Rev. Mat. Iberoam. 42 (2026), no. 3, 831–854
DOI 10.4171/RMI/1617

Balayage of measures: behavior near a corner

Christophe Charlier and Jonatan Lenells

Abstract. We consider the balayage of a measure μ defined on a domain Ω onto its boundary $\partial\Omega$. Assuming that Ω has a corner of opening $\pi\alpha$ at a point $z_0 \in \partial\Omega$ for some $0 < \alpha \leq 2$ and that $d\mu(z) \asymp |z - z_0|^{2b-2}\, d^2 z$ as $z \to z_0$ for some $b > 0$, we obtain the precise rate of vanishing of the balayage of μ near z_0. The rate of vanishing is universal in the sense that it only depends on α and b. We also treat the case when the domain has multiple corners at the same point. Moreover, when $2b \leq 1/\alpha$, we provide explicit constants for the upper and lower bounds.

1. Introduction

Balayage measures were introduced by Henri Poincaré in the late 19th century as a tool to solve the Laplace equation [21]. Given a measure μ on a domain Ω, the balayage (sweeping) of μ onto $\partial\Omega$ is a measure ν on $\partial\Omega$ whose potential outside Ω coincides with that of μ (up to a constant if Ω is unbounded). More precisely, given a bounded Jordan domain Ω and a non-negative measure μ with compact support in Ω, the balayage measure $\nu := \mathrm{Bal}(\mu, \partial\Omega)$ is defined as the unique measure supported on $\partial\Omega$ such that $\nu(\partial\Omega) = \mu(\Omega)$, $\nu(P) = 0$ for every Borel set P of zero capacity, and such that

$$\int_{\partial\Omega} \log \frac{1}{|z - w|}\, d\nu(w) = \int_{\Omega} \log \frac{1}{|z - w|}\, d\mu(w)$$

holds for quasi-every $z \in \mathbb{C} \setminus \Omega$ (see, e.g., Theorem II.4.7 in [23]).

Balayage measures can also be defined in terms of the harmonic measure (see (2.2) below). The concept of balayage plays a significant role in potential theory, see, for instance, [11, 19]. Various applications and generalizations of balayage theory can be found in [2, 3, 6, 9, 14–18, 20, 25–27].

Let $B_r(z)$ denote the open disk of radius r centered at z. In this paper we deal with the following question, which is relevant for example in the study of two-dimensional Coulomb gases as explained in Section 7.

Mathematics Subject Classification 2020: 31A15 (primary); 31A20, 31A05 (secondary).
Keywords: balayage measure, harmonic measure, boundary behavior.

Question. *If*

- *Ω has a Hölder-C^1 corner of opening $\pi\alpha$ at a point $z_0 \in \partial\Omega$ for some $0 < \alpha \le 2$,*
- *and $d\mu(z) \asymp |z - z_0|^{2b-2} d^2z$ as $z \to z_0$ $(z \in \Omega)$ for some $b > 0$, where d^2z is the two-dimensional Lebesgue measure,*

what is the behavior of $\nu(\partial\Omega \cap B_r(z_0))$ as $r \to 0$?

It has been conjectured in [9] that

$$\nu(\partial\Omega \cap B_r(z_0)) \asymp \begin{cases} r^{\min\{2b, 1/\alpha\}}, & \text{if } 2b \neq 1/\alpha, \\ r^{2b} \log \frac{1}{r}, & \text{if } 2b = 1/\alpha, \end{cases} \qquad \text{as } r \to 0. \tag{1.1}$$

Here we prove this conjecture. In particular, we establish that the rate of vanishing of $\nu(\partial\Omega \cap B_r(z_0))$ as $r \to 0$ only depends on α and b, but is otherwise universal, in the sense that it is independent of other properties of Ω and μ. In fact, we will strengthen (1.1) in several ways:

(1) We show that (1.1) holds for more general open sets Ω than Jordan domains. In particular, Ω does not have to be connected, and the connected components of Ω do not have to be simply connected.

(2) We obtain bounds on the implicit constants in (1.1) whenever $2b \le 1/\alpha$. For example, if $2b < 1/\alpha$ and

$$d\mu(z) = (1 + o(1))|z - z_0|^{2b-2} d^2z \quad \text{as } z \to z_0,$$

then we show that, for any $\varepsilon > 0$,

$$(1-\varepsilon)\frac{\tan(\pi\alpha b)}{2b^2} r^{2b} \le \nu(\partial\Omega \cap B_r(z_0)) \le (1+\varepsilon)\frac{\pi\alpha}{2b}\left(1 + \frac{16b}{\pi(1/\alpha - 2b)}\right) r^{2b}$$

for all small enough $r > 0$. Since

$$\frac{\tan(\pi\alpha b)}{2b^2} = \frac{\pi\alpha}{2b} + O(b) \quad \text{as } b \to 0,$$

this formula shows that $\nu(\partial\Omega \cap B_r(z_0))$ behaves like $\frac{\pi\alpha}{2b} r^{2b}$ in the limit of small b, i.e., not only is the order of vanishing of the balayage measure ν near the corner a universal quantity depending only on α and b, but the constant prefactor is also universal in the small b limit.

(3) We show that (1.1) remains true also in the case when Ω has multiple corners at a single point z_0, provided that $\pi\alpha$ is the largest of the opening angles. More precisely, if Ω is an open set such that $\partial\Omega \cap B_r(z_0)$ for small $r > 0$ is a disjoint union of m domains with Hölder-C^1 corners at z_0 with opening angles $\pi\alpha_j \in (0, 2\pi]$, for $j = 1, \dots, m$, and $\Omega \setminus B_r(z_0)$ is a disjoint union of finitely connected Jordan domains, then (1.1) holds with $\alpha := \max_j \alpha_j$.

The situation where Ω has an outward-pointing cusp (i.e., the case $\alpha = 0$) deserves a separate analysis and is treated in the companion paper [10].

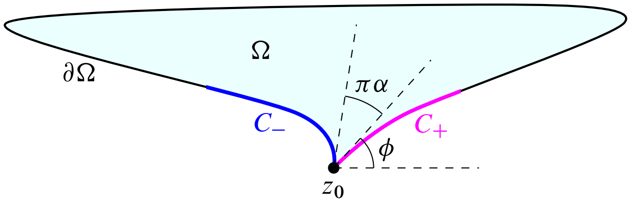

Figure 1. Illustration of a corner of opening $\pi\alpha$ at z_0.

2. Main results

Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. A *Jordan curve in* $\mathbb{C}^*$ is the image of the unit circle under an injective continuous function $\partial\mathbb{D} \to \mathbb{C}^*$; in other words, it is a non-self-intersecting loop in $\mathbb{C}^*$. If Ω is a simply connected open subset of $\mathbb{C}^*$, then Ω is a *Jordan domain* if $\partial\Omega$ is a Jordan curve in $\mathbb{C}^*$. If Ω is an open connected subset of $\mathbb{C}^*$ such that $\partial\Omega$ is a finite union of pairwise disjoint Jordan curves, then we say that Ω is a *finitely connected Jordan domain* in $\mathbb{C}^*$.

A *Jordan arc in* $\mathbb{C}^*$ is the image of a closed interval $I \subset \mathbb{R}$ under an injective continuous function $I \to \mathbb{C}^*$. A Jordan arc $C \subset \mathbb{C}$ is of class $C^{1,\gamma}$, $0 < \gamma \le 1$, if it has a parametrization $C\colon w(t)$, $0 \le t \le 1$, such that the derivative $w'(t)$ exists, is nonzero, and is Hölder continuous with exponent γ on $[0, 1]$.

Let $\partial\Omega$ be the boundary of a finitely connected Jordan domain Ω in $\mathbb{C}^*$. We say that Ω has a *corner* of opening $\pi\alpha$, $0 < \alpha \le 2$, at $z_0 \in \partial\Omega \cap \mathbb{C}$ if there are closed Jordan arcs $C_\pm \subset \partial\Omega$ ending at z_0 and lying on different sides of z_0 such that

$$\arg(z - z_0) \to \begin{cases} \phi & \text{as } C_+ \ni z \to z_0, \\ \phi + \pi\alpha & \text{as } C_- \ni z \to z_0, \end{cases} \tag{2.1}$$

for some ϕ, and such that $\{z_0 + \rho e^{i(\phi+\pi\alpha/2)} : \rho \in (0, \rho_0)\} \subset \Omega$ for some small enough $\rho_0 > 0$, see Figure 1. In (2.1), the branch is chosen so that $\arg(\cdot - z_0)$ is continuous in $(\bar{\Omega} \cap B_{\rho_0}(z_0)) \setminus \{z_0\}$. The situation where Ω has an inward-pointing cusp is covered by the case $\alpha = 2$. We say that the corner is *Hölder-C^1* if the Jordan arcs $C_\pm$ can be chosen to be of class $C^{1,\gamma}$ for some $\gamma > 0$.

If $\Omega \subset \mathbb{C}^*$ is an open set and μ is a non-negative measure on Ω, then the balayage $\nu := \mathrm{Bal}(\mu, \partial\Omega)$ of μ onto $\partial\Omega$ is the measure on $\partial\Omega$ defined, whenever it exists, by

$$\nu(E) = \int_\Omega \omega(z, E, \Omega)\, d\mu(z) \quad \text{for Borel subsets } E \text{ of } \partial\Omega, \tag{2.2}$$

where $\omega(z, E, \Omega)$ is the harmonic measure of E at z in Ω. We assume that $\omega(\cdot; E, \Omega)$ is μ-measurable, so that the integral (2.2) is well defined. (We recall that $\omega(z, E, \Omega)$ can be interpreted as the probability that a Brownian motion starting at z exits Ω at a point in E, see, e.g., p. 73 in [12].)

Let $d^2z = dx\,dy$ denote the Lebesgue measure on $\mathbb{C}$. If μ is a non-negative measure on Ω and $b > 0$, then we write

$$d\mu(z) = (1 + o(1))\,|z - z_0|^{2b-2}\, d^2z, \quad \text{as } z \to z_0 \in \partial\Omega \tag{2.3}$$

if, for every $\varepsilon > 0$, there exists a radius $\rho_0 > 0$ such that

$$\left|\mu(A) - \int_A |z - z_0|^{2b-2}\, d^2z\right| \leq \varepsilon \int_A |z - z_0|^{2b-2}\, d^2z \tag{2.4}$$

for all measurable subsets A of $\Omega \cap B_{\rho_0}(z_0)$. Similarly, we write

$$d\mu(z) \asymp |z - z_0|^{2b-2}\, d^2z \quad \text{as } z \to z_0 \in \partial\Omega$$

if there exist a $\rho_0 > 0$ and constants $c_1, c_2 > 0$ such that

$$c_1 \int_A |z - z_0|^{2b-2}\, d^2z \leq \mu(A) \leq c_2 \int_A |z - z_0|^{2b-2}\, d^2z \tag{2.5}$$

for all measurable subsets A of $\Omega \cap B_{\rho_0}(z_0)$. If $f(r)$ and $g(r)$ are two positive functions defined for all small enough $r > 0$, then we write

$$f(r) \asymp g(r) \quad \text{as } r \to 0$$

if there exist an $r_0 > 0$ and constants $c_1, c_2 > 0$ such that

$$c_1 g(r) \leq f(r) \leq c_2 g(r) \quad \text{for all } r \in (0, r_0).$$

2.1. A single corner

We first state our main result in the case of a single corner at z_0. The result will then be generalized in Section 2.2 to the case of an arbitrary finite number of corners at z_0. In the case of a single corner, our main result is the following. We use $C > 0$ and $c > 0$ to denote generic strictly positive constants.

Theorem 2.1 (A single corner). *Let Ω be a finitely connected Jordan domain in $\mathbb{C}^*$. Let $0 < \alpha \leq 2$ and suppose Ω has a Hölder-C^1 corner of opening $\pi\alpha$ at a point $z_0 \in \partial\Omega \cap \mathbb{C}$. Let μ be a non-negative measure of finite total mass on Ω such that*

$$d\mu(z) = (1 + o(1))\, |z - z_0|^{2b-2}\, d^2z \quad \text{as } z \to z_0 \text{ for some } b > 0.$$

For every $\varepsilon > 0$, the balayage $\nu := \mathrm{Bal}(\mu, \partial\Omega)$ of μ onto $\partial\Omega$ obeys the following inequalities for all sufficiently small $r > 0$.

If $2b < 1/\alpha$,

$$(1 - \varepsilon)\, \frac{\tan(\pi\alpha b)}{2b^2}\, r^{2b} \leq \nu(\partial\Omega \cap B_r(z_0)) \leq (1 + \varepsilon)\, \frac{\pi\alpha}{2b} \Big(1 + \frac{16b}{\pi(1/\alpha - 2b)}\Big) r^{2b}. \tag{2.6}$$

If $2b = 1/\alpha$,

$$(1 - \varepsilon)\, \frac{2}{\pi b}\, r^{2b} \log(\tfrac{1}{r}) \leq \nu(\partial\Omega \cap B_r(z_0)) \leq (1 + \varepsilon)\, \frac{4}{b}\, r^{2b} \log(\tfrac{1}{r}). \tag{2.7}$$

If $2b > 1/\alpha$,

$$c\, r^{1/\alpha} \leq \nu(\partial\Omega \cap B_r(z_0)) \leq C\, r^{1/\alpha}. \tag{2.8}$$

In particular, if $d\mu(z) \asymp |z - z_0|^{2b-2}\, d^2 z$ *as* $z \to z_0$*, then*

$$\nu(\partial\Omega \cap B_r(z_0)) \asymp \begin{cases} r^{2b} & \text{if } 2b < 1/\alpha, \\ r^{2b}\log\frac{1}{r} & \text{if } 2b = 1/\alpha, \\ r^{1/\alpha} & \text{if } 2b > 1/\alpha, \end{cases} \quad \text{as } r \to 0. \tag{2.9}$$

The proof of Theorem 2.1 is presented in Section 5.

Remark 2.2. Theorem 2.1 implies in particular that the density of ν never blows up or vanishes at "standard" points where $\partial\Omega$ admits a tangent ($\alpha = 1$) and where $d\mu(z)/d^2 z \asymp 1$ ($b = 1$).

Remark 2.3. In Theorem 2.16(i) and Remark 2.18 of [9], the balayage measure ν was computed exactly in the case when $\Omega = \{re^{i\theta} : 0 < r < a, 0 < \theta < \pi\alpha\}$ is a circular sector of radius $a > 0$ and opening angle $\pi\alpha \in (0, 2\pi]$, and when μ is the measure $d\mu(z) = |z|^{2b-2} d^2 z$ with $b > 0$. We will use this fact to obtain the lower bounds on $\nu(\partial\Omega \cap B_r(z_0))$ in (2.6)–(2.8).

Remark 2.4. Let c_1 and c_2 be the largest possible constants and C_1 and C_2 the smallest possible constants such that, for every $\varepsilon > 0$ and ν as in Theorem 2.1, there exists $r_0 > 0$ such that for all $r \in (0, r_0)$, the inequalities

$$(1-\varepsilon) c_1 r^{2b} \le \nu(\partial\Omega \cap B_r(z_0)) \le (1+\varepsilon) C_1 r^{2b} \qquad \text{if } 2b < 1/\alpha,$$
$$(1-\varepsilon) c_2 r^{2b} \log(\tfrac{1}{r}) \le \nu(\partial\Omega \cap B_r(z_0)) \le (1+\varepsilon) C_2 r^{2b} \log(\tfrac{1}{r}) \qquad \text{if } 2b = 1/\alpha,$$

hold. The inequalities (2.6) and (2.7) imply that

$$\frac{\tan(\pi\alpha b)}{2b^2} \le c_1, \quad C_1 \le \frac{\pi\alpha}{2b}\Big(1 + \frac{16b}{\pi(1/\alpha - 2b)}\Big), \quad \frac{2}{\pi b} \le c_2 \quad \text{and} \quad C_2 \le \frac{4}{b}.$$

On the other hand, it follows from Remark 2.18 in [9] that

$$c_1 \le \frac{\tan(\pi\alpha b)}{2b^2} \quad \text{and} \quad c_2 \le \frac{2}{\pi b}.$$

This implies that the constants $\frac{\tan(\pi\alpha b)}{2b^2}$ and $\frac{2}{\pi b}$ appearing on the left-hand sides in (2.6) and (2.7) are the best possible. Finding C_1 and C_2 is an interesting problem for future research.

2.2. Multiple corners

The following theorem treats the case when Ω has multiple corners at z_0, see Figure 2.

Theorem 2.5 (Multiple corners at the same point). *Let* Ω *be an open subset of* $\mathbb{C}^*$*, let* $z_0 \in \partial\Omega \cap \mathbb{C}$*, and let* $m \ge 1$ *be an integer. Suppose there is a radius* $\rho_0 > 0$ *such that*

(i) *the open set* $\Omega \cap B_{\rho_0}(z_0)$ *has* m *connected components* $\{U_j\}_1^m$ *such that* $\bar{U}_j \cap \bar{U}_k = \{z_0\}$ *whenever* $j \ne k$ *and each component* U_j *has a Hölder-*C^1 *corner at* z_0 *of opening angle* $\pi\alpha_j \in (0, 2\pi]$*, and*

(ii) $\Omega \setminus \overline{B_{\rho_0}(z_0)}$ *is a disjoint union of finitely connected Jordan domains.*

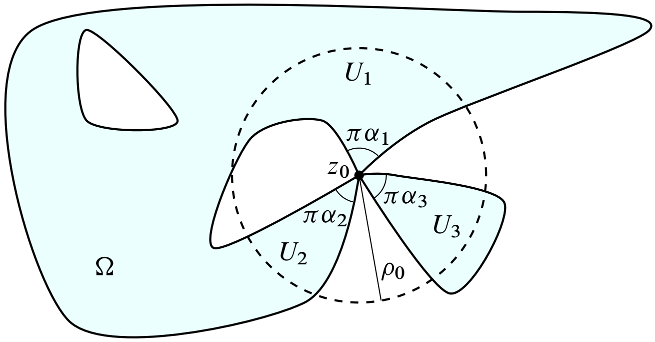

Figure 2. Illustration of an open set Ω with three corners at z_0.

Let μ be a non-negative measure of finite total mass on Ω such that

$$d\mu(z) = (1 + o(1))\, |z - z_0|^{2b-2}\, d^2z, \quad \text{as } z \to z_0,$$

for some $b > 0$. Let

$$\alpha := \max_{1 \le j \le m} \alpha_j.$$

For every $\varepsilon > 0$, the balayage $\nu := \mathrm{Bal}(\mu, \partial\Omega)$ of μ onto $\partial\Omega$ obeys the following inequalities for all sufficiently small $r > 0$.

If $2b < 1/\alpha$,

$$\begin{aligned}(1-\varepsilon)\sum_{j=1}^{m} \frac{\tan(\pi\alpha_j b)}{2b^2}\, r^{2b} &\le \nu(\partial\Omega \cap B_r(z_0)) \\ &\le (1+\varepsilon)\sum_{j=1}^{m} \frac{\pi\alpha_j}{2b}\Big(1 + \frac{16b}{\pi(1/\alpha_j - 2b)}\Big)\, r^{2b}.\end{aligned} \tag{2.10}$$

If $2b = 1/\alpha$,

$$m_\alpha (1-\varepsilon)\, \frac{2}{\pi b}\, r^{2b} \log(\tfrac{1}{r}) \le \nu(\partial\Omega \cap B_r(z_0)) \le m_\alpha (1+\varepsilon)\, \frac{4}{b}\, r^{2b} \log(\tfrac{1}{r}). \tag{2.11}$$

If $2b > 1/\alpha$,

$$c\, r^{1/\alpha} \le \nu(\partial\Omega \cap B_r(z_0)) \le C\, r^{1/\alpha}. \tag{2.12}$$

Here, m_α is the number of α_j such that $\alpha_j = \alpha$, i.e., the number of corners with the largest opening angle. In particular, if $d\mu(z) \asymp |z - z_0|^{2b-2}\, d^2z$ as $z \to z_0$, then (2.9) holds.

The proof of Theorem 2.5 is presented in Section 6. The main idea of the proof is to show that the various corners decouple up to terms of $O(r^{1/\alpha})$ and then apply Theorem 2.1 to each of the corners.

3. Local structure of Ω near z_0

We consider the structure of Ω near the corner at z_0. We first treat the case of a single corner.

3.1. A single corner

Fix $\alpha \in (0, 2]$ and let Ω be a finitely connected Jordan domain in $\mathbb{C}^*$ such that Ω has a Hölder-C^1 corner of opening $\pi\alpha$ at $z_0 \in \mathbb{C}$. After applying a translation and a rigid rotation, we may assume that $z_0 = 0$ and that

$$\arg z \to \begin{cases} 0 & \text{as } C_+ \ni z \to 0, \\ \pi\alpha & \text{as } C_- \ni z \to 0, \end{cases} \tag{3.1}$$

where $C_\pm \subset \partial\Omega$ are curves of class $C^{1,\gamma}$ for some $\gamma > 0$ ending at 0 and lying on different sides of 0.

The next lemma shows that, for sufficiently small $\rho > 0$, $\Omega \cap B_\rho(0)$ is a small deformation of S_ρ, where

$$S_\rho := \{z \in \mathbb{C} : 0 < \arg z < \pi\alpha,\ 0 < |z| < \rho\} \tag{3.2}$$

is a circular sector of angle $\pi\alpha$ and radius ρ.

Lemma 3.1. *There exists a radius $\rho_0 > 0$ such that the following hold, possibly after shortening the curves $C_\pm$:*

(i) $\overline{B_{\rho_0}(0)} \cap \partial\Omega = C_+ \cup C_-$.

(ii) $C_+ \cap C_- = \{0\}$.

(iii) $C_+ = w_+([0, \rho_0])$ *and* $C_- = w_-([0, \rho_0])$, *where the curves* $w_\pm\colon [0, \rho_0] \to \mathbb{C}$ *are* C^1 *on* $(0, \rho_0)$ *and such that* $|w_\pm(r)| = r$ *for* $r \in [0, \rho_0]$, *i.e.,* $w_\pm$ *are parametrizations of* $C_\pm$ *with the distance to the origin as parameter. Moreover,*

$$\arg w_+(r) = O(r^\gamma), \qquad \text{as } r \to 0, \tag{3.3a}$$

$$\arg w_-(r) = \pi\alpha + O(r^\gamma), \qquad \text{as } r \to 0, \tag{3.3b}$$

where the branch is chosen so that $\arg(\cdot)$ *is continuous in* $(\bar{\Omega} \cap B_{\rho_0}(0)) \setminus \{0\}$, *and*

$$w_+'(r) = 1 + O(r^\gamma), \qquad \text{as } r \to 0, \tag{3.4a}$$

$$w_-'(r) = e^{i\pi\alpha} + O(r^\gamma), \qquad \text{as } r \to 0. \tag{3.4b}$$

(iv) *For each* $0 < r \le \rho_0$, *it holds that* $\Omega \cap \{|z| = r\} = J_r$ *where* J_r *is the circular arc*

$$J_r := \{re^{i\theta} : \arg w_+(r) < \theta < \arg w_-(r)\}. \tag{3.5}$$

(v) J_r *has length* $r\Theta(r)$ *where, for some constant* $C_1 > 0$,

$$\Theta(r) \le \pi\alpha(1 + C_1 r^\gamma), \quad 0 < r \le \rho_0, \tag{3.6}$$

Proof. Since C_+ is $C^{1,\gamma}$ and tangent at 0 to the ray $\{x \geq 0\}$, we can write C_+ as

$$C_+ = \{x(t) + iy(t) : 0 \leq t \leq t_1\},$$

where $t_1 > 0$, and with $x, y \in C^{1,\gamma}([0, t_1])$ satisfying $x'(t) = x'(0) + O(t^\gamma)$, $y'(t) = O(t^\gamma)$, and $x'(0) > 0$. Rescaling t if necessary, we may assume that $x'(0) = 1$. Integration of $x'(t) = 1 + O(t^\gamma)$ and $y'(t) = O(t^\gamma)$ then gives $x(t) = t(1 + O(t^\gamma))$ and $y(t) = O(t^{1+\gamma})$. Defining

$$r_+(t) := |x(t) + iy(t)| = x(t)\sqrt{1 + \left(\frac{y(t)}{x(t)}\right)^2},$$

we have

$$r'_+(t) = \frac{x'(t) + y(t)y'(t)/x(t)}{\sqrt{1 + y(t)^2/x(t)^2}} = 1 + O(t^\gamma), \quad \text{as } t \to 0.$$

Hence, shrinking $t_1 > 0$ if necessary, we may assume that $r_+(t)$ is strictly increasing for $t \in [0, t_1]$. Let $t(r)$ for $r \in [0, r_1]$, where $r_1 = r_+(t_1)$, be the inverse of this function. The parametrization

$$w_+(r) := x(t(r)) + iy(t(r)), \quad r \in [0, r_1],$$

satisfies $|w_+(r)| = r$ for all $r \in [0, r_1]$. Moreover,

$$\arg w_+(r) = \arctan \frac{y(t(r))}{x(t(r))} = O(t(r)^\gamma) = O(r^\gamma), \qquad \text{as } r \to 0,$$
$$w'_+(r) = x'(t(r))t'(r) + iy'(t(r))t'(r) = 1 + O(r^\gamma), \qquad \text{as } r \to 0.$$

This completes the proof of the existence of a parametrization w_+ satisfying assertion (iii); the proof for w_- is similar and we omit it. Moreover, $\partial\Omega \setminus (w_+([0, \rho_0)) \cup w_-([0, \rho_0)))$ is a closed set in $\mathbb{C}^*$ which does not contain 0. Therefore, shrinking ρ_0 if necessary, we may assume that $\partial\Omega \cap \overline{B_{\rho_0}(0)} = C_+ \cup C_-$ and that $C_+ \cap C_- = \{0\}$. Furthermore, for each $0 < r < \rho_0$, $\Omega \cap \{|z| = r\} = J_r$, where J_r is defined by (3.5). This completes the proof of assertions (i)–(iv). From (3.3), we infer that the circular arc J_r has length $r\Theta(r)$, where $\Theta(r) = \pi\alpha + O(r^\gamma)$ as $r \to 0$, so shrinking ρ_0 if necessary, we obtain also (v). ■

3.2. Multiple corners

We next consider the case when Ω has $m \geq 1$ corners at $z_0 = 0$. Suppose Ω is an open subset of $\mathbb{C}^*$ satisfying (i) and (ii) of Theorem 2.5 with $z_0 = 0$. By applying Lemma 3.1 to each of the components U_j of $\Omega \cap B_{\rho_0}(0)$, we see that after shrinking ρ_0 if necessary, the following hold (see Figure 3):

(i) $\Omega \cap B_{\rho_0}(0) = \bigsqcup_{j=1}^m U_j$ and $\bar{\Omega} \cap \overline{B_{\rho_0}(0)} = \bigcup_{j=1}^m \bar{U}_j$, where U_j has a Hölder-C^1 corner of opening $\pi\alpha_j \in (0, 2\pi]$ at 0.

(ii) $\overline{B_{\rho_0}(0)} \setminus \Omega = \bigcup_{j=1}^m \bar{V}_j$, where $\{V_j\}_1^m$ are the m connected components of $B_{\rho_0}(0) \setminus \bar{\Omega}$.

(iii) The sets U_j and V_j are ordered, so that $C_{j,+} := \bar{U}_j \cap \bar{V}_{j-1}$ ($V_0 \equiv V_m$) and $C_{j,-} := \bar{U}_j \cap \bar{V}_j$ are Jordan arcs of class C^{1,γ_j}, $0 < \gamma_j \leq 1$, with $C_{j,+} \cap C_{j,-} = \{0\}$, and such that if $C_{j,\pm}$ are oriented outwards, then U_j lies to the left of $C_{j,+}$ and to the right of $C_{j,-}$ for $j = 1, \dots, m$.

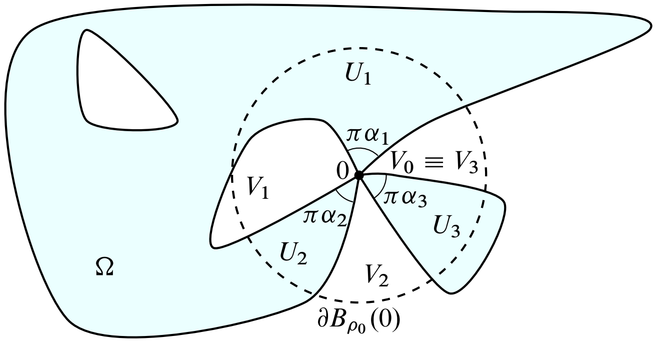

Figure 3. Illustration of Section 3.2 with $m = 3$.

(iv) For each $j = 1, \dots, m$, $J_{j,r} := U_j \cap \partial B_r(0)$ is an arc of length $r\Theta_j(r)$, where

$$\Theta_j(r) \le \pi\alpha_j(1 + C_j r^{\gamma_j}), \quad 0 < r \le \rho_0, \tag{3.7}$$

for some constant $C_j > 0$.

We henceforth assume that $\rho_0 > 0$ has been chosen so small that the above properties hold.

4. Estimates of harmonic measure

In this section, we derive upper bounds on $\omega(z, \partial U_j \cap B_r(0), \Omega)$ that will be used in the proofs of Theorem 2.1 and Theorem 2.5.

Let Ω be an open connected subset of $\mathbb{C}^*$. A *metric* ρ on Ω is a non-negative Borel measurable function ρ on Ω such that the ρ-area of Ω,

$$A(\Omega, \rho) = \int_\Omega \rho^2 \, d^2 z$$

satisfies $0 < A(\Omega, \rho) < \infty$. Let E and F be subsets of $\bar{\Omega}$, and let Γ be the family of all connected arcs in Ω joining E and F. The *extremal distance* $d_\Omega(E, F)$ from E to F is defined by

$$d_\Omega(E, F) = \sup_\rho \frac{L(\Gamma, \rho)^2}{A(\Omega, \rho)}, \quad \text{where } L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} \int_\gamma \rho |dz|. \tag{4.1}$$

We will need the following result, which is Theorem H.7 in [12].

Lemma 4.1 (Theorem H.7 in [12]). *Let $\tilde{\Omega}$ be a finitely connected Jordan domain, and let E be a finite union of arcs contained in one component $\tilde{\Gamma}$ of $\partial\tilde{\Omega}$. Suppose σ is a Jordan arc in $\mathbb{C}$ connecting $z_1 \in \tilde{\Omega}$ to $\tilde{\Gamma} \setminus E$. Then*

$$\omega(z_1, E, \tilde{\Omega}) \le \frac{8}{\pi} e^{-\pi d_{\tilde{\Omega}\setminus\sigma}(\sigma, E)}.$$

The next lemma will be used to obtain upper bounds on $\omega(z, \partial U_j \cap B_r(0), \Omega)$. The lemma treats the case when Ω has any finite number of corners at $z_0 = 0$; the case of a single corner is included as a special case. The proof is basically a combination of the proofs of Theorem IV.6.2 in [12] and Theorem H.8 in [12]. Theorem H.8 in [12] treats the cartesian case, whereas we are interested in the polar case. In Theorem IV.6.2 of [12], the estimate is for $\omega(z_1, E, \Omega)$ with $|z_1|$ small and E outside a large disk. We need the opposite situation: E inside a small disk and $|z_1|$ large. We therefore provide a proof. We assume that ρ_0 is chosen as in Section 3.2.

Lemma 4.2. *Suppose Ω is an open subset of $\mathbb{C}^*$ fulfilling* (i) *and* (ii) *of Theorem 2.5 with $z_0 = 0$ and some integer $m \geq 1$. Fix $j \in \{1, \dots, m\}$ and let $0 < r_0 < R_0 \leq \rho_0$. Let $z_1 \in \Omega$ be such that $|z_1| \geq R_0$. If $r\Theta_j(r)$ is the length of $U_j \cap \partial B_r(0)$, then*

$$\omega(z_1, \partial U_j \cap B_{r_0}(0), \Omega) \leq \frac{8}{\pi} e^{-\pi \int_{r_0}^{R_0} \frac{dr}{r\Theta_j(r)}}. \tag{4.2}$$

Proof. If $m \leq 2$, let $\Omega' := \Omega$. If $m \geq 3$, let $\Omega' := \Omega \cup (B_{\rho_0}(0) \cap \bigcup_{i \neq j-1,j} \bar{V}_i) \setminus \{0\}$, where the union is over all $i = 1, \dots, m$ with $i \neq j-1$ and $i \neq j$, i.e., over all i for which V_i is not adjacent to U_j. Let $E := U_j \cap \partial B_{r_0}(0)$ and let $\tilde{\Omega}$ be the component of $\Omega' \setminus E$ that contains z_1. The definition of Ω' implies that $\tilde{\Omega}$ has at most a single corner at 0; in particular, $\tilde{\Omega}$ is a finitely connected Jordan domain.

Let $\tilde{\Gamma}$ be the component of $\partial\tilde{\Omega}$ containing E, see Figure 4. Note that E separates $\partial U_j \cap B_{r_0}(0)$ from z_1 in Ω'. Hence, by the maximum principle,

$$\omega(z_1, \partial U_j \cap B_{r_0}(0), \Omega) \leq \omega(z_1, \partial U_j \cap B_{r_0}(0), \Omega') \leq \omega(z_1, E, \tilde{\Omega}).$$

Let $\sigma \subset \{z : |z| = |z_1|\}$ be a curve (not necessarily contained in Ω) connecting z_1 to $\tilde{\Gamma}$. By Lemma 4.1,

$$\omega(z_1, E, \tilde{\Omega}) \leq \frac{8}{\pi} e^{-\pi d_{\tilde{\Omega}\setminus\sigma}(\sigma, E)}.$$

Therefore it only remains to show that

$$d_{\tilde{\Omega}\setminus\sigma}(\sigma, E) \geq \int_{r_0}^{R_0} \frac{dr}{r\Theta(r)}. \tag{4.3}$$

Let $J_{j,r} := U_j \cap \partial B_r(0)$. For $z \in \tilde{\Omega} \setminus \sigma$, we define the metric

$$\rho(z) = \begin{cases} \frac{1}{r\Theta_j(r)} & \text{for } z \in J_{j,r}, \\ 0 & \text{for } z \in \tilde{\Omega} \setminus \bigcup_{r_0 < r < R_0} J_{j,r}, \end{cases}$$

where $r = |z|$. Let Γ be the family of all arcs in $\tilde{\Omega} \setminus \sigma$ connecting E and σ. For each $r \in (r_0, R_0)$, $J_{j,r}$ separates σ from E in $\tilde{\Omega}$. Hence, if $\gamma \in \Gamma$, then

$$\int_\gamma \rho(z)\,|dz| \geq \int_{r_0}^{R_0} \frac{dr}{r\Theta_j(r)},$$

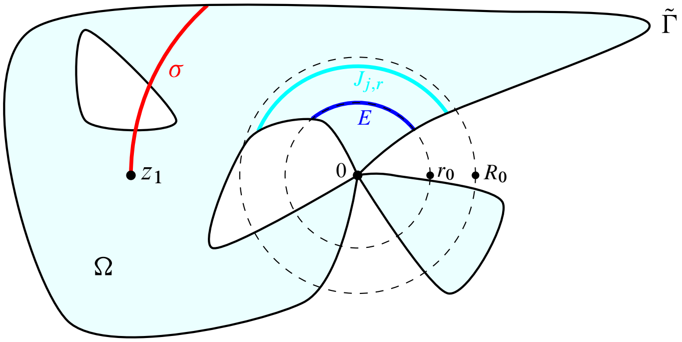

Figure 4. Illustration of Lemma 4.2.

and so

$$L(\Gamma,\rho) \geq \int_{r_0}^{R_0} \frac{dr}{r\Theta_j(r)}.$$

Furthermore, the ρ-area of $\tilde{\Omega} \setminus \sigma$ is given by

$$A(\tilde{\Omega} \setminus \sigma, \rho) = \int_{r_0}^{R_0} \int_{\{\theta : re^{i\theta} \in J_{j,r}\}} \frac{1}{r^2 \Theta_j(r)^2} \, r \, d\theta dr = \int_{r_0}^{R_0} \frac{dr}{r\Theta_j(r)}.$$

Hence, in view of (4.1),

$$d_{\tilde{\Omega}\setminus\sigma}(\sigma, E) \geq \frac{L(\Gamma,\rho)^2}{A(\tilde{\Omega} \setminus \sigma, \rho)} \geq \int_{r_0}^{R_0} \frac{dr}{r\Theta_j(r)},$$

which proves (4.3) and thus completes the proof of the lemma. ∎

By applying Lemma 4.2, we can prove the next lemma which provides upper bounds on the harmonic measure $\omega(z, \partial U_j \cap B_r(0), \Omega)$ for any $0 < r < \min(\rho_0, |z|)$.

Lemma 4.3. *Suppose Ω is an open subset of $\mathbb{C}^*$ fulfilling* (i) *and* (ii) *of Theorem* 2.5 *with $z_0 = 0$ and $m \geq 1$. Fix $j \in \{1, \ldots, m\}$ and let α_j, C_j, and γ_j be as in* (3.7). *If $r > 0$ and $z \in \Omega$ are such that $0 < r < |z| \leq \rho_0$, then*

$$\omega(z, \partial U_j \cap B_r(0), \Omega) \leq \frac{8}{\pi} \left(\frac{r}{|z|} \right)^{1/\alpha_j} (1 + C_j |z|^{\gamma_j})^{1/(\alpha_j \gamma_j)}. \tag{4.4}$$

Moreover, if $r > 0$ and $z \in \Omega$ are such that $0 < r < \rho_0 \leq |z|$, then

$$\omega(z, \partial U_j \cap B_r(0), \Omega) \leq \frac{8}{\pi} \left(\frac{r}{\rho_0} \right)^{1/\alpha_j} (1 + C_j \rho_0^{\gamma_j})^{1/(\alpha_j \gamma_j)}. \tag{4.5}$$

Proof. Applying Lemma 4.2 with $R_0 = \min(\rho_0, |z|)$, we find that if $r_0 \in [0, \rho_0)$, then (see Figure 5)

$$\omega(z, \partial U_j \cap B_{r_0}(0), \Omega) \leq \frac{8}{\pi} e^{-\pi \int_{r_0}^{\min(\rho_0,|z|)} \frac{dr}{r\Theta_j(r)}} \quad \text{for all } z \in \Omega \text{ with } |z| > r_0. \tag{4.6}$$

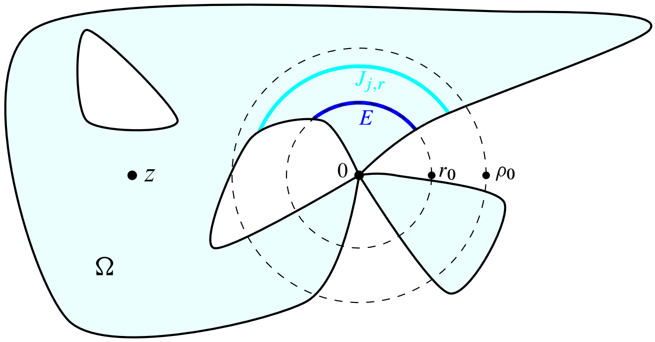

Figure 5. Illustration of the argument leading to (4.6) in the case of $|z| \geq \rho_0$.

Using (3.7), we see that, whenever $z \in \Omega$ and $0 < r_0 < |z| \leq \rho_0$,

$$\int_{r_0}^{\min(\rho_0,|z|)} \frac{dr}{r\Theta_j(r)} \geq \int_{r_0}^{|z|} \frac{dr}{r\pi\alpha_j(1+C_j r^{\gamma_j})} = \frac{\log(|z|/r_0)}{\pi\alpha_j} - \frac{\log\left(\frac{1+C_j|z|^{\gamma_j}}{1+C_j r_0^{\gamma_j}}\right)}{\pi\alpha_j\gamma_j}$$
$$\geq \frac{\log(|z|/r_0)}{\pi\alpha_j} - \frac{\log(1+C_j|z|^{\gamma_j})}{\pi\alpha_j\gamma_j}.$$

Employing this inequality in (4.6) and replacing r_0 by r, we arrive at (4.4). The proof of (4.5) follows similarly using that

$$\int_{r_0}^{\min(\rho_0,|z|)} \frac{dr}{r\Theta_j(r)} \geq \frac{\log(\rho_0/r_0)}{\pi\alpha_j} - \frac{\log(1+C_j\rho_0^{\gamma_j})}{\pi\alpha_j\gamma_j}$$

whenever $z \in \Omega$ and $0 < r_0 < \rho_0 \leq |z|$. ∎

5. Proof of Theorem 2.1

Let Ω be a finitely connected Jordan domain in $\mathbb{C}^*$ such that Ω has a Hölder-C^1 corner of opening $\pi\alpha$ at 0. Fix $b > 0$ and let μ be a non-negative measure on Ω of finite total mass such that $d\mu(z) = (1+o(1))|z|^{2b-2}d^2z$ as $z \to 0$. Let $\nu = \mathrm{Bal}(\mu, \partial\Omega)$. Shrinking ρ_0 if necessary, we may assume that (2.4) holds for all measurable subsets A of $\Omega \cap \overline{B_{\rho_0}(0)}$.

Applying Lemma 4.3 with $m = 1$, we find the following estimates: if $r > 0$ and $z \in \Omega$ are such that $0 < r < |z| \leq \rho_0$, then

$$\omega(z, \partial\Omega \cap B_r(0), \Omega) \leq \frac{8}{\pi}\left(\frac{r}{|z|}\right)^{1/\alpha}(1+C_1|z|^{\gamma})^{1/(\alpha\gamma)}, \tag{5.1}$$

while if $r > 0$ and $z \in \Omega$ are such that $0 < r < \rho_0 \leq |z|$, then

$$\omega(z, \partial\Omega \cap B_r(0), \Omega) \leq \frac{8}{\pi}\left(\frac{r}{\rho_0}\right)^{1/\alpha}(1+C_1\rho_0^{\gamma})^{1/(\alpha\gamma)}, \tag{5.2}$$

where $C_1 > 0$ and $0 < \gamma \leq 1$.

The proof of the following lemma is based on integration of the inequalities (5.1) and (5.2).

Lemma 5.1. *For every $\varepsilon > 0$, we have*

$$\nu(\partial\Omega \cap B_r(0)) \le \begin{cases} (1+\varepsilon)\,\frac{\pi\alpha}{2b}\,(1+\frac{16b}{\pi(1/\alpha-2b)})\,r^{2b} & \text{if } 2b < 1/\alpha, \\ (1+\varepsilon)\,8\alpha\,r^{2b}\log(\frac{1}{r}) & \text{if } 2b = 1/\alpha, \\ C\,r^{1/\alpha} & \text{if } 2b > 1/\alpha, \end{cases}$$

for all sufficiently small $r > 0$.

Proof. Recall that, by definition, $\nu(\partial\Omega \cap B_r(0)) = \int_\Omega \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z)$. For $r \in (0, \rho_0)$, we write $\nu(\partial\Omega \cap B_r(0))$ as the sum of three integrals:

$$\nu(\partial\Omega \cap B_r(0)) = I_{1,r} + I_{2,r} + I_{3,r},$$

where

$$I_{1,r} := \int_{\Omega\cap\overline{B_r(0)}} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z),$$
$$I_{2,r} := \int_{(\Omega\cap B_{\rho_0}(0))\setminus\overline{B_r(0)}} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z),$$
$$I_{3,r} := \int_{\Omega\setminus B_{\rho_0}(0)} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z).$$

Let $\varepsilon > 0$. Shrinking ρ_0 if necessary, we may assume that

$$(1 + C_1\rho_0^\gamma)^{1/(\alpha\gamma)+1} \le 1 + \varepsilon. \tag{5.3}$$

Using the fact that the harmonic measure of any set is ≤ 1, (2.4), and (3.6), we find

$$\begin{aligned} I_{1,r} &\le \int_{\Omega\cap\overline{B_r(0)}} d\mu(z) \le (1+\varepsilon)\int_{\Omega\cap\overline{B_r(0)}} |z|^{2b-2}\, d^2z \\ &\le (1+\varepsilon)\,\pi\alpha(1 + C_1 r^\gamma)\int_0^r \rho^{2b-1}\, d\rho \le (1+\varepsilon)^2\,\frac{\pi\alpha}{2b}\,r^{2b} \end{aligned} \tag{5.4}$$

for all sufficiently small $r > 0$. To estimate $I_{2,r}$, we use (2.4), (5.1), and (3.6) to write, for all sufficiently small $r > 0$,

$$\begin{aligned} I_{2,r} &\le \int_{(\Omega\cap B_{\rho_0}(0))\setminus\overline{B_r(0)}} \frac{8}{\pi}\left(\frac{r}{|z|}\right)^{1/\alpha}(1 + C_1|z|^\gamma)^{1/(\alpha\gamma)}(1+\varepsilon)\,|z|^{2b-2}\, d^2z \\ &\le \int_r^{\rho_0} \frac{8}{\pi}\left(\frac{r}{\rho}\right)^{1/\alpha}(1 + C_1\rho^\gamma)^{1/(\alpha\gamma)}(1+\varepsilon)\,\pi\alpha(1 + C_1\rho^\gamma)\,\rho^{2b-1}\, d\rho \\ &\le 8(1+\varepsilon)\,\alpha\,(1 + C_1\rho_0^\gamma)^{1/(\alpha\gamma)+1}\, r^{1/\alpha}\int_r^{\rho_0} \rho^{2b-1-1/\alpha}\, d\rho. \end{aligned}$$

In light of (5.3), this gives

(5.5)
$$I_{2,r} \leq \begin{cases} (1+\varepsilon)^2\, 8\alpha\, r^{1/\alpha}\, \frac{\rho_0^{2b-1/\alpha}-r^{2b-1/\alpha}}{2b-1/\alpha} \leq C\, r^{1/\alpha} + (1+\varepsilon)^2 \frac{8\alpha}{|2b-1/\alpha|}\, r^{2b} & \text{if } 2b \neq 1/\alpha, \\ (1+\varepsilon)^2\, 8\alpha\, r^{2b} \log(\frac{\rho_0}{r}) & \text{if } 2b = 1/\alpha, \end{cases}$$

for all sufficiently small $r > 0$. Finally, using (5.2) and the fact that μ has finite total mass, we obtain

$$I_{3,r} \leq \frac{8}{\pi}\left(\frac{r}{\rho_0}\right)^{1/\alpha}(1 + C_1\rho_0^{\gamma})^{1/(\alpha\gamma)} \int_{\Omega\setminus B_{\rho_0}(0)} d\mu(z) \leq C\, r^{1/\alpha} \tag{5.6}$$

for all sufficiently small $r > 0$. Since $\nu(\partial\Omega \cap B_r(0)) = I_{1,r} + I_{2,r} + I_{3,r}$ and $\varepsilon > 0$ was arbitrary, the desired conclusion follows from (5.4), (5.5), and (5.6). ∎

Lemma 5.1 establishes the upper bounds on $\nu(\partial\Omega \cap B_r(0))$ stated in (2.6)–(2.8). In what follows, we establish the lower bounds on $\nu(\partial\Omega \cap B_r(0))$ stated in (2.6)–(2.8).

Let Γ_1 be the component of $\partial\Omega$ containing 0. Let $\Omega_1 \supset \Omega$ be the component of $\mathbb{C}^* \setminus \Gamma_1$ that contains Ω. Then $\partial\Omega_1 = \Gamma_1$ is a Jordan curve and Ω_1 has a Hölder-C^1 corner of opening $\pi\alpha$ at 0. By the Jordan curve theorem and the Riemann mapping theorem, there is a conformal map f of the open upper half-plane $\mathbb{H}$ onto Ω_1 such that $f(0) = 0$, $C_+ \subset f(\mathbb{R}_+)$, and $C_- \subset f(\mathbb{R}_-)$. Decreasing γ if necessary, in what follows we assume that $\gamma < 1/2$. By Lemma A.1, since 0 is a Hölder-C^1 corner, $f(z) = cz^{\alpha}(1 + O(z^{\alpha\gamma}))$ as $z \to 0$, $z \in \bar{\mathbb{H}}$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant and $z^{\alpha} := |z|^{\alpha} e^{i\alpha \arg z}$, $\arg z \in (-\pi/2, 3\pi/2)$. From (3.1), we infer that $\arg c = 0$, so that replacing $f(z)$ with $f(c^{-1/\alpha} z)$, we have

$$f(z) = z^{\alpha}(1 + O(z^{\alpha\gamma})) \quad \text{as } z \to 0,\ z \in \bar{\mathbb{H}}. \tag{5.7}$$

Let a be so small that the image under f of the half-disk $B_a(0) \cap \mathbb{H}$ of radius a is contained in $\Omega \cap B_{\rho_0}(0)$, where ρ_0 is as in Lemma 3.1. Let $S_{a^{\alpha}}$ be the circular sector of angle $\pi\alpha$ and radius a^{α} defined by (3.2). Let h be the conformal map of the half-disk $B_a(0) \cap \mathbb{H}$ onto $S_{a^{\alpha}}$ given by $h(z) = z^{\alpha}$. Then $f \circ h^{-1}$ is a conformal map of $S_{a^{\alpha}}$ onto $f(B_a(0) \cap \mathbb{H})$.

Lemma 5.2. *For every $\varepsilon > 0$, there exists $a > 0$ such that, for all sufficiently small $r > 0$,*

$$\begin{aligned} &\nu(\partial\Omega \cap B_r(0)) \\ &\quad \geq (1-\varepsilon)\int_{S_{a^{\alpha}}} \omega((f\circ h^{-1})(w), \partial\Omega \cap B_r(0), f(B_a(0)\cap\mathbb{H}))\, |w|^{2b-2}\, d^2w. \end{aligned} \tag{5.8}$$

Proof. Since $f(B_a(0) \cap \mathbb{H}) \subset \Omega$, we have

$$\nu(\partial\Omega \cap B_r(0)) = \int_{\Omega} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) \geq \int_{f(B_a(0)\cap\mathbb{H})} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z).$$

Utilizing that $f(B_a(0) \cap \mathbb{H})$ is contained in $\Omega \cap B_{\rho_0}(0)$ and (2.4), we obtain

$$\nu(\partial\Omega \cap B_r(0)) \geq (1-\varepsilon)\int_{f(B_a(0)\cap\mathbb{H})} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, |z|^{2b-2}\, d^2z \tag{5.9}$$

for all sufficiently small $r > 0$. Whenever r is so small that $\partial\Omega \cap B_r(0)$ is a subset of $\partial f(B_a(0) \cap \mathbb{H})$, the maximum principle yields

$$\omega(z, \partial\Omega \cap B_r(0), \Omega) \geq \omega(z, \partial\Omega \cap B_r(0), f(B_a(0) \cap \mathbb{H}))$$

and thus (5.9) implies

$$\nu(\partial\Omega \cap B_r(0)) \geq (1-\varepsilon) \int_{f(B_a(0)\cap\mathbb{H})} \omega(z, \partial\Omega \cap B_r(0), f(B_a(0) \cap \mathbb{H}))\, |z|^{2b-2}\, d^2 z. \tag{5.10}$$

The final step is to perform the change of variables $z = (f \circ h^{-1})(w) = f(w^{1/\alpha})$ in the integral in (5.10). It follows from (5.7) that

$$z = w(1 + O(w^{\gamma})) \quad \text{as } w \to 0,\ w \in \bar{S}_{a^\alpha}. \tag{5.11}$$

Moreover, by Theorem 3.9 in [22],

$$f'(w^{1/\alpha}) = \alpha w^{1-1/\alpha}(1 + o(1)) \quad \text{as } w \to 0,\ w \in \bar{S}_{a^\alpha}.$$

Therefore, shrinking a if necessary, we have

$$|z|^{2b-2}\, d^2 z = |z|^{2b-2}\, |f'(w^{1/\alpha})|^2 \left|\frac{w^{1/\alpha-1}}{\alpha}\right|^2 d^2 w \geq (1-\varepsilon)\, |w|^{2b-2}\, d^2 w$$

for all $z \in f(B_a(0) \cap \mathbb{H})$. Hence, changing variables from z to w in (5.10) and recalling that $\varepsilon > 0$ was arbitrary, we conclude that (5.8) holds. ■

Let μ_b be the restriction of the measure $|w|^{2b-2} d^2 w$ to S_{a^α}. Let $\nu_b := \mathrm{Bal}(\mu_b, \partial S_{a^\alpha})$ be the balayage of μ_b onto ∂S_{a^α} so that

$$\nu_b(E) = \int_{S_{a^\alpha}} \omega(w, E, S_{a^\alpha})\, |w|^{2b-2}\, d^2 w \quad \text{for Borel subsets } E \text{ of } \partial S_{a^\alpha}.$$

By the conformal invariance of harmonic measure, we have

$$\omega((f \circ h^{-1})(w), \partial\Omega \cap B_r(0), f(B_a(0) \cap \mathbb{H})) = \omega(w, (h \circ f^{-1})(\partial\Omega \cap B_r(0)), S_{a^\alpha}).$$

Moreover, by (5.11), the set $(h \circ f^{-1})(\partial\Omega \cap B_r(0))$ contains $\partial S_{a^\alpha} \cap B_{(1-\varepsilon)r}(0)$ for all small enough $r > 0$. Consequently, we deduce from (5.8) that, for all sufficiently small $r > 0$,

$$\begin{aligned} \nu(\partial\Omega \cap B_r(0)) &\geq (1-\varepsilon)\, \nu_b\big((h \circ f^{-1})(\partial\Omega \cap B_r(0))\big) \\ &\geq (1-\varepsilon)\, \nu_b(\partial S_{a^\alpha} \cap B_{(1-\varepsilon)r}(0)). \end{aligned} \tag{5.12}$$

On the other hand, by Remark 2.18 in [9], we have, for $R \in (0, a^\alpha)$, the following.

- If $2b \notin \frac{1}{\alpha} + \frac{2}{\alpha}\mathbb{N}_{\geq 0}$,

$$\nu_b(\partial S_{a^\alpha} \cap B_R(0)) = 2\int_0^R \frac{4a^{\alpha 2b}}{\alpha\pi r} \sum_{j=0}^{\infty} \frac{(\frac{r}{a^\alpha})^{2b} - (\frac{r}{a^\alpha})^{\frac{2j}{\alpha}+\frac{1}{\alpha}}}{(\frac{2j}{\alpha}+\frac{1}{\alpha})^2 - (2b)^2}\, dr.$$

- If $2b = 1/\alpha$,

$$\nu_b(\partial S_{a^\alpha} \cap B_R(0)) = 2\int_0^R \frac{1}{\pi}\Big(2r^{2b-1}\log(\frac{a^\alpha}{r}) + \sum_{j=1}^{\infty} \alpha r^{2b-1}\frac{1-(\frac{r}{a^\alpha})^{\frac{2j}{\alpha}}}{j(j+1)}\Big)dr.$$

- If $2b = \frac{1+2k}{\alpha}$, $k \in \mathbb{N}_{\geq 1}$,

$$\nu_b(\partial S_{a^\alpha} \cap B_R(0)) = 2\int_0^R \frac{1}{\pi}\Big(\frac{2r^{2b-1}}{1+2k}\log(\frac{a^\alpha}{r}) + \frac{4a^{\alpha 2b}}{\alpha r}\sum_{\substack{j=0\\ j\neq k}}^{\infty} \frac{(\frac{r}{a^\alpha})^{2b} - (\frac{r}{a^\alpha})^{\frac{2j}{\alpha}+\frac{1}{\alpha}}}{(\frac{2j}{\alpha}+\frac{1}{\alpha})^2 - (2b)^2}\Big)dr.$$

If $2b < 1/\alpha$, then, as $R \to 0$,

$$\begin{aligned}
&2\int_0^R \frac{4a^{\alpha 2b}}{\alpha\pi r}\sum_{j=0}^{\infty}\frac{(r/a^\alpha)^{2b} - (r/a^\alpha)^{2j/\alpha+1/\alpha}}{(2j/\alpha+1/\alpha)^2-(2b)^2}dr \\
&= 2\int_0^R \frac{4a^{\alpha 2b}}{\alpha\pi r}\sum_{j=0}^{\infty}\frac{(r/a^\alpha)^{2b}}{((2j+1)/\alpha)^2-(2b)^2}dr + O(R^{1/\alpha}) \\
&= \frac{8}{\alpha\pi}\sum_{j=0}^{\infty}\frac{1}{((2j+1)/\alpha)^2-(2b)^2}\int_0^R r^{2b-1}dr + O(R^{1/\alpha}) = \frac{\tan(\pi\alpha b)}{2b^2}R^{2b} + O(R^{1/\alpha}),
\end{aligned}$$

where for the last step we have used equation 1.421.1 in [13]. Hence, for any $\varepsilon > 0$,

$$\nu_b\big(\partial S_{a^\alpha} \cap B_R(0)\big) \geq \begin{cases} (1-\varepsilon)\frac{\tan(\pi\alpha b)}{2b^2}R^{2b} & \text{if } 2b < 1/\alpha, \\ (1-\varepsilon)\frac{2}{\pi b}R^{2b}\log(\frac{1}{R}), & \text{if } 2b = 1/\alpha, \\ cR^{1/\alpha} & \text{if } 2b > 1/\alpha, \end{cases}$$

for all small enough $R > 0$. Employing this estimate in (5.12), we conclude that, for each $\varepsilon > 0$,

$$\nu(\partial\Omega \cap B_r(0)) \geq \begin{cases} (1-\varepsilon)\frac{\tan(\pi\alpha b)}{2b^2}r^{2b} & \text{if } 2b < 1/\alpha, \\ (1-\varepsilon)\frac{2}{\pi b}r^{2b}\log(\frac{1}{r}), & \text{if } 2b = 1/\alpha, \\ cr^{1/\alpha} & \text{if } 2b > 1/\alpha, \end{cases}$$

for all small enough $r > 0$. This establishes also the desired lower bounds in (2.6)–(2.8).

Finally, assume that $d\mu(z) \asymp |z-z_0|^{2b-2}d^2z$ as $z \to z_0$ so that (2.5) holds for some c_1, c_2, ρ_0. The estimate (2.9) follows by applying (2.6)–(2.8) to $\nu_1 := \mathrm{Bal}(\mu_1, \partial\Omega)$ and $\nu_2 := \mathrm{Bal}(\mu_2, \partial\Omega)$, where the measures μ_1 and μ_2 are defined by

$$d\mu_j(z) = \begin{cases} |z|^{2b-2}d^2z & \text{if } |z| < \rho_0, \\ \frac{1}{c_j}d\mu(z) & \text{if } |z| \geq \rho_0, \end{cases} \qquad j = 1, 2,$$

and noting that

$$c_1\nu_1(\partial\Omega \cap B_r(z_0)) \leq \nu(\partial\Omega \cap B_r(z_0)) \leq c_2\nu_2(\partial\Omega \cap B_r(z_0)).$$

The proof of Theorem 2.1 is complete.

6. Proof of Theorem 2.5

Suppose Ω is an open subset of $\mathbb{C}^*$ satisfying (i) and (ii) of Theorem 2.5 with $z_0 = 0$ and some $m \geq 1$. Let μ be a non-negative measure of finite total mass on Ω such that $d\mu(z) = (1 + o(1))\,|z - z_0|^{2b-2}\, d^2 z$ as $z \to z_0$. Let $\alpha := \max_{1 \leq j \leq m} \alpha_j$ and let $\nu := \mathrm{Bal}(\mu, \partial\Omega)$. Let $\nu_j := \mathrm{Bal}(\mu|_{U_j}, \partial U_j)$ be the balayage of the restriction of μ to the component U_j of $\Omega \cap B_{\rho_0}(z_0)$.

The next lemma shows that up to terms of order $O(r^{1/\alpha})$, $\nu(\partial\Omega \cap B_r(0))$ is given by the sum of the contributions $\nu_j(\partial U_j \cap B_r(0))$ from the m corners. In other words, the contributions from the m corners decouple and can be computed locally up to terms of order $O(r^{1/\alpha})$.

Lemma 6.1 (Decoupling and localization). *There is a constant $C > 0$ such that*

$$\nu(\partial\Omega \cap B_r(0)) - C r^{1/\alpha} \leq \sum_{j=1}^{m} \nu_j(\partial U_j \cap B_r(0)) \leq \nu(\partial\Omega \cap B_r(0)) \tag{6.1}$$

for all sufficiently small $r > 0$.

Proof. By definition,

$$\nu(\partial\Omega \cap B_r(0)) = \int_\Omega \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z)$$

and, for $j = 1, \dots, m$,

$$\nu_j(\partial U_j \cap B_r(0)) = \int_{U_j} \omega(z, \partial U_j \cap B_r(0), U_j)\, d\mu(z).$$

Since U_j, $j = 1, \dots, m$, are the connected components of $U := \Omega \cap B_{\rho_0}(0) = \bigcup_{j=1}^m U_j$, we have

$$\begin{aligned}\sum_{j=1}^{m} \nu_j(\partial U_j \cap B_r(0)) &= \sum_{j=1}^{m} \int_{U_j} \omega(z, \partial U_j \cap B_r(0), U_j)\, d\mu(z) \\ &= \sum_{j=1}^{m} \int_{U_j} \omega(z, \partial U \cap B_r(0), U)\, d\mu(z) = \int_U \omega(z, \partial\Omega \cap B_r(0), U)\, d\mu(z).\end{aligned}$$

Hence, using twice that $U \subset \Omega$,

$$\begin{aligned}\sum_{j=1}^{m} \nu_j(\partial U_j \cap B_r(0)) &\leq \int_U \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) \\ &\leq \int_\Omega \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) = \nu(\partial\Omega \cap B_r(0))\end{aligned} \tag{6.2}$$

for all $r \in (0, \rho_0)$, which is the second inequality in (6.1).

Let $r \in (0, \rho_0)$. By Kakutani's theorem (see, e.g., Theorem F.6 and page 477 of [12]),

$$\omega(z, \partial\Omega \cap B_r(0), \Omega) = \mathbb{P}(W_\infty(z) \in \partial\Omega \cap B_r(0))$$

where $W_\infty(z) \in \partial\Omega \cap B_r(0)$ is the event that a Brownian motion starting at z exits Ω at a point in $\partial\Omega \cap B_r(0)$.

Suppose $z \in U_j$ for some $j = 1, \dots, m$, and let E_z be the event that the Brownian motion $W_t(z)$ starting at z hits the set $A := \Omega \cap \partial B_{\rho_0}(0)$. We split the event $W_\infty(z) \in \partial\Omega \cap B_r(0)$ into two depending on whether the set A is hit or not:

$$\begin{aligned}&\mathbb{P}(W_\infty(z) \in \partial\Omega \cap B_r(0)) \\ &\qquad = \mathbb{P}(\{W_\infty(z) \in \partial\Omega \cap B_r(0)\} \setminus E_z) + \mathbb{P}(\{W_\infty(z) \in \partial\Omega \cap B_r(0)\} \cap E_z).\end{aligned}$$

If $W_t(z)$ does not hit the arc A, then the Brownian motion stays in U_j for all times. So, using Kakutani's theorem again,

$$\mathbb{P}(\{W_\infty(z) \in \partial\Omega \cap B_r(0)\} \setminus E_z) = \omega(z, \partial U_j \cap B_r(0), U_j).$$

Thus

$$\omega(z, \partial U_j \cap B_r(0), U_j) \geq \omega(z, \partial\Omega \cap B_r(0), \Omega) - \mathbb{P}(\{W_\infty(z) \in \partial\Omega \cap B_r(0)\} \cap E_z). \tag{6.3}$$

On the other hand, if the Brownian motion starting at $z \in U_j$ hits A, then in order to exit Ω in $\partial\Omega \cap B_r(0)$, it must make it from some point in A to $\partial\Omega \cap B_r(0)$. The probability for a Brownian motion starting at a point $z_1 \in A$ to exit Ω in $\partial\Omega \cap B_r(0)$ is $\omega(z_1, \partial\Omega \cap B_r(0), \Omega)$. Thus,

$$\mathbb{P}(\{W_\infty(z) \in \partial\Omega \cap B_r(0)\} \cap E_z) \leq \mathbb{P}(E_z) \sup_{z_1 \in A} \omega(z_1, \partial\Omega \cap B_r(0), \Omega). \tag{6.4}$$

Since

$$\omega(z_1, \partial\Omega \cap B_r(0), \Omega) = \sum_{k=1}^{m} \omega(z_1, \partial U_k \cap B_r(0), \Omega),$$

the estimate (4.5) yields

$$\begin{aligned}\sup_{z_1 \in A} \omega(z_1, \partial\Omega \cap B_r(0), \Omega) &\leq \sup_{z_1 \in A} \sum_{k=1}^{m} \omega(z_1, \partial U_k \cap B_r(0), \Omega) \\ &\leq \sum_{k=1}^{m} \frac{8}{\pi} \Big(\frac{r}{\rho_0}\Big)^{1/\alpha_k} (1 + C_k\, \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)}.\end{aligned}$$

Using this estimate and the fact that $\mathbb{P}(E_z) \leq 1$ in (6.4), and then substituting the resulting inequality into (6.3), we arrive at

$$\omega(z, \partial U_j \cap B_r(0), U_j) \geq \omega(z, \partial\Omega \cap B_r(0), \Omega) - \sum_{k=1}^{m} \frac{8}{\pi} \Big(\frac{r}{\rho_0}\Big)^{1/\alpha_k} (1 + C_k\, \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)}$$

for each $z \in U_j$. Integrating with respect to $d\mu(z)$ over U_j, we obtain

$$\begin{aligned}
&\nu_j(\partial U_j \cap B_r(0)) \\
&\quad \geq \int_{U_j} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) - \sum_{k=1}^{m} \frac{8}{\pi} \left(\frac{r}{\rho_0}\right)^{1/\alpha_k} (1 + C_k \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)} \mu(U_j).
\end{aligned}$$

Summing over j from 1 to m and then using that

$$\bigcup_{j=1}^{m} U_j = \Omega \setminus (\Omega \setminus B_{\rho_0}(0)) = \Omega \cap B_{\rho_0}(0),$$

we find

$$\begin{aligned}
&\sum_{j=1}^{m} \nu_j(\partial U_j \cap B_r(0)) \\
&\qquad \geq \int_{\bigcup_{j=1}^{m} U_j} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) \\
&\qquad\quad - \sum_{k=1}^{m} \frac{8}{\pi} \left(\frac{r}{\rho_0}\right)^{1/\alpha_k} (1 + C_k \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)} \sum_{j=1}^{m} \mu(U_j) \\
&\qquad = \nu(\partial\Omega \cap B_r(0)) - \int_{\Omega \setminus B_{\rho_0}(0)} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) \\
&\qquad\quad - \sum_{k=1}^{m} \frac{8}{\pi} \left(\frac{r}{\rho_0}\right)^{1/\alpha_k} (1 + C_k \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)} \mu(\Omega \cap B_{\rho_0}(0)).
\end{aligned}$$

The integral on the right-hand side can be estimated using (4.5):

$$\begin{aligned}
\int_{\Omega \setminus B_{\rho_0}(0)} \omega(z, \partial\Omega \cap B_r(0), \Omega)\, d\mu(z) &= \sum_{k=1}^{m} \int_{\Omega \setminus B_{\rho_0}(0)} \omega(z, \partial U_k \cap B_r(0), \Omega)\, d\mu(z) \\
&\leq \sum_{k=1}^{m} \frac{8}{\pi} \left(\frac{r}{\rho_0}\right)^{1/\alpha_k} (1 + C_k \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)} \mu(\Omega \setminus B_{\rho_0}(0))
\end{aligned}$$

for all $r \in (0, \rho_0)$. It transpires that

$$\begin{aligned}
&\sum_{j=1}^{m} \nu_j(\partial U_j \cap B_r(0)) \\
&\qquad \geq \nu(\partial\Omega \cap B_r(0)) - \sum_{k=1}^{m} \frac{8}{\pi} \left(\frac{r}{\rho_0}\right)^{1/\alpha_k} (1 + C_k \rho_0^{\gamma_k})^{1/(\alpha_k \gamma_k)} \mu(\Omega).
\end{aligned} \tag{6.5}$$

Since $\alpha = \max_{1 \leq j \leq m} \alpha_j$, the first inequality in (6.1) follows. ■

For $j = 1, \dots, m$, we can estimate $\nu_j(\partial U_j \cap B_r(0))$ by applying Theorem 2.1 to the domain U_j. Summing over j from 1 to m, this yields, for all sufficiently small $r > 0$,

$$(1-\varepsilon)\sum_{j=1}^{m}\frac{\tan(\pi\alpha_j b)}{2b^2}r^{2b} \le \sum_{j=1}^{m}\nu_j(\partial U_j \cap B_r(0)) \le (1+\varepsilon)\sum_{j=1}^{m}\frac{\pi\alpha_j}{2b}\Big(1+\frac{16b}{\pi(\frac{1}{\alpha_j}-2b)}\Big)r^{2b},$$

$$m_\alpha(1-\varepsilon)\frac{2r^{2b}}{\pi b}\log(\tfrac{1}{r}) - cr^{2b} \le \sum_{j=1}^{m}\nu_j(\partial U_j \cap B_r(0)) \le m_\alpha(1+\varepsilon)\frac{4r^{2b}}{b}\log(\tfrac{1}{r}) + Cr^{2b},$$

$$cr^{1/\alpha} \le \sum_{j=1}^{m}\nu_j(\partial U_j \cap B_r(0)) \le Cr^{1/\alpha},$$

where the first, second and third lines correspond to $2b < 1/\alpha$, $2b = 1/\alpha$ and $2b > 1/\alpha$, respectively, and m_α is the number of α_j such that $\alpha_j = \alpha$. Combining these inequalities with Lemma 6.1, the estimates in (2.10)–(2.12) follow. The fact that (2.9) holds if $d\mu(z) \asymp |z - z_0|^{2b-2}\, d^2z$ as $z \to z_0$ then follows in the same way as in the proof of Theorem 2.1.

7. Application

In this section, we highlight the relevance of Theorems 2.1 and 2.5 in the study of two-dimensional Coulomb gases. The planar Coulomb gas model for n points with external potential $Q\colon \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is the probability measure

$$\frac{1}{Z_n}\prod_{1\le j<k\le n}|\mathrm{z}_j - \mathrm{z}_k|^\beta \prod_{j=1}^{n} e^{-n\frac{\beta}{2}Q(\mathrm{z}_j)}\, d^2\mathrm{z}_j, \qquad \mathrm{z}_1, \dots, \mathrm{z}_n \in \mathbb{C}, \tag{7.1}$$

where Z_n is the normalization constant, and $\beta > 0$ is the inverse temperature.

For $\beta = 2$, (7.1) is also the law of the complex eigenvalues of a class of $n \times n$ random normal matrices (see, e.g., [7]). Standard equilibrium convergence theorems imply, under quite general assumptions on Q, that the points $\mathrm{z}_1, \dots, \mathrm{z}_n$ will condensate (as $n \to \infty$ and with high probability) on the support S of an equilibrium measure μ. The measure μ is defined as the unique measure minimizing

$$\sigma \mapsto \int \log\frac{1}{|z-w|}\, d\sigma(z)\, d\sigma(w) + \int Q(z)\, d\sigma(z)$$

among all Borel probability measures σ on $\mathbb{C}$, see [23]. For example, if $Q(z) = |z|^{2b}$, then

$$d\mu(z) = \frac{b^2}{\pi}|z|^{2b-2}\chi_S(z)\, d^2z, \qquad S = \{z \in \mathbb{C} : |z| \le b^{-1/(2b)}\},$$

where χ_S is the indicator function of S, see also Figure 6 (row 1). More generally, if Q is smooth on S, then by Theorem II.1.3 in [23], μ is absolutely continuous with respect to d^2z and given by $\frac{1}{4\pi}\Delta Q(z)\chi_S(z)\, d^2z$. The situation is more complicated if $Q = +\infty$ on a certain subset $\Omega \subset \mathbb{C}$: this so-called "hard wall constraint" confines the points to lie in $\mathbb{C} \setminus \Omega$. Recent studies on Coulomb gases with hard edges include [1,2,5,8,9]. Suppose

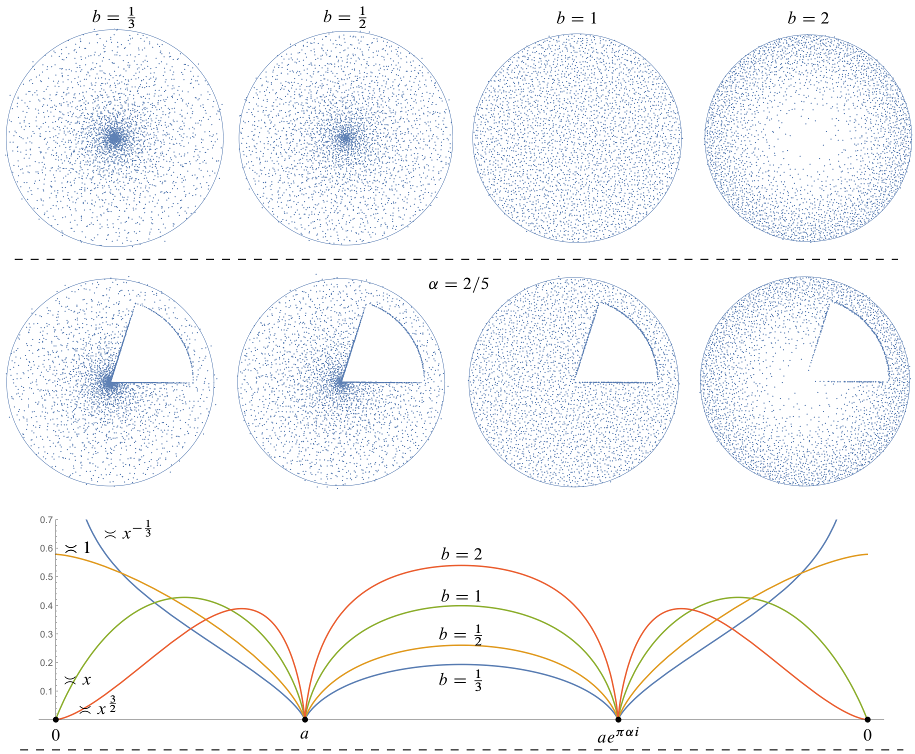

Figure 6. (Taken from [9]) Row 1: the point process (7.1) with $Q(z) = |z|^{2b}$ and the indicated values of b. In each plot, the thin blue circle is $\partial S = \{z : |z| = b^{-1/(2b)}\}$. Row 2: the point process (7.1) with Q as in (7.2), $Q_0(z) = |z|^{2b}$ and $\Omega = \{re^{i\theta} : 0 < r < a,\ 0 < \theta < \pi\alpha\}$, $\alpha = 2/5$, and $a = 0.8b^{-\frac{1}{2b}}$. Row 3: the normalized density $\frac{d\nu(z)/|dz|}{\nu(\partial\Omega)}$ for $\alpha = 2/5$ and $a = 0.8b^{-1/(2b)}$.

for example that Q_0 is smooth on $\mathbb{C}$, let S_0 be the support of the associated equilibrium measure μ_0, let $\Omega \subset \mathbb{C}^*$ be a finitely connected Jordan domain such that $\partial\Omega \subset S_0$, and define

$$Q(z) := \begin{cases} Q_0(z), & \text{if } z \notin \Omega, \\ +\infty, & \text{if } z \in \Omega. \end{cases} \tag{7.2}$$

It is proved in [1, 2, 9] that the associated equilibrium measure is $\mu := \mu_0|_{S_0 \setminus \Omega} + \nu$, where $\nu := \mathrm{Bal}(\mu_0|_\Omega, \partial\Omega)$. The point process (7.1) with Q as in (7.2), $Q_0(z) := |z|^{2b}$, and $\Omega := \{re^{i\theta} : 0 < r < a,\ 0 < \theta < \pi\alpha\}$, $\alpha = 2/5$, $a = 0.8b^{-1/(2b)}$ is illustrated in Figure 6 (row 2) for several values of b, and the balayage measure ν is illustrated in Figure 6 (row 3).

The universality conjecture asserts that as $n \to \infty$ the limiting local statistical properties of the random points $z_1, \dots, z_n$ around a given $z_0 \in \mathbb{C}$ depend only on β and on the behavior of μ at z_0. Consider the hard wall case (7.2), and suppose that $d\mu_0(z) \asymp$

$|z-z_0|^{2b-2}d^2z$ as $z \to z_0$ for some $b > 0$, and that Ω has a Hölder-C^1 corner of opening $\pi\alpha$ at z_0 for some $\alpha \in (0, 2]$. Theorem 2.1 implies that

$$(7.3)\qquad d\mu(z)/|dz| \asymp \begin{cases} |z-z_0|^{\min\{2b,1/\alpha\}-1}, & \text{if } 2b \neq 1/\alpha, \\ |z-z_0|^{2b-1}\log\frac{1}{|z-z_0|}, & \text{if } 2b = 1/\alpha, \end{cases} \qquad \text{as } z \to z_0,\ z \in \partial\Omega,$$

where $|dz|$ is the arclength measure on ∂U. Note that the rate of convergence (or blow-up) in (7.3) depends on Ω only through α. If Ω has multiple corners at z_0, then similar estimates can be obtained from Theorem 2.5. For $\beta = 2$, universality results on local statistics near hard edges can be found in [4, 24] in the case where $\alpha = 1$ and $b = 1$. In view of the above, new universality classes are expected for other values of α and b.

A. A technical lemma

Lemma A.1 stated below is used in Section 5 to obtain (5.7). The lemma is a special case of Exercise 3.4.1 in [22]. For the reader's convenience, we include a proof.

Let Ω_1 be a connected Jordan domain such that $\partial\Omega_1$ has a Hölder-C^1 corner of opening $\pi\alpha$ for some $0 < \alpha \leq 2$ at $0 \in \partial\Omega_1$, and such that one of the boundary arcs forming the corner is tangent to $(0, +\infty)$ at 0. Hence, there exist $\gamma \in (0, 1]$ and curves $C_\pm \subset \partial\Omega_1$ of class $C^{1,\gamma}$ ending at 0 and lying on different sides of 0 such that (3.1) holds.

Lemma A.1. *Suppose f is a conformal map from the open upper half-plane $\mathbb{H}$ onto Ω_1 such that $f(0) = 0$, $C_+ \subset f(\mathbb{R}_+)$, and $C_- \subset f(\mathbb{R}_-)$. Suppose also that $\gamma < 1/2$. Then*

$$(A.1)\qquad f(z) = cz^\alpha(1 + O(z^{\alpha\gamma})), \quad \text{as } z \to 0,\ z \in \bar{\mathbb{H}},$$

for some $c > 0$.

Proof. Let $\Phi(w) = w^{1/\alpha}$, where the principal branch is used for the root, so that Φ maps the sector $\{z : \arg z \in (0, \pi\alpha)\}$ conformally onto the upper-half plane $\mathbb{H}$. Let $w_\pm$ be the parametrizations of $C_\pm$ given in Lemma 3.1. By (3.4),

$$(\Phi \circ w_+)'(r) = \Phi'(w_+(r))w_+'(r) = \frac{1}{\alpha}r^{1/\alpha-1}(1 + O(r^\gamma)), \qquad \text{as } r \to 0,$$
$$(\Phi \circ w_-)'(r) = \Phi'(w_-(r))w_-'(r) = -\frac{1}{\alpha}r^{1/\alpha-1}(1 + O(r^\gamma)), \qquad \text{as } r \to 0.$$

Integrating the above, and then replacing r by r^α, we find

$$(\Phi \circ w_+)(r^\alpha) = r(1 + O(r^{\alpha\gamma})), \quad (\Phi \circ w_-)(r^\alpha) = -r(1 + O(r^{\alpha\gamma})), \quad \text{as } r \to 0,$$

Using also that

$$\frac{d}{dr}(\Phi \circ w_+)(r^\alpha) = 1 + O(r^{\alpha\gamma}), \quad \frac{d}{dr}(\Phi \circ w_-)(r^\alpha) = -1 + O(r^{\alpha\gamma}), \quad \text{as } r \to 0,$$

it follows that

$$r \mapsto \begin{cases} (\Phi \circ w_+)(r^\alpha), & r \geq 0, \\ (\Phi \circ w_-)((-r)^\alpha), & r < 0, \end{cases}$$

is a Jordan curve of class $C^{1,\gamma^\star}$ parametrizing $\partial\Phi(\Omega_1)$ in a small neighborhood of 0, where $\gamma^\star = \min\{\alpha\gamma, 1\}$. The assumption $\gamma < 1/2$, together with $\alpha \leq 2$, implies $\gamma^\star = \alpha\gamma < 1$.

By the Kellogg–Warschawski theorem (Theorem 3.6 in [22]), the function $g = \Phi \circ f : \mathbb{H} \to \Phi(\Omega_1)$ satisfies

$$g(z) = c^{1/\alpha} z (1 + O(z^{\alpha\gamma})), \quad \text{as } z \to 0,\ z \in \bar{\mathbb{H}}, \tag{A.2}$$

for some constant $c \in \mathbb{C} \setminus \{0\}$. Hence $f(z) = g(z)^\alpha$ satisfies (A.1). Moreover, since $C_+ \subset f(\mathbb{R}_+)$ and $w'_+(0) = 1$, we have $c > 0$, which finishes the proof. ■

Funding. C. Charlier acknowledges support from the Swedish Research Council, Grant no. 2021-04626, and J. Lenells acknowledges support from the Swedish Research Council, Grant no. 2021-03877.

References

[1] Adhikari, K.: Hole probabilities for β-ensembles and determinantal point processes in the complex plane. *Electron. J. Probab.* **23** (2018), article no. 48, 21 pp. Zbl 1410.60048 MR 3814242

[2] Adhikari, K. and Reddy, N. K.: Hole probabilities for finite and infinite Ginibre ensembles. *Int. Math. Res. Not. IMRN* (2017), no. 21, 6694–6730. Zbl 1405.60066 MR 3719476

[3] Aleksanyan, H. and Shahgholian, H.: Discrete Balayage and boundary sandpile. *J. Anal. Math.* **138** (2019), no. 1, 361–403. Zbl 1429.35186 MR 3996043

[4] Ameur, Y., Charlier, C. and Cronvall, J.: Random normal matrices: eigenvalue correlations near a hard wall. *J. Stat. Phys.* **191** (2024), no. 8, article no. 98, 49 pp. Zbl 1553.60006 MR 4783708

[5] Berezin, S.: Functional central limit theorems for constrained Mittag-Leffler ensemble in hard edge scaling. *Electron. J. Probab.* **30** (2025), article no. 138, 27 pp. Zbl 08102216 MR 4955064

[6] Björn, A., Björn, J., Mäkäläinen, T. and Parviainen, M.: Nonlinear balayage on metric spaces. *Nonlinear Anal.* **71** (2009), no. 5-6, 2153–2171. Zbl 1166.31004 MR 2524427

[7] Byun, S.-S. and Forrester, P. J.: *Progress on the study of the Ginibre ensembles*. KIAS Springer Series in Mathematics 3, Springer, Singapore, 2025. Zbl 1561.15001

[8] Byun, S.-S. and Park, S.: Large gap probabilities of complex and symplectic spherical ensembles with point charges. *J. Funct. Anal.* **290** (2026), no. 4, article no. 111260, 63 pp. Zbl 08141081 MR 4983380

[9] Charlier, C.: Hole probabilities and balayage of measures for planar Coulomb gases. Preprint 2023, arXiv:2311.15285v2.

[10] Charlier, C. and Lenells, J.: Balayage of measures: behavior near a cusp. *Potential Anal.* **63** (2025), no. 3, 1407–1439. Zbl 08127320 MR 4984411

[11] Doob, J. L.: *Classical potential theory and its probabilistic counterpart*. Classics Math., Springer, Berlin, 2001. Zbl 0990.31001 MR 1814344

[12] Garnett, J. B. and Marshall, D. E.: *Harmonic measure*. New Math. Monogr. 2, Cambridge University Press, Cambridge, 2005. Zbl 1077.31001 MR 2150803

[13] Gradshteyn, I. S. and Ryzhik, I. M.: *Table of integrals, series, and products*. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007. Zbl 1208.65001 MR 2360010

[14] Groot, A. and Kuijlaars, A. B. J.: Matrix-valued orthogonal polynomials related to hexagon tilings. *J. Approx. Theory* **270** (2021), article no. 105619, 36 pp. Zbl 1518.33006 MR 4293701

[15] Gustafsson, B.: Direct and inverse balayage – some new developments in classical potential theory. *Nonlinear Anal., Theory Methods Appl.* **30** (1997), no. 5, 2557–2565. Zbl 0889.31001 MR 1602951

[16] Gustafsson, B.: Lectures on balayage. In *Clifford algebras and potential theory*, pp. 17–63. Univ. Joensuu Dept. Math. Rep. Ser. 7, University of Joensuu, Joensuu, 2004. Zbl 1088.31001 MR 2103705

[17] Gustafsson, B. and Roos, J.: Partial balayage on Riemannian manifolds. *J. Math. Pures Appl. (9)* **118** (2018), 82–127. Zbl 1404.31019 MR 3852470

[18] Gustafsson, B. and Sakai, M.: Properties of some balayage operators, with applications to quadrature domains and moving boundary problems. *Nonlinear Anal.* **22** (1994), no. 10, 1221–1245. Zbl 0852.35144 MR 1279981

[19] Landkof, N. S.: *Foundations of modern potential theory*. Grundlehren Math. Wiss. 180, Springer, New York-Heidelberg, 1972. Zbl 0253.31001 MR 0350027

[20] Nishry, A. and Wennman, A.: The forbidden region for random zeros: appearance of quadrature domains. *Comm. Pure Appl. Math.* **77** (2024), no. 3, 1766–1849. Zbl 1539.60058 MR 4692882

[21] Poincaré, H.: *Theorie du potentiel Newtonien*. Carré et Naud, Paris, 1899. Zbl 30.0692.01

[22] Pommerenke, C.: *Boundary behaviour of conformal maps*. Grundlehren Math. Wiss. 299, Springer, Berlin, 1992. Zbl 0762.30001 MR 1217706

[23] Saff, E. B. and Totik, V.: *Logarithmic potentials with external fields*. Grundlehren Math. Wiss. 316, Springer, Berlin, 1997. Zbl 0881.31001 MR 1485778

[24] Seo, S.-M.: Edge behavior of two-dimensional Coulomb gases near a hard wall. *Ann. Henri Poincaré* **23** (2022), no. 6, 2247–2275. Zbl 07541500 MR 4420573

[25] Zorii, N.: Balayage of measures on a locally compact space. *Anal. Math.* **48** (2022), no. 1, 249–277. Zbl 1524.31012 MR 4388578

[26] Zorii, N.: On the theory of balayage on locally compact spaces. *Potential Anal.* **59** (2023), no. 4, 1727–1744. Zbl 1531.31016 MR 4684374

[27] Zorii, N.: On the theory of capacities on locally compact spaces and its interaction with the theory of balayage. *Potential Anal.* **59** (2023), no. 3, 1345–1379. Zbl 1527.31015 MR 4647953

Received July 11, 2024; revised December 14, 2025.

Christophe Charlier
Centre for Mathematical Sciences, Lund University
22100 Lund, Sweden;
christophe.charlier@math.lu.se

Jonatan Lenells
Department of Mathematics, KTH Royal Institute of Technology
10044 Stockholm, Sweden;
jlenells@kth.se