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Lépingle inequality and martingale Hardy spaces

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Abstract. We prove that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a filtration $(\mathcal{F}_n)_{n\geq 1}$ and $E(\Omega)$ is a quasi-Banach Köthe function space with the property that the Lépingle inequality is satisfied for adapted sequences in $E(\Omega)$, then the couple of martingale Hardy spaces $(H_E^S(\Omega), H_\infty^S(\Omega))$ is K-closed in the couple of Köthe–Bochner spaces $(E(\Omega;\ell_2), L_\infty(\Omega;\ell_2))$. This extends the commutative form of a recent result of Moyart from L_1 to general Köthe function spaces and provides a lifting of real interpolation of function spaces to corresponding martingale Hardy spaces. As applications, we obtain new type of interpolation results for Musielak–Orlicz martingale Hardy spaces and variable martingale Hardy spaces. We also prove an analogous result on automatic transfer (without any assumption) of real interpolation of couple of quasi-Banach Köthe function spaces $(E(\Omega), L_\infty(\Omega))$ to the couple of corresponding conditional martingale Hardy spaces $(H_E^s(\Omega), H_\infty^s(\Omega))$.

1. Introduction

Various types of Hardy spaces arising from martingales theory have played significant role in the development of probability and other fields of mathematics. As is now well established, real interpolation method is a powerful tool in many aspects of analysis. Thus, interpolations of martingale Hardy spaces is a natural direction to pursue. In fact, the study of interpolation of martingale Hardy spaces has a long history for which we refer to [13, 32, 33, 35] for more in depth discussions. Recently, the real interpolation theory for martingale Hardy spaces has seen rapid growth. More specifically, martingale Hardy type spaces associated with general function spaces. For instance, Ho [11] obtained that the martingale Hardy–Lorentz–Karamata spaces are real interpolation spaces of martingale Hardy space under the action of a new interpolation functor. Jiao et al. [14] considered the real interpolation for variable conditional martingale Hardy spaces. Other recent articles dealing with this direction are [6, 8, 23, 24]. The results in the articles cited above are for conditional martingale Hardy spaces and it is only recently ([25, 30]) that a general result has been established for the real interpolation of the couple (H_1^S, H_∞^S). We refer to the next section for the unexplained notation here. Prior to [25, 30], the interpolations of the family of martingale Hardy spaces $\{H_p^S\}_{0<p\leq\infty}$ were only studied under the

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assumption that the filtration is regular. The natural next step in this direction is to systematically investigate if the vast amount of available literature on conditional martingale Hardy spaces remain valid for martingale Hardy spaces.

The primary objective of this paper is to investigate the compatible couple (H_E^S, H_∞^S) when E is a general quasi-Banach Köthe function space in the sense of [22]. Although, through standard reiteration techniques on K-functionals, Moyart's result ([25]) extends to martingale Hardy spaces associated with rearrangement invariant Banach function spaces, no specific tool is available to handle other classes of function spaces that are not rearrangement invariants. Examples of such classes of function spaces are variable Lebesgue spaces and more generally, Musielak–Orlicz spaces. Our main result is Theorem 3.1 below which can be roughly stated that under some natural assumptions on the function space E, the real interpolation lifts from the couple of function spaces (E, L_∞) to the couple of corresponding Hardy spaces (H_E^S, H_∞^S). More precisely, we obtain that the couple (H_E^S, H_∞^S) is K-closed in the couple of Köthe–Bochner spaces $(E(\ell_2), L_\infty(\ell_2))$ in the sense of Pisier ([28]). This generalizes in the commutative setting the recent result of Moyart discussed earlier to abstract quasi-Banach Köthe function spaces.

Our method of proof is constructive and therefore provides an alternative approach to Moyart's result for the commutative L_1. It is inspired by ideas from [29]. However, we have to significantly deviate from [29], since properties involving the notion of decreasing rearrangement of functions were used there in very crucial ways, but they are not necessarily available for the present situation.

It turns out that our method can also be modified to provide a more concise and unified approach to the couple of conditional martingale Hardy spaces (H_E^s, H_∞^s) for any arbitrary quasi-Banach Köthe function space E. We refer to Theorem 3.6 below for the exact formulation. It complements and unifies all previously known results on real interpolations for conditional martingale Hardy spaces.

The paper is organized as follows. In Section 2, we give a brief introduction of various concepts needed for the presentation. More specifically, we discuss Doob's maximal inequality and its dual form for general Köthe function spaces along with the Lépingle inequality. Section 3 is where we provide the formulation and proof of our primary result in the form of K-closed couples together with some extensions and consequences. The section also contains simpler approach to real interpolations of conditional Hardy spaces. In Section 4, we discuss Musielak–Orlicz Hardy spaces as concrete illustration of the previous section. We conclude the paper with discussions on a related topic. Namely, Stein's inequality in Banach Köthe function spaces.

2. Definitions and preliminary results

Throughout, we write $A \lesssim B$ if there is some absolute constant c such that $A \leq cB$. We say that A is equivalent to B if $A \lesssim B$ and $B \lesssim A$. In this case, we write $A \approx B$.

2.1. Function spaces

In this subsection, we collect some definitions and properties from function spaces that are relevant for the paper.

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. We denote by $L_0(\Omega, \mathcal{F}, \mathbb{P})$ (or simply $L_0(\Omega)$) the vector space of all (equivalence classes of) real-valued $\mathcal{F}$-measurable functions on Ω. A *Köthe function space* over $(\Omega, \mathcal{F}, \mathbb{P})$ is a linear subspace $E(\Omega)$ (or simply E) of $L_0(\Omega)$ such that if $f \in E(\Omega)$, $g \in L_0(\Omega)$, and $|g| \leq |f|$, then $g \in E(\Omega)$. We say that the Köthe space E is quasi-normed if there is a lattice quasi-norm $\|\cdot\|_E$ defined on $E(\Omega)$. A complete quasi-normed Köthe function space is called a quasi-Banach Köthe function space. We refer to [22] for unexplained terminology from (quasi-) Banach function theory.

Assume that X is a Banach space. We say that a function $f : \Omega \to X$ is *strongly $\mathcal{F}$-measurable* if there is a sequence of simple functions $(f_n)_{n\geq 1}$ with $f_n : \Omega \to X$ for all $n \geq 1$ and such that $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_X = 0$ for $\mathbb{P}$-almost all ω. As in the scalar case, we denote by $L_0(\Omega; X)$ the space of all (equivalence classes of) strongly $\mathcal{F}$-measurable X-valued functions defined on Ω.

The *Köthe–Bochner space* $E(\Omega; X)$ is defined by setting

$$E(\Omega; X) := \big\{ f \in L_0(\Omega; X) : \|f(\cdot)\|_X \in E(\Omega) \big\}.$$

Since $\|\cdot\|_X$ is a norm, one can easily see that the mapping

$$\begin{gathered} f \mapsto \| \|f(\cdot)\|_X \|_E, \\ E(\Omega; X) \to [0, \infty), \end{gathered}$$

is a quasi-norm whenever $E(\Omega)$ is a quasi-Banach Köthe function space, and that $E(\Omega; X)$ becomes a quasi-Banach space when equipped with the mixed-norm

$$\|f\|_{E(\Omega;X)} := \| \|f(\cdot)\|_X \|_E, \quad f \in E(\Omega; X).$$

We remark that if $E(\Omega) = L_p(\Omega)$ for $0 < p \leq \infty$, then $E(\Omega; X)$ is the Lebesgue–Bochner space $L_p(\Omega; X)$. We refer to [5, 21] for more background concerning vector-valued functions and Köthe–Bochner spaces.

2.2. Martingale Hardy spaces

We introduce standard notations from martingale theory. We refer to [7, 33] for the theory of martingale space. Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing filtration of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$ and for $n \geq 1$, denote by $\mathbb{E}_n$ the conditional expectation operator relative to the σ-subalgebra $\mathcal{F}_n$.

We recall that a sequence $f = (f_n)_{n\geq 1} \subset L_1(\Omega)$ is called *a martingale* with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$ if

$$\mathbb{E}_n(f_{n+1}) = f_n, \quad n \geq 1.$$

We denote by $\mathcal{M}$ the collection of all martingales relative to the filtration $(\mathcal{F}_n)_{n\geq 1}$.

For a given martingale $f = (f_n)_{n\geq 1}$, the martingale difference sequence of f is given by

$$df_n = f_n - f_{n-1}, \quad n \geq 1,$$

with the convention that $f_0 = 0$. If in addition, the f_n's belong to $E(\Omega)$ for a quasi-Banach Köthe function space $E(\Omega)$, then f is called an E-martingale. In this case, we set

$$\|f\|_E = \sup_{n\geq 1} \|f_n\|_E.$$

If $\|f\|_E < \infty$, then f is called a bounded E-martingale and is denoted by $f \in E(\Omega)$. For any given martingale $f = (f_n)_{n\geq 1}$, we define the square function and the conditional square function as follows:

$$S_m(f) = \Big(\sum_{n=1}^{m} |df_n|^2\Big)^{1/2}, \qquad S(f) = \Big(\sum_{n=1}^{\infty} |df_n|^2\Big)^{1/2}, \quad m \geq 1;$$
$$s_m(f) = \Big(\sum_{n=1}^{m} \mathbb{E}_{n-1}|df_n|^2\Big)^{1/2}, \quad s(f) = \Big(\sum_{n=1}^{\infty} \mathbb{E}_{n-1}|df_n|^2\Big)^{1/2}, \quad m \geq 1.$$

The martingale Hardy space and martingale conditioned Hardy space associated with a Köthe function space $E(\Omega)$ are respectively defined as follows:

$$H_E^S(\Omega) := \{f \in \mathcal{M} : \|f\|_{H_E^S} = \|S(f)\|_E < \infty\}$$

and

$$H_E^s(\Omega) := \{f \in \mathcal{M} : \|f\|_{H_E^s} = \|s(f)\|_E < \infty\}.$$

Clearly, when $E = L_p$ for some $0 < p \leq \infty$, then we have the familiar martingale Hardy spaces $H_p^S(\Omega)$ and the conditional martingale Hardy space $H_p^s(\Omega)$, respectively. The case where E is an interpolation of the couple (L_p, L_q), for $0 < p, q \leq \infty$, is well studied in the literature. Our focus will be on Hardy spaces associated with (quasi-) Banach Köthe function spaces that are not necessarily rearrangement invariant in the sense of [22].

It clearly follows from the definition that the martingale Hardy space $H_E^S(\Omega)$ embeds isometrically into the Köthe–Bochner space $E(\Omega; \ell_2)$. The isometric embedding is given by $f = (f_n)_{n\geq 1} \mapsto (df_n)_{n\geq 1}$. This fact is essential in the subsequent discussion below.

It is important to note that some standard tools from martingale theory such as Doob maximal inequality may not be available for general function spaces (see, e.g., [34]). We consider the next two definitions as working conditions.

Definition 2.1. We say that a quasi-Banach Köthe function space $E(\Omega)$ satisfies *the dual form of the Doob maximal inequality* (with respect to $(\mathcal{F}_n)_{n\geq 1}$) if there exists a constant $C > 0$ such that if for any sequence of non-negative functions $(f_n)_{n\geq 1}$ in $E(\Omega)$,

$$\Big\|\sum_{n\geq 1} \mathbb{E}_n(f_n)\Big\|_E \leq C\Big\|\sum_{n\geq 1} f_n\Big\|_E.$$

A companion property that we will need in the sequel is for adapted sequences. Recall that a sequence $(a_n)_{n\geq 1}$ is called an *adapted sequence* if for every $n \geq 1$, a_n is $\mathcal{F}_n$-measurable.

Definition 2.2. We say that a quasi-Banach Köthe function space $E(\Omega)$ satisfies *the Lépingle inequality* if there exists a positive constant C such that for every adapted sequence $(a_n)_{n\geq 1}$ in $E(\Omega)$,

$$\Big\|\Big(\sum_{n\geq 1} |\mathbb{E}_{n-1}(a_n)|^2\Big)^{1/2}\Big\|_E \leq C\Big\|\Big(\sum_{n\geq 1} |a_n|^2\Big)^{1/2}\Big\|_E.$$

Recall that the original Lépingle inequality ([20]) is for $L_1(\Omega)$. Other classes of Banach function spaces that are interpolations of (L_1, L_∞) are also known to satisfy such property. We refer to the next section for more discussions on these two definitions.

Our first result is a Davis type decompositions of adapted sequences in quasi-Banach Köthe function spaces satisfying the dual form of the Doob maximal inequality.

Proposition 2.3. *Let $E(\Omega)$ be a quasi-Banach Köthe function space that satisfies the dual form of the Doob maximal inequality. There exists a positive constant C such that if $a = (a_n)_{n\geq 1}$ is an adapted sequence in $E(\Omega;\ell_2)$, then there exist two adapted sequences b and c such that $a = b + c$,*

$$\Big\| \sum_{n\geq 1} |b_n| \Big\|_E \leq C \Big\| \Big(\sum_{n\geq 1} |a_n|^2 \Big)^{1/2} \Big\|_E,$$

and

$$\Big\| \Big(\sum_{n\geq 1} \mathbb{E}_{n-1}(|c_n|^2) \Big)^{1/2} \Big\|_E \leq C \Big\| \Big(\sum_{n\geq 1} |a_n|^2 \Big)^{1/2} \Big\|_E.$$

Proof. For $n \geq 1$, set

$$\lambda_n := \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2}.$$

Clearly, $(\lambda_n)_{n\geq 1}$ is an increasing sequence and is adapted. Set

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n \quad \text{a.e.}$$

By assumption, $\lambda_\infty \in E(\Omega)$. Define

$$b_n = a_n \chi_{\{\lambda_n > 2\lambda_{n-1}\}} \quad \text{and} \quad c_n = a_n \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}}.$$

Then, $b = (b_n)_{n\geq 1}$ and $c = (c_n)_{n\geq 1}$ are adapted sequences satisfying $a = b + c$. On the set $\{\lambda_n > 2\lambda_{n-1}\}$, we have $\lambda_n \leq 2(\lambda_n - \lambda_{n-1})$, hence

$$\sum_{n\geq 1} |b_n| \leq \sum_{n\geq 1} \lambda_n \leq 2 \sum_{n\geq 1} (\lambda_n - \lambda_{n-1}).$$

This implies that

$$\Big\| \sum_{n\geq 1} |b_n| \Big\|_E \leq 2 \, \|\lambda_\infty\|_E.$$

On the other hand, for $n \geq 1$,

$$\begin{aligned} |a_n|^2 \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}} &= (\lambda_n^2 - \lambda_{n-1}^2) \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}} = (\lambda_n + \lambda_{n-1})(\lambda_n - \lambda_{n-1}) \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}} \\ &\leq 3\lambda_{n-1} (\lambda_n - \lambda_{n-1}) \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}} \leq 3\lambda_{n-1} (\lambda_n - \lambda_{n-1}). \end{aligned}$$

Thus, we have the following estimates:

$$\sum_{n\geq 1} \mathbb{E}_{n-1}(|c_n|^2) \leq 3 \sum_{n\geq 1} \lambda_{n-1} \, \mathbb{E}_{n-1}(\lambda_n - \lambda_{n-1}) \leq 3\lambda_\infty \sum_{n\geq 1} \mathbb{E}_{n-1}(\lambda_n - \lambda_{n-1}).$$

It follows that

$$\Big\|\Big(\sum_{n\geq 1}\mathbb{E}_{n-1}(|c_n|^2)\Big)^{1/2}\Big\|_E \lesssim \|\lambda_\infty\|_E^{1/2}\Big\|\sum_{n\geq 1}\mathbb{E}_{n-1}(\lambda_n-\lambda_{n-1})\Big\|_E^{1/2},$$

and from the dual form of Doob maximal inequality, we conclude that

$$\Big\|\Big(\sum_{n\geq 1}\mathbb{E}_{n-1}(|c_n|^2)\Big)^{1/2}\Big\|_E \lesssim \|\lambda_\infty\|_E.$$

The proof is complete. ■

The next result shows that the Lépingle inequality follows from the dual form of the Doob maximal inequality.

Theorem 2.4. *If $E(\Omega)$ is a quasi-Banach Köthe function space that satisfies the dual form of the Doob maximal inequality, then it satisfies the Lépingle inequality.*

Proof. Let $(a_n)_{n\geq 1}$ be an adapted sequence in $E(\Omega;\ell_2)$. Consider the decomposition $a=b+c$ given by Proposition 2.3. Then

$$\Big\|\Big(\sum_{n\geq 1}|\mathbb{E}_{n-1}(a_n)|^2\Big)^{1/2}\Big\|_E \lesssim \Big\|\Big(\sum_{n\geq 1}|\mathbb{E}_{n-1}(b_n)|^2\Big)^{1/2}\Big\|_E + \Big\|\Big(\sum_{n\geq 1}|\mathbb{E}_{n-1}(c_n)|^2\Big)^{1/2}\Big\|_E.$$

We estimate the two terms on the right-hand side separately. For the first term,

$$\Big\|\Big(\sum_{n\geq 1}|\mathbb{E}_{n-1}(b_n)|^2\Big)^{1/2}\Big\|_E \leq \Big\|\sum_{n\geq 1}\mathbb{E}_{n-1}(|b_n|)\Big\|_E \lesssim \Big\|\sum_{n\geq 1}|b_n|\Big\|_E \lesssim \Big\|\Big(\sum_{n\geq 1}|a_n|^2\Big)^{1/2}\Big\|_E.$$

In the first inequality above, we have used the facts that ℓ_2-norm is smaller than ℓ_1-norm. The second inequality is the dual form the Doob maximal inequality. Now we consider the second term. Using the Jensen inequality for conditional expectations,

$$\Big\|\Big(\sum_{n\geq 1}|\mathbb{E}_{n-1}(c_n)|^2\Big)^{1/2}\Big\|_E \leq \Big\|\Big(\sum_{n\geq 1}\mathbb{E}_{n-1}(|c_n|^2)\Big)^{1/2}\Big\|_E \lesssim \Big\|\Big(\sum_{n\geq 1}|a_n|^2\Big)^{1/2}\Big\|_E.$$

We have the desired inequality by combining both estimates. ■

2.3. Real interpolations

Let us now review the basics of real interpolations. Assume that (A_0, A_1) is a compatible couple of quasi-Banach spaces in the sense that both A_0 and A_1 embed continuously into some topological vector space $\mathcal{Z}$. This allows us to define the spaces $A_0\cap A_1$ and A_0+A_1. These are quasi-Banach spaces when equipped with the quasi-norms

$$\|x\|_{A_0\cap A_1} = \max\big\{\|x\|_{A_0}, \|x\|_{A_1}\big\}$$

and

$$\|x\|_{A_0+A_1} = \inf\big\{\|x_0\|_{A_0}+\|x_1\|_{A_1} : x=x_0+x_1,\ x_0\in A_0,\ x_1\in A_1\big\},$$

respectively.

Definition 2.5. A quasi-Banach space A is called an *interpolation space* for the couple (A_0, A_1) if $A_0 \cap A_1 \subseteq A \subseteq A_0 + A_1$ and whenever a bounded linear operator $T\colon A_0 + A_1 \to A_0 + A_1$ is such that $T(A_0) \subseteq A_0$ and $T(A_1) \subseteq A_1$, we have $T(A) \subseteq A$ and

$$\|T : A \to A\| \le c \max\left\{\|T\colon A_0 \to A_0\|, \|T\colon A_1 \to A_1\|\right\}$$

for some constant c.

We are primarily interested in an interpolation method generally referred to as the real method. A fundamental notion for the construction of real interpolation spaces is the *K-functional* which we now describe. For $x \in A_0 + A_1$, we define the K-functional by setting for $t > 0$,

$$K(x,t) = K(x,t;A_0,A_1) = \inf\left\{\|x_0\|_{A_0} + t\,\|x_1\|_{A_1} : x = x_0 + x_1,\ x_0 \in A_0,\ x_1 \in A_1\right\}.$$

Note that for each $t > 0$, the functional $x \mapsto K(x,t)$ gives an equivalent quasi-norm on $A_0 + A_1$.

If $0 < \theta < 1$ and $0 < q < \infty$, we recall that the real interpolation space $A_{\theta,q} = (A_0, A_1)_{\theta,q}$ by $x \in A_{\theta,\gamma}$ if and only if

$$\|x\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(x,t;A_0,A_1)\right)^q \frac{dt}{t}\right)^{1/q} < \infty.$$

If $q = \infty$, we define $x \in A_{\theta,\infty}$ if and only if

$$\|x\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{t>0} t^{-\theta} K(x,t;A_0,A_1) < \infty.$$

For $0 < \theta < 1$ and $0 < q \le \infty$, the functional $\|\cdot\|_{\theta,q}$ is a quasi-norm and $(A_{\theta,\gamma}, \|\cdot\|_{\theta,q})$ is a quasi-Banach space. We have

$$A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta,q} \subseteq A_0 + A_1,$$

and it is an interpolation space of the couple (A_0, A_1) in the sense of Definition 2.5.

The following concept on K-functionals was formally introduced by Pisier in [28] and will be essential in the subsequent discussions.

Definition 2.6. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces and let B_0 (respectively, B_1) be a subspace of A_0 (respectively, A_1). The couple (B_0, B_1) is said to be *K-closed in the couple* (A_0, A_1) if there exists an absolute constant C such that for every $y \in B_0 + B_1$ and $t > 0$,

$$K(y,t;B_0,B_1) \le CK(y,t;A_0,A_1).$$

Since the reverse inequality is always valid (with constant 1), K-closedness means that K-functionals of the couples (B_0, B_1) and (A_0, A_1) are equivalent on $B_0 + B_1$ uniformly on $t > 0$. Clearly, if (B_0, B_1) is K-closed in (A_0, A_1), then for every indices $0 < \theta < 1$ and $0 < q \le \infty$,

$$(B_0, B_1)_{\theta,\gamma} = (B_0 + B_1) \cap (A_0, A_1)_{\theta,q}.$$

That is, one can deduce interpolation results for the couple (B_0, B_1) from the corresponding results on the larger couple. For more information and background on K-closed couples, we refer to [12, 18, 19, 28].

To conclude this section, we record two elementary but useful statements on K-functionals of couples of quasi-Banach Köthe function spaces and couples of Köthe–Bochner function spaces. We include both arguments for completeness.

Lemma 2.7. *Let $E(\Omega)$ be a quasi-Banach Köthe function space. For $f \in E(\Omega)$ and $t > 0$,*

$$K(f, t; E(\Omega), L_\infty(\Omega)) \approx \inf_{\mu>0} \left\{ \| |f| \chi_{\{|f|>\mu\}} \|_E + t\mu \right\}.$$

Proof. One inequality is trivial. Indeed, let $f \in E(\Omega)$ and $t > 0$. For $\mu > 0$, write

$$f = f_1 + f_2, \quad \text{with } f_1 = f \chi_{\{|f|>\mu\}} \text{ and } f_2 = f \chi_{\{|f|\leq\mu\}}.$$

Clearly, $\|f_2\|_\infty \leq \mu$. We have

$$K(f, t; E(\Omega), L_\infty(\Omega)) \leq \|f_1\|_E + t\mu = \| |f| \chi_{\{|f|>\mu\}} \|_E + t\mu.$$

Taking infimum over all $\mu > 0$, we have one inequality.

For the reverse inequality, take a decomposition $f = f_1 + f_2$ so that

$$2K`(f, t; E(\Omega), L_\infty(\Omega)) \geq \|f_1\|_E + t\|f_2\|_\infty.$$

Let $\gamma = \|f_2\|_\infty$. Since $(|f| - \gamma) \leq |f_1|$, taking the positive part, we have

$$|f_1| \geq (|f| - \gamma)^+ = (|f| - \gamma)\chi_{\{|f|>\gamma\}}.$$

Next, we estimate the last function from below:

$$\begin{aligned}
(|f| - \gamma)\chi_{\{|f|>\gamma\}} &= \sum_{j\geq 1} (|f| - \gamma)\chi_{\{2^{j-1}\gamma<|f|\leq 2^j\gamma\}} \\
&\geq \sum_{j\geq 2} (2^{j-1} - 1)\gamma\chi_{\{2^{j-1}\gamma<|f|\leq 2^j\gamma\}} \\
&\geq \sum_{j\geq 2} (2^{j-1} - 1)2^{-j}|f|\chi_{\{2^{j-1}\gamma<|f|\leq 2^j\gamma\}}.
\end{aligned}$$

The key point is that for $j \geq 2$, $(2^{j-1} - 1)2^{-j} \geq 1/4$. Thus, we obtain further that

$$|f_1| \geq (|f| - \gamma)\chi_{\{|f|>\gamma\}} \geq \frac{1}{4}|f|\chi_{\{|f|>2\gamma\}}.$$

Therefore, for $\mu = 2\gamma$, we have

$$2K(f, t; E(\Omega), L_\infty(\Omega)) \geq \frac{1}{4}\| |f| \chi_{\{|f|>\mu\}} \|_E + \frac{1}{2}t\mu.$$

This concludes that

$$\| |f| \chi_{\{|f|>\mu\}} \|_E + t\mu \leq 8K(f, t; E(\Omega), L_\infty(\Omega)).$$

This clearly implies the desired inequality and therefore concludes the proof. ∎

Lemma 2.8. *Let $E(\Omega)$ be a quasi-Banach Köthe function space and let X be a Banach space. For $v \in E(\Omega; X)$ and $t > 0$,*

$$K(v, t; E(\Omega; X), L_\infty(\Omega; X)) \approx \inf_{\mu>0} \big\{ \| \|v(\cdot)\|_X \chi_{\{\|v(\cdot)\|_X > \mu\}} \|_E + t\mu \big\}.$$

Proof. Let $v \in E(\Omega; X)$ and $t > 0$. Set

$$f = \|v(\cdot)\|_X.$$

For $\mu > 0$, write

$$v = v_1 + v_2, \quad \text{with } v_1 = v\chi_{\{f>\mu\}} \text{ and } v_2 = v\chi_{\{f\le\mu\}}.$$

Clearly, $\|v_2\|_{L_\infty(X)} \le \mu$ and

$$K(v, t; E(\Omega; X), L_\infty(\Omega; X)) \le \|v_1\|_{E(X)} + t\mu = \|f\chi_{\{f>\mu\}}\|_E + t\mu.$$

Taking infimum over $\mu > 0$, we have one inequality.

For the reverse inequality, let $\varepsilon > 0$, and fix a decomposition $v = v_1 + v_2$, with $v_1 \in E(\Omega; X)$ and $v_2 \in L_\infty(\Omega; X)$, and such that

$$\|v_1\|_{E(X)} + t\|v_2\|_{L_\infty(X)} \le K(v, t; E(\Omega; X), L_\infty(\Omega; X)) + \varepsilon.$$

Since

$$\|v(\cdot)\|_X \le \|v_1(\cdot)\|_X + \|v_2(\cdot)\|_X,$$

we have from Lemma 2.7 that

$$\begin{aligned}
&\inf_{\mu>0}\big\{ \| \|v(\cdot)\|_X \chi_{\{\|v(\cdot)\|_X>\mu\}} \|_E + t\mu \big\} \\
&\quad \le \inf_{\mu>0} \big\{ \|(\|v_1(\cdot)\|_X + \|v_2(\cdot)\|_X)\chi_{\{\|v_1(\cdot)\|_X + \|v_2(\cdot)\|_X > \mu\}}\|_E + t\mu \big\} \\
&\quad \lesssim K(\|v_1(\cdot)\|_X + \|v_2(\cdot)\|_X, t; E(\Omega), L_\infty(\Omega)) \le \| \|v_1(\cdot)\|_X\|_E + t\| \|v_2(\cdot)\|_X\|_\infty \\
&\quad = \|v_1\|_{E(X)} + t\|v_2\|_{L_\infty(X)} \le K(v, t; E(\Omega; X), L_\infty(\Omega; X)) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we have the desired estimate. ∎

3. Main results

In this section, we present two interpolation results on martingale Hardy spaces and conditional Hardy spaces. The following is our first principal result.

Theorem 3.1. *Let $E(\Omega)$ be a quasi-Banach Köthe function space satisfying the Lépingle inequality. Then the couple of martingale Hardy spaces $(H_E^S(\Omega), H_\infty^S(\Omega))$ is K-closed in the couple of Köthe–Bochner spaces $(E(\Omega, \ell_2), L_\infty(\Omega, \ell_2))$.*

For $E = L_1$, this is a commutative version of a recent result of Moyart ([25]). Our method of proof is constructive and is very different from [25]. If $E(\Omega)$ is a rearrangement invariant Banach function space, then it is an interpolation space of $(L_p(\Omega), L_q(\Omega))$ for some $1 \le p \le q \le \infty$. In this case, the statement can be deduced from the L_1-case using general reiteration of K-functionals. The main interest is for the case where $E(\Omega)$ is not a rearrangement invariant space. Motivating examples are presented in the next section.

According to Lemma 2.8, Theorem 3.1 is equivalent to the following.

Proposition 3.2. *Let $E(\Omega)$ be as in Theorem 3.1. For $f \in H_E^S(\Omega)$ and $t > 0$,*

$$K(f, t; H_E^S(\Omega), H_\infty^S(\Omega)) \lesssim \inf_{\mu>0} \big\{\|S(f)\chi_{\{S(f)>\mu\}}\|_E + t\mu\big\}.$$

Proof. Fix $\mu > 0$. For $k \geq 1$, set

$$U_k := \{S_k(f) \leq \mu\}.$$

Then, U_k is $\mathcal{F}_k$-measurable and the sequence $(U_k)_{k\geq 1}$ is decreasing, with

$$U = \bigcap_{k\geq 1} U_k = \{S(f) \leq \mu\}.$$

Define a martingale g by setting for $k \geq 1$,

$$dg_k = df_k\,\chi_{U_k} - \mathbb{E}_{k-1}(df_k\,\chi_{U_k}).$$

Clearly, $(dg_k)_{k\geq 1}$ is a martingale difference sequence that induces the martingale g. We will need the following properties of g.

Lemma 3.3. (1) *For $k \geq 1$, $|dg_k| \leq 2\mu$.*

(2) $\Big\|\sum_{k\geq 1} |df_k|^2 \chi_{U_k}\Big\|_\infty \leq \mu^2.$

Proof of Lemma 3.3. The first item clearly follows from

$$|df_k\,\chi_{U_k}|^2 \leq S_k^2(f)\chi_{U_k} \leq \mu^2$$

and the fact that $\mathbb{E}_{k-1}$ is a contractive projection on $L_\infty(\Omega)$.

For the second item, we write, for a given $n \geq 1$,

$$\sum_{k=1}^n |df_k|^2\chi_{U_k} = \sum_{k=1}^n (S_k^2(f) - S_{k-1}^2(f))\chi_{U_k} = \sum_{k=1}^n S_k^2(f)\chi_{U_k} - \sum_{k=1}^n S_{k-1}^2(f)\chi_{U_k}.$$

By performing an indexing shift on the second summation, we can rewrite the preceding identity in the following form:

$$\sum_{k=1}^n |df_k|^2\chi_{U_k} = S_n^2(f)\chi_{U_n} + \sum_{k=1}^{n-1} S_k^2(f)(\chi_{U_k} - \chi_{U_{k+1}}).$$

From the definition of the sets U_n's, we have

$$\sum_{k=1}^n |df_k|^2\chi_{U_k} \leq \mu^2\chi_{U_n} + \mu^2\sum_{k=1}^{n-1}(\chi_{U_k} - \chi_{U_{k+1}}) \leq \mu^2.$$

Since n is arbitrary, we have

$$\sum_{k\geq 1} |df_k|^2\chi_{U_k} \leq \mu^2,$$

proving the second item of the lemma. ∎

Next, for $k \geq 1$, we set

$$R_k := \{S_k(g) \leq 2\mu\}.$$

As before, R_k is $\mathcal{F}_k$-measurable and the sequence $(R_k)_{k\geq 1}$ is decreasing with

$$R = \bigcap_{k\geq 1} R_k = \{S(g) \leq 2\mu\}.$$

We define two martingales $f^{(1)}$ and $f^{(2)}$ by setting for $k \geq 1$,

$$\begin{aligned} df_k^{(1)} &:= dg_k\, \chi_{R_{k-1}}; \\ df_k^{(2)} &:= df_k - df_k^{(1)}. \end{aligned} \tag{3.1}$$

Note that since R_{k-1} is $\mathcal{F}_{k-1}$-measurable, $(df_k^{(1)})_{k\geq 1}$ is a martingale difference sequence and therefore $f^{(1)}$ is a well-defined martingale. Moreover,

$$f = f^{(1)} + f^{(2)}.$$

We will estimate the K-functional using this decomposition.

We will show first that

$$\|f^{(1)}\|_{H_\infty^S} \leq \sqrt{8}\,\mu.$$

For this, we built on the idea used in the proof of Lemma 3.3. For a given $n \geq 1$, we have

$$\begin{aligned} S_n^2(f^{(1)}) &= \sum_{k=1}^n |df_k^{(1)}|^2 = \sum_{k=1}^n |dg_k|^2 \chi_{R_{k-1}} = \sum_{k=1}^n (S_k^2(g) - S_{k-1}^2(g))\, \chi_{R_{k-1}} \\ &= \sum_{k=1}^n S_k^2(g)\, \chi_{R_{k-1}} - \sum_{k=1}^n S_{k-1}^2(g)\, \chi_{R_{k-1}}. \end{aligned}$$

After an indexing shift on the second sum,

$$S_n^2(f^{(1)}) = S_n^2(g)\, \chi_{R_{n-1}} + \sum_{k=1}^{n-1} S_k^2(g)(\chi_{R_{k-1}} - \chi_{R_k}).$$

We remark that from the first item of Lemma 3.3, we have for $1 \leq k \leq n$,

$$S_k^2(g)\chi_{R_{k-1}} = [|dg_k|^2 + S_{k-1}^2(g)]\, \chi_{R_{k-1}} \leq [4\mu^2 + S_{k-1}^2(g)]\, \chi_{R_{k-1}} \leq 8\mu^2.$$

With this estimate, we have

$$S_n^2(f^{(1)}) \leq 8\mu^2 \chi_{R_{n-1}} + 8\mu^2 \sum_{k=1}^{n-1} (\chi_{R_{k-1}} - \chi_{R_k}) \leq 8\mu^2.$$

Since n is arbitrary, we have $S^2(f^{(1)}) \leq 8\mu^2$, which is equivalent to

$$\|f^{(1)}\|_{H_\infty^S} \leq \sqrt{8}\mu.$$

Next, we will estimate $\|f^{(2)}\|_{H_E^S}$.

First, fix $k \geq 1$. From the definition,

$$\begin{aligned} df_k^{(2)} &= df_k - \big[df_k\,\chi_{U_k} - \mathbb{E}_{k-1}(df_k\chi_{U_k})\big]\chi_{R_{k-1}} \\ &= df_k\chi_{\Omega\setminus U_k} + df_k\,\chi_{U_k}\chi_{\Omega\setminus R_{k-1}} + \mathbb{E}_{k-1}(df_k\chi_{U_k})\,\chi_{R_{k-1}} \\ &= df_k\chi_{\Omega\setminus U_k} + df_k\,\chi_{U_k}\chi_{\Omega\setminus R_{k-1}} - \mathbb{E}_{k-1}(df_k\chi_{U_{k-1}\setminus U_k})\,\chi_{R_{k-1}}. \end{aligned}$$

Then, we have the initial estimate

$$|df_k^{(2)}|^2 \lesssim |df_k|^2\chi_{\Omega\setminus U} + |df_k|^2\chi_{U_k}\chi_{\Omega\setminus R} + |\mathbb{E}_{k-1}(df_k\,\chi_{U_{k-1}\setminus U_k})|^2.$$

Taking summations over all $k \geq 1$, we have

$$S^2(f^{(2)}) \lesssim S^2(f)\,\chi_{\Omega\setminus U} + \sum_{k\geq 1}|df_k|^2\chi_{U_k}\chi_{\Omega\setminus R} + \sum_{k\geq 1}|\mathbb{E}_{k-1}(df_k\,\chi_{U_{k-1}\setminus U_k})|^2.$$

We recall from Lemma 3.3 that

$$\sum_{k\geq 1}|df_k|^2\chi_{U_k} \leq \mu^2.$$

This implies further that

$$S^2(f^{(2)}) \lesssim S^2(f)\,\chi_{\Omega\setminus U} + \mu^2\chi_{\Omega\setminus R} + \sum_{k\geq 1}|\mathbb{E}_{k-1}(df_k\,\chi_{\Omega\setminus U}\,\chi_{U_{k-1}\setminus U_k})|^2. \tag{3.2}$$

On the other hand, since

$$dg_k = df_k\chi_{U_k} - \mathbb{E}_{k-1}(df_k\chi_{U_k}) = df_k\,\chi_{U_k} + \mathbb{E}_{k-1}(df_k\,\chi_{U_{k-1}\setminus U_k}),$$

we have

$$|dg_k|^2 \leq 2|df_k|^2 + 2|\mathbb{E}_{k-1}(df_k\,\chi_{U_{k-1}\setminus U_k})|^2.$$

This implies that

$$S^2(g) \leq 2S^2(f) + 2\sum_{k\geq 1}|\mathbb{E}_{k-1}(df_k\,\chi_{U_{k-1}\setminus U_k})|^2.$$

Denote

$$W(f) := \Big(\sum_{k\geq 1}|\mathbb{E}_{k-1}(df_k\,\chi_{U_{k-1}\setminus U_k})|^2\Big)^{1/2}.$$

The preceding inequality on $S^2(g)$ gives

$$\Omega \setminus R = \{S^2(g) \geq 4\mu^2\} \subseteq \{S^2(f) \geq \mu^2\} \cup \{(W(f))^2 \geq \mu^2\}.$$

Using this inclusion, we further get from (3.2) that

$$\begin{aligned} S^2(f^{(2)}) &\lesssim S^2(f)\,\chi_{\Omega\setminus U} + \mu^2\chi_{\{S^2(f)\geq\mu^2\}} + \mu^2\chi_{\{(W(f))^2\geq\mu^2\}} + (W(f))^2 \\ &\lesssim 2S^2(f)\,\chi_{\Omega\setminus U} + 2(W(f))^2. \end{aligned}$$

Therefore, we have

$$S(f^{(2)}) \lesssim S(f)\chi_{\Omega\setminus U} + W(f).$$

Taking quasi-norms in $E(\Omega)$,

$$\|f^{(2)}\|_{H_E^S} \lesssim \|S(f)\chi_{\Omega\setminus U}\|_E + \|W(f)\|_E. \tag{3.3}$$

Observe that $(df_k\chi_{U_{k-1}\setminus U_k})_{k\geq 1}$ is an adapted sequence. By the assumption that Lépingle inequality is valid for $E(\Omega)$, we deduce that

$$\begin{aligned}\|W(f)\|_E &= \Big\|\Big(\sum_{k\geq 1}|\mathbb{E}_{k-1}(df_k\chi_{U_{k-1}\setminus U_k})|^2\Big)^{1/2}\Big\|_E \lesssim \Big\|\Big(\sum_{k\geq 1}|df_k|^2\chi_{U_{k-1}\setminus U_k}\Big)^{1/2}\Big\|_E \\ &\lesssim \Big\|\Big(\sum_{k\geq 1}S^2(f)\chi_{U_{k-1}\setminus U_k}\Big)^{1/2}\Big\|_E \approx \|S(f)\chi_{\Omega\setminus U}\|_E.\end{aligned}$$

Thus, (3.3) gives the estimate

$$\|f^{(2)}\|_{H_E^S} \lesssim \|S(f)\chi_{\Omega\setminus U}\|_E.$$

We can now deduce that

$$K(f,t;H_E^S(\Omega),H_\infty^S(\Omega)) \leq \|f^{(2)}\|_{H_E^S} + t\|f^{(1)}\|_{H_\infty^S} \lesssim \|S(f)\chi_{\{S(f)>\mu\}}\|_E + t\mu.$$

By taking infimum over all $\mu > 0$, we arrive at the desired conclusion. ∎

The following interpolation result is now immediate.

Corollary 3.4. *Let $E(\Omega)$ be a quasi-Banach Köthe function space satisfying the Lépingle inequality, and let $0 < \theta < 1$ and $0 < q \leq \infty$. If $F(\Omega) = (E(\Omega), L_\infty(\Omega))_{\theta,q}$, then*

$$(H_E^S(\Omega), H_\infty^S(\Omega))_{\theta,q} = H_F^S(\Omega).$$

We should note that in the statement above, $F(\Omega)$ as a linear subset of the function space $E(\Omega) \subset L_0(\Omega)$ is a quasi-Banach Köthe function space. This can be easily seen from the definition of interpolation quasi-norms.

As expected, the automatic lifting also applies to more general interpolation functor arising from K-functionals. Recall that a quasi-Banach Köthe function space Φ on $(0,\infty)$ is called a K- *method parameter* if it contains the function $t \mapsto \min(1,t)$. If Φ is a K-method parameter, then for every quasi-Banach couple (E_0, E_1), the set

$$K_\Phi(E_0,E_1) := \{x \in E_0 + E_1 : t \mapsto K(x,t;E_0,E_1) \in \Phi\}$$

is a linear subset of $E_0 + E_1$ and is a quasi-Banach space under the quasi-norm

$$\|x\|_{K_\Phi(E_0,E_1)} = \|t \mapsto K(x,t)\|_\Phi.$$

This yields an exact interpolation functor K_Φ called the K*-method* with parameter Φ. As in Corollary 3.4, Theorem 3.1 implies the following more general statement.

Remark 3.5. Let $E(\Omega)$ be a quasi-Banach Köthe function space satisfying the Lépingle inequality. If Φ is a K-method parameter and $F(\Omega) = K_\Phi(E(\Omega), L_\infty(\Omega))$, then

$$K_\Phi(H_E^S(\Omega), H_\infty^S(\Omega)) = H_F^S(\Omega).$$

Our next result is for conditional Hardy spaces. It provides an automatic lifting from quasi-Banach Köthe function spaces to corresponding conditional Hardy spaces without any restrictions.

Theorem 3.6. *Let $E(\Omega)$ be a quasi-Banach Köthe function space and let Φ be a K-parameter. If $F(\Omega) = K_\Phi(E(\Omega), L_\infty(\Omega))$, then*

$$K_\Phi(H_E^s(\Omega), H_\infty^s(\Omega)) = H_F^s(\Omega).$$

The special case of real interpolations of conditional martingale Hardy space associated with Musielak–Orlicz spaces can be found in Theorem 3.1 of [24] (see the next section for definitions). We also refer to Theorem 1.1 in [14] for the case of real interpolations of variable martingale conditional Hardy spaces. Our proof is much simpler than those of [14, 24], taking advantage of the idea used for Theorem 3.1.

As in the proof of Theorem 3.1, it will be deduced from the following lemma.

Lemma 3.7. *Let $E(\Omega)$ be as in Theorem 3.6. For $f \in H_E^s$ and $t > 0$,*

$$K(f, t; H_E^s(\Omega), H_\infty^s(\Omega)) \leq \inf_{\mu>0} \big\{\|s(f)\chi_{\{s(f)>\mu\}}\|_E + t\mu\big\}.$$

Proof. The proof follows the idea of Proposition 3.2, but is much simpler and shorter since it does not require two steps. We include the details for completeness.

Fix $\mu > 0$. For $k \geq 1$, set

$$Q_k = \{s_k(f) \leq \mu\}.$$

Then, Q_k is $\mathcal{F}_{k-1}$-measurable and $(Q_k)_{k\geq 1}$ is decreasing with

$$Q = \bigcap_{k\geq 1} Q_k = \{s(f) \leq \mu\}.$$

Define two martingales g and h by setting for $k \geq 1$,

$$\begin{aligned} dg_k &:= df_k\,\chi_{Q_k}; \\ dh_k &:= df_k\,\chi_{\Omega\setminus Q_k}. \end{aligned} \tag{3.4}$$

Clearly, $(dg_k)_{k\geq 1}$ and $(dg_k)_{k\geq 1}$ are martingale difference sequences and $f = g + h$.

We will show that

$$\|g\|_{H_\infty^s} \leq \mu.$$

The verification of this is a simple adaptation of Lemma 3.3. Indeed, for a given $n \geq 1$, we have

$$\begin{aligned} s_n^2(g) &= \sum_{k=1}^n \mathbb{E}_{k-1}(|dg_k|^2) = \sum_{k=1}^n \mathbb{E}_{k-1}(|df_k|^2)\chi_{Q_k} = \sum_{k=1}^n (s_k^2(f) - s_{k-1}^2(f))\chi_{Q_k} \\ &= \sum_{k=1}^n s_k^2(f)\chi_{Q_k} - \sum_{k=1}^n s_{k-1}^2(f)\chi_{Q_k}. \end{aligned}$$

As in the proof of Proposition 3.2, this leads to

$$s_n^2(g) = s_n^2(f)\chi_{Q_n} + \sum_{k=1}^{n-1} s_k^2(f)(\chi_{Q_k} - \chi_{Q_{k+1}}).$$

From the definition of the sets Q_n's, we have

$$s_n^2(g) \le \mu^2\chi_{Q_n} + \mu^2\sum_{k=1}^{n-1}(\chi_{Q_k} - \chi_{Q_{k+1}}) \le \mu^2.$$

Since n is arbitrary, we have $s^2(g) \le \mu^2$ and thus proving that

$$\|g\|_{H_\infty^s} \le \mu.$$

Next, we will estimate $\|h\|_{H_E^s}$. This is done using the following simple estimate of its conditional square function:

$$s^2(h) = \sum_{k\ge 1}\mathbb{E}_{k-1}(|df_k|^2)\chi_{\Omega\backslash Q_k} \le \sum_{k\ge 1}\mathbb{E}_{k-1}(|df_k|^2)\chi_{\Omega\backslash Q} = s^2(f)\chi_{\Omega\backslash Q}.$$

This implies that

$$\|h\|_{H_E^s} \le \|s(f)\chi_{\{s(f)>\mu\}}\|_E.$$

We can now use the decomposition $f = g + h$ to deduce that

$$K(f,t;H_E^s(\Omega),H_\infty^s(\Omega)) \le \|h\|_{H_E^s} + t\|g\|_{H_\infty^s} \le \|s(f)\chi_{\{s(f)>\mu\}}\|_E + t\mu.$$

Since $\mu > 0$ is arbitrary, the lemma is verified. ∎

Proof of Theorem 3.6. We will verify first that

$$K_\Phi(H_E^s(\Omega),H_\infty^s(\Omega)) \subseteq H_F^s(\Omega).$$

Consider the mapping $T\colon f \mapsto s(f)$. By definitions,

$$T\colon H_E^s(\Omega) \to E(\Omega) \quad \text{and} \quad T\colon H_\infty^s(\Omega) \to L_\infty(\Omega)$$

are isometries. Moreover, one can easily verify that it is sublinear. By Theorem 5.2.3 in [2] and the definition of $F(\Omega)$, we have

$$\|f\|_{H_F^s} = \|s(f)\|_F = \|T(f)\|_F \approx \|Tf\|_{K_\Phi(E,L_\infty)} \lesssim \|f\|_{K_\Phi(H_E^s,H_\infty^s)}.$$

For the reverse inclusion, we note that for given $f \in H_F^s(\Omega)$, Lemma 3.7 while combined with Lemma 2.7 can be reformulated as:

$$K(f,t;H_E^s(\Omega),H_\infty^s(\Omega)) \lesssim K(s(f),t;E(\Omega),L_\infty(\Omega)), \quad t > 0.$$

It follows that

$$\|f\|_{K_\Phi(H_E^s,H_\infty^s)} \lesssim \|s(f)\|_{K_\Phi(E,L_\infty)} \approx \|s(f)\|_F = \|f\|_{H_F^s}.$$

This completes the proof. ∎

We remark that Lemma 3.7 is in spirit a form of K-closedness. However, we could not formulate it as such since the embedding of conditioned spaces developed by Junge in [16] even in the case of conditional expectation acting on function spaces appears to involve noncommutative von Neumann algebras. But, at the time of this writing, theory of noncommutative spaces associated with non-symmetric spaces such as variable Lebesgue spaces is not available.

4. Application to concrete function spaces

4.1. Musielak–Orlicz spaces

A function $\Phi\colon [0,\infty) \to [0,\infty]$ is called an *Orlicz function* if it is increasing, $\Phi(0) = 0$, $\lim_{t\to 0^+} \Phi(t) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$.

For $p_0, p_1 \in (0,\infty)$, we say that an Orlicz function Φ satisfies:

$(\mathrm{aInc})_{p_0}$ if there exists a constant $C \geq 1$ such that

$$\Phi(st) \leq C s^{p_0} \Phi(t), \qquad \forall s \in (0,1), t \in (0,\infty);$$

$(\mathrm{aDec})_{p_1}$ if there exists a constant $C \geq 1$ such that

$$\Phi(st) \leq C s^{p_1} \Phi(t), \qquad \forall s \in (1,\infty), t \in (0,\infty).$$

Definition 4.1. A function $\varphi\colon \Omega \times [0,\infty) \to [0,\infty]$ is called a *Musielak–Orlicz* function if

(i) $w \mapsto \varphi(w, |f(w)|)$ is measurable for every $f \in L_0(\Omega)$;

(ii) for a.e $w \in \Omega$, $t \mapsto \varphi(w,t)$ is an Orlicz function.

We say that φ satisfies $(\mathrm{aInc})_{p_0}$ (respectively, $(\mathrm{aDec})_{p_1}$) if there is a constant $C \geq 1$ so that for a.e $w \in \Omega$, the function $t \mapsto \varphi(w,t)$ satisfies $(\mathrm{aInc})_{p_0}$ (respectively, $(\mathrm{aDec})_{p_1}$) with the constant C.

Definition 4.2. The *Musielak–Orlicz space* $L_\varphi(\Omega)$ is defined to be the collection of all function $f \in L_0(\Omega)$ such that

$$\|f\|_{L_\varphi} := \inf\Big\{\lambda \in (0,\infty) : \int_\Omega \varphi\Big(w, \frac{|f(w)|}{\lambda}\Big)\, d\mathbb{P}(w) \leq 1\Big\} < \infty.$$

One can easily see that $L_\varphi(\Omega)$ is a Köthe function space. Moreover, if the Musielak–Orlicz function φ satisfies $(\mathrm{aInc})_{p_0}$ for some $p_0 \in (0,\infty)$, then $\|\cdot\|_{L_\varphi}$ is a quasi-norm on $L_\varphi(\Omega)$ and $(L_\varphi(\Omega), \|\cdot\|_{L_\varphi})$ is then a quasi-Banach Köthe function space. We refer to [9, 26, 27] for more information.

For $q \in (0,\infty)$, the *Musielak–Orlicz Lorentz space* $L_{\varphi,q}(\Omega)$ is defined to be the set all functions f such that

$$\|f\|_{L_{\varphi,q}} := \begin{cases} \Big(\displaystyle\int_0^\infty \lambda^q \|\chi_{\{|f|>\lambda\}}\|_{L_\varphi}^q \frac{d\lambda}{\lambda}\Big)^{1/q}, & q < \infty, \\ \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{L_\varphi}, & q = \infty, \end{cases}$$

is finite. As in the case of $L_\varphi(\Omega)$, if the Musielak–Orlicz function φ satisfies $(\mathrm{aInc})_{p_0}$ for some $p_0 \in (0,\infty)$, then $L_{\varphi,q}(\Omega)$ is a quasi-Banach Köthe function space.

The Musielak–Orlicz Lorentz spaces are general framework that encompass various classes of function spaces. Some concrete examples are:

(i) $\varphi(w,t) = \Phi(t)$ where Φ is an Orlicz function. Then, $L_\varphi(\Omega)$ (respectively, $L_{\varphi,q}(\Omega)$) is the Orlicz (respectively, Orlicz–Lorentz) space $L_\Phi(\Omega)$ (respectively, $L_{\Phi,q}(\Omega)$);

(ii) $\varphi(w,t) = t^{p(w)}$ for some measurable function $p\colon \Omega \to (0,\infty)$. Then, $L_\varphi(\Omega)$ is the variable Lebesgue space $L_{p(\cdot)}(\Omega)$;

(iii) $\varphi(w,t) = \Phi(t^{p(w)})$. Then, $L_\varphi(\Omega)$ is the variable Orlicz space.

(iv) $\varphi(w,t) := t^{p(w)} + v(w)t^{q(w)}$ where $p(\cdot)$ and $q(\cdot)$ are measurable function and $v(\cdot)$ is a non-negative integrable function. Then, $L_\varphi(\Omega)$ is the Lebesgue space for double phase functionals with variable exponents.

It is worth pointing out that in general, Musielak–Orlicz spaces and Musielak–Orlicz Lorentz spaces are not necessarily rearrangement invariant spaces.

Real interpolations of variable Lebesgue spaces and variable Lorentz spaces have been well studied in the literature. We refer to [17] for more background. These results on variable spaces have been generalized to the more general setting of Musielak–Orlicz spaces and Musielak–Orlicz Lorentz spaces. We record one here for further use.

Proposition 4.3 (Lemma 2.7 in [24]). *Let φ be a Musielak–Orlicz function satisfying* $(\mathrm{aInc})_{p_0}$ *and* $(\mathrm{aDec})_{p_1}$ *for some $p_0, p_1 \in (0,\infty)$, $0 < q \le \infty$, $0 < \theta < \infty$, and*

$$\varphi_{1/1-\theta}(w,t) := \varphi(w, t^{1/1-\theta}).$$

Then, the following holds:

$$(L_\varphi(\Omega), L_\infty(\Omega))_{\theta,q} = L_{\varphi_{1/1-\theta},q}(\Omega).$$

In order to take full advantage of the general approach in the previous section, we will only work with the natural situation where the Doob maximal inequality or its dual form is valid for the space $L_\varphi(\Omega)$. For this, we assume that for every $n \ge 1$, the σ-algebra $\mathcal{F}_n$ is generated by countably many atoms. Recall that $A \in \mathcal{F}_n$ is called an atom, if for any $B \subset A$ with $B \in \mathcal{F}_n$ satisfying $\mathbb{P}(B) < \mathbb{P}(A)$, we have $\mathbb{P}(B) = 0$. We denote by $\mathcal{A}(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. It is clear that for $f \in L_1(\Omega)$, we have

$$\mathbb{E}_n(f) = \sum_{A \in \mathcal{A}(\mathcal{F}_n)} \Big(\frac{1}{\mathbb{P}(A)} \int_A f \, d\mathbb{P}\Big) \chi_A, \quad n \in \mathbb{N}.$$

We refer to [34] for examples of filtrations satisfying the above assumption.

Recall that for a Musielak–Orlicz function φ, its left inverse φ^{-1} is given by

$$\varphi^{-1}(w,t) := \inf\{u \in [0,\infty) : \varphi(w,u) \ge t\}.$$

We now consider the following two conditions on φ:

(A0) There exists a constant $\beta \in (0,1]$ such that for a.e. $w \in \Omega$,

$$\beta \le \varphi^{-1}(w,1) \le 1/\beta;$$

(A1) There exists a constant $\beta \in (0,1)$ such that

$$\beta\varphi^{-1}(w_1,t) \le \varphi^{-1}(w_2,t)$$

for any $A \in \bigcup_{n\ge1} \mathcal{A}(\mathcal{F}_n)$, $t \in [1, 1/\mathbb{P}(A)]$, and almost every $w_1, w_2 \in A$.

Our initial motivation is the case of variable martingales. For this special case, recall that a measurable function $p\colon \Omega \to (0, \infty)$ is called a *variable exponent*. For $A \in \mathcal{F}$, we denote

$$p_+(A) := \operatorname*{ess\,sup}_{w \in A} p(w), \quad p_-(A) := \operatorname*{ess\,inf}_{w \in A} p(w)$$

and

$$p_+ := p_+(\Omega), \quad p_- := p_-(\Omega).$$

Denote by $\mathcal{P}(\Omega)$ the collection of all variable exponents $p(\cdot)$ such that

$$0 < p_- \leq p_+ \leq \infty.$$

We refer to the monograph [4] for more background on variable Lebesgue spaces.

In this special case, the conditions (A0) and (A1) are equivalent to the probabilistic version of the log-Hölder continuity condition on the exponent considered in [15] and [34]. This is formulated as follows: there exists a constant $K_{p(\cdot)} \geq 1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(A)^{p_-(A)-p_+(A)} \leq K_{p(\cdot)}, \quad \forall A \in \bigcup_{n\geq 1} \mathcal{A}(\mathcal{F}_n). \tag{4.1}$$

We refer to [34] for further details. Our main interests in the above conditions are the validity of the Doob maximal inequality and its dual form taken from [10].

Theorem 4.4 (Theorem 1.1 in [10]). *Let φ be a Musielak–Orlicz function satisfying* (A0)*,* (A1)*, and* $(\mathrm{aInc})_p$ *with $p \in (1, \infty)$. Then, there exists a positive constant $C > 0$ so that for every $f \in L_\varphi(\Omega)$,*

$$\Big\| \sup_{n\geq 1} |\mathbb{E}_n(f)| \Big\|_{L_\varphi} \leq C \|f\|_{L_\varphi}.$$

The dual form of the Doob maximal inequality is also valid under similar type conditions.

Theorem 4.5 (Theorem 3.8 in [10]). *Let φ be a Musielak–Orlicz function satisfying* $(\mathrm{aInc})_{p_0}$ *and* $(\mathrm{aDec})_{p_1}$ *with $1 \leq p_0 \leq p_1 < \infty$. If φ satisfies both* (A0) *and* (A1)*, then there exists a positive constant C such that for any sequence of non-negative functions $(f_n)_{n\geq 1}$ in $L_\varphi(\Omega)$,*

$$\Big\| \sum_{n\geq 1} \mathbb{E}_n(f_n) \Big\|_{L_\varphi} \leq C \Big\| \sum_{n\geq 1} f_n \Big\|_{L_\varphi}.$$

According to Theorem 2.4, we may deduce the following statement for the Lépingle inequality on Musielak–Orlicz spaces.

Proposition 4.6. *Let φ be a Musielak–Orlicz function satisfying* $(\mathrm{aInc})_{p_0}$ *and* $(\mathrm{aDec})_{p_1}$ *with $1 \leq p_0 \leq p_1 < \infty$. If φ satisfies both* (A0) *and* (A1)*, then $L_\varphi(\Omega)$ satisfies the Lépingle inequality.*

Below, we use the notation $H_\varphi^S(\Omega)$ and $H_{\varphi,q}^S(\Omega)$ for martingale Hardy spaces associated with $L_\varphi(\Omega)$ and $L_{\varphi,q}(\Omega)$ respectively.

As an immediate consequence of Theorem 3.1, Proposition 4.3, and Proposition 4.6, we obtain the following.

Corollary 4.7. *Let φ be a Musielak–Orlicz function satisfying* $(\mathrm{aInc})_{p_0}$ *and* $(\mathrm{aDec})_{p_1}$ *with* $1 \leq p_0 \leq p_1 < \infty$. *If φ satisfies both* (A0) *and* (A1)*, then*

(i) $(H_\varphi^S(\Omega), H_\infty^S(\Omega))$ *is K-closed in* $(L_\varphi(\Omega; \ell_2), L_\infty(\Omega; \ell_2))$;

(ii) *the following holds*:

$$(H_\varphi^S(\Omega), H_\infty^S(\Omega))_{\theta,q} = H^S_{\varphi_{1/1-\theta},q}(\Omega).$$

Corresponding results for variable martingale Hardy spaces can be stated under the condition (4.1). This should be compared with interpolation results for couples of conditional Hardy spaces from [6, 14, 23, 24].

5. Concluding remark

A closely related to the Lépingle inequality is the Stein inequality, which is originally stated for L_p-spaces when $1 < p < \infty$. Stein's inequality plays a significant role in martingale theory. We refer to [1, 31] for more on Stein's inequality. For variable Lebesgue space $L_{p(\cdot)}$, Stein's inequality was obtained in [15] under the condition (4.1), but also subjected to the extra assumption $1 \leq r < p_- \leq p_+ < \infty$. Thus, it is a natural question to consider if it has an extension to the general context of Banach Köthe function spaces. In the next result, we show that it is in fact a consequence of the Doob maximal inequality and its dual form. This may be of independent interest.

Theorem 5.1. *Assume that $E(\Omega)$ is a σ-order continuous Banach Köthe function space such that both the Doob maximal inequality and its dual form are satisfied in $E(\Omega)$. For given $1 < r < \infty$, there exists a constant C such that for every sequence $(\theta_n)_{n\geq 1}$ in $E(\Omega)$,*

$$\Big\| \Big(\sum_{n\geq 1} |\mathbb{E}_n(\theta_n)|^r \Big)^{1/r} \Big\|_E \leq C \Big\| \Big(\sum_{n\geq 1} |\theta_n|^r \Big)^{1/r} \Big\|_E.$$

Proof. Consider the operator T that takes every sequence $(\theta_n)_{n\geq 1}$ in the Köthe–Bochner space $E(\Omega; \ell_1)$ into the sequence $(\mathbb{E}_n(\theta_n))_{n\geq 1} \subset E(\Omega)$. Then from the dual form of the Doob maximal inequality, we have

$$\Big\| \sum_{n\geq 1} |\mathbb{E}_n(\theta_n)| \Big\|_E \leq \Big\| \sum_{n\geq 1} \mathbb{E}_n(|\theta_n|) \Big\|_E \lesssim \Big\| \sum_{n\geq 1} |\theta_n| \Big\|_E.$$

This translates to $T\colon E(\Omega; \ell_1) \to E(\Omega; \ell_1)$ being a bounded linear operator. Similarly, if $(a_n)_{n\geq 1}$ is a sequence in $E(\Omega)$ such that $\xi = \sup_{n\geq 1} |a_n| \in E(\Omega)$, then

$$\| \sup_{n\geq 1} |\mathbb{E}_n(a_n)| \|_E \leq \| \sup_{n\geq 1} |\mathbb{E}_n(\xi)| \|_E \lesssim \|\xi\|_E$$

where in the second inequality we have used the Doob maximal inequality. This means that $T\colon E(\Omega; \ell_\infty) \to E(\Omega; \ell_\infty)$ is bounded.

Now we appeal to complex interpolation of Köthe–Bochner spaces. By assumption, $E(\Omega)$ is a σ-order continuous function space. According to [3], p. 125, for $0 < \theta < 1$, the following holds isometrically:

$$[E(\Omega;\ell_1), E(\Omega;\ell_\infty)]_\theta = E(\Omega;[\ell_1,\ell_\infty]_\theta).$$

Here, $[\cdot,\cdot]_\theta$ denotes the complex interpolation with exponent θ. Taking $\theta = 1 - r^{-1}$, we have by interpolation that $T\colon E(\Omega;\ell_r) \to E(\Omega;\ell_r)$ is bounded, which implies the needed inequality. The proof is complete. ∎

Examples of spaces satisfying the assumptions of Theorem 5.1 are the Musielak–Orlicz spaces $L_\varphi(\Omega)$ where φ satisfies (A0), (A1), (aInc)$_{p_0}$, and (aDec)$_{p_1}$ with $1 < p_0 \le p_1 < \infty$.

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