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Charmenability and stiffness of arithmetic groups

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Abstract. We characterize charmenability among arithmetic groups and deduce dichotomy statements pertaining to normal subgroups, characters, dynamics, representations and associated operator algebras. We do this by studying the stationary dynamics on the space of characters of the amenable radical, and in particular we establish stiffness: any stationary probability measure is invariant. This generalizes a classical result of Furstenberg for dynamics on the torus. Under a higher rank assumption, we show that any action on the space of characters of a finitely generated virtually nilpotent group is stiff.

Keywords: arithmetic groups, stationary dynamics, characters.

1. Introduction

In this paper we are concerned with the dynamics of the conjugation action of a group Λ on the compact convex space $\text{PD}_1(\Lambda)$ of all normalized positive-definite functions on Λ . In particular, we are interested in the structure of invariant compact convex subsets and faces of $\text{PD}_1(\Lambda)$. A special attention will be given to the subset of fixed points $\text{Tr}(\Lambda) := \text{PD}_1(\Lambda)^\Lambda$, and the corresponding set of extreme points $\text{Ch}(\Lambda) := \text{ext}(\text{Tr}(\Lambda))$. An element of $\text{Tr}(\Lambda)$ will be called a *trace* of Λ and an element of $\text{Ch}(\Lambda)$ will be called a *character*. (Note that in some texts, e.g. [4], the term character refers to what we call here a trace.)

Definition 1.1. A countable group Λ is said to be *charmenable* if it satisfies the following two properties:

- (1) (Ubiquity of traces) Every compact convex Λ -invariant subset of $\text{PD}_1(\Lambda)$ contains a trace.
- (2) (Character dichotomy) Every character of Λ is either supported on $\text{Rad}(\Lambda)$ or von Neumann amenable (see Definition 3.8).

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Here $\text{Rad}(\Lambda)$ denotes the *amenable radical* of Λ , the unique maximal normal amenable subgroup, and an element of $\text{PD}_1(\Lambda)$ is *supported on* $\text{Rad}(\Lambda)$ if it vanishes outside this subgroup. We note that amenable groups are charmenable in a strong way: they satisfy both conditions in (2) above. For non-amenable groups these two conditions are mutually exclusive.

Our main result is a characterization of charmenability among arithmetic groups, which in this paper are defined as follows.

Definition 1.2. Given a \mathbb{Q} -algebraic group \mathbf{G} , a subgroup $\Lambda < \mathbf{G}(\mathbb{Q})$ is said to be an *arithmetic subgroup of \mathbf{G}* if for some (equivalently, any) faithful \mathbb{Q} -rational representation $\mathbf{G} \rightarrow \text{GL}_n$, Λ is commensurable with the preimage of $\text{GL}_n(\mathbb{Z})$ in $\mathbf{G}(\mathbb{Q})$.

Rigidity phenomena in arithmetic groups have been extensively studied since the foundational work of Margulis on normal subgroups [36]. However, it appears that all of the major advancements thus far have been restricted to cases where the algebraic group is semisimple. The primary aim of this paper is to extend such results, via the notion of charmenability, to general arithmetic groups.

Theorem A. *Let \mathbf{G} be a connected \mathbb{Q} -algebraic group. Then either all arithmetic subgroups of \mathbf{G} are charmenable or none of them are. Denoting by \mathbf{R} the solvable radical of \mathbf{G} , the first case occurs if and only if the real rank of \mathbf{G}/\mathbf{R} is not equal to 1 and \mathbf{G}/\mathbf{R} has at most one \mathbb{R} -isotropic \mathbb{Q} -simple factor.*

The following proposition describes a variety of properties that charmenable groups satisfy, most of which are proved in [4]. An exception is the third property which is novel (see Theorem 6.6). The notation involved will be clarified and the proof will be discussed in Section 6.

Proposition 1.3. *For a charmenable group Λ the following hold:*

- (1) *Every normal subgroup is either amenable or co-amenable.*
- (2) *Every trace is either amenable or supported on $\text{Rad}(\Lambda)$.*
- (3) *For every probability measure preserving action with a spectral gap, the point stabilizers are either a.e. amenable or a.e. co-amenable. The associated IRS is thus either supported on subgroups of $\text{Rad}(\Lambda)$, or supported on co-amenable subgroups.*
- (4) *Every URS either is contained in the space of subgroups of $\text{Rad}(\Lambda)$ or admits a Λ -invariant probability measure (in which case the previous item applies).*
- (5) *Every unitary representation either is amenable or weakly contains a representation induced from $\text{Rad}(\Lambda)$.*
- (6) *If M is a finite factor von Neumann algebra admitting a representation $\Lambda \rightarrow \mathfrak{U}(M)$ with ultraweak-dense image, then M is either hyperfinite, or isomorphic to the von Neumann algebra of a representation induced from $\text{Rad}(\Lambda)$.*
- (7) *The C^* -algebra A of any non-amenable unitary representation of Λ admits a unital surjective $*$ -homomorphism onto the C^* -algebra of a representation induced from*

$\text{Rad}(\Lambda)$. Furthermore, any maximal ideal of A is the kernel of such a surjection, and any tracial state on A factorizes through such a surjection.

The acronyms IRS and URS stand for the dynamical notions of “invariant random subgroup” and “uniformly recurrent subgroup”, first coined in [1] and in [22]. See Section 6.2 for their precise definitions.

When property (T) is assumed, stronger statements are obtained. For example, amenable (resp. co-amenable) normal subgroups must be finite (resp. of finite index), amenable representations must contain a finite-dimensional subrepresentation, etc. With further assumptions on the amenable radical we obtain the following result.

Proposition 1.4. *Let Λ be a charmenable group with property (T). Assume that $\text{Rad}(\Lambda)$ has only countably many subgroups. Then every ergodic IRS is finitely supported and every URS is finite.*

The amenable radical of an arithmetic group is virtually polycyclic (Corollary 4.9) and thus has countably many subgroups. We therefore deduce the following corollary.

Corollary B. *Let \mathbf{G} be a connected \mathbb{Q} -algebraic group and let $\Lambda < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Denoting by \mathbf{R} the solvable radical of \mathbf{G} , if the real rank of \mathbf{G}/\mathbf{R} is not equal to 1 and \mathbf{G}/\mathbf{R} has at most one \mathbb{R} -isotropic \mathbb{Q} -simple factor then Λ satisfies all properties in Proposition 1.3. If moreover Λ has property (T), then every ergodic IRS is finitely supported and every URS is finite.*

The proof of Theorem A will be given in Section 5.2 and the proofs of Propositions 1.3 and 1.4 are discussed in Section 6.

Theorem A in case the solvable radical is trivial, namely when \mathbf{G} is semisimple, was proved in [4, 15]. Our main concern in this paper is dealing with the case where the solvable radical is non-trivial. In that case [4, Proposition 4.19] enables us to reduce the proof of Theorem A to a certain stiffness result. A prototypical example is the proof of [4, Theorem C] which establishes the charmenability of $\text{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ using Furstenberg’s classical theorem [21] regarding the stiffness of the standard action of $\text{SL}_n(\mathbb{Z})$ on \mathbb{T}^n . The stiffness result that we need is Theorem C below. In order to state it, we need some preparation.

Definition 1.5. Let Λ be a countable group acting on a topological space X by homeomorphisms. Given a probability measure μ on Λ , we say that the Λ -action on X is μ -stiff if any μ -stationary probability measure¹ ν on X , that is, ν for which $\mu * \nu := \sum_{g \in \Lambda} \mu(g)g.\nu = \nu$, is Λ -invariant.

We note that since we are not assuming X to be compact, there may be no μ -stationary measures at all on X , in which case the action is stiff in a vacuous sense. This is never the case in the type of dynamical systems we consider.

¹In this paper, all measures on topological spaces are assumed to be Borel measures, i.e. defined on the σ -algebra generated by open subsets.

Proposition 1.6. *Let Λ be a countable group acting by automorphisms on a countable group Γ . Then for every probability measure μ on Γ , the set of μ -stationary probability measures on $\text{Ch}(\Gamma)$ is non-empty.*

Note that $\text{Ch}(\Gamma)$ is Polish, but may be non-compact (see Example 1.9 below). The proof of Proposition 1.6 will be given in Section 3.1.

A general group has no canonical probability measure, and stiffness may very well depend on different choices of μ . However, for arithmetic groups we have preferred measures.

Definition 1.7. Let \mathbf{G} be an \mathbb{R} -algebraic group, $\mathbf{P} < \mathbf{G}$ a minimal \mathbb{R} -parabolic subgroup and $\mathbf{N} < \mathbf{G}$ the normalizer of \mathbf{P} (note that \mathbf{P} contains the solvable radical of \mathbf{G} , $\mathbf{P} < \mathbf{N}$ is of finite index and if \mathbf{G} is connected then $\mathbf{P} = \mathbf{N}$). A probability measure μ on $\mathbf{G}(\mathbb{R})$ (not necessarily fully supported) is called a *Furstenberg measure* if in the $\mathbf{G}(\mathbb{R})$ -invariant measure class on $\mathbf{G}(\mathbb{R})/\mathbf{N}(\mathbb{R})$ there exists a probability measure with which this space is a Furstenberg–Poisson boundary² for $(\mathbf{G}(\mathbb{R}), \mu)$. If $\Lambda < \mathbf{G}(\mathbb{R})$ is a closed subgroup which is generated as a semigroup by the support of the Furstenberg measure μ , we will say that μ is a *Furstenberg measure on Λ* .

Furstenberg proved that lattices in semisimple groups carry Furstenberg measures; see Proposition 2.24 for an elaboration. The following is a corollary of the above fact.

Proposition 1.8. *For every \mathbb{Q} -algebraic group \mathbf{G} and an arithmetic subgroup Λ , there exists a Furstenberg measure on Λ .*

The proof of Proposition 1.8 will be given in Section 2.6. We are now ready to state our main stiffness result.

Theorem C. *For an arithmetic group Λ endowed with a Furstenberg measure μ , the action of Λ on $\text{Ch}(\text{Rad}(\Lambda))$ is μ -stiff.*

Note that this theorem, unlike Theorems A and B, involves no rank assumption. A main ingredient in the proof of Theorem C is Theorem 4.1 which regards stiffness of the action of lattices in a semisimple Lie group on the space of characters of a nilpotent group. Another consequence of the latter theorem, which might be of independent interest, is the following.

Theorem D. *Let Λ be an irreducible lattice in a connected semisimple Lie group G with finite center and let μ be a Furstenberg measure on Λ . Assume that G is not isogenous to a group of the form $\text{SO}(1, m) \times K$ or $\text{SU}(1, m) \times K$, with K compact. Then for any action of Λ by automorphisms on a finitely generated virtually nilpotent group Γ , the corresponding action on $\text{Ch}(\Gamma)$ is μ -stiff.*

The proofs of Theorems C, D and 4.1 will be given in Section 4. A crucial ingredient is a theorem of Mozes on epimorphic subgroups, which in turn relies on Ratner’s theory.

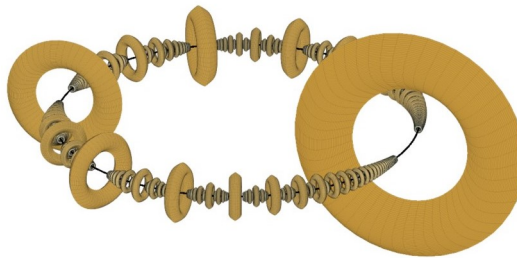
²For the definition of a Furstenberg–Poisson boundary see Theorem 2.17.

If Γ is a finitely generated abelian group then $\text{Ch}(\Gamma)$, which coincides with $\hat{\Gamma}$, the Pontryagin dual of Γ , is a finite union of tori. However, Theorems C, D and 4.1 are not to be confused with the results of Benoist-Quint regarding stiffness for actions on nilmanifolds [12]. Indeed, the space $\text{Ch}(\Gamma)$ for a finitely generated virtually nilpotent group which is not virtually abelian is neither homogeneous nor a manifold.

Example 1.9. The space $\text{Ch}(H)$ of characters of the discrete Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

is homeomorphic to the subset of \mathbb{R}^3 illustrated in the following picture:



Here points on the horizontal circle represent the space of characters of the center $Z < H$. The inclusion $Z \subseteq H$ induces a surjective continuous map $\text{Ch}(H) \rightarrow \text{Ch}(Z) \simeq \hat{Z} \cong \mathbb{R}/\mathbb{Z}$ (see Lemma 3.10). The fiber over an irrational point of \mathbb{R}/\mathbb{Z} consists of a single point, whereas the fiber of a reduced fraction $p/q \in \mathbb{R}/\mathbb{Z}$ could be identified with $\mathbb{R}^2/q^{-1}\mathbb{Z}^2$, the Pontryagin dual of $q\mathbb{Z}^2$, which could be identified with the center of H/qZ (see Theorem 4.6). We note that $\text{Ch}(H)$ is not compact: the rational points of the horizontal circle in the picture above are excluded.

Related works

An excellent introduction to the theory of characters is [10]. A comprehensive study of characters of nilpotent groups has been carried out since the 1970s (see e.g. [16, 27, 31, 32]). This study was recently generalized to polycyclic groups, and thus to arithmetic subgroups of solvable algebraic groups [34].

A classification of the characters of arithmetic subgroups of higher rank semisimple algebraic groups was completed in [42] following [8]. The situation in the rank 1 case is complicated and somewhat less understood (see e.g. [41]). We know of no reference dealing with character theory of general arithmetic groups, though the paper [11] is closely related.

Dynamical aspects of spaces of positive-definite functions on lattices in semisimple groups were considered recently in [3, 4, 15]; see also [26]. The notion of “charmenability” is highlighted in [4]. Dynamical questions regarding spaces of measures on tori are

classical. In particular, the question of “stiffness” is dealt with in [12, 14], following the pioneering work [21]. Surprisingly, it seems that questions regarding dynamics on the spaces of characters of non-abelian nilpotent groups have never been studied before. For our main goal in this paper, proving Theorem A, the stiffness result of Theorem C is sufficient. However, it seems to us that the latter theorem is not optimal, as we deal only with Furstenberg measures, and we hope to improve it in a future work.

Another aspect in which we hope to improve this work in the future is by considering S -arithmetic groups. This will force us to tackle characters on nilpotent groups which are not finitely generated, but we do not foresee a major problem in doing so.

The structure of this paper

Sections 2 and 3 are devoted to introducing the needed preliminaries and basic results regarding stationary dynamics and characters respectively. In Section 4 we state and prove Theorem 4.1 and use it to prove Theorems C and D. In Section 5 we prove Theorem A. Section 6 is devoted to proving Propositions 1.3 and 1.4.

2. Stationary dynamics

In this section we define and discuss general statements regarding stationary dynamics associated to a group G . For almost all of our purposes it suffices to consider the case where G is countable. However, as some of our claims might be of independent interest, we shall mostly work in the setting where G is a general locally compact second countable group.

2.1. Ergodic-theoretical preliminaries

By a *Borel space* we mean a set endowed with a σ -algebra. Morphisms between Borel spaces are just measurable maps, and they are called *Borel maps*. A *standard Borel space* is a Borel space which is isomorphic as such to a Polish space with its Borel σ -algebra. By a *Lebesgue space* we understand a standard Borel space endowed with a measure class. A *Lebesgue map* from a Lebesgue space to a Borel space is a class of a.e. equal Borel maps, defined on a conull subset of the domain. Morphisms between Lebesgue spaces are measure class preserving Lebesgue maps. We note that the category of Lebesgue spaces is equivalent to the opposite category of abelian von Neumann subalgebras of bounded operators on separable Hilbert spaces, by the functor $X \mapsto L^\infty(X) \subseteq \mathcal{B}(L^2(X))$.

Let G be a locally compact second countable group. An action of G on a Borel space X is a *Borel action* if the corresponding action map $G \times X \rightarrow X$ is a Borel map. A *Borel G -space* is a Borel space endowed with a Borel action of G . A *Lebesgue G -space* is a Lebesgue space endowed with a Borel G -action that preserves the measure class. The Lebesgue G -space is called *ergodic* if every measurable G -invariant subset is either null or conull.

For a topological space X we denote by $\text{Prob}(X)$ the space of all Radon probability measures endowed with the weak- $*$ topology. If X is Polish (resp. compact) then so is $\text{Prob}(X)$, and all Borel probability measures are Radon. Suppose X and Y are Polish spaces and $p : X \rightarrow Y$ is a continuous surjective map. Let $\eta \in \text{Prob}(Y)$ and consider a Lebesgue map $Y \rightarrow \text{Prob}(X)$ that takes y to a measure ν_y supported on the fiber $p^{-1}(y)$. Then one may integrate these ν_y into a probability measure $\nu = \int_Y \nu_y d\eta$, that is, for a bounded continuous function f on X , $\int_X f d\nu = \int_Y (\int_{p^{-1}(y)} f d\nu_y) d\eta$. The *disintegration theorem* states that every $\nu \in \text{Prob}(X)$ is given by an integration as above with respect to a unique Lebesgue map $Y \rightarrow \text{Prob}(X) : y \mapsto \nu_y$. This is called the *disintegration of ν along p* .

Suppose X and Y are Polish G -spaces and $p : X \rightarrow Y$ is a continuous and surjective G -equivariant map. Consider $\nu \in \text{Prob}(X)$ whose measure class is preserved by G , and the pushforward measure $p_*\nu$ on Y . Then the triple (X, ν, p) is said to be *measure preserving relative to Y* if the disintegration map $Y \rightarrow \text{Prob}(X)$ associated with ν and p is G -equivariant. We shall often say briefly that X is measure preserving relative to Y , provided that ν and p are given.

2.2. Choquet simplices and stationary dynamics

A general reference for Choquet theory is [45]. Recall that any compact convex set C in a locally convex topological vector space is the closed convex hull of its extreme points $\text{Ext}(C)$, by the Krein–Milman theorem. Assuming that C is metrizable, $\text{Ext}(C)$ is a G_δ subset of C , thus a Polish space. For a probability measure ν on $\text{Ext}(C)$ we define the barycenter, $\text{bar}(\nu)$, to be the unique point $c \in C$ such that $\phi(c) = \int \phi d\nu$ for every continuous linear functional on the ambient space. We thus obtain an affine continuous map $\text{bar} : \text{Prob}(\text{Ext}(C)) \rightarrow C$. Choquet’s theorem says that this map is surjective.

Definition 2.1. The metrizable compact convex set C is said to be a *Choquet simplex* if the map bar is bijective.

For a metrizable Choquet simplex C , the space $\text{Prob}(\text{Ext}(C))$ is a Polish space, thus the map bar is a Borel isomorphism by Suslin’s theorem. It thus makes sense to consider the category whose objects are Polish convex spaces and whose morphisms are affine Borel maps. The space $\text{Prob}(X)$, for a Polish space X , is a prominent example and a Polish convex space is called a *Polish simplex* if it is affine Borel isomorphic to a space of the form $\text{Prob}(X)$. Note that a Polish simplex is a Choquet simplex iff it is compact.

Given a Polish space X and a continuous action of a topological group G on X by homeomorphisms, we have the standard identification $\text{Erg}_G(X) = \text{Ext}(\text{Prob}(X)^G)$, where $\text{Erg}_G(X)$ denote the space of ergodic G -invariant probability measures on X . Integration gives a continuous affine map $\text{Prob}(\text{Erg}_G(X)) \rightarrow \text{Prob}(X)^G$, and this map is a bijection by the existence and uniqueness of the disintegration map, considering for a given $\nu \in \text{Prob}(X)^G$ the ν -a.e. defined Borel map dualizing the von Neumann algebra inclusion $L^\infty(X, \nu)^G \subset L^\infty(X, \nu)$. Thus $\text{Prob}(X)^G$ is a Polish simplex.

Proposition 2.2. *Let C be a metrizable Choquet simplex and assume the group G acts continuously on C by affine transformations. Then C^G is a metrizable Choquet simplex.*

Proof. Denote $X = \text{Ext}(C)$. Then the affine G -equivariant Borel isomorphism $\text{bar} : \text{Prob}(X) \rightarrow C$ induces an affine Borel isomorphism $\text{Prob}(X)^G \rightarrow C^G$. Since $\text{Prob}(X)^G$ is a Polish simplex, it follows that C^G is a Polish simplex. It is a Choquet simplex as it is compact. ■

For another proof of Proposition 2.2, see [33, Corollary 12.13].

Next, we provide an analogue of stationary measures. We start by giving some background. Consider a continuous (in the strong operator topology) action of locally compact second countable group G on a Banach space E by linear isometries. This induces a continuous action on the dual space E^* endowed with the weak- $*$ topology. We define the convolution between a probability measure μ on G and an element $\varphi \in E^*$ by

$$(\mu * \varphi)(v) := \int_G \varphi(g^{-1}.v) d\mu(g) \quad (v \in E). \tag{2.1}$$

It is easy to see that this integral is absolutely convergent, and $\mu * \varphi$ is again an element of E^* . We say that φ is μ -stationary if $\mu * \varphi = \varphi$. Equation (2.1) thus defines a continuous action of the topological semigroup $\text{Prob}(G)$, with the convolution operation and the weak- $*$ topology, on E^* .

We are interested in $\text{Prob}(G)$ -invariant subsets $C \subseteq E^*$, as well as in the subset $C^\mu \subseteq C$ consisting of all μ -stationary points in C , for various $\mu \in \text{Prob}(G)$. If C is convex, weak- $*$ compact and G -invariant, then C is definitely $\text{Prob}(G)$ -invariant. Indeed, the convolution operation is given by the following composition:

$$\begin{aligned} \text{Prob}(G) \times C &\rightarrow \text{Prob}(G \times C) \rightarrow \text{Prob}(C) \rightarrow C, \\ (\mu, c) &\mapsto (\mu \times \delta_c) \mapsto a_*(\mu \times \delta_c) \mapsto \text{bar}(a_*(\mu \times \delta_c)), \end{aligned}$$

where $a : G \times C \rightarrow C$ is the action map. Another important example is when $C = \text{Prob}(X)$ where X is a Polish space and the action on C comes from a continuous action of G on X . In that case C is a convex subset of the dual of the Banach space of all bounded continuous functions on X and the convolution operation is given by the pushforward of the action map

$$\text{Prob}(G) \times \text{Prob}(X) \hookrightarrow \text{Prob}(G \times X) \rightarrow \text{Prob}(X).$$

Definition 2.3. Let G be a locally compact group. $\mu \in \text{Prob}(G)$ is said to be *admissible* if it is absolutely continuous with respect to the Haar measure class on G , and it is not supported on a closed proper subsemigroup of G .

It is well known and easy to prove that, for μ admissible, any μ -stationary probability measure has a G -invariant class.

Lemma 2.4. *Let X be a Polish space and G a locally compact second countable group that acts on it continuously by homeomorphisms. Fix an admissible probability measure μ on G . Then a μ -stationary probability measure ν on X is ergodic if and only if it is an extreme point of $\text{Prob}(X)^\mu$.*

Proof. This is [6, Corollary 2.7]. In this reference, X is assumed to be compact, but the proof works just as well if X is Polish. ■

We denote by $\text{Erg}_\mu(X) \subset \text{Prob}(X)^\mu$ the subset of ergodic measures. In view of Lemma 2.4, the integration map $\text{Prob}(\text{Erg}_\mu(X)) \rightarrow \text{Prob}(X)^\mu$ establishes a continuous affine bijection, again by the existence and uniqueness of the disintegration map, considering for a given $\nu \in \text{Prob}(X)^\mu$ the ν -a.e. defined Borel map dualizing the von Neumann algebra inclusion $L^\infty(X, \nu)^\mu \subset L^\infty(X, \nu)$. Thus $\text{Prob}(X)^\mu$ is a Polish simplex.

Lemma 2.5. *Let G act on a non-empty metrizable Choquet simplex C continuously by affine homeomorphisms. Let X denote the set of extreme points of C and consider the induced action of $\text{Prob}(G)$ on $\text{Prob}(X)$. Then the barycenter map $\text{Prob}(X) \rightarrow C$ is a $\text{Prob}(G)$ -equivariant continuous bijection and thus induces a continuous bijection of non-empty spaces $\text{Prob}(X)^\mu \rightarrow C^\mu$, for any $\mu \in \text{Prob}(G)$.*

Proof. The map $\text{Prob}(X) \rightarrow C$ is a continuous affine bijection. It is obviously G -equivariant, thus also μ -equivariant for any $\mu \in \text{Prob}(G)$. Therefore it induces a bijection $\text{Prob}(X)^\mu \rightarrow C^\mu$. By Kakutani’s fixed point theorem, C^μ is non-empty. It follows that $\text{Prob}(X)^\mu$ is non-empty as well. ■

Arguing as in the proof of Proposition 2.2, we deduce the following.

Proposition 2.6. *Let C be a metrizable Choquet simplex and assume the locally compact second countable group G acts on C continuously by affine transformations. Let μ be an admissible probability measure on G and consider its convolution action on C . Then C^μ is a metrizable Choquet simplex.*

2.3. Stiffness

We retain the setting of a locally compact second countable group G acting on a Banach space E by linear isometries, along with the corresponding continuous action of the semi-group $\text{Prob}(G)$ on E^* as given in (2.1).

Definition 2.7. Let C be a $\text{Prob}(G)$ -invariant subset of E^* . We say that the action of G on C is *affinely μ -stiff* for a probability measure $\mu \in \text{Prob}(G)$ if any μ -stationary point in C is G -invariant. A continuous action of G on a Polish space X is said to be *topologically μ -stiff* if the corresponding action on $\text{Prob}(X)$ is affinely μ -stiff.

When there is no risk of confusion, e.g. when a Polish G -space X does not admit any a priori affine structure, we shall simply say *μ -stiff*. We shall also often say that a Polish

space X is stiff as long as the action of G as well as the probability measure $\mu \in \text{Prob}(G)$ are implied by context.

We remark that in Definition 2.7 it could very well be the case that C^μ , the set of μ -stationary points in C , is empty, in which case the action is μ -stiff in a vacuous sense. This is never the case when C is weak- $*$ compact due to the standard Kakutani argument. By Lemma 2.5, this is also not the case for $\text{Prob}(X)$, where $X = \text{Ext}(C)$ for some metrizable Choquet simplex C .

Lemma 2.8. *Let $\mu \in \text{Prob}(G)$ be admissible. The action of G on a Polish space X is μ -stiff if and only if any ergodic $\nu \in \text{Prob}(X)^\mu$ is invariant.*

Proof. This is an immediate application of Lemma 2.4 and the fact that $\text{Prob}(X)^\mu$ is a Polish simplex. ■

We shall now state a few general and simple lemmas on stiffness.

Lemma 2.9. *Assume G is a finite group and μ is an admissible probability measure on G . Then every G -invariant convex subset $C \subset E^*$ is affinely μ -stiff.*

Proof. Observe that $\text{Prob}(G)^\mu = \{\lambda\}$, where λ is the uniform measure on G . As a result $\frac{1}{n} \sum_{k=1}^n \mu^k$ converges to λ and therefore $C^\mu = C^\lambda = C^G$. ■

Lemma 2.10. *Let C be a convex $\text{Prob}(G)$ -invariant subset of E^* . Let \bar{G} be the quotient of G by the kernel of $G \curvearrowright C$ and let $\bar{\mu}$ be the pushforward of μ under the quotient map $G \rightarrow \bar{G}$. Then C is affinely μ -stiff if and only if it is affinely $\bar{\mu}$ -stiff.*

Proof. Since the G -action on C factorizes through \bar{G} , it is easy to see that the corresponding $\text{Prob}(G)$ -action on C factorizes through $\text{Prob}(\bar{G})$. This shows that $C^\mu = C^{\bar{\mu}}$. As it is clear that $C^G = C^{\bar{G}}$, the statement follows. ■

Lemma 2.11. *Let $p : C_1 \rightarrow C_2$ be a continuous affine surjective G -equivariant map between two $\text{Prob}(G)$ -invariant convex sets and assume that the fibers of p are compact. If C_1 is affinely μ -stiff then so is C_2 .*

Proof. For $c_2 \in C_2^\mu$, the fiber $p^{-1}(c_2)$ is non-empty, compact, convex and closed under convolution by μ . Thus we can choose a μ -stationary element $c_1 \in p^{-1}(c_2)$. Since C_1 is stiff, c_1 must be G -invariant, and as a result, $c_2 = p(c_1)$ is G -invariant. ■

2.4. Relative stiffness

We introduce a relative version of stiffness for actions of a locally compact second countable group G endowed with an admissible probability measure μ .

Definition 2.12. Let $p : X \rightarrow Y$ be a G -equivariant continuous map between Polish G -spaces. The space X is said to be μ -stiff relative to Y if X is measure preserving relative to Y with respect to any choice of an ergodic μ -stationary probability measure ν on X .

Lemma 2.13. *If Y is stiff and X is stiff relative to Y then X is stiff. Conversely, if X is stiff then it is stiff relative to Y and, assuming further that the fibers of p are compact, Y is stiff.*

Proof. Fix some $\nu \in \text{Prob}(X)^\mu$ ergodic, and let $\nu = \int \nu_y dp_*\nu$ be the disintegration of ν along p .

If X is stiff relative to Y and Y is stiff then $p_*\nu$ is G -invariant, and

$$g.\nu = \int_Y g.\nu_y dp_*\nu = \int_Y \nu_{g.y} dp_*\nu = \int_Y \nu_y dp_*\nu = \nu \quad (\forall g \in G).$$

Hence ν is G -invariant. This shows X is stiff in light of Lemma 2.8.

The fact that if X is stiff then it is stiff relative to Y follows from the uniqueness of the disintegration, as $\nu = \int_Y g.\nu_{g^{-1}y} dp_*\nu$ implies $\nu_y = g.\nu_{g^{-1}y}$ a.e. Further, if the fibers of p are compact and X is stiff then Y is stiff by Lemma 2.11. ■

We introduce two cases in which relative stiffness is implied.

Proposition 2.14. *Let $p : X \rightarrow Y$ be a G -equivariant continuous map between Polish G -spaces and assume all fibers are countable (or finite). Then X is stiff relative to Y .*

This generalizes the well known fact that actions on countable spaces are stiff.

Proof. Let $\nu \in \text{Prob}(X)^\mu$ ergodic and denote $\eta = p_*\nu$. Disintegrate ν along p , namely write $\nu = \int_Y \nu_y d\eta$ where $\nu_y \in \text{Prob}(X)$ is defined for η -a.e. y and is supported on the fiber $p^{-1}(y)$. For each $y \in Y$ for which ν_y is defined, let M_y be the (finite) set of all $x \in p^{-1}(y)$ that attain a maximal ν_y -measure (recall that $p^{-1}(y)$ is countable). Denote by M the union of all such M_y , and suppose first that we have managed to show that $\nu(M) = 1$. In that case, ν_y is the uniform measure supported on M_y for a.e. y . The fact that G preserves the measure class of ν implies that M is essentially G -invariant. Thus writing ν according to its integration formula we get that for all $g \in G$, $\nu_y(M_y \cap gM_{g^{-1}y}) = 1$ for a.e. $y \in Y$, or equivalently $g\nu_{g^{-1}y} = \nu_y$. By Fubini, for a.e. $y \in Y$, the equality $g\nu_{g^{-1}y} = \nu_y$ holds for a.e. $g \in G$. This equality must hold in fact for all $g \in G$ due to the continuity of the action on the space $L(Y, \text{Prob}(X))$ of Lebesgue maps in the weak-* topology induced by its containment in $L^1(Y, C_b(X))^*$. Relative stiffness therefore follows.

In order to show that indeed $\nu(M) = 1$, choose some Borel section $s : Y \rightarrow M$ of $p \upharpoonright_M$. The image $s(Y)$ is a Borel subset of X (Suslin’s theorem), and its ν -measure is positive (as its ν_y -measure is positive for a.e. $y \in Y$). For any $g \in G$ denote by $s^g : Y \rightarrow X$ the section defined by $s^g(y) = gs(g^{-1}y)$. By the way M was constructed, we have $\nu_y(s(y)) \geq \nu_y(s^g(y))$, where equality holds if and only if $s^g(y) \in M$. In particular, $\nu(s(Y)) \geq \nu(s^g(Y))$, and equality holds if and only if $s^g(Y) \subseteq M$, up to null sets. The equality $\nu(s(Y)) = \nu(s^g(Y))$ indeed holds for μ -a.e. $g \in G$ due to the stationarity of ν :

$$\nu(s(Y)) = \int_G \nu(g.s(Y)) d\mu = \int_G \nu(s^g(Y)) d\mu.$$

We see that for Haar-a.e. $g \in G$, $s^g(Y) \subseteq M$ up to null sets, because μ is admissible. This must hold for all $g \in G$ due to the continuity of the G -action on $L^\infty(X, \nu)$ with the weak- $*$ topology, along with the observation that the set of all $f \in L^\infty(X, \nu)$ for which $\chi_M \cdot f = f$ is closed.

Fix a countable dense subgroup $G_0 \leq G$. The above shows that the set $M_0 := \bigcup_{g \in G_0} s^g(Y)$ is contained in M , up to null sets. But M_0 has positive measure as it contains $s(Y)$, and it is clearly G_0 -invariant. By the ergodicity of ν we conclude that M_0 has full ν -measure. In particular, M is of full measure, as desired. ■

The second setting where we show relative stiffness is for extension by compact groups:

Proposition 2.15. *Let X be a Polish G -space and let K be a compact group acting continuously on X . Assume that the K -action and the G -action commute with each other, so that G naturally acts on the orbit space X/K . Then X is stiff relatively to X/K .*

The proof uses Zimmer’s notion of minimal cocycles which we now introduce (see [51] for further details). Let Ω be an ergodic Lebesgue G -space and let K be a second countable locally compact group. A cocycle of G over Ω taking values in K is a Lebesgue map $\alpha : G \times \Omega \rightarrow K$ satisfying

$$\alpha(g_1 g_2, \omega) = \alpha(g_1, g_2 \omega) \alpha(g_2, \omega), \quad (g_1, g_2 \in G, \omega \in \Omega).$$

Given α one can construct the Lebesgue G -space $\Omega \times_\alpha K$ as follows: as a Lebesgue space it is just $\Omega \times K$ with the product σ -algebra and with the product of the measure class of Ω with the Haar measure class on K . The G -action on this space is defined by $g \cdot (\omega, k) = (g \cdot \omega, \alpha(g, \omega)k)$. It is not hard to see that this turns $\Omega \times_\alpha K$ into a Lebesgue G -space. Two cocycles α and β are said to be *cohomologous* if there exists an a.e. defined measurable map $\varphi : \Omega \rightarrow K$ for which

$$\beta(g, \omega) = \varphi(g \cdot \omega)^{-1} \alpha(g, \omega) \varphi(\omega)$$

for all $g \in G$ and for a.e. $\omega \in \Omega$. In that case $\Phi : (\omega, k) \mapsto (\omega, \varphi(\omega)k)$ defines an isomorphism of Lebesgue G -space between $\Omega \times_\alpha K$ and $\Omega \times_\beta K$.

While α takes values in K , it could very well be that α , or perhaps some cocycle β cohomologous to α , takes values in a proper subgroup of K . To see this, define $K_\alpha := \overline{\langle \text{Im}(\alpha) \rangle}$, which is the minimal closed subgroup of K containing $\{\alpha(g, \omega)\}_{g \in G, \omega \in \Omega}$. Then α is called a *minimal cocycle* if there is no β cohomologous to α for which $K_\beta \subsetneq K_\alpha$.

Theorem 2.16 ([51, Corollary 3.8]). *Let Ω be an ergodic Lebesgue G -space, and let K be a second countable compact group. Then:*

- (1) *Any cocycle $\alpha : G \times \Omega \rightarrow K$ is cohomologous to a minimal cocycle.*
- (2) *If α and β are cohomologous minimal cocycles then K_α and K_β are conjugate in K .*
- (3) *α is a minimal cocycle with $K = K_\alpha$ if and only if $\Omega \times_\alpha K$ is ergodic.*

Proof of Proposition 2.15. Let $\nu \in \text{Prob}(X)^\mu$ be ergodic. We also assume as we may, upon restriction, that $X = K \cdot \text{supp } \nu$. We consider the Borel map $X \rightarrow \text{Sub}(K)$ taking each x to its stabilizer K_x in K . Since the action of G commutes with that of K , this map is G -invariant, hence (as ν is ergodic) essentially constant on some subgroup K_0 . Moreover, this map is K -equivariant when K acts on $\text{Sub}(K)$ by conjugation, so K_0 is normal in K . As $X = K \cdot \text{supp } \nu$ we deduce that K_0 acts trivially on X , thus we assume as we may that K_0 is trivial. Replacing X with a conull subset which is $G \times K$ -invariant, we assume as we may that K acts freely on X .

Denote $Y = X/K$ and let $p : X \rightarrow Y$ be the quotient map and endow Y with the measure $p_*\nu$. Using a choice of an a.e. defined Borel section $s : Y \rightarrow X$ of p we may identify $X \cong Y \times K$ as Borel spaces, and endow X with the product measure

$$\tilde{\nu} = p_*\nu \times \lambda \in \text{Prob}(Y \times K) \cong \text{Prob}(X),$$

where λ is the Haar measure on K . This makes X a Lebesgue G -space. Define $\alpha : G \times Y \rightarrow K$ by setting $\alpha(g, y)$ to be the unique element in K for which $g.s(y) = \alpha(g, y).s(g.y)$. Then it is easy to check that α is a cocycle and that $\rho : (y, k) \mapsto \sigma(y)k$ defines an isomorphism between the Lebesgue G -spaces $Y \times_\alpha K$ and X . Using Theorem 2.16, we assume as we may that α is a minimal cocycle, whose image is denoted by $K_\alpha \leq K$. We get a G -invariant and K -equivariant Borel map $X \rightarrow K/K_\alpha$. By ν -ergodicity, this map is essentially constant. Upon conjugating α by an element of K , we assume as we may that the essential image is the basic coset eK_α . It follows that ν is supported on the subset $Y \times_\alpha K_\alpha$ of $Y \times_\alpha K = X$. We thus assume as we may that $K = K_\alpha$ and α is a minimal cocycle.

Using Theorem 2.16 once again we see that $\tilde{\nu}$ is G -ergodic. Noticing that $\tilde{\nu}$ is μ -stationary, we deduce by Lemma 2.4 that it is extremal. Observing that $\tilde{\nu} = \int_K k.\nu d\lambda(k)$, we conclude that $\nu = \tilde{\nu}$. It follows that the disintegration map $Y \rightarrow \text{Prob}(Y \times_\alpha K)$ is given by $y \mapsto \delta_y \times \lambda$, which is G -equivariant. Thus, indeed, X is relatively measure preserving with respect to $Y = X/K$. ■

2.5. Furstenberg–Poisson boundaries

The following result is the fundamental theorem of Furstenberg–Poisson theory (see [6, Section 2] for a detailed overview). In fact, it could be taken as the definition of Furstenberg–Poisson boundary.

Theorem 2.17 (Furstenberg). *Consider a locally compact second countable group G and an admissible probability measure μ on G . The Furstenberg–Poisson boundary of (G, μ) is a Lebesgue G -space $\Pi(G, \mu)$ endowed with a μ -stationary probability measure ν which satisfies the following property: for every metrizable compact convex G -set C , the correspondence that associates with a Lebesgue G -map $\theta : \Pi(G, \mu) \rightarrow C$ the barycenter $\text{bar } \theta_*\nu \in C$ is a bijection between the collection of all such Lebesgue G -maps and the space C^μ of stationary points in C .*

Moreover, every Lebesgue G -space Ω endowed with a stationary measure η which satisfies this property admits a unique morphism of Lebesgue G -spaces $\Omega \rightarrow \Pi(G, \mu)$ and this morphism pushes η to ν . The above uniquely defines $\Pi(G, \mu)$ up to a unique isomorphism of Lebesgue G -spaces.

An immediate consequence of the above result is the following useful criterion for invariance of a measure.

Corollary 2.18. *With the notation above, a point $c \in C^\mu$ is G -fixed iff the corresponding map $\Pi(G, \mu) \rightarrow C$ is essentially constant.*

We next discuss passage to finite index subgroups.

Definition 2.19. Consider a locally compact second countable group G and an open subgroup of finite index $G' < G$. Let μ be an admissible probability measure on G . The associated *hitting time* is the function

$$\tau : G^{\mathbb{N}} \rightarrow \mathbb{N} \cup \infty, \quad (g_1, g_2, \dots) \mapsto \inf \{n \in \mathbb{N} \mid g_n \cdot \dots \cdot g_1 \in G'\}.$$

By the law of large numbers, this function is $\mu^{\mathbb{N}}$ -a.e. finite, thus the *hitting map*

$$h : G^{\mathbb{N}} \rightarrow G', \quad w = (g_1, g_2, \dots) \mapsto g_{\tau(w)} \cdot \dots \cdot g_1,$$

is $\mu^{\mathbb{N}}$ -a.e. defined. The *hitting measure* on G' is defined to be $\mu' = h_*(\mu^{\mathbb{N}})$.

The hitting measure is discussed in a general context in [24]. In particular, the following is proved.

Proposition 2.20. *Consider a locally compact second countable group G and an open subgroup $G' < G$ of finite index. Let μ be an admissible probability measure on G and let μ' be the corresponding hitting measure on G' . Then the restriction to G' of the Furstenberg–Poisson boundary of (G, μ) is (uniquely isomorphic to) the Furstenberg–Poisson boundary of (G', μ') . Moreover, for every metrizable compact convex G -set C , $C^\mu \subset C^{\mu'}$.*

Proof. The first part of the proposition is proved in [24, Corollary 2.14] under the assumption that G is discrete, but an examination of the proof clearly reveals that the discreteness assumption is superfluous. The second part of the proposition follows from the first by Theorem 2.17. ■

Lemma 2.21. *Consider a locally compact second countable group G and an open normal subgroup $G' \triangleleft G$ of finite index. Let μ be an admissible probability measure on G and let μ' be the corresponding hitting measure on G' . If a Polish G -space X is μ' -stiff then it is also μ -stiff.*

Proof. Suppose X is μ' -stiff, denote $C = \text{Prob}(X)$ and let $c \in C^\mu$. By Theorem 2.20, $c \in C^\mu \subseteq C^{\mu'} = C^{G'}$. Let $\bar{\mu}$ be the pushforward measure of μ under the quotient map $G \rightarrow \bar{G} := G/G'$. The set $C^{G'}$ is G -invariant, and the action of G on it factorizes

through \bar{G} . It follows by Lemma 2.10 that $c \in (C^{G'})^{\bar{\mu}}$. But \bar{G} is finite so by Lemma 2.9, $c \in (C^{G'})^{\bar{G}} = C^G$. ■

Lemma 2.22. *Consider a locally compact second countable group G with an admissible measure μ on it. Let $p : X \rightarrow Y$ be a G -equivariant continuous and surjective map between Polish G -spaces. Assume that $G' := \ker(G \curvearrowright Y)$ is an open subgroup of finite index in G and let μ' denote the hitting measure on it. If every fiber of p is μ' -stiff then X is μ -stiff.*

Proof. By Lemma 2.21 it suffices to show that X is μ' -stiff. Given $\nu \in \text{Prob}(X)$ we may disintegrate it along p , that is, write $\nu = \int_Y \nu_y d(p_*\nu)$ where $\nu_y \in \text{Prob}(X)$ is defined for a.e. $y \in Y$ and is supported on the fiber $p^{-1}(y)$. Hence $\mu' * \nu = \int_Y \mu' * \nu_y d(p_*\nu)$ and each $\mu' * \nu_y$ is again a probability measure on X which is supported on $p^{-1}(y)$. Therefore if ν is μ' -stationary then we have found two disintegrations of ν along p , so by uniqueness of the disintegration measures, ν_y is μ' -stationary for $p_*\nu$ -a.e. y . Since the action of G' on each fiber is μ' -stiff, we see that the measures ν_y are G' -invariant a.e. Thus ν is G' -invariant as desired. ■

The following result of Margulis is a generalization of [20, Theorem 3].

Proposition 2.23. *Let G be a compactly generated locally compact group with left Haar measure λ , and let $\Gamma < G$ be a lattice. Let $\psi : G \rightarrow [0, \infty)$ be a continuous function with $\int_G \psi d\lambda = 1$, which is supported on a compact subset that generates G as a semigroup, and consider the probability measure $\mu = \psi \cdot \lambda$ on G . Then there exists a fully supported probability measure μ' on Γ such that the restriction to Γ of the Furstenberg–Poisson boundary of (G, μ) is μ' -stationary and for every metrizable compact convex G -set C , $C^\mu \subset C^{\mu'}$. Moreover, given any function $\phi : \Gamma \rightarrow (0, 1]$, we may choose $\mu' \leq \phi$.*

Proof. This is essentially the content of [37, Proposition VI.4.1]. This reference deals with an S -arithmetic setting, but this is superfluous and it serves as a complete proof that the μ -stationary measure on $\Pi(G, \mu)$ is μ' -stationary for some fully supported $\mu' \in \text{Prob}(\Gamma)$. Moreover, given any function $\phi : \Gamma \rightarrow (0, 1]$ one can impose throughout the construction the condition $\mu'(\gamma) \leq \phi(\gamma)$. The fact that $C^\mu \subset C^{\mu'}$ follows by Theorem 2.17. ■

2.6. Furstenberg measures

We have defined the notion of a “Furstenberg measure” in the introduction (see Definition 1.7). The following is a slight generalization of a classical result of Furstenberg [20, Theorem 3], which regards the case in which Γ is contained in the identity component of $\mathbf{G}(\mathbb{R})$.

Proposition 2.24. *Let \mathbf{G} be an \mathbb{R} -algebraic group with a semisimple identity component and let $\Gamma < \mathbf{G}(\mathbb{R})$ be a lattice. Then there exists a Furstenberg measure μ on Γ .*

Proof. Denote $G = \mathbf{G}(\mathbb{R})$ and let G° be its identity component. Let $P < G^\circ$ be a minimal parabolic subgroup and $K < G^\circ$ be a maximal compact subgroup of this connected semisimple Lie group. Since all minimal parabolic subgroups, as well as all maximal compact subgroups, are conjugate and P and K are self-normalizers in G° , we may identify the spaces G°/P and G°/K with the corresponding sets of minimal parabolic and maximal compact subgroups in G° . Thus, the conjugation action of G on G° induces transitive actions on G°/P and G°/K correspondingly and we may identify $G^\circ/P \simeq G/N_G(P)$ and $G^\circ/K \simeq G/N_G(K)$.

Endow G with the left Haar measure λ , and let $\psi : G \rightarrow [0, \infty)$ be a continuous function with $\int_G \psi \, d\lambda = 1$ whose support is compact and generates G as a semigroup. Consider the probability measure $\bar{\mu} = \psi \cdot \lambda$ on G . Find a $\bar{\mu}$ -stationary probability measure ν on the compact G -space $G/N_G(P)$. By Proposition 2.23, there exists a fully supported probability measure μ on Γ such that ν is μ -stationary. We choose $\bar{\mu} \leq \phi$ for $\phi : \Gamma \rightarrow (0, 1]$ small enough so that μ has finite logarithmic first moment and random walk entropy. We now use [28, Theorem 10.7] to deduce that $(G/N_G(P), \nu)$ is the Furstenberg–Poisson boundary of (Γ, μ) . Strictly speaking, it is assumed in this reference that G is connected, but since G acts by isometries on the symmetric space G°/K , the use of the strip approximation [28, Theorem 6.5] is intact. Since $N_G(P) = \mathbf{N}(\mathbb{R})$ where \mathbf{N} is the normalizer of the Zariski closure of P , we conclude that

$$G/N_G(P) = \mathbf{G}(\mathbb{R})/\mathbf{N}(\mathbb{R}),$$

as required to show that μ is a Furstenberg measure on Γ . ■

Proof of Proposition 1.8. Let \mathbf{G} be a \mathbb{Q} -algebraic group and Λ an arithmetic subgroup. Let $\mathbf{R} \triangleleft \mathbf{G}$ be the solvable radical of the identity component subgroup in \mathbf{G} and denote $\bar{\mathbf{G}} = \mathbf{G}/\mathbf{R}$. Let $\bar{\Lambda}$ be the image of Λ in $\bar{\mathbf{G}}(\mathbb{Q})$. By [46, Proposition 4.1], $\bar{\Lambda}$ is an arithmetic subgroup of $\bar{\mathbf{G}}$, thus by a classical theorem of Borel and Harish-Chandra it is a lattice in $\bar{\mathbf{G}}(\mathbb{R})$. Using Proposition 2.24, we find a Furstenberg measure $\bar{\mu}$ on $\bar{\Lambda}$. Then the Furstenberg–Poisson boundary of $(\bar{\Lambda}, \bar{\mu})$ is realized by $\bar{\mathbf{G}}(\mathbb{R})/\bar{\mathbf{N}}(\mathbb{R})$, where $\bar{\mathbf{N}} < \bar{\mathbf{G}}$ is the normalizer of some minimal \mathbb{R} -parabolic subgroup $\bar{\mathbf{P}}$ of $\bar{\mathbf{G}}$. We denote by \mathbf{P} and \mathbf{N} the preimages of $\bar{\mathbf{P}}$ and $\bar{\mathbf{N}}$ in \mathbf{G} correspondingly and observe that \mathbf{P} is a minimal \mathbb{R} -parabolic subgroup and \mathbf{N} is its normalizer in \mathbf{G} . Since the kernel of $\Lambda \rightarrow \bar{\Lambda}$ is solvable, hence amenable, we can use [29, Theorem 2] to find a probability measure μ on Λ such that the Furstenberg–Poisson boundary of (Λ, μ) is the same as the Furstenberg–Poisson boundary of $(\bar{\Lambda}, \bar{\mu})$, that is, $\bar{\mathbf{G}}(\mathbb{R})/\bar{\mathbf{N}}(\mathbb{R})$. Identifying the latter with $\mathbf{G}(\mathbb{R})/\mathbf{N}(\mathbb{R})$, we conclude that μ is a Furstenberg measure on Λ . ■

3. Characters

Throughout this section we fix a countable group Λ . All representations are assumed to be unitary. For a comprehensive overview of the theory of characters and more generally of dual spaces in harmonic analysis we refer to [10].

3.1. Basics

A function $\varphi : \Lambda \rightarrow \mathbb{C}$ is called *positive-definite* if for any $n \in \mathbb{N}$, and for any elements $g_1, \dots, g_n \in \Lambda$, the matrix $(\varphi(g_i^{-1}g_j))_{i,j}$ is positive-semidefinite, that is, $\sum_{i,j} \bar{\alpha}_i \alpha_j \varphi(g_i^{-1}g_j) \geq 0$ for any choice of complex numbers $\alpha_1, \dots, \alpha_n$. The space of such functions is denoted by $\text{PD}(\Lambda)$. It is not hard to see that $\varphi(e) = \sup_{g \in \Lambda} |\varphi(g)|$ and in particular $\text{PD}(\Lambda) \subseteq l^\infty(\Lambda)$. We may therefore speak of the weak- $*$ topology on $\text{PD}(\Lambda)$, which makes $\text{PD}(\Lambda)$ a close convex set. The space $\text{PD}_1(\Lambda)$ obtained by adding the normalization condition $\varphi(e) = 1$ is then compact and convex. The weak- $*$ topology on $\text{PD}_1(\Lambda)$ coincides with the topology of pointwise convergence. The restriction to Λ of any state on the (maximal) C^* -algebra $C^*(\Lambda)$ is positive-definite and conversely any $\varphi \in \text{PD}_1(\Lambda)$ can be extended to $C^*(\Lambda)$. This establishes a bijection between $\text{PD}_1(\Lambda)$ and the space of states on $C^*(\Lambda)$.

An element $\varphi \in \text{PD}_1(\Lambda)$ which is conjugation invariant is called a *trace*. Traces on Λ correspond to tracial states on $C^*(\Lambda)$. The collection $\text{Tr}(\Lambda)$ of all traces is again compact and convex, so it is natural to speak of extreme points in $\text{Tr}(\Lambda)$, that is, traces that cannot be written as a proper convex sum of two traces. Any extremal point of this set is called a *character* of Λ , and the collection of all characters is denoted by $\text{Ch}(\Lambda)$. Observe that $\text{Tr}(\Lambda)$ is metrizable, as Λ is countable, thus its G_δ -subset $\text{Ch}(\Lambda)$ is Polish (see [45, Proposition 1.3]), but it is in general not compact; see e.g. Example 1.9. The space $\text{Ch}(\Lambda)$ is sometimes referred to as the *Thoma dual* of Λ as it serves as a dual space of Λ for harmonic-analytic purposes. The following theorem may be thought of as a Fourier transform on Λ .

Theorem 3.1 ([49]). *$\text{Tr}(\Lambda)$ is a Choquet simplex, that is, the barycenter map*

$$\mathcal{F}_\Lambda : \text{Prob}(\text{Ch}(\Lambda)) \rightarrow \text{Tr}(\Lambda), \quad \nu \mapsto \int_{\text{Ch}(\Lambda)} \chi \, d\nu,$$

is a continuous affine bijection.

Proof of Proposition 1.6. The proposition follows from Lemma 2.5 in view of Theorem 3.1 and the metrizability of $\text{Tr}(\Lambda)$. ■

Definition 3.2. Let $\varphi \in \text{PD}_1(\Lambda)$. The *support* of φ is $\text{supp } \varphi = \Lambda \setminus \varphi^{-1}(0)$. The *kernel* of φ is $\ker \varphi := \varphi^{-1}(1)$; and φ is called *faithful* if $\ker \varphi = \{e\}$.

The following lemma is standard, and its proof is not hard with the use of the GNS construction which will be recalled momentarily. For details, see [10, Lemma 11.C.8].

Lemma 3.3. *For any $\varphi \in \text{PD}_1(\Lambda)$, $\ker \varphi$ is a subgroup of Λ , and $\varphi(k_1 g k_2) = \varphi(g)$ for all $k_1, k_2 \in \ker \varphi$ and $g \in \Lambda$. In particular, if φ is a trace then $\ker \varphi$ is a normal subgroup, and φ descends to a faithful trace of $\Lambda/\ker \varphi$.*

A *GNS triplet* of Λ is a triple (\mathcal{H}, π, ξ) where \mathcal{H} is a Hilbert space, $\pi : \Lambda \rightarrow \mathcal{U}(\mathcal{H})$ is a representation, and ξ is a cyclic vector. Given such a triplet, the function

$$\varphi : g \mapsto \langle \pi(g)\xi, \xi \rangle \quad (g \in \Lambda) \tag{3.1}$$

is in $\text{PD}(\Lambda)$. Conversely, the GNS construction associates to any $\varphi \in \text{PD}(\Lambda)$ a GNS triplet (\mathcal{H}, π, ξ) such that (3.1) holds, and this triplet is unique up to an equivalence of representations that takes the respective cyclic vectors to one another [10, Proposition 1.B.10]. Furthermore, $\varphi \in \text{PD}_1(\Lambda)$ if and only if ξ is a unit vector, and φ is extremal if and only if π is irreducible [10, Proposition 1.B.14].

Let Γ be a subgroup of Λ . Given $\varphi \in \text{PD}_1(\Gamma)$, its *trivial extension* $\tilde{\varphi} : \Lambda \rightarrow \mathbb{C}$ is

$$\tilde{\varphi}(g) = \begin{cases} \varphi(g), & g \in \Gamma, \\ 0, & g \notin \Gamma. \end{cases}$$

The fact that $\tilde{\varphi}$ is positive-definite follows from the following lemma:

Lemma 3.4. *Let $\Gamma \leq \Lambda$, and let $\varphi \in \text{PD}_1(\Gamma)$ with GNS triplet (\mathcal{H}, π, ξ) . Consider the Hilbertian tensor product $\tilde{\mathcal{H}} = l^2(\Lambda/\Gamma) \otimes \mathcal{H}$, the induced representation $\tilde{\pi} = \text{Ind}_\Gamma^\Lambda \pi$, and the vector $\tilde{\xi} = \delta_{e\Gamma} \otimes \xi$ in $\tilde{\mathcal{H}}$. Then $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\xi})$ is a GNS triplet whose corresponding positive-definite function is $\tilde{\varphi}$.*

Proof. The proof is straightforward (see [10, Proposition 1.F.10]). ■

Lemma 3.5. *Let N be a normal subgroup of index $d < \infty$ in Λ , and let φ be an extreme point of $\text{PD}_1(N)$. Then the trivial extension $\tilde{\varphi} \in \text{PD}_1(\Lambda)$ is a convex combination of d (or fewer) extreme points of $\text{PD}_1(\Lambda)$.*

Proof. Let (\mathcal{H}, π, ξ) be a GNS triplet associated with φ . Then to $\tilde{\varphi}$ corresponds the GNS triplet $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\xi})$ as described in Lemma 3.4. As N is normal, $\text{Res}_N^\Lambda \text{Ind}_N^\Lambda \pi = \bigoplus_{i=1}^d \pi^{t_i}$ where t_1, \dots, t_d are coset representatives for N in Λ , and π^{t_i} is the representation of N defined by $n \mapsto \pi(t_i^{-1}nt_i)$. Since φ is extremal, π is irreducible and thus so is each π^{t_i} . It follows that $\tilde{\pi} = \text{Ind}_N^\Lambda \pi$ is a direct sum of at most d irreducible representations because this is true even when restricting to N . This direct sum decomposition gives rise to a decomposition of $\tilde{\varphi}$ as a convex combination of at most d extreme points of $\text{PD}_1(\Lambda)$. ■

3.2. Tracial representations

A *tracial von Neumann algebra* is a pair (M, τ) , where M is a von Neumann algebra and τ is a normal faithful trace on M . A *finite factor* is a tracial von Neumann algebra whose center consists only of the scalar multiples of the identity. A *tracial representation* of a group Λ is a triple (M, τ, π) where (M, τ) is a tracial von Neumann algebra and $\pi : \Lambda \rightarrow \mathcal{U}(M)$ is a representation of Λ in the group of unitary elements of M such that $\pi(\Lambda)$ generates M as a von Neumann algebra. Two tracial representations are said to be equivalent if there is an isomorphism between the two von Neumann algebras which intertwines the traces and the representations. The following is a fundamental result in the theory of characters.

Theorem 3.6 (Thoma). *There is a one-to-one correspondence between*

- (1) traces $\varphi : \Lambda \rightarrow \mathbb{C}$, and
- (2) equivalence classes of tracial representations of Λ .

Moreover, φ is a character if and only if the corresponding von Neumann algebra is a factor.

For the proof see [10, Proposition 11.C.4, Theorem 11.C.5], which are formulated in a different, yet equivalent language, in view of [10, Proposition 6.B.5]. Briefly, given a tracial representation (M, τ, π) , $\phi = \tau \circ \pi$ is a trace on Λ and we can retrieve (M, τ, π) from ϕ by the GNS construction.

Corollary 3.7. *For an abelian group Λ , the space $\text{Ch}(\Lambda)$ coincides with the Pontryagin dual $\hat{\Lambda}$.*

Proof. Given $\varphi \in \text{Ch}(\Lambda)$ let (M, τ, π) be the associated tracial representation. Then M is a factor, and it is commutative, thus isomorphic to \mathbb{C} . It follows that π is a homomorphism into the unit circle. ■

Definition 3.8. A von Neumann algebra is called *amenable* if for some (equivalently, any) embedding $M \subseteq \mathcal{B}(\mathcal{H})$, there exists a linear unital positive map $E : \mathcal{B}(\mathcal{H}) \rightarrow M$ satisfying $E(xTy) = xE(T)y$ for all $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in M$ (such an E is called a *conditional expectation*). A trace $\varphi \in \text{Tr}(\Lambda)$, with corresponding tracial representation (M, τ, π) , is called *von Neumann amenable* if M is amenable.

Any finite-dimensional von Neumann algebra is amenable. It is well known that all separable infinite-dimensional amenable finite factors are isomorphic to a single von Neumann algebra \mathcal{R} , often referred to as the hyperfinite II_1 -factor [17, 40]. We refer to [2] for more on amenable von Neumann algebras.

3.3. Relative characters and stiffness

Traces and characters generalize to a relative setting. Let Γ be a group acting on Λ by group automorphisms. Then Γ acts on $\text{PD}_1(\Lambda)$ by $\varphi \mapsto \varphi^\gamma$, where $\varphi^\gamma(g) = \varphi(\gamma^{-1}(g))$, and $\text{Tr}(\Lambda)$ is an invariant subset. We define the space of Γ -traces of Λ , $\text{Tr}_\Gamma(\Lambda)$, to be the Γ -fixed points of $\text{Tr}(\Lambda)$, and the space of Γ -characters of Λ , $\text{Ch}_\Gamma(\Lambda)$, to be the extreme points of $\text{Tr}_\Gamma(\Lambda)$.

Theorem 3.9. *The compact convex space $\text{Tr}_\Gamma(\Lambda)$ is a Choquet simplex, that is,*

$$\text{Prob}(\text{Ch}_\Gamma(\Lambda)) \rightarrow \text{Tr}_\Gamma(\Lambda)$$

is a continuous affine bijection.

Proof. By Theorem 3.1, $\text{Tr}(\Lambda)$ is a Choquet simplex, thus by Proposition 2.2 so is its subspace of Γ -fixed points. ■

We now restrict to the case where N is a normal subgroup of Λ so that Λ acts on it by conjugation. For any $\varphi \in \text{Tr}(\Lambda)$ the restriction $\varphi \upharpoonright_N$ is easily seen to be in $\text{Tr}_\Lambda(N)$. We call the mapping $r : \varphi \rightarrow \varphi \upharpoonright_N$ the *restriction map*. Obviously it is continuous, and it surjects onto $\text{Tr}_\Lambda(N)$. Indeed, the trivial extension $\tilde{\varphi}$ is a trace of Λ whose restriction to N is φ .

Lemma 3.10. *The restriction map $r : \text{Tr}(\Lambda) \rightarrow \text{Tr}_\Lambda(N)$ maps $\text{Ch}(\Lambda)$ surjectively onto $\text{Ch}_\Lambda(N)$. In particular, if N is in the center of Λ , then r maps $\text{Ch}(\Lambda)$ onto the Pontryagin dual \hat{N} of N .*

Proof. The fact that r takes $\text{Ch}(\Lambda)$ to $\text{Ch}_\Lambda(N)$ is shown in [49, Lemma 14]. The fact that r is surjective is shown in [49, Lemma 16]. Finally, if N is central then Λ acts trivially on $\text{Tr}(N)$ and therefore $\text{Ch}_\Lambda(N)$ is equal to $\text{Ch}(N)$, which is simply \hat{N} by Corollary 3.7. ■

Proposition 3.11. *Suppose N is a normal subgroup of finite index in Λ . Then each fiber of the restriction map $\text{Ch}(\Lambda) \rightarrow \text{Ch}_\Lambda(N)$ is of size at most $[\Lambda : N]$.*

The proof of Proposition 3.11 will be given in Section 3.4. The following is an important conclusion regarding stiffness of actions on spaces of characters, and will be used often.

Proposition 3.12. *Fix a group Γ and an admissible measure μ on Γ . Assume that Γ acts by automorphisms on a group Λ and let N be a Γ -invariant normal subgroup of Λ . Consider the following statements regarding the corresponding Γ -actions on $\text{Ch}(\Lambda)$, $\text{Ch}(N)$ and $\text{Ch}_\Lambda(N)$:*

- (1) $\Gamma \curvearrowright \text{Ch}(N)$ is stiff.
- (2) $\Gamma \curvearrowright \text{Ch}_\Lambda(N)$ is stiff.
- (3) $\Gamma \curvearrowright \text{Ch}(\Lambda)$ is stiff.

Then (1) implies (2), (3) implies (2) if the fibers of $\text{Ch}(\Lambda) \rightarrow \text{Ch}_\Lambda(N)$ are compact, and all three are equivalent if $[\Lambda : N] < \infty$.

Proof. By Theorem 3.9, $\text{Tr}(N)$ and $\text{Tr}_\Lambda(N)$ are Choquet simplices and thus we have Γ -equivariant bijections

$$\text{Prob}(\text{Ch}_\Lambda(N)) \simeq \text{Tr}_\Lambda(N) = \text{Tr}(N)^\Lambda \simeq \text{Prob}(\text{Ch}(N))^\Lambda. \tag{3.2}$$

Note, however, that these are not in general homeomorphisms. Nevertheless, all these spaces are (or are not) stiff simultaneously by Lemma 2.5.

Stiffness of the Polish space $\text{Ch}(N)$ is by definition equivalent to stiffness of the convex space $\text{Prob}(\text{Ch}(N))$, which implies the stiffness of its Γ -invariant convex subspace $\text{Prob}(\text{Ch}(N))^\Lambda$, hence by (3.2), of $\text{Prob}(\text{Ch}_\Lambda(N))$, which is equivalent (by definition) to the stiffness of the Polish space $\text{Ch}_\Lambda(N)$. This shows that (1) implies (2).

By Lemma 3.10, the map $\text{Ch}(\Lambda) \rightarrow \text{Ch}_\Lambda(N)$ is surjective, thus by Lemma 2.11, if its fibers are compact then (3) implies (2).

Assume that $[\Lambda : N] < \infty$. Then (2) implies (3) by Propositions 3.11 and 2.14. Finally, to see that (2) implies (1) note that (3.2) identifies $\text{Ch}_\Lambda(N)$ with the ergodic Λ -invariant probability measures on $\text{Ch}(N)$. All such measures are uniform measures on finite orbits and thus $\text{Ch}_\Lambda(N)$ is identified with the space of orbits $\text{Ch}(N)/\Lambda$. The statement then follows from Proposition 2.14. ■

Corollary 3.13. *Fix a group Γ and an admissible measure μ on Γ . Assume that Γ acts by automorphisms on a group Λ . If Λ is finitely generated and $\Lambda_0 < \Lambda$ is a Γ -invariant subgroup of finite index then $\Gamma \curvearrowright \text{Ch}(\Lambda)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(\Lambda_0)$ is stiff. If, further, Λ is residually finite and $F \triangleleft \Lambda$ is a Γ -invariant finite normal subgroup then $\Gamma \curvearrowright \text{Ch}(\Lambda)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(\Lambda/F)$ is stiff.*

Proof. Assume Λ is finitely generated and let $\Lambda_0 < \Lambda$ be a Γ -invariant subgroup. Since Λ is finitely generated, it has only finitely many subgroups of index $[\Lambda : \Lambda_0]$, thus their intersection N is a characteristic normal subgroup of finite index in Λ which is contained in Λ_0 . In particular, N is Γ -invariant. By (1) \Leftrightarrow (3) in Proposition 3.12 we see that $\Gamma \curvearrowright \text{Ch}(\Lambda)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(N)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(\Lambda_0)$ is stiff.

Assume further that Λ is residually finite and let $F \triangleleft \Lambda$ be a Γ -invariant finite normal subgroup. Since Λ is residually finite, we can find a normal subgroup $N \triangleleft \Lambda$ of finite index such that F injects into Λ/N . Arguing as above, we assume as we may that N is characteristic, thus Γ -invariant. Therefore, the Γ -action descent to an action on the finite group Λ/N . We consider the Γ -equivariant injection $\Lambda \rightarrow \Lambda/F \times \Lambda/N$ and identify Λ with its image, which is of finite index. Applying the first part of the corollary twice, we find that $\Gamma \curvearrowright \text{Ch}(\Lambda)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(\Lambda/F \times \Lambda/N)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(\Lambda/F \times \{e\})$ is stiff. We conclude that $\Gamma \curvearrowright \text{Ch}(\Lambda)$ is stiff iff $\Gamma \curvearrowright \text{Ch}(\Lambda/F)$ is stiff. ■

3.4. Relative traces as positive-definite functions with kernels

If $\varphi \in \text{Tr}(\Lambda)$ is an extreme point of $\text{PD}_1(\Lambda)$, then it is also an extreme point of $\text{Tr}(\Lambda)$, but the converse does not hold in general. We introduce a perspective to bridge this gap, and in turn to complete the proof of Proposition 3.11. We start with the following general statement:

Proposition 3.14. *Consider a group Γ acting on Λ by group automorphisms and the associated semidirect product $\Gamma \ltimes \Lambda$. Denote by $p : \Gamma \ltimes \Lambda \rightarrow \Lambda$ the map $(\gamma, g) \mapsto g$. Then $\varphi \mapsto \varphi \circ p$ defines an injective affine continuous map $\text{PD}_1(\Lambda)^\Gamma \rightarrow \text{PD}_1(\Gamma \ltimes \Lambda)$. The image is the set of all $\psi \in \text{PD}_1(\Gamma \ltimes \Lambda)$ with $\Gamma \subseteq \ker \psi$, and in particular it is a face of $\text{PD}_1(\Gamma \ltimes \Lambda)$.*

Proof. Let $\varphi \in \text{PD}_1(\Lambda)^\Gamma$. Given $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$, $g_1, \dots, g_n \in \Lambda$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ write

$$\begin{aligned} \sum_{i,j} \overline{\alpha_i} \alpha_j (\varphi \circ p)((\gamma_i, g_i)^{-1} \cdot (\gamma_j, g_j)) &= \sum_{i,j} \overline{\alpha_i} \alpha_j \varphi(\gamma_i^{-1}(g_i^{-1}g_j)) \\ &= \sum_{i,j} \overline{\alpha_i} \alpha_j \varphi(g_i^{-1}g_j) \geq 0. \end{aligned}$$

Hence $\varphi \circ p \in \text{PD}_1(\Gamma \ltimes \Lambda)$ (a more conceptual argument is obtained via the associated GNS representation). The fact that the mapping $\varphi \mapsto \varphi \circ p$ is injective, affine and continuous is straightforward.

Let F denote the set of all $\psi \in \text{PD}_1(\Gamma \rtimes \Lambda)$ with $\Gamma \subseteq \ker \psi$. As ψ attains values in the unit disk in \mathbb{C} , it follows that F is a face of $\text{PD}_1(\Gamma \rtimes \Lambda)$. Moreover, $\varphi \circ p \in F$ for any $\varphi \in \text{PD}_1(\Lambda)^\Gamma$, so it remains to show surjectivity onto F . If $\psi \in \text{PD}_1(\Gamma \rtimes \Lambda)$ then clearly $\psi \upharpoonright_\Lambda \in \text{PD}_1(\Lambda)$. Assuming moreover that $\psi \in F$ we deduce by Lemma 3.3 that for all $\gamma \in \Gamma$ and $g \in \Lambda$,

$$\psi \upharpoonright_\Lambda(\gamma(g)) = \psi(e_\Gamma, \gamma(g)) = \psi((\gamma, e_\Lambda)(e_\Gamma, g)(\gamma^{-1}, e_\Lambda)) = \psi(e_\Gamma, g) = \psi \upharpoonright_\Lambda(g),$$

showing that $\psi \upharpoonright_\Lambda \in \text{PD}_1(\Lambda)^\Gamma$, and

$$\psi(\gamma, g) = \psi((e_\Gamma, g)(\gamma, e_\Lambda)) = \psi(e_\Gamma, g) = \psi \upharpoonright_\Lambda \circ p(\gamma, g),$$

showing that $\psi = \psi \upharpoonright_\Lambda \circ p$. ■

Example 3.15. Fix a group Λ and let $\Gamma = \Lambda$ act on it by inner automorphisms. We get the identification $\text{PD}_1(\Lambda)^\Gamma = \text{Tr}(\Lambda)$. Note that in this case we may also identify $\Gamma \rtimes \Lambda \simeq \Lambda \times \Lambda$, by $(\gamma, \lambda) \mapsto (\gamma, \lambda\gamma)$. Under these identifications, the map $\text{PD}_1(\Lambda)^\Gamma \rightarrow \text{PD}_1(\Gamma \rtimes \Lambda)$ considered in Proposition 3.14 identifies $\text{Tr}(\Lambda)$ with the subset of elements of $\text{PD}_1(\Lambda \times \Lambda)$ which are trivial on the diagonal subgroup. This is a well known identification.

The following is a generalization of the previous example.

Example 3.16. Consider a group Γ acting on Λ by group automorphisms and assume that the image of Γ in $\text{Aut}(\Lambda)$ contains all inner automorphisms. Then $\text{PD}_1(\Lambda)^\Gamma = \text{Tr}_\Gamma(\Lambda)$, so that $\text{Tr}_\Gamma(\Lambda)$ embeds as a face of $\text{PD}_1(\Gamma \rtimes \Lambda)$. In particular, $\varphi \in \text{Tr}_\Gamma(\Lambda)$ belongs to $\text{Ch}_\Gamma(\Lambda)$ if and only if $\varphi \circ p$ is an extreme point of $\text{PD}_1(\Gamma \rtimes \Lambda)$.

We will use the last example in the course of the proof of the following lemma.

Lemma 3.17. *Let N be a normal subgroup of index $d < \infty$ in Λ , and let $\varphi \in \text{Ch}_\Lambda(N)$. Then the trivial extension $\tilde{\varphi} \in \text{Tr}(\Lambda)$ decomposes as a convex combination of d (or fewer) characters of Λ .*

Proof. Consider the inner action of Λ on itself, as well as on the invariant subset N . With respect to these actions form the semidirect products $\Lambda \rtimes \Lambda$ and $\Lambda \rtimes N$ and let $p_\Lambda : \Lambda \rtimes \Lambda \rightarrow \Lambda$ and $p_N : \Lambda \rtimes N \rightarrow N$ denote the (set-theoretic) projections onto the second coordinates. By Proposition 3.14, showing that $\tilde{\varphi}$ is a convex combination of d characters is the same as showing that $\tilde{\varphi} \circ p_\Lambda$ is a convex combination of d extreme points of $\text{PD}_1(\Lambda \rtimes \Lambda)$.

Now φ is an extreme point of $\text{Tr}_\Lambda(N)$ and so by Proposition 3.14, $\varphi \circ p_N$ is an extreme point of $\text{PD}_1(\Lambda \rtimes N)$. Hence, by Lemma 3.5, $\widetilde{\varphi \circ p_N}$, the trivial extension of $\varphi \circ p_N$ from $\Lambda \rtimes N$ to $\Lambda \rtimes \Lambda$, is a convex combination of d extreme points of $\text{PD}_1(\Lambda \rtimes \Lambda)$. But clearly $\tilde{\varphi} \circ p_\Lambda = \widetilde{\varphi \circ p_N}$, which completes the proof. ■

Proof of Proposition 3.11. Let $\varphi \in \text{Ch}_\Lambda(N)$ and let $\tilde{\varphi}$ be its trivial extension. By Lemma 3.17, $\tilde{\varphi}$ is a convex combination of $[\Lambda : N]$ characters of Λ . But according to

[49, Lemma 15], each $\psi \in \text{Ch}(\Lambda)$ with $\psi \upharpoonright_N = \varphi$ appears as one of those $[\Lambda : N]$ characters in the decomposition of $\tilde{\varphi}$. It follows that the fiber of φ under the restriction map $\text{Ch}(\Lambda) \rightarrow \text{Ch}_\Lambda(N)$ is of size at most $[\Lambda : N]$. ■

4. Stiffness of arithmetic groups

The goal of this section is to prove Theorem 4.1 below and to deduce from it Theorems C and D.

Theorem 4.1. *Let U be a connected, simply connected nilpotent real Lie group and $H \leq U$ a lattice. Let S be a reductive group with compact center acting continuously by automorphisms on U and let $\Gamma < S$ be a lattice preserving H . Let μ be a Furstenberg measure on Γ . Then the action of Γ on $\text{Ch}(H)$ is μ -stiff.*

Due to Lemma 2.21, there is no loss of generality in assuming that S is connected. We denote by S_{nc} the subgroup of S generated by unipotents. This is a semisimple group with no compact simple factors. We denote by S_c the maximal compact normal subgroup of S . We obtain a finite central extension $S_{\text{nc}} \times S_c \rightarrow S$. Upon replacing S with this central extension, replacing Γ with its preimage and replacing μ with an arbitrary lift, we assume as we may that $S = S_{\text{nc}} \times S_c$. Note that indeed, μ is still a Furstenberg measure on Γ .

We note that S has a canonical real algebraic structure and every continuous linear representation of it is automatically real algebraic. By the Borel density theorem, the image of Γ is Zariski dense in S_{nc} . Upon replacing S_c with the closure of the image of Γ , we assume as we may that this image is dense. We thus find that Γ is Zariski dense in S .

We let $P < S$ be a minimal parabolic and note that $P = P_{\text{nc}} \times S_c$, where P_{nc} is a minimal parabolic in S_{nc} . Thus $S/P \simeq S_{\text{nc}}/P_{\text{nc}}$. Endowing this space with the Haar measure class, it is Γ -isomorphic to $\Pi(\Gamma, \mu)$ as a Lebesgue space to, by the assumption that μ is a Furstenberg measure.

4.1. The abelian case

We start by proving the following special case.

Proposition 4.2 (cf. [21]). *Theorem 4.1 holds when U is abelian.*

This formulation is merely a slight generalization of Furstenberg’s original result, yet this generalization is essential to our discussion, so we give here its full proof. We will need the following definition.

Definition 4.3. Let k be a field and let \mathbf{L} be a k -algebraic group. An algebraic subgroup \mathbf{H} defined over k is said to be *epimorphic* in \mathbf{L} if for any k -representation V of \mathbf{L} , a vector $v \in V$ is fixed by \mathbf{L} as long as it is fixed by \mathbf{H} .

Any k -parabolic subgroup \mathbf{P} of a connected k -algebraic group \mathbf{L} is epimorphic. To see that, suppose $\rho : \mathbf{L} \rightarrow \text{GL}(V)$ is a representation and that $v \in V$ is \mathbf{P} -fixed. Thus the

orbit map $\mathbf{L} \rightarrow V$ given by $g \mapsto \rho(g)v$ factorizes through a map $\mathbf{L}/\mathbf{P} \rightarrow V$. But this map must be constant as \mathbf{L}/\mathbf{P} is a projective variety, meaning that v is \mathbf{L} -fixed.

The proof of Proposition 4.2 is based on the following measure rigidity theorem due to Mozes.

Theorem 4.4 ([39, Theorem 2]). *Let \mathbf{G} be an \mathbb{R} -algebraic group, let $\mathbf{L} \leq \mathbf{G}$ be an \mathbb{R} -algebraic subgroup generated by connected unipotent \mathbb{R} -subgroups and let $\mathbf{H} < \mathbf{L}$ be a connected epimorphic \mathbb{R} -algebraic subgroup of \mathbf{L} . Consider a discrete subgroup Γ of $\mathbf{G}(\mathbb{R})$. Then any probability measure on $\mathbf{G}(\mathbb{R})/\Gamma$ is $\mathbf{L}(\mathbb{R})$ -invariant as long as it is $\mathbf{H}(\mathbb{R})$ -invariant.*

Proof of Proposition 4.2. In this case $U \cong \mathbb{R}^m$ and $\text{Ch}(H) \cong U/H$ is a torus which we denote by X . Let $\nu \in \text{Prob}(X)^\mu$ and consider the corresponding boundary map given by Theorem 2.17,

$$\Theta : S/P \simeq S_{\text{nc}}/P_{\text{nc}} \simeq \Pi(\Gamma, \mu) \rightarrow \text{Prob}(X).$$

We will show that Θ is essentially constant, thus ν is Γ -invariant (Corollary 2.18).

We let ds be the Haar measure on S , normalized so that it induces a probability measure on S/Γ . We consider the quotient map $q : S \rightarrow S/P \simeq S_{\text{nc}}/P_{\text{nc}}$ and set $\theta := \Theta \circ q$. This is an a.e. defined Γ -equivariant measurable map.

We consider the space $S \times_\Gamma X$, defined as $S \times X$ quotiented by the Γ -action $\gamma.(s, x) = (s\gamma^{-1}, \gamma x)$, and we let S act on this space by left multiplication on the first coordinate. We define the measure $\tilde{\sigma}$ on $S \times X$ by

$$\tilde{\sigma} = \int_S \delta_s \times \theta_{s^{-1}} ds.$$

It is easy to see that $\tilde{\sigma}$ is both Γ -invariant, for the action considered above, and P -invariant, for the restriction of the left S -action. Hence $\tilde{\sigma}$ descends to a P -invariant probability measure σ on $S \times_\Gamma X$.

We define $G = S \times U$. Since every linear representation of S is real algebraic, we find that G is a real algebraic group. We set $\Lambda = \Gamma \times H$ and notice that this is a lattice in G . In fact, we may identify

$$G/\Lambda = (S \times U)/\Lambda \simeq ((S \times U)/H)/\Gamma \simeq (S \times X)/\Gamma = S \times_\Gamma X.$$

Using this identification, we endow G/Λ with the measure σ constructed above.

We are now in the setting of Theorem 4.4: G is a real algebraic group, $\Lambda \leq G$ is discrete, S_{nc} is a connected real algebraic subgroup of G generated by unipotents, and P_{nc} is an epimorphic subgroup of S_{nc} that fixes σ . We conclude that σ is S_{nc} -invariant. In turn, $\tilde{\sigma}$ is S_{nc} -invariant. Since for $t \in S_{\text{nc}}$ we have

$$t.\tilde{\sigma} = \int_S \delta_{ts} \times \theta_{s^{-1}} ds = \int_S \delta_s \times \theta_{s^{-1}t} ds.$$

The S_{nc} -invariance of $\tilde{\sigma}$ implies that $\theta_{s^{-1}t} = \theta_{s^{-1}}$ almost surely, meaning that θ , and therefore also Θ , is S_{nc} -invariant. It follows that $\Theta : S_{\text{nc}}/P_{\text{nc}} \rightarrow \text{Prob}(X)$ is essentially constant, as desired. ■

4.2. Proof of Theorem 4.1

The path from the abelian case to the nilpotent case is based on central induction of characters.

Definition 4.5. The *FC-center* of a group H is the union of all of its finite conjugacy classes, and is denoted by $\text{FC}(H)$. The group H is called *centrally inductive* if every character φ , viewed as a faithful character of $H/\ker \varphi$, is supported on $\text{FC}(H/\ker \varphi)$.

The FC-center is a characteristic subgroup that plays an important role in the character theory of countable groups (see for example the recent result in [9]), and in greater extent for nilpotent groups:

Theorem 4.6 ([16, 27]). *Any finitely generated nilpotent group is centrally inductive.*

The proof of Theorem 4.1 is by induction on the dimension of U . Let ν be an ergodic measure in $\text{Prob}(\text{Ch}(H))^\mu$. Consider the map $k : \text{Ch}(H) \rightarrow \text{Sub}(Z)$ that takes each character to the intersection of its kernel with the center Z of H . Then $k_*\nu$ is an ergodic μ -stationary probability measure on $\text{Sub}(Z)$. But $\text{Sub}(Z)$ is countable and therefore by Lemma 2.14 (applied for $X = \text{Sub}(Z)$ and Y being a singleton) $k_*\nu$ must be Γ -invariant, thus by ergodicity, a uniform measure on some finite orbit $\mathcal{D} \subseteq \text{Sub}(Z)$.

Case a: \mathcal{D} contains a non-trivial subgroup. Fix a non-trivial subgroup $K \in \mathcal{D}$. By the assumptions on U , the exponential map $\exp : \text{Lie}(U) \rightarrow U$ is bijective. Let \tilde{K} denote the Lie subgroup of U corresponding to the \mathbb{R} -span of $\log(K)$. The group Z is finitely generated free abelian [47, Theorem 2.18] so that \tilde{K} is infinite. It follows that \tilde{K} is a Lie subgroup of the center of U of positive dimension. Moreover, K is a finite index subgroup of $H \cap \tilde{K}$ and is a lattice in \tilde{K} .

Let Γ' be the kernel of the action of Γ on \mathcal{D} , endowed with the hitting measure μ' . By Proposition 2.20, μ' is a Furstenberg measure. As K is Γ' -invariant so is \tilde{K} [47, Theorem 2.11]. By the Zariski density of Γ' in S , \tilde{K} is also S -invariant and therefore S acts on $U_0 := U/\tilde{K}$ by continuous automorphisms. Now $H_0 := H/(H \cap \tilde{K})$ is discrete in U_0 and moreover a lattice [47, Theorem 2.1]. By induction hypothesis, $\text{Ch}(H_0)$ is μ' -stiff. Since this holds for each $K \in \mathcal{D}$, we may apply Proposition 2.22 to the map $k : \text{supp } \nu \rightarrow \mathcal{D}$ to conclude that the action of Γ on $\text{supp } \nu$ is μ -stiff. It follows that ν is Γ -invariant.

Case b: \mathcal{D} consists only of the trivial subgroup. A non-trivial normal subgroup of a nilpotent group intersects the center non-trivially. Thus, in this case, ν -a.e. character of H is faithful. Since H is a lattice in U , it is nilpotent, finitely generated and torsion-free [47, Theorem 2.18]. Therefore it is centrally inductive, and its FC-center is its center, Z (see [48, Theorem 4.37]). Thus ν -a.e. $\chi \in \text{Ch}(H)$ is supported on Z . Writing $\varphi = \mathcal{F}_H(\nu)$ for the corresponding trace (Theorem 3.1), we see that φ is supported on Z :

$$\forall h \in H \setminus Z : \quad \varphi(h) = \int_{\text{Ch}(H)} \chi(h) d\nu(\chi) = 0. \quad (4.1)$$

The pushforward measure $r_*\nu$ under the Γ -equivariant restriction map $r : \text{Ch}(H) \rightarrow \text{Ch}(Z) \simeq \hat{Z}$ is a μ -stationary probability measure. Now Z is a lattice in the center of U [47, Proposition 2.17]. Since Z is Γ -invariant and the center of U is S -invariant, Proposition 4.2 implies that $r_*\nu$ is Γ -invariant. The trace corresponding to $r_*\nu$ via the Fourier transform is $\varphi \upharpoonright_Z$ and so we see that

$$\forall \gamma \in \Gamma, h \in Z : \quad \varphi^\gamma(h) = \varphi(h). \tag{4.2}$$

From (4.1) and (4.2) we see that φ , and therefore ν , is Γ -invariant.

4.3. Proof of Theorem D

By pulling back via a central extension, replacing Λ with its preimage and μ with an arbitrary lift (which is necessarily still a Furstenberg measure), we assume as we may that G is algebraically simply connected. We may also assume that G is connected due to Lemma 2.21. We consider an action of Λ on a finitely generated virtually nilpotent group Γ . By Corollary 3.13 we assume as we may that Γ is actually nilpotent. Recall that for a finitely generated nilpotent group the set of torsion elements is a characteristic finite subgroup. Applying again Corollary 3.13, we assume as we may that Γ is torsion-free.

There exists a simply connected nilpotent Lie group U that has Γ embedded as a lattice [47, Theorem 2.18]. The group of continuous automorphisms $\text{Aut}(U)$ is a real algebraic subgroup of $\text{GL}(\text{Lie}(U))$.

If the action is finite, then the statement holds by Lemma 2.9. Otherwise, the image of the action map $\rho : \Lambda \rightarrow \text{Aut}(U)$ is unbounded in $\text{Aut}(U)$ because $\text{Aut}(U)$ is discrete. Margulis’s super-rigidity theorem [38, Theorem 16.1.4] then guarantees the existence of a finite index subgroup $\Lambda' \leq \Lambda$ and a continuous representation $\hat{\rho} : G \rightarrow \text{Aut}(U)$ such that $\hat{\rho} \upharpoonright_{\Lambda'} = \rho$. By Theorem 4.1, Λ' acts μ' -stiffly on $\text{Ch}(\Gamma)$, where μ' is the hitting measure coming from μ . It follows by Lemma 2.21 that the action of Λ on $\text{Ch}(\Gamma)$ is μ -stiff.

4.4. The amenable radical of arithmetic groups

Let Λ be an arithmetic subgroup of a \mathbb{Q} -algebraic group \mathbf{G} . For a \mathbb{Q} -algebraic subgroup \mathbf{H} of \mathbf{G} denote $\Lambda_{\mathbf{H}} := \Lambda \cap \mathbf{H}(\mathbb{Q})$. Then $\Lambda_{\mathbf{H}}$ is clearly an arithmetic subgroup of \mathbf{H} . If $\bar{\mathbf{G}}$ is a quotient of \mathbf{G} by a \mathbb{Q} -algebraic normal subgroup, denote by $\Lambda_{\bar{\mathbf{G}}}$ the image of Λ under the quotient map $\mathbf{G} \rightarrow \bar{\mathbf{G}}$. Then $\Lambda_{\bar{\mathbf{G}}}$ is an arithmetic subgroup of $\bar{\mathbf{G}}$ [46, Theorem 4.1].

Lemma 4.7. *Let Λ be an arithmetic subgroup of a \mathbb{Q} -algebraic semisimple group \mathbf{G} . Then $\text{Rad}(\Lambda)$ is finite, and it is moreover trivial assuming that \mathbf{G} has no finite normal subgroups, and no \mathbb{Q} -simple \mathbb{R} -anisotropic factors.*

Proof. Assume first that \mathbf{G} has no \mathbb{Q} -simple \mathbb{R} -anisotropic factors, in which case Λ is Zariski dense in $\mathbf{G}(\mathbb{R})$ by the Borel density theorem. Let \mathbf{H} denote the \mathbb{R} -Zariski closure of $\text{Rad}(\Lambda)$ in \mathbf{G} . Then $\mathbf{H}(\mathbb{R})$ is an amenable normal subgroup of $\mathbf{G}(\mathbb{R})$ and thus must

be compact [50, Section 4.1]. But $\text{Rad}(\Lambda)$ is discrete and so it must be finite, making \mathbf{H} finite as well. Hence, if \mathbf{G} is in addition assumed to have no finite normal subgroups, then $\text{Rad}(\Lambda)$ is trivial.

Now allow \mathbf{G} to have \mathbb{Q} -simple \mathbb{R} -anisotropic factors. Let \mathbf{G}_c denote the product of all \mathbb{R} -anisotropic \mathbb{Q} -simple factors and set $\bar{\mathbf{G}} := \mathbf{G}/\mathbf{G}_c$. Then the image $\bar{\Lambda}$ of Λ under this quotient is an arithmetic subgroup of $\bar{\mathbf{G}}$. By the previous paragraph, $\text{Rad}(\bar{\Lambda})$ is finite and therefore so is $\overline{\text{Rad}(\Lambda)}$ as it is clearly contained in $\text{Rad}(\bar{\Lambda})$. But $\Lambda_{\mathbf{G}_c}$ is finite as it is a discrete subgroup of the compact group $\mathbf{G}_c(\mathbb{R})$, so it follows that $\text{Rad}(\Lambda)$ is finite. ■

We now consider a general arithmetic subgroup Λ of some \mathbb{Q} -algebraic group \mathbf{G} . Let \mathbf{R} be the solvable radical of \mathbf{G} , and consider the semisimple group $\mathbf{S} = \mathbf{G}/\mathbf{R}$. Observe that the size of any finite normal subgroup of \mathbf{S} is bounded from above by $|\mathbf{Z}| \cdot |\mathbf{S} : \mathbf{S}^\circ|$ where \mathbf{S}° is the identity component of \mathbf{S} and \mathbf{Z} is the (finite) center of \mathbf{S}° . It follows that \mathbf{S} admits a unique maximal finite normal subgroup, which coincides with the center when \mathbf{S} is connected. Let $\bar{\mathbf{S}}$ be the quotient of \mathbf{S} by all \mathbb{R} -anisotropic \mathbb{Q} -simple factors, as well as by the remaining maximal finite normal subgroup. We deduce the following facts regarding the arithmetic subgroup $\Lambda_{\mathbf{R}}$ of \mathbf{R} and $\Lambda_{\bar{\mathbf{S}}}$ of $\bar{\mathbf{S}}$.

Proposition 4.8. *Let Λ be an arithmetic subgroup of \mathbf{G} . Then:*

- (1) $\Lambda_{\mathbf{R}}$ is a finite index subgroup of $\text{Rad}(\Lambda)$.
- (2) $\Lambda_{\bar{\mathbf{S}}}$ is equal to $\Lambda/\text{Rad}(\Lambda)$.

Proof. It is clear that $\Lambda_{\mathbf{R}}$ is contained in $\text{Rad}(\Lambda)$. Since $\Lambda_{\mathbf{S}} = \Lambda/\Lambda_{\mathbf{R}}$ is an arithmetic subgroup of the semisimple group \mathbf{S} , $\text{Rad}(\Lambda_{\mathbf{S}})$ is finite by Lemma 4.7. Item (1) follows as $\text{Rad}(\Lambda)/\Lambda_{\mathbf{R}}$ is contained in the finite group $\text{Rad}(\Lambda/\Lambda_{\mathbf{R}}) = \text{Rad}(\Lambda_{\mathbf{S}})$.

For (2), consider the quotient map $\sigma : \mathbf{G} \rightarrow \bar{\mathbf{S}}$, and let $K \leq \Lambda$ be the kernel of $\sigma \upharpoonright_{\Lambda}$. As $\sigma(\text{Rad}(\Lambda))$ is a normal amenable subgroup of the arithmetic subgroup $\Lambda_{\bar{\mathbf{S}}} = \sigma(\Lambda)$ of $\bar{\mathbf{S}}$, Lemma 4.7 shows that $\sigma(\text{Rad}(\Lambda))$ is trivial, meaning that $\text{Rad}(\Lambda) \subseteq K$. But K is clearly normal and amenable and so actually $\text{Rad}(\Lambda) = K$. It follows that $\Lambda/\text{Rad}(\Lambda) = \Lambda_{\bar{\mathbf{S}}}$. ■

As any solvable arithmetic group is polycyclic [35], we get the following consequence:

Corollary 4.9. *The amenable radical of an arithmetic group is virtually polycyclic.*

In order to account for non-connected algebraic groups in Theorem C, we will need the following lemma:

Lemma 4.10. *If Λ_1, Λ_2 are two commensurable subgroups of a group Λ , then $\text{Rad}(\Lambda_1)$ and $\text{Rad}(\Lambda_2)$ are commensurable as well.*

Proof. For two subgroups H_1 and H_2 of Λ write $H_1 \lesssim H_2$ if H_1 has a finite index subgroup which is contained in H_2 . We make the following observations:

- (1) \lesssim is a transitive relation.
- (2) H_1 and H_2 are commensurable if and only if $H_1 \lesssim H_2$ and $H_2 \lesssim H_1$.

- (3) If H_1 is a finite index subgroup of H_2 then $\text{Rad}(H_2) \lesssim \text{Rad}(H_1)$.
- (4) If H_1 is a finite index normal subgroup of H_2 then $\text{Rad}(H_1) \leq \text{Rad}(H_2)$.

The last property uses the fact that the amenable radical of a group is a characteristic subgroup. Now, to prove the lemma it is enough to consider the case where one of the subgroups, say Λ_1 , is a finite index subgroup in the other, Λ_2 . Set $\Lambda'_1 = \bigcap_{g \in \Lambda_2} g^{-1} \Lambda_1 g$. By the four observations above we deduce that

$$\text{Rad}(\Lambda_2) \lesssim \text{Rad}(\Lambda_1) \lesssim \text{Rad}(\Lambda'_1) \lesssim \text{Rad}(\Lambda_2)$$

so that $\text{Rad}(\Lambda_1)$ and $\text{Rad}(\Lambda_2)$ are commensurable. ■

4.5. Proof of Theorem C

The following lemma and its proof can be found in a more general form in [16]:

Lemma 4.11 ([16, Lemma 2.2]). *Let H be a discrete group, $N \leq H$ a normal subgroup, and assume that $A := H/N$ is abelian. The continuous action of \hat{A} on $\text{Ch}(H)$ given by pointwise multiplication, $(\chi, \varphi) \mapsto \chi \cdot \varphi$, preserves the fibers of the restriction map $\text{Ch}(H) \rightarrow \text{Ch}_H(N)$ and acts on each of them transitively.*

Let \mathbf{G} be an arbitrary \mathbb{Q} -algebra group, and let Λ be an arithmetic subgroup of \mathbf{G} . To show the statement of Theorem C, namely the stiffness of the action $\Lambda \curvearrowright \text{Ch}(\text{Rad}(\Lambda))$, it is enough to consider the case when \mathbf{G} is connected. Indeed, denote by $\mathbf{G}^\circ \leq \mathbf{G}$ the connected component of the identity and set $\Lambda_0 = \Lambda \cap \mathbf{G}^\circ(\mathbb{Q})$. By Lemma 2.21, it suffices to show that the action $\Lambda_0 \curvearrowright \text{Ch}(\text{Rad}(\Lambda))$ is stiff (with respect to the hitting measure). By Lemma 4.10, the arithmetic subgroups Λ and Λ_0 of $\mathbf{G}(\mathbb{Q})$ have commensurable amenable radicals, and so by Corollary 3.13 it is enough to show that the action $\Lambda_0 \curvearrowright \text{Ch}(\text{Rad}(\Lambda_0))$ is stiff. The non-connected case is therefore reduced to the connected case.

Thus, assume that \mathbf{G} is a connected \mathbb{Q} -algebraic group with solvable radical \mathbf{R} , unipotent radical \mathbf{U} and semisimple part \mathbf{S} . By general theory, \mathbf{R}/\mathbf{U} is abelian and is centralized by \mathbf{S} . Let Λ be an arithmetic subgroup of \mathbf{G} and let μ be a Furstenberg measure on Λ . We use the notations introduced in Section 4.4. By the classical theorem of Borel and Harish-Chandra, Λ_S is a lattice in the real semisimple Lie group $S = \mathbf{S}(\mathbb{R})$. We denote by $\bar{\mu}$ the measure on Λ_S pushed from μ . Clearly, it is a Furstenberg measure. By Mal'tsev theory, Λ_U is a lattice in the simply connected nilpotent Lie group $U = \mathbf{U}(\mathbb{R})$. Thus, by Theorem 4.1, the action of Λ_S on $\text{Ch}(\Lambda_U)$ is stiff and by Proposition 3.12 so is its action on $\text{Ch}_{\Lambda_R}(\Lambda_U)$. Consider the restricting map $r : \text{Ch}(\Lambda_R) \rightarrow \text{Ch}_{\Lambda_R}(\Lambda_U)$. The group $A := \Lambda_R/\Lambda_U$ is abelian and it is centralized by Λ_S . Hence by Lemma 4.11, the compact group \hat{A} acts transitively on each fiber of r , commuting with the action of Λ_S . Using Proposition 2.15 with $X = \text{Ch}(\Lambda_R)$, $Y = \text{Ch}_{\Lambda_R}(\Lambda_U)$ and $K = \hat{A}$, we find that r is relatively stiff with respect to the Λ_S -action. We deduce by Lemma 2.13 that the Λ_S -action on $\text{Ch}(\Lambda_R)$ is stiff and, using Lemma 2.10, we find that the Λ -action on $\text{Ch}(\Lambda_R)$ is stiff. Finally, using Proposition 4.8, we conclude by Corollary 3.13 that the Λ -action on $\text{Ch}(\text{Rad}(\Lambda))$ is stiff.

5. Charmenability of arithmetic groups

5.1. A criterion for charmenability

Definition 5.1. Let Λ be a countable group and let Ω be a Lebesgue Λ -space.

- (1) Ω is called *metrically ergodic* if for any action of Λ by isometries on a separable metric space Z , any Lebesgue Λ -map $\Omega \rightarrow Z$ is essentially constant.
- (2) Ω is called *Zimmer amenable* if for any Borel Λ -space W with an essentially surjective Borel Λ -map $p : W \rightarrow \Omega$ whose fibers admit compact convex structures on which Λ acts affinely, there exists a Lebesgue Λ -map which is a section of p .
- (3) Ω is called a Λ -*boundary* if it is both metrically ergodic and Zimmer amenable.

The following is a strengthening of a theorem by Kaimanovich [30].

Theorem 5.2 ([5, Theorem 2.7]). *Let Λ be a countable group and let μ be an admissible probability measure on Λ . Then the Furstenberg–Poisson boundary $\Pi(\Lambda, \mu)$ is a Λ -boundary.*

We will make use of the following simple lemma:

Lemma 5.3. *Let Λ be a countable group, let $N \triangleleft \Lambda$ be a normal amenable subgroup and set $\Gamma = \Lambda/N$. If B is a Γ -boundary then it is also a Λ -boundary.*

Proof. We show that B is metrically ergodic as a Λ -space assuming it is so as a Γ -space (in fact regardless of whether N is amenable), and that B is Zimmer amenable as a Λ -space assuming it is so as a Γ -space.

Suppose Λ acts by isometries on a separable metric space Z , and let $f : B \rightarrow Z$ be a Lebesgue Λ -map. Since N acts trivially on B , the essential image of f is supported on N -fixed points of Z . We may therefore view f as a Lebesgue Γ -map $B \rightarrow Z^N$. By metric ergodicity of B as a Γ -space, f must be essentially constant. This shows that B is metrically ergodic as a Λ -space.

Consider now a Borel Λ -space W with an essentially surjective Borel Λ -map $p : W \rightarrow B$ whose fibers admit compact convex structures on which Λ acts affinely. Each fiber of p is N -invariant, and therefore has N -fixed points. It follows that the map $p \upharpoonright_{W^N} : W^N \rightarrow B$ is an essentially surjective Borel Γ -map with compact convex fibers on which Γ acts affinely. By Zimmer amenability of B as a Lebesgue Γ -space, there exists a Lebesgue Γ -map which is a section of $p \upharpoonright_{W^N}$, and this map is obviously also a Lebesgue Λ -map and a section of p . ■

The proof of Theorem A is based on a criterion for charmenability established in [4]. We shall first state it, and then explain the notions involved.

Criterion 5.4 ([4, Proposition 4.20]). *Let Λ be a countable group and define $\Gamma := \Lambda/\text{Rad}(\Lambda)$. Assume that Γ admits a boundary B satisfying the following two conditions:*

- (1) *Every separable, ergodic, faithful $(\Gamma, L^\infty(B))$ -von Neumann algebra (M, E) is either invariant or Γ -singular.*

(2) Every Lebesgue Λ -map $B \rightarrow \text{PD}_1(\text{Rad}(\Lambda))$ is essentially constant.

Then Λ is charmenable.

Note that when Λ is arithmetic, the first condition of the criterion deals only with the semisimple part. The fact that it holds in the higher rank setting is essentially established in [4, 15]; see Proposition 5.7 below. As the main focus of this work is in cases where the amenable radical is non-trivial, it is the second condition that is of our main interest. We shall nevertheless briefly clarify the first condition, but we emphasize that the notions involved do not play an important conceptual role in this work. See [4] for further details.

Definition 5.5. Let A be a C^* -algebra, and let $\mathcal{S}(A)$ denote the space of states on A . Then $\phi_1, \phi_2 \in \mathcal{S}(A)$ are called *singular* if $\|\phi_1 - \phi_2\| = 2$.

Given a Lebesgue space Ω , two Lebesgue maps $\theta_1, \theta_2 : \Omega \rightarrow \mathcal{S}(A)$ are called *singular* if $\theta_1(\omega)$ and $\theta_2(\omega)$ are singular for a.e. $\omega \in \Omega$.

Two unital completely-positive maps $E_1, E_2 : A \rightarrow L^\infty(\Omega)$ are called *singular* if the corresponding dual maps $\theta_1, \theta_2 : \Omega \rightarrow \mathcal{S}(A)$ (given by $\theta_i(\omega)(a) = E_i(a)(\omega)$) are singular.

Given a separable von Neumann algebra M , two normal unital completely-positive maps $E_1, E_2 : M \rightarrow L^\infty(\Omega)$ are called *singular* if M admits a separable unital C^* -subalgebra A such that the restrictions $E_1 \upharpoonright_A, E_2 \upharpoonright_A$ are singular.

Definition 5.6. Let Λ be a countable group. A Λ -von Neumann algebra is a von Neumann algebra N endowed with an ultraweak continuous action of Λ by automorphisms. It is said to be *ergodic* if the subalgebra N^Λ of fixed points is trivial (i.e. $= \mathbb{C}$).

Given a Λ -von Neumann algebra N , a (Λ, N) -von Neumann algebra is a pair (M, E) where M is a Λ -von Neumann algebra, and E is a unital completely-positive Λ -equivariant map $E : M \rightarrow N$; (M, E) is said to be *faithful* or *extremal* if E has the corresponding properties. If $E(M) \subset N^\Lambda$, we say that (M, E) is Λ -invariant.

Assume $N = L^\infty(\Omega)$ where Ω is a Lebesgue Λ -space. Then E is said to be Λ -singular if the normal unital completely-positive maps $E_g : M \ni x \mapsto E(gx) \in L^\infty(\Omega)$ are pairwise singular.

The only important example for our concern is when N is the commutative von Neumann algebra of L^∞ -functions on the Furstenberg–Poisson boundary of Λ with respect to some measure μ . In that case (Λ, N) -von Neumann algebras may be seen as a non-commutative version of the boundary maps appearing in Theorem 2.17.

5.2. Proof of Theorem A

The first condition of Criterion 5.4 holds for arithmetic subgroups of semisimple groups of higher rank:

Proposition 5.7. Let Γ be an arithmetic subgroup of a connected \mathbb{Q} -algebraic group \mathbf{G} , and let B be its Furstenberg–Poisson boundary with respect to a Furstenberg measure μ . Assume that \mathbf{G} is \mathbb{Q} -simple, center-free, without \mathbb{R} -anisotropic \mathbb{Q} -simple factors, and with \mathbb{R} -rank at least 2. Then the first condition of Criterion 5.4 holds true.

Proof. Let $G = \mathbf{G}(\mathbb{R})$. By Lemma 4.7, $\text{Rad}(\Gamma)$ is trivial. The intersection of Γ with any compact simple factor of G is central, hence trivial. We may therefore assume that G has no compact simple factors. Let (M, E) be a separable, ergodic, faithful $(\Gamma, L^\infty(B))$ -von Neumann algebra which is not Γ -invariant. We explain how to deduce singularity of E from the work of [15] or of [4], depending on whether G is simple or a product of a few simple factors.

Assume first G is simple. Let P be a minimal parabolic subgroup of G so that $B = G/P$. Then by [15, Theorem B] there is a proper closed subgroup $P \leq Q \not\leq G$ and a unital normal Γ -equivariant $*$ -embedding $\iota : L^\infty(G/Q) \rightarrow M$ such that $E \circ \iota$ is equal to the natural embedding $p_Q^* : L^\infty(G/Q) \rightarrow L^\infty(G/P)$. Note that the theorem in this reference is formalized in a somewhat dual way, though it is equivalent (see [26, Theorem E]). Let $M_0 = \iota(L^\infty(G/Q))$ and consider $(M_0, E \upharpoonright_{M_0})$ as a $(G, L^\infty(G/P))$ -von Neumann algebra. It is not G -invariant as Q is proper, and it is extremal because M_0 is ergodic (see [4, Lemma 4.16]). Due to [4, Example 2.14, Lemma 6.2], the singularity criterion of [4, Proposition 4.15] applies so that $(M_0, E \upharpoonright_{M_0})$ is G -singular. It is therefore in particular Γ -singular. Since M contains M_0 as a subalgebra, it follows that (M, E) is singular as well (see [4, Remark 4.12]).

Now assume G has at least two (non-compact) simple factors. Then, in the terminology of [4], Γ is an arithmetic group of product type and is therefore charmenable by [4, Proposition 6.1]. More specifically, it is transparent that the proof of this proposition all boils down to showing that E and E_γ are singular for any non-central element $\gamma \in \Gamma$. But in our case Γ is center-free, and so E is Γ -singular. ■

The dynamical study of Section 4, which eventually led to the proof of Theorem C, was aimed towards the establishment of the second condition of Criterion 5.4 for arbitrary arithmetic groups:

Proposition 5.8. *Let Λ be an arithmetic group and let (B, ν_B) be its Furstenberg–Poisson boundary with respect to a Furstenberg measure μ . Then the second condition of Criterion 5.4 holds true.*

Proof. Let $\theta : B \rightarrow \text{PD}_1(\text{Rad}(\Lambda))$ be a Lebesgue Λ -map. Since $\text{Rad}(\Lambda)$ acts trivially on B , the essential image of θ consists of traces, namely, we may view θ as a Lebesgue Λ -map $\theta : B \rightarrow \text{Tr}(\text{Rad}(\Lambda))$. By pushing forward ν_B through θ and taking its barycenter, we get a μ -stationary trace $\varphi \in \text{Tr}(\text{Rad}(\Lambda))^\mu$. The μ -stationary probability measure on $\text{Ch}(\text{Rad}(\Lambda))$ corresponding to φ (Theorem 3.1) is Λ -invariant by Theorem C. It follows that φ is Λ -invariant, which means that θ is essentially constant by Corollary 2.18. ■

Proof of Theorem A. Let \mathbf{G} be a \mathbb{Q} -algebraic group with solvable radical \mathbf{R} and $\mathbf{S} = \mathbf{G}/\mathbf{R}$, and let Λ_S be the image of the arithmetic subgroup Λ under the quotient map $\mathbf{G} \rightarrow \mathbf{S}$. We first observe that if \mathbf{S} has more than one \mathbb{R} -isotropic \mathbb{Q} -simple factor then Λ_S has a normal subgroup which is neither amenable nor co-amenable. Also, if the real rank of \mathbf{S} is 1 then Λ_S has such a normal subgroup, as it is relatively hyperbolic. In both cases we see that Λ admits a normal subgroup which is neither amenable nor co-amenable, which implies

that Λ is not charmenable [4, Propositions 3.3]. This shows that the conditions imposed on \mathbf{S} in the statement of the theorem are necessary for charmenability of Λ .

We now turn to showing that they are sufficient as well. Note that as Λ is an arbitrary arithmetic subgroup of \mathbf{G} , this will in particular show that either all arithmetic subgroups of \mathbf{G} are charmenable, or none is.

If the real rank of \mathbf{S} is 0 then $\mathbf{S}(\mathbb{R})$ is compact, in which case $\mathbf{G}(\mathbb{R})$ is amenable, hence Λ is amenable and a fortiori charmenable. We thus assume from now on that \mathbf{S} has at most one \mathbb{R} -isotropic \mathbb{Q} -simple factor and that the real rank of \mathbf{S} is at least 2, and we argue that Λ is charmenable. We will show that it satisfies the first condition of Criterion 5.4 (the second condition holds in general by Proposition 5.8).

Let $\bar{\mathbf{S}}$ denote the quotient of \mathbf{S} by all \mathbb{R} -anisotropic \mathbb{Q} -simple factors, as well as by the remaining center. Let $\Lambda_{\bar{\mathbf{S}}}$ denote the image of Λ under the quotient map $\mathbf{G} \rightarrow \bar{\mathbf{S}}$. Fix a Furstenberg measure μ on Λ and let $\bar{\mu}$ be the pushforward measure onto $\Lambda_{\bar{\mathbf{S}}}$, which is a Furstenberg measure as well. The Furstenberg–Poisson boundary of Λ with respect to μ is the same as the Furstenberg–Poisson boundary $\Lambda_{\bar{\mathbf{S}}}$ with respect to $\bar{\mu}$ and we denote it by B . By Theorem 5.2, B is a $\Lambda_{\bar{\mathbf{S}}}$ -boundary and therefore also a Λ -boundary by Lemma 5.3. By Proposition 4.8, we have $\Lambda/\text{Rad}(\Lambda) = \Lambda_{\bar{\mathbf{S}}}$. In particular, this is an arithmetic subgroup of $\bar{\mathbf{S}}$. This first condition of Criterion 5.4 therefore follows from Proposition 5.7. ■

6. Properties of charmenable groups

This section is dedicated to proving Propositions 1.3 and 1.4 from the introduction. Here too, Λ denotes a countable group.

6.1. Amenable traces

Let $\pi : \Lambda \rightarrow \mathcal{U}(\mathcal{H})$ be a representation. A sequence of vectors $v_n \in \mathcal{H}$ of norm 1 is said to be *almost Λ -invariant* if

$$\forall g \in \Lambda : \quad \|\pi(g)v_n - v_n\| \rightarrow 0.$$

If such a sequence exists, we say that π has *almost invariant vectors*. The representation π is called *amenable* if $\pi \otimes \bar{\pi}$ has almost invariant vectors, where $\bar{\pi}$ is the dual representation. See [7] for more on this notion, including properties and equivalent definitions. A trace φ on Λ is called *amenable* if its GNS representation is amenable. This should not be confused with the stronger notion of a von Neumann amenable trace given in Definition 3.8. The second condition in the definition of charmenability implies the dichotomy stated on traces:

Proposition 6.1 ([4, Proposition 3.2]). *Assume that every character of Λ is either von Neumann amenable or supported on $\text{Rad}(\Lambda)$. Then any trace of Λ is either amenable or supported on $\text{Rad}(\Lambda)$.*

Note that a representation π of a product of groups $\Gamma_1 \times \Gamma_2$ may have no almost invariant vectors even if both restrictions $\pi \upharpoonright_{\Gamma_1}, \pi \upharpoonright_{\Gamma_2}$ do. Indeed, if π_i is a representation of Γ_i without almost invariant vectors then we may extend each π_i to $\Gamma_1 \times \Gamma_2$ trivially, and consider $\pi = \pi_1 \oplus \pi_2$. The following proposition shows that this is the only obstruction.

Proposition 6.2. *Suppose that Γ_1, Γ_2 are two subgroups of Λ which commute with each other and satisfy $\Lambda = \Gamma_1\Gamma_2$. Let $\pi : \Lambda \rightarrow \mathcal{U}(\mathcal{H})$ be a representation of Λ . Assume that the restrictions $\pi \upharpoonright_{\Gamma_1}$ and $\pi \upharpoonright_{\Gamma_2}$ have almost invariant vectors, whereas π does not. Then π decomposes as a direct sum $\pi = \pi_1 \oplus \pi_2$ such that $\pi_1 \upharpoonright_{\Gamma_1}$ and $\pi_2 \upharpoonright_{\Gamma_2}$ have almost invariant vectors but $\pi_1 \upharpoonright_{\Gamma_2}$ and $\pi_2 \upharpoonright_{\Gamma_1}$ do not.*

Proof. We first recall a general fact about almost invariant vectors. Consider a countable group Γ and fix an admissible probability measure μ on Γ satisfying $\mu(\gamma^{-1}) = \mu(\gamma)$ for all $\gamma \in \Gamma$. Given a unitary representation $\rho : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$, there is the associated Markov operator

$$\mathcal{B}(\mathcal{H}) \ni A_\mu : v \mapsto \sum_{\gamma \in \Gamma} \mu(\gamma)\rho(\gamma)v.$$

Then A_μ is a positive contractive operator, and its norm is 1 if and only if ρ has almost invariant vectors [43, Proposition 7.1.4]. We apply this reasoning to Γ_1 and Γ_2 , that is, we fix generating probability measures on each of those subgroups, and consider the corresponding Markov operators denoted by A_1 and A_2 . Then 1 is in the spectrum of both A_1 and A_2 , and yet there is some $\varepsilon > 0$ such that

$$\max_{i=1,2} \|A_i v - v\| > \varepsilon, \quad \forall v \in \mathcal{H} \text{ with } \|v\| = 1. \tag{6.1}$$

For $i = 1, 2$, let $p_i \in \mathcal{B}(\mathcal{H})$ denote the spectral projection of A_i associated with the interval $[1 - \varepsilon, \varepsilon]$, and denote by $\mathcal{K}_i := p_i \mathcal{H} \subseteq \mathcal{H}$ the corresponding subspace. Then \mathcal{K}_i is an A_i -invariant subspace which contains almost Γ_i -invariant vectors, whereas its orthogonal complement \mathcal{K}_i^\perp does not. Moreover, it follows from (6.1) that \mathcal{K}_1 and \mathcal{K}_2 are orthogonal subspaces, i.e. $\mathcal{K}_2 \subseteq \mathcal{K}_1^\perp$. Finally, note that \mathcal{K}_1 is Γ_2 -invariant (and vice versa). Indeed, this can be seen as a consequence of the fact that von Neumann algebras are closed under taking spectral projections, so that p_1 is in the double commutant of A_1 and thus commutes with $\pi(\Gamma_2)$.

Thus for each $\gamma_1 \in \Gamma_1$, $\pi(\gamma_1)\mathcal{K}_1^\perp$ is a Γ_2 -invariant subspace that contains \mathcal{K}_2 . The intersection

$$\mathcal{H}_2 := \bigcap_{\gamma_1 \in \Gamma_1} \pi(\gamma_1)\mathcal{K}_1^\perp$$

is therefore a $\Gamma_1 \times \Gamma_2$ -invariant subspace which contains \mathcal{K}_2 , and as a result has almost Γ_2 -invariant vectors. However, \mathcal{H}_2 does not have almost Γ_1 -invariant vectors because it is contained in \mathcal{K}_1^\perp . The orthogonal complement $\mathcal{H}_1 := \mathcal{H}_2^\perp$ on the other hand has almost Γ_1 -invariant vectors because it contains \mathcal{K}_1 , but does not have almost Γ_2 -invariant vectors because it is contained in \mathcal{K}_2^\perp . The subrepresentations π_1, π_2 corresponding to the invariant subspaces $\mathcal{K}_1, \mathcal{K}_2$ therefore give the desired decomposition. ■

Recall that to any positive-definite function φ corresponds a GNS triplet (\mathcal{H}, π, ξ) . If φ is moreover a trace then π extends to a representation $\pi \times \rho : \Lambda \times \Lambda \rightarrow \mathcal{U}(\mathcal{H})$ such that $\pi(g)\xi = \rho(g^{-1})\xi$ for all $g \in G$ (see Example 3.15). In particular, $\varphi(g) = \langle \pi(g)\xi, \xi \rangle = \langle \xi, \rho(g)\xi \rangle$. We call ρ the *right GNS representation* of φ . To avoid confusion, we shall call π the *left GNS representation*.

Proposition 6.3. *Let φ be a trace of Λ , and let π and ρ be the left and right GNS representations of φ . If π has almost invariant vectors then so does $\pi \times \rho : \Lambda \times \Lambda \rightarrow \mathcal{U}(\mathcal{H})$.*

Proof. Let (\mathcal{H}, π, ξ) be a GNS triplet associated with φ . Observe that

$$\varphi(g) = \langle \xi, \rho(g)\xi \rangle = \langle \rho(g)^*\xi, \xi \rangle, \quad g \in \Lambda. \tag{6.2}$$

Thus π is equivalent to the adjoint representation $\bar{\rho}$, by the uniqueness of the GNS. Since π has almost invariant vectors, so does ρ . Assuming towards a contradiction that $\pi \times \rho$ has no almost invariant vectors implies, by Proposition 6.2, that there is a subrepresentation $\pi_1 \times \rho_1$ of $\pi \times \rho$ such that π_1 has almost invariant vectors but ρ_1 does not. Let \mathcal{H}_1 denote the subspace corresponding to this subrepresentation. Since ξ is cyclic with respect to π as well as to ρ , its projection onto \mathcal{H}_1 is non-zero and we denote its normalization by $\xi_1 \in \mathcal{H}_1$. It is not hard to see that ξ_1 must be cyclic with respect to the representations π_1 and ρ_1 .

Let φ_1 denote the positive-definite function corresponding to the GNS triplet $(\mathcal{H}_1, \pi_1, \xi_1)$. Then a similar computation to (6.2) shows that $(\mathcal{H}_1, \bar{\rho}_1, \xi_1)$ is a GNS triplet associated with φ_1 . By uniqueness of the GNS we conclude that π_1 and $\bar{\rho}_1$ are equivalent, and in particular ρ_1 has almost invariant vectors. This is a contradiction. ■

Proposition 6.4. *Let φ be a trace of Λ , and let π and ρ be the left and right GNS representations corresponding to φ . If φ is amenable then so is $\pi \times \rho : \Lambda \times \Lambda \rightarrow \mathcal{U}(\mathcal{H})$.*

Proof. The function $|\varphi|^2 = \varphi \cdot \bar{\varphi}$ is a trace on Λ . Indeed, $(\mathcal{H} \otimes \mathcal{H}, \pi \otimes \bar{\pi}, \xi \otimes \xi)$ is the corresponding GNS triplet, and $\rho \otimes \bar{\rho}$ is the right GNS. As φ is amenable, $\pi \otimes \bar{\pi}$ has almost invariant vectors, and so by Proposition 6.3 so does $(\pi \otimes \bar{\pi}) \times (\rho \otimes \bar{\rho})$. But the latter is canonically equivalent to $(\pi \times \rho) \otimes \overline{(\pi \times \rho)}$ so that $\pi \times \rho$ is amenable. ■

6.2. *Invariant random subgroups and p.m.p. actions*

For a group Λ denote by $\text{Sub}(\Lambda)$ the space of all subgroups of Λ endowed with the Chabauty topology. This topology is induced by the natural embedding $\text{Sub}(\Lambda) \subseteq \{0, 1\}^\Lambda$ with the product topology, and in particular it is compact. We say that $H \in \text{Sub}(\Lambda)$ is *co-amenable* if Λ/H admits a Λ -invariant mean (see [25]).

Lemma 6.5. *The set of co-amenable subgroups of a countable group Λ is a Borel subset of $\text{Sub}(\Lambda)$.*

Proof. Let F_k be an enumeration of all finite subsets of Λ . For a subgroup $H \leq \Lambda$ and a set $S \subseteq \Lambda$ let S/H denote the image of S under the quotient map $\Lambda \rightarrow \Lambda/H$. Clearly,

$\{F_k/H\}_{k \in \mathbb{N}}$ exhausts all possible finite subsets of Λ/H . For $g \in \Lambda$ and $k \in \mathbb{N}$ define the map $f_{k,g} : \text{Sub}(\Lambda) \rightarrow \mathbb{Q}$ by

$$f_{k,g}(H) = \frac{|(F_k \triangle g.F_k)/H|}{|F_k/H|}.$$

Suppose a sequence of subgroups H_n converges to H . Since Λ is discrete, this just means that a given $x \in \Lambda$ is in H if and only if it is eventually in H_n . Therefore, $|S/H_n|$ eventually coincides with $|S/H|$ for any given finite set $S \subseteq \Lambda$. This shows that $f_{k,g}$ is locally constant and in particular continuous. It follows that

$$\bigcap_{i,j \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcap_{g \in F_i} f_{k,g}^{-1}([0, 2^{-j})).$$

is a Borel set. It remains to note that this set is precisely the set of all co-amenable subgroups (see the Følner condition for co-amenable [23, Theorem 2.9]). ■

We consider the continuous action of Λ on $\text{Sub}(\Lambda)$ by conjugation. An *invariant random subgroup* (IRS for short) of Λ is a Λ -invariant Borel probability measure on $\text{Sub}(\Lambda)$. A *uniformly recurrent subgroup* (URS) is a minimal (non-empty) Λ -invariant closed subset of $\text{Sub}(\Lambda)$. We refer to [1, 22] for more on these notions.

By a *p.m.p. action* we mean a probability measure space (Ω, ν) endowed with a measure preserving action of Λ . We say that such an action has a *spectral gap* if the associated Koopman representation $\Lambda \rightarrow \mathcal{U}(L_0^2(\Omega, \nu))$ (where L_0^2 is the orthogonal complement of the constants) has no almost invariant vectors. We note that having a spectral gap implies ergodicity. The pushforward of ν under the stabilizer map $\omega \mapsto \text{stab}_\Lambda(\omega)$ is an IRS on Λ . In fact, any IRS arises in this manner, and any ergodic IRS arises from an ergodic p.m.p. action [1].

Theorem 6.6 (Compare with [44, Theorem 3.2] and [18, Theorem 2.12]). *Let Λ be a countable group, and suppose that any trace of Λ is either amenable or supported on the amenable radical. Then for any p.m.p. action (Ω, ν) of Λ with spectral gap, the stabilizer of a random point $\omega \in \Omega$ either is contained in $\text{Rad}(\Lambda)$ almost surely, or is co-amenable almost surely. Thus, denoting by μ the corresponding IRS, either the set of all subgroups of $\text{Rad}(\Lambda)$ is of full μ -measure, or the set of all co-amenable subgroups is of full μ -measure.*

We shall need some preparation.

Lemma 6.7. *Consider a measure space (Ω, ν) and a measure preserving action of a group Λ on it, and fix $1 \leq p < \infty$. Let \mathcal{F}_p denote the set of functions $\Omega \rightarrow \mathbb{R}_{\geq 0}$ in the unit sphere of $L^p(\Omega, \nu)$. Let \mathcal{P} denote the set of probability measure on Ω which are absolutely continuous with respect to ν , endowed with the total variation metric. Then the mapping*

$$\phi_p : \mathcal{F}_p \rightarrow \mathcal{P} : f \mapsto f^p \nu$$

is a Λ -equivariant bijection, and both ϕ_p and ϕ_p^{-1} are uniformly continuous. In particular, for a sequence $f_n \in \mathcal{F}_p$ the following are equivalent:

(1) f_n are almost invariant in the L^p -norm:

$$\forall g \in \Lambda : \|g \cdot f_n - f_n\|_p \rightarrow 0.$$

(2) $\nu_n = \phi_p(f_n)$ are almost invariant in the total variation norm:

$$\forall g \in \Lambda : \|g \cdot \nu_n - \nu_n\|_{\text{tot}} \rightarrow 0.$$

Proof. Consider first the map $\mathcal{F}_p \rightarrow \mathcal{F}_1 : f \mapsto f^p$. In the literature it is referred to as the *Mazur map*, and it is well known that both this map and its inverse are uniformly continuous [13, Chapter 9.1]. It is also clearly a Λ -equivariant bijection. We may therefore assume that $p = 1$ and consider the map ϕ_1 .

ϕ_1 is of course bijective and it is evidently Λ -equivariant. It is a standard fact that the total variation distance between two absolutely continuous probability measures is precisely half of the L^1 -distance of the corresponding probability density functions, so in particular ϕ_1 and ϕ_1^{-1} are uniformly continuous.

The equivalence between almost invariance of the L^p -functions and the corresponding measures follows from equivariance and uniform continuity. ■

Let $\Lambda \curvearrowright (\Omega, \nu)$ be a p.m.p. action. We define $\varphi : \Lambda \rightarrow \mathbb{C}$ by

$$\varphi(g) = \nu(\text{Fix}(g))$$

where $\text{Fix}(g) = \{\omega \in \Omega : g \cdot \omega = \omega\}$, and note that this is a trace on Λ . We recall a construction of a representation associated with this p.m.p. action that utilizes the GNS representation of φ . Consider the orbit equivalence relation

$$\mathcal{R} := \{(g \cdot \omega, \omega) : \omega \in \Omega, g \in \Lambda\} \subseteq \Omega \times \Omega$$

endowed with the subspace σ -algebra of the product. Denote by p_1 the projection of \mathcal{R} onto the first coordinate, and define the measure $\tilde{\nu}$ on \mathcal{R} by setting, for any measurable $E \subseteq \mathcal{R}$,

$$\tilde{\nu}(E) = \int_{\Omega} |p_1^{-1}(\omega) \cap E| d\nu.$$

In other words, $\tilde{\nu}$ is obtained by integrating the counting measures on the fibers of p_1 along ν .

Λ acts on \mathcal{R} via the left coordinate, as well as via the right coordinate, and consider the corresponding Koopman representations $\pi, \rho : \Lambda \rightarrow \mathcal{U}(L^2(\mathcal{R}, \tilde{\nu}))$. Then π and ρ obviously commute with each other and satisfy $\pi(g)\rho(g)\mathbf{1}_{\Delta} = \mathbf{1}_{\Delta}$ where $\mathbf{1}_{\Delta}$ is the characteristic function of the diagonal of $\Omega \times \Omega$. Moreover, $\varphi(g) = \langle \pi(g)\mathbf{1}_{\Delta}, \mathbf{1}_{\Delta} \rangle$. This means that the left and right GNS representations of φ are the cyclic subrepresentation of π generated by $\mathbf{1}_{\Delta}$. See [10, Theorem 15.F.6] for further details.

Lemma 6.8. *Let (Ω, ν) be a p.m.p. action and let $\varphi : g \mapsto \nu(\text{Fix}(g))$ be the corresponding trace of Λ . If (Ω, ν) has a spectral gap and φ is amenable then the stabilizer of ν -a.e. point in Ω is co-amenable.*

Proof. Consider the representation $\pi \times \rho : \Lambda \times \Lambda \rightarrow \mathcal{U}(L^2(\mathcal{R}, \tilde{\nu}))$ where $(\mathcal{R}, \tilde{\nu})$ is the orbit equivalence relation space associated with (Ω, ν) , as constructed above. By Proposition 6.4, the combined left and right GNS representation $\pi_\varphi \times \rho_\varphi$ of φ is amenable. Hence $\pi \times \rho$ is amenable as it contains $\pi_\varphi \times \rho_\varphi$. This means that there are functions $f_n \in L^2(\mathcal{R})^{\otimes 2} \subseteq L^2(\mathcal{R}^2)$ of norm 1 which are almost invariant for the $\Lambda \times \Lambda$ -action. In particular, $|f_n|$ are almost invariant and so by Lemma 6.7, the probability measures $|f_n|^2 \tilde{\nu}^2$ on \mathcal{R}^2 are almost invariant (in the total variation norm). Thus so are the pushforward probability measure η_n under the projection $\mathcal{R}^2 \rightarrow \mathcal{R}$ (onto, say, the first coordinate).

The projection $p_1 : \mathcal{R} \rightarrow \Omega$ onto the first coordinate is $\Lambda \times \{e\}$ -equivariant. Thus the pushforward probability measures $\nu_n := p_{1*} \eta_n$ on Ω are absolutely continuous with respect to ν and are almost invariant. By slightly changing ν_n (e.g. taking a convex combination with ν) we may assume that they are all in the same measure class of ν . Using Lemma 6.7 once again, we deduce that the L^2 -functions $\sqrt{d\nu_n/d\nu}$ are almost invariant. By assumption, $\Lambda \curvearrowright (\Omega, \nu)$ has a spectral gap, so $\sqrt{d\nu_n/d\nu}$ must converge to 1 in L^2 , which means that (again by Lemma 6.7) ν_n converges to ν in the total variation norm.

We shall from now on consider only the action of Λ on \mathcal{R} via the second coordinate, that is, the action of $\{e\} \times \Lambda$. This action preserves the fibers of p_1 . Let $\eta_n = \int_\Omega \eta_n^\omega d\nu_n$ be the disintegration of η_n along p_1 , and set $\eta'_n = \int_\Omega \eta_n^\omega d\nu$. Then $\|\eta'_n - \eta_n\|_{\text{tot}} \rightarrow 0$ and therefore by the triangle inequality, η'_n are almost invariant.

Choose an enumeration $\Lambda = \{g_i\}_{i \in \mathbb{N}}$ and let $F_n = \{g_1, \dots, g_n\}$. By perhaps replacing η'_n with a subsequence, we may assume that $\max_{g \in F_n} \|g \cdot \eta'_n - \eta'_n\| < 5^{-n}$. For each $n \in \mathbb{N}$, the measures η_n^ω are supported on pairwise distinct fibers of p_1 and as a result,

$$\int_\Omega \|g \cdot \eta_n^\omega - \eta_n^\omega\| d\nu(\omega) = \left\| \int_\Omega (g \cdot \eta_n^\omega - \eta_n^\omega) d\nu(\omega) \right\| = \|g \cdot \eta'_n - \eta'_n\| < 5^{-n} \quad (\forall g \in F_n).$$

Let $A_n := \{\omega \in \Omega : \max_{g \in F_n} \|g \cdot \eta_n^\omega - \eta_n^\omega\| \geq 2^{-n}\}$. Then by Markov's inequality $\nu(A_n) < (2/5)^n$, and therefore $\Omega_0 = \bigcap_{n \in \mathbb{N}} \Omega \setminus A_n$ has positive measure. For any $\omega \in \Omega_0$, η_n^ω is a sequence of almost invariant measures on the orbit Λ/Λ_ω of ω , and as a result, Λ_ω is co-amenable [19]. It follows that the set of $\omega \in \Omega$ for which Λ_ω is co-amenable is a Borel (Lemma 6.5) Λ -invariant set of positive ν -measure, and so by ergodicity it has full measure. ■

Proof of Theorem 6.6. Let (Ω, ν) be a p.m.p. action of Λ with spectral gap. Let $\varphi : g \mapsto \nu(\text{Fix}(g))$ be the corresponding trace, and note that $\varphi(g) = \mu(H \leq \Lambda : g \in H)$ where μ is the IRS of Λ obtained by pushing ν forward under the stabilizer map $\Omega \rightarrow \text{Sub}(\Lambda)$. By assumption, either φ is amenable or it is supported on the amenable radical. If the former case occurs then μ -a.e. subgroup is co-amenable by Lemma 6.8. The latter case means that $\mu(H \leq \Lambda : g \notin H) = 1 - \varphi(g) = 1$ for any $g \in \Lambda \setminus \text{Rad}(\Lambda)$. Thus, as Λ is countable,

$$\mu(H \leq \Lambda : H \subseteq \text{Rad}(\Lambda)) = \mu\left(\bigcap_{g \in \Lambda \setminus \text{Rad}(\Lambda)} \{H \leq \Lambda : g \notin H\}\right) = 1,$$

meaning that μ -a.e. subgroup is contained in $\text{Rad}(\Lambda)$. ■

6.3. Representations and their C^* -algebras

The C^* -algebra of a representation $\pi : \Lambda \rightarrow \mathcal{U}(\mathcal{H})$ is by definition the C^* -subalgebra $C_\pi^*(\Lambda) \subseteq \mathcal{B}(\mathcal{H})$ generated by the image of π .

Proposition 6.9. *Let Λ be a charmenable group and let π be a non-amenable representation. Denote by \mathcal{S} the collection of all representations of Λ that are induced from a representation of $\text{Rad}(\Lambda)$ and are weakly contained in π . Then:*

- (1) \mathcal{S} is not empty.
- (2) For any $\rho \in \mathcal{S}$, the assignment $\pi(g) \mapsto \rho(g)$ extends to a unital surjective $*$ -homomorphism $\Theta_\rho : C_\pi^*(\Lambda) \rightarrow C_\rho^*(\Lambda)$.
- (3) Any maximal ideal of $C_\pi^*(\Lambda)$ is the kernel of Θ_ρ for some $\rho \in \mathcal{S}$.
- (4) Any tracial state on $C_\pi^*(\Lambda)$ factorizes through Θ_ρ for some $\rho \in \mathcal{S}$.

This proposition is analogous to [4, Propositions 3.6, 3.7], and the proof is in spirit the same. For completeness, we write out the details.

Proof of Proposition 6.9. Let $C \subseteq \text{PD}_1(\Lambda)$ be the collection of all positive-definite functions of the form $\phi \circ \pi$ for a state ϕ on $C_\pi^*(\Lambda)$. This is a closed convex Λ -invariant subset. By charmenability it contains at least one trace φ . The GNS representation π_φ of φ is weakly contained in π and therefore cannot be amenable. This means φ is not amenable and so by Proposition 6.1, it is supported on the amenable radical. By Lemma 3.4, π_φ is induced from the amenable radical, i.e. $\pi_\varphi \in \mathcal{S}$. This proves (1), and (2) follows as it is one of the equivalent definitions of weak containment of representations [10, Proposition 8.B.4].

For (3), let I be a proper ideal of $C_\pi^*(\Lambda)$ with quotient map $q : C_\pi^*(\Lambda) \rightarrow C_\pi^*(\Lambda)/I$ and choose some faithful $*$ -embedding $C_\pi^*(\Lambda)/I \subseteq \mathcal{B}(\mathcal{H}_0)$. The representation $q \circ \pi$ of Λ is not amenable as it is weakly contained in π . By the above, there exists a representation $\rho_0 \in \mathcal{S}$ and a unital surjective $*$ -homomorphism $\sigma : C_\pi^*(\Lambda)/I \rightarrow C_{\rho_0}^*(\Lambda)$. In particular, $\Theta_{\rho_0} = \sigma \circ q$ is a unital surjective $*$ -homomorphism and I is contained in the kernel of Θ_{ρ_0} .

Finally, for (4), let ϕ be a tracial state of $C_\pi^*(\Lambda)$, let $\varphi := \phi \circ \pi \in \text{Tr}(\Lambda)$, and let π_φ be the GNS representation associated to φ . Then ϕ factorizes through $\Theta_{\pi_\varphi} : C_\pi^*(\Lambda) \rightarrow C_{\pi_\varphi}^*(\Lambda)$. As π_φ is weakly contained in π it cannot be amenable, so φ must be supported on $\text{Rad}(\Lambda)$ (Proposition 6.1). It follows from Lemma 3.4 that π_φ is induced from the amenable radical, i.e. $\pi_\varphi \in \mathcal{S}$. ■

6.4. Concluding the proofs

Proof of Proposition 1.3. (1), (2) and (4) are respectively established in [4, Propositions 3.3, 3.2, 3.5]; (6) is an immediate consequence of Theorem 3.6 in light of the classical fact that all amenable finite factors are hyperfinite [2, Chapter 11]; (5) and (7) follow from Proposition 6.9; and (3) follows from (2) due to Theorem 6.6. ■

Note that the second condition of charmenability alone implies (1)–(3).

Proof of Proposition 1.4. We start with the statement about IRS. Let X be an ergodic p.m.p. action of Λ , and let μ be the corresponding IRS. Since Λ has property (T), X has a spectral gap. We may therefore apply Theorem 6.6. If μ -a.e. subgroup is contained in $\text{Rad}(\Lambda)$ then it is finitely supported as it is ergodic and $\text{Sub}(\text{Rad}(\Lambda))$ is countable. If μ -a.e. subgroup is co-amenable, then μ -a.e. subgroup is in fact of finite index as Λ has property (T) [25, Claim 5.9].

Now for the statement about URS, let X be a URS of Λ and apply Proposition 1.3(4). If X admits an invariant probability measure then it in particular admits an ergodic one, so X is an ergodic IRS. We may therefore apply the above to conclude that this IRS is supported on a finite orbit which must be equal to X by minimality. Suppose now that $X \subseteq \text{Sub}(\text{Rad}(\Lambda))$. The derived space X' , i.e. the set of accumulation points of the compact space X , is a closed subset of X which is invariant under all homeomorphisms. By minimality, X' is either empty or all X . The latter is not possible since perfect Polish spaces have the cardinality of the continuum. It follows that X is discrete and compact and therefore finite. ■

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