



Matthew Stover

Algebraic fundamental groups of fake projective planes

Received September 13, 2022; revised December 8, 2023

Abstract. Fundamental groups of fake projective planes fall into fifty distinct isomorphism classes, one for each complex conjugate pair. We prove that this is not the case for their algebraic fundamental groups: there are only forty-six isomorphism classes. We show that there are four pairs of complex conjugate pairs of fake projective planes that are $\text{Aut}(\mathbb{C})$ -equivalent and hence have mutually isomorphic algebraic fundamental groups. All other pairs of algebraic fundamental groups are shown to be distinct through explicit finite étale covers. As a by-product, this provides the first examples of commensurable but nonisomorphic lattices in a rank one semisimple Lie group that have isomorphic profinite completions.

Keywords: fake projective planes, algebraic fundamental groups, ball quotients, profinite completions.

1. Introduction

If X is a smooth complex projective surface, then $\pi_1(X)$ will denote its topological fundamental group and $\pi_1^{\text{alg}}(X)$ its algebraic fundamental group. Then $\pi_1^{\text{alg}}(X)$ is the profinite completion of $\pi_1(X)$. This paper classifies the isomorphism classes of algebraic fundamental groups of fake projective planes, proving that there are exactly forty-six classes. It is known that there are fifty isomorphism classes of topological fundamental groups. Indeed, Cartwright and Steger completed the classification of fake projective planes [14], proving that there are one-hundred isomorphism classes that divide into fifty distinct complex conjugate pairs. Each complex conjugate pair determines a lattice in the Lie group $\text{PU}(2, 1)$, and Mostow rigidity implies that distinct complex conjugate pairs have nonisomorphic fundamental groups. The main result of this paper is the following.

Theorem 1.1. *There are exactly forty-six isomorphism classes of algebraic fundamental groups of fake projective planes. Specifically, suppose that X and Y are fake projective planes and that $\pi_1(X) = \Gamma_j$, $\pi_1(Y) = \Gamma_k$ for $j < k$ under the numbering given in the*

Matthew Stover: Department of Mathematics, Temple University, Philadelphia, PA 19122, USA; mstover@temple.edu

Mathematics Subject Classification 2020: 14J29 (primary); 22E40, 32J15, 14G35, 14J25, 14Q10, 20E18 (secondary).

Appendix. Then $\pi_1^{\text{alg}}(X)$ is isomorphic to $\pi_1^{\text{alg}}(Y)$ if and only if $\{j, k\}$ is one of the pairs

$$\{34, 35\}, \quad \{43, 44\}, \quad \{47, 49\}, \quad \{48, 50\}.$$

To prove that certain fake projective planes have nonisomorphic algebraic fundamental groups, we exhibit explicit finite quotients of their topological fundamental groups that preclude the existence of an isomorphism of algebraic fundamental groups. In terms of étale covers, we produce explicit connected, étale, Galois covers that distinguish between the two algebraic fundamental groups. This part of the proof is the content of Section 4. In fact, the proof is effective. If two fake projective planes have nonisomorphic algebraic fundamental groups, we show that this can be detected by an étale cover of degree at most 248; see Corollary 4.13. See Remark 4.14 for an alternate argument found by a referee that, while ineffective and hence not sufficient to prove Corollary 4.13, cuts a much faster path to reducing the proof of Theorem 1.1 to proving that the four given pairs have isomorphic algebraic fundamental groups.

The more technical and difficult part of the paper, completed in Section 7, is proving that the remaining pairs have isomorphic algebraic fundamental groups. This is done following Serre's observation that $\text{Aut}(\mathbb{C})$ -equivalent complex projective varieties have the same algebraic fundamental group [31]. Here, we use conjugation of Shimura varieties to prove that four pairs of fake projective planes are $\text{Aut}(\mathbb{C})$ -equivalent. Specifically, we use the close relationship between fake projective planes and Shimura varieties, which have canonical models defined over a number field, to define the action of $\text{Aut}(\mathbb{C})$ on fake projective planes and then apply work of Shih [32] on conjugation of Shimura varieties; this approach has the added feature of working directly with connected components, avoiding a minor but annoying technicality that would arise using the adelic definition. This requires careful analysis of the arithmetic lattices associated with these fake projective planes. One case should also be covered by work of Borisov and Fatighenti [19] on explicit equations for fake projective planes, but this would require verification that we did not pursue (see Remark 7.8). The proof of Theorem 1.1 therefore allows us to give the complete $\text{Aut}(\mathbb{C})$ classification of fake projective planes.

Corollary 1.2. *There are exactly forty-six $\text{Aut}(\mathbb{C})$ -equivalence classes of fake projective planes.*

Using $\text{Aut}(\mathbb{C})$ conjugation to find locally symmetric varieties with nonisomorphic topological fundamental groups but isomorphic algebraic fundamental groups has some history. Examples are found in Milne and Suh [25], Bauer, Catanese, and Grunewald [2], and Rajan [29]. In the language of discrete subgroups of Lie groups, one constructs nonisomorphic lattices in (possibly distinct) Lie groups with isomorphic profinite completions. See for example [1, 21] for more on higher-rank lattices with the same profinite completion.

The rank one case has recently been of particular interest, especially in light of the remarkable breakthrough by Bridson, McReynolds, Reid, and Spitler [4], who found hyperbolic 3-manifold groups Γ that are *profinutely rigid*, i.e., if Λ is any other finitely

generated, residually finite group with profinite completion $\widehat{\Lambda} \cong \widehat{\Gamma}$, then $\Lambda \cong \Gamma$. The first examples of nonisomorphic real rank one lattices with isomorphic profinite completions were given in [34], where we proved using $\text{Aut}(\mathbb{C})$ -conjugation that, for all $n \geq 2$, there are torsion-free cocompact lattices in $\text{PU}(n, 1)$ that are not isomorphic but have isomorphic profinite completions. A distinct feature of the examples in [34] is that the lattices are not commensurable (the higher-rank lattices in [29] are commensurable, but those in [25] are not). In this paper, we provide the first commensurable examples in rank one, confirming in this case the suspicion in [25, Remark 2.7] echoed in [34, Remark 2.3].

Corollary 1.3. *There are commensurable lattices in $\text{PU}(2, 1)$ that are not isomorphic but have isomorphic profinite completions.*

We note, however, that it is still an open problem whether or not there are distinct nonuniform lattices in $\text{PU}(n, 1)$ with isomorphic profinite completions. Some specific examples where certain methods used in this paper could be of use are the following:

Question 1.4. *Consider neat nonuniform lattices Γ_1, Γ_2 in $\text{PU}(n, 1)$ such that the associated ball quotients admit biholomorphic smooth toroidal compactifications; see for instance [15, 16, 18] and references therein for many (often commensurable) examples. Is it ever the case that the profinite completions $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic?*

It is noted in the Appendix to [33] that for any example in [17] of a pair of lattices with biholomorphic smooth toroidal compactifications, the lattices are distinguished from one another by the number of subgroups of index 4. It follows that the associated lattices in $\text{PU}(2, 1)$ have nonisomorphic profinite completions. Thus having biholomorphic smooth toroidal compactifications does not necessarily imply isomorphism of profinite completions.

Question 1.5. *Suppose that $\Gamma_1, \Gamma_2 < \text{PU}(n, 1)$ are neat nonuniform lattices with isomorphic profinite completions. Are the smooth toroidal compactifications of the ball quotients $\Gamma_j \backslash \mathbb{B}^n$ biholomorphic? Do the compactifications have isomorphic algebraic fundamental groups?*

Remark 1.6. There are many smooth projective surfaces not birational to a fake projective plane with (topological, hence algebraic) fundamental group isomorphic to the fundamental group of a fake projective plane. See [36, Section 1] for a discussion of some known constructions and [36, Section 7] for one method of producing examples. On the other hand, see [36, Section 6] for a proof that an *aspherical* smooth projective surface with fundamental group isomorphic to the fundamental group of a smooth compact ball quotient must in fact be biholomorphic to a ball quotient. In particular, an aspherical smooth projective surface with the same fundamental group as a fake projective plane must be a fake projective plane. It would be interesting to know if an aspherical smooth projective surface with $\pi_1^{\text{alg}}(V)$ isomorphic to the algebraic fundamental group of a fake projective plane is necessarily a fake projective plane.

Remark 1.7. A negative answer to [36, Question 3] would provide smooth projective surfaces V with $\pi_1(V)$ not isomorphic to the fundamental group of a fake projective plane but with $\pi_1^{\text{alg}}(V)$ isomorphic to the algebraic fundamental group of a fake projective plane. Our later work [35] answering the analogue of that question for arithmetic lattices in $\text{PU}(n, 1)$ of “simple type” does not apply to the commensurability classes that contain fake projective planes.

Remark 1.8 (Remark on use of the computer). This paper relies on computations conducted in the computer algebra program Magma [3]. Code is available on the author’s website [23]. The only functions used were those related to finite index subgroups of finitely presented groups. However, the classification of fake projective planes by Cartwright and Steger relies so heavily on computer calculation that, even if the relatively simple (if prohibitively long) computations necessary to confirm the results of this paper were done by hand, there is no reasonable sense in which this paper could be considered completely independent of the use of computer algebra software.

A hierarchy for fake projective plane groups

We close the introduction by describing a hierarchy for classifying fake projective plane groups by their finite quotients, suggested by Jakob Stix. See Figure 1, where $\pi_1^{\text{alg, solv}}$ denotes the solvable algebraic fundamental group and $\pi_1^{\text{alg, nil}}$ the nilpotent. The equality between fake projective planes up to π_1^{alg} versus $\pi_1^{\text{alg, solv}}$ follows from the fact that all groups used in Section 4 to differentiate between algebraic fundamental groups are solvable.

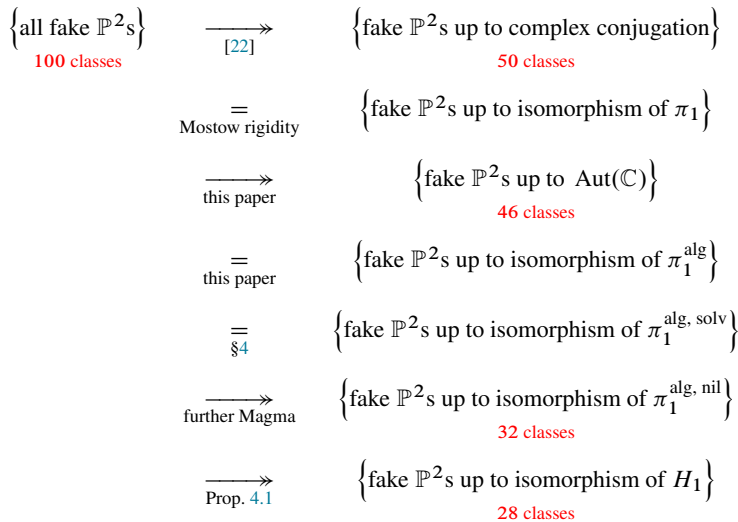


Fig. 1. A hierarchy for identifying fake projective planes.

This is not discussed elsewhere, but one can also show using routine Magma calculations that there are 32 classes distinguished by their nilpotent quotients. Specifically,

among groups Γ_j with the same abelianization, Γ_{41} is distinguished from Γ_4 and Γ_5 by nilpotent quotients, and similarly Γ_{17} is distinguished from Γ_{48} and Γ_{50} , Γ_{13} is distinguished from Γ_{23} , and Γ_{36} is distinguished from Γ_{37} and Γ_{45} .

This paper is structured as follows. An Appendix at the end of the paper gives a numbering system to the fifty fundamental groups of fake projective planes that is referenced throughout the paper. We begin in Section 2 with some basics on the action of $\text{Aut}(\mathbb{C})$ on smooth projective varieties. In Section 3 we briefly present some facts about profinite completions of discrete groups that are used in Section 4 to distinguish between all nonisomorphic algebraic fundamental groups. Then Section 5 recaps conjugation of Shimura varieties and Section 6 briefly presents some facts about certain groups over nonarchimedean fields and their buildings. These sections are used in Section 7 to produce the nontrivial isomorphisms between algebraic fundamental groups that combine with the results in Section 4 to prove Theorem 1.1.

2. The action of $\text{Aut}(\mathbb{C})$

This section follows [2, Section 4]. To start, an element $\tau \in \text{Aut}(\mathbb{C})$ acts on an element $f(x) = \sum \alpha_I x^I$ of the polynomial ring $\mathbb{C}[x_0, \dots, x_N]$ by

$$\tau(f)(x) = \sum_{I=(i_0, \dots, i_N)} \tau(\alpha_I) x^I,$$

where we use the usual multi-index notation for polynomials. Given a projective variety $X \subseteq \mathbb{P}^N$ cut out by homogeneous polynomials $\{f_1(x), \dots, f_r(x)\}$, the coordinate action of $\tau \in \text{Aut}(\mathbb{C})$ on \mathbb{P}^n maps X to a new variety

$$X^\tau = \{x \in \mathbb{P}^N : \tau(f_j)(x) = 0 \text{ for } 1 \leq j \leq r\}.$$

Indeed, $f_j(x) = 0$ if and only if $\tau(f_j)(\tau(x)) = 0$.

Applying the above to the graph of an element $\psi \in \text{Aut}(X)$, we obtain an element $\psi^\tau \in \text{Aut}(X^\tau)$. A G -marked variety is then a triple (X, G, η) , where X is a projective variety and $\eta : G \rightarrow \text{Aut}(X)$ is an injective homomorphism. For each G -marking (X, G, η) of X , we obtain the G -marking

$$(X, G, \eta)^\tau = (X^\tau, G, \tau \circ \eta \circ \tau^{-1}) \tag{1}$$

of X^τ . Set $\eta^\tau = \tau \circ \eta \circ \tau^{-1}$. We record some essential properties of the $\text{Aut}(\mathbb{C})$ -action on projective varieties.

Lemma 2.1. *Let $X \subseteq \mathbb{P}^N$ be a projective variety. For $\tau \in \text{Aut}(\mathbb{C})$ and a marking (X, G, η) , the following hold:*

- (1) *If $\eta(g)$ has a fixed point for some $g \in G$, then $\eta^\tau(g)$ has a fixed point on X^τ .*
- (2) *$(X/\eta(G))^\tau = X^\tau/\eta^\tau(G)$.*
- (3) *The variety X is smooth if and only if X^τ is smooth.*

Proof. The first property follows from the definition of η^τ and the fact that $X \rightarrow X^\tau$ is a bijection on points. The second property is [2, Remark 4.4 (4)]. The third follows from the fact that smoothness is defined by the rank of the Jacobian matrix for the system of polynomials, which is invariant under the action of $\text{Aut}(\mathbb{C})$. ■

We will use the following corollary in Section 7.

Corollary 2.2. *Let G be a finite group and (Y, G, η) be a smooth projective surface such that $\eta(G)$ has no fixed points (i.e., $Y \rightarrow X$ is finite étale) and $X = Y/\eta(G)$ is a fake projective plane. Then X^τ is a fake projective plane.*

Proof. Lemma 2.1 implies that $X^\tau = Y^\tau/\eta^\tau(G)$ is also a smooth projective surface. Betti numbers are invariant under τ by Serre’s GAGA principle, so X^τ also has the same complex cohomology as \mathbb{P}^2 . We must rule out the possibility that X^τ is isomorphic to \mathbb{P}^2 . Since X is a ball quotient, so is X^τ by a theorem of Kazhdan [20]. In particular, X^τ has infinite fundamental group, and therefore it is a fake projective plane. ■

3. Profinite completions of discrete groups

Let X be a smooth complex projective variety. Throughout this paper, $\pi_1(X)$ will denote the topological fundamental group, considered as a finitely generated discrete group, and $\pi_1^{\text{alg}}(X)$ the algebraic fundamental group. Then $\pi_1^{\text{alg}}(X)$ is isomorphic to the profinite completion of $\pi_1(X)$. As explained in the introduction, there are now many known examples of pairs X, Y such that $\pi_1(X)$ is not isomorphic to $\pi_1(Y)$ but $\pi_1^{\text{alg}}(X) \cong \pi_1^{\text{alg}}(Y)$. Here and throughout, the profinite completion of a group Γ is denoted $\widehat{\Gamma}$.

This section is concerned with tools to prove that $\pi_1^{\text{alg}}(X)$ is not isomorphic to $\pi_1^{\text{alg}}(Y)$ through finite quotients of $\pi_1(X)$ and $\pi_1(Y)$. These tools come from the study of profinite completions of arbitrary finitely generated discrete groups. See [30] for a thorough survey; we will only need the following two fundamental facts found there:

Lemma 3.1 ([30, Proposition 3.2]). *If Γ and Δ are discrete groups with $\widehat{\Gamma} \cong \widehat{\Delta}$, then Γ and Δ have isomorphic abelianizations.*

Theorem 3.2 ([30, Corollary 4.2]). *If Γ, Δ are finitely generated groups with $\widehat{\Gamma} \cong \widehat{\Delta}$, then one has equalities*

$$|\text{Hom}(\Gamma, Q)| = |\text{Hom}(\Delta, Q)|,$$

$$|\text{Epi}(\Gamma, Q)| = |\text{Epi}(\Delta, Q)|$$

for all finite groups Q , where $\text{Epi}(\Lambda, Q)$ is the set of epimorphisms of a group Λ onto Q .

In the language of smooth complex projective varieties, Lemma 3.1 says that if $\pi_1^{\text{alg}}(X) \cong \pi_1^{\text{alg}}(Y)$, then $H_1(X, \mathbb{Z}) \cong H_1(Y, \mathbb{Z})$. Theorem 3.2 says that if X and Y admit a different number of isomorphism classes of connected étale coverings with Galois group some fixed finite group Q , then they have nonisomorphic algebraic fundamental groups.

Note that Theorem 3.2 is slightly stronger than the more elementary statement that if X admits a connected étale cover with group Q and Y does not, then their algebraic fundamental groups are not isomorphic. (Here we call étale covers isomorphic if they are associated with the same conjugacy class of subgroups of the fundamental group.)

4. Distinct isomorphism classes

In this section, we prove that the distinct algebraic fundamental groups of fake projective planes are indeed distinct. Throughout this section, we use the numbering of fake projective plane fundamental groups given in the Appendix. Applying Lemma 3.1 after using computer algebra software to compute abelianizations (see also [5]) immediately gives us the following start to proceedings.

Proposition 4.1. *If Γ_j and Γ_k are fake projective plane fundamental groups with $\widehat{\Gamma}_j \cong \widehat{\Gamma}_k$, then j and k appear in the same line of Table 1.*

$H_1(\Gamma_j, \mathbb{Z})$	j	$H_1(\Gamma_j, \mathbb{Z})$	j
$(\mathbb{Z}/2) \times (\mathbb{Z}/26)$	4, 5, 41	$\mathbb{Z}/18$	19, 34, 35
$\mathbb{Z}/14$	9, 11, 25, 32, 39	$\mathbb{Z}/42$	22, 30
$(\mathbb{Z}/2) \times (\mathbb{Z}/14)$	12, 20	$\mathbb{Z}/6$	24, 43, 44
$(\mathbb{Z}/2)^2 \times (\mathbb{Z}/6)$	13, 23	$\mathbb{Z}/21$	27, 31
$\mathbb{Z}/28$	16, 47, 49	$(\mathbb{Z}/3) \times (\mathbb{Z}/6)$	36, 37, 45
$(\mathbb{Z}/2) \times (\mathbb{Z}/12)$	17, 48, 50	$\mathbb{Z}/7$	38, 40
$(\mathbb{Z}/2) \times (\mathbb{Z}/18)$	18, 26		

Tab. 1. Fake projective planes with isomorphic first homology over \mathbb{Z} .

Note that Proposition 4.1 omits those $1 \leq j \leq 50$ where Γ_j is the unique fake projective plane fundamental group with abelianization isomorphic to Γ_j^{ab} . Consequently, for any j that does not appear in a line of Table 1, Lemma 3.1 immediately gives us that $\widehat{\Gamma}_j \cong \widehat{\Gamma}_k$ implies $k = j$; these $\widehat{\Gamma}_j$ are therefore completely classified with regard to the proof of Theorem 1.1.

We now focus on finite quotients that distinguish between various lines of Table 1. The following sequence of facts is proved with computer algebra software by computing the number of homomorphisms onto the appropriate finite group G . We use the following notation for finite groups: S_n is the symmetric group on n letters, A_n the alternating group on n letters, and D_n the dihedral group of order $2n$. In what follows, two finite quotients of a group are considered distinct if and only if they have distinct kernels.

Fact 4.2. *If Γ_j is a fake projective plane group admitting S_3 as a quotient, then*

$$j \in \{1, 2, 4, 5, 11, 12, 14, 16, 20, 21, 25, 26, 30, 32, 33, 34, 35, 36, 37, 42, 43, 44, 45, 47, 49\}.$$

The S_3 quotient is unique except in the following cases:

$$\left\{ \begin{array}{ll} j = 20 & 4 \text{ quotients} \\ j = 21 & 2 \text{ quotients} \\ j = 25 & 4 \text{ quotients} \\ j = 26 & 2 \text{ quotients} \\ j = 32 & 4 \text{ quotients.} \end{array} \right.$$

Fact 4.3. *If Γ_j is a fake projective plane group admitting an A_4 quotient, then*

$$j \in \{22, 24, 27, 29, 45\}$$

and each such Γ_j admits exactly two distinct A_4 quotients.

Fact 4.4. *The only fake projective plane group with a $GL_2(\mathbb{F}_3)$ quotient is Γ_5 , which admits four such quotients.*

Fact 4.5. *If Γ_j is a fake projective plane group with the affine linear group $AGL_1(\mathbb{F}_8)$ in one variable over \mathbb{F}_8 as a quotient, then*

$$j \in \{9, 11, 12, 14, 16, 20, 22, 25, 27, 30, 31, 32, 38, 39, 40, 47, 49\}.$$

There are twenty-seven $AGL_1(\mathbb{F}_8)$ quotients for $j \in \{20, 32, 38\}$ and three otherwise.

Fact 4.6. *If Γ_j is a fake projective plane admitting D_8 as a quotient, then $j \in \{8, 48, 50\}$ and Γ_j admits precisely four D_8 quotients.*

Fact 4.7. *The only fake projective plane group admitting D_9 as a quotient is Γ_{32} , which admits nine such quotients.*

Fact 4.8. *The only fake projective plane group admitting D_{13} as a quotient is Γ_{39} , which admits six such quotients.*

Fact 4.9. *If Γ_j is a fake projective plane group with the quaternion group Q of order 8 as a quotient, then $j \in \{3, 23, 28, 41\}$. For $j = 3, 23$, Γ_j admits twelve Q quotients and for $j = 28, 41$ there are only six.*

Fact 4.10. *If Γ_j is a fake projective plane group having the unique nonabelian group G of order 27 that is a semidirect product $(\mathbb{Z}/9) \rtimes (\mathbb{Z}/3)$ as a quotient, then $j \in \{37, 45\}$ and Γ_j admits six such quotients.*

Fact 4.11. *If Γ_j is a fake projective plane group with the unique semidirect product $(\mathbb{Z}/13) \rtimes (\mathbb{Z}/4)$ with faithful action of $\mathbb{Z}/4$ as a quotient, then $j \in \{1, 47, 49\}$ and Γ_j admits three such quotients.*

We now combine the above information to reduce the proof of Theorem 1.1 to considering four pairs.

Theorem 4.12. *Let X and Y be fake projective planes with $\pi_1(X) = \Gamma_j$ and $\pi_1(Y) = \Gamma_k$ for $j < k$. If the pair $\{j, k\}$ is not among*

$$\{34, 35\}, \quad \{43, 44\}, \quad \{47, 49\}, \quad \{48, 50\},$$

then $\pi_1^{\text{alg}}(X)$ and $\pi_1^{\text{alg}}(Y)$ are not isomorphic.

Proof. We must prove $\widehat{\Gamma}_j$ is not isomorphic to $\widehat{\Gamma}_k$ with the exception of the four pairs in the statement of the theorem. First, note that j and k must appear in the same line of Table 1. We now apply Theorem 3.2 to the quotients given in Facts 4.2–4.11 to eliminate the remaining pairs.

Case 1: $\{4, 5, 41\}$ (Facts 4.2 and 4.4). The group Γ_{41} does not have S_3 as a quotient, but Γ_4 and Γ_5 do. Then Γ_4 does not have $\mathrm{GL}_2(\mathbb{F}_3)$ as a quotient, but Γ_5 does. Thus all three have distinct profinite completions.

Case 2: $\{9, 11, 25, 32, 39\}$ (Facts 4.2, 4.7, and 4.8). The groups Γ_{11} and Γ_{25} have an S_3 quotient, but Γ_9 , Γ_{32} , and Γ_{39} do not. Moreover, Γ_{25} has four S_3 quotients, while Γ_{11} only has one. Then Γ_{32} has a D_9 quotient, unlike Γ_9 and Γ_{25} , and Γ_{25} has a D_{13} quotient, unlike Γ_9 or Γ_{32} . Thus all five groups have distinct profinite completions.

Case 3: $\{12, 20\}$ (Fact 4.2). The groups Γ_{12} and Γ_{20} both have S_3 quotients, but Γ_{12} has only one, whereas Γ_{20} has four. Therefore the groups have distinct profinite completions.

Case 4: $\{13, 23\}$ (Fact 4.9). The group Γ_{23} has the quaternion group of order 8 as a quotient, but Γ_{13} does not, hence the two have nonisomorphic profinite completions.

Case 5: $\{16, 47, 49\}$ (Fact 4.11). The unique semidirect product $(\mathbb{Z}/13) \rtimes (\mathbb{Z}/4)$ with faithful $\mathbb{Z}/4$ -action is a quotient of Γ_{47} and Γ_{49} , but not of Γ_{16} . Therefore $\widehat{\Gamma}_{16}$ cannot be isomorphic to $\widehat{\Gamma}_j$ for $j \in \{47, 49\}$.

Case 6: $\{17, 48, 50\}$ (Fact 4.6). The groups Γ_{48} and Γ_{50} both admit D_8 as a quotient, but Γ_{17} does not. Therefore $\widehat{\Gamma}_{17}$ cannot be isomorphic to $\widehat{\Gamma}_j$ for $j \in \{48, 50\}$.

Case 7: $\{18, 26\}$ (Fact 4.2). The group Γ_{18} does not admit S_3 as a quotient, but Γ_{26} does. Therefore the pair have distinct profinite completions.

Case 8: $\{19, 34, 35\}$ (Fact 4.2). The groups Γ_{34} and Γ_{35} both have S_3 as a quotient, but Γ_{19} does not. Therefore $\widehat{\Gamma}_{19}$ cannot be isomorphic to $\widehat{\Gamma}_j$ for $j \in \{34, 35\}$.

Case 9: $\{22, 30\}$ (Fact 4.2). The group Γ_{30} has an S_3 quotient, unlike Γ_{22} , so the pair have nonisomorphic profinite completions.

Case 10: $\{24, 43, 44\}$ (Fact 4.2). The groups Γ_{43} and Γ_{44} both admit S_3 as a quotient, but Γ_{24} does not. Therefore $\widehat{\Gamma}_{24}$ cannot be isomorphic to $\widehat{\Gamma}_j$ for $j \in \{43, 44\}$.

Case 11: $\{27, 31\}$ (Fact 4.3). The group Γ_{27} has A_4 as a quotient, unlike Γ_{31} , so their profinite completions are not isomorphic.

Case 12: $\{36, 37, 45\}$ (Facts 4.3 and 4.10). The group Γ_{45} admits an A_4 quotient, but Γ_{36} and Γ_{37} do not. Also Γ_{37} has the nontrivial semidirect product $(\mathbb{Z}/9) \rtimes (\mathbb{Z}/3)$ as a quotient, but Γ_{36} does not. Therefore all three have distinct profinite completions.

Case 13: $\{38, 40\}$ (Fact 4.5). Each group has $\mathrm{AGL}_1(\mathbb{F}_8)$ as a quotient, but Γ_{38} has twenty-seven distinct $\mathrm{AGL}_1(\mathbb{F}_8)$ quotients, whereas Γ_{40} has only three. It follows that their profinite completions are not isomorphic.

We have ruled out all possible isomorphisms between algebraic fundamental groups of fake projective planes except the four pairs in the statement of the theorem. Therefore the proof of the theorem is complete. ■

Define two connected étale covers $Z \rightarrow X$ between smooth projective varieties to be isomorphic when the associated subgroups of $\pi_1(X)$ are conjugate. In particular, under this definition a fixed X can admit nonisomorphic covers by a fixed Z . In this language, considering the orders of the finite groups in Facts 4.2–4.11 and possible orders of first homology groups with \mathbb{Z} coefficients, we obtain the following corollary to Theorem 4.12.

Corollary 4.13. *Let X and Y be fake projective planes for which the number of isomorphism classes of étale covers with group G are exactly the same for all groups G with $|G| \leq 248$. Then X and Y are either isomorphic, complex conjugate, or $\pi_1(X) = \Gamma_j$ and $\pi_1(Y) = \Gamma_k$ where (up to reordering) the pair $\{j, k\}$ is one of the four given in the statement of Theorem 4.12.*

In other words, away from the four pairs of fundamental groups we will soon show have isomorphic profinite completions, one can effectively determine through étale covers whether or not two fake projective planes are isomorphic up to complex conjugation.

Remark 4.14. A referee noticed the following shorter path to proving Theorem 4.12. Let X^{ab} be the universal étale abelian covering of the fake projective plane X . If $\pi_1^{\text{alg}}(X)$ is isomorphic to $\pi_1^{\text{alg}}(Y)$, then also $H_1(X^{\text{ab}}, \mathbb{Z}) \cong H_1(Y^{\text{ab}}, \mathbb{Z})$. A short computer program combining this observation with Proposition 4.1 reduces one to studying the pairs

$$\{4, 5\}, \quad \{34, 35\}, \quad \{43, 44\}, \quad \{47, 49\}, \quad \{48, 50\}.$$

One then rules out the pair $\{4, 5\}$ using Fact 4.4. However, this argument does not imply Corollary 4.13, and the small nonabelian étale covers produced in the current proof of Theorem 4.12 may be of independent interest, so the longer proof was retained.

5. Conjugation of Shimura varieties

This section recaps work of Milne and Suh [25, Section 1] and earlier work of Miyake [26] on conjugation of Shimura varieties and generally follows the notation from [25]. Let F be a totally real number field and H an algebraic group over F . The restriction of scalars H_* is the \mathbb{Q} -algebraic group so that

$$H_*(R) = H(F \otimes_{\mathbb{Q}} R)$$

for any \mathbb{Q} -algebra R and

$$H_*(\mathbb{R}) = \prod_{\nu:F \rightarrow \mathbb{R}} H_{\nu}(\mathbb{R}),$$

where H_{ν} is the real algebraic group defined by extension of scalars from $\nu(F)$ to \mathbb{R} . Notice that $H(F)$ is a subgroup of $H_*(\mathbb{R})$, and we then define

$$H(F)^{\dagger} = H(F) \cap H_*(\mathbb{R})^{\dagger},$$

where $H_*(\mathbb{R})^+$ is the connected component of $H_*(\mathbb{R})$ containing the identity. We assume that H is a simply connected semisimple F -algebraic group, which implies that $H(\mathbb{R})$ is connected. If v is a nonarchimedean place of F with completion F_v , then H_{F_v} will denote the group $H(F_v)$ obtained by extension of scalars from F to F_v .

Let \mathbb{A}_F^∞ denote the finite adèles of F . A *congruence subgroup* of $H(F)$ is a subgroup of the form

$$\tilde{\Gamma} = H(F) \cap K$$

for an open compact subgroup $K < H(\mathbb{A}_F^\infty)$, where $H(F)$ is diagonally embedded in $H(\mathbb{A}_F^\infty)$. If $H^{\text{ad}}(F)^+$ is the adjoint group of $H(F)^+$, then a subgroup $\Gamma < H^{\text{ad}}(F)^+$ is a congruence subgroup if its preimage $\tilde{\Gamma}$ in $H(F)$ with respect to the natural central isogeny is a congruence subgroup. When we need to differentiate between different congruence subgroups, we will use $\tilde{\Gamma}(K)$ and $\Gamma(K)$ to emphasize the open compact subgroup.

Suppose that X is a bounded symmetric domain and $\text{Hol}(X)^+$ is the connected component of the identity in the group of holomorphic automorphisms of X . The case of interest to us is where there is a surjective homomorphism $H(\mathbb{R})^+ \rightarrow \text{Hol}(X)^+$ with compact kernel. If $\Gamma < H^{\text{ad}}(F)^+$ is a congruence subgroup and $V = \Gamma \backslash X$ the associated algebraic variety, then we say that V has *type* (H, X) .

The following is Theorem 1.3 of [25] combined with the more refined version of the theorem’s conclusion provided by [25, Remark 1.6].

Theorem 5.1. *Suppose that H is a simply connected and semisimple F -algebraic group and that V is a smooth algebraic variety over \mathbb{C} of type (H, X) . For any $\tau \in \text{Aut}(\mathbb{C})$, the variety V^τ is of type (H', X') for some semisimple F -algebraic group H' such that*

$$\begin{cases} H'_v \cong H_{\tau \circ v} & \text{for all real archimedean } v \\ H'_{F_v} \cong H_{F_v} & \text{for all nonarchimedean } v. \end{cases}$$

Moreover, suppose that $V = \tilde{\Gamma}(K) \backslash X$ for $K \leq H(\mathbb{A}_F^\infty)$ with $\tilde{\Gamma}(K)$ torsion-free. The isomorphisms $H'_{F_v} \cong H_{F_v}$ for all nonarchimedean v induce an isomorphism $H'(\mathbb{A}_F^\infty) \cong H(\mathbb{A}_F^\infty)$ so that

$$V^\tau = \tilde{\Gamma}'(K') \backslash X',$$

where K' is the image of K in $H'(\mathbb{A}_F^\infty)$ under the isomorphism.

In fact, we will need even more concrete information about the induced isomorphism $H(\mathbb{A}_F^\infty) \rightarrow H'(\mathbb{A}_F^\infty)$, so we sketch the ideas behind the proof of Theorem 5.1. We follow Miyake [26, Section 5], which specifically considers the case of interest for us.

Let F be a totally real number field with $[F : \mathbb{Q}] = d$, \mathcal{O}_F be the ring of integers of F , and E/F be a totally imaginary quadratic extension. Fix a cubic division algebra D with center E and involution of second kind, that is, an anti-involution σ such that the restriction of σ to E is the nontrivial element of $\text{Gal}(E/F)$. A nonzero element $h \in D^*$ for which $\sigma(h) = h$ defines a nondegenerate hermitian form on D by

$$h(x, y) = \sigma(x)hy.$$

One then modifies h to obtain a symplectic form ψ on D (as an F -vector space) in a standard way described in [24, Example 8.5].

Set $n = 9d$. Considering $D_{\mathbb{R}} = D \otimes_{\mathbb{Q}} \mathbb{R}$ as a complex vector space, the action of D on itself defines a representation $\Phi : D \rightarrow M_n(\mathbb{C})$ so that $\Phi \oplus \bar{\Phi}$ is a rational representation, where $\bar{\Phi}$ is the complex conjugate representation. Consider a \mathbb{Z} -lattice $\mathcal{M} \subset D$. Scaling ψ by a rational number, which does not change its unitary group, we can assume that

$$\mathrm{tr}_{D/\mathbb{Q}}(\psi(\mathcal{M}, \mathcal{M})) = \mathbb{Z}.$$

We call $\Omega = (D, \Phi, \sigma; \psi, \mathcal{M})$ a *PEL-type*.

A *PEL-structure* $(\mathcal{A}, \mathcal{C}, \theta)$ of type Ω consists of an abelian variety \mathcal{A} with polarization \mathcal{C} and $\theta : D \rightarrow \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$ satisfying the following properties:

- There is a complex torus \mathbb{C}^n/Λ , an isomorphism $\eta : D_{\mathbb{R}} \rightarrow \mathbb{C}^n$, and a homomorphism $\iota : \mathbb{C}^n \rightarrow \mathcal{A}$ such that ι induces a biregular isomorphism from \mathbb{C}^n/Λ to \mathcal{A} under which

$$\iota(\Phi(a)w) = \theta(a)\iota(w) \quad \text{for all } a \in D \text{ and } w \in \mathbb{C}^n.$$

- For all $a \in D$ and $x \in D_{\mathbb{R}}$, one has $\eta(\mathcal{M}) = \Lambda$ and $\eta(ax) = \Theta(a)\eta(x)$.
- The polarization \mathcal{C} contains a divisor that determines a Riemann form E on \mathbb{C}^n/Λ such that

$$E(\eta(x), \eta(y)) = \mathrm{tr}_{D/\mathbb{Q}}(\psi(x, y)) \quad \text{for all } x, y \in D_{\mathbb{R}}.$$

Any $\tau \in \mathrm{Aut}(\mathbb{C})$ acts on PEL-structures by

$$(\mathcal{A}, \mathcal{C}, \theta)^\tau = (\mathcal{A}^\tau, \mathcal{C}^\tau, \theta^\tau).$$

The PEL-type of $(\mathcal{A}, \mathcal{C}, \theta)^\tau$ is then $\Omega^\tau = (D^\tau, \Phi^\tau, \sigma^\tau; \psi^\tau, \mathcal{M}^\tau)$.

Now, suppose that τ is the extension to $\mathrm{Aut}(\mathbb{C})$ of an element of $\mathrm{Aut}(F)$ and that \mathcal{M} is in fact an order in D . We see that one can take $h' = \tau(h)$. For any nonarchimedean place v of F and \mathcal{O}_v the integers of F_v ,

$$\mathcal{M}_v = \mathcal{M} \otimes_{\mathcal{O}_F} \mathcal{O}_v \tag{2}$$

is a lattice in D_v , considered as an F_v -vector space, and the elements P_v of H_{F_v} stabilizing \mathcal{M}_v define an open compact subgroup. Then, considering P_v as a subgroup of $H_{F_v} \leq H(\mathbb{A}_F^\infty)$, its image in $H'(\mathbb{A}_F^\infty)$ under the isomorphism from Theorem 5.1 must stabilize the lattice

$$\mathcal{M}_{\tau(v)}^\tau = \mathcal{M}^\tau \otimes_{\mathcal{O}_F} \mathcal{O}_{\tau(v)}, \tag{3}$$

hence it maps to the corresponding open compact subgroup $P_{\tau(v)}$ of $H'_{F_{\tau(v)}}$ stabilizing $\mathcal{M}_{\tau(v)}^\tau$. This proves the following.

Proposition 5.2. *With the notation established in this section, suppose that τ is the extension to $\mathrm{Aut}(\mathbb{C})$ of an element in $\mathrm{Aut}(F)$. Then, up to similarity, one can take $h' = \tau(h)$. Moreover, for any order \mathcal{M} of D , the isomorphism $H(\mathbb{A}_F^\infty) \rightarrow H'(\mathbb{A}_F^\infty)$ in Theorem 5.1 sends the open compact subgroup of H_{F_v} stabilizing the local lattice \mathcal{M}_v defined in (2) to the open compact subgroup of $H'_{F_{\tau(v)}}$ stabilizing the local lattice $\mathcal{M}_{\tau(v)}^\tau$ defined in (3).*

6. Buildings and their vertices

This section continues with the notation established in Section 5. Our proof that certain fake projective plane groups have isomorphic profinite completions requires very specific information about the maximal compact subgroups in the arithmetic definition of each group. This section briefly introduces the building X_v associated with the groups H_{F_v} over the nonarchimedean completions F_v of F and the open compact subgroups of H_{F_v} stabilizing vertices of X_v .

The algebraic groups H_{F_v} we will need to understand are locally isomorphic to either $SL_3(F_v)$ or the special unitary group $SU(h, E_\mu)$ of a hermitian form h with respect to a quadratic extension E_μ/F_v . We handle each case individually.

6.1. The building for $SL_3(F_v)$

This case is quite classical, and we refer to either [37, Section 2.8] or [13, Section 2] for details. The vertices of X_v are homothety classes of lattices $\mathcal{L} \subset F_v^3$. We denote the homothety class of \mathcal{L} by $[\mathcal{L}]$. Here, a lattice means an \mathcal{O}_v -submodule of rank 3, where \mathcal{O}_v is the valuation ring of F_v . Choosing a uniformizer π for \mathcal{O}_v , distinct vertices $[\mathcal{L}_1]$ and $[\mathcal{L}_2]$ are adjacent if there are representatives \mathcal{L}_1 and \mathcal{L}_2 for the homothety classes such that $\pi\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_1$. Chambers of X_v consist of triples of mutually adjacent vertices.

Then $PGL_3(F_v)$ acts transitively on the vertices of X_v . This implies that all vertices are *hyperspecial*, and the stabilizer $P_{\mathcal{L}} < SL_3(F_v)$ of $[\mathcal{L}]$ is a maximal open compact subgroup. We call $P_{\mathcal{L}}$ a *hyperspecial parahoric subgroup*, and note that all maximal parahoric subgroups are conjugate in $GL_3(F_v)$. For consistency with the unitary case, we say that all vertices of the building are *Type 1* vertices.

6.2. The building for $SU(h, E_\mu)$

See [13, Section 2] for an excellent detailed construction of the building from our point of view. In this case, the building X_v is a tree. As with $SL_3(F_v)$, the vertices are related to homothety classes of lattices. Given a lattice \mathcal{L} , let \mathcal{L}^* denote its dual with respect to the form h . It ends up that either $[\mathcal{L}^*] = [\mathcal{L}]$ or the two homothety classes define adjacent vertices in the building for $SL_3(E_\mu)$. The vertices of X_v fall into two classes:

Type 1: Classes $[\mathcal{L}]$ for which $[\mathcal{L}] = [\mathcal{L}^*]$.

Type 2: Unordered pairs $([\mathcal{L}], [\mathcal{L}^*])$ of neighbors in the building for $SL_3(E_\mu)$.

A Type 1 vertex $[\mathcal{L}']$ is adjacent to a Type 2 vertex $([\mathcal{L}], [\mathcal{L}^*])$ if and only if the triple forms a chamber in the building for $SL_3(E_\mu)$. A vertex is never adjacent to another of the same type. Type 1 vertices are hyperspecial and Type 2 vertices are special, but not hyperspecial. The action of $SU(h, E_\mu)$ preserves the type of a vertex, and the action is transitive on vertices of a given type. The maximal open compact subgroups of $SU(h, E_\mu)$ are vertex stabilizers, hence there are two conjugacy classes.

7. Isomorphic algebraic fundamental groups

We now prove the isomorphisms between algebraic fundamental groups that, combined with Theorem 4.12, complete the proof of Theorem 1.1. The general progression for each pair $\{j, k\}$ is:

- (1) Describe the number field F and F -algebraic group H associated with the pair.
- (2) Relate Γ_j and Γ_k to certain congruence subgroups of $H(F)$.
- (3) Study the action of the extension to $\text{Aut}(\mathbb{C})$ of a generator τ for $\text{Gal}(F/\mathbb{Q})$ on the congruence quotients of the ball related to Γ_j and Γ_k using Theorem 5.1.
- (4) When Γ_j and Γ_k do not themselves arise from congruence subgroups of $H(F)$, use additional covering space arguments to study the action of τ on the relevant fake projective planes.

It follows in each case that $\widehat{\Gamma}_j \cong \widehat{\Gamma}_k$, as desired. When Theorem 5.1 does not apply directly to the lattices at hand we must use nearby auxiliary lattices, which introduces some technical annoyances. In an attempt to make each case relatively self-contained, there is some near-verbatim repetition in the presentation of each case. However, each case also requires at least one ingredient not seen in the other three.

7.1. The pair $\{34, 35\}$

We follow [27, Section 9.3]. See [6, 7] for a more concrete description of these groups as matrix groups. This is the most complicated pair of the four, so a first-time reader may want to read the later cases first.

7.1.1. The number fields and some prime ideals. Consider $F = \mathbb{Q}(\alpha)$ with $\alpha^2 = 5$ and $E = F(\beta)$ with $\beta^2 = -3$. There is a unique prime ideal \mathfrak{p}_2 of F dividing 2, and \mathfrak{p}_2 splits as a product of two primes \mathfrak{q}_{2+} and \mathfrak{q}_{2-} of E . All three primes have residue field \mathbb{F}_4 .

7.1.2. The algebraic group and open compact subgroups. The simply connected F -algebraic group H is defined in [27, Section 9.1] using the unique cubic division algebra D over E with nontrivial invariants at the two primes of E dividing 2 and a hermitian element $h \in D$. We refer to [27] for details of the construction, which will not concern us. We now define open compact subgroups of H_{F_ν} for each nonarchimedean place ν .

If $|\nu|$ denotes the residue characteristic of ν and $|\nu| \geq 3$, we take P_ν to be a hyperspecial parahoric subgroup, that is, the stabilizer of a Type 1 vertex of X_ν . It remains to define the subgroup $P_2 < H_{F_2}$, where F_2 is the completion associated with \mathfrak{p}_2 . For each prime $\mathfrak{q}_{2\pm}$ of E dividing \mathfrak{p}_2 , the completion $D_{2\pm}$ of D is a division algebra. The units of norm 1 in $D_{2\pm}$ are isomorphic as abstract groups and are both isomorphic to H_{F_2} . Specifically,

$$H_{F_2} \cong \{(x, y) \in D_{2+}^1 \times D_{2-}^1 : \sigma(x, y)(h, h)(x, y) = (h, h)\},$$

where σ extends to the factor-swapping isomorphism, so σ -hermitian elements of D map in diagonally. It follows that $y = h x h^{-1}$ defines an isomorphism between H_{F_2} and D_{2+}^1 . Under the above isomorphism of H_{F_2} with D_{2+}^1 , let \mathcal{O}_{2+} be the unique maximal order of D_{2+} . Then $P_2 \cong \mathcal{O}_{2+}^1$.

7.1.3. *The arithmetic groups.* Let $K = \prod P_v < H(\mathbb{A}_F^\infty)$ with P_v defined in Section 7.1.2 and set $\Gamma = H(F) \cap K$. Let $\Lambda < \text{SU}(2, 1)$ be the normalizer of Γ and $\bar{\Gamma}, \bar{\Lambda}$ denote the images of these groups in $\text{PU}(2, 1)$. The group Γ_{32} is the index 3 normal congruence subgroup of $\bar{\Gamma}$ described in [27, Remark 9.4]. In particular, Γ_{32} is the congruence subgroup of $H^{\text{ad}}(F)^+$ associated with the open compact subgroup

$$K' = P'_2 \times \prod_{|v| \geq 3} P_v < H(\mathbb{A}_F^\infty), \tag{4}$$

where P'_2 is the kernel of the unique homomorphism from P_2 to $\mathbb{Z}/3$.

The groups Γ_j for $j = 32, 34, 35$ then fit into the diagram of Figure 2. All neighboring inclusions in the diagram are index 3. We now give a sequence of facts that combine information from [27, Section 9.3], Cartwright and Steger’s presentation for $\bar{\Lambda}$ [8], and Magma to give a more detailed description of the groups in Figure 2.

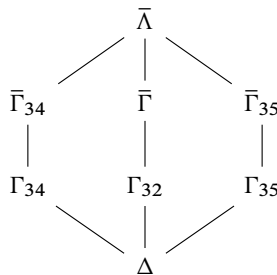


Fig. 2. The diagram of fake projective plane groups for $j = 32, 34, 35$.

Fact 7.1. *The group $\bar{\Lambda}$ contains a unique subgroup of index 3 admitting a homomorphism onto $\mathbb{Z}/21$. This must be $\bar{\Gamma}$.*

Fact 7.2. *The group $\bar{\Gamma}$ contains a unique normal subgroup of index 3. This must be Γ_{32} .*

Fact 7.3. *The group $\Delta = \Gamma_{32} \cap \Gamma_{34}$ equals $\Gamma_{32} \cap \Gamma_{35}$. Then Δ is a normal index 3 subgroup of both Γ_{34} and Γ_{35} , but is not normal in Γ_{32} . Among all conjugacy classes of index 3 subgroups of Γ_{32} , Δ is in the unique conjugacy class that admits four distinct homomorphisms onto S_3 .*

Fact 7.4. *The lift of any element of $\bar{\Gamma}$ to D^* has reduced norm contained in $(E^*)^3$.*

Fact 7.5. *In terms of Cartwright and Steger’s generators AA, BB, ZZ for $\bar{\Gamma}$ in [8], coset representatives for Δ in Γ_{34} are given by $(\text{BBZZ}^{-1})^{\pm 2}$.*

Fact 7.6. *In terms of Cartwright and Steger’s generators AA, BB, ZZ for $\bar{\Gamma}$ in [8], coset representatives for Δ in Γ_{35} are given by $(BBZZ)^{\pm 2}$.*

Let $\xi = \frac{1}{2}(-1 + \beta)$ be a primitive 3rd root of unity and set

$$b = \det(BB) = \frac{1 - \alpha\beta}{4}.$$

By Fact 7.4, any lift to D^* of an element of Δ has reduced norm contained in $(E^*)^3$. Then $\det(Z) = \xi^2$, and Facts 7.5–7.6 imply that Γ_{34} is distinguished from Γ_{35} by the property that every preimage $\hat{\gamma}$ of $\gamma \in \Gamma_j$ in D^* has reduced norm contained in the subgroup

$$N_j = \langle b^{2m_j} \xi (E^*)^3 \rangle \subset E^*$$

where $m_{34} = 1$ and $m_{35} = 2$. The identification of Γ_j/Δ with $\mathbb{Z}/3$ is given by the isomorphism $N_j/(E^*)^3 \cong \mathbb{Z}/3$.

7.1.4. *The isomorphism of profinite completions for $j = 34, 35$.*

Proposition 7.7. *Let X_{34}, X_{35} be fake projective planes with fundamental groups Γ_{34}, Γ_{35} and choose a representative $\tau \in \text{Aut}(\mathbb{C})$ for the generator of $\text{Gal}(F/\mathbb{Q})$. Then, up to complex conjugation, X_{34}^τ is isomorphic to X_{35} . Therefore $\hat{\Gamma}_{34} \cong \hat{\Gamma}_{35}$.*

Proof. Choosing the appropriate complex structures, the diagram of subgroups in Figure 2 implies that we have the diagram of $\mathbb{Z}/3$ -étale coverings in Figure 3, where Y is the ball quotient associated with Δ . Let $p_j : Y \rightarrow X_j$ be the covering and ψ_j a generator for its deck group. We will prove, as indicated in Figure 3, that $Y^\tau \cong Y$, which implies that τ acts on $\text{Aut}(Y)$. We then show that, up to the correct choice of generators for the deck groups, $\psi_{34}^\tau = \psi_{35}$ in the sense of Section 2. This implies that $p_{34}^\tau = p_{35}$, hence $X_{34}^\tau \cong X_{35}$, which proves the proposition.

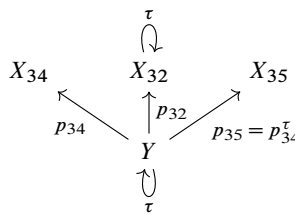


Fig. 3. The common $\mathbb{Z}/3$ cover of X_{32}, X_{34} , and X_{35} .

To prove that $Y^\tau \cong Y$, we first show that $X_{34}^\tau \cong X_{32}$, which is a consequence of Proposition 5.2. To see this, Γ_{32} is the group associated with the open compact subgroup K' of $H(\mathbb{A}_F^\infty)$ defined in (4). For places ν with $|\nu| \geq 3$, P_ν is the stabilizer of a Type 1 vertex of the building X_ν (i.e., P_ν is hyperspecial), hence the image of K under the self-isomorphism of $H(\mathbb{A}_F^\infty)$ induced by τ is hyperspecial at all such ν . For p_2 , the image of P_2 is the unique maximal order of $D_{2+} \cong D_{2-}$, so it is again P_2 under the isomorphism induced by τ . Since P'_2 is the kernel of the unique homomorphism from P_2 to $\mathbb{Z}/3$, the

image of P'_2 is also isomorphic to P'_2 . This implies that, up to conjugacy in $H(\mathbb{A}_F^\infty)$, the self-isomorphism takes K' to K' . In other words, X_{32}^τ has fundamental group Γ_{32} and so $X_{32}^\tau \cong X_{32}$.

Then, applying Fact 7.3 to Y^τ , it is the unique connected étale cover of $X_{32}^\tau = X_{32}$ whose fundamental group admits four distinct homomorphisms onto S_3 . By Theorem 3.2 we see that, up to conjugacy in Γ_{32} , $\pi_1(Y^\tau)$ equals $\pi_1(Y)$. Therefore, $Y^\tau \cong Y$, which proves the first assertion.

We now study ψ_{34}^τ by lifting to a self-isometry of the universal cover. A choice of lift of ψ_{34} is given by choosing a coset representative γ_{34} for $\bar{\Gamma}$, where the choice of nontrivial coset representative is equivalent to a choice of generator for the deck group. Following the notation in Section 7.1.3, we choose γ_{34} so that any lift of γ_{34} to D^* has determinant contained in $b^2\xi(E^*)^3$.

As in [26, Section 5.2], we identify complex hyperbolic space \mathbb{B}^2 with the space $\{\mathcal{Q}_z\}_{z \in \mathbb{B}^2}$ of PEL-structures of type Ω , with Ω defined from the algebra D and hermitian form h as in Section 5. The lattice \mathcal{M} associated with the PEL-type Ω is stabilized by $\Gamma \leq D^1$. The lift of ψ_{34}^τ to the universal cover of X_{34}^τ is then given by

$$\mathcal{Q}_z^\tau \mapsto (\tau \circ \gamma_{34} \circ \tau^{-1})\mathcal{Q}_z^\tau = \gamma_{34}^\tau \mathcal{Q}_z^\tau,$$

where γ_{34}^τ is in the sense of the $\text{Aut}(\mathbb{C})$ -action as in (1), not the $\text{Gal}(F/\mathbb{Q})$ -action on D .

By Corollary 2.2, γ_{34}^τ is the lift to \mathbb{B}^2 of an element $\text{Aut}(Y)$ of order 3 with no fixed points and quotient a fake projective plane. We can then choose a lift $\hat{\gamma}_{34}$ of γ_{34} to D^* , and the action of γ_{34}^τ on PEL-types takes $\Omega^\tau = (D^\tau, \Phi^\tau, \sigma^\tau; \psi^\tau, \mathcal{M}^\tau)$ to the equivalent PEL-type

$$\Omega' = (D^\tau, \Phi^\tau, \sigma^\tau; \psi^\tau, (\mathcal{M}\hat{\gamma}_{34})^\tau)$$

(see [26, Section 5.4, Corollary]), where now we can consider $(\mathcal{M}\hat{\gamma}_{34})^\tau$ as the τ -conjugate for the Galois action on $\mathcal{M} \subset D$ by identifying $D_\mathbb{R}$ with \mathbb{C}^9 . Taking the determinant, this implies that a lift $\hat{\gamma}_{34}^\tau$ of γ_{34}^τ to D^* has reduced norm contained in $\tau(\det(\hat{\gamma}_{34}))(E^*)^3 = \tau(b^2\xi)(E^*)^3$.

Then $\tau(b^2) = b^{-2}$ and $\tau(\xi) = \xi$, so any representative for γ_{34}^τ in D^* has reduced norm contained in $b\xi(E^*)^3$. This is a nontrivial coset representative for Δ in Γ_{35} by Fact 7.6, hence (up to the correct choice of generator) $\psi_{34}^\tau = \psi_{35}$. This completes the second and final step required to prove the proposition. ■

Remark 7.8. Proposition 7.7 should also follow from work of Borisov and Fatighenti. See [19] and the auxiliary files referenced there. Their work gives equations for the relevant pair of fake projective planes that are invariant under $\text{Gal}(F/\mathbb{Q})$. It seems difficult to rigorously verify that the $\text{Gal}(F/\mathbb{Q})$ -orbit of equations derived from [19] are indeed for two distinct fake projective planes, so we give the above proof instead.

7.2. The pair {43, 44}

This case is arguably the easiest of the four. We follow [28, Section A.6]. See [9, 10] for a more concrete description of these groups as matrix groups.

7.2.1. *The number fields and some prime ideals.* Consider $F = \mathbb{Q}(\alpha)$ with $\alpha^2 = 6$ and $E = F(\beta)$ with $\beta^2 = -3$. There is a unique prime ideal \mathfrak{p}_2 of F dividing 2, and \mathfrak{p}_2 is divisible by a unique prime \mathfrak{q}_2 of E . The residue fields are \mathbb{F}_2 and \mathbb{F}_4 , respectively. There is also a unique prime \mathfrak{p}_3 of F dividing 3, which is ramified over \mathbb{Q} with residue field \mathbb{F}_3 . Then \mathfrak{p}_3 splits into two primes $\mathfrak{q}_{3\pm}$ of E , which also have residue field \mathbb{F}_3 .

7.2.2. *The algebraic group and open compact subgroups.* The simply connected F -algebraic group H is defined in [27, Section 9.1] using the unique cubic division algebra D over E with nontrivial invariants at the two primes of E dividing 3 and a hermitian element $h \in D$. We refer to [27] for details of the construction, which will not concern us. We now define open compact subgroups of H_{F_ν} for each nonarchimedean place ν .

Let F_2 be the completion of F associated with \mathfrak{p}_2 . Then H_{F_2} is the special unitary group of h with respect to the extension E_2/F_2 , where E_2 is the associated completion of E . We take P_2 to be the nonhyperspecial maximal parahoric subgroup stabilizing a Type 2 vertex of the building X_2 for H_{F_2} . If $|\nu|$ denotes the residue characteristic of ν and $|\nu| \geq 5$, we take P_ν to be a hyperspecial parahoric subgroup, that is, the stabilizer of a Type 1 vertex of X_ν .

It remains to define the subgroup $P_3 < H_{F_3}$, where F_3 is the completion associated with \mathfrak{p}_3 . For each prime $\mathfrak{q}_{3\pm}$ of E dividing \mathfrak{p}_3 , the completion $D_{3\pm}$ of D is a division algebra. The units of norm 1 in $D_{3\pm}$ are isomorphic as abstract groups and are mutually isomorphic to H_{F_3} . Then

$$H_{F_3} \cong \{(x, y) \in D_{3+}^1 \times D_{3-}^1 : \sigma(x, y)(h, h)(x, y) = (h, h)\},$$

where σ extends to the factor-swapping isomorphism, so σ -hermitian elements of D map in diagonally. It follows that $y = h x h^{-1}$ defines an isomorphism between H_{F_3} and D_{3+}^1 . Under the above isomorphism of H_{F_3} with D_{3+}^1 , let \mathcal{O}_{3+} be the unique maximal order of D_{3+} . Then $P_3 \cong \mathcal{O}_{3+}^1$.

7.2.3. *The arithmetic groups.* Let $K = \prod P_\nu \leq H(\mathbb{A}_F^\infty)$ with P_ν defined in Section 7.2.2 and set $\Gamma = H(F) \cap K$. Let $\Lambda < \text{SU}(2, 1)$ be the normalizer of Γ and $\bar{\Gamma}, \bar{\Lambda}$ denote the images of these groups in $\text{PU}(2, 1)$. The groups Γ_j for $j = 42, 43, 44$ then fit into the diagram of Figure 4, where all subgroups are normal. In fact, $\bar{\Lambda}/\Gamma_j$ and $\Gamma_j/\bar{\Gamma}$ are isomorphic to $\mathbb{Z}/3$ for all j .

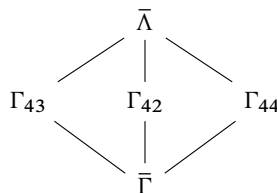


Fig. 4. The diagram of fake projective plane groups for $j = 42, 43, 44$.

Let $\xi = \frac{1}{2}(-1 + \beta)$ be a primitive 3rd root of unity and define

$$a = (1 + \alpha\beta)(1 - \alpha\beta)^{-1}.$$

Then, as described in [28, Section A.6], Γ_j is characterized by the property that every preimage $\hat{\gamma}$ of $\gamma \in \Gamma_j$ in D^* has reduced norm contained in the subgroup

$$N_j = \langle a^{m_j} \xi (E^*)^3 \rangle \subset E^*$$

for some $m_j \in \{0, 1, 2\}$. The identification of $\Gamma_j/\bar{\Gamma}$ with $\mathbb{Z}/3$ is given by the isomorphism $N_j/(E^*)^3 \cong \mathbb{Z}/3$. The notation in [5] (see Table 9) means that $m_{42} = 0$, $m_{43} = 1$, and $m_{44} = 2$.

7.2.4. The isomorphism of profinite completions for $j = 43, 44$.

Proposition 7.9. *Let X_{43}, X_{44} be fake projective planes with fundamental groups Γ_{43}, Γ_{44} and choose a representative $\tau \in \text{Aut}(\mathbb{C})$ for the generator of $\text{Gal}(F/\mathbb{Q})$. Then, up to complex conjugation, X_{43}^τ is isomorphic to X_{44} . Therefore $\hat{\Gamma}_{43} \cong \hat{\Gamma}_{44}$.*

Proof. Choosing the appropriate complex structures, the diagram of subgroups in Figure 4 implies that we have the diagram of $\mathbb{Z}/3$ -étale coverings in Figure 5, where Y is the ball quotient associated with $\bar{\Gamma}$. Let $p_j : Y \rightarrow X_j$ be the covering and ψ_j a generator for its deck group. We will prove, as indicated in Figure 5, that $Y^\tau \cong Y$, which implies that τ acts on $\text{Aut}(Y)$. We then show that, up to the correct choice of generators for the deck groups, $\psi_{43}^\tau = \psi_{44}$ in the sense of Section 2. This implies that $p_{43}^\tau = p_{44}$, hence $X_{43}^\tau \cong X_{44}$, which proves the proposition.

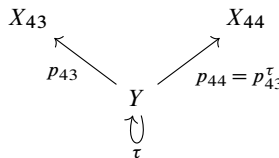


Fig. 5. The common $\mathbb{Z}/3$ cover of X_{43} and X_{44} .

The fact that $Y^\tau \cong Y$ is a consequence of Proposition 5.2. To see this, note that $\bar{\Gamma} = \pi_1(Y)$ is the group associated with the open compact subgroup K of $H(\mathbb{A}_F^\infty)$ defined in Section 7.2.3. For places v with $|v| \geq 5$, P_v is the stabilizer of a Type 1 vertex of the building X_v (i.e., P_v is hyperspecial), hence the image of K under the self-isomorphism of $H(\mathbb{A}_F^\infty)$ induced by τ is hyperspecial at every v . For p_3 , the image open compact subgroup is defined by the unique maximal order of $D_{3+} \cong D_{3-}$, so it is again P_3 under the isomorphism induced by τ . Lastly, P_2 is the stabilizer of a Type 2 vertex of the associated building, hence so is the image group. This implies that, up to conjugacy in $H(\mathbb{A}_F^\infty)$, the self-isomorphism takes K to K . In other words, Y^τ has fundamental group $\bar{\Gamma}$ and so $Y^\tau \cong Y$. This proves the first assertion.

We now study ψ_{43}^τ by lifting to a self-isometry of the universal cover. A choice of lift of ψ_{43} is given by choosing a coset representative γ_{43} for $\bar{\Gamma}$ in Γ_{43} , where the choice of nontrivial coset representative is equivalent to a choice of generator for the Deck group. Following the notation in Section 7.2.3, we choose γ_{43} so that any lift of γ_{43} to D^* has determinant contained in $a\xi(E^*)^3$.

As in [26, Section 5.2], we identify complex hyperbolic space \mathbb{B}^2 with the space $\{\mathcal{Q}_z\}_{z \in \mathbb{B}^2}$ of PEL-structures of type Ω , with Ω defined from the algebra D and hermitian form h as in Section 5. The lattice \mathcal{M} associated with the PEL-type Ω is stabilized by $\Gamma \leq D^1$. The lift of ψ_{43}^τ to the universal cover of Y_{43}^τ is then given by

$$\mathcal{Q}_z^\tau \mapsto (\tau \circ \gamma_{43} \circ \tau^{-1})\mathcal{Q}_z^\tau = \gamma_{43}^\tau \mathcal{Q}_z^\tau,$$

where γ_{43}^τ is in the sense of the $\text{Aut}(\mathbb{C})$ -action as in (1), not the $\text{Gal}(F/\mathbb{Q})$ -action on D .

By Corollary 2.2, γ_{43}^τ is the lift to \mathbb{B}^2 of an element $\text{Aut}(Y)$ of order 3 with no fixed points and quotient a fake projective plane. We can then choose a lift $\hat{\gamma}_{43}$ of γ_{43} to D^* , and the action of γ_{43}^τ on PEL-types takes $\Omega^\tau = (D^\tau, \Phi^\tau, \sigma^\tau; \psi^\tau, \mathcal{M}^\tau)$ to the equivalent PEL-type

$$\Omega' = (D^\tau, \Phi^\tau, \sigma^\tau; \psi^\tau, (\mathcal{M}\hat{\gamma}_{43})^\tau)$$

(see [26, Section 5.4, Corollary]), where now we can consider $(\mathcal{M}\hat{\gamma}_{43})^\tau$ as the τ -conjugate for the Galois action on $\mathcal{M} \subset D$ by identifying $D_\mathbb{R}$ with \mathbb{C}^9 . Taking the determinant, this implies that a lift $\hat{\gamma}_{43}^\tau$ of γ_{43}^τ to D^* has reduced norm contained in $\tau(\det(\hat{\gamma}_{43}))(E^*)^3 = \tau(a\xi)(E^*)^3$.

Then $\tau(a) = a^{-1}$ and $\tau(\xi) = \xi$, so any representative for γ_{43}^τ in D^* has reduced norm contained in $a^2\xi(E^*)^3$. This is a nontrivial coset representative for $\bar{\Gamma}$ in Γ_{44} , hence (up to the correct choice of generator) $\psi_{43}^\tau = \psi_{44}$. This completes the second and final step required to prove the proposition. ■

7.3. The pairs {47, 49} and {48, 50}

These pairs are commensurable and best handled together. The argument for these cases benefits from slightly more refined information about the commensurability class than was needed for previous cases. We follow [11], but change some notation.

7.3.1. The number fields. Let $F = \mathbb{Q}(\alpha)$ with $\alpha^2 = 7$ and $E = F(\beta)$ with $\beta^2 = -1$. If ζ is a primitive 7th root of unity and $L = E(\zeta)$, then L/E is a cubic Galois extension with Galois group generated by the element ϕ characterized by

$$\phi(\zeta) = \zeta^2.$$

The maximal totally real subfield of L is $L^+ = F(\omega)$, where

$$\omega = \zeta + \zeta^{-1},$$

which has minimal polynomial $t^3 + t^2 - 2t - 1$. Then $\phi(\omega) = \omega^2 - 2$ and the restriction of ϕ to L^+ generates the Galois group of L^+/F .

7.3.2. *Some prime ideals.* There is a unique prime ideal \mathfrak{p}_2 of F dividing 2, and \mathfrak{p}_2 splits as a product of two primes \mathfrak{q}_{2+} and \mathfrak{q}_{2-} of E . All three primes have residue field \mathbb{F}_2 , and all three ideals are principal; we can take $\pi_2 = 3 + \alpha$ as a generator for \mathfrak{p}_2 , $\sigma_{2+} = 1 + \frac{\alpha-\beta}{2}$ as a generator for \mathfrak{q}_{2+} , and $\sigma_{2-} = 1 - \frac{\alpha+\beta}{2}$ as a generator for \mathfrak{q}_{2-} . Both of $\mathfrak{q}_{2\pm}$ remain prime in L , hence $\sigma_{2\pm}$ are generators of prime ideals of L with residue field \mathbb{F}_8 .

There are two primes $\mathfrak{p}_{3\pm}$ of F dividing 3, and each defines a unique prime ideal $\mathfrak{q}_{3\pm}$ of E . The residue fields are \mathbb{F}_3 and \mathbb{F}_9 , respectively. One can take $\pi_{3\pm} = 2 \pm \alpha$ as a generator for either $\mathfrak{p}_{3\pm}$ or $\mathfrak{q}_{3\pm}$. This ordering is chosen so that α reduces modulo $\mathfrak{p}_{3\pm}$ to ± 1 . The completion of F with respect to either place $v_{3\pm}$ associated with $\mathfrak{p}_{3\pm}$ is isomorphic to \mathbb{Q}_3 and the corresponding completion of E is isomorphic to the unique unramified quadratic extension E_3 of \mathbb{Q}_3 . There are two primes of L dividing 3 with residue field \mathbb{F}_{729} , and the completion of L with respect to either prime is the unique unramified cubic extension L_3 of E_3 .

There is a unique prime \mathfrak{p}_7 of F dividing 7, which has generator $\pi_7 = \alpha$ and residue field \mathbb{F}_7 . Then π_7 continues to generate the unique prime ideal \mathfrak{q}_7 of E dividing \mathfrak{p}_7 , which has residue field \mathbb{F}_{49} . The unique prime ideal $\hat{\mathfrak{q}}_7$ of L dividing \mathfrak{q}_7 has generator $\hat{\pi}_7 = \zeta - 1$ and residue field \mathbb{F}_{49} (i.e., the prime \mathfrak{q}_7 is totally ramified in L).

7.3.3. *The algebra.* Consider the cubic division algebra D over E generated by L and an element y subject to the relations

$$\begin{aligned}
 yxy^{-1} &= \phi(x) \quad \text{for all } x \in L, \\
 y^3 &= \frac{3 + \alpha\beta}{4}.
 \end{aligned}$$

Set $\delta = y^3$. Then

$$\iota(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & \phi(x) & 0 \\ 0 & 0 & \phi^2(x) \end{pmatrix} \tag{5}$$

$$\iota(y) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{pmatrix} \tag{6}$$

is an embedding of D in $M_3(L)$ such that the subgroup D^1 of elements of reduced norm 1 consists precisely of the elements that map to $SL_3(L)$. The assignments

$$\begin{aligned}
 \sigma(\alpha) &= \alpha & \sigma(\omega) &= \omega \\
 \sigma(\zeta) &= \zeta^{-1} & \sigma(y) &= y^{-1}
 \end{aligned}$$

extend to an anti-involution of D that is the restriction of conjugate transposition on $M_3(L)$ with respect to $\text{Gal}(L/L^+)$ under either embedding.

Using the factorization

$$\delta = (3\alpha - 8)\sigma_{2+}^2\sigma_{2-}^{-2}, \tag{7}$$

with $3\alpha - 8$ a unit of F , one checks that D remains a division algebra over the localization of E at each of $\mathfrak{q}_{2\pm}$ and that these are the unique places of E where D remains a division algebra; see [11]. Let F_2 be the completion of F at the place associated with \mathfrak{p}_2 , and let $D_{2\pm}$ be the completion of D with respect to the completion of E associated with $\mathfrak{q}_{2\pm}$. Then D_{2-} is the opposite algebra of D_{2+} . For every other place μ of E with completion E_μ , $D \otimes_E E_\mu$ is isomorphic to $M_3(E_\mu)$.

Remark 7.10. If one wants y^3 to be an algebraic integer, δ could be replaced with $(3\alpha - 8)\sigma_{2+}^2\sigma_{2-} = \sigma_{2-}^3\delta$.

7.3.4. *The hermitian element and algebraic group.* The element

$$\begin{aligned} h &= \frac{3 - \alpha - 3\beta + \alpha\beta}{2} + \frac{3 - \alpha - 3\beta + \alpha\beta}{2}\zeta + \left(1 - \frac{\alpha + \beta}{2}\right)\zeta^2 \\ &= \left(2 - \frac{4\alpha}{7}\right) + \left(1 - \frac{3\alpha}{7}\right)\omega - \frac{\alpha}{7}\omega^2 \end{aligned}$$

of $L^+ \subset D$ maps to a hermitian element of $M_3(L)$ with respect to the Galois involution of L/L^+ . Let $v_j : F \rightarrow \mathbb{R}$ denote the embedding generated by

$$v_j(\alpha) = (-1)^j\sqrt{7}$$

for $j \in \{0, 1\}$. Then v_j extends to a complex conjugate pair of embeddings of E into \mathbb{C} . Fix one such extension (e.g., $v_j(\beta) = i$). Defining $v_j(\omega) = 2\cos(2\pi/7)$ further extends v_j to a complex embedding of L (the other extensions are obtained by precomposition with ϕ). Applying v_j to matrices then gives us a pair of embeddings, still denoted by v_j , of D into $M_3(\mathbb{C})$. One checks that $v_0(h)$ is a hermitian form of signature $(1, 2)$ and $v_1(h)$ has signature $(3, 0)$.

Our simply connected semisimple F -algebraic group H is then the special unitary group of h . That is,

$$H(F) = \{z \in D^1 : \sigma(z)hz = h\}$$

and $H_*(\mathbb{R}) \cong \text{SU}(1, 2) \times \text{SU}(3)$. We now define the open compact subgroups K_j of $H(\mathbb{A}_F^\infty)$ for $j \in \{47, 49\}$.

7.3.5. *The open compact subgroup for \mathfrak{p}_2 .* The units of norm 1 in $D_{2\pm}$ are isomorphic as abstract groups and are both isomorphic to H_{F_2} . Then

$$H_{F_2} \cong \{(x, y) \in D_{2+}^1 \times D_{2-}^1 : \sigma(x, y)(h, h)(x, y) = (h, h)\},$$

where σ extends to the factor-swapping isomorphism, so σ -hermitian elements of D map in diagonally. It follows that $y = h x h^{-1}$ defines an isomorphism between H_{F_2} and D_{2+}^1 .

Under the above isomorphism of H_{F_2} with D_{2+}^1 , let \mathcal{O}_{2+} be the unique maximal order of D_{2+} . Then $P_2 \cong \mathcal{O}_{2+}^1$.

7.3.6. *The open compact subgroups for $|v| \geq 5$.* Recall that $|v|$ denotes the residue characteristic of a place v of F , and assume $|v| \geq 5$. For every place μ of E dividing v with completion E_μ , $D \otimes_E E_\mu$ is isomorphic to $M_3(E_\mu)$. For all places v of F other than \mathfrak{p}_2 , this implies

$$H_{F_v} \cong \begin{cases} \text{SU}(h, E_\mu) & v \text{ inert in } E \\ \text{SL}_3(F_v) & v \text{ splits in } E \end{cases}$$

where $\text{SU}(h, E_\mu)$ denotes the special unitary group of h with respect to the quadratic extension E_μ/F_v . If $|v| \neq 7$, take P_v to be the open compact subgroup consisting of all elements of H_{F_v} that are v -adically integral with respect to the above basis. Then P_v is hyperspecial for all such v .

Now, consider the unique place v_7 dividing 7, which is associated with the prime \mathfrak{p}_7 defined in Section 7.3.2. There is a unique place μ_7 of E dividing v_7 and the associated local field extension E_7/F_7 is an unramified quadratic extension. Note that the factorization in (7) implies that δ is a unit of E_7 (i.e., has valuation zero). Similarly, there is a unique place $\hat{\mu}_7$ of L dividing μ_7 , and the completion L_7 is a totally ramified cubic extension of E_7 . Even more, we will need the unique prime \hat{v}_7 of L^+ divisible by $\hat{\mu}_7$ and the associated completion L_7^+ . The group P_7 is the hyperspecial parahoric subgroup stabilizing a certain Type 1 vertex of the building X_7 described in detail in [12].

While we do not need such specific information, we briefly describe the lattice stabilized by P_7 . One can find a unit $\eta \in L_7^\times$ such that $N_{L_7/E_7}(\eta) = \delta$ and $N_{L_7/L_7^+}(\eta) = 1$. The element

$$c_\eta = \begin{pmatrix} \eta\phi(\eta) & 0 & 0 \\ 0 & \phi(\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

centralizes $\iota(x)$ for $x \in L$ with $\iota(x)$ as in (5) and conjugates $\iota(y)$ to

$$\begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & \phi(\eta) \\ \phi^2(\eta) & 0 & 0 \end{pmatrix}$$

with $\iota(y)$ as in (6). Define

$$c = \begin{pmatrix} 1 & 1 & 1 \\ \xi - 1 & \phi(\xi - 1) & \phi^2(\xi - 1) \\ \alpha^{-1}(\xi - 1)^2 & \alpha^{-1}\phi(\xi - 1)^2 & \alpha^{-1}\phi^2(\xi - 1)^2 \end{pmatrix} \tag{9}$$

and the conjugate embedding

$$\iota_\eta(z) = (cc_\eta)\iota(z)(cc_\eta)^{-1}$$

of D into $M_3(L_7)$. The hermitian element h_c for this conjugate embedding is

$$\begin{aligned} & \sigma(cc_\eta)^{-1}h(cc_\eta)^{-1} \\ &= \sigma(c)^{-1}hc^{-1} \\ &= \begin{pmatrix} 13 - 5\alpha & 13 - 5\alpha + 5\beta - 2\alpha\beta & \frac{-21+5\alpha\beta}{2} + 4\alpha - 7\beta \\ 13 - 5\alpha - 5\beta + 2\alpha\beta & 16 - 6\alpha & -14 + \frac{11\alpha-5\beta}{2} + \alpha\beta \\ \frac{-21-5\alpha\beta}{2} + 4\alpha + 7\beta & -14 + \frac{11\alpha+5\beta}{2} - \alpha\beta & 14 - 5\alpha \end{pmatrix}, \end{aligned} \tag{10}$$

which we note is in fact a hermitian matrix in $M_3(E)$. Then $P_7 < H_{F_7}$ is the open compact subgroup of consisting of the $g \in H_{F_7}$ such that $(cc_\eta)g(cc_\eta)^{-1}$ is E_7 -adically integral. In other words, if \mathcal{O}_7 is the ring of integral elements in E_7 , then

$$P_7 \cong \text{GL}_3(\mathcal{O}_7) \cap (cc_\eta)H_{F_7}(cc_\eta)^{-1}.$$

It remains to define the open compact subgroups for the primes dividing 3.

7.3.7. The open compact subgroups for $\mathfrak{p}_{3\pm}$. The construction uses several of the elements used to define P_7 in the previous subsection. We have $F_{3\pm} \cong \mathbb{Q}_3$ the completion associated with $\mathfrak{p}_{3\pm}$ and extensions $E_{3\pm}$, $L_{3\pm}$ and $L_{3\pm}^+$ all associated with the various extensions of F . When we only care about a particular field up to isomorphism, we drop the \pm from the notation. We work in the product $H_{F_{3+}} \times H_{F_{3-}}$ and define maximal compact subgroups of each factor simultaneously.

A similar argument to $|\nu| = 7$, now where all fields are unramified extensions, shows that there exists an element $\eta \in L_3$ such that $N_{L_3/E_3}(\eta) = \delta$ and $N_{L_3/L_3^+}(\eta) = 1$. From this η , we define (c_η, c_η) analogously to (8) and c exactly as in (9). We then define

$$c_+ = \text{diag}(1, 1, \pi_{3+}), \quad c_- = \text{diag}(\pi_{3-}, 1, 1),$$

and consider the two hermitian forms

$$h_\pm = \sigma(c_\pm cc_\eta)^{-1}h(c_\pm cc_\eta)^{-1},$$

which, as with the form in (10), has matrix in $M_3(E)$.

We first study h_+ . Let P'_3 be the elements in $H_{F_{3+}}$ whose conjugate by c_+cc_η preserve the standard E_{3+} lattice and P_3 be the elements in $H_{F_{3-}}$ whose conjugate by c_+cc_η preserves the standard E_{3-} lattice. Then $\pi_{3+}h_+$ has integral matrix entries over E and determinant $2\alpha - 5 \in \mathfrak{p}_{3+}$, where $\nu_{3+}(2\alpha - 5) = 1$ and $\nu_{3-}(2\alpha - 5) = 0$. As described in [12], this implies that the standard lattice in E_{3+} is not self-dual for h_+ but its dual is a neighbor in the Bruhat–Tits building for $\text{PGL}_3(E_3)$. However, $2\alpha - 5$ is a unit of F_{3-} , which implies that the standard lattice is self-dual. These two cases precisely describe the two types of vertex in the Bruhat–Tits tree for $\text{SU}(h_+, E_3)$, where P'_3 stabilizes a Type 2 vertex and P_3 stabilizes a Type 1 vertex (see Section 6.2).

For h_- , the roles are reversed. Indeed, it is now $\pi_{3-}h_-$ that is integral over E , and it has determinant $37 - 14\alpha \in \mathfrak{p}_{3-}$, which has $\nu_{3-}(37 - 14\alpha) = 1$ and $\nu_{3+}(37 - 14\alpha) = 0$.

All Type 1 (resp. Type 2) vertex stabilizers in $SU(h_+, E_{3+}) \cong SU(h_-, E_{3-})$ are conjugate, hence isomorphic (see Section 6.2). In particular, we can identify the elements in $H_{F_{3+}}$ whose conjugate by $c_{-}cc_{\eta}$ preserves the standard E_{3+} lattice with P_3 and P'_3 with the elements in $H_{F_{3-}}$ whose conjugate by $c_{-}cc_{\eta}$ preserves the standard E_{3-} lattice.

7.3.8. *The arithmetic lattices for $j = 47, 49$.* We now define

$$K_j = \begin{cases} P_2 \times P'_3 \times P_3 \times \prod_{|v| \geq 5} P_v & j = 47 \\ P_2 \times P_3 \times P'_3 \times \prod_{|v| \geq 5} P_v & j = 49. \end{cases}$$

Here, the second factor of the product corresponds to the place of F associated with \mathfrak{p}_{3+} and the third factor corresponds to the place for \mathfrak{p}_{3-} . Then the image Γ_j of $\Gamma(K_j)$ in $H^{\text{ad}}(F)^+$ is the fundamental group of the j th complex conjugate pair of fake projective planes. The groups $\Gamma(K_{47})$ and Γ_{47} are the examples called Λ' and $\bar{\Lambda}'$ in [28, Section A.9], and $\Gamma(K_{49}), \Gamma_{49}$ are $\Lambda'', \bar{\Lambda}''$ (this case was mistakenly omitted in [27], hence is not found there).

7.3.9. *The isomorphism of profinite completions for $j = 47, 49$.*

Proposition 7.11. *Let X_{47}, X_{49} be fake projective planes with fundamental groups Γ_{47}, Γ_{49} and choose a representative $\tau \in \text{Aut}(\mathbb{C})$ for the generator of $\text{Gal}(F/\mathbb{Q})$. Then, up to complex conjugation, X_{47}^{τ} is isomorphic to X_{49} . Therefore $\hat{\Gamma}_{47} \cong \hat{\Gamma}_{49}$.*

Proof. By Proposition 5.2, the algebraic group associated with X_{47}^{τ} is the algebraic group with hermitian form $\tau(h)$, which is again H . By Theorem 5.1, we must describe the open compact subgroup at each localization under the isomorphism of algebraic groups over the finite adèles of F . In particular, we must show that the self-isomorphism of $H(\mathbb{A}_F^{\infty})$ induced by τ takes K_{47} to K_{49} . By Proposition 5.2, we only need to understand the τ -conjugate of the local lattice used to defined each open compact subgroup P_v of H_{F_v} .

For $|v| \neq 2, 3, 7$, P_v is defined to be the stabilizer in H_{F_v} of the standard E_{μ} lattice for μ dividing v . Since $\det(h)$ is a unit of F , the standard lattice is self-dual for h , P_v is the stabilizer of a Type 1 vertex of the building for H_{F_v} . Then $\tau(h)$ also has determinant a unit, so the stabilizer of the standard lattice is again P_v . Thus, up to conjugacy in $H_{F_{\tau(v)}}$ we send P_v to $P_v \cong P_{\tau(v)}$ under the isomorphism. For \mathfrak{p}_7 , the determinant of the conjugate h_c defined in (10) is again a unit of F , so the same argument implies that P_7 is mapped to a conjugate of P_7 . For \mathfrak{p}_2 , the open compact subgroup is defined by the unique maximal order of $D_{2+} \cong D_{2-}$, so it is again P_2 under the isomorphism induced by τ .

It remains to consider $|v| = 3$. Here, we have

$$\nu_{3\pm}(\det(\tau(h))) = 1 - \nu_{3\pm}(\det(h)).$$

This means that replacing h with $\tau(h)$ swaps whether or not the standard lattice is self-dual with respect to the conjugate form described in the definition of the open compact subgroups for each of the two 3-adic places in Section 7.3.7. It follows from Proposition 5.2

that $P'_3 \times P_3$ is mapped to $P_3 \times P'_3$ by swapping factors under the Galois action of τ . This combines with the assertions for the other P_ν to prove that the isomorphism of algebraic groups over \mathbb{A}_F^∞ takes K_{47} to K_{49} , which completes the proof of the proposition. ■

7.3.10. The isomorphism of profinite completions for $j = 48, 50$.

Proposition 7.12. *Let X_{48}, X_{50} be fake projective planes with fundamental groups Γ_{48}, Γ_{50} and choose a representative $\tau \in \text{Aut}(\mathbb{C})$ for the generator of $\text{Gal}(F/\mathbb{Q})$. Then, up to complex conjugation, X_{48}^τ is isomorphic to X_{50} . Therefore $\widehat{\Gamma}_{48} \cong \widehat{\Gamma}_{50}$.*

Proof. For $j = 47, 49$, using computer algebra software, we see that there is a fake projective plane X_j with fundamental group Γ_j such that there is a unique connected Galois étale cover $Y_j \rightarrow X_j$ with group S_3 so that Y_j is also a Galois étale cover of X_{j+1} with group $\mathbb{Z}/6$. By Lemma 2.1, the map $Y_{47} \rightarrow Y_{47}^\tau$ induces $X_{47} \rightarrow X_{47}^\tau$. Proposition 7.11 implies that, up to replacing X_{49} with its complex conjugate, $X_{47}^\tau \cong X_{49}$, so $Y_{47}^\tau \cong Y_{49}$ by uniqueness of each Y_j as a 6-fold Galois cover of X_j that also covers X_{j+1} . We then obtain the diagram of maps in Figure 6, where Corollary 2.2 implies the existence of the dashed arrow. Therefore, up to the choice of complex structure on X_{50} determined by the choice of complex structure on X_{49} , this proves that $X_{50} \cong X_{48}^\tau$, which proves the proposition. ■

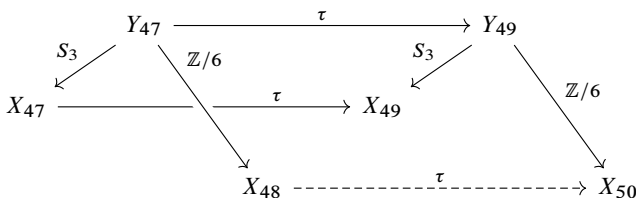


Fig. 6. The cover Y_j for $j = 47, 49$.

Appendix: Fake projective plane groups

We label the fifty (topological) fundamental groups of fake projective planes following notation used by Cartwright–Steger [14] and Prasad–Yeung [27,28]. First, fake projective planes are grouped by the totally imaginary field used to define the relevant commensurability classes of arithmetic lattices. Within each commensurability class, we number the lattices based on their order of appearance in the register of presentations of fake projective plane groups given on Cartwright’s website [5]; this is the file that gives each fake projective plane group the ‘identifier’ given below. Specifically, each table below consists of a number j and the name from [5], meaning that throughout this paper, Γ_j denotes the fundamental group of the given complex conjugate pair of fake projective planes.

j	Identifier from [5]
1	$p = 5, \emptyset, D_3$
2	$p = 5, \{2\}, D_3$
3	$p = 5, \{2i\}$

Tab. 2. The cases $a = 1$ from [5]: $\mathbb{Q}(\sqrt{-1})$.

j	Identifier from [5]
4	$p = 3, \emptyset, D_3$
5	$p = 3, \{2\}, D_3$
6	$p = 3, \{2i\}$

Tab. 3. The cases $a = 2$ from [5]: $\mathbb{Q}(\sqrt{-2})$.

j	Identifier from [5]	j	Identifier from [5]
7	$p = 2, \emptyset, D_3 2_7$	14	$p = 2, \{3\}, D_3$
8	$p = 2, \emptyset, 7_{21}$	15	$p = 2, \{3\}, 3_3$
9	$p = 2, \emptyset, D_3 X_7$	16	$p = 2, \{3, 7\}, D_3$
10	$p = 2, \{7\}, D_3 2_7$	17	$p = 2, \{3, 7\}, 3_3$
11	$p = 2, \{7\}, D_3 7_7$	18	$p = 2, \{5\}$
12	$p = 2, \{7\}, D_3 7'_7$	19	$p = 2, \{5, 7\}$
13	$p = 2, \{7\}, 7_{21}$		

Tab. 4. The cases $a = 7$ from [5]: $\mathbb{Q}(\sqrt{-7})$.

j	Identifier from [5]	j	Identifier from [5]
20	$p = 2, \emptyset, D_3$	25	$p = 2, \{5\}, D_3$
21	$p = 2, \emptyset, 3_3$	26	$p = 2, \{5\}, 3_3$
22	$p = 2, \{3\}, D_3$	27	$p = 2, \{3, 5\}, D_3$
23	$p = 2, \{3\}, 3_3$	28	$p = 2, \{3, 5\}, 3_3$
24	$p = 2, \{3\}, (D_3)_3$	29	$p = 2, \{3, 5\}, (D_3)_3$

Tab. 5. The cases $a = 15$ from [5]: $\mathbb{Q}(\sqrt{-15})$.

j	Identifier from [5]
30	$p = 2, \emptyset$
31	$p = 2, \{23\}$

Tab. 6. The cases $a = 23$ from [5]: $\mathbb{Q}(\sqrt{-23})$.

j	Identifier from [5]	j	Identifier from [5]
32	$p = 2, \emptyset, d_3 D_3$	36	$p = 2, \emptyset, d_3 X'_3$
33	$p = 2, \emptyset, D_3 X_3$	37	$p = 2, \emptyset, X_9$
34	$p = 2, \emptyset, (dD)_3 X_3$	38	$p = 2, \{3\}, d_3 D_3$
35	$p = 2, \emptyset, (d^2D)_3 X_3$		

Tab. 7. The cases \mathcal{E}_2 from [5]: $\mathbb{Q}(\sqrt{5}, e^{2\pi i/3})$.

j	Identifier from [5]
39	$p = 2, \emptyset, D_3$
40	$p = 2, \{17-\}, D_3$

Tab. 8. The cases \mathcal{C}_{10} from [5]: $\mathbb{Q}(\sqrt{-7 + 4\sqrt{2}})$.

j	Identifier from [5]
41	$p = 3, \emptyset, d_3 D_3$
42	$p = 3, \{2\}, D_3$
43	$p = 3, \{2\}, (dD)_3$
44	$p = 3, \{2\}, (d^2D)_3$
45	$p = 3, \{2i\}$

Tab. 9. The cases \mathcal{C}_{18} from [5]: $\mathbb{Q}(\sqrt{6}, e^{2\pi i/3})$.

j	Identifier from [5]
46	$p = 2, \emptyset, D_3 2_7$
47	$p = 2, \{3+\}, D_3$
48	$p = 2, \{3+\}, (3+)_3$
49	$p = 2, \{3-\}, D_3$
50	$p = 2, \{3-\}, (3-)_3$

Tab. 10. The cases \mathcal{C}_{20} from [5]: $\mathbb{Q}(\sqrt{7}, i)$.

Acknowledgments. Thanks are due to Alan Reid for conversations about commensurability and isomorphisms between profinite completions, to Lev Borisov for correspondence about equations for fake projective planes, Jakob Stix for interesting correspondence, and the referees for helpful comments, particularly the shortened path to Theorem 4.12 recorded in Remark 4.14.

Funding. This material is based upon work supported by Grants DMS-1906088 and DMS-2203555 from the National Science Foundation.

References

- [1] Aka, M.: [Arithmetic groups with isomorphic finite quotients](#). J. Algebra **352**, 322–340 (2012) Zbl 1254.20026 MR 2862189
- [2] Bauer, I., Catanese, F., Grunewald, F.: [Faithful actions of the absolute Galois group on connected components of moduli spaces](#). Invent. Math. **199**, 859–888 (2015) Zbl 1318.14034 MR 3314516
- [3] Bosma, W., Cannon, J., Playoust, C.: [The Magma algebra system. I. The user language](#). J. Symbolic Comput. **24**, 235–265 (1997) Zbl 0898.68039 MR 1484478
- [4] Bridson, M. R., McReynolds, D. B., Reid, A. W., Spitler, R.: [Absolute profinite rigidity and hyperbolic geometry](#). Ann. of Math. (2) **192**, 679–719 (2020) Zbl 1455.57028 MR 4172619
- [5] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/bargammapresentations.txt>, presentations of fake projective plane groups (visited on September 2, 2024, and similarly for [6]–[12])

- [6] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C2p2/D.txt>, supplementary file for the case \mathcal{C}_2 , $p = 2$
- [7] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C2p2/padic.gap>, supplementary file for special primes in the case \mathcal{C}_2 , $p = 2$
- [8] Cartwright, D. I., Steger, T.: https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C2p2/gpc2p2generators_reducesyntax.txt, supplementary file for matrix generators in the case \mathcal{C}_2 , $p = 2$
- [9] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C18p3/D.txt>, supplementary file for the case \mathcal{C}_{18} , $p = 3$
- [10] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C18p3/padic.gap>, supplementary file for special primes in the case \mathcal{C}_{18} , $p = 3$
- [11] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C20p2/D.txt>, supplementary file for the case \mathcal{C}_{20} , $p = 2$
- [12] Cartwright, D. I., Steger, T.: <https://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/C20p2/padic.gap>, supplementary file for special primes in the case \mathcal{C}_{20} , $p = 2$
- [13] Cartwright, D. I., Steger, T.: Application of the Bruhat–Tits tree of $SU_3(h)$ to some \tilde{A}_2 groups. *J. Austral. Math. Soc. Ser. A* **64**, 329–344 (1998) Zbl 0915.51005 MR 1623286
- [14] Cartwright, D. I., Steger, T.: Enumeration of the 50 fake projective planes. *C. R. Math. Acad. Sci. Paris* **348**, 11–13 (2010) Zbl 1180.14039 MR 2586735
- [15] Di Cerbo, L. F., Stover, M.: Multiple realizations of varieties as ball quotient compactifications. *Michigan Math. J.* **65**, 441–447 (2016) Zbl 1351.32026 MR 3510915
- [16] Di Cerbo, L. F., Stover, M.: Bielliptic ball quotient compactifications and lattices in $PU(2, 1)$ with finitely generated commutator subgroup. *Ann. Inst. Fourier (Grenoble)* **67**, 315–328 (2017) Zbl 1394.32020 MR 3612333
- [17] Di Cerbo, L. F., Stover, M.: Classification and arithmeticity of toroidal compactifications with $3\bar{c}_2 = \bar{c}_1^2 = 3$. *Geom. Topol.* **22**, 2465–2510 (2018) Zbl 1391.32038 MR 3784527
- [18] Di Cerbo, L. F., Stover, M.: Punctured spheres in complex hyperbolic surfaces and bielliptic ball quotient compactifications. *Trans. Amer. Math. Soc.* **372**, 4627–4646 (2019) Zbl 1435.32029 MR 4009437
- [19] Fatighenti, E.: Surfaces of general type with $p_g = 1$, $q = 0$, $K^2 = 6$ and Grassmannians. *Math. Nachr.* **293**, 88–100 (2020) Zbl 1451.14120 MR 4060364
- [20] Kajdan, D.: Arithmetic varieties and their fields of quasi-definition. In: *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 2, Gauthier-Villars, Paris, 321–325 (1971) Zbl 0223.14025 MR 0435081
- [21] Kammeyer, H., Kionke, S., Raimbault, J., Sauer, R.: Profinite invariants of arithmetic groups. *Forum Math. Sigma* **8**, article no. e54, 22 pp. (2020) Zbl 1456.20023 MR 4176758
- [22] Kulikov, V. S., Kharlamov, V. M.: On real structures on rigid surfaces. *Izv. Ross. Akad. Nauk Ser. Mat.* **66**, 133–152 (2002) (in Russian) Zbl 1055.14060 MR 1917540
- [23] Magma code associated with this paper. <https://github.com/mtstover/FakeP2pi1>, visited on September 2, 2024
- [24] Milne, J. S.: Introduction to Shimura Varieties. <https://www.jmilne.org/math/xnotes/svi.html> (2017), visited on September 2, 2024
- [25] Milne, J. S., Suh, J.: Nonhomeomorphic conjugates of connected Shimura varieties. *Amer. J. Math.* **132**, 731–750 (2010) Zbl 1194.14034 MR 2666906
- [26] Miyake, K.: Models of certain automorphic function fields. *Acta Math.* **126**, 245–307 (1971) Zbl 0212.53203 MR 0563067
- [27] Prasad, G., Yeung, S.-K.: Fake projective planes. *Invent. Math.* **168**, 321–370 (2007) Zbl 1253.14034 MR 2289867
- [28] Prasad, G., Yeung, S.-K.: Addendum to “Fake projective planes” *Invent. Math.* **168**, 321–370 (2007). *Invent. Math.* **182**, 213–227 (2010) Zbl 1388.14074 MR 2672284

- [29] Rajan, C. S.: [An example of non-homeomorphic conjugate varieties](#). *Math. Res. Lett.* **18**, 937–942 (2011) [Zbl 1262.14023](#) [MR 2875866](#)
- [30] Reid, A. W.: [Profinite properties of discrete groups](#). In: *Groups St Andrews 2013*, London Math. Soc. Lecture Note Ser. 422, Cambridge University Press, Cambridge, 73–104 (2015) [Zbl 1354.20018](#) [MR 3445488](#)
- [31] Serre, J.-P.: [Exemples de variétés projectives conjuguées non homéomorphes](#). *C. R. Acad. Sci. Paris* **258**, 4194–4196 (1964) [Zbl 0117.38003](#) [MR 0166197](#)
- [32] Shih, K.-y.: [Conjugations of arithmetic automorphic function fields](#). *Invent. Math.* **44**, 87–102 (1978) [Zbl 0405.14023](#) [MR 0572986](#)
- [33] Stover, M.: [Volumes of Picard modular surfaces](#). *Proc. Amer. Math. Soc.* **139**, 3045–3056 (2011) [Zbl 1301.22005](#) [MR 2811261](#)
- [34] Stover, M.: [Lattices in \$PU\(n, 1\)\$ that are not profinitely rigid](#). *Proc. Amer. Math. Soc.* **147**, 5055–5062 (2019) [Zbl 1453.20040](#) [MR 4021068](#)
- [35] Stover, M., Toledo, D.: [Residual finiteness for central extensions of lattices in \$PU\(n, 1\)\$ and negatively curved projective varieties](#). *Pure Appl. Math. Quart.* **18**, 1771–1797 (2022) [Zbl 1509.22006](#) [MR 4504038](#)
- [36] Stover, M., Toledo, D.: [Residually finite lattices in \$\widetilde{PU}\(2, 1\)\$ and fundamental groups of smooth projective surfaces](#). *Michigan Math. J.* **72**, 559–597 (2022) [Zbl 1506.14042](#) [MR 4460264](#)
- [37] Tits, J.: [Reductive groups over local fields](#). In: *Automorphic forms, representations and L -functions (Corvallis, OR, 1977)*, Part 1, *Proc. Sympos. Pure Math.* 33, American Mathematical Society, Providence, RI, 29–69 (1979) [Zbl 0415.20035](#) [MR 0546588](#)