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On NSOP₂ theories

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Abstract. Answering a question of Džamonja and Shelah, we show that every NSOP₂ theory is NSOP₁.

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1. Introduction

One of the most exciting areas of research in modern model theory is the classification along various dividing lines of nonsimple but otherwise tame theories, especially NSOP_n theories (theories satisfying the negation of the *n*th strict order property) for $1 \leq n \leq 3$. The first two of these properties, introduced in [12], requires the nonexistence of certain trees:

Definition 1.1. A theory T is SOP₁ if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ such that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n < \omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \sqsupseteq \eta_1 \frown \langle 0 \rangle$,

$$\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \frown \langle 1 \rangle})\}$$

is inconsistent. Otherwise it is NSOP₁ (not SOP₁).

Definition 1.2. A theory T is SOP₂ if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ such that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n < \omega}$ is consistent for any $\sigma \in 2^\omega$, but for incomparable η_1 and η_2 , $\{\varphi(x, b_{\eta_1}), \varphi(x, b_{\eta_2})\}$ is inconsistent. Otherwise it is NSOP₂.

The property NSOP₃ was introduced in [32] as part of a family of notions NSOP_n for $n \geq 3$:

Definition 1.3. A theory T is SOP_n if there is a definable relation $R(x_1, x_2)$ with no n -cycles, but with tuples $\{a_i\}_{i < \omega}$ with $\models R(a_i, a_j)$ for $i < j$. Otherwise it is NSOP_n.

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For $n \geq 1$, all NSOP_n theories have NSOP , the negation of the strict order property. Moreover, NSOP_2 is the same as NTP_1 , the negation of the tree property of the first kind (see [8] for a proof), and the class of simple theories is the intersection of the class of NTP_2 theories and the class of $\text{NTP}_1 = \text{NSOP}_2$ theories [31]. The tree properties of the first and second kind, TP_1 and TP_2 , as well as the strict order property SOP , were defined by Shelah [31] (note that the two tree properties were defined in [31] in the language of the cardinal invariants $\kappa_{\text{set}}(T)$ and $\kappa_{\text{inp}}(T)$, and TP_1 is the property that $\kappa_{\text{set}}(T) = \infty$ while TP_2 is the property that $\kappa_{\text{inp}}(T) = \infty$, though explicit definitions can be found in, say, [8]). The properties NSOP_n form a hierarchy that generalizes the property of model-theoretic simplicity, which is itself weaker than stability:

Fact 1.4 ([12, 32]). *Simple theories are NSOP_1 , and NSOP_n theories are NSOP_m for $n \leq m$.*

In [33] it is shown that T_{feq}^* , the model companion of the theory of parametrized equivalence relations, is NSOP_1 but not simple; a number of further examples have since been found by various authors. Examples of theories with SOP_n and NSOP_{n+1} , for $n \geq 3$, were originally given in [32].

Yet the main problem, posed by Džamonja and Shelah [12], has remained unsolved:

Problem 1.5. *Are all NSOP_3 theories NSOP_2 ? Are all NSOP_2 theories NSOP_1 ?*

In this paper we answer the latter question in the positive:

Theorem 1.6. *All NSOP_2 theories are NSOP_1 .*

One reason for the significance of this problem comes from Shelah and Usvyatsov's proposal [33] to characterize classes of theories both internally in terms of the structure of their sufficiently saturated models, and externally in terms of orders on theories. The NSOP_2 theories have a deep external characterization: under the generalized continuum hypothesis, Džamonja and Shelah [12] showed that maximality in \triangleleft^* , an order related to the Keisler order, implies a combinatorial property related to SOP_2 , which Shelah and Usvyatsov [33] then showed to be the same as SOP_2 ; later, Malliaris and Shelah [27] showed the equivalence between SOP_2 and \triangleleft^* -maximality under the generalized continuum hypothesis. On the other hand, NSOP_1 theories can be characterized internally not only in terms of trees, but through the theory of independence, in analogy with stability theory. It is well-known that simple theories, which include the stable theories, are characterized as those theories where forking and dividing behave in certain ways as they do in stable theories; for example, symmetry of forking characterizes simple theories. In [18], Kaplan and Ramsey showed that *Kim-forking*, or forking witnessed by invariant Morley sequences, is the correct way of extending the theory of forking to NSOP_1 theories from simple theories. By relaxing the requirement of base monotonicity, they extended the Kim–Pillay characterization of simple theories in terms of the existence of abstract independence relations to NSOP_1 theories, and, more concretely, characterized NSOP_1 theories by the symmetry of Kim-independence, by the independence theorem for Kim-independence, and by a variant of Kim's lemma in simple theories, asserting

that *Kim-dividing* of a formula, rather than dividing, is witnessed by any invariant Morley sequence. Our result that NSOP₁ theories coincide with NSOP₂ theories therefore shows a satisfying agreement between dividing lines related to Keisler's order and dividing lines related to independence.

We outline the paper and give a word on the strategy for the proof. In Section 3, we develop in general theories a version of a construction used by Chernikov and Kaplan [7] to study forking and dividing in NTP₂ theories. In [1], Adler initiated the study of abstract relations between sets in a model, generalizing some of the properties of forking-independence, coheirs, and other concrete relations from model theory, and provided a set of potential axioms for these relations.¹ We notice that the construction of Chernikov and Kaplan can be relativized to relations between sets satisfying certain axioms, obtaining new relations between sets from old ones, and iterating this construction one can obtain a canonical class of coheirs in any theory.

In Section 4, we study this canonical class of coheirs in NSOP₂ theories. Before the development of Kaplan and Ramsey's theory [18] of Kim-independence in NSOP₁ theories, Chernikov [6] proposed finding a theory of independence for NSOP₂ theories, and the proof of our main result comes from our efforts to answer this proposal. Just as in [7], where Chernikov and Kaplan's construction gives maximal classes in the *dividing order* of Ben Yaacov and Chernikov [4], we show that in NSOP₂ theories our variant of this construction gives *minimal* classes in the restriction of this order to coheir Morley sequences, proving an analogue of Kim's lemma. As a by-product of this construction, we also initiate the theory of independence in a class related to the NATP theories of Ahn and Kim [2], the study of which was further developed by Ahn, Kim and Lee [3], showing that under this assumption Kim-forking and Kim-dividing coincide for coheir Morley sequences. (See [25] for the question of finding an analogue for NSOP₁ theories of the role that NTP₂ theories play relative to simple theories, and developing Kim-independence in that analogue; that Kim-forking coincides with Kim-dividing for coheir Morley sequences in a related class gives us preliminary evidence that NATP completes this analogy.)

¹Other than Adler's work [1] and Conant's work [10] on free amalgamation theories, an additional observation which ultimately led us to the proof of this result is found in [13], where d'Elbée proposes the problem of explaining the apparent ubiquity of additional independence relations with no known abstract model-theoretic definitions in NSOP₁ theories, such as *strong independence* existing alongside Kim-independence in the theory ACFG (introduced as part of a more general class in [14]) of algebraically closed fields with a generic additive subgroup. He observes that just as in the case of free amalgamation of generic functional structures in [24] or generic incidence structures in [11], these stronger independence relations can be used to prove the equivalence of forking and dividing for complete types in many known NSOP₁ theories. Before proving Theorem 1.6, we gave some very weak axioms (including *stationarity*, a feature of the examples considered by [13]) for abstract relations between sets over a model, which appeared to be very common in NSOP theories including strictly NSOP₁ theories and NSOP₄ theories, and proved that theories with such a relation could not be NSOP₂; instead of considering Morley sequences in canonical coheirs as below, we used \downarrow -independent sequences for the abstract relation \downarrow , in the sense of [10, Definition 7.5]. Note also that the *quasi-strong finite character* property considered below is a property of the examples in [13].

In Section 5, we investigate behavior similar to NSOP₁ theories in NSOP₂ theories. We introduce the notion of *Conant-independence*, which will generalize the relation $A \downarrow_M^a B$ defined by $\text{acl}(MA) \cap \text{acl}(MB) = M$ in the *free amalgamation theories* introduced by Conant [10] (based on concepts used to study the isometry groups of Urysohn spheres in [36]); see the following section. While it will end up coinciding with Kim-independence in our case, we studied a version of Conant-independence in a potentially strictly NSOP₁, potentially SOP₃ generalization of free amalgamation theories in [29]. Conant-independence in NSOP₂ theories can be defined as Kim-independence relative to canonical Morley sequences, just as \downarrow^a is Kim-independence relative to free amalgamation Morley sequences (as in [10, Lemma 7.7]); it can also be defined by forcing Kim's lemma on Kim-independence, requiring a formula to divide with respect to *every* Morley sequence instead of just one, as suggested in tentative remarks of Kim [21] in his discussion of *strong dividing in subtle theories*. We show that many of Ramsey and Kaplan's [18] arguments on Kim-independence in NSOP₁ theories can be generalized to Conant-independence in NSOP₂ theories, including a chain condition, symmetry and a weak independence theorem. (But as is apparent in [10, 29], similar behavior can occur in a SOP₃ theory, which is why the following section is essential to the proof of our main result.)

In Section 6, we conclude the proof of Theorem 1.6. One consequence of Conant's free amalgamation axioms (say, the freedom, closure and stationarity axioms, in [10, Definition 2.1]) is the following:

Let \downarrow denote free amalgamation and $A_1 \downarrow_M^a B$, $A_2 \downarrow_M^a C$, and $B \downarrow_M C$ with $A_1 \equiv_M A_2$. Then there is some $A \downarrow_M^a BC$ with $A \equiv_{MB} A_1$ and $A \equiv_{MC} A_2$.

We will have shown in the prior section that Conant-independence is symmetric, and that a similar fact holds, roughly, when replacing free amalgamation with canonical coheirs and \downarrow^a with Conant-independence. Conant [10] showed that *modular* free amalgamation theories must be either simple or SOP₃ (see [15] for a related result on countably categorical Hrushovski constructions), starting with a failure of forking-independence to coincide with \downarrow^a (because forking-independence cannot be symmetric unless a theory is simple) and using the above fact to build up a configuration giving SOP₃. Starting, analogously, with the assumption that an NSOP₂ theory T is SOP₁, so Kim-dividing independence is not symmetric and therefore fails to coincide with Conant-independence, we simulate Conant's construction of an instance of SOP₃. In short, we show that an NSOP₂ theory is either NSOP₁ or SOP₃. But an NSOP₂ theory is of course not SOP₃, so it must be NSOP₁.

2. Preliminaries

We let a, b, c, d, e, A, B, C denote sets, potentially with an enumeration depending on context, and x, y, z, X, Y, Z denote tuples of variables. We let \mathbb{M} denote a sufficiently saturated and sufficiently strongly homogeneous model of a theory T and let M denote a small elementary submodel. We write AB to denote the union (or concatenation) of the

sets A and B , and write I, J , etc. for infinite sequences (or sometimes trees) of tuples or an infinite linearly ordered sets.

Relations between sets

Roughly following the axioms for abstract independence relations in [1], as well as others that are standard in the literature, we define the following axioms for relations $A \downarrow_M B$ between sets over a model:

- *Invariance:* For all $\sigma \in \text{Aut}(\mathbb{M})$, $A \downarrow_M B$ implies $\sigma(A) \downarrow_{\sigma(M)} \sigma(B)$.
- *Existence:* For all A and M , $A \downarrow_M M$.
- *Full existence:* For all A, B and M , there is always some $A' \equiv_M A$ with $A \downarrow_M B$.
- *Left extension:* If $A \downarrow_M B$ and $A \subseteq C$, there is some $B' \equiv_{MA} B$ with $C \downarrow_M B'$.
- *Right extension:* If $A \downarrow_M B$ and $B \subseteq C$, there is some $A' \equiv_{MB} A$ with $A' \downarrow_M C$.
- *Left monotonicity:* If $A \downarrow_M B$ and $A' \subseteq A$, then $A' \downarrow_M B$
- *Right monotonicity:* If $A \downarrow_M B$ and $B' \subseteq B$, then $A \downarrow_M B'$.

(We will refer to the last two properties, taken together, as *monotonicity*.)

- *Symmetry:* If $A \downarrow_M B$ then $B \downarrow_M A$.

Note that existence and right extension together imply full existence.

Coheirs and Morley sequences

A global type p is a complete type over \mathbb{M} . For $p \in S(A)$ for $M \subseteq A$, we say p is *finitely satisfiable* over M or a *coheir extension* of its restriction to M if every formula in p is satisfiable in M . Global types p finitely satisfiable in M are *invariant* over M : whether $\varphi(x, b)$ belongs to p for φ a formula without parameters depends only on the type of the parameter b over M . We write $a \downarrow^u_M b$ to denote that $\text{tp}(a/Mb)$ is finitely satisfiable in M . We let $a \downarrow^h_M b$ denote $b \downarrow^u_M a$. The relation \downarrow^u (over models) is well-known to satisfy all of the above properties other than symmetry. We say $\{b_i\}_{i \in I}$, for I potentially finite, is a *coheir sequence* over M if $b_i \downarrow^u_M b_{<i}$ for $i \in I$. We say a coheir sequence $\{b_i\}_{i \in I}$, for I infinite, is moreover a *coheir Morley sequence* over M if there is a fixed global type $p(x)$ finitely satisfiable in M such that $b_i \models p(x)|_{Mb_{<i}}$ for $i \in I$. The type of a coheir Morley sequence over M (indexed by a given set) is well-known to depend only on $p(x)$, and coheir Morley sequences are known to be indiscernible; the type of a coheir sequence over M depends only on the global coheirs over M extending the $\text{tp}(b_i/Mb_{<i})$.

NSOP₁ theories and Kim-dividing

The following definition is due to Kaplan and Ramsey [18]:

Definition 2.1. A formula $\varphi(x, b)$ *Kim-divides* over M if there is a coheir Morley sequence $\{b_i\}_{i < \omega}$ starting with b such that $\{\varphi(x, b_i)\}_{i < \omega}$ is inconsistent (equivalently,

k -inconsistent for some k : any subset of size k is inconsistent). A formula $\varphi(x, b)$ *Kim-forks* over M if it implies a (finite) disjunction of formulas Kim-dividing over M . We write $a \downarrow_M^{Kd} b$, and say that a is *Kim-dividing independent* from b over M if $\text{tp}(a/Mb)$ does not include any formulas Kim-dividing over M .

In this paper we use nonstandard terminology: in [18], Kim-dividing etc. are defined in terms of Morley sequences in *invariant* types over M rather than *finitely satisfiable* types over M . In NSOP₁ theories, the definition of Kim-independence in terms of invariant types agrees with the definition in terms of finitely satisfiable types, by Kim’s Lemma, Theorem 3.16 of [18]. Moreover, many of the characterizations of NSOP₁ in terms of Kim-independence (such as those given in [18–20]) go through using the nonstandard Definition 2.1. The characterization of NSOP₁ we will use in modified form will be the one in terms of symmetry; see Fact 2.2 below.² The reason for the nonstandard definition is that \downarrow^u is known to satisfy left extension. This will do us no harm for our main result, though when we briefly consider Kim-forking in some NATP theories, we will note the nonstandard usage.

The following follows directly from [8, Proposition 5.2]; see also [18, Proposition 3.22] (where the evident argument for the version for invariant types is given) and [18, Theorem 5.16] for the full symmetry characterization of NSOP₁.

Fact 2.2. *Suppose that \downarrow^{Kd} is symmetric, that is, for sets A, B and M a model, $A \downarrow_M^{Kd} B$ implies $B \downarrow_M^{Kd} A$. Then T is NSOP₁.*

Proof. Suppose T is SOP₁, witnessed by a formula $\varphi(x, y)$; then [8, Proposition 5.2] gives c_0, c_1, b_0 such that $c_0 \downarrow_M^u b_0$ and therefore $c_0 \downarrow_M^{Kd} b_0$, but $\models \varphi(b_0, c_0)$, c_0, c_1 begin a coheir Morley sequence over M , and $\{\varphi(x, c_0), \varphi(x, c_1)\}$ is inconsistent. So $\varphi(x, c_0)$ Kim-divides (according to our nonstandard coheir definition) over M , and therefore $b_0 \not\downarrow_M^{Kd} c_0$. ■

NSOP₂ theories

A characterization of SOP₂ as k -TP₁ was proven by Kim and Kim [22], who also introduced the notion of *weak k -TP₁*, proved that it implies SOP₁, and conjectured that it also implies SOP₂:

Definition 2.3. The theory T has *weak k -TP₁* if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in \omega < \omega}$ such that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n < \omega}$ is consistent for any $\sigma \in \omega^\omega$, but for pairwise incomparable $\eta_1, \dots, \eta_k \in \omega^{<\omega}$ with common meet, $\{\varphi(x, b_{\eta_i})\}_{i=1}^k$ is inconsistent.

Later, Chernikov and Ramsey [8, Theorem 4.1] claimed to show that weak k -TP₁ implies SOP₂, but their proof is incorrect; the embedded tree $\{b_\eta\}_{\eta \in \omega < \omega}$ in the proof of

²We could also have used the Kim’s lemma characterization, which, by [18, Theorem 3.16 (2)], can be stated with coheir Morley sequences rather than invariant ones. With more work, we could also have used the standard definition of Kim-independence and stated everything in terms of invariant types; see footnote 3 of this paper.

that theorem is not actually strongly indiscernible over the parameter set C . In an earlier version of our paper, we used this result. In this section, we will introduce a new property, called k -DCTP₁, which we use in place of weak k -TP₁ in our argument. Using a method similar to that in [8], we show that k -DCTP₁ is equivalent to SOP₂.

- Definition 2.4.** (1) Let $\eta, \eta' \in \omega^\omega$. Then $\eta \triangleleft \eta'$ denotes the standard extension order on trees (i.e. $\eta \subset \eta'$), and $\eta \wedge \eta'$ denotes the meet of η and η' in the extension order (i.e. $\eta \cap \eta'$). Let $\eta, \eta' \in \omega^{<\omega}$ be such that $\eta \not\subseteq \eta'$ and $\eta' \not\subseteq \eta$. Then $\eta <_{\text{lex}} \eta'$ if η' is greater than η in the lexicographic order: for $n < \omega$ least such that $\eta(n) \neq \eta'(n)$, $\eta(n) < \eta'(n)$.
- (2) ([34, Proposition 2.51 (IIIb)]) A list $\eta_1, \dots, \eta_n \in \omega^{<\omega}$ is a *descending comb* if it is an antichain such that $\eta_1 <_{\text{lex}} \dots <_{\text{lex}} \eta_n$, and, for $1 \leq k < n$, $\eta_1 \wedge \dots \wedge \eta_{k+1} \triangleleft \eta_1 \wedge \dots \wedge \eta_k$.

So for example, all descending combs of length n have the same quantifier-free type in the language $\{<_{\text{lex}}, \triangleleft, \wedge\}$ as the descending comb $\langle 0 \rangle^{n-1} \frown \langle 1 \rangle, \dots, \langle 1 \rangle$; meanwhile, $\langle 00 \rangle, \langle 01 \rangle, \langle 10 \rangle, \langle 11 \rangle$ is an example of a lexicographically ordered antichain that is not a descending comb.

Definition 2.5 ([35, Definitions 11, 12]). For tuples $\bar{\eta}, \bar{\eta}'$ of elements of $\omega^{<\omega}$, we write $\bar{\eta} \sim_0 \bar{\eta}'$ to mean that $\bar{\eta}$ has the same quantifier-free type in the language $\{<_{\text{lex}}, \triangleleft, \wedge\}$ as $\bar{\eta}'$. For $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ a tree-indexed set of tuples and $\bar{\eta} = \eta_1, \dots, \eta_n$ an n -tuple of elements of $\omega^{<\omega}$, we write $b_{\bar{\eta}} =: b_{\eta_1} \dots b_{\eta_n}$, and call $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ *strongly indiscernible* over a set A if for all tuples $\bar{\eta}, \bar{\eta}'$ of elements of $\omega^{<\omega}$ with $\bar{\eta} \sim_0 \bar{\eta}'$, we have $b_{\bar{\eta}} \equiv_A b_{\bar{\eta}'}$.

Fact 2.6 ([35, Theorem 16]; see [30] for an alternate proof). *Let $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ be a tree-indexed set of tuples, and A a set. Then there is $\{c_\eta\}_{\eta \in \omega^{<\omega}}$ strongly indiscernible over A such that for any tuple $\bar{\eta}$ of elements of $\omega^{<\omega}$ and $\varphi(x) \in L(A)$, if $\models \varphi(b_{\bar{\eta}'})$ for all $\bar{\eta}' \sim_0 \bar{\eta}$, then $\models \varphi(c_{\bar{\eta}})$.*

Definition 2.7. The theory T has k -DCTP₁ if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ such that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n < \omega}$ is consistent for any $\sigma \in \omega^\omega$, but for any descending comb $\eta_1, \dots, \eta_k \in \omega^{<\omega}$, $\{\varphi(x, b_{\eta_i})\}_{i=1}^k$ is inconsistent.

Lemma 2.8. *For any $k > 1$, a theory has SOP₂ if and only if it has k -DCTP₁.*

Proof. (\Rightarrow) The property 2-DCTP₁ follows directly from [8, Fact 4.1]. In the language of [8], this fact says that SOP₂ implies TP₁. But TP₁ is just 2-DCTP₁, which implies k -DCTP₁ for all $k > \omega$.

(\Leftarrow) We follow the proof of [8, Theorem 4.1]. Let $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ witness k -DCTP₁ with the formula $\varphi(x, y)$. By Fact 2.6, we can assume $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ is strongly indiscernible (as paths and descending combs are preserved under \sim_0 -equivalence). We will produce a witness to SOP₂. Let $\eta_i = \langle 0 \rangle^i \frown \langle 1 \rangle$ (so that, say, η_n, \dots, η_0 will form a descending comb), and let n be maximal such that

$$\{\varphi(x, b_{\eta_i \frown \langle 0 \rangle^\alpha}) : i < n, \alpha < \omega\}$$

is consistent; by consistency of the paths, n will be at least 1, and by inconsistency of descending combs of size k , n will be at most k . Let $C = \{b_{\eta_i \wedge (0)^\alpha} : i < n - 1, \alpha < \omega\}$. We see that, say, $\mu = \langle 0 \rangle^{n-1}$ sits strictly above the meets of any two or more of the η_i for $i < n - 1$ in the order \triangleleft , and is incomparable to and lexicographically to the left of η_{n-2} when $n > 1$, so the appropriately tree-indexed subset $\{c_\eta\}_{\eta \in \omega < \omega}$ of $\{b_\eta\}_{\eta \in \omega < \omega}$ consisting of those b_η with $\mu \trianglelefteq \eta$ (that is, where $c_\eta = b_{\mu \wedge \eta}$) really is strongly indiscernible over C . By strong indiscernibility of $\{b_\eta\}_{\eta \in \omega < \omega}$ and the fact that $\{\varphi(x, b_{\eta_i \wedge (0)^\alpha}) : i < n, \alpha < \omega\}$ is consistent, $\{\varphi(x, c_{(0) \wedge (0)^\alpha}) : \alpha < \omega\} \cup \{\varphi(x, c) : c \in C\}$ is consistent; let d realize it, and by Ramsey, compactness and an automorphism over C , we can assume $\{c_{(0) \wedge (0)^\alpha}\}_{\alpha < \omega}$ is indiscernible over dC . On the other hand, for $p(y, \bar{z}) = \text{tp}(d, \{c_{(0) \wedge (0)^\alpha}\}_{\alpha < \omega} / C)$, we see that $p(y, \{c_{(0) \wedge (0)^\alpha}\}_{\alpha < \omega}) \cup p(y, \{c_{(1) \wedge (0)^\alpha}\}_{\alpha < \omega})$ is inconsistent, by strong indiscernibility of $\{b_\eta\}_{\eta \in \omega < \omega}$ and inconsistency (by maximality of n) of $\{\varphi(x, b_{\eta_i \wedge (0)^\alpha}) : i \leq n, \alpha < \omega\}$ (noting that, say, $p(y, \{c_{(0) \wedge (0)^\alpha}\}_{\alpha < \omega})$ contains $\{\varphi(x, b_{\eta_i \wedge (0)^\alpha}) : i < n - 1, \alpha < \omega\}$). This is exactly what the “path collapse lemma” [8, Lemma 4.3] tells us that we need to obtain SOP_2 . ■

Though the proof of [8, Theorem 4.1] is incorrect, that theorem (albeit, not a “local” version) will be a corollary of our main result, Theorem 1.6.

Corollary 2.9 (to Theorem 1.6). *For any k , a theory has weak k -TP₁ if and only if it has SOP_2 .*

Proof. If T has SOP_2 , which is the same as weak 2-TP₁, it follows that T has weak k -TP₁ for any k . If T has weak k -TP₁ for some k , then, because [22] shows that weak k -TP₁ implies SOP_1 , T has SOP_1 . Therefore, T has SOP_2 , by Theorem 1.6. ■

3. Canonical coheirs in any theory

The following section will require no assumptions on T . Iterating a similar construction to the one used by Chernikov and Kaplan [7] to prove the equivalence of forking and dividing for formulas in NTP_2 theories, we will construct a canonical class of coheir extensions in any theory. In Section 4, we will show that NSOP_2 theories satisfy a variant of Kim’s lemma for NSOP_1 theories [18, Theorem 3.16] relative to coheir sequences for this canonical class of coheirs.

Proposition 3.1. *Let T be any theory. Consider relations \downarrow between sets over a model that are stronger than \downarrow^h , satisfy invariance, monotonicity, full existence and right extension, and satisfy the coheir chain condition: if $a \downarrow_M b$ and $I = \{b_i\}_{i < \omega}$ is a coheir Morley sequence starting with b , then there is some $I' \equiv_M I$ with $a \downarrow_M I'$ and each term of I' satisfying $\text{tp}(b/Ma)$. There is a weakest such relation \downarrow^{CK} .*

The “weakest” clause is not necessary for the main result, but we include it anyway to show our construction is canonical.

We start by relativizing the notions of Kim-dividing, Kim-forking, and *quasi-dividing* [7, Definition 3.2] to an M -invariant ideal on the definable subsets of \mathbb{M} .

Definition 3.2. Let \mathcal{I} be an M -invariant ideal on the definable subsets of \mathbb{M} .

- A formula $\varphi(x, b)$ \mathcal{I} -Kim-divides over M if there is a coheir Morley sequence $\{b_i\}_{i < \omega}$ over M starting with b such that for some k , the intersection of some (or equivalently any) k -element subset of $\{\varphi(\mathbb{M}, b_i)\}_{i < \omega}$ belongs to \mathcal{I} .
- A formula $\varphi(x, b)$ \mathcal{I} -Kim-forks over \mathbb{M} if it implies a (finite) disjunction of formulas \mathcal{I} -Kim-dividing over M .
- A formula $\varphi(x, b)$ \mathcal{I} -quasi-divides over M if there are b_1, \dots, b_n with $b \equiv_M b_i$ for all i such that $\bigcap_{i=1}^n \varphi(\mathbb{M}, b_i) \in \mathcal{I}$.
- We write $\varphi(x, b) \vdash^{\mathcal{I}} \psi(x, c)$ if $\varphi(\mathbb{M}, b) \setminus \psi(\mathbb{M}, c) \in \mathcal{I}$.

The proof of the following lemma is adapted straightforwardly from the proof of the “broom lemma” of Chernikov and Kaplan [7, Lemma 3.1].³ For the convenience of the reader we give a simplified proof of the modified version; note that this version is just a rephrasing in terms of ideals of [5, Lemma 4.19]:

Lemma 3.3 (“ \mathcal{I} -broom lemma”). *Suppose*

$$\alpha(x, e) \vdash^{\mathcal{I}} \psi(x, c) \vee \bigvee_{i=1}^N \varphi_i(x, a_i)$$

with $\varphi_i(x, a_i)$ \mathcal{I} -Kim-dividing over M with respect to $P(x)$ and $c \downarrow_M^u a_1 \dots a_N$. Then there are some e_1, \dots, e_m with $e_i \equiv_M e$ such that $\bigwedge_{i=1}^m \alpha(x, e_i) \vdash^{\mathcal{I}} \psi(x, c)$. In particular, \mathcal{I} -Kim-forking implies \mathcal{I} -quasi-dividing over M .

Proof. We need the following claim:

Claim 3.4. *Let a^1, \dots, a^n begin a coheir Morley sequence over M , in a global type q finitely satisfiable over M . Let $a \equiv_M a^i$ and let b be any tuple. Then there are b^1, \dots, b^n such that $b^1 a^1, \dots, b^n a^n$ begin a coheir Morley sequence and $b^i a^i \equiv_M b a$. (The same is true for coheir Morley sequences themselves, rather than just their initial segments.)*

Proof. Left extension for \downarrow^u gives a global type r finitely satisfiable over M extending both q and $\text{tp}(ab/M)$. Now take a coheir Morley sequence in r and apply an automorphism. The parenthetical is similar. ■

³Alex Kruckman, in a personal communication with the author, discussed an alternative to this proof for showing the properness of the ideal corresponding to the independence relation \downarrow^{CK} , with the broom lemma as a corollary, which works for invariant Morley sequences as well as coheir Morley sequences; it is based on unpublished work of James Hanson on the concept of “fracturing,” a generalization of quasi-forking and quasi-dividing. With additional work, we could have used these arguments to prove the results of this paper using invariant types rather than coheirs. See also [17, Theorem 2.14] for some related work on “reliably invariant types,” and [23, Remark 5.10] for a generalization of the applications at the end of Section 4 to the standard Kim-forking and Kim-dividing using a new result of that paper on NATP theories.

Now we can prove the lemma by induction on N . Write $\bigvee_{i=1}^{N-1} \varphi_i(x, a_i)$ as $\varphi(x, b)$, and let $a = a_N$. Let p be a global coheir extension of $\text{tp}(c/Mba)$. Let $\{a^i\}_{i=1}^n$ be such that $a^i \downarrow^u a^{i-1}, \dots, a^1$ and $a^i \equiv_M a$ for $1 \leq i \leq n$ and $\bigwedge_{i=1}^n \varphi_N(x, a^i) \vdash^{\mathcal{J}} \perp$. By the claim, find b^1, \dots, b^n such that

$$a^i b^i \downarrow^u a^{i-1} b^{i-1} \dots a^1 b^1$$

and $a^i b^i \equiv_M ab$ for $1 \leq i \leq n$. Then we can assume $c \models p|_{Maba^1 b^1 \dots a^n b^n}$. From $c \downarrow^u_M a^1 b^1 \dots a^n b^n$, together with $a^i b^i \downarrow^u a^{i-1} b^{i-1} \dots a^1 b^1$ for $1 \leq i \leq n$, it is easy to check

$$ca^{i+1} b^{i+1} \dots a^n b^n \downarrow^u_M a^i b^i$$

for $1 \leq i \leq n$, and therefore

$$cb^{i+1} \dots b^n \downarrow^u_M b^i$$

for $1 \leq i < n$.

Now for $1 \leq i \leq n$ we have $ca^i b^i \equiv_M cab$. Let $e_i ca^i b^i \equiv_M ecab$ for $1 \leq i \leq n$. Then

$$\bigwedge \alpha(x, e_i) \vdash^{\mathcal{J}} \psi(x, c) \vee \bigvee_{i=1}^n \varphi(x, b^i) \vee \bigwedge_{i=1}^n \varphi_N(x, a^i).$$

But by choice of the a^i ,

$$\bigwedge \alpha(x, e_i) \vdash^{\mathcal{J}} \psi(x, c) \vee \bigvee_{i=1}^n \varphi(x, b^i).$$

Now for $1 \leq i \leq n$, because $b^i \equiv_M b$, $\varphi(x, b^i)$ will be of the form $\bigvee_{j=1}^{N-1} \varphi_j(x, a'_j)$ for $\varphi_j(x, a'_j) \mathcal{J}$ -Kim-dividing over M . So, as the first of n steps, we can apply $cb^2 \dots b^n \downarrow^u b^1$ and the inductive hypothesis on N to find some conjunction $\beta(x, \bar{e})$ of conjugates of $\bigwedge \alpha(x, e_i)$ (which will therefore be a conjunction of conjugates of $\alpha(x, e)$) such that

$$\beta(x, \bar{e}) \vdash^{\mathcal{J}} \psi(x, c) \vee \bigvee_{i=2}^n \varphi(x, b^i).$$

Repeating $n - 1$ more times, we are done. ■

We now begin our construction. The following terminology comes from the notion of strong finite character (used in e.g. [8]).

Definition 3.5. Let \downarrow be an invariant relation between sets over a model. We say that \downarrow satisfies *quasi-strong finite character* if for p, q complete types over some model M , $\{a, b \models p(x) \cup q(y) : a \downarrow b\}$ is type-definable.

Definition 3.6. Let \downarrow be an invariant relation between sets over a model satisfying monotonicity, right extension and quasi-strong finite character, and fix a complete type $P(x)$ over a model M .

- (1) A set of formulas $\{\varphi_i(x, b_i)\}_{i \in I}$ is h^\perp -inconsistent⁴ with respect to $P(x)$ if there is no $a \models P(x)$ with $a \downarrow_M \{b_i\}_{i \in I}$ and $\models \varphi_i(a, b_i)$ for all $i \in I$.
- (2) A formula $\varphi(x, b)$ h^\perp -Kim-divides with respect to $P(x)$ if there is a coheir Morley sequence $\{b_i\}_{i < \omega}$ starting with b such that $\{\varphi(x, b_i)\}_{i < \omega}$ is h^\perp -inconsistent with respect to $P(x)$.
- (3) A formula h^\perp -Kim-forks with respect to $P(x)$ if it implies a disjunction of formulas h^\perp -Kim-dividing with respect to $P(x)$.
- (4) A formula $\varphi(x, b)$ h^\perp -quasi-divides over M with respect to $P(x)$ if there are b_1, \dots, b_n with $b_i \equiv_M b$ and $\{\varphi(x, b_i)\}_{i=1}^n$ h^\perp -inconsistent with respect to $P(x)$.

Lemma 3.7. (1) *The sets defined by formulas $\varphi(x, b)$ such that $\{\varphi(x, b)\}$ is h^\perp -inconsistent with respect to $P(x)$ form an M -invariant ideal $\mathcal{I}_{P(x)}^\perp$.*

(2) *A set $\{\varphi_i(x, b_i)\}_{i \in I}$ is h^\perp -inconsistent with respect to $P(x)$ if and only if some finite subset is (so its conjunction defines a set in the ideal $\mathcal{I}_{P(x)}^\perp$).*

Proof. For (1), it suffices to show that (a) if $\models \forall x (\varphi(x, b) \rightarrow \psi(x, c))$, and $\psi(x, c)$ is h^\perp -inconsistent with respect to $P(x)$, then $\varphi(x, b)$ is h^\perp -inconsistent with respect to $P(x)$, and (b) if both $\varphi(x, b)$ and $\psi(x, c)$ are h^\perp -inconsistent with respect to $P(x)$ then so is $\varphi(x, b) \vee \psi(x, c)$. For (a), suppose otherwise; then there is some realization a of $P(x)$ with $\models \varphi(a, b)$ and $a \downarrow_M b$. By right extension, we can assume $a \downarrow_M bc$. But then $\models \psi(a, c)$, and by right monotonicity, $a \downarrow_M c$, contradicting that $\psi(x, c)$ is h^\perp -inconsistent with respect to $P(x)$. For (b), suppose otherwise; then there is some realization a of $P(x)$ with $\models \varphi(a, b) \vee \psi(a, c)$ and $a \downarrow_M bc$; without loss of generality, $\models \varphi(a, b)$, and by right monotonicity, $a \downarrow_M b$, contradicting that $\varphi(x, b)$ is h^\perp -inconsistent with respect to $P(x)$.

Applying monotonicity as in the proof of (a) also gives us the fact that a set $\{\varphi_i(x, b_i)\}_{i \in I}$ is h^\perp -inconsistent with respect to $P(x)$ if some finite subset is (so its conjunction defines a set in the ideal $\mathcal{I}_{P(x)}^\perp$). To complete the proof of (2), we show the “only if” direction. If $\{\varphi_i(x, b_i)\}_{i \in I}$ is h^\perp -inconsistent with respect to $P(x)$ then there is no realization a of $P(x) \cup \{\varphi_i(x, b_i)\}_{i \in I}$ with $a \downarrow_M \{b_i\}_{i \in I}$. But the set of realizations a of $P(x)$ that satisfy $a \downarrow_M \{b_i\}_{i \in I}$ is, by quasi-strong finite character, type-definable. So by compactness, there must be some finite $I_0 \subseteq I$ such there is no realization a of $P(x) \cup \{\varphi_i(x, b_i)\}_{i \in I_0}$ with $a \downarrow_M \{b_i\}_{i \in I}$. But if there is a realization a

⁴The notation h^\perp comes from the fact that the first relation $\downarrow^{(0)}$ in the sequence of independence relations $\downarrow^{(n)}$ we define below is the heir relation \downarrow^h . In previous unpublished notes which eventually led to the writing of this article, the author gives a more ad hoc approach to obtaining a class of coheirs satisfying Kim’s lemma in NSOP₂ theories, using what is there called “tree h -dividing.” This earlier approach only uses the ideal corresponding to \downarrow^h , instead of an iteratively defined sequence of ideals; however, the version of dividing defined in this earlier approach must be with respect to trees, rather than sequences.

of $P(x) \cup \{\varphi_i(x, b_i)\}_{i \in I_0}$ with $a \downarrow_M \{b_i\}_{i \in I_0}$, then we can even get $a \downarrow_M \{b_i\}_{i \in I}$ by right extension, so $\{\varphi_i(x, b_i)\}_{i \in I_0}$ will be as desired. ■

Corollary 3.8. *For all formulas, h^\perp -Kim-forking with respect to $P(x)$ implies h^\perp -quasi-dividing with respect to $P(x)$.*

Proof. By Lemma 3.7, h^\perp -Kim-dividing with respect to $P(x)$ is just $\mathcal{I}_{P(x)}^\perp$ -Kim-dividing, and also h^\perp -quasi-dividing with respect to $P(x)$ is just $\mathcal{I}_{P(x)}^\perp$ -quasi-dividing. Apply Lemma 3.3 to $\mathcal{I}_{P(x)}^\perp$. ■

Lemma 3.9. *If a formula $\varphi(x, b)$ is h^\perp -inconsistent with respect to $P(x)$, then it is h^\perp -inconsistent with respect to any complete type $Q(x, y)$ extending $P(x)$. So the same is true for h^\perp -Kim-dividing and h^\perp -Kim-forking.*

Proof. Suppose otherwise. Then there is a realization ac of $Q(x, y) \cup \{\varphi(x, b)\}$ with $ac \downarrow_M b$. So by left monotonicity, $a \downarrow_M b$, but a realizes $P(x) \cup \{\varphi(x, b)\}$, a contradiction. ■

We are now in a position to study derived independence relations:

Definition 3.10. Let \downarrow be an invariant relation between sets over a model satisfying monotonicity, right extension and quasi-strong finite character. Then we define $a \downarrow'_M b$ to mean that $\text{tp}(a/Mb)$ does not contain any formulas h^\perp -Kim-forking with respect to $\text{tp}(a/M)$.

Lemma 3.11. *Suppose \downarrow is an invariant relation between sets over a model satisfying monotonicity, right extension, quasi-strong finite character, and full existence. Then so is \downarrow' .*

Proof. Invariance is obviously inherited from \downarrow . Quasi-strong finite character is by construction and right extension is also standard from the construction: if $a \downarrow'_M b$ but for some $c \in \mathbb{M}$ there is no $a' \equiv_{Mb} a$ with $a' \downarrow'_M bc$, then $\text{tp}(a/Mb)$ must imply a disjunction of formulas with parameters in Mbc , each of which h^\perp -Kim-forks with respect to $\text{tp}(a/M)$; some formula in $\text{tp}(a/Mb)$ must then imply this disjunction, which will then h^\perp -Kim-fork with respect to $\text{tp}(a/M)$, contradicting $a \downarrow'_M b$. Right monotonicity is by definition. Left monotonicity is Lemma 3.9. It remains to show full existence; the proof is a straightforward generalization of the proof of [7, Proposition 3.7]. By right extension, it suffices to show existence: that $b \downarrow'_M M$ for any tuple b . Suppose otherwise; then $\text{tp}(b/M)$ contains a formula $\varphi(x, m)$ for $m \in M$ that h^\perp -Kim-forks over M . By Corollary 3.8, $\varphi(x, m)$ h^\perp -quasi-divides over M . Since $m \in M$, this just means that $\varphi(x, m) \in \mathcal{I}_{\text{tp}(b/M)}^\perp$. But since $\varphi(x, m) \in \text{tp}(b/M)$, this contradicts full existence for \downarrow . ■

The next observation is required to produce a relation with the coheir chain condition:

Lemma 3.12. *Let \downarrow be as in Lemma 3.11 and suppose $a \downarrow'_M b$. Then for $I = \{b_i\}_{i < \omega}$ a coheir Morley sequence starting with b , there is $I' \equiv_M I$ with $a \downarrow_M I'$ and each term of I' satisfying $\text{tp}(b/Ma)$. In particular, \downarrow' implies \downarrow , so h^\downarrow -Kim-forking implies h^\downarrow -Kim-forking.*

Proof. Suppose otherwise: then for $q = \text{tp}(a, b/M)$, $\bigcup_{i < \omega} q(x, b_i)$ is h^\downarrow -inconsistent with respect to $\text{tp}(a/M)$, so by Lemma 3.7 (2), some finite subset must be h^\downarrow -inconsistent with respect to $\text{tp}(a/M)$. This gives us a formula in $q(x, b)$ that h^\downarrow -Kim divides with respect to $\text{tp}(a/M)$, a contradiction. ■

Note that \downarrow^h satisfies the assumptions of Lemma 3.11. Now define inductively, $\downarrow^{(0)} = \downarrow^h$, $\downarrow^{(n+1)} = (\downarrow^{(n)})'$. Let $\downarrow^{\text{CK}} = \bigcap_{i=0}^\infty \downarrow^{(i)}$.⁵ Then because h^\downarrow -Kim-forking implies $h^{\downarrow^{(n+1)}}$ -Kim-forking, and $a \downarrow_M^{\text{CK}} b$ means that $\text{tp}(a/Mb)$ does not contain an $h^{\downarrow^{(n)}}$ -Kim-forking formula for any n , right extension and quasi-strong finite character are standard. Monotonicity and invariance follow from monotonicity and invariance of the $\downarrow^{(n)}$. By right extension for \downarrow^{CK} , full existence for \downarrow^{CK} would follow from the existence property $b \downarrow_M^{\text{CK}} M$ for any b , but this just follows from full existence for each $\downarrow^{(n)}$. Finally, the coheir chain condition follows from Lemma 3.12 together with quasi-strong finite character for the $\downarrow^{(n)}$ and compactness.

It remains to show that \downarrow^{CK} is the weakest relation implying \downarrow^h and satisfying these properties. Let \downarrow be some other such relation and assume by induction that \downarrow implies $\downarrow^{(n)}$. Assume $a \downarrow_M b$; we show $a \downarrow_M^{(n+1)} b$. Suppose otherwise; by right extension for \downarrow , we can assume $\text{tp}(a/Mb)$ contains a formula $\varphi(x, b)$ that $h^{\downarrow^{(n)}}$ -Kim-divides with respect to $\text{tp}(a/M)$. Let $I = \{b_i\}_{i < \omega}$ be a coheir Morley sequence starting with b witnessing this. Then by the coheir chain condition for \downarrow , there is some a' with $a' \downarrow_M I$, so in particular $a' \downarrow_M^{(n)} I$ by induction, and with $a'b_i \equiv_M ab$ for $i < \omega$, so in particular with a' satisfying $\{\varphi(x, b_i)\}_{i < \omega}$, a contradiction.

This completes the proof of Proposition 3.1.

Remark 3.13. If $\mathbb{M}' \succ \mathbb{M}$ is a very large (sufficiently saturated) model, then \downarrow^{CK} as computed in \mathbb{M}' restricts to \downarrow^{CK} as computed in \mathbb{M} . We can see that \downarrow^{CK} has this property as it is true for \downarrow^h and is preserved by going from $\downarrow^{(n)}$ to $\downarrow^{(n+1)}$. However, it is also immediate that invariance, monotonicity, full existence, right extension, and the coheir chain condition are preserved on passing from \mathbb{M}' to \mathbb{M} .

4. Canonical coheirs in NSOP₂ theories

The goal of this section is to prove a version of “Kim’s lemma for Kim-dividing” for Morley sequences coming from the canonical relation \downarrow^{CK} in NSOP₂ theories.

⁵CK stands for “Chernikov–Kaplan.”

Lemma 4.1. *Let $p(x)$ be a type over M . Then it has a global extension $q(x)$ such that for all tuples $b \in \mathbb{M}$, if $c \models q|Mb$, then $b \downarrow_M^{\text{CK}} c$. So in particular, $q(x)$ is a global coheir of $p(x)$.*

Proof. In a very large $\mathbb{M}' \succ \mathbb{M}$, full existence and invariance for \downarrow^{CK} , and an automorphism, give us a realization c' of $p(x)$ with $\mathbb{M} \downarrow_M^{\text{CK}} c'$. Now take $q(x)$ to be $\text{tp}(c'/\mathbb{M})$, and the lemma follows by monotonicity on the left. ■

Definition 4.2. We call $q(x)$ as in Lemma 4.1 a *canonical coheir*, and a coheir Morley sequence in it a *canonical Morley sequence*.

Canonical coheirs and canonical Morley sequences are not, strictly speaking, truly canonical, because there can be many choices for an extension of a type to a canonical coheir. The reason for the choice of the word “canonical” is that the relation \downarrow^{CK} from which these coheirs are obtained is in fact canonical, being the weakest relation satisfying the properties stated in Proposition 3.1.

We aim to prove a version of Kim’s lemma from [18] for canonical Morley sequences in NSOP₂ theories. To start, we introduce the notion of a *coheir tree* in a general theory T .

Definition 4.3. Let p be any type over M . We say that a tree $\{b_\eta\}_{\eta \in \omega \leq n}$ of realizations of p is a *coheir tree* in p if

- (1) for each $\mu \in \omega^{<n}$, $(\{b_\eta\}_{\eta \geq \mu \smallfrown (i)})_{i=0}^\infty$ (the sequence consisting of the subtrees above a fixed node) is a coheir Morley sequence over M ;
- (2) there are global coheir extensions q_0, \dots, q_n of p such that for each $\mu \in \omega^{n-m}$, $b_\mu \models q_m|_{\{b_\eta\}_{\eta \triangleright \mu}}$.

The key lemma of this section allows us to construct coheir trees in any theory so that sequences of nodes with common meet are canonical Morley sequences. The term “descending comb” will have a similar meaning in a tree of finite height as it does in $\omega^{<\omega}$. Before proving this key lemma, Lemma 4.5 below, we prove a preliminary lemma:

Lemma 4.4. *If $a \downarrow_M^{\text{CK}} b$ and I is a coheir tree in $\text{tp}(b/M)$, then there is some $I' \equiv_M I$ with $a \downarrow_M^{\text{CK}} I'$ (so in particular $I' \downarrow_M^u a$) each term of which satisfies $\text{tp}(b/Ma)$.*

Proof. Let $I = \{b_\eta\}_{\eta \in \omega \leq n}$; we find $I' = \{b'_\eta\}_{\eta \in \omega \leq n}$ as desired. Define $\zeta_m := \langle 0 \rangle^m$. The proof is by downward induction. For the base case, take $b'_\zeta = b$. Now suppose $\{b'_\eta\}_{\eta \geq \xi_{n-k}}$ is already constructed with $a \downarrow_M^{\text{CK}} \{b'_\eta\}_{\eta \geq \xi_{n-k}}$, $\{b'_\eta\}_{\eta \geq \xi_{n-k}} \equiv_M \{b_\eta\}_{\eta \geq \xi_{n-k}}$ and each term satisfying $\text{tp}(b/Ma)$; we construct $\{b'_\eta\}_{\eta \geq \xi_{n-(k+1)}}$ with $a \downarrow_M^{\text{CK}} \{b'_\eta\}_{\eta \geq \xi_{n-(k+1)}}$, $\{b'_\eta\}_{\eta \geq \xi_{n-(k+1)}} \equiv_M \{b_\eta\}_{\eta \geq \xi_{n-(k+1)}}$ and each term satisfying $\text{tp}(b/Ma)$. First, $\{b'_\eta\}_{\eta \triangleright \xi_{n-(k+1)}}$, with $a \downarrow_M^{\text{CK}} \{b'_\eta\}_{\eta \triangleright \xi_{n-(k+1)}}$, $\{b'_\eta\}_{\eta \triangleright \xi_{n-(k+1)}} \equiv_M \{b_\eta\}_{\eta \triangleright \xi_{n-(k+1)}}$ and each term satisfying $\text{tp}(b/Ma)$ comes directly from the chain condition; define $J_0 =: \{b'_\eta\}_{\eta \triangleright \xi_{n-(k+1)}}$. Now find $b' \models q_{k+1}(x)|_{MJ_0}$, and use left extension for \downarrow^u to find $J_1 \equiv_M J_0$ with $J_1 \downarrow_M^u J_0$ such that b' is one of the terms of J_1 . Finally, apply the chain condition to J_0, J_1 , which begins a coheir Morley sequence over M , to

obtain $J'_1 \equiv_{MJ_0} J_1$ with $J'_1 \equiv_{Ma} J_0$ and $a \downarrow_M^{\text{CK}} J_0 J'_1$, and define $b'_{\zeta_{n-(k+1)}}$ to be the term of J'_1 corresponding to b' to complete this stage of the induction. ■

Lemma 4.5. *Let $p(x)$ be any type over M . Let $q(x)$ be a canonical coheir extension of $p(x)$. Let b_0, \dots, b_n be a coheir sequence over M of realizations of p . Then there is a coheir tree indexed by $\omega^{\leq n}$, any path of which, read in the direction of the root, realizes $\text{tp}(b_0 \dots b_n/M)$, and any descending comb of which, read in lexicographic order, begins a canonical Morley sequence in $q(x)$.*

Proof. By induction, it suffices to show this for a coheir sequence $b_0 \dots, b_{n+1}$ assuming $I_n = \langle b_\eta \rangle_{\eta \in \omega^{\leq n}}$ is already constructed for $b_0 \dots, b_n$. First, we find a coheir sequence $\{I_n^i\}_{i=0}^\alpha$, long enough to apply the Erdős–Rado theorem, of realizations of $\text{tp}(I_n/M)$, such that each node of I_n^γ satisfies $q(x)|_{M\{I_n^i\}_{i < \gamma}}$. Suppose $\{I_n^i\}_{i < \gamma}$ is already constructed; taking $a = \{I_n^i\}_{i < \gamma}$ in the previous lemma and $b \models q(x)|_{M\{I_n^i\}_{i < \gamma}}$, we can choose I_n^γ to be the I' given by the previous lemma. Having taken $\{I_n^i\}_{i=0}^\alpha$ long enough, we can find a coheir Morley sequence $\{I_n^i\}_{i=0}^\omega$ which also has the property that each node of I_n^m satisfies $q(x)|_{M\{I_n^i\}_{i < m}}$.

Now let q_{n+1} be a global extension, finitely satisfiable in M , of $\text{tp}(b_{n+1}/M b_0 \dots b_n)$. Then we take $b \models q_{n+1}(x)|_{M\{I_n^i\}_{i=0}^\infty}$ as the new root, guaranteeing the condition on paths. The condition on descending combs is then also preserved: any descending comb of size at least 2 either will lie inside of the I_n^i , so will of course begin a Morley sequence in $q(x)$ by the induction hypothesis, or will consist of a descending comb inside one copy I_n^i followed by an additional node of a later copy I_n^j for $i < j$, which will indeed continue the Morley sequence in $q(x)$ begun by the previous nodes. Now reindex accordingly to obtain the desired coheir tree. ■

We can now state and prove our version of Kim’s lemma in NSOP₂ theories:

Theorem 4.6. *Let T be NSOP₂. Suppose a canonical Morley sequence $\{b^i\}_{i < \omega}$, $b^0 = b$, witnesses Kim-dividing of a formula $\varphi(x, b)$ over M . Then there is a finite bound (depending only on $\varphi(x, y)$ and the value k such that $\{\varphi(x, b^i)\}_{i < \omega}$ is k -inconsistent) on the length of a coheir sequence $\{b_i\}_{i=1}^n$ over M of realizations of $\text{tp}(b/M)$ such that $\{\varphi(x, b_i)\}_{i=1}^n$ is consistent. In particular, every coheir Morley sequence starting with b witnesses Kim-dividing of $\varphi(x, b)$ over M .*

Proof. Let $q(x)$ be a canonical coheir extension of $\text{tp}(b/M)$ and k the degree of Kim-dividing for $\varphi(x, b)$ witnessed by a canonical Morley sequence in $q(x)$. Let $\{b_i\}_{i=0}^n$ be a coheir sequence over M of realizations of $\text{tp}(b/M)$ such that $\{\varphi(x, b_i)\}_{i=0}^n$ is consistent. Then the coheir tree given by the previous lemma gives the first $n + 1$ levels of an instance of k -DCTP₁: the k -dividing witnessed by canonical Morley sequences in $q(x)$ gives the inconsistency condition for descending combs of size k , and the consistency of $\{\varphi(x, b_i)\}_{i=0}^n$ gives the consistency of the paths. So if n is without bound, we must have k -DCTP₁ for $\varphi(x, y)$ by compactness, and thus SOP₂ by Lemma 2.8. ■

We have some applications of this proof to a notion related to the NATP theories introduced by Ahn and Kim [2], and studied in greater depth by Ahn, Kim and Lee [3]. These results would apply to NATP theories if we could prove an NATP analogue of Lemma 4.5. The result for NATP theories would be interesting because the class of NATP theories contains not only all NSOP₁ theories, but also all NTP₂ theories [2]. The following is the original definition from [2]:

Definition 4.7. The theory T has NATP (*the negation of the antichain tree property*) if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ such that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n < \omega}$ is 2-inconsistent for any $\sigma \in 2^\omega$, but for pairwise incomparable $\eta_1, \dots, \eta_l \in 2^{<\omega}$, $\{\varphi(x, b_{\eta_i})\}_{i=1}^l$ is consistent.

In [3], Ahn, Kim and Lee define a theory to have k -ATP if the above fails replacing 2-inconsistency with k -inconsistency, and show that for any $k \geq 2$, a theory fails to be NATP (that is, has 2-ATP) if and only if it has k -ATP. That is, they show the analogue for NATP theories of results of Kim and Kim [22] on NSOP₂ theories, but not of those claimed by Chernikov and Ramsey [8], nor of the above Lemma 2.8. Since by that lemma, k -DCTP₁ is equal to NSOP₂, and NATP is in some sense “dual” to NSOP₂ (as discussed in [2]), one might ask whether, for any k , the following definition is equivalent to the failure of NATP:

Definition 4.8. The theory T has k -DCTP₂ if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ such that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n < \omega}$ is k -inconsistent for any $\sigma \in 2^\omega$, but for any descending comb $\eta_1, \dots, \eta_l \in 2^{<\omega}$, $\{\varphi(x, b_{\eta_i})\}_{i=1}^l$ is consistent.⁶ The theory T is NDCTP₂ if it does not have k -DCTP₂ for any $k \geq 2$.

If so, then the following applies to NATP theories:

Theorem 4.9. *Let T be NDCTP₂. Let M be any model and b any tuple. Then there is a global type extending $\text{tp}(b/M)$, finitely satisfiable in M , such that for any formula $\varphi(x, y)$ with parameters in M , if coheir Morley sequences in this type do not witness Kim-dividing of $\varphi(x, b)$, no coheir Morley sequence over M starting with b witnesses Kim-dividing of $\varphi(x, b)$ over M .*

This follows from the same construction. The following corollary is standard (see [7, Corollary 3.16] for a similar argument):

Corollary 4.10. *Let T be NDCTP₂. Then Kim-forking (with respect to coheir Morley sequences) coincides with Kim-dividing (with respect to coheir Morley sequences).*

5. Conant-independence in NSOP₂ theories

We introduce a notion of independence which will generalize, in the proof of the main result of this paper, the role played by \downarrow^a in the *free amalgamation theories* introduced

⁶Hanson [17] uses the term k -CTP₂, the k -comb tree property, to refer to k -DCTP₂.

in [10]. The notation \downarrow^{K^*} comes from the related notion of Kim-independence from [18], \downarrow^K ; a similar notion involving dividing with respect to *all* (invariant) Morley sequences is suggested in tentative remarks of Kim [21].

Definition 5.1. Let M be a model and $\varphi(x, b)$ a formula. We say $\varphi(x, b)$ *Conant-divides* over M if for every coheir Morley sequence $\{b_i\}_{i < \omega}$ over M starting with b , $\{\varphi(x, b_i)\}_{i < \omega}$ is inconsistent. We say $\varphi(x, b)$ *Conant-forks* over M if it implies a disjunction of formulas Conant-dividing over M . We say a is *Conant-independent* from b over M , written $a \downarrow_M^{K^*} b$, if $\text{tp}(a/Mb)$ does not contain any formulas Conant-forking over M .

Note that this definition differs from the standard definition of Conant-independence given in [29], in that it uses coheir Morley sequences rather than invariant Morley sequences. In a personal communication, Alex Kruckman and the author show how to carry out this proof with the standard Conant-independence; see footnote 3 of this paper. To show directly that the coheir and invariant type definitions of Conant-dividing agree in NSOP₂ theories, we may also dualize Theorem 3.10 of [23], which says that Kim-dividing and Kim-dividing with respect to coheir Morley sequences agree in NATP theories.

Proposition 5.2. *In any theory T , Conant-forking coincides with Conant-dividing for formulas, and \downarrow^{K^*} has right extension.*

Proof. We see first of all that Conant-dividing is preserved under adding and removing unused parameters: it suffices to show that if $\models \forall x (\varphi(x, a) \leftrightarrow \varphi'(x, ab))$ then $\varphi(x, a)$ Conant-divides over M if and only if $\varphi'(x, ab)$ Conant-divides over M . Let $\{a_i b_i\}_{i < \omega}$ be a coheir Morley sequence starting with ab witnessing the failure of Conant-dividing of the latter; then $\{a_i\}_{i < \omega}$ witnesses the failure of Conant-dividing of the former. Conversely, let $\{a_i\}_{i < \omega}$ be a coheir Morley sequence starting with a witnessing the failure of Conant-dividing of $\varphi(x, a)$; then by Claim 3.4 and an automorphism there are $\{b_i\}_{i < \omega}$ such that $\{a_i b_i\}_{i < \omega}$ is a coheir Morley sequence starting with ab , and this will witness the failure of Conant-dividing of $\varphi'(x, ab)$. The result is now standard, following, say, the proof in [18] of the analogous fact for Kim-dividing under Kim’s lemma. Suppose $\varphi(x, b)$ Conant-forks over M but does not Conant-divide over M ; by the above we can assume it implies a disjunction of the form $\bigvee_{i=1}^n \varphi_i(x, b)$ where $\varphi_i(x, b)$ Conant-divides over M . Let $\{b_i\}_{i < \omega}$ be a coheir Morley sequence starting with b witnessing the failure of Conant-dividing, so there is some a realizing $\{\varphi(x, b_i)\}_{i < \omega}$. Then by the pigeonhole principle, there is some $1 \leq k \leq n$ such that a realizes infinitely many of the $\varphi_k(x, b_i)$. By an automorphism this contradicts Conant-dividing of $\varphi_k(x, b)$.

Right extension is standard and exactly as in Lemma 3.11: if $a \downarrow_M^{K^*} b$ but there is no $a' \equiv_{Mb} a$ with $a' \downarrow_M^{K^*} bc$, then $\text{tp}(a/Mb)$ must imply a disjunction of formulas with parameters in Mbc Conant-forking over M ; some formula in $\text{tp}(a/Mb)$ must then imply this disjunction, which will then Conant-fork over M , contradicting $a \downarrow_M^{K^*} b$. ■

The following is immediate from Theorem 4.6:

Corollary 5.3. *Let T be NSOP₂. Then a formula Conant-divides (so Conant-forks) over M if and only if it Kim-divides with respect to some (any) canonical Morley sequence.*

We develop the theory of Conant-independence in NSOP₂ theories in analogy with the theory of Kim-independence in NSOP₁ theories.

Proposition 5.4 (Canonical Chain Condition). *Let T be NSOP₂ and suppose $a \downarrow_M^{K^*} b$. Then for any canonical Morley sequence I starting with b , we can find some $I' \equiv_{Mb} I$ indiscernible over a ; any such I' will satisfy $a \downarrow_M^{K^*} I'$.*

Proof. This is similar to the proof of, say, the analogous fact about Kim-independence in NSOP₁ theories [18, Proposition 3.21]. The existence of such an I' follows from the previous corollary by Ramsey and compactness. To get $a \downarrow_M^{K^*} I'$, let $I' = \{b_i\}_{i < \omega}$; it suffices to show $a \downarrow_M^{K^*} b_0 \dots b_{n-1}$ for any n . But $\{b_{in} b_{i+1} \dots b_{i+n-1}\}_{i < \omega}$ is a coheir Morley sequence over M starting with $b_0 \dots b_{n-1}$, each term of which satisfies $\text{tp}(b_0 \dots b_{n-1}/Ma)$, so $a \downarrow_M^{K^*} b_0 \dots b_{n-1}$ follows. ■

Theorem 5.5. *Let T be NSOP₂. Then Conant-independence is symmetric.*

Proof. Suppose otherwise, so for some $a, b \in \mathbb{M}$, $a \downarrow_M^{K^*} b$ but $b \not\downarrow_M^{K^*} a$ over M . We use $a \downarrow_M^{K^*} b$ to build trees as in the proof of symmetry of Kim-independence for NSOP₁ theories (the construction is [18, Lemma 5.11]). Specifically, what we want is, for any n , a tree $(I_n, J_n) = (\{a_\eta\}_{\eta \in \omega^{\leq n}}, \{b_\sigma\}_{\sigma \in \omega^n})$, infinitely branching at the first n levels and then with each a_σ for $\sigma \in \omega^n$ at level $n + 1$ followed by a single additional leaf b_σ at level $n + 2$, satisfying the following properties:

- (1) For $\eta \preceq \sigma$, $a_\eta b_\sigma \equiv_M ab$.
- (2) For $\eta \in \omega^{<n}$, the subtrees above η , i.e. the sequence

$$(\{a_\mu\}_{\mu \succeq \eta \frown \langle i \rangle, \mu \in \omega^{\leq n}} \{b_\mu\}_{\mu \succeq \eta \frown \langle i \rangle, \mu \in \omega^n})_{i < \omega},$$

form a canonical Morley sequence indiscernible over a_η , so by Proposition 5.4, a_η is Conant-independent over M from those branches taken together.

For the base case, take $I_0 = a$ and $J_0 = b$. Suppose (I_n, J_n) is already constructed; we construct (I_{n+1}, J_{n+1}) . We see that the root a_\emptyset of (I_n, J_n) is Conant-independent from the rest of the tree, $(I_n, J_n)^*$: for $n = 0$ this is just the assumption $a \downarrow_M^{K^*} b$, while for $n > 0$ this is (2). So by extension we find $a'_\emptyset \equiv_{M(I_n J_n)^*} a_\emptyset$ (so guaranteeing (1)), to be the root of (I_{n+1}, J_{n+1}) , with $a \downarrow_M^{K^*} I_n J_n$. Then by Proposition 5.4, find some canonical Morley sequence $\{(I_n, J_n)^i\}_{i < \omega}$ starting with (I_n, J_n) indiscernible over Ma'_\emptyset , guaranteeing (2), and reindex accordingly.

Now let $\varphi(x, a) \in \text{tp}(b/Ma)$ (so $\varphi(x, y)$ is assumed to have parameters in M) witness the Conant-dependence of b on a over M and let k be the (strict) bound supplied by Theorem 4.6. We show I_n gives the first $n + 1$ levels of an instance of k -DCTP₁ for $\varphi(x, y)$, giving a contradiction to NSOP₂ by compactness and Lemma 2.8. Consistency

of the paths comes from (1). As for the inconsistency of a descending comb of size k , it follows from (2) (and the same reasoning as in the proof of Lemma 4.5) that a descending comb forms a coheir sequence, so the inconsistency follows by choice of k . ■

Note that by constructing a tree of size κ , for sufficiently large κ , and using an Erdős–Rado version of Fact 2.6 (see [18, Lemma 5.10] for a result of this kind for a similar kind of indiscernible tree, itself based on [16, Theorem 1.13]), we could have assumed that the tree we constructed in the above proof is *strongly indiscernible*. It follows that we could have only used the fact that if a canonical Morley sequence witnesses Kim-dividing of a formula, then so does any coheir Morley sequence; the statement of Theorem 4.6 is somewhat stronger.⁷

We next aim to prove a version of the “weak independence theorem.” To formulate this, we need the following strengthening of Lemma 4.1:

Lemma 5.6. *Let $p(x)$ be a type over M . Then there is some global extension $q(x)$ of $p(x)$ such that, for all tuples $b \in \mathbb{M}$, if $c \in \mathbb{M}$ with $c \models q(x)|_{Mb}$, then for any $a \in \mathbb{M}$ there is $a' \equiv_{Mc} a$ with $a' \in \mathbb{M}$ such that $\text{tp}(a'/Mb)$ extends to a canonical coheir of $\text{tp}(a'/M) = \text{tp}(ac/M)$ that also extends $q(x)$. So in particular, $q(x)$ is a canonical coheir of $p(x)$.*

Proof. Working in a very large $\mathbb{M}' \succ \mathbb{M}$, find $\mathbb{M}_1 \equiv_M \mathbb{M}$ with $\mathbb{M} \downarrow_M^{\text{CK}} \mathbb{M}_1$ using full existence for \downarrow^{CK} . Find a realization c'' of $p(x)$ in \mathbb{M}_1 and let $q(x)$ be its type over \mathbb{M} . Now suppose $b \in \mathbb{M}$ and $c \in \mathbb{M}$ with $c \models q(x)|_{Mb}$, and let $a \in \mathbb{M}$. Then there is some $a'' \in \mathbb{M}_1$ with $a''c'' \equiv_M ac$. As $c'' \equiv_{Mb} c$, there is some $a' \in \mathbb{M}$ with $a''c'' \equiv_{Mb} a'c$. Together with $a''c'' \equiv_M ac$, it follows that $a' \equiv_{Mc} a$. And $\text{tp}(a'/Mb)$ extends to $\text{tp}(a''c''/\mathbb{M})$, which it remains to show is canonical. But by right monotonicity, $\mathbb{M} \downarrow_M^{\text{CK}} a''c''$, so the result follows by left monotonicity (see also the proof of Lemma 4.1). ■

Definition 5.7. We call $q(x)$ as in Lemma 5.6 a *strong canonical coheir*, and a coheir Morley sequence in it a *strong canonical Morley sequence*.

The proof of the following is as in [18, Proposition 6.1]:

Proposition 5.8 (Weak Independence Theorem). *Assume T is NSOP₂. Let $a_1 \downarrow_M^{K^*} b_1$, $a_2 \downarrow_M^{K^*} b_2$, and $a_1 \equiv_M a_2$, and let $\text{tp}(b_2/Mb_1)$ extend to a strong canonical coheir $q(x)$ of $\text{tp}(b_2/M)$. Then there exists a realization a of $\text{tp}(a_1/Mb_1) \cup \text{tp}(a_2/Mb_2)$ with $a \downarrow_M^{K^*} b_1b_2$.*

⁷In fact, by using a local version of the chain condition—if $a \downarrow_M^{K^*} b$ and $\models \varphi(a, b)$, then there is some coheir Morley sequence $I = \{b_i\}_{i < \omega}$ such that $b_i \equiv_M b$, $\models \varphi(a, b_i)$ for $i < \omega$, and $a \downarrow_M^{K^*} I$ —we could have avoided Theorem 4.6 altogether up to this point, but we have not yet found a suitable replacement for the “weak independence theorem” below that does not require it. We leave the details to the reader.

Proof. We start with the following claim, proven exactly as in [18] but with canonical rather than invariant Morley sequences:

Claim 5.9. *There exists some b'_2 with $a_1 b'_2 \equiv_M a_2 b_2$ and $a_1 \downarrow_M^{K^*} b_1 b'_2$.*

Proof. It is enough by symmetry of \downarrow^{K^*} to find b'_2 with $a_1 b'_2 \equiv_M a_2 b_2$ and $b_1 b'_2 \downarrow_M^{K^*} a_1$. If $p(x, a_2) = \text{tp}(b_2/Ma_2)$ (leaving implied, throughout the proof of this claim, any parameters in M in types and formulas), then by $a_2 \downarrow_M^{K^*} b_2$ and symmetry we have $b_2 \downarrow_M^{K^*} a_2$, so because $a_1 \equiv_M a_2$ we know that $p(x, a_1)$ contains no formulas Conant-forking over M . It suffices to show consistency of

$$p(x, a_1) \cup \{\neg\varphi(x, b_1, a_1) : \varphi(x, y, a_1) \text{ Conant-forks over } M\}.$$

Otherwise, by compactness and equivalence of Conant-forking with Conant-dividing, we must have $p(x, a_1) \vdash \varphi(x, b_1, a_1)$ for some $\varphi(x, y, z)$ with $\varphi(x, y, a_1)$ Conant-dividing over M . By symmetry, $b_1 \downarrow_M^{K^*} a_1$. So Proposition 5.4 yields a canonical Morley sequence $\{a_1^i\}_{i < \omega}$ starting with a_1 and indiscernible over Mb_1 . So

$$\bigcup_{i=0}^{\infty} p(x, a_1^i) \vdash \{\varphi(x, b_1, a_1^i)\}_{i < \omega}.$$

But because $p(x, a_1)$ contains no formulas Conant-dividing over M and $\{a_1^i\}_{i < \omega}$ is a canonical Morley sequence, $\bigcup_{i=0}^{\infty} p(x, a_1^i)$ is consistent, so $\{\varphi(x, b_1, a_1^i)\}_{i < \omega}$ and therefore $\{\varphi(x, y, a_1^i)\}_{i < \omega}$ is consistent. But this contradicts the fact that $\varphi(x, y, a_1)$ Conant-divides over M . ■

We now complete the proof of the proposition. Let $p_2(x, b_2) = \text{tp}(a_2/Mb_2)$ (with parameters in M left implied); we have to show that $\text{tp}(a_1/Mb_1) \cup p_2(x, b_2)$ has a realization a with $a \downarrow_M^{K^*} b_1 b_2$. So for $b''_2 \equiv_{Mb_1} b_2$ with $b''_2 \models q(x)|_{Mb_1 b'_2}$, it suffices to show that $\text{tp}(a_1/Mb_1) \cup p_2(x, b''_2)$ has a realization a with $a \downarrow_M^{K^*} b_1 b''_2$. Using $b''_2 \equiv_M b_2 \equiv_M b'_2$, we find b'_1 with $b'_1 b''_2 \equiv_M b_1 b'_2$; using the fact that $q(x)$ is a strong canonical coheir, we can assume $\text{tp}(b'_1 b''_2/Mb_1 b'_2)$ extends to a canonical coheir of its restriction to M . So $b'_1 b''_2, b_1 b'_2$ begins a canonical Morley sequence I over M , and by Proposition 5.4 and an automorphism, there is some $a \equiv_{Mb_1 b'_2} a_1$ with $a \downarrow_M^{K^*} I$ and therefore $a \downarrow_M^{K^*} b_1 b''_2$, and with I indiscernible over Ma . By $a \equiv_{Mb_1} a_1$ we see that a realizes $\text{tp}(a_1/Mb_1)$, and by $ab''_2 \equiv_M ab'_2 \equiv_M a_1 b'_2 \equiv_M a_2 b_2$ we conclude that a realizes $p_2(x, b''_2)$. ■

6. NSOP₂ and NSOP₁ theories

We are now ready to prove that if T is NSOP₂, then it is NSOP₁. The proof follows Conant’s proof [10, Theorem 7.17] that certain *free amalgamation* theories are either simple or SOP₃. As anticipated in Section 5, \downarrow^{K^*} will play the role of \downarrow^a , while (strong) canonical Morley sequences will play the role of Morley sequences in the free amal-

gamation relation. This makes sense, as Lemma 7.6 of [10] shows that \downarrow^a is just Kim-independence with respect to Morley sequences in the free amalgamation relation, while Conant-independence in an NSOP₂ theory is Kim-independence with respect to canonical Morley sequences. Similarly to how Conant uses free amalgamation and \downarrow^a to show that a (modular) free amalgamation theory is either simple or SOP₃, we will show by strong canonical types and \downarrow^{K^*} that if T is NSOP₂, then

$$T \text{ is either NSOP}_1 \text{ or SOP}_3$$

and therefore must be NSOP₁.⁸

We begin our proof.

Assume T is NSOP₂ and suppose T is SOP₁. Obviously Kim-dividing independence, \downarrow^{Kd} , implies \downarrow^{K^*} ; the reverse implication would imply that \downarrow^{Kd} is symmetric, contradicting SOP₁ by Fact 2.2. So there are $a \downarrow_M^{K^*} b$ with $a \not\downarrow_M^{Kd} b$ over M ; let $r(x, y) = \text{tp}(a, b/M)$, and let $\{b_i\}_{i < \omega}$ be a coheir Morley sequence over M starting with b such that $\{r(x, b_i)\}_{i < \omega}$ is k -inconsistent for some k . The following corresponds to Claim 1 of the proof of [10, Theorem 7.17], but requires a different argument.⁹

Claim 6.1. *We can assume $k = 2$. More precisely, there are $\tilde{a}, \tilde{b} \in \mathbb{M}$ with $\tilde{a} \downarrow_M^{K^*} \tilde{b}$ and some coheir Morley sequence $\{\tilde{b}_i\}_{i < \omega}$ over M starting with \tilde{b} such that, for $\tilde{r}(\tilde{x}, \tilde{y}) =: \text{tp}(\tilde{a}, \tilde{b}/M)$, $\{\tilde{r}(\tilde{x}, \tilde{b}_i)\}_{i < \omega}$ is 2-inconsistent.*

Proof. Because $\{r(x, b_i)\}_{i < \omega}$ is k -inconsistent, there is no realization a' of $\{r(x, b_i)\}_{i < k}$ with $a' \downarrow_M^{K^*} b_0 \dots b_{k-1}$. Let $k^* < k$ be the maximal value of k such that there is a realization a' of $\{r(x, b_i)\}_{i < k}$ with $a' \downarrow_M^{K^*} b_0 \dots b_{k^*-1}$, and let $\check{b} = b_0 \dots b_{k^*-1}$. Then

$$\{\check{b}_i\}_{i < \omega} = \{b_{ik^*} \dots b_{ik^*+k^*-1}\}_{i \in \mathbb{N}}$$

is a coheir Morley sequence starting with \check{b} . Let $a' \downarrow_M^{K^*} \check{b}$ realize $\{r(x, b_i)\}_{i < k^*}$, and let $r'(x, y) = \text{tp}(a', \check{b}/M)$. Then by maximality and symmetry, there is no realization a'' of $r'(x, b_0) \cup r'(x, \check{b}_1)$ with $\check{b}_0 \check{b}_1 \downarrow_M^{K^*} a''$. So there is no coheir Morley sequence $\{a'_i\}_{i \in \mathbb{N}}$ starting with a' , every term of which realizes $r'(x, \check{b}_0) \cup r'(x, \check{b}_1)$. But by $a' \downarrow_M^{K^*} \check{b}$, symmetry and Proposition 5.4, there is some $M\check{b}$ -indiscernible canonical Morley sequence I starting with a such that $I \downarrow_M^{K^*} \check{b}$. So let \tilde{a} be I and \tilde{b} be \check{b} . Since $\tilde{r}(\tilde{x}, \tilde{b}) = \text{tp}(I/M\check{b})$ contains $\bigcup_{i < \omega} r'(x_i, \tilde{b})$, \tilde{a} and \tilde{b} are as desired. ■

Now replace a with \tilde{a} and b with \tilde{b} , as in Claim 6.1; let $\rho(x, y) \in r(x, y) = \text{tp}(a, b/M)$ be such that $\{\rho(x, b_i)\}_{i < \omega}$ is 2-inconsistent, by compactness. We have $b_1 \downarrow_M^{K^*} b_0$, in

⁸In [29], we generalize Conant’s work by studying abstract independence relations in potentially strictly NSOP₁ or SOP₃ theories, finding a more general set of axioms for these relations than Conant’s free amalgamation axioms under which the NSOP₁-SOP₃ dichotomy holds and showing relationships with Conant-independence for *invariant* rather than coheir Morley sequences—note that in Conant’s free amalgamation theories, this is just \downarrow^a .

⁹See also [28, footnote 1] for another argument involving the proof of [18, Proposition 3.14], which is suggested in a private communication of Hyoyoon Lee.

analogy with Claim 2 of the proof of [10, Theorem 7.17], because $b_1 \downarrow_M^u b_0$ and clearly \downarrow^u implies \downarrow^{K^*} .

Fix a strong canonical coheir extension $q(x)$ of $p(x) = \text{tp}(b/M)$. We wish to construct, by induction, a configuration $\{b_i^1 b_i^2\}_{i < \omega}$ with the following properties:

- (1) For all $n < \omega$, for J_n the sequence beginning with b_i^2 for $i < n$ and then continuing with b_i^1 for $i \geq n$, J_n is a strong canonical Morley sequence in $q(x)$.
- (2) For $i \leq j$, $b_i^1 b_j^2 \equiv_M b_0 b_1$.
- (3) $b_1^1 \dots b_n^1 \downarrow_M^{K^*} b_0^2 \dots b_n^2$ for any $n < \omega$.

Then by $a \downarrow_M^{K^*} b$, (1) gives consistent sequences of instances of $r(x, y)$, while (2) gives inconsistent pairs by Claim 6.1, so we can get an instance of SOP₃ from this configuration exactly as in the argument at the end of the proof of [10, Theorem 7.17], which we will reproduce for the convenience of the reader.

We make repeated use of symmetry for \downarrow^{K^*} throughout. Let $b_0^1 b_0^2 = b_0 b_1$, which will satisfy (the relevant versions of) (1) through (3). Now suppose $\{b_i^1 b_i^2\}_{i \leq n}$ already constructed, satisfying (1) through (3); we show we can extend this to $\{b_i^1 b_i^2\}_{i \leq n+1}$ also satisfying (1) through (3). We start by adding b_{n+1}^1 , and then add b_{n+1}^2 . If we take $b_{n+1}^1 \models q(x)|_{M b_0^1 b_0^2 \dots b_n^1 b_n^2}$ then (1) and (2) are preserved up to this point, and (3) is preserved by the following claim (which also holds of Kim-independence in NSOP₁ theories):

Claim 6.2. *If $a \downarrow_M^{K^*} b$ and $\text{tp}(c/Mab)$ extends to an M -invariant type $q(x)$, then $ac \downarrow_M^{K^*} b$.*

Proof. By Proposition 5.4, let $I = \{b_i\}_{i < \omega}$ be an Ma -indiscernible canonical Morley sequence over M starting with b . Choose $c^* \models q|_{M I a}$, so for $i < \omega$, $b_i a \equiv_{M c^*} b_0 a = ba$. Since I is a coheir Morley sequence with $b_i \equiv_{M a c^*} b$ for $i < \omega$, $ac^* \downarrow_M^{K^*} b$ by Proposition 5.2, so $ac \downarrow_M^{K^*} b$ as $c^* \models \text{tp}(c/ab)$. ■

Now by $b_1 \downarrow_M^{K^*} b_0$ and the fact that J_0 is still a (strong) canonical Morley sequence up to this point, we can find a realization $b_* \downarrow_M^{K^*} b_0^1 \dots b_{n+1}^1$ of $\{t(b_i^1, y)\}_{i=1}^{n+1}$ for $t(x, y) = \text{tp}(b_0 b_1/M)$ by Proposition 5.4 and an automorphism. Take $b^* \models q(x)|_{M b_0^2 \dots b_n^2}$, so $b^* \equiv_M b_*$. Because $\text{tp}(b^*/M b_0^2 \dots b_n^2)$ extends to a strong canonical coheir,

$$b_0^1 \dots b_{n+1}^1 \downarrow_M^{K^*} b_0^2 \dots b_n^2,$$

and $b_0^1 \dots b_{n+1}^1 \downarrow_M^{K^*} b_*$, by Proposition 5.8 we may find $(b_0^1)' \dots (b_{n+1}^1)'$ such that $(b_0^1)' \dots (b_{n+1}^1)' \equiv_{M b_0^2 \dots b_n^2} b_0^1 \dots b_{n+1}^1$, $(b_0^1)' \dots (b_{n+1}^1)' b^* \equiv_M b_0^1 \dots b_{n+1}^1 b_*$, and $(b_0^1)' \dots (b_{n+1}^1)' \downarrow_M^{K^*} b_0^2 \dots b_n^2 b^*$. Then, because $(b_0^1)' \dots (b_{n+1}^1)' \equiv_{M b_0^2 \dots b_n^2} b_0^1 \dots b_{n+1}^1$, we can find b_{n+1}^2 such that $(b_0^1)' \dots (b_{n+1}^1)' b^* \equiv_{M b_0^2 \dots b_n^2} b_0^1 \dots b_{n+1}^1 b_{n+1}^2$. Then $b_0^1 \dots b_{n+1}^1 b_{n+1}^2 \equiv_M (b_0^1)' \dots (b_{n+1}^1)' b^* \equiv_M b_0^1 \dots b_{n+1}^1 b_*$, so (2) is preserved; $b_{n+1}^2 \equiv_{M b_0^2 \dots b_n^2} b^*$, so (1) is preserved; and $b_1^1 \dots b_{n+1}^1 \downarrow_M^{K^*} b_0^2 \dots b_{n+1}^2$, so (3) is preserved, as desired.

Now having constructed the configuration, let a_n realize the consistent set of instances of $r(x, y)$ coming from J_n , and let $d_i = (b_i^1, b_i^2)$, $z = (y^1, y^2)$, $\varphi(x, y) = \rho(x, y_1)$, $\psi(x, z) = \rho(x, y_2)$. As in the proof of [10, Theorem 7.17], these satisfy the hypotheses of the following fact:

Fact 6.3 (Corrected version of [10, Proposition 7.2]¹⁰). *Suppose there are sequences $\{a_i\}_{i < \omega}$, $\{d_i\}_{i < \omega}$, and $\varphi(x, y)$, $\psi(x, y)$ such that*

- (1) $\models \varphi(a_i, d_j)$ for all $i < j$ and $\psi(a_i, d_j)$ for all $i \geq j$,
- (2) for all $i < j$, $\varphi(x, b_i) \cup \psi(x, b_j)$ is inconsistent.

Then T is SOP₃.

So T is SOP₃.

This concludes the proof of the main result of this paper.

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¹⁰Gabriel Conant, in a personal communication with the author, noted this correction; the version in [9] includes this correction. See also [26, Conclusion 6.15] for an earlier version of this fact, which can also be used here.

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