n-1)$-dimensional subspace $X \subset W_1$ such that* $\operatorname{span}\{X, AA^*(X)\}$ *is an n-dimensional isotropic subspace of* (W_1, b_1). *Then A is k-reducible.*

Proof. If A is not regular semisimple then A is k-reducible by definition, so assume that A has invariants $b \in B^{\mathrm{rs}}(k)$. (Recall that $B = V /\!\!/ G$.) The Grassmannian of n-dimensional isotropic subspaces of (W_1, b_1) has two connected components called rulings of (W_1, b_1), and the subspaces spanned by $\{e_1^*, \dots, e_n^*\}$ and $\{e_1^*, \dots, e_{n-1}^*, e_n\}$ lie in distinct rulings [83, §2.2]; let $\mathcal{R}$ be the ruling containing the subspace

$$X_1 := \operatorname{span}\{e_1^*, \dots, e_n^*\}.$$

Using the Ω-action we may assume that $\operatorname{span}\{X, AA^*(X)\}$ lies in $\mathcal{R}$. To prove the proposition, it suffices to prove the claim that $G(k)$ acts simply transitively on the set of pairs (D, Y), where $D \in V_b(k)$ and $Y \subset W_1$ is an $(n-1)$-dimensional subspace such that $\operatorname{span}\{Y, DD^*(Y)\}$ is an n-dimensional isotropic subspace of (W_1, b_1) contained in the ruling $\mathcal{R}$. Indeed, the description of the Kostant section κ_b from (A.8) shows that (κ_b, X_1) is such a pair, so the claim implies that (A, X) and (κ_b, X_1) are $G(k)$-conjugate. The proof of the claim is identical to the proof of [70, Proposition 4.4] using the results of [83, §2.2.2]; we omit the details. ∎

Corollary A.4. *Suppose that M contains the top right $(n-1) \times (n+1)$ and $n \times (n-1)$ blocks. Then every element of $V(M)(\mathbb{Q})$ is $\mathbb{Q}$-reducible.*

Proof. If $A \in V(M)(\mathbb{Q})$, a computation shows that the entries of AA^* in the top right $(n-1) \times n$ and $n \times (n-1)$ blocks are zero. In other words, AA^* looks like

$$\left(\begin{array}{cc:c|c:cc} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ \hdashline * & * & * & * & 0 & 0 \\ \hline * & * & * & * & * & * \\ \hdashline * & * & * & * & * & * \\ * & * & * & * & * & * \end{array}\right).$$

It follows that the subspace $X = \operatorname{span}\{e_{n-1}^*, \ldots, e_1^*\}$ satisfies the assumptions of Proposition A.3. ■

Recall that $\mathcal{C}$ denotes the collection of subsets M of Φ_V with the property that for all $a, b \in \Phi_V$ with $b \in M$ and $a \geq b$, it follows that $a \in M$. Also recall the description of the partial ordering on Φ_V in Appendix A.2.

Proposition A.5. *Let $M \in \mathcal{C}$ and suppose that $V(M)(\mathbb{Q})$ contains $\mathbb{Q}$-irreducible elements. Then the following properties hold:*

(1) $\{t_i - s_i, s_i - t_i\} \subset \Phi_V \setminus M$ *for all* $1 \leq i \leq n-1$;

(2) $\{t_n - s_n, t_n + s_n, -t_n + s_n, -t_n - s_n\} \subset \Phi_V \setminus M$;

(3) *for every* $1 \leq i \leq n-2$, *either* $t_i - s_{i+1}$ *or* $s_i - t_{i+1}$ *lies in* $\Phi_V \setminus M$;

(4) $\#(\{t_{n-1} - s_n, t_{n-1} + s_n, t_n + s_{n-1}, -t_n + s_{n-1}\} \cap M) \leq 2$.

Proof. Note that if $\omega \in \Omega$, then $V(M)(\mathbb{Q})$ contains $\mathbb{Q}$-irreducible elements if and only if $V(\omega(M))(\mathbb{Q})$ does. The first three parts follow from applying Corollary A.2 to $\omega(M)$ for all $\omega \in \Omega$ and properties of the partial ordering of Φ_V. Part (4) follows from applying Corollary A.4 to $\omega(M)$ for $\omega \in M$. ■

The reader is invited to visualise the conditions of Proposition A.5 using the organisation of the weights Φ_V of (A.7).

A.4. Bounding the remaining cusp integrals

Let $\mathcal{C}^{\mathrm{good}}$ be the subset of $\mathcal{C}$ consisting of those $M \in \mathcal{C}$ that satisfy parts (1)–(4) of Proposition A.5. Note that $\omega(M) \in \mathcal{C}^{\mathrm{good}}$ if $M \in \mathcal{C}^{\mathrm{good}}$ and $\omega \in \Omega$.

Lemma A.6. *If $M \in \mathcal{C}^{\mathrm{good}}$, then every element of S_G is of the form $a_1 + a_2$ for some $a_1, a_2 \in \Phi_V \setminus M$.*

Proof. Using the Ω-action, it suffices to consider $\beta_1, \ldots, \beta_{n-1}$. For $1 \leq i \leq n-2$, we have identities

$$\begin{aligned} \beta_i = t_i - t_{i+1} &= (t_i - s_i) + \boxed{(s_i - t_{i+1})} \\ &= \boxed{(t_i - s_{i+1})} + (s_{i+1} - t_{i+1}). \end{aligned}$$

At least one of the two boxed terms is in $\Phi_V \setminus M$ by part (3) of Proposition A.5, and the unboxed terms are always in $\Phi_V \setminus M$ by part (1) of that proposition. To treat β_{n-1}, consider the identities

$$\begin{aligned}\beta_{n-1} = t_{n-1} - t_n &= \boxed{(t_{n-1} - s_n)} + (s_n - t_n) \\ &= \boxed{(t_{n-1} + s_n)} + (-s_n - t_n) \\ &= (t_{n-1} - s_{n-1}) + \boxed{(s_{n-1} - t_n)}.\end{aligned}$$

One of the three boxed terms must be contained in $\Phi_V \setminus M$ by part (4) of Proposition A.5, and all the unboxed terms are contained in $\Phi_V \setminus M$ by parts (1) and (2) of that proposition. ∎

The discussion in Section 8.11 and Proposition A.5 shows that in order to prove Proposition 8.12 when H is of type D_{2n} for all $n \geq 2$, it suffices to prove the following proposition.

Proposition A.7. *For every $M \in \mathscr{C}^{\mathrm{good}}$, there exists a function $f\colon \Phi_V \setminus M \to \mathbb{R}_{\geq 0}$ with the following properties:*

(1) $\sum_{a \in \Phi_V \setminus M} f(a) < \#M$;

(2) *the vector*

$$\sum_{\beta \in \Phi_G^+} \beta - \sum_{a \in M} a + \sum_{a \in \Phi_V \setminus M} f(a)a$$

has strictly positive coefficients with respect to the basis S_G.

We prove Proposition A.7 using induction on n. The base case $n = 2$ is easy to check explicitly, and also follows from Case 1 of the proof of Proposition A.8 below (which only assumes $n \geq 2$). See [80, p. 1217] which considers the D_4 case in detail and also proves this base case.

To perform the induction step, let $\Phi_V^{[1]}$ be the subset of Φ_V of vectors of the form $\{\pm t_1 \pm s_i\} \cup \{\pm t_i \pm s_1\}$; in other words, $\Phi_V^{[1]}$ consists of the first and last rows and columns of (A.7). Similarly, let $\Phi_G^{[1]}$ be the subset of roots of Φ_G that have a nonzero coordinate at β_1 or γ_1 in the root basis S_G. Write $\Phi_V = \Phi_V^{[1]} \sqcup \Phi_V^{[n-1]}$ and $\Phi_G = \Phi_G^{[1]} \sqcup \Phi_G^{[n-1]}$. Then $\Phi_V^{[n-1]}$ and $\Phi_G^{[n-1]}$ arise from the constructions of Appendix A.2 with n replaced by $n-1$. Moreover, $\sum_{\beta \in \Phi_G^{[1],+}} \beta = (2n-2)t_1 + (2n-2)s_1$. To prove Proposition A.7, it therefore suffices to prove the following statement.

Proposition A.8. *Let $n \geq 3$ be an integer, let $M \in \mathscr{C}^{\mathrm{good}}$ and write $M^{[1]} := M \cap \Phi_V^{[1]}$. Then there exists a function $f^{[1]}\colon \Phi_V \setminus M \to \mathbb{R}_{\geq 0}$ with the following properties:*

(1) $\sum_{a \in \Phi_V \setminus M} f^{[1]}(a) < \#M^{[1]}$;

(2) *the vector*

$$(2n-2)t_1 + (2n-2)s_1 - \sum_{a \in M^{[1]}} a + \sum_{a \in \Phi_V \setminus M} f^{[1]}(a)a \tag{A.11}$$

has strictly positive coefficients with respect to the basis S_G.

Proof. Note that if the proposition is true for M, it is also true for $\omega(M)$ for every $\omega \in \Omega$. We may therefore replace M by a Ω-conjugate in what follows. We also note that it suffices to find for each $M \in \mathcal{C}^{\mathrm{good}}$ a function $f^{[1]}\colon \Phi_V \setminus M \to \mathbb{R}_{\geq 0}$ that satisfies the first property and such that (A.11) has nonnegative (instead of positive) coefficients with respect to S_G. Indeed, by Lemma A.6, every element of $\beta \in S_G$ is a sum $a_1 + a_2$ of two elements of $\Phi_V \setminus M$, so by adding to $f^{[1]}$ the function $a_1 \mapsto \epsilon$, $a_2 \mapsto \epsilon$ for some very small ϵ, we may ensure that $f^{[1]}$ has strictly positive coefficient at every element of S_G.

We will distinguish three cases, after introducing some notation. We say $M \in \mathcal{C}^{\mathrm{good}}$ is *bounded* if there exists a function $f^{[1]}\colon \Phi_V \setminus M \to \mathbb{R}_{\geq 0}$ satisfying the conclusions of Proposition A.8. If $M \in \mathcal{C}^{\mathrm{good}}$, we write $w_1(M) := (2n-2)t_1 + (2n-2)s_1 - \sum_{a \in M^{[1]}} a$. Recall that if $n+1 \leq i \leq 2n$, then we write $t_i := -t_{2n+1-i}$ and $s_i := -s_{2n+1-i}$. We use $O(\geq 0)$ as a shorthand for an element of $X^*(T')$ that has nonnegative coordinates with respect to S_G. We also recall the useful formulae (A.3).

Case 1. Suppose that $M^{[1]} \subset \{t_1 - s_n, t_1 + s_n, \dots, t_1 + s_1, \dots, t_n + s_1, -t_n + s_1\}$. Let a (resp. b) be the number of elements of $M^{[1]}$ contained in the first row (resp. last column). Then $1 \leq a, b \leq n+1$, and since we may switch the roles of t_n and $-t_n$ and similarly for $\pm s_n$, we may assume that $M^{[1]} = \{t_1 + s_a, \dots, t_1 + s_1, \dots, t_b + s_1\}$. Using the Ω-action, we may assume that $a \geq b$. We have

$$w_1(M) = (2n-2-a)t_1 - t_2 - \cdots - t_b + (2n-2-b)s_1 - s_2 - \cdots - s_a.$$

If $a, b \leq n-1$, then (A.3) shows that $w_1(M)$ has positive S_G coefficients, so M is evidently bounded. We may therefore assume that $a = n$ or $n+1$. A computation shows that the coefficients of $w_1(M)$ at $\{\beta_1, \dots, \beta_{n-2}\} \cup \{\gamma_1, \dots, \gamma_{n-2}\}$ are nonnegative (in fact at least $n+1-a$), so we focus on the coefficients at $\beta_{n-1}, \beta_n, \gamma_{n-1}, \gamma_n$. If $b \leq n-1$, then $w_1(M) = O(\geq 0) + (n-a)(\beta_{n-1} + \beta_n)/2$. This is negative only when $a = n+1$, so assume that this is the case. Using Lemma A.6, write $\beta_{n-1} = a_1 + a_2$, $\beta_n = a_3 + a_4$ for some $a_i \in \Phi_V \setminus M$. Choose a function $f^{[1]}\colon \Phi_V \setminus M \to \mathbb{R}_{\geq 0}$ such that $f^{[1]}$ supported on $\{a_1, \dots, a_4\}$, such that $\sum_{a \in \Phi_V \setminus M} f^{[1]}(a)a = (\beta_{n-1} + \beta_n)/2$ and such that $\sum_{a \in \Phi_V \setminus M} f^{[1]}(a) < \#M^{[1]}$. Such a function exists since $2 < \#M^{[1]}$ and shows that M is bounded in this case.

It remains to treat the case where $n \leq a, b \leq n+1$. A calculation shows that

$$w_1(M) = \begin{cases} O(\geq 0) - \dfrac{1}{2}\beta_n - \dfrac{1}{2}\gamma_n & \text{if } (a,b) = (n,n), \\ O(\geq 0) - \beta_n & \text{if } (a,b) = (n+1,n), \\ O(\geq 0) - \dfrac{1}{2}(\beta_{n-1} + \beta_n + \gamma_{n-1} + \gamma_n) & \text{if } (a,b) = (n+1,n+1). \end{cases}$$

In each of these cases, we can use Lemma A.6 to show that M is bounded.

Case 2. Suppose that $M^{[1]}$ contains an element of the form $t_1 - s_a$ with $2 \leq a \leq n-1$ and is contained in the set $\{t_1 - s_a, \dots, t_1 + s_1, \dots, t_n + s_1, -t_n + s_1\}$. Let b be the number of elements of $M^{[1]}$ contained in the last column, so that $\#M^{[1]} = (n-a) + n + b$. Then

$1 \leq b \leq n+1$, and by using the Ω-action we may assume that $M^{[1]} = \{t_1 - s_a, \dots, t_1 + s_1, \dots, t_b + s_1\}$. We have

$$w_1(M) = (a-3)t_1 - t_2 - \dots - t_b + (2n-2-b)s_1 - s_2 - \dots - s_{a-1}.$$

By assumption, $t_1 - s_{a-1} \in \Phi_V \setminus M$ and $2n - a - b \geq 0$. We compute that

$$w_1(M) + (2n-a-b)(t_1 - s_{a-1}) = (2n-b-3)t_1 - t_2 - \dots - t_b + O(\geq 0). \quad \text{(A.12)}$$

All the S_G coefficients of the above expression are nonnegative unless $2n-b-2-b<0$, in other words unless $b \geq n$. Therefore, if $b < n$, the function $f^{[1]}$ mapping $t_1 - s_{a-1}$ to $2n-a-b$ and all other elements of $\Phi_V \setminus M$ to zero shows that M is bounded, since $2n-a-b < \#M^{[1]} = 2n-a+b$. If $b = n$, (A.12) equals $O(\geq 0) - \beta_n$. Therefore, Lemma A.6 and the inequality $(2n-a-b)+2 < \#M^{[1]}$ show that M is bounded in this case. If $b = n+1$, (A.12) equals $O(\geq 0) - \beta_{n-2} - \beta_{n-1} - \beta_n$. Therefore, Lemma A.6 and the inequality $(2n-a-b)+6 < \#M^{[1]}$ (which holds since $n \geq 3$) show that M is again bounded in this case.

Case 3. Suppose that $M^{[1]}$ is of the form $\{t_1 - s_a, \dots, t_1 - s_n, t_1 + s_n, \dots, t_1 + s_1, \dots, t_n + s_1, -t_n + s_1, \dots, -t_b + s_1\}$ for some $2 \leq a, b \leq n-1$. Using the Ω-action, we may assume that $a \geq b$. A calculation using (A.3) shows that

$$\begin{aligned} w_1(M) &= (a-3)t_1 - t_2 - \dots - t_{b-1} + (b-3)s_1 - s_2 - \dots - s_{a-1} \\ &= (a-3)\beta_1 + \dots + (a-b-1)\beta_{b-1} + \dots + (a-b-1)\beta_{n-2} \\ &\quad + \frac{1}{2}(a-b-1)(\beta_{n-1} + \beta_n) + (b-3)\gamma_1 + \dots + (b-a-1)\gamma_{a-1} + \dots \\ &\quad + (b-a-1)\gamma_{n-2} + \frac{1}{2}(b-a-1)(\gamma_{n-1} + \gamma_n). \end{aligned}$$

By assumption, $s_1 - t_{b-1} \in \Phi_V \setminus M$ and $a \geq b$, and we compute that

$$\begin{aligned} w_1(M) + (a-b)(s_1 - t_{b-1}) = O(\geq 0) - \beta_{b-1} - \dots - \beta_{n-2} - \frac{1}{2}(\beta_{n-1} + \beta_n) \\ + O(\geq 0) - \gamma_{a-1} - \dots - \gamma_{n-2} - \frac{1}{2}(\gamma_{n-1} + \gamma_n). \end{aligned}$$

Therefore, to prove that M is bounded it suffices to prove (using Lemma A.6) that

$$(a-b) + 2((n-b+1) + (n-a+1)) < \#M^{[1]} = 4n - a - b + 1.$$

This inequality is equivalent to $2b > 3$, which is true since $b \geq 2$.

Conclusion. Since $M \in \mathscr{C}^{\text{good}}$, every M has an Ω-conjugate that falls under one of the above three cases. ■

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