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On complete cohomogeneity one hypersurfaces in the hyperbolic space

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Abstract. We study isometric immersions $f\colon M^n \to \mathbb{H}^{n+1}$ into hyperbolic space of dimension $n+1$ of a complete Riemannian manifold of dimension n on which a compact connected group of intrinsic isometries acts with principal orbits of codimension one. We provide a characterization if either $n \geq 3$ and M^n is compact, or $n \geq 5$ and the connected components of the set where the sectional curvature is constant and equal to -1 are bounded.

1. Introduction

The study of homogeneous n-dimensional submanifolds M^n, i.e., where the isometry group $\mathrm{Iso}(M^n)$ acts transitively on M^n, was initiated by Kobayashi in [14], where the author proved that every compact homogeneous Euclidean hypersurface must be a round sphere. The complete classification of homogeneous hypersurfaces in simply connected Riemannian manifolds with constant sectional curvature $c \in \mathbb{R}$, denoted by $\mathbb{Q}_c^{n+1}$, without the compactness hypothesis, was later achieved in the works [13, 18, 22–24].

Subsequent studies relaxed the homogeneity assumption, focusing on isometric immersions $f\colon M^n \to \mathbb{Q}_c^{n+1}$ where a *compact*, *connected* subgroup G of $\mathrm{Iso}(M^n)$ acts with orbits of codimension one. Such f is referred to as a *hypersurface of G-cohomogeneity one*. In this work, we will always assume that G is compact unless stated otherwise. In [20], it was shown that compact Euclidean hypersurfaces of G-cohomogeneity one with umbilical principal orbits must be rotational (cf. Definition 2.3). Subsequent works, such as [3, 15], have introduced other conditions on the orbits, also leading to the hypersurfaces being rotational.

A more complete understanding of Euclidean hypersurfaces of G-cohomogeneity one was obtained in [16], where they dropped the restrictive assumption that principal orbits be umbilical and generalized the previous results on the topic (cf. [25] and references therein for a discussion), and it can be stated as follows.

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Theorem 1.1 (Theorem 1.1 in [16]). *Let $f: M^n \to \mathbb{R}^{n+1}$ be a complete hypersurface of G-cohomogeneity one. Assume either that $n \geq 3$ and M^n is compact or that $n \geq 5$ and the connected components of the flat part of M^n are bounded. Then f is either rigid or a hypersurface of revolution.*

Note that the assumption concerning the boundedness of the connected components of the flat part is necessary. Otherwise, it is possible to construct flat cylinders over planar curves that do not meet the conclusions of the theorem. In their result, they made use of an unpublished version of the main result in [21] due to Ferus (cf. Theorem 13.2 in [5]). To be able to use such a result, they needed to understand the hypersurface of G-cohomogeneity one that has at least one complete $(n-2)$-dimensional leaf of the relative nullity distribution, i.e., the kernel of the second fundamental form. They showed that, as in the case of homogeneous hypersurfaces, they must be isometric to $M^n = \mathbb{S}^2 \times \mathbb{R}^{n-2}$ and the isometric immersion is canonical in $\mathbb{R}^{n+1}$, i.e., as a product of a hypersurface and the Euclidean factor.

Hypersurfaces of G-cohomogeneity one in the hyperbolic space with umbilical principal orbits were studied in [2]. They showed that, when G is compact, the hypersurface must be rotational. In this work, we address complete hypersurfaces of G-cohomogeneity one in hyperbolic space without assuming the umbilicity in the orbits. As these orbits will be homogeneous submanifolds with codimension two in the hyperbolic space, we employ the tools developed in [7] to show that the main result in [16], as stated below, holds in the hyperbolic setting.

Theorem A. *Let $f: M^n \to \mathbb{H}^{n+1}$ be a complete hypersurface of G-cohomogeneity one. Assume either that $n \geq 3$ and M^n is compact or that $n \geq 5$ and the connected components of the set $\{x \in M^n : \nu(x) \geq n-1\}$ are bounded. Then f is either rigid or a rotational submanifold.*

Similar to the work that inspired this one, it is necessary to understand the hypersurfaces that admit a complete leaf of dimension $n-2$ from the distribution of relative nullity. Unlike the homogeneous case, where such hypersurfaces do not exist as shown in [23], examples appear in the scenario of hypersurfaces with cohomogeneity one. These are necessarily rotational submanifolds, not over a profile curve, but over a 3-dimensional Riemannian manifold as the profile.

Theorem B. *Let $f: M^n \to \mathbb{H}^{n+1}$, with $n \geq 5$, be a complete hypersurface of G-cohomogeneity one. If there exists a complete leaf of relative nullity with dimension $n-2$, then M^n is isometric to $\mathbb{R} \times_{\rho_0} \mathbb{S}^2_{c_1} \times_{\rho_1} \mathbb{S}^{n-3}_{c_2}$, for some constants $c_1 > 0$ and $c_2 > 0$ and warping functions ρ_0 and ρ_1 defined on $\mathbb{R}$. Additionally, there exists a warped product representation $\Psi: V^{1+i_0} \times_{\sigma_0} \mathbb{Q}^{2+i_1}_{\tilde{c}_1} \times_{\sigma_1} \mathbb{S}^{n-3}_{c_2} \to \mathbb{H}^{n+1}$, where $i_0 + i_1 = 1$ and V^{1+i_0} is an open subset of a totally geodesic submanifold, with warping functions σ_0 and σ_1 defined on V^{1+i_0}, in which f admits an extrinsic warped product structure. More precisely, there exist isometric immersions $f_0: \mathbb{R} \to V^{1+i_0}$ and $f_1: \mathbb{S}^2_{c_1} \to \mathbb{Q}^{2+i_1}_{\tilde{c}_1}$ such that $\rho_0 = \sigma_0 \circ f_0$, $\rho_1 = \sigma_1 \circ f_1$, and $f = \Psi \circ (f_0 \times f_1 \times \mathrm{Id})$, where* Id *is the identity map defined on $\mathbb{S}^{n-3}_{c_2}$. Moreover, $\nu = n-2$ throughout the submanifold and the leaves of relative nullity are given by $\mathbb{R} \times_{\rho_0} \{p\} \times_{\rho_1} \mathbb{S}^{n-3}_{c_2}$ for $p \in \mathbb{S}^2_{c_1}$. In particular, f is a rotational submanifold.*

The manuscript is organized as follows: in Section 2 we introduce the notation to be used, as well as results from the literature that play a key role in this work. In Section 3, we present the proofs of the main theorems.

2. Preliminaries

Let $f: M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion of a Riemannian manifold. For a point $x \in M^n$ and a normal vector $\xi \in T_x M^\perp$, we denote by $\alpha_f(x)$ the second fundamental form of f, and by $A_\xi^f(x)$ the shape operator of f with respect to ξ, respectively. These are related by the equation

$$\langle A_\xi^f X, Y \rangle = \langle \alpha_f(X, Y), \xi \rangle, \quad \text{for all } X, Y \in T_x M.$$

The *relative nullity subspace* $\Delta(x) \subset T_x M$ of f at $x \in M^n$ is the kernel of its second fundamental form at $x \in M^n$, specifically,

$$\Delta(x) = \{X \in T_x M : \alpha_f(X, Y) = 0, \text{ for all } Y \in T_x M\}.$$

The dimension $\nu(x)$ of the subspace $\Delta(x)$ is called the *index of relative nullity* at $x \in M^n$.

We say that $x \in M^n$ is a *totally geodesic point* of M^n if $\nu(x) = n$. If f has constant index of relative nullity on an open subset $U \subset M^n$, then Δ is an autoparallel distribution, i.e.,

$$\nabla_T S \in \Gamma(\Delta), \quad \text{for all } T, S \in \Gamma(\Delta).$$

Moreover, its leaves are totally geodesic submanifolds of M^n and their images under f are totally geodesic submanifolds of $\mathbb{Q}_c^{n+1}$. The index of relative nullity ν is upper semi-continuous and, in particular, the subset $M_0 = \{x \in M^n : \nu(x) = \nu_0\}$ where ν attains its minimum value ν_0 is open. Moreover, if M^n is complete, then the leaves of relative nullity are also complete (cf. [9]).

We are now in a position to state the version of *Sacksteder's theorem* that will be used in this work. Recall that f is said to be *rigid* if any other isometric immersion $g: M^n \to \mathbb{Q}_c^{n+p}$ is congruent to f by an isometry of the ambient space $\mathbb{Q}_c^{n+p}$. That is, there exists an isometry $\mathcal{J} \in \mathrm{Iso}(\mathbb{Q}_c^{n+p})$ such that $g = \mathcal{J} \circ f$.

Theorem 2.1 (Theorem 13.2 in [5]). *Let $f: M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion of a complete Riemannian manifold with $n \geq 3$ and $c \leq 0$. If there exists no complete leaf of relative nullity of dimension $n-1$ or $n-2$, then for any other isometric immersion $\tilde{f}: M^n \to \mathbb{Q}_c^{n+1}$, the following holds*:

(i) *The set B^f of totally geodesic points of f coincides with that of $\tilde{f}$.*

(ii) *On each connected component of $M \setminus B^f$, the shape operators of f and $\tilde{f}$ satisfy $A^f = \pm A^{\tilde{f}}$.*

In particular, if B^f does not disconnect M^n, then f is rigid.

Remark 2.2. Theorem 2.1 also holds when the ambient space is the sphere $\mathbb{S}^{n+1}$ without any restriction on the leaves of relative nullity, but it requires $n \geq 4$ (cf. Lemma 32 in [10]).

We briefly recall the notion of a warped product of isometric immersions into a space form $\mathbb{Q}_c^l$. This relies on the warped product representations of $\mathbb{Q}_c^l$, that is, isometries of warped products onto open subsets of $\mathbb{Q}_c^l$. All such isometries were described by Nölker in [19] for warped products with arbitrarily many factors. In particular, any isometry of a warped product with two factors onto an open subset of $\mathbb{Q}_c^l$ arises as a restriction of an explicitly constructible isometry

$$\Psi : V^{l-m} \times_\sigma \mathbb{Q}_{\tilde{c}}^m \to \mathbb{Q}_c^l$$

onto an open dense subset of $\mathbb{Q}_c^l$, where $\mathbb{Q}_{\tilde{c}}^m$ is a complete spherical submanifold of $\mathbb{Q}_c^l$ and V^{l-m} is an open subset of the unique totally geodesic submanifold $\mathbb{Q}_c^{l-m}$ of $\mathbb{Q}_c^l$, whose tangent space at some point $\bar{z} \in \mathbb{Q}_{\tilde{c}}^m$ is the orthogonal complement of the tangent space of $\mathbb{Q}_{\tilde{c}}^m$ at $\bar{z}$. The isometry Ψ is completely determined by the choice of $\mathbb{Q}_{\tilde{c}}^m$ and of a point $\bar{z} \in \mathbb{Q}_{\tilde{c}}^m$, and it is called the *warped product representation* of $\mathbb{Q}_c^l$ determined by these data.

Definition 2.3. An isometric immersion $f\colon N_0^{n_0} \times_\rho N_1^{n_1} \to \mathbb{Q}_c^{n+p}$ is said to be an *extrinsic warped product* if there exist isometric immersions $f_0\colon N_0^{n_0} \to V^{n_0+l_0}$ and $f_1\colon N_1^{n_1} \to \mathbb{Q}_{\tilde{c}}^{n_1+l_1}$, with $n_0 + n_1 = n$, and a warped product representation $\Psi\colon V^{n_0+l_0} \times_\sigma \mathbb{Q}_{\tilde{c}}^{n_1+l_1} \to \mathbb{Q}_c^{n+p}$, with $l_0 + l_1 = p$ and $\rho = \sigma \circ f_0$, and such that $f = \Psi \circ (f_0 \times f_1)$. Equivalently, if the following diagram commutes:

$$\begin{array}{ccc}
V^{n_0+l_0} \times_\sigma \mathbb{Q}_{\tilde{c}}^{n_1+l_1} & \xrightarrow{\quad \Psi \quad} & \\
\uparrow f_0 \qquad \uparrow f_1 & \circlearrowright & \\
N_0{}^{n_0} \times_\rho N_1{}^{n_1} & \xrightarrow[f = \Psi \circ (f_0 \times f_1)]{} & \mathbb{Q}_c^{n+p}.
\end{array}$$

If $l_1 = 0$, then f is called a *rotational submanifold* with profile f_0. We point out that this definition differs subtly from the definition of a *hypersurface of revolution*, which requires the profile submanifold f_0 to be a curve (cf. Definition 2.2 in [8]).

Remark 2.4. A method for constructing examples of rotational submanifolds involves polar actions (cf. [11,17]) on the ambient space. If the principal orbits of such an action are spheres or contain a spherical factor (cf. Remark 3.4), then any hypersurface contained in a totally geodesic section, when acted upon by the group, yields a rotational submanifold whose profile is precisely that hypersurface.

The following lemma was established in Lemma 3.8 of [20] for the case in which the ambient space is the Euclidean space, and the same ideas can be applied to other space forms. Since the result will be used in the proof of Theorem B, we present a proof for the general case here.

Lemma 2.5. *Let $I \subset \mathbb{R}$ be an open interval, let N^{n-1} be a Riemannian manifold with constant sectional curvature $\tilde{c}$, and consider the warped product $M^n = I \times_\rho N^{n-1}$ with warping function ρ. Given a hypersurface $f\colon M^n \to \mathbb{Q}_c^{n+1}$, where $c \in \mathbb{R}$ and $n \geq 3$, its shape operator A satisfies one of the following conditions*:

(i) *rank* $A(p) \geq n-1$,

(ii) *rank* $A(p) \leq 1$,

at each point $p \in M^n$.

Proof. It is known that the curvature tensor of a warped product Riemannian manifold satisfies

$$\begin{aligned} R^M(X,\xi)\xi &= -\frac{\mathrm{Hess}_\rho(\xi,\xi)}{\rho}X, \\ R^M(X,Y)\xi &= 0, \\ R^M(X,Y)Z &= R^N(X,Y)Z - \frac{\|\nabla\rho\|^2}{\rho^2}(\langle Y,Z\rangle X - \langle X,Z\rangle Y), \end{aligned} \tag{2.1}$$

for all $X,Y,Z \in \Gamma(TN)$, where R^M and R^N denote the curvature tensor of M^n and N^{n-1}, respectively, and ξ is a vector field tangent to the factor I. We will divide the proof into two cases.

First, suppose $A\xi = 0$ at a point p. This implies that $AX \in T_pN$ and that $\operatorname{rank} A = \operatorname{rank} A|_{T_pN}$, where $A|_{T_pN}$ is the restriction of A to T_pN. From the Gauss equation, we have

$$R^M(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY.$$

Using (2.1) and the fact that N has constant sectional curvature $\tilde{c}$, it follows that

$$\langle AY,Z\rangle AX - \langle AX,Z\rangle AY = \Big(\tilde{c} - c - \frac{\|\nabla\rho\|^2}{\rho^2}\Big)(\langle Y,Z\rangle X - \langle X,Z\rangle Y).$$

Thus, at the point p, we have $\operatorname{rank} A = n-1$ or $\operatorname{rank} A \leq 1$ depending on whether the expression $(\tilde{c} - c - \|\nabla\rho\|^2/\rho^2)$ is different from zero or equal to zero, respectively.

Now, suppose that $A\xi \neq 0$. If $A\xi = \mu\xi$ at a point p, then by (2.1) and the Gauss equation, we have

$$\begin{aligned} -\frac{\mathrm{Hess}_\rho(\xi,\xi)}{\rho}X &= R^M(X,\xi)\xi = c(\langle\xi,\xi\rangle X - \langle X,\xi\rangle\xi) + \langle A\xi,\xi\rangle AX - \langle AX,\xi\rangle A\xi \\ &= c\|\xi\|^2 X + \mu\|\xi\|^2 AX, \end{aligned}$$

for all $X \in T_pN$. This is equivalent to

$$\mu AX = -\Big(c\|\xi\|^2 + \frac{\mathrm{Hess}_\rho(\xi,\xi)}{\rho}\Big)X.$$

Thus, $\operatorname{rank} A|_{T_pN} = n-1$ or $\operatorname{rank} A|_{T_pN} = 0$, i.e., $\operatorname{rank} A = n$ or $\operatorname{rank} A = 1$. Consider now the case

$$A\xi = \mu\xi + V_0,$$

with $0 \neq V_0 \in T_pN$ and $\mu \in \mathbb{R}$. Given $W \in \operatorname{span}\{\xi, V_0\}^\perp$, using (2.1) and the Gauss equation, we get

$$R^M(W,V_0)\xi = \langle AV_0,\xi\rangle AW - \langle AW,\xi\rangle AV_0 = 0,$$

which implies

$$\langle V_0, V_0\rangle AW = \langle A\xi, W\rangle AV_0 = 0.$$

Hence, $\operatorname{span}\{V_0, \xi\}^\perp \subset \ker A$. Therefore, $\operatorname{rank} A \le 2$. We will show now that AV_0 and $A\xi$ are linearly dependent. Since $n \ge 3$, taking $W \in \ker A \cap T_pN$, the warped metric and the Gauss equation give us

$$-\frac{\operatorname{Hess}_\rho(\xi,\xi)}{\rho} W = R^M(W,\xi)\xi = c\langle \xi,\xi\rangle W,$$

which implies

$$-\frac{\operatorname{Hess}_\rho(\xi,\xi)}{\rho} = c\|\xi\|^2$$

at the point p. It follows from the expression of the curvature tensor R^M and the Gauss equation

$$-\frac{\operatorname{Hess}_\rho(\xi,\xi)}{\rho} V_0 = R^M(V_0,\xi)\xi = c\|\xi\|^2 V_0 + \langle A\xi,\xi\rangle AV_0 - \langle AV_0,\xi\rangle A\xi,$$

that

$$\langle A\xi,\xi\rangle AV_0 - \langle AV_0,\xi\rangle A\xi = 0.$$

This implies $\operatorname{rank} A = 1$, which concludes the proof of the lemma. ∎

Finally, we present here the characterization of the objects studied in this paper when the principal orbits are umbilical and G is compact.

Theorem 2.6. *Let $f\colon M^n \to \mathbb{H}^{n+1}$ be a complete hypersurface of G-cohomogeneity one whose principal orbits are umbilical in M^n. Assume either that $n \ge 3$ and M^n is compact, or that $n \ge 4$ and the connected components of the set $\{x \in M^n : \nu(x) \ge n-1\}$ are bounded. Then f is a rotational submanifold.*

The case where M^n is complete and $n \ge 4$ follows from Theorem 6.1 in [2]. Note that they allowed G to be non-compact, which can lead to non-rotational examples. However, in this case, the orbits are necessarily non-compact. The case where the manifold is compact and $n \ge 3$ follows from the same arguments as in Corollary 4 of [17] (see also Proposition 2.2 in [16]). These arguments involve a group homomorphism that naturally arises in this setting. Due to its inherent interest, we present the statement of this homomorphism for the case when the ambient space is also hyperbolic. The proof is the same as in the Euclidean case, but uses Theorem 2.1.

Theorem 2.7. *Let $f\colon M^n \to \mathbb{Q}_c^{n+1}$ be a complete hypersurface with $n \ge 3$ and $c \le 0$. If there exists no complete leaf of relative nullity of dimension $n-1$ or $n-2$, then the identity component $\operatorname{Iso}^0(M^n)$ of the isometry group of M^n admits an orthogonal representation $\rho\colon \operatorname{Iso}^0(M^n) \to \operatorname{Iso}^0(\mathbb{Q}_c^{n+1})$ such that $f \circ g = \rho(g) \circ f$ for all $g \in \operatorname{Iso}^0(M^n)$.*

3. The proofs

Proof of Theorem B. The beginning of the proof closely follows the proof of Proposition 2.3 in [16], establishing the objects to be used (such a structure was also provided in Proposition 3.1 of [2]). Since M^n carries a complete leaf of relative nullity $\mathcal{F}$, it cannot be compact. Thus the orbit space $\Omega = M^n/G$ is homeomorphic to either $\mathbb{R}$ or $[0,\infty)$. Moreover, if $\pi: M^n \to \Omega$ denotes the canonical projection and $\gamma: \mathbb{R} \to M^n$ is a normal geodesic parameterized by arc-length, then $\pi \circ \gamma$ maps $\mathbb{R}$ homeomorphically onto Ω in the first case, and it is a covering map of $\mathbb{R} \setminus \{0\}$ onto the subset Ω^0 of interior points of Ω in the latter. Set $I = \gamma^{-1}(G(\mathcal{F}))$. Since $G(\mathcal{F})$ is a closed unbounded connected subset, and using that $G(\mathcal{F}) = G(\gamma(I))$, it follows easily that if $I \neq \mathbb{R}$ then $I = [a,\infty)$ for some $a \in \mathbb{R}$ in the first case, and $I = (-\infty, b] \cup [a,\infty)$ for some $a, b > 0$ in the latter. Note that the shape operator has constant rank two on $G(\mathcal{F})$. This is because at points where $\nu^f = n-2$, the relative nullity distribution Δ equals the kernel of the tensor $R^M - cR_0$, where $R_0(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$ for all $X,Y,Z \in TM$ and $c = -1$. Therefore, the set where the second fundamental form has rank two is invariant under isometries. Let $p = \gamma(t_0)$ and consider $\varepsilon > 0$ such that $(t_0-\varepsilon, t_0+\varepsilon) \subset I$ and such that the map

$$\Phi : (t_0-\varepsilon, t_0+\varepsilon) \times \Sigma_p \to \pi^{-1}(t_0-\varepsilon, t_0+\varepsilon)$$

given by

$$\Phi(t, g(p)) = g(\gamma(t))$$

is a G-equivariant diffeomorphism, where $\Sigma_p = G \cdot p$ denotes the orbit of p under the action of G. We call $\Gamma = \pi^{-1}(t_0-\varepsilon, t_0+\varepsilon)$ a tube around Σ_p. We have a well-defined vector field ξ on Γ given by $\xi(y) = g_*(\gamma(t))\gamma'(t)$ for $y = g(\gamma(t))$, $t \in (t_0-\varepsilon, t_0+\varepsilon)$, and $\xi(y)$ is orthogonal to $\Sigma_{\gamma(t)}$ at y.

Now let η be a local unit normal vector field to f on Γ and let A^f_η be the shape operator of f with respect to η. Given a principal orbit $\Sigma_p = G \cdot p \subset \Gamma$ of G, the vector fields $\tilde{\xi} = f_*(\xi|_{\Sigma_p})$ and $\tilde{\eta} = \eta|_{\Sigma_p}$ determine an orthonormal normal frame of the restriction $\tilde{f}: \Sigma_p \to \mathbb{H}^{n+1}$ of f to Σ_p. Denote by $A_{\tilde{\eta}}$ and $A_{\tilde{\xi}}$ the corresponding shape operators (here we omit the superscript for clarity, as there is no ambiguity). By construction, we have $A_{\tilde{\xi}} \circ g_* = g_* \circ A_{\tilde{\xi}}$ for any $g \in G$, hence the eigenvalues of $A_{\tilde{\xi}}$ are constant. On the other hand, $A_{\tilde{\eta}} = \Pi \circ A^f_\eta$, where Π is the orthogonal projection of TM^n onto $T\Sigma_p$. In particular, $\operatorname{rank} A_{\tilde{\eta}} \leq \operatorname{rank} A^f_\eta$, so we have $\operatorname{rank} A_{\tilde{\eta}} \leq 2$ on Σ_p. Since $A_{\tilde{\xi}}$ is G-invariant and Σ_p^{n-1} is homogeneous, it follows from the Gauss equation that we have the following two cases to consider:

(1) rank $A_{\tilde{\eta}} \leq 1$ on each principal orbit contained in Γ;
(2) rank $A_{\tilde{\eta}} = 2$ on some principal orbit contained in Γ.

First, we show that (1) does not occur. Assume the opposite. Since Σ_p^{n-1} is a compact submanifold of $\mathbb{H}^{n+1}$, there is a point $x \in \Sigma_p^{n-1}$ where $\nu^{\tilde{f}}(x) = 0$ (the index of relative nullity of $\tilde{f}$). From Theorem 6 in [7], the universal cover $\tilde{\Sigma}_p^{n-1}$ of Σ_p^{n-1} is isometrically immersed in $\mathbb{H}^n$ as a compact isoparametric hypersurface. This means $\tilde{\Sigma}_p^{n-1}$ is isometric to $\mathbb{S}_c^{n-1}$ for some $c > 0$. Additionally, the shape operator $A_{\tilde{\xi}}$ of Σ_p^{n-1} will be a constant multiple of the identity. In particular, the principal orbits in Γ are umbilical in M^n. Given that the integral curve of ξ is the normal geodesic, we derive from the main result of [12] that there is an open subset $U \subset M^n$, $U = I \times_\rho N_c^{n-1}$, where N is a Riemannian manifold with constant curvature $c > 0$, and $f(U) \subset \Gamma$ (see also Proposition 3.2 in [2]). Considering $n \geq 5$ and $\nu^f|_\Gamma = n - 2$, this leads to a contradiction with Lemma 2.5.

If rank $A_{\tilde{\eta}} = 2$ along a principal orbit $\Sigma_p^{n-1} \subset \Gamma$, then the same rank condition holds on a smaller tube around Σ_p^{n-1} within Γ, which we continue to denote by Γ. Our goal now is to obtain a decomposition of the tangent space of M^n compatible with a warped structure as stated in the theorem, in order to use the main result in [19]. To achieve this, we need to show that ν^f is constant throughout M^n and that the eigenspaces of $A_{\tilde{\xi}}$ are autoparallel distributions in Σ_p^{n-1}. This will follow from the series of assertions proved below. In the proof, ∇ will denote the connection on Σ_p^{n-1} and ${}^M\nabla$ will denote the connection on M^n.

Claim 3.1. *The vector field $\tilde{\eta}$ is parallel along the distribution* $\ker A_{\tilde{\eta}}$ *with respect to the normal connection of $\tilde{f}$, represented here simply by $\nabla^\perp$.*

It follows from the compactness of Σ_p^{n-1} that there exist a point $q \in \Sigma_p^{n-1}$ and a direction in the normal space $\zeta \in T_{\tilde{f}}^\perp \Sigma(q)$ such that the shape operator A_ζ is positive definite. In particular, it follows that there exist $a, b \in \mathbb{R}$ such that

$$A_\zeta = aA_{\tilde{\eta}} + bA_{\tilde{\xi}},$$

and since the rank of $A_{\tilde{\eta}}$ is 2 and $n \geq 5$, it follows that $b \neq 0$ and, consequently, $A_{\tilde{\xi}}|_{\ker A_{\tilde{\eta}}}$ is a definite matrix at the point q. Since $A_{\tilde{\xi}}$ is G-equivariant, we have that $A_{\tilde{\xi}}$ is a definite operator when restricted to $\ker A_{\tilde{\eta}}$ along Σ_q^{n-1}. Now consider the Codazzi equation

$$A_{\nabla_Y^\perp \tilde{\eta}} X + A_{\tilde{\eta}}(\nabla_Y X) = A_{\nabla_X^\perp \tilde{\eta}} Y + A_{\tilde{\eta}}(\nabla_X Y),$$

for vectors $Y, X \in \Gamma(\ker A_{\tilde{\eta}})$. Since $n \geq 5$, the kernel of the linear form $\langle \nabla_. \tilde{\eta}, \tilde{\xi} \rangle$ restricted to $\ker A_{\tilde{\eta}}$ has dimension at least $n - 4 \geq 1$, so we can choose $Y \in \ker \langle \nabla_. \tilde{\eta}, \tilde{\xi} \rangle|_{\ker A_{\tilde{\eta}}}$. Taking the inner product of the previous Codazzi equation with Y, we obtain the relation

$$\langle \nabla_X^\perp \tilde{\eta}, \tilde{\xi} \rangle \langle A_{\tilde{\xi}} Y, Y \rangle = 0,$$

which holds for any $X \in \Gamma(\ker A_{\tilde{\eta}})$ and implies the parallelism of $\tilde{\eta}$ along $\ker A_{\tilde{\eta}}$. $\triangle$

Let $h \colon N_p^{n-3} \to \mathbb{H}^{n+1}$ be the restriction of $\tilde{f}$ to the leaf N_p^{n-3} of $\ker A_{\tilde{\eta}}$ passing through the point p. It is not difficult to see that $\ker A_{\tilde{\eta}}$ is G-invariant and the leaves of the distribution are homogeneous submanifolds, hence we omit the point in the notation. Therefore, if one leaf of the distribution $\ker A_{\tilde{\eta}}$ is autoparallel, then all are.

Claim 3.2. $\ker A_{\tilde{\eta}}$ *is an autoparallel distribution in Σ_p^{n-1}.*

Consider the Codazzi equation

$$-A_{\nabla_Z^\perp \tilde{\eta}} X - A_{\tilde{\eta}}(\nabla_Z X) = \nabla_X A_{\tilde{\eta}} Z - A_{\nabla_X^\perp \tilde{\eta}} Z - A_{\tilde{\eta}}(\nabla_X Z),$$

for $X \in \Gamma(\ker A_{\tilde{\eta}})$ and $Z \in \Gamma(\mathrm{Im} A_{\tilde{\eta}})$.

Taking the inner product with $Y \in \ker A_{\tilde\eta}$ and using the parallelism of $\tilde\eta$ along $\ker A_{\tilde\eta}$, we obtain

$$\langle \nabla^\perp_Z \tilde\eta, \tilde\xi\rangle\langle \alpha_{\tilde f}(X,Y), \tilde\xi\rangle = \langle \nabla_X Y, A_{\tilde\eta} Z\rangle. \tag{3.1}$$

Define ω as the one-form where $\omega(X) = \langle \nabla^\perp_X \tilde\xi, \tilde\eta\rangle$ for $X \in T\Sigma$. Let $\{Z_1, Z_2\}$ be a local orthonormal frame for $\mathrm{Im} A_{\tilde\eta}$, and let $W_1, W_2 \in (\ker A_{\tilde\eta})^\perp$ be the vector fields such that $A_{\tilde\eta} W_1 = Z_1$ and $A_{\tilde\eta} W_2 = Z_2$, with $W_2 \in \ker\omega$. From (3.1), the second fundamental form of h, denoted by α_h, evaluated on $X, Y \in TN$ is

$$\alpha_h(X,Y) = \begin{cases} \langle A_{\tilde\xi} X, Y\rangle(\tilde\xi + \langle \nabla^\perp_{W_1}\tilde\eta, \tilde\xi\rangle Z_1) & \text{if } \ker A_{\tilde\eta} \text{ is not autoparallel,}\\ \langle A_{\tilde\xi} X, Y\rangle \tilde\xi & \text{if } \ker A_{\tilde\eta} \text{ is autoparallel.}\end{cases} \tag{3.2}$$

Therefore, the first normal bundle $\mathcal{N}_1^h = \{\alpha_h(X,Y) : X, Y \in \ker A_{\tilde\eta}\}$ of h is locally generated by $\tilde\xi + \langle \nabla^\perp_{W_1}\tilde\eta, \tilde\xi\rangle Z_1$ and therefore one-dimensional. Given $n \geq 5$, we apply Theorem A in [4] (or Proposition 2.7 in [5]) to obtain that $\tilde\xi + \langle \nabla^\perp_{W_1}\tilde\eta, \tilde\xi\rangle Z_1$ is parallel and we have a reduction of codimension for $h\colon N^{n-3} \to \mathbb{H}^{n-2}$. Since the image of h is bounded, the classification of homogeneous hypersurfaces implies that h is umbilical and N^{n-3} is isometric to a sphere $\mathbb{S}^{n-3}_{c_2}$ for some constant $c_2 > 0$. Since N^{n-3} is not isometric to $\mathbb{R}^{n-3}$, it follows from the same arguments in Lemma 4.8 of [6] (or Lemma 17 in [7]) that $\ker A_{\tilde\eta}$ must be autoparallel in Σ^{n-1}_p. △

Our goal now is to show that the shape operator $A_{\tilde\xi}$ has only two distinct eigenvalues, and the associated eigenspaces will give rise to the distributions that allow us to obtain the warped structure mentioned in the theorem statement.

Claim 3.3. *We have that $\xi \in \Delta^f$, $\nu^f = n-2$, and the operator $A_{\tilde\xi}$ has only two distinct eigenvalues, with the eigenspaces being autoparallel distributions in Σ^{n-1}_p.*

Using the umbilicity of h, equation (3.2) simplifies to

$$\alpha_h(X,Y) = \langle A_{\tilde\xi} X, Y\rangle\tilde\xi = \lambda\langle X, Y\rangle\tilde\xi \quad \text{for all } X, Y \in TN,$$

for some constant $\lambda \neq 0$. The constancy of λ follows from the G-invariance of $A_{\tilde\xi}$, while $\lambda \neq 0$ follows from the Gauss equation and the compactness of N^{n-3}_p (more precisely, we have that $|\lambda| > 1$). Moreover, since $\ker A_{\tilde\eta}$ is autoparallel, (3.1) implies that

$$\langle \nabla^\perp_W \tilde\eta, \tilde\xi\rangle = 0 \quad \text{for all } W \in \mathrm{Im}\, A_{\tilde\eta}. \tag{3.3}$$

This is equivalent to $\langle A^f_\eta W, \xi\rangle = 0$ for all $W \in \mathrm{Im}\, A_{\tilde\eta}$, since $\tilde\eta$ and $\tilde\xi$ are just the restrictions of η and ξ to Σ^{n-1}_p, and therefore $\langle \nabla^\perp_W \tilde\eta, \tilde\xi\rangle = \langle {}^{\mathbb{H}}\nabla_W \eta, \xi\rangle$. Given that $\operatorname{rank} A_{\tilde\eta} = \operatorname{rank} A_\eta$, it follows that $\xi \in \Delta^f(q)$ for all $q \in \Sigma^{n-1}_p$. Since ξ is the field generated by the normal geodesic, we conclude that $\xi \in \Delta^f$ everywhere. Therefore, the segments of normal geodesics in Γ are contained in the leaves of $\Delta^f = \ker A_\eta$. Since these leaves are assumed to be complete, we obtain that $\nu^f = n-2$ throughout M^n.

It follows from equation (3.3) that $\tilde f$ has a flat normal bundle, and therefore the shape operators $A_{\tilde\xi}$ and $A_{\tilde\eta}$ are simultaneously diagonalizable in a local orthonormal frame $\{X_1, X_2, X_3, \ldots, X_{n-1}\}$ of $T\Sigma_p$, where X_1 and X_2 generate $(\ker A_{\tilde\eta})^\perp$. Let δ_1 and δ_2 be the eigenvalues of $A_{\tilde{\eta}}$, and let λ_i be the constant eigenvalues of $A_{\tilde{\xi}}$. The Codazzi equation for X_1 and X_2 in the direction $\tilde{\eta}$ gives

$$\langle \nabla_{X_1} X_2, X\rangle \delta_2 = \langle \nabla_{X_2} X_1, X\rangle \delta_1,$$

for all $X \in \ker A_{\tilde{\eta}}$. Since $\delta_i \neq 0$, the orthogonal projections of $\nabla_{X_1} X_2$ and $\nabla_{X_2} X_1$ onto $\ker A_{\tilde{\eta}}$ are linearly dependent. From the previous discussion on umbilicity, it follows that $\lambda_i = \lambda$ for $i \geq 3$. The goal is to show that $\lambda_1 = \lambda_2 \neq \lambda$. If $\lambda_1 = \lambda_2 = \lambda$, the distribution $(\text{span}\{\xi\})^\perp$ is spherical in M^n, since at each point $p \in M^n$, $(\text{span}\{\xi\})^\perp$ is identified with the tangent space of the orbit Σ_p^{n-1}. It follows from [12] that $M^n = I \times_\rho P_c^{n-1}$ is a warped product of $I \subset \mathbb{R}$ and some manifold P of constant sectional curvature $c > 0$, which contradicts the dimension of relative nullity as stated in Lemma 2.5. Assuming that $\lambda_1 \neq \lambda_2$, we apply the Codazzi equation for $A_{\tilde{\xi}}$ to the vectors $X \in \ker A_{\tilde{\eta}}$ and X_i (for $i = 1, 2$) to obtain

$$(\lambda_i - \lambda)\langle X, \nabla_{X_i} X_i\rangle = 0. \tag{3.4}$$

Thus $\lambda_i \neq \lambda$ implies $\langle X, \nabla_{X_i} X_i\rangle = 0$. Moreover, from the Gauss equation and the homogeneity of Σ_p^{n-1}, the sectional curvature $K(X_1, X_2)$ is constant. Since $K(X_1, X_2) = \lambda_1\lambda_2 + \delta_1\delta_2 - 1$ and λ_1 and λ_2 are constant, it follows that $\delta_1\delta_2$ is constant. Now, using the Codazzi equation for $A_{\tilde{\eta}}$ with $X \in \ker A_{\tilde{\eta}}$ and X_i, we obtain

$$X(\delta_i) = \delta_i \langle X, \nabla_{X_i} X_i\rangle.$$

Since $\delta_1\delta_2$ is constant, $X(\delta_1\delta_2) = \delta_2 X(\delta_1) + \delta_1 X(\delta_2) = 0$, which implies

$$\delta_1\delta_2(\langle X, \nabla_{X_1} X_1\rangle + \langle X, \nabla_{X_2} X_2\rangle) = 0.$$

As $\delta_1\delta_2 \neq 0$, we have $\langle X, \nabla_{X_1} X_1\rangle + \langle X, \nabla_{X_2} X_2\rangle = 0$. Since $\lambda_1 \neq \lambda_2$, at least one of λ_1 and λ_2 is different from λ. From (3.4), the corresponding term $\langle X, \nabla_{X_i} X_i\rangle$ is zero, and hence both are zero. Choosing $X \in \ker A_{\tilde{\eta}}$ orthogonal to the projections $\nabla_{X_1} X_2$ and $\nabla_{X_2} X_1$, we conclude

$$\langle \nabla_X \nabla_{X_i} X_i, X\rangle = 0, \quad \text{for } i \in \{1, 2\},$$

since $\langle \nabla_{X_i} X_i, X\rangle = 0$ and $\ker A_{\tilde{\eta}}$ is an autoparallel distribution. As $\text{Im}A_{\tilde{\eta}}$ is parallel along $\ker A_{\tilde{\eta}}$ and by our choice of X, we obtain

$$\langle \nabla_{X_i} \nabla_X X_i, X\rangle = 0, \quad \text{for } i \in \{1, 2\}.$$

Using all the previous information, we also get

$$\langle \nabla_{[X, X_i]} X_i, X\rangle = 0, \quad \text{for } i \in \{1, 2\}.$$

From these relations, it follows that

$$\langle R(X, X_i)X_i, X\rangle = 0, \quad \text{for } i \in \{1, 2\}.$$

From the Gauss equation, we have

$$0 = K(X, X_i) = -1 + \lambda_i \langle A_{\tilde{\xi}} X, X\rangle, \quad \text{for } i \in \{1, 2\},$$

contradicting $\lambda_1 \neq \lambda_2$.

Now we have $\lambda_0 := \lambda_1 = \lambda_2$, and $A_{\tilde{\xi}}$ is expressed in this basis as

$$A_{\tilde{\xi}} = \left(\begin{array}{cc|c} \lambda_0 & 0 & \multirow{2}{*}{0} \\ 0 & \lambda_0 & \\ \hline \multicolumn{2}{c|}{0} & \lambda I \end{array}\right). \tag{3.5}$$

A direct application of the Codazzi equation, together with the constancy of λ_0 and λ, implies that the eigenspaces E_{λ_0} and E_λ of $A_{\tilde{\xi}}$ are autoparallel distributions in Σ_p^{n-1}. $\triangle$

We thus have the natural decomposition $TM = \operatorname{span}\{\xi\} \oplus V_0 \oplus V_1$ everywhere, where $V_0 = (\Delta^f)^\perp$ and V_1 is the orthogonal complement of $\operatorname{span}\{\xi\}$ in Δ^f. Since expression (3.5) holds for all principal orbits of G, it follows that V_0 at the point $q \in M^n$ is identified with the eigenspace E_{λ_0} of $A_{\tilde{\xi}}$, and is therefore an umbilical distribution whose mean curvature vector is parallel to ξ with constant norm along the principal orbits of G, making it a spherical distribution in M^n. Similarly, V_1 is also a spherical distribution in M^n. In summary, it follows from the claims proven above that

$$^M\nabla_\xi \xi = 0, \quad {}^M\nabla_{X_i} Y_i \subset V_i \oplus \operatorname{span}\{\xi\}, \quad {}^M\nabla_\xi Y_i \subset V_i \quad \text{and} \quad {}^M\nabla_{X_i}\xi \subset V_i$$

for all $X_i, Y_i \in V_i$, $i \in \{0,1\}$. Therefore, V_0 and V_1 are spherical distributions in M^n, whose orthogonal complements are autoparallel distributions in M^n. It follows from the results of [19] (see also Theorem 10.21 in [5]) that M^n is isometric to $\mathbb{R} \times_{\rho_0} P^2_{c_1} \times_{\rho_1} N^{n-3}_{c_2}$, where $P^2_{c_1}$ and $N^{n-3}_{c_2}$ are manifolds with constant sectional curvatures $c_1 > 0$ and $c_2 > 0$ respectively, and ρ_0, ρ_1 are warping functions defined on $\mathbb{R}$. Moreover, f admits an extrinsic warped product structure. More precisely, there exists a warped product representation $\Psi\colon V^{1+i_0} \times_{\sigma_0} \mathbb{Q}^{2+i_1}_{\tilde{c}_1} \times_{\sigma_1} \mathbb{Q}^{n-3+i_2}_{\tilde{c}_2} \to \mathbb{H}^{n+1}$, where $i_0 + i_1 + i_2 = 1$ and V^{1+i_0} is an open subset of a totally geodesic submanifold, with warping functions σ_0 and σ_1 defined on V^{1+i_0}. Furthermore, there exist isometric immersions $f_0\colon \mathbb{R} \to V^{1+i_0}$, $f_1\colon P^2_{c_1} \to \mathbb{Q}^{2+i_1}_{\tilde{c}_1}$, and $f_2\colon N^{n-3}_{c_2} \to \mathbb{Q}^{n-3+i_2}_{\tilde{c}_2}$ such that $\rho_0 = \sigma_0 \circ f_0$, $\rho_1 = \sigma_1 \circ f_1$, and $f = \Psi \circ (f_0 \times f_1 \times f_2)$. In particular, the following diagram commutes:

$$\begin{array}{ccccccc}
V^{1+i_0} & \times_{\sigma_0} & \mathbb{Q}^{2+i_1}_{\tilde{c}_1} & \times_{\sigma_1} & \mathbb{Q}^{n-3+i_2}_{\tilde{c}_2} & \xrightarrow{\quad\Psi\quad} & \\
\uparrow f_0 & & \uparrow f_1 & & \uparrow f_2 & \circlearrowleft & \\
\mathbb{R} & \times_{\rho_0} & P^2_{c_1} & \times_{\rho_1} & N^{n-3}_{c_2} & \xrightarrow[f = \Psi \circ (f_0 \times f_1 \times f_2)]{} & \mathbb{H}^{n+1}.
\end{array}$$

If $i_2 = 1$, using that $TN^{n-3} \subset \Delta^f$ at all points, it is straightforward to verify from the curvature tensor equations of a warped product that $\tilde{c}_2 = c_2 > 0$, which implies that f_2 is a totally geodesic immersion (cf. Corollary 7.13 in [5]). As a result, f would be totally geodesic. Consequently, $i_2 = 0$ and $i_0 + i_1 = 1$. This places us precisely in the scenario of the statement of the theorem. In particular, $P^2_{c_1} = \mathbb{S}^2_{c_1}$, $N^{n-3}_{c_2} = \mathbb{S}^{n-3}_{c_2}$ and we have a rotational submanifold whose profile submanifold is $\mathbb{R} \times_{\rho_0} \mathbb{S}^2_{c_1}$, the immersion is determined by f_0 and f_1, and the warped product representation Ψ, thus concluding the proof. ■

Remark 3.4. In the proof of Theorem B, the decomposition $TM = \operatorname{span}\{\xi\} \oplus V_0 \oplus V_1$ yields a totally geodesic distribution $\operatorname{span}\{\xi\} \oplus V_0$, and a spherical distribution V_1, satisfying $\alpha_f(X,Y)=0$ for all $X\in\operatorname{span}\{\xi\}\oplus V_0$ and $Y\in V_1$. This structure implies that M^n is isometric to a warped product $L^3\times_\rho \mathbb{S}^{n-3}_{c_2}$ for some warping function $\rho\colon L^3\to\mathbb{R}$, where L^3 is a profile manifold, and thus the submanifold is rotational. The proof actually provides a more precise characterization by showing that the submanifold arises from the action of $\mathrm{SO}(3)\times\mathrm{SO}(n-2)$ on a complete geodesic within a totally geodesic section $\mathbb{H}^2$. Consequently, the profile submanifold is foliated by spheres and the principal orbits of G are products of spheres.

Proof of Theorem A. It follows from our assumption on the set $\{x\in M^n : \nu(x)\geq n-1\}$ that there does not exist a complete leaf of relative nullity with $\nu^f=n-1$ or with $\nu^f=n$. If there exists a complete leaf of relative nullity with $\nu^f=n-2$, then, by Theorem B, M^n must be a rotational submanifold. Thus, we can assume that there are no complete leaves of relative nullity with $\nu^f>n-3$. Let B^f denote the set of totally geodesic points of f. According to Theorem 2.1, f is either rigid or B^f disconnects M. We will show that if we are in the latter case, f is a rotational submanifold. First we verify that B^f is G-invariant. By applying Theorem 2.1 to each pair f and $f\circ g$, where $g\in G$, we find

$$A^f=\pm A^{f\circ g}. \tag{3.6}$$

Moreover, for any vector fields $X,Y\in\Gamma(TM)$ and normal vector field $\eta\in\Gamma(T^\perp M)$,

$$\langle A^{f\circ g}_{\eta\circ g}X,Y\rangle=\langle {}^{\mathbb{H}}\nabla_{f_*(g_*X)}f_*(g_*Y),\eta\circ g\rangle=\langle A^f_\eta(g_*X),g_*Y\rangle=\langle g_*^{-1}A^f_\eta(g_*X),Y\rangle.$$

This leads to $A^{f\circ g}_{\eta\circ g}=g_*^{-1}\circ A^f_\eta\circ g_*$, for all $g\in G$, along with (3.6), which shows that B^f is G-invariant. Since B^f disconnects M^n and the singular orbits have dimension less than $n-1$, we can take a point $p\in B^f\cap M_{\mathrm{reg}}$, where M_{reg} denotes the set of regular points of M^n with respect to the action, and consider the orbit $\Sigma^{n-1}_p=G\cdot p$. Let $g=f\circ i:\Sigma^{n-1}_p\to\mathbb{H}^{n+1}$, where $i:\Sigma^{n-1}_p\to M^n$ is the inclusion. The strategy is to show that Σ^{n-1}_p is an umbilical hypersurface in M^n and so the theorem will follow from Theorem 2.6. Since $\Sigma^{n-1}_p\subset B^f$, the second fundamental form of f vanishes on Σ^{n-1}_p, so we have

$$\alpha_g(X,Y)=f_*\alpha_i(X,Y)+\alpha_f(i_*X,i_*Y)=f_*\alpha_i(X,Y)\quad\text{for all } X,Y\in T\Sigma. \tag{3.7}$$

As in the beginning of Theorem B, we have the unit normal vectors $\tilde\xi,\tilde\eta\in T^\perp_g\Sigma$ given by the restriction of the normal geodesic to the regular orbits and the unit normal vector η from the isometric immersion f. By construction, $A^g_{\tilde\xi}$ is G-invariant. The Codazzi equation for vectors $X,Y\in T\Sigma$ in the direction of $\tilde\eta$ simplifies to

$$\omega(X)A_{\tilde\xi}Y=\omega(Y)A_{\tilde\xi}X, \tag{3.8}$$

where $\omega(X)=\langle\nabla^\perp_X\tilde\eta,\tilde\xi\rangle$ for $X\in T\Sigma$. Using ideas analogous to those in the proof of Claim 3.1, we will prove that $\tilde\xi$ is parallel along Σ^{n-1}_p, which is equivalent to $T\Sigma_p=\ker\omega$. Assuming otherwise, let $x\in\Sigma^{n-1}_p$ with $\ker\omega(x)\neq T_x\Sigma_p$. Then (3.8) implies $\ker\omega(x)\subset\ker A_{\tilde\xi}$, leading to $\nu_g(x)\geq n-2\geq 2$. Since $A^g_{\tilde\xi}$ is G-invariant, it follows that $\nu_g\geq 2$ everywhere, contradicting the compactness of Σ^{n-1}_p because complete leaves of relative nullity for submanifolds in $\mathbb{H}^{n+1}$ are unbounded. Hence, $\ker\omega=T\Sigma_p$. From (3.7), the first normal bundle $\mathcal{N}^g_1:=\operatorname{span}\{\alpha_g(X,Y):X,Y\in T\Sigma\}$ of g is a one-dimensional distribution spanned by $\tilde\xi$, therefore $\mathcal{N}^g_1$ is a parallel distribution along Σ^{n-1}_p. By Proposition 2.1 in [5] (see also [1]), we have a reduction of codimension, that is, g lies in a totally geodesic submanifold $\mathbb{H}^n \subset \mathbb{H}^{n+1}$. Given the compactness and classification of homogeneous hypersurfaces in hyperbolic space, Σ_p^{n-1} is isometric to $\mathbb{S}_{c_1}^{n-1}$ and the isometric immersion g is totally umbilical, as desired. This, together with (3.7), implies that Σ_p^{n-1} is umbilical in M^n. Since the principal orbits are totally umbilical, Theorem 2.6 implies that f is a rotational submanifold. ∎

Remark 3.5. Although there are versions of Theorem 2.1, and consequently of Theorem 2.7, where the ambient space is the sphere, the principal orbits of G will be homogeneous hypersurfaces of some totally geodesic submanifold of the ambient space, and in particular, they will be isoparametric. However, it cannot be guaranteed that they will be intrinsically umbilical (see also Remark 12 in [17]).

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