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Enriques surfaces of non-degeneracy 3

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Abstract. We classify all non-extendable 3-sequences of half-fibers on Enriques surfaces. If the characteristic is different from 2, we prove in particular that every Enriques surface admits a 4-sequence, which implies that every Enriques surface is the minimal desingularization of an Enriques sextic, and that every Enriques surface is birational to a Castelnuovo quintic.

Keywords: Enriques surface, non-degeneracy, genus one fibration, half-fiber, Enriques sextic, Castelnuovo quintic.

1. Introduction

In this paper, we continue our investigation of c-sequences on Enriques surfaces [20]. Let X be an Enriques surface defined over an algebraically closed field of arbitrary characteristic p.

Recall that a (non-degenerate) *c-sequence* on X is a c-tuple $(F_1, \dots, F_c)$ of half-fibers such that $F_i.F_j = 1 - \delta_{ij}$. We say that a c-sequence is *extendable* if there is a c'-sequence with $c' > c$ such that the former is contained in the latter, disregarding the order. We define the *non-degeneracy* $\mathrm{nd}(F_1, \dots, F_c)$ of a c-sequence $(F_1, \dots, F_c)$ to be the maximal c' such that $(F_1, \dots, F_c)$ extends to a c'-sequence. We define the *maximal* and *minimal non-degeneracy* of the surface X to be

$$\max \mathrm{nd}(X) := \max \mathrm{nd}(F_1, \dots, F_c), \quad \min \mathrm{nd}(X) := \min \mathrm{nd}(F_1, \dots, F_c),$$

where the maximum and minimum are taken over all c-sequences on X for all possible c.

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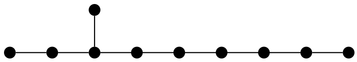
In [10, Definition 6.1.9], the maximal non-degeneracy of X is called *non-degeneracy invariant* of X and denoted by $\mathrm{nd}(X)$.

Our work is motivated by the known results on Enriques surfaces of non-degeneracy 1 and 2. Recall from [20, Definition 1.1] that an Enriques surface X is of *type* $\widetilde{E}_8$, if X contains (-2)-curves with the following dual graph:



type $\widetilde{E}_8$

Then, the classification of Enriques surfaces of non-degeneracy 1 is as follows.

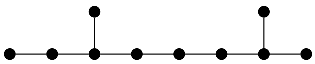
Theorem 1.1 ([8, Theorem 3.4.1] or [10, Theorem 6.1.10]). *For an Enriques surface X, the following are equivalent:*

(1) $\min \mathrm{nd}(X) = 1$,

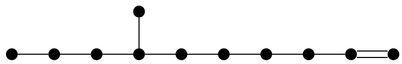
(2) $\max \mathrm{nd}(X) = 1$,

(3) *X is of type $\widetilde{E}_8$.*

For non-degeneracy 2, recall from [20, Definition 1.1] that an Enriques surface X is of *type* $\widetilde{D}_8$, respectively *of type* $\widetilde{E}_7$, if X contains (-2)-curves with the following dual graphs:



type $\widetilde{D}_8$



type $\widetilde{E}_7$

Then, the classification of Enriques surfaces of non-degeneracy 2 is as follows.

Theorem 1.2 ([20, Theorem 1.3]). *For an Enriques surface X, the following are equivalent:*

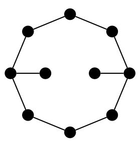
(1) $\min \mathrm{nd}(X) = 2$,

(2) $\max \mathrm{nd}(X) = 2$,

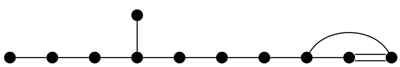
(3) *X is of type $\widetilde{D}_8$ or $\widetilde{E}_7$.*

Enriques surfaces of type $\widetilde{E}_8$, $\widetilde{D}_8$ or $\widetilde{E}_7$ only exist in characteristic $p = 2$, and are either classical or supersingular. They are collectively called *extra-special*. Thus, X is extra-special if and only if $\min \mathrm{nd}(X) \leq 2$ or, equivalently, $\max \mathrm{nd}(X) \leq 2$.

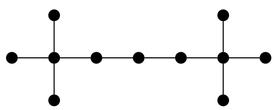
Here we study Enriques surfaces of non-degeneracy 3. We say that X is of *type* $\widetilde{A}_7$, *type* $\widetilde{E}_7^{(2)}$, *type* $2\widetilde{D}_4$, respectively, if X contains (-2)-curves with the following dual graphs:



type $\widetilde{A}_7$



type $\widetilde{E}_7^{(2)}$



type $2\widetilde{D}_4$

We will see in Examples 4.1, 4.7 and 4.10 that each of these surfaces admits a non-extendable 3-sequence. Our main achievement is the following theorem.

Theorem 1.3. *For an Enriques surface X, the following are equivalent:*

(1) $\min \operatorname{nd}(X) = 3$,

(2) *X is of type $\widetilde{A}_7$, $\widetilde{E}_7^{(2)}$, or $2\widetilde{D}_4$.*

Its proof can be obtained by combining Theorems 4.14 and 4.16, where, in fact, we classify not only Enriques surfaces with $\min \operatorname{nd}(X) = 3$ but also the structure of all non-extendable 3-sequences on them. In Section 4.1, we describe all the non-extendable 3-sequences explicitly.

Remark 1.4. We remark that, by [14, Proposition 2.16], for an Enriques surface X of type $\widetilde{E}_8$, $\widetilde{D}_8$, $\widetilde{E}_7$, $\widetilde{E}_7^{(2)}$, and $2\widetilde{D}_4$, the graph of (-2)-curves appearing in the definition of X is in fact the graph of *all* (-2)-curves on X. This is not true for Enriques surfaces of type $\widetilde{A}_7$ and, by [3, Section 4], a general Enriques surface of type $\widetilde{A}_7$ contains infinitely many (-2)-curves.

Enriques surfaces of type $\widetilde{E}_7^{(2)}$ or $2\widetilde{D}_4$ only exist in characteristic $p = 2$, and are either classical or supersingular. In contrast, Enriques surfaces of type $\widetilde{A}_7$ exist in any characteristic (see, e.g., [19]). In characteristic $p = 2$, they are ordinary Enriques surfaces. Additionally, by Remarks 4.3, 4.9 and 4.12, there is always a 4-sequence on surfaces of type $\widetilde{A}_7$ or $2\widetilde{D}_4$, while surfaces of type $\widetilde{E}_7^{(2)}$ admit only three genus one fibrations. Therefore, we deduce the following corollary.

Corollary 1.5. *For an Enriques surface X, the following are equivalent:*

(1) $\max \operatorname{nd}(X) = 3$,

(2) *X is of type $\widetilde{E}_7^{(2)}$.*

In particular, if $p \neq 2$, or if $p = 2$ and X is ordinary, then $\max \operatorname{nd}(X) \geq 4$.

Our results have deep implications on the theory of projective models of Enriques surfaces.

Historically, Enriques surfaces arose towards the end of the 19th century in the context of Castelnuovo's criterion for rationality of surfaces as examples of non-rational smooth projective surfaces whose arithmetic and geometric genus vanish. The original construction by Enriques gives a 10-dimensional family of Enriques surfaces arising as minimal resolutions of sextic surfaces in $\mathbb{P}^3$ which are non-normal along the edges of a tetrahedron [12, §39]. We call these sextics *Enriques sextics*. Castelnuovo observed that every Enriques sextic can be transformed into a normal quintic with two tacnodes and two triple points [5, §15], which we call *Castelnuovo quintic*. Later, Enriques proved that every complex Enriques surface arises as a minimal resolution of a degeneration of an Enriques sextic [13]. Modern proofs of this result were given by Artin [1] and Averbukh [23, Chapter X]. This result was extended to classical Enriques surfaces which are not of type $\widetilde{E}_8$ in arbitrary characteristic (see [6, Theorem 7.4] and [17, Theorem 3.1]).

In [20, Corollary 1.7], as a consequence of Theorem 1.2, we were able to restrict which degenerations of Enriques sextics one has to allow in order to obtain all Enriques surfaces as their minimal resolutions. Now, as a consequence of Corollary 1.5, we obtain the following result, which shows that if the characteristic is different from 2, then no degeneration is necessary at all. In other words, every Enriques surface arises via Enriques' original construction and, consequently, every Enriques surface is birational to a Castelnuovo quintic.

Theorem 1.6. *For $p \neq 2$, every Enriques surface is the minimal resolution of an Enriques sextic*

$$x_0^2x_1^2x_2^2 + x_0^2x_1^2x_3^2 + x_0^2x_2^2x_3^2 + x_1^2x_2^2x_3^2 + x_0x_1x_2x_3Q = 0, \tag{1.1}$$

and birationally equivalent to the corresponding Castelnuovo quintic

$$x_0(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_0^2x_1^2) + x_1Q', \tag{1.2}$$

where Q is a quadric and $Q' = Q(x_2x_3, x_0x_1, x_0x_2, x_0x_3)$.

We refer the reader to Section 5 for a proof and for the corresponding statement for classical Enriques surfaces in characteristic 2. We leave it to the reader to formulate the analogous consequences of Corollary 1.5 for ordinary and supersingular Enriques surfaces in characteristic 2.

The paper is structured as follows. In Section 2, we recall and prove basic results about c-sequences, with special focus on the cases $c = 1, 2, 3$. The fundamental geometrical insight for the proof of Theorem 1.3 is the distinction between special and non-special 3-sequences (see Definition 3.1). We study general properties of special 3-sequences in Section 3. Section 4 on non-extendable 3-sequences constitutes the technical core of this paper. Examples of all non-extendable 3-sequences are provided in Section 4.1. We then classify special and non-special non-extendable 3-sequences in Sections 4.2 and 4.3, respectively. Finally, we discuss in Section 5 the implications of our theorem on the theory of projective models of Enriques surfaces.

2. c-sequences

We use exactly the same notations and conventions as in our previous paper [20, §2]. We begin by recalling two results that show that the geometry of genus one fibrations and their half-fibers governs the cone of effective curves on Enriques surfaces. Linear equivalence is denoted by $\sim$, and the Weyl group of X, which is generated by reflections along classes of (-2)-curves on X, is denoted by W_X^{nod}.

Lemma 2.1 ([8, Theorem 3.2.1] or [9, Theorem 2.3.3]). *If D is an effective divisor on X with $D^2 \geq 0$, then there exist non-negative integers a_i and (-2)-curves R_i such that*

$$D \sim D' + \sum_i a_i R_i,$$

where D' is the unique nef divisor in the W_X^{nod}-orbit of D. In particular, $D^2 = D'^2$.

Lemma 2.2 ([9, Corollary 2.2.9]). *An effective divisor D on X is a half-fiber of a genus one fibration on X if and only if D is nef, primitive, and $D^2 = 0$.*

As in [20, p. 137], we define a *c-degenerate (canonical isotropic) n-sequence on X* as an n-tuple of the form

$$\left(F_1, F_1 + R_{1,1}, \dots, F_1 + \sum_{j=1}^{r_1} R_{1,j}, F_2, \dots, F_2 + \sum_{j=1}^{r_2} R_{2,j}, \dots, F_c, \dots, F_c + \sum_{j=1}^{r_c} R_{c,j}\right),$$

where the F_i are half-fibers of genus one fibrations on X and the $R_{i,j}$ are (-2)-curves satisfying the following conditions:

(1) $F_i.F_j = 1 - \delta_{ij}$,

(2) $R_{i,j}.R_{i,j+1} = 1$,

(3) $R_{i,j}.R_{k,l} = 0$ unless $(k,l) = (i,j)$ or $(k,l) = (i, j \pm 1)$,

(4) $F_i.R_{i,1} = 1$ and $F_i.R_{k,l} = 0$ if $(k,l) \neq (i,1)$.

If $c = n$, we simply call the above sequence a *c-sequence*. We say that a c-degenerate n-sequence *extends* to a c'-degenerate n'-sequence if the former is contained in the latter, disregarding the ordering.

Theorem 2.3 ([7, Lemma 1.6.1, Theorem 3.3]). *If $n \neq 9$, then every c-degenerate n-sequence on X can be extended to a c'-degenerate 10-sequence for some $c' \geq c$.*

The question of whether c-sequences extend to c'-sequences with $c' > c$ is the question treated in [20] and in the present article.

Numerical equivalence is denoted by $\equiv$, and the numerical lattice of X is denoted by $\mathrm{Num}(X)$. Recall from [9, Proposition 1.5.1] that $\mathrm{Num}(X)$ is isomorphic to E_{10}, the unique even unimodular hyperbolic lattice of rank 10. By [7, §1], the classes of the components of a 10-sequence generate a sublattice of index 3 in $\mathrm{Num}(X)$. To generate the whole numerical lattice, one needs the classes $e_{i,j}$ described in the following lemma.

Lemma 2.4 ([7, Lemma 1.6.2]). *If $(f_1, \dots, f_{10})$ is a 10-tuple of vectors in E_{10} such that $f_i.f_j = 1 - \delta_{ij}$, then for each i, j with $1 \leq i \neq j \leq 10$, there exists an isotropic vector $e_{i,j} \in E_{10}$ such that $e_{i,j}.f_i = e_{i,j}.f_j = 2$ and $e_{i,j}.f_k = 1$ for $k \neq i, j$.*

In particular, the following corollary shows that the intersection behavior of a divisor on X with the components of a 10-sequence cannot be arbitrary.

Corollary 2.5. *If $(f_1, \dots, f_{10})$ is a 10-tuple of vectors in E_{10} such that $f_i.f_j = 1 - \delta_{ij}$, then for each vector $v \in E_{10}$, the intersection number $v.\sum f_i$ is divisible by 3 and if $v \in \langle f_1, \dots, f_{10} \rangle$, then $v.\sum f_i$ is divisible by 9.*

Proof. The lattice $L = \langle f_1, \dots, f_{10} \rangle$ has index 3 in E_{10} by [7, §1]. For every element $v \in L$, we have $9 \mid (\sum f_i.v)$, since this holds for $v = f_i$. For any of the $e_{j,k}$ in Lemma 2.4, we have $9 \nmid (\sum f_i.e_{j,k}) = 12$. Hence, together with the f_i, any such $e_{j,k}$ generates E_{10}. Since $3 \mid (\sum f_i.e_{j,k})$, we obtain the claim. ∎

A 1-sequence is nothing but a half-fiber of a genus one fibration. We recall some basic results on genus one fibrations on Enriques surfaces. Adopting Kodaira's notation, we denote by I_n the fibers of *multiplicative* type, and by I_n^*, II, III, IV, II^*, III^*, IV^* we denote the fibers of *additive* type. To state the result, we recall that if $p = 2$, there are three different types of Enriques surfaces, distinguished by the torsion component Pic^τ_X of the identity of their Picard scheme: *classical* ($\mathrm{Pic}^\tau_X(k) = \mathbb{Z}/2\mathbb{Z} = \langle \omega_X \rangle$), *ordinary* ($\mathrm{Pic}^\tau_X = \mu_2$), and *supersingular* ($\mathrm{Pic}^\tau_X = \alpha_2$).

Lemma 2.6 ([9, Theorem 4.10.3]). *Let $f: X \to \mathbb{P}^1$ be a genus one fibration on X.*

- *If $p \neq 2$, then f is an elliptic fibration with two half-fibers, and each of them is either non-singular, or singular of multiplicative type.*
- *If $p = 2$ and X is classical, then f is an elliptic or quasi-elliptic fibration with two half-fibers, and each of them is either an ordinary elliptic curve, or singular of additive type.*
- *If $p = 2$ and X is ordinary, then f is an elliptic fibration with one half-fiber, which is either an ordinary elliptic curve, or singular of multiplicative type.*
- *If $p = 2$ and X is supersingular, then f is an elliptic or quasi-elliptic fibration with one half-fiber, which is either a supersingular elliptic curve, or singular of additive type.*

In particular, every genus one fibration $f: X \to \mathbb{P}^1$ admits a half-fiber F, so it is of the form $|2F|$, and (F) is a 1-sequence. Next, we give a bound on the number of vertical components of genus one fibrations. Recall that a genus one fibration $|2F|$ is called *special*, if there exists a (-2)-curve R, called *special bisection* of $|2F|$, with $F.R = 1$.

Proposition 2.7. *If $|2F|$ is a genus one fibration on an Enriques surface, then the following hold:*

(1) *The number of irreducible curves contained in s fibers of $|2F|$ is at most $8 + s$.*

(2) *The sum of the root lattices associated to fibers of $|2F|$ embeds into E_8.*

Proof. The Jacobian fibration of $|2F|$ is a fibration of a rational surface by [9, Proposition 4.10.1]. Moreover, $|2F|$ and its Jacobian have the same types of fibers by [9, Theorem 4.3.20]. Hence, both claims follow from the corresponding statements for rational elliptic and quasi-elliptic surfaces, which are well known (see, e.g., [22]). ∎

Next, we turn to 2-sequences (F_1, F_2). By [9, Theorem 3.3.4], the linear system $|2F_1 + 2F_2|$ associated to a 2-sequence (F_1, F_2) yields a morphism of degree 2 onto a quartic symmetroid del Pezzo surface in $\mathbb{P}^4$ (in the sense of [9, Chapter 0.6]). Depending on whether X is classical or not, there are four or two half-fibers in a 2-sequence and the following proposition restricts the possible combinations of half-fibers in a 2-sequence.

Proposition 2.8. *If (F_1, F_2) is a 2-sequence, then the following hold:*

(1) *The half-fibers of $|2F_1|$ and $|2F_2|$ do not share components.*

(2) *The sum of the root lattices associated to the half-fibers of $|2F_1|$ and $|2F_2|$ embeds into E_8.*

(3) *There is a simple fiber $G_2 \in |2F_2|$ containing all components of F_1 except a simple one.*

Proof. Claim (1) is [11, Lemma 3.5].

If X is classical, let F_1' and F_2' be the second half-fibers of $|2F_1|$ and $|2F_2|$, respectively. To prove claim (2), note that the sum of the root lattices associated to F_1, F_1', F_2, and F_2' embeds into $\langle F_1, F_2\rangle^\perp \subseteq \mathrm{Num}(X)$. Since $\mathrm{Num}(X) \cong U \oplus E_8$ and $\langle F_1, F_2\rangle \cong U$, the lattice $\langle F_1, F_2\rangle^\perp$ is even, unimodular, and negative definite, hence isomorphic to E_8.

Claim (3) is trivial if F_1 is irreducible. Suppose $F_1 = \sum a_i R_i$ for some (-2)-curves R_i and $a_i > 0$. For each i, we have $a_i R_i.F_2 \leq F_1.F_2 = 1$ and $R_i.F_2 \geq 0$. Hence, $R_i.F_2 \neq 0$ for exactly one i and for this i we have $R_i.F_2 = 1$ and $a_i = 1$. Since $a_i = 1$, the (-2)-curve R_i is a simple component of F_1 and $F_1 - R_i$ is a connected curve with $(F_1 - R_1).F_2 = 0$, so there is a fiber G_2 of $|2F_2|$ containing $F_1 - R_i$. By claim (1), the fiber G_2 is simple. ■

3. Special 3-sequences

Let (F_1, F_2, F_3) be a 3-sequence on an Enriques surface X. The associated linear system $|F_1 + F_2 + F_3|$ induces a morphism $\varphi\colon X \to \mathbb{P}^3$. We recall here the properties of this linear system, referring the reader to [9, §3.5] for further details. Restricting φ to a general member of the pencil $|F_1 + F_2|$, one checks that φ is birational if and only if $|F_1 + F_2 - F_3| = \emptyset$, in which case the image of φ is a non-normal sextic S in $\mathbb{P}^3$. If X is classical, the non-normal locus of S is a tetrahedron of lines if and only if $|F_1 + F_2 - F_3 + K_X| = \emptyset$. The divisors in $|F_1 + F_2 - F_3|$ consist of (-2)-curves, so every 3-sequence on a general Enriques surface (which does not contain (-2)-curves) leads to a sextic model as above.

In this section, we provide an in-depth analysis of the so-called *special* 3-sequences introduced by Cossec (cf. [7, Definition 5.3.1]), for which one of the above conditions is violated.

Definition 3.1. A 3-sequence (F_1, F_2, F_3) is said to be *special* if $F_1 + F_2 - F_3$ is numerically equivalent to an effective divisor, i.e.,

$$|F_1 + F_2 - F_3| \neq \emptyset \quad \text{or} \quad |F_1 + F_2 - F_3 + K_X| \neq \emptyset.$$

This definition is actually independent of the chosen order, as the following proposition shows.

Proposition 3.2. *If (F_1, F_2, F_3) is a 3-sequence, then the following hold:*

(1) *We have $|F_1 + F_2 - F_3| \neq \emptyset$ if and only if $|F_i + F_j - F_k| \neq \emptyset$ for every permutation (i, j, k) of $(1, 2, 3)$.*

(2) *If $|F_1 + F_2 - F_3| \neq \emptyset$ and X is classical, then $|F_1 + F_2 - F_3 + K_X| = \emptyset$.*

Proof. For claim (1), it suffices to show that $|F_i + F_j - F_k| \neq \emptyset$ if $|F_1 + F_2 - F_3| \neq \emptyset$. Let $S_3 \in |F_1 + F_2 - F_3|$. Since $S_3.(F_1 + F_2) = 0$ and the linear system $|F_1 + F_2|$ is a pencil without fixed components by [9, Proposition 2.6.1], we have

$$\mathcal{O}_{S_3}(F_1 + F_2) \cong \mathcal{O}_{S_3}.$$

Now, consider the exact sequence

$$0 \to \mathcal{O}_X(F_3) \to \mathcal{O}_X(F_1 + F_2) \to \mathcal{O}_{S_3} \to 0. \tag{3.1}$$

The two sheaves on the left have 1-dimensional (resp. 2-dimensional) space of global sections, hence no higher cohomology by the Riemann–Roch theorem. Therefore,

$$h^1(S, \mathcal{O}_{S_3}) = 0 \quad \text{and} \quad h^0(S, \mathcal{O}_{S_3}) = 1.$$

Now, $|2F_2|$ is a pencil without fixed components and $S_3.F_2 = 0$, so we have the exact sequence

$$0 \to \mathcal{O}_X(F_2 + F_3 - F_1) \to \mathcal{O}_X(2F_2) \to \mathcal{O}_{S_3} \to 0.$$

Since

$$h^0(X, \mathcal{O}_X(2F_2)) = 2 \quad \text{and} \quad h^0(S_3, \mathcal{O}_{S_3}) = 1,$$

this shows that $|F_2 + F_3 - F_1| \neq \emptyset$. The same type of argument applies to the other permutations of $(1, 2, 3)$, thus proving claim (1).

For claim (2), we tensor sequence (3.1) with $\mathcal{O}_X(K_X - F_3)$ to obtain

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + F_1 + F_2 - F_3) \to \mathcal{O}_{S_3}(K_X - F_3) \to 0.$$

So, it suffices to show that $h^0(S, \mathcal{O}_{S_3}(K_X - F_3)) = 0$. For this, consider the exact sequence

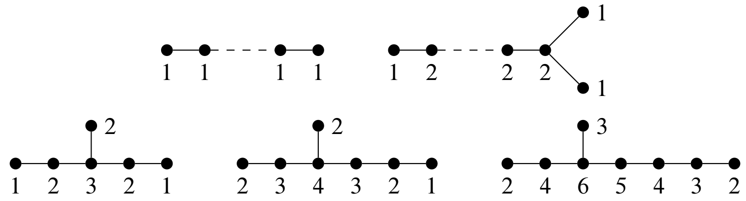
$$0 \to \mathcal{O}_X(K_X - F_1 - F_2) \to \mathcal{O}_X(K_X - F_3) \to \mathcal{O}_{S_3}(K_X - F_3) \to 0.$$

By Serre duality, H^0 and H^1 of the first two sheaves vanish, so

$$h^0(S, \mathcal{O}_{S_3}(K_X - F_3)) = 0. \qquad \blacksquare$$

Since $|F_i + F_j|$ is 1-dimensional and without fixed components, the linear system $|F_i + F_j - F_k|$, if non-empty, contains a unique element S_k. In the following proposition, we collect basic properties of S_1, S_2, S_3 and, in particular, we see that they are determined by their support.

Recall that for (simply laced) Dynkin diagrams, the coefficients of the highest root are given as follows [4, Chapter VI, Proposition 25, pp. 165 and 250ff.]:



Proposition 3.3. *If $S_k \in |F_i + F_j - F_k|$, then the following hold:*

(1) $S_k^2 = -2$, $S_k.F_i = S_k.F_j = 0$, *and* $S_k.F_k = 2$.

(2) $S_1.S_2 = S_1.S_3 = S_2.S_3 = 2$.

(3) *There exist simple fibers G_1, G_2, G_3 with $G_l \in |2F_l|$ such that $S_j + S_k = G_i$.*

(4) *The dual graph of the components of each S_k is a Dynkin diagram, and their coefficients are given by the corresponding highest root.*

Proof. Claims (1) and (2) are obvious, so let us prove claims (3) and (4). Since $S_k^2 = -2$ and $S_k.F_i = S_k.F_j = 0$, the divisor S_k is a connected sum of (-2)-curves whose support is contained in a single fiber of $|2F_i|$ and $|2F_j|$. We have $S_j + S_k \sim 2F_i$, so the supports of S_k and S_j are contained in the same fiber $G_i \in |2F_i|$.

We claim first that the support of S_k is strictly contained in the support of G_i. If this were not the case, the support of the fiber $G_j = S_i + S_k$ would coincide with the support of S_k, since its dual graph already contains an extended Dynkin diagram. Hence, the supports of G_i and G_j would coincide, and since the multiplicities of the components of G_i and G_j are uniquely determined by the dual graph and the multiplicity of the fiber, we would have $mG_i = nG_j$ for some $m, n \in \{1, 2\}$, a contradiction. Therefore, the dual graph of S_k is a connected subgraph of an extended Dynkin diagram, and, thus, it is a (simply laced) Dynkin diagram.

Now, notice that $S_k.R \leq 0$ for any component R of S_k. Indeed, assume by contradiction that $S_k.R > 0$. Then, the equations $(S_j + S_k).R = G_i.R = 0$ imply that $S_i.R = S_j.R < 0$, whence $R.G_k = R.(S_i + S_j) < 0$, contradicting the fact that G_k is nef. Therefore, S_k is a multiple of the so-called fundamental cycle by [2, Proposition 2], whose coefficients are given by the corresponding highest root (by uniqueness of the fundamental cycle). As $S_k^2 = -2$, S_k must in fact coincide with the fundamental cycle.

Finally, we need to show that G_i is simple. Assume that $G_i = 2F_i$. Then, it holds that $S_j + S_k = 2F_i$. Since $F_k.F_i = 1$ and $F_k.S_k = 2$, there is a simple component R of F_i contained in S_k (with multiplicity 2) but not contained in S_j. Hence, $2F_i - 2R = (S_k - 2R) + S_j$. Since $(F_i - R).R' \leq 0$ for all components R' of S_j and since S_j is the smallest effective divisor supported on $\mathrm{Supp}(S_j)$ with this property, we have that $(F_i - R) - S_j$ is effective. Now, the equation $(S_k - 2R) = (F_i - R) + (F_i - R - S_j)$ implies that $S_k - 2R$ contains the fundamental cycle of the support of $F_i - R$, but this is absurd, as then S_k and F_i would have the same support. Hence, G_i is simple. ∎

The following remark explains the geometric significance of the curves $S_k \in |F_i + F_j - F_k|$.

Remark 3.4. If $|F_1 + F_2 - F_3| \neq \emptyset$, then by [9, Theorem 3.3.5], the morphism φ induced by $|F_1 + F_2 + F_3|$ is generically of degree 2 onto a normal cubic symmetroid surface $\mathscr{C}$ in $\mathbb{P}^3$ (in the sense of [9, Chapter 0.7]). More precisely,

(1) If $p \neq 2$, or $p = 2$ and X is classical, then $\mathscr{C}$ is the *Cayley cubic* given by

$$x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3 + x_0x_1x_2 = 0,$$

which has four nodes at the four coordinate points. There are nine lines on $\mathscr{C}$. Six of them form a tetrahedron whose vertices are the nodes. The other three lines are $\ell_1 = \{x_0 + x_1 = x_2 + x_3 = 0\}$, $\ell_2 = \{x_0 + x_2 = x_1 + x_3 = 0\}$, and $\ell_3 = \{x_0 + x_3 = x_1 + x_2 = 0\}$. If $p \neq 2$, then the ℓ_i form a triangle on X in the plane $\{x_0 + x_1 + x_2 + x_3 = 0\}$, while if $p = 2$, they are coplanar concurrent lines which meet at the point $[1 : 1 : 1 : 1]$.

(2) If $p = 2$ and X is ordinary, then $\mathscr{C}$ is the cubic surface with a D_4^1-singularity given by the equation

$$x_1x_2x_3 + x_0x_3^2 + x_1^2x_2 + x_1x_2^2 = 0.$$

The surface contains six lines, three of which pass through the D_4^1-singularity at $[1 : 0 : 0 : 0]$. The other three lines are $\ell_1 = \{x_0 = x_1 = 0\}$, $\ell_2 = \{x_0 = x_2 = 0\}$, and $\ell_3 = \{x_0 = x_1 + x_2 + x_3 = 0\}$. These three lines form a triangle in the plane $\{x_0 = 0\}$.

(3) If $p = 2$ and X is supersingular, then $\mathscr{C}$ is the cubic surface with a D_4^0-singularity given by the equation

$$x_0x_3^2 + x_1^2x_2 + x_1x_2^2 = 0.$$

The surface contains six lines, three of which pass through the D_4^0-singularity at $[1 : 0 : 0 : 0]$. The other three lines are $\ell_1 = \{x_0 = x_1 = 0\}$, $\ell_2 = \{x_0 = x_2 = 0\}$, and $\ell_3 = \{x_0 = x_1 + x_2 = 0\}$. These three lines are contained in the plane $\{x_0 = 0\}$ and meet at the single point $[0 : 0 : 0 : 1]$.

By studying the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_X(2F_k) \to \mathcal{O}_X(F_1 + F_2 + F_3) \to \mathcal{O}_{S_k}(F_1 + F_2 + F_3) \to 0,$$

one sees that

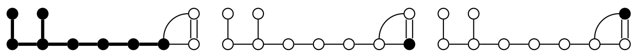
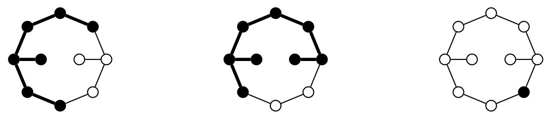
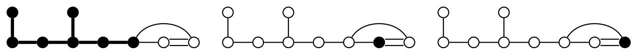
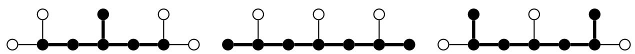
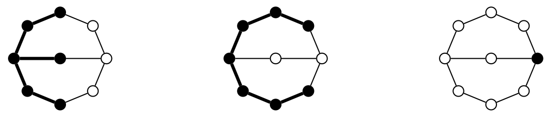
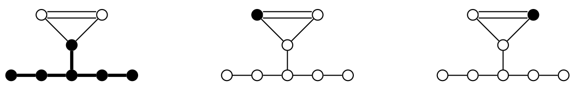
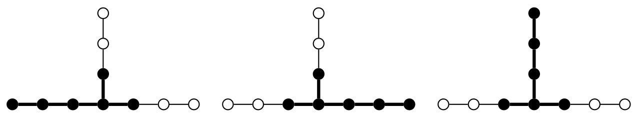
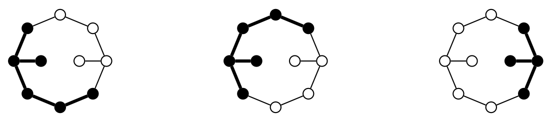
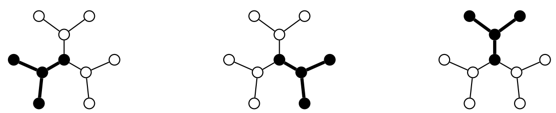
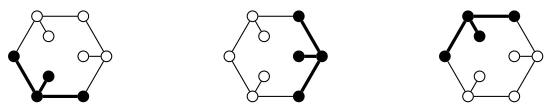
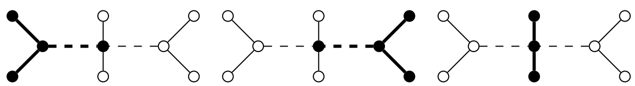
$$h^0(S_k, \mathcal{O}_{S_k}(F_1 + F_2 + F_3)) = 2,$$

so S_k maps to a line via φ. Similarly, each half-fiber of each $|2F_i|$ maps to a line. By [9, §3.3], the images of the half-fibers are the lines through the singular points on $\mathscr{C}$. Since the S_i are not contained in half-fibers by Proposition 3.3, they map to the lines ℓ_i described above. In particular, if $p \neq 2$, or $p = 2$ and X is ordinary, then the divisor $S_1 + S_2 + S_3$ maps to a triangle on $\mathscr{C}$.

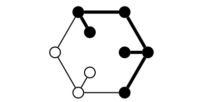
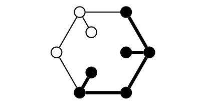
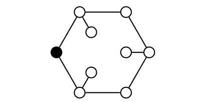
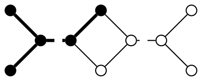
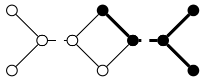
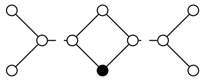
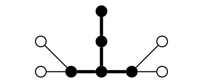
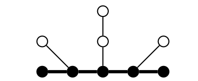
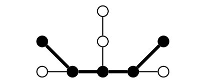
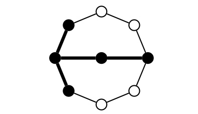
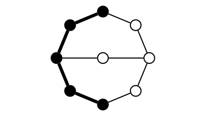
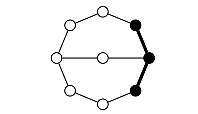
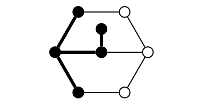
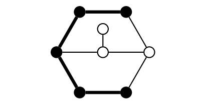
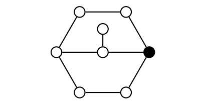
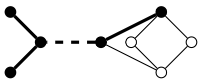
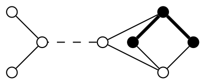
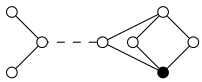
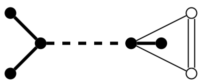
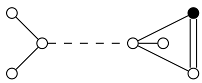
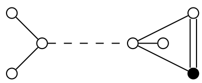
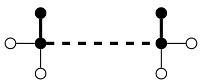
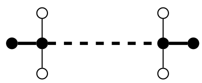
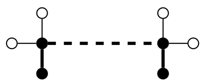
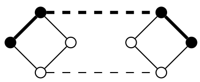
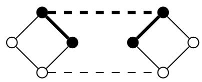
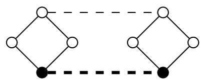
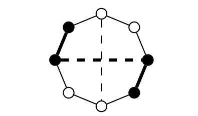
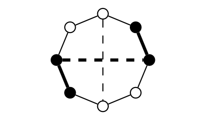
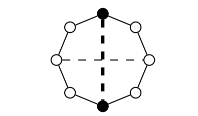
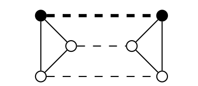
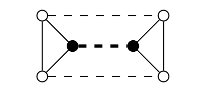
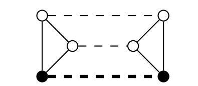
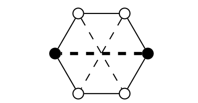
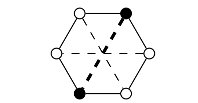
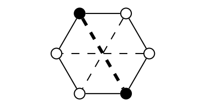
By Proposition 3.3, the divisors S_k are uniquely determined by their dual graphs. This observation and the geometric picture, explained in Remark 3.4, motivate the following definition and notation.

Definition 3.5. Given a special 3-sequence (F_1, F_2, F_3) and $S_k \in |F_i + F_j - F_k|$, the dual graph of S_k is denoted by Γ_k. The dual graph of $S_1 + S_2 + S_3$ is denoted by Γ and is called the *triangle graph* of (F_1, F_2, F_3).

As it turns out, there exist only finitely many possibilities for the shape of the triangle graph Γ, listed in Table 1. The rest of this section is dedicated to the proof that this list is complete.

(S_1, S_2, S_3)	Γ_1	Γ_2	Γ_3	$\lvert\Gamma\rvert$
(E_8, A_1, A_1)				10
(E_7, D_8, A_1)				10
(E_7, A_1, A_1)				9
(E_6, A_7, A_7)				10
(E_6, A_7, A_1)				9
(E_6, A_1, A_1)				8
(D_6, D_6, D_6)				10
(D_6, D_6, D_4)				10
(D_4, D_4, D_4)				10
(D_4, D_4, D_4)				9
(D_m, D_n, A_3) $m, n \geq 3$ $m + n \leq 9$				$m + n + 1$

Tab. 1 (First part). Possible graphs Γ_i as subgraphs of the triangle graph Γ of a special 3-sequence, see Proposition 3.7. We use the notation $D_3 = A_3$.

(S_1, S_2, S_3)	Γ_1	Γ_2	Γ_3	$\vert\Gamma\vert$
(D_6, D_6, A_1)				9
(D_m, D_n, A_1) $m, n \geq 4$ $m + n \leq 10$				$m + n$
(D_5, A_5, A_5)				9
(D_5, A_5, A_3)				9
(D_5, A_5, A_1)				8
(D_m, A_3, A_1) $4 \leq m \leq 7$				$m + 3$
(D_m, A_1, A_1) $4 \leq m \leq 8$				$m + 2$
(A_m, A_m, A_m) $3 \leq m \leq 7$				$m + 4$
(A_m, A_m, A_n) $m \geq 3, n \geq 1$ $m + n \leq 9$				$m + n + 2$
(A_m, A_m, A_n) $m \geq 3,\ n \geq 1$ $m + n \leq 9$				$m + n + 2$
(A_m, A_n, A_l) $m, n, l \geq 1$ $m + n + l \leq 11$				$m + n + l$
(A_m, A_n, A_l) $m, n, l \geq 1$ $m + n + l \leq 11$				$m + n + l$

Tab. 1 (Second part). Possible graphs Γ_i as subgraphs of the triangle graph Γ of a special 3-sequence, see Proposition 3.7. We use the notation $D_3 = A_3$.

Lemma 3.6. *For $G_i = S_j + S_k$, the possible types of G_i, S_j and S_k are listed in the following table, up to permutation of S_j and S_k:*

G_i	(S_j, S_k)
II*	(E_8, A_1) or (E_7, D_8)
III*	(E_7, A_1), (E_6, A_7) *or* (D_6, D_6)
IV*	(E_6, A_1) or (D_5, A_5)
IV	(A_2, A_1)
III	(A_1, A_1)
I_n^*	(D_{n+4}, A_1), (D_k, D_{n-k+6}), (D_{n+3}, A_3) or (A_{n+3}, A_{n+3})
I_n	(A_k, A_{n-k})

Proof. Choose a simple component R of G_i. Then, without loss of generality, we may assume that R is in S_k and not in S_j. Fix the type of G_i and S_k. A combination G_i, S_j, S_k occurs in the table if and only if $G_i - S_k$ is the fundamental cycle of its support. From here, compiling the list in the lemma is straightforward. ■

Proposition 3.7. *Up to permutation, the only possibilities for the types of* (S_1, S_2, S_3) *and their dual graphs are listed in Table 1.*

Proof. We use Lemma 3.6 throughout. Once we have determined the types of S_i and G_i, drawing the dual graph is straightforward.

If S_1 is of type E_8, then both S_2 and S_3 must be of type A_1. Here, G_2 and G_3 are of type II*, and G_1 is of type I_2 or III.

If S_1 is of type E_7, then S_2 and S_3 are of type D_8 or A_1. Since X admits no fiber of type I_n^* with $n \geq 5$ by Proposition 2.7, S_2 and S_3 cannot be both of type D_8, so we may assume that S_3 is of type A_1. If S_2 is of type D_8, then the type of (G_1, G_2, G_3) is $(\mathrm{I}_4^*, \mathrm{III}^*, \mathrm{II}^*)$. If S_2 is of type A_1, then G_2 and G_3 are of type III*, and G_1 is of type I_2 or III.

If S_1 is of type E_6, then S_2 and S_3 are of type A_7 or A_1. If both are of type A_7, then the type of (G_1, G_2, G_3) is $(\mathrm{I}_4^*, \mathrm{III}^*, \mathrm{III}^*)$, because X admits no fiber of type I_{14} by Proposition 2.7. If S_2 is of type A_7 and S_3 is of type A_1, then the type of (G_1, G_2, G_3) is $(\mathrm{I}_8, \mathrm{IV}^*, \mathrm{III}^*)$. If S_2 and S_3 are of type A_1, then G_2 and G_3 are of type IV* and G_1 is of type I_2 or III.

Next, assume that the type of (S_1, S_2, S_3) is (D_m, D_n, D_l) with $m \geq n \geq l \geq 4$. If two of the G_i are of type III*, then so is the third, since, by Proposition 2.7, X does not admit a fiber of type I_6^*. Hence, we get $(m, n, l) = (6, 6, 6)$. If only one of the G_i is of type III*, then $l = 4$, since X does not admit a fiber of type I_n^* with $n \geq 5$. Hence, we get $(m, n, l) = (6, 6, 4)$. Now, assume that all G_i are of type I_k^*. If $m \geq 5$, then all the S_i share a component, namely the leaf of the long branch of S_1. Thus, there are $l - 2$ components of S_3 disjoint from G_3 and they span a root lattice of rank $l - 2$. The root lattice associated to G_3 has rank $m + n - 2$. By Proposition 2.7, we obtain $m + n + l - 4 \leq 8$, contradicting $m \geq 5$. Hence, we have $(m, n, l) = (4, 4, 4)$ and we get two possible graphs, according to whether the S_i share a component or not.

Now, assume that the type of (S_1, S_2, S_3) is (D_m, D_n, A_l), with $m \geq n \geq 4$. First, we claim that G_1 and G_2 are of type I_k^* for a suitable k. If one of them is not, then $l = 5$, G_1 and G_2 are of type IV^* and G_3 is of type I_4^*. But then the central vertex of S_3 is the vertex of valency 3 in S_1 and S_2, which is absurd. Next, if G_3 is of type III^*, then $l \in \{1, 3\}$. If $l = 3$, then the central vertex of S_3 would have to be the simple vertex on the long tail of S_1 and S_2. But these are two distinct curves on X, since $S_1 + S_2 = G_3$ is of type III^*, so this case does not occur. Thus, $l = 1$ and the type of (G_1, G_2, G_3) is $(\mathrm{I}_2^*, \mathrm{I}_2^*, \mathrm{III}^*)$. Finally, if G_3 is of type I_{m+n-6}^*, then both $l = 3$ and $l = 1$ are possible and the type of (G_1, G_2, G_3) is $(\mathrm{I}_{n-3}^*, \mathrm{I}_{m-3}^*, \mathrm{I}_{m+n-6}^*)$ in the former case and $(\mathrm{I}_{n-4}^*, \mathrm{I}_{m-4}^*, \mathrm{I}_{m+n-6}^*)$ in the latter.

Next, assume that the type of (S_1, S_2, S_3) is (D_m, A_n, A_l), with $m \geq 4$. We assume $n \geq l$. We have $n, l \in \{1, 3, 5\}$. If $n = 5$, then necessarily $m = 5$ and $l \in \{1, 3, 5\}$. The type of (G_1, G_2, G_3) is $(\mathrm{I}_6, \mathrm{I}_1^*, \mathrm{IV}^*)$, $(\mathrm{I}_8, \mathrm{I}_2^*, \mathrm{IV}^*)$, and $(\mathrm{I}_2^*, \mathrm{IV}^*, \mathrm{IV}^*)$, respectively. Moreover, the cases where $n, l \in \{1, 3\}$ are all possible and, up to the ambiguity between I_2 and III, the type of (S_1, S_2, S_3) determines the type of (G_1, G_2, G_3).

Finally, assume that the type of (S_1, S_2, S_3) is (A_m, A_n, A_l). We assume $m \geq n \geq l$. If the G_i are all of type I_k^*, then $m = n = l \geq 3$. If G_3 is of type I_k^*, then $m = n \geq 3$. Notice that, if $l \neq 1$, then there are two different ways in which the cycle S_3 can meet the simple components of G_3. Finally, if the G_i are all of type I_k (or III or IV), then $m, n, l \geq 1$. Notice again that, if $l \neq 1$, then there are two different ways in which the cycle S_3 can meet G_3. ∎

Remark 3.8. The triangle graph Γ has at most 11 components. Indeed, we have that $G_1.S_1 = 2$, so there are at most 2 components in S_1 that are not orthogonal to F_1. The other k components of the triangle graph lie in fibers of $|2F_1|$, and G_1 is the only whole fiber of $|2F_1|$ contained in the triangle graph, so $k \leq 9$ by Proposition 2.7. Similarly, if the triangle graph contains a unique component not lying in fibers of $|2F_1|$, then $|\Gamma| \leq 10$. From this, we deduce the upper bounds on m, n, and l in Table 1.

Remark 3.9. By Remark 3.4, the divisors S_i map to lines on the cubic surface which is the image of the morphism induced by $|F_1 + F_2 + F_3|$. If there exists a curve simultaneously contained in S_1, S_2, and S_3, then the images of the S_i have to meet in a single point. By Remark 3.4, this implies that $p = 2$ and X is classical or supersingular. The interested reader can use this criterion to compile a list of possible triangle graphs if $p \neq 2$ or X is ordinary from the list in Table 1.

Remark 3.10. In [20, Proposition 3.1], we proved that every 2-sequence (F_1, F_2) such that there exist simple fibers $G_1 \in |2F_1|$ and $G_2 \in |2F_2|$ that share a component extends to a 3-sequence (F_1, F_2, F_3). The half-fiber F_3 we constructed satisfies the property that there exists a configuration of (-2)-curves $\sum a_i R_i$ such that $F_3 + \sum a_i R_i \in |F_1 + F_2|$. In other words, $F_1 + F_2 - F_3 \equiv \sum a_i R_i$, so (F_1, F_2, F_3) is special. Therefore, a 2-sequence (F_1, F_2) extends to a special 3-sequence if and only if there are two simple fibers $G_1 \in |2F_1|$, $G_2 \in |2F_2|$ that share a component. (For the converse, S_3 provides such a component.)

Finally, we note that most Enriques surfaces that contain a (-2)-curve contain a triangle graph.

Theorem 3.11. *If X contains a (-2)-curve and is not extra-special, then X admits a special 3-sequence. In particular, X is a double cover of one of the cubic surfaces in Remark 3.4 and the dual graph of X contains one of the triangle graphs listed in Table 1.*

Proof. If there exist a (-2)-curve R and three half-fibers F_1, F_2, F_3 on X with $F_i.R = 0$ for all $i = 1, 2, 3$, then we can conclude that R is contained in two simple fibers by Proposition 2.8, so X admits a special 3-sequence by Remark 3.10. In particular, we may assume that every c-degenerate 10-sequence on X satisfies either $c \leq 2$ or $c = 10$.

By Theorems 1.1 and 1.2, every half-fiber F on X extends to a non-degenerate 3-sequence, hence to a non-degenerate 10-sequence $(F_1, \ldots, F_{10})$ with $F_1 = F$. Every component R of a fiber $G \in |2F|$ satisfies $R.F_i \leq G.F_i = 2$ for every $i = 2, \ldots, 10$. In particular, if G has at least three components, then one of them satisfies $R.F_i = 0$ for at least three i, and then we can argue as before. Hence, we may assume that all fibers of all genus one fibrations on X have at most two components.

By assumption, X contains a (-2)-curve. Therefore, by [18, Theorem A.3], there exist a half-fiber F and a (-2)-curve R on X with $F.R = 1$. Extend $(F, F + R)$ to a c-degenerate 10-sequence. Since $F + R$ occurs in the 10-sequence, we have $c \neq 10$, hence $c \leq 2$, but then F has a fiber with more than two components, contradicting the conclusion of the previous paragraph. ∎

4. Non-extendable 3-sequences

This section is structured as follows. In Section 4.1, we present four examples of non-extendable 3-sequences. The surfaces on which these 3-sequences occur are of type $\widetilde{A}_7$, $\widetilde{E}_7^{(2)}$, and $2\widetilde{D}_4$. In Section 4.2, we prove Theorem 4.14, which shows that the first three examples are in fact the only special non-extendable 3-sequences that can occur on an Enriques surface. In Section 4.3, we prove Theorem 4.16, which shows that the fourth example is the only type of non-special non-extendable 3-sequence.

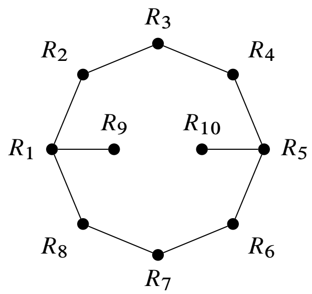
Taken together, the results of this section give a complete classification of all non-extendable 3-sequences and of all Enriques surfaces of non-degeneracy 3.

Notation. Throughout this section, thick lines and dashed lines in the dual graphs indicate simple fibers and half-fibers, respectively.

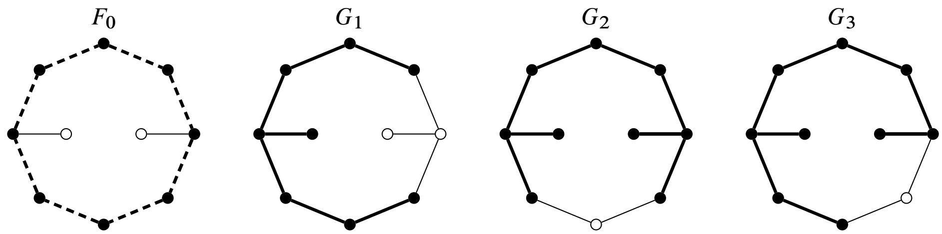
4.1. Examples

We will now present the four types of non-extendable 3-sequences. The first two types appear on Enriques surfaces of type $\widetilde{A}_7$. The other two types appear only in characteristic 2.

Example 4.1. Let X be an Enriques surface of type $\widetilde{A}_7$. Recall that this means that X contains ten (-2)-curves giving rise to the following dual graph:



Consider the following half-fiber F_0 and fibers G_1, G_2, G_3 on X:



By Lemma 2.6, the existence of F_0 implies that $p \neq 2$, or $p = 2$ and X is ordinary. The same lemma shows that G_1, G_2, and G_3 are simple. Choose half-fibers F_1, F_2, F_3 such that $G_i \in |2F_i|$.

Claim 4.2. *The half-fibers* (F_1, F_2, F_3) *form a special non-extendable* 3*-sequence with triangle graph of type* (E_7, D_8, A_1).

Proof. First, one checks that

$$F_i.F_j = \frac{1}{4}G_i.G_j = 1 - \delta_{ij},$$

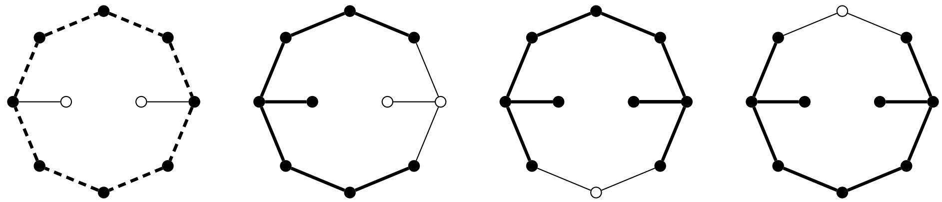
so (F_1, F_2, F_3) is indeed a 3-sequence. Next, $G_1 + G_2 - G_3 = 2R_6$, so (F_1, F_2, F_3) is special. Similarly, one checks that the supports of $G_2 + G_3 - G_1$ and $G_1 + G_3 - G_2$ are of type D_8 and E_7, respectively, hence the type of the special 3-sequence is (E_7, D_8, A_1). It remains to prove that (F_1, F_2, F_3) is non-extendable.

Seeking a contradiction, assume that there exists a half-fiber F_4 with $F_4.F_i = 1$ for $i \in \{1, 2, 3\}$. Since

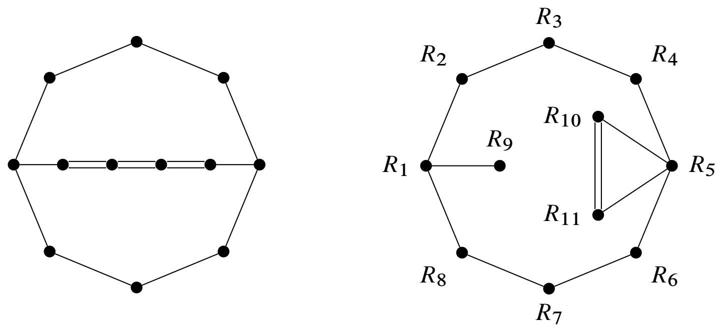
$$F_4.(F_1 + F_3 - F_2) = 1 \quad \text{and} \quad S_2 \in |F_1 + F_3 - F_2|$$

is supported on a configuration of (-2)-curves of type E_7, then F_4 necessarily intersects the only simple component of S_2, namely R_4. This, however, contradicts the facts that $F_4.(F_2 + F_3 - F_1) = 1$ and R_4 appears with multiplicity 2 in $S_1 \in |F_2 + F_3 - F_1|$. ■

Remark 4.3. If X is of type $\widetilde{A}_7$, then $\max \operatorname{nd}(X) \geq 4$. Indeed, the following four half-fibers and simple fibers form a 4-sequence:

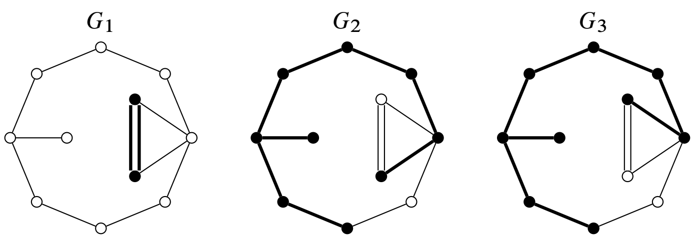


Example 4.4. Let X be an Enriques surface of type $\widetilde{A}_7$. Using the notation of Example 4.1, the curve R_{10} is a component of a fiber of $|G_1|$. By [19, Theorem 3.1], we get one of the following two graphs according to whether this fiber is a half-fiber or a simple fiber:



type I type BP

An Enriques surface containing the left configuration of (−2)-curves has finite automorphism group and occurs as type I in Kondo's list [16]. Enriques surfaces containing the right configuration of (−2)-curves were first studied by Barth and Peters [3] as examples of Enriques surfaces with small (but infinite) automorphism group. Now, assume that X is of type BP and, overriding previous notation, consider the following three simple fibers G_1, G_2 and G_3:



Choose half-fibers F_1, F_2, F_3 such that $G_i \in |2F_i|$.

Claim 4.5. *The half-fibers* (F_1, F_2, F_3) *form a special non-extendable* 3*-sequence with triangle graph of type* (E_8, A_1, A_1).

Proof. First, one checks that $F_i.F_j = \frac{1}{4}G_i.G_j = 1 - \delta_{ij}$, so (F_1, F_2, F_3) is indeed a 3-sequence. Next, $G_1 + G_2 - G_3 = 2R_{11}$, so (F_1, F_2, F_3) is special. Similarly, one checks that the supports of $G_2 + G_3 - G_1$ and $G_1 + G_3 - G_2$ are of type E_8 and A_1, respectively, hence the type of the special 3-sequence is (E_8, A_1, A_1). It remains to prove that (F_1, F_2, F_3) is non-extendable.

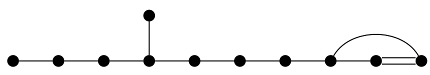
If F_4 is a half-fiber with $F_4.F_i = 1$ for $i \in \{1, 2, 3\}$, then we have

$$F_4.(F_2 + F_3 - F_1) = 1.$$

However, this contradicts the fact that $S_1 \in |F_2 + F_3 - F_1|$ has no simple component, since the support of S_1 is a configuration of (-2)-curves of type E_8. ■

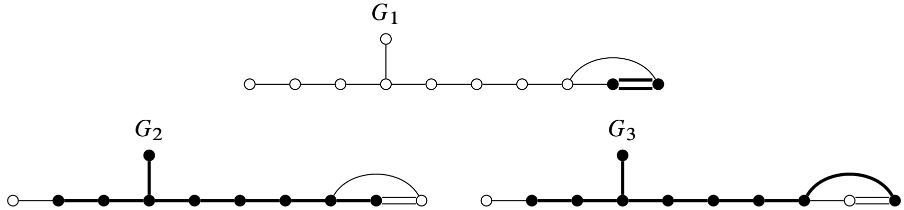
Remark 4.6. An Enriques surface X of type BP is in particular of type $\widetilde{A}_7$, therefore $\max \operatorname{nd}(X) \geq 4$, cf. Remark 4.3.

Example 4.7. Let X be an Enriques surface of type $\widetilde{E}_7^{(2)}$. Recall that this means that X contains eleven (-2)-curves giving rise to the following dual graph:



Such a surface is a classical or supersingular Enriques surface defined over a field of characteristic 2, with finite automorphism group [14, Theorems 11.4 and 11.12]. There are no other (-2)-curves on X. In particular, there are only three genus one fibrations on X.

Choose fibers G_1, G_2, G_3 as follows:



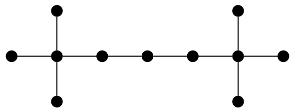
Choose half-fibers F_1, F_2, F_3 such that the corresponding fibrations have the G_i as fibers.

Claim 4.8. *The half-fibers (F_1, F_2, F_3) form a special non-extendable 3-sequence with triangle graph of type (E_8, A_1, A_1).*

Proof. Since X is not extra-special, and $|2F_1|$, $|2F_2|$, $|2F_3|$ are the only genus one fibrations on X, they must form a non-extendable 3-sequence. In particular, the G_i are simple fibers because $G_i.G_j = 4$. From $G_1 + G_2 - G_3 = 2R$, we infer that (F_1, F_2, F_3) is special. ■

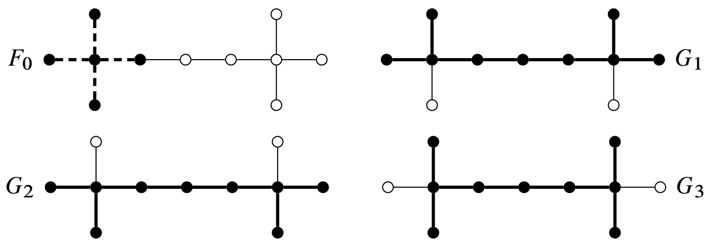
Remark 4.9. If X is of type $\widetilde{E}_7^{(2)}$, then $\min \operatorname{nd}(X) = \max \operatorname{nd}(X) = 3$. Indeed, the surface X contains only 3 genus one fibrations and the corresponding half-fibers form a 3-sequence.

Example 4.10. Let X be an Enriques surface of type $2\widetilde{D}_4$. Recall that this means that X contains eleven (-2)-curves giving rise to the following dual graph:

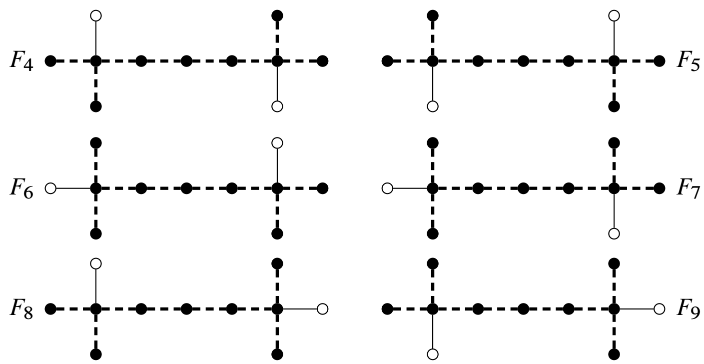


Such a surface is a classical Enriques surface defined over a field of characteristic 2, with finite automorphism group [14, Theorem 13.4]. There are no other (-2)-curves on X. There are exactly 10 genus one fibrations on X, one with two half-fibers F_0, F_0' of type I_0^*, and the others with a (half-)fiber of type I_4^*.

By Theorem 1.2, F_0 extends to a 3-sequence (F_0, F_1, F_2). Since all configurations of type I_4^* on X have intersection number 2 with F_0, this means that there exist simple fibers $G_1 \in |2F_1|$ and $G_2 \in |2F_2|$ of type I_4^*. The sequence (F_0, F_1, F_2) is non-special by Proposition 3.3, since $|2F_0|$ contains no reducible simple fibers. Hence, we can apply Remark 3.10 to find a half-fiber F_3 different from F_0 and F_0' and such that (F_1, F_2, F_3) is a special 3-sequence. In particular, $|2F_3|$ must admit a reducible simple fiber G_3, necessarily of type I_4^*. There is a unique configuration of type I_4^* which has intersection number 4 with both G_1 and G_2, so this is our G_3, and it turns out that (F_0, F_1, F_2, F_3) is in fact a 4-sequence:



Every other configuration of type I_4^* on X has intersection number 2 with at least one of the G_i. Hence, the following configurations are all half-fibers:



Claim 4.11. *For each* $4 \leq i \leq 9$*, there is a unique* 3*-sequence containing* F_i*. This* 3*-sequence is non-special and non-extendable. In particular,* $\mathrm{nd}(F_i) = 3$.

Proof. By the symmetry of the graph, it suffices to prove the statement for F_4. We have $F_4.F_j \geq 2$ for $j \neq 1, 2$ and $F_4.F_1 = F_4.F_2 = 1$. Hence, (F_1, F_2, F_4) is the unique 3-sequence that extends F_4. Since $(F_1 + F_2 - F_4).F_5 = -2 < 0$ and F_5 is nef, (F_1, F_2, F_4) is non-special. ■

Remark 4.12. If X is of type $2\widetilde{D}_4$, then $\min \mathrm{nd}(X) = 3$ and $\max \mathrm{nd}(X) = 4$. Indeed, the half-fibers (F_0, F_1, F_2, F_3) form a 4-sequence.

Example 4.13. Enriques surfaces with finite automorphism group are classified into 14 types, according to the dual graph associated to the (-2)-curves that they contain, some of which have already been introduced in the previous examples [14, 16, 19]. Every genus one fibration on an Enriques surface X with $|\mathrm{Aut}(X)| < \infty$ has at least one reducible fiber. Consequently, one can list all genus one fibrations on X and compute the exact values of $\min \mathrm{nd}(X)$ and $\max \mathrm{nd}(X)$ (cf. [10, 21]). For the sake of completeness, these values are displayed in Table 2, although we will never use this table in the proofs of the present paper.

$\mathbf{nd}(X)$	I	II	III	IV	V	VI	VII	VIII	$\widetilde{E}_8$	$\widetilde{D}_8$	$\widetilde{E}_7$	$\widetilde{E}_7^{(2)}$	$2\widetilde{D}_4$	$\widetilde{E}_6$
min	3	4	5	6	5	7	7	4	1	2	2	3	3	4
max	4	7	8	10	7	10	10	7	1	2	2	3	4	4

Tab. 2. Non-degeneracy of Enriques surfaces with finite automorphism group.

4.2. *Special non-extendable* 3*-sequences*

In Section 4.1, we provided three types of special non-extendable 3-sequences. The aim of this section is to prove the following result, which says that our list exhausts all possible examples.

Theorem 4.14. *If (F_1, F_2, F_3) is a special non-extendable 3-sequence and $S_k \in |F_i + F_j - F_k|$, then, after possibly permuting the indices, one of the following holds:*

(1) *(S_1, S_2, S_3) is of type (E_8, A_1, A_1), and X is of type* BP.

(2) *(S_1, S_2, S_3) is of type (E_8, A_1, A_1), and X is of type $\widetilde{E}_7^{(2)}$.*

(3) *(S_1, S_2, S_3) is of type (E_7, D_8, A_1), and X is of type $\widetilde{A}_7$.*

In particular, X admits a special non-extendable 3-sequence if and only if X is of type $\widetilde{A}_7$ or $\widetilde{E}_7^{(2)}$.

Recall that any non-extendable 3-sequence (F_1, F_2, F_3) is contained by Theorem 2.3 in a 3-degenerate 10-sequence of the following form:

$$\left(F_1, \ldots, F_1 + \sum_{j=1}^{r_1} R_{1,j}, F_2, \ldots, F_2 + \sum_{j=1}^{r_2} R_{2,j}, F_3, \ldots, F_3 + \sum_{j=1}^{r_3} R_{3,j}\right). \tag{4.1}$$

Proposition 4.15. *If* (F_1, F_2, F_3) *is a special non-extendable* 3*-sequence, then the components of the triangle graph of* (F_1, F_2, F_3) *span a lattice of rank* 10 *and discriminant* 1, 4 *or* 16.

Proof. Let Λ be the lattice spanned by the components of the triangle graph, which we recall is the dual graph of components of the divisor $S_1 + S_2 + S_3$. By Theorem 2.3, there exists a 3-degenerate 10-sequence as in (4.1). The lattice L spanned by the divisors in the 10-sequence has index 3 in $\mathrm{Num}(X)$.

We first show that the $R_{i,j}$ and $2F_i$ are contained in Λ. We argue by induction on j. Since S_i is effective and $R_{i,1}.S_i = -1$, the curve $R_{i,1}$ is a component of S_i. Assume that $R_{i,j}$ is contained in S_i for $1 \leq j \leq n$. Then, $(S_i - \sum_{j=1}^{n} R_{i,j})$ is effective and

$$R_{i,n+1}.\left(S_i - \sum_{j=1}^{n} R_{i,j}\right) = -1.$$

Hence, also $R_{i,n+1}$ is contained in S_i. Since $2F_i \equiv G_i = S_j + S_k$, this shows that all the $R_{i,j}$ and $2F_i$ are in Λ, as claimed.

Now, we have

$$F_1 + F_2 + F_3 \equiv S_1 + S_2 + S_3 \in \Lambda,$$

so $\Lambda \cap L \subseteq L$ has index 1, 2 or 4. Thus, it suffices to show that $\Lambda[F_1, F_2, F_3]$ contains an element of $\mathrm{Num}(X) \setminus L$.

Without loss of generality, we may assume $r_3 \geq 3$. Consider the vector $e_{9,10} \in \mathrm{Num}(X)$ given by Lemma 2.4. Note that $e_{9,10} \notin L$ because of Corollary 2.5. By the Riemann–Roch theorem, there exists an effective divisor $D \in \mathrm{Pic}(X)$, whose numerical class is $e_{9,10}$, such that $D.F_i = 1$ for $i \in \{1, 2, 3\}$, and $D.R_{i,j} = 0$ for all (i, j) except for $D.R_{3,r_3-1} = 1$. According to Lemma 2.1, there exist D' and C with $D \sim D' + C$, such that D' is nef, $D^2 = D'^2 = 0$, and C is a linear combination of (-2)-curves with non-negative coefficients. Since $D'.F_i \leq 1$ for all i and (F_1, F_2, F_3) is non-extendable, $D' \equiv F_i$ for some $i \in \{1, 2, 3\}$. In particular, $D.D' = 1$ and, thus, $C^2 = -2$.

Assume $D' \equiv F_i$ for some $i \in \{1, 2, 3\}$. Then, $C.F_j = 0$ for $j \neq i$, so C is a connected configuration of (-2)-curves contained in a fiber G_j of $|2F_j|$. Choose $j \neq i, 3$. We also have $C.R_{3,r_3-1} = 1$, so G_j is the fiber containing R_{3,r_3-1}. In the second paragraph of the proof, we saw that R_{3,r_3-1} is contained in S_3, so G_j is the fiber containing S_3. By Proposition 3.3, all components of G_j are in Λ, so

$$D = D' + C \in \Lambda[F_1, F_2, F_3],$$

as desired. ∎

We can now prove the theorem.

Proof of Theorem 4.14. Any Enriques surface of type $\widetilde{A}_7$ or $\widetilde{E}_7^{(2)}$ admits a special non-extendable 3-sequence (Examples 4.1 and 4.7), so it suffices to prove the first statement.

By Corollary 4.15, the sublattice of $\mathrm{Num}(X)$ spanned by the components of S_1, S_2, S_3 has rank 10 and discriminant $d \in \{1, 4, 16\}$. In particular, the triangle graph Γ of (F_1, F_2, F_3) has at least 10 vertices. By inspection of Table 1, we obtain the following 15 possible types of (S_1, S_2, S_3) such that $|\Gamma| \geq 10$:

- (E_8, A_1, A_1), (E_7, D_8, A_1), (E_6, A_7, A_7), (D_6, D_6, D_6), (D_6, D_6, D_4),
- the first type of (D_4, D_4, D_4),
- (D_m, D_n, A_3) with $m + n = 9$, (D_m, D_n, A_1) with $m + n = 10$,
- (D_7, A_3, A_1), (D_8, A_1, A_1), (A_m, A_m, A_m) with $6 \leq m \leq 7$,
- two types of (A_m, A_m, A_n) with $8 \leq m + n \leq 9$,
- two types of (A_m, A_n, A_l) with $10 \leq m + n + l \leq 11$.

For seven of these types, namely for (D_4, D_4, D_4), (D_m, D_n, A_3), (D_m, D_n, A_1), (D_m, A_3, A_1), (D_m, A_1, A_1), the second case of type (A_m, A_m, A_n) and the second case of type (A_m, A_n, A_l), we have $d > 16$, which is absurd. We then analyze the remaining eight dual graphs.

In case (E_8, A_1, A_1), consider the genus one fibration $|2F|$ with a fiber with two components that occurs in the graph. There is a fiber $G \in |2F|$ of type III^* such that a simple component of G does not belong to Γ. If G is not a simple fiber, then we obtain the graph of type $\tilde{E}_7^{(2)}$. If G is a simple fiber, we obtain the graph of type BP, and in particular X is of type $\widetilde{A}_7$.

In case (E_7, D_8, A_1), Γ is the dual graph of type $\widetilde{A}_7$, so X is of type $\widetilde{A}_7$.

In case (E_6, A_7, A_7), any half-fiber of type I_2^* in Γ extends the sequence.

In case (D_6, D_6, D_6), the half-fiber of type IV^* in Γ extends the sequence.

In case (D_6, D_6, D_4), the half-fiber of type I_8 in Γ extends the sequence.

In case (A_m, A_m, A_m), $6 \leq m \leq 7$, any half-fiber of type I_0^* in Γ extends the sequence.

In the first case of type (A_m, A_m, A_n), $8 \leq m + n \leq 9$, any half-fiber of type I_4 in Γ extends the sequence as soon as $n \geq 2$. If $n = 1$, then $m = 7$ (since X cannot contain a fiber of type I_5^* by Proposition 2.7), and it holds that $d = 64 > 16$.

In the first case of type (A_m, A_n, A_l), $10 \leq m + n + l \leq 11$, assume up to symmetry that $m \geq n \geq l$. If $l \geq 2$, then any half-fiber of type I_3 in Γ extends the sequence. If $l = 1$, then $m + n = 9$ (since X cannot contain a fiber of type I_{10}), and a computation shows that $d = 144 > 16$. ∎

4.3. Non-special non-extendable 3-sequences

In Example 4.10, we provided examples of non-special non-extendable 3-sequences, all of which occurred on Enriques surfaces of type $2\widetilde{D}_4$. The aim of this section is to prove that these are all the examples that exist.

Theorem 4.16. *If (F_1, F_2, F_3) is a non-special non-extendable 3-sequence, then X is of type $2\widetilde{D}_4$ and, after possibly permuting the indices, both $|2F_1|$ and $|2F_2|$ admit a simple fiber of type I_4^*, and $|2F_3|$ admits a half-fiber of type I_4^*. In particular, X admits a non-special non-extendable 3-sequence if and only if X is of type $2\widetilde{D}_4$.*

As in the special case, any non-extendable 3-sequence (F_1, F_2, F_3) is contained in a 3-degenerate 10-sequence as in (4.1).

Lemma 4.17. *Let* (F_1, F_2, F_3) *be a non-special non-extendable* 3*-sequence. If* R *is a* (-2)*-curve such that*

$$(F_1 + F_2 + F_3).R = 1,$$

then R *lies in a half-fiber of one of the* $|2F_i|$*. In particular, all the components* $R_{i,j}$ *in* (4.1) *lie in half-fibers of* $|2F_1|$*,* $|2F_2|$ *or* $|2F_3|$*.*

Proof. Since the F_i are nef, we can assume that $F_1.R = F_2.R = 0$ and $F_3.R = 1$ up to switching the indices. Consider the exact sequence

$$0 \to \mathcal{O}_X(F_1 + F_2 - R) \to \mathcal{O}_X(F_1 + F_2) \to \mathcal{O}_R(F_1 + F_2) \to 0.$$

By [9, Proposition 2.6.1], the linear system $|F_1 + F_2|$ has no fixed components. Hence, according to [9, Corollary 2.6.5], we have $h^0(X, \mathcal{O}_X(F_1 + F_2)) = 2$. On the other hand, $\mathcal{O}_R(F_1 + F_2) = \mathcal{O}_R$, since $(F_1 + F_2).R = 0$. Thus, $h^0(R, \mathcal{O}_R(F_1 + F_2)) = 1$ and the long exact sequence in cohomology shows that $|F_1 + F_2 - R|$ is non-empty.

Let $D \in |F_1 + F_2 - R|$. Then, $D^2 = 0$ and $D.F_1 = D.F_2 = D.F_3 = 1$. By Lemma 2.1, we can write $D \sim D' + \sum_i a_i R_i$ with $D' \neq 0$ nef, $D'^2 = 0$, and the R_i being (-2)-curves. Since the F_i are nef, we have $D'.F_i \leq 1$ for $i = 1, 2, 3$, so D' is a half-fiber F on X by [9, Corollary 2.2.9]. As (F_1, F_2, F_3) is non-extendable, D' must be a half-fiber of $|2F_1|$, $|2F_2|$ or $|2F_3|$.

Assume first that $F \equiv F_3$. Then, $R + \sum a_i R_i \equiv F_1 + F_2 - F_3$, contradicting the assumption that (F_1, F_2, F_3) is non-special.

Assume instead that $F \equiv F_1$ or $F \equiv F_2$. After a suitable permutation of these half-fibers, we may assume $F = F_1$. Then, $|D - F| = |F_2 - R| = |\sum_i a_i R_i|$, so $F_2 = R + \sum_i a_i R_i$ and R belongs to the half-fiber F_2.

For the second part of the statement, note that $R_{1,j}$ is in the same fiber of $|2F_2|$ and $|2F_3|$ as $R_{1,1}$. By the first part of the statement, $R = R_{1,1}$ is in a half-fiber of $|2F_2|$ or $|2F_3|$. Hence, $R_{1,j}$ is in a half-fiber of $|2F_2|$ or $|2F_3|$. A similar argument applies to the other $R_{i,j}$. ∎

Lemma 4.18. *Let* $(F_1, F_1 + R_1, F_2, F_2 + R_2)$ *be a* 2*-degenerate* 4*-sequence. If* R_2 *is a component of a half-fiber of* $|2F_1|$*, then* R_1 *is not a component of a half-fiber of* $|2F_2|$*.*

Proof. Suppose without loss of generality that R_2 is a component of F_1 and that R_1 is a component of F_2. Then, both half-fibers F_1 and F_2 are reducible. Recall that F_1 and F_2 do not share components by Proposition 2.8. The half-fiber F_1 has exactly one component which is a special bisection of $|2F_2|$, and that component must be R_2, since $R_2.F_2 = 1$. Analogously, R_1 is the unique component of F_2 which is a special bisection of $|2F_1|$. This implies that R_1 and R_2 must meet, which contradicts the fact that $R_1.R_2 = 0$. ∎

We can now prove the theorem.

Proof of Theorem 4.16. Any Enriques surface of type $2\widetilde{D}_4$ admits a non-special non-extendable 3-sequence (see Example 4.10), so it suffices to prove the first statement.

Fix a non-special non-extendable 3-sequence (F_1, F_2, F_3). By Theorem 2.3, there exists a 3-degenerate 10-sequence as in (4.1). Without loss of generality, we may assume $r_1 \geq 3$. By Lemma 4.17, we may assume that $R_{1,1}$ is contained in F_3.

Consider the vector $e_{2,3} \in \mathrm{Num}(X)$ given in Lemma 2.4. By the Riemann–Roch theorem, there exists an effective divisor $D \in \mathrm{Pic}(X)$, whose numerical class is $e_{2,3}$, such that $D.F_i = 1$, $D.R_{1,1} = 1$, $D.R_{1,3} = -1$ and $D.R_{i,j} = 0$ for all other i, j. By Lemma 2.1, there exist D' and C with $D \sim D' + C$, such that D' is nef, $D^2 = D'^2 = 0$, and C is a linear combination of (-2)-curves with non-negative coefficients. Since $D'.F_i \leq 1$ for all i and (F_1, F_2, F_3) is non-extendable, $D' \equiv F_i$ for some $i \in \{1, 2, 3\}$.

Let $E := F_1 + F_2 + F_3 - D$. Then, $E^2 = 0$ and $E.F_i = 1$. Again by Lemma 2.1, we can write $E \sim E' + C'$ with $E' \equiv F_j$ for some $j \in \{1, 2, 3\}$. Notice that $i \neq j$, since otherwise $F_1 + F_2 + F_3 - 2F_i$ would be numerically equivalent to the effective divisor $C + C'$, contradicting the fact that the 3-sequence is non-special. Therefore, $C + C'$ is numerically equivalent to one of the F_i, so it coincides with one of the half-fibers of $|2F_1|$, $|2F_2|$ or $|2F_3|$. Since $C.R_{1,3} = -1$, C contains $R_{1,3}$, so $C + C' = F_3$. In particular, either $D' \equiv F_1$ and $E' \equiv F_2$, or vice versa.

Claim 4.19. *There exists a* 3*-degenerate* 10*-sequence extending* (F_1, F_2, F_3) *as in* (4.1)*, such that, after possibly permuting the half-fibers,* $r_3 = 0$ *and all* $R_{i,j}$ *are contained in* F_3.

Proof. If $r_2 = r_3 = 0$, the desired property is already satisfied. Hence, assume that $r_2 > 0$ or $r_3 > 0$.

Suppose first $D' \equiv F_1$ and $E' \equiv F_2$. Since $C'.F_2 = 1$, C' contains a special bisection R of $|2F_2|$. Now, notice that $R_{1,1}$ is not a component of C', since it is simple in F_3 and already contained in C. Moreover, $C'.R_{1,1} = (E - F_2).R_{1,1} = 0$, so $R_{1,1}$ is disjoint from C'. In particular, $R_{1,1}.R = 0$, and any extension of $(F_1, F_1 + R_{1,1}, F_2, F_2 + R, F_3)$ satisfies the desired property by Lemma 4.18.

Suppose instead that $D' \equiv F_2$ and $E' \equiv F_1$. If $r_2 > 0$, then $C.R_{2,1} = -1$, so $R_{2,1}$ is contained in F_3. In particular, $r_3 = 0$ by Lemma 4.18 and we are done. Hence, we may assume $r_3 > 0$, and that $R_{3,1}$ is in F_2 by Lemma 4.18. Since

$$C'.R_{1,1} = (E - F_1).R_{1,1} = -1,$$

C' contains $R_{1,1}$. On the other hand, $C.F_2 = (D - F_2).F_2 = 1$, so C contains a special bisection R of $|2F_2|$. However, $C.R_{3,1} = 0$ and C does not contain $R_{3,1}$, so $R_{3,1}$ is disjoint from C. In particular, $R.R_{3,1} = 0$. Then, $(F_2, F_2 + R, F_3, F_3 + R_{3,1})$ is a 2-degenerate 4-sequence, whose existence contradicts Lemma 4.18. ∎

We replace our original 3-degenerate 10-sequence by one satisfying the properties of Claim 4.19. In particular, $r_1 + r_2 = 7$, and we can suppose that $r_1 \geq 4$. Denote by $G_1 \in |2F_1|$ and $G_2 \in |2F_2|$ the fibers containing $R_{1,3}$. By Proposition 2.8, both G_1 and G_2 are simple fibers.

Claim 4.20. *F_3 is a half-fiber of type* I_4^*.

Proof. As F_3 contains two disjoint configurations of type A_{r_1} and A_{r_2} with $r_1 + r_2 = 7$, we deduce that F_3 is of type I_8, I_9, I_4^*, III^* or II^*.

Since $C.R_{1,3} = -1$, the curve $R_{1,3}$ is a component of C. Repeating the argument with the other $R_{i,j}$, we have that $C - R_{1,2} - \cdots - R_{1,r_1}$ is effective. However, this divisor intersects $R_{1,3}$ negatively, so reasoning similarly we obtain that the divisor

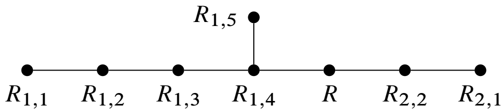
$$Z := C - R_{1,2} - 2R_{1,3} - \cdots - 2R_{1,r_1-1} - R_{1,r_1}$$

is effective. In particular, $R_{1,3}$ has multiplicity ≥ 2 in C, and, therefore, $R_{1,3}$ has multiplicity ≥ 2 in F_3. So, F_3 is additive. Also, since C and C' both intersect one of the half-fibers with multiplicity 1 and $C + C' = F_3$, each of the curves C and C' contains a simple component of F_3, so F_3 cannot be of type II^*. It remains to exclude the case that F_3 is of type III^*.

Seeking a contradiction, assume that F_3 is of type III^*. Denote by R the only component of F_3 different from the $R_{i,j}$. The only possible way of fitting the $R_{i,j}$ in F_3 is if $r_1 = 7$ or $r_1 = 5$. The former is impossible, since otherwise G_2 would contain both simple components of F_3, contradicting Proposition 2.8. Hence, we may assume that $r_1 = 5$, $r_2 = 2$, and that the divisor

$$Z = C - R_{1,2} - 2R_{1,3} - 2R_{1,4} - R_{1,5}$$

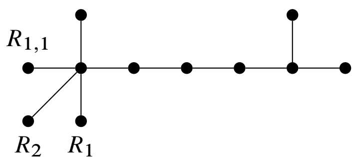
is effective. Since $R_{1,1}$ and $R_{2,1}$ are simple components of F_3, we obtain the following dual graph:



We have $R.R_{1,4} = R.R_{2,2} = 1$ and by writing R as linear combination of F_3 and the $R_{i,j}$, we obtain $D.R = 1$.

Since $e_{2,3}$ is not contained in the lattice spanned by the 3-degenerate 10-sequence by Corollary 2.5, neither is Z, so Z must contain R. Thus, we can recursively remove the following curves from Z while staying effective: R, $R_{2,2}$, $R_{2,1}$, $R_{2,2}$, R, $R_{1,4}$, $R_{1,3}$, $R_{1,2}$ and $R_{1,1}$. But then C contains both simple components of F_3, which is absurd. ■

By Proposition 2.8, both G_1 and G_2 contain a configuration of type D_8, so they are of type II^* or I_4^*. In the former case, the component of G_i not contained in F_3 would be a special bisection of $|2F_3|$ and X would be of type $\widetilde{D}_8$, which is absurd. Hence, both G_1 and G_2 are of type I_4^*. If the extra components R_i of G_i lie on the same side of F_3, then we get the following graph:



There is a half-fiber of type I_0^* and a configuration of type D_5 disjoint from this half-fiber. This contradicts Proposition 2.7. Therefore, R_1 and R_2 lie on opposite sides of F_3 and we get the dual graph of type $2\widetilde{D}_4$. This finishes the proof. ■

5. Projective models of Enriques surfaces

In this section, we describe an application of Corollary 1.5 to the theory of projective models of Enriques surfaces. In particular, we show that every Enriques surface in characteristic different from 2 arises via Enriques' original construction. For this, we note first that every Enriques surface of non-degeneracy 4 admits a non-special 3-sequence.

Proposition 5.1. *If (F_1, F_2, F_3, F_4) is a 4-sequence, then at most one of the 3-sequences (F_i, F_j, F_k) with $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$ is special.*

Proof. Seeking a contradiction, assume that two of the 3-sequences are special. By Proposition 3.2, we may assume without loss of generality that there exist

$$S_3 \in |F_1 + F_2 - F_3| \quad \text{and} \quad S_4 \in |F_1 + F_2 - F_4|.$$

Then,

$$S_3.S_4 = (F_1 + F_2 - F_3).(F_1 + F_2 - F_4) = -1.$$

Hence, S_3 and S_4 share a component. This is absurd, because $F_3 + S_3, F_4 + S_4 \in |F_1 + F_2|$, $F_3 + S_3 \neq F_4 + S_4$, and $|F_1 + F_2|$ has no base components by [9, Proposition 2.6.1]. ■

Corollary 5.2. *For an Enriques surface X, the following are equivalent:*

(1) *X is not of type $\widetilde{E}_8$, $\widetilde{D}_8$, $\widetilde{E}_7$, or $\widetilde{E}_7^{(2)}$.*

(2) *X admits a non-special 3-sequence.*

Proof. If X admits a non-special 3-sequence, then X is not of type $\widetilde{E}_8$, $\widetilde{D}_8$, $\widetilde{E}_7$, or $\widetilde{E}_7^{(2)}$, since the first three types of surfaces satisfy $\max \mathrm{nd}(X) \leq 2$ and the unique 3-sequence on the latter type of surface is special, as we have seen in Example 4.7.

Conversely, if X is not of one of the excluded types, then Theorems 1.1 and 1.2, and Corollary 1.5 show that X admits a 4-sequence. By Proposition 5.1, this implies that X admits a non-special 3-sequence. ■

By [9, Theorem 3.5.1], a non-special 3-sequence (F_1, F_2, F_3) on a classical Enriques surface X induces, via the linear system $|F_1 + F_2 + F_3|$, a realization of X as the minimal desingularization of an Enriques sextic (1.1). Thus, the following is an immediate consequence of Corollary 5.2.

Theorem 5.3. *Any classical Enriques surface which is not of type $\widetilde{E}_8$, $\widetilde{D}_8$, $\widetilde{E}_7$ or $\widetilde{E}_7^{(2)}$ is the minimal resolution of an Enriques sextic (1.1). In particular, if $p \neq 2$, then every Enriques surface arises via this construction.*

Following an observation of Castelnuovo, we note that the Cremona transformation

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_2x_3 : x_0x_1 : x_0x_2 : x_0x_3]$$

maps an Enriques sextic (1.1) into a surface of the form

$$x_0^4x_1^2x_2^4x_3^2 + x_0^4x_1^2x_2^2x_3^4 + x_0^4x_2^4x_3^4 + x_0^6x_1^2x_2^2x_3^2 + x_0^3x_1x_2^2x_3^2Q'.$$

Upon dividing by $x_0^3x_2^2x_3^2$, we obtain the Castelnuovo quintic (1.2), which has elliptic singularities at the four vertices of the coordinate tetrahedron (see [9, Example 1.6.4]). In terms of linear systems, and after suitably permuting the coordinates, the above quintic model corresponds to the linear subsystem of $|2F_1 + F_2 + F_3|$ spanned by the four divisors $2F_1' + F_2' + F_3'$, $2F_1 + F_2' + F_3'$, $F_1 + F_1' + F_2 + F_3'$, and $F_1 + F_1' + F_2' + F_3$, where F_i' denotes the second half-fiber of $|2F_i|$.

Theorem 5.4. *Any classical Enriques surface which is not of type $\widetilde{E}_8$, $\widetilde{D}_8$, $\widetilde{E}_7$ or $\widetilde{E}_7^{(2)}$ is birational to a normal Castelnuovo quintic* (1.2)*. In particular, if $p \neq 2$, then every Enriques surface arises via this construction.*

Remark 5.5. Quintic models of Enriques surfaces in characteristic $p \neq 2$ were previously studied by Castelnuovo [5], Kim [15], Stagnaro [24] and Umezu [25]. Applying [25, Corollary 2.3] together with our Corollary 1.5 yields Theorem 5.4 in these characteristics, since Stagnaro's first construction of quintic Enriques surfaces [24] coincides with the one obtained via Castelnuovo's transformation.

Remark 5.6. It was known already to Enriques [13] that every complex Enriques surface is birational to a double cover of $\mathbb{P}^2$ branched over (a degeneration of) an *Enriques octic*, which is the union of two lines and a singular sextic, such that the sextic has two tacnodes whose tacnodal tangents coincide with the lines and a node at the intersection of the two lines. Such a double plane model can be obtained by choosing a 2-sequence (F_1, F_2) on X and composing the map from X to a 4-nodal quartic del Pezzo induced by the complete linear system $|2F_1 + 2F_2|$ with the projection from a line on the del Pezzo surface through two of the nodes. The resulting rational map to $\mathbb{P}^2$ corresponds, up to permutation of the half-fibers, to the linear system $|2F_1 + F_2|$.

Our results also show that we can assume these Enriques octics to satisfy a certain non-degeneracy condition, in every characteristic different from 2. Choose a non-special 3-sequence (F_1, F_2, F_3) on X. The composition of the map from X to the Castelnuovo quintic with the projection from one of the two elliptic singularities of multiplicity 3 is given, after a suitable permutation of coordinates, by the linear system $|2F_1 + F_2|$, so this double plane model coincides with the one described in the previous paragraph. In terms of equations, we can write the quintic as

$$x_3^2C_1 + x_0x_1x_3Q'' + x_0x_1C_2,$$

where the C_i are equations of cubics passing through $[0 : 0 : 1]$, tangent to $\{x_0 = 0\}$ at $[0 : 1 : 0]$, and tangent to $\{x_1 = 0\}$ at $[1 : 0 : 0]$, and Q'' is the equation of a conic passing through $[1:0:0]$ and $[0:1:0]$. The corresponding Enriques octic is given by $x_0x_1(x_0x_1Q''^2 - 4C_1C_2)$. In particular, we get the existence of two cubics and a conic that are in special position with respect to the Enriques octic. The two cubics given by C_1 and C_2 are the images of F_3 and F_3'. Hence, our results show that every Enriques surface in characteristic different from 2 is birational to a double cover of $\mathbb{P}^2$ branched over an Enriques octic that admits two cubics C_1 and C_2 as above.

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