Rev. Mat. Iberoam. 42 (2026), no. 3, 1189–1200
DOI 10.4171/RMI/1605

Mather–Yau type theorem for higher Nash blowup algebras

Hong Duc Nguyen

Abstract. In this paper, we establish a Mather–Yau theorem for higher Nash blowup algebras, demonstrating that the isomorphism type of the local ring of any hypersurface singularity, defined over an arbitrary field, is fully determined by its higher Nash blowup algebras. The classical Mather–Yau theorem (1982) asserts that for isolated complex hypersurface singularities, the isomorphism type of the local ring is determined by the Tjurina algebra. In positive characteristic, this result was extended by considering the higher Tjurina algebras by Greuel and Pham (2017) under the assumptions of an algebraically closed ground field and isolated singularities. Our work begins by proving the stability of higher Nash blowup algebras under contact equivalence in a very general framework. Specifically, we show that the higher Nash blowup algebras of any system of elements in an analytic or geometric ring remain invariant under contact equivalence. For complex hypersurface singularities, this stability was conjectured by Hussain, Ma, Yau, and Zuo, and was recently verified by Le and Yasuda. Finally, the converse is established using a classical result of Samuel (1956).

Dedicated to Professor Ha Huy Vui on the occasion of his 75th birthday

1. Introduction

We study in this paper higher Nash blowup algebras. The study of higher Nash blowup algebras was motivaed by the notion of higher Nash blowup, which was introduced by Yasuda [17] to study the problem of desingularization of varieties. They are closely related to higher Jacobian matrices and Jacobian ideals, which are fundamental objects in the study of singularities of varieties as well as singularities of morphisms [4–6].

In their paper [11], Hussain, Ma, Yau, and Zuo proposed a conjecture predicting that the higher Nash blowup algebras $\mathcal{T}_n(f)$ of an isolated complex hypersurface singularity f are contact invariants. They verified this for plane curve singularities and $n = 2$, see Theorem A in [11]. The conjecture was proved recently by Le and Yasuda in [13] for (not

Mathematics Subject Classification 2020: 14B05 (primary); 13N10, 14J17, 32S05, 32S10 (secondary).
Keywords: Mather–Yau theorem, higher Nash blowups, higher Tjurina algebras, higher Jacobian ideals.

necessary isolated) complex hypersurface singularities. In this paper, we overcome these challenges by developing a more general framework, proving the conjecture for cases where $\mathbf{f}$ is a system of elements in an analytic or geometric ring R. More precisely, let R be an analytic or geometric ring (see Definition 2.1), and let $\mathbf{f} = (f_1, \dots, f_s)$ be a system of elements in R. We prove that the n-th Nash blowup algebra $\mathcal{T}_n(\mathbf{f})$ of $\mathbf{f}$ is invariant under contact equivalence (Theorem 2.10).

Motivated by the Mather–Yau theorem, we explore the inverse of the conjecture, which asks whether the contact equivalence class of f can be uniquely determined by its higher Nash blowup algebras. For hypersurface singularities, we provide an affirmative answer to this question. The classical Mather–Yau theorem [14] establishes that for isolated complex hypersurface singularities, the isomorphism type of its local ring is determined by its Tjurina algebra. However, this result does not hold in positive characteristic, as pointed out by Mather and Yau. Greuel and Pham [10] and Kerner [12] addressed this limitation by proving that the isomorphism type of the local ring of an isolated hypersurface singularity over a field of positive characteristic can be determined by sufficiently high Tjurina algebras. The analogous question for non-isolated hypersurface singularities remains unresolved.

In this paper, we extend this investigation by replacing higher Tjurina algebras with higher Nash blowup algebras. We demonstrate that the isomorphism type of the local ring of a hypersurface singularity f is uniquely determined by its higher Nash blowup algebra of degree at least 2, without imposing any assumptions on f or on the ground field (Theorem 3.2).

2. Higher Jacobian ideals and contact equivalence

2.1. Higher Jacobian ideals

Let $\mathbb{k}$ be an arbitrary field. We recall in this section the notion of higher Jacobian ideals of a system of elements of an analytic or geometric $\mathbb{k}$-algebra.

Definition 2.1. A $\mathbb{k}$-algebra R is called is called *geometric* if it is either a finite type $\mathbb{k}$-algebra, or a localization or a completion of a finite type $\mathbb{k}$-algebra. In this paper, we denote by $\mathbb{k}\{\mathbf{x}\}$ the ring of convergent power series over a quasi-complete valued field $\mathbb{k}$, where $\mathbf{x} = (x_1, \dots, x_d)$. Notice that a valued field $\mathbb{k}$ is quasi-complete if the completion $\bar{\mathbb{k}}$ of $\mathbb{k}$ is a separable field extension of $\mathbb{k}$. In characteristic 0 every real valued field is already quasi-complete. We call R *an analytic* ring if it is isomorphic to a quotient of the ring $\mathbb{k}\{\mathbf{x}\}$.

From now on, we always assume that R is either analytic or geometric. Let $\mathbf{t} = (t_1, \dots, t_s)$ and let S be either $\mathbb{k}[\mathbf{t}]$ or $\mathbb{k}\{\mathbf{t}\}$ or $\mathbb{k}[\![\mathbf{t}]\!]$, corresponding to the structure of R. Let $\mathbf{f} = (f_1, \dots, f_s)$ be a system of nonzero divisors in an analytic or geometric $\mathbb{k}$-algebra R so that the corresponding $t_i \mapsto f_i$ defines a morphism of $\mathbb{k}$-algebras $S \to R$. We denote by $R \otimes_{\mathbf{f}} R$ the tensor product over S and consider it as R-module via the map

$$\delta_{\mathbf{f}} : R \to R \otimes_{\mathbf{f}} R, \quad \alpha \mapsto \alpha \otimes_{\mathbf{f}} 1.$$

Let $\rho_{\mathbf{f}} : R \otimes_{\mathbf{f}} R \to R$ be a morphism of $\mathbb{k}$-algebras defined as $\rho_{\mathbf{f}}(\alpha \otimes \beta) = \alpha\beta$ and let $\mathcal{I}_{\mathbf{f}}$ be the kernel of $\rho_{\mathbf{f}}$. For any $n \in \mathbb{N}^*$, we define the *R-module of Kähler differentials of* *order n* of $\mathbf{f}$ by

$$\Omega_{\mathbf{f}}^{(n)} := \mathcal{I}_{\mathbf{f}}/\mathcal{I}_{\mathbf{f}}^{n+1}.$$

Lemma 2.2. *Assume that R is analytic or geometric. Then the R-module $\Omega_{\mathbf{f}}^{(n)}$ is finite.*

Proof. We prove this for the geometric case, since the analytic case is similar. Assume that R is a localization or a completion of the ring $\mathbb{k}[\mathbf{x}]/I$. Then it is easily seen that

$$\mathcal{I}_{\mathbf{f}} = (x_i \otimes 1 - 1 \otimes x_i \mid i = 1, \dots, d),$$

where x_i denotes the image of x_i in R. Hence the R-module $\Omega_{\mathbf{f}}^{(n)}$ is generated by the elements $(\mathbf{x} \otimes 1 - 1 \otimes \mathbf{x})^{\alpha}$ with $1 \le |\alpha| \le n$, where

$$(\mathbf{x} \otimes 1 - 1 \otimes \mathbf{x})^{\alpha} := (x_1 \otimes 1 - 1 \otimes x_1)^{\alpha_1} \cdots (x_d \otimes 1 - 1 \otimes x_d)^{\alpha_d}$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. ∎

Let M be any finite R-module. Choose a presentation

$$\bigoplus_{j \in J} R \to R^{\oplus n} \to M \to 0$$

of M. Let $A = (a_{ij})_{i=1,\dots,n, j \in J}$ be the matrix of the map $\bigoplus_{j \in J} R \to R^{\oplus n}$. The k-th Fitting ideal of M is the ideal $\mathbf{Fitt}_k(M)$ generated by the $(n-k) \times (n-k)$ minors of A, which is independent of the choice of the presentation.

Definition 2.3. Let R be an analytic or geometric $\mathbb{k}$-algebra and let $e = \dim R/(\mathbf{f})$, $r = \binom{n+e}{e} - 1$, where $(\mathbf{f})$ is the ideal generated by $f_1, \dots, f_s$. Then the r-th Fitting ideal $\mathbf{Fitt}_r(\Omega_{\mathbf{f}}^{(n)})$ of the R-module $\Omega_{\mathbf{f}}^{(n)}$ is called the *n-th Jacobian ideal of* $\mathbf{f}$, and it is denoted by $\mathcal{J}_n(\mathbf{f})$.

Let us give an explicit description of $\mathcal{J}_n(f)$ in the case when f is an element of $R = \mathbb{k}[\mathbf{x}]$ or $\mathbb{k}\{\mathbf{x}\}$ or $\mathbb{k}[\![\mathbf{x}]\!]$, where $\mathbf{x} = (x_1, \dots, x_d)$. To an $f \in R$ and an $n \in \mathbb{N}^*$, we associate a matrix described as follows:

$$\mathrm{Jac}_n(f) := (r_{\beta,\alpha})_{0 \le |\beta| \le n-1, 1 \le |\alpha| \le n},$$

where the ordering for row and column indices is graded lexicographical,

$$r_{\beta,\alpha} = r_{\beta,\alpha}(f) := \begin{cases} 0 & \text{if } \alpha_i < \beta_i \text{ for some } 1 \le i \le d, \\ 0 & \text{if } \alpha = \beta, \\ \frac{\partial^{\alpha-\beta} f}{(\alpha-\beta)!} & \text{if } \alpha > \beta, \end{cases} \tag{2.1}$$

where $\alpha > \beta$ if $\alpha \ne \beta$ and $\alpha_i \ge \beta_i$ for all i. For $\mathbb{k}$ of positive characteristic, we will consider $\partial^{\gamma} f/\gamma!$ just as a formal symbol which stands for the coefficient of y^{γ} in the expansion of $f(x+y) - f(x)$. Clearly, $\mathrm{Jac}_n(f)$ is a matrix of type $\binom{d-1+n}{d} \times \left(\binom{d+n}{d} - 1\right)$ with entries in R.

Definition 2.4. Let $R = \mathbb{k}[\mathbf{x}]$, $\mathbb{k}\{\mathbf{x}\}$ or $\mathbb{k}[\![\mathbf{x}]\!]$. For $f \in R$, the matrix $\mathrm{Jac}_n(f)$ is called the *Jacobian matrix of order n of f*, or the *n-th Jacobian matrix of f*.

Remark 2.5. There are a few versions of higher Jacobian matrix which are slightly different from one another. Our definition above follows the one adopted in [3]. Another version considered in [5, 11] differs in that the diagonal entries $r_{\alpha,\alpha}$ are f instead of 0. These two versions coincide modulo f. The one in [4], which the authors call the *Jacobi–Taylor matrix*, has one extra column by allowing $|\alpha| = 0$.

Proposition 2.6 (Proposition 2.15 in [13]). *Let* $R = \Bbbk[\mathbf{x}]$, $\Bbbk\{\mathbf{x}\}$ *or* $\Bbbk[\![\mathbf{x}]\!]$. *The* n*-th Jacobian ideal* $\mathcal{J}_n(f)$ *of* f *is generated by all the maximal minors of the matrix* $\mathrm{Jac}_n(f)$.

Proposition 2.7 (Proposition 2.19 in [13]). *Let* $R = \Bbbk[\mathbf{x}]$, $\Bbbk\{\mathbf{x}\}$ *or* $\Bbbk[\![\mathbf{x}]\!]$. *We have the inclusions*

$$\mathcal{J}_n(f) \subseteq \mathcal{J}_1(f)^{\binom{d-2+n}{d-1}}.$$

In particular, $\mathcal{J}_n(f) \subseteq \mathcal{J}_1(f)^3$ *if either*

- $d \geq 3$ *and* $n \geq 2$, *or*
- $d = 2$ *and* $n \geq 3$.

2.2. Higher Nash blowup algebras and contact equivalence

Let $\mathbf{f}$ be a system of nonzero elements in R. The n-th Jacobian and the Nash blowup local algebra of f are defined respectively as

$$\mathcal{M}_n(\mathbf{f}) := R/\mathcal{J}_n(\mathbf{f}) \quad \text{and} \quad \mathcal{T}_n(\mathbf{f}) := R/(\mathbf{f}) + \mathcal{J}_n(\mathbf{f}).$$

In the following, we will show that these algebras are stable under right and contact equivalence, respectively. Recall that two systems $\mathbf{f} = (f_1, \ldots, f_s)$ and $\mathbf{g} = (g_1, \ldots, g_s)$ of s elements in R are called *right equivalent*, denoted by $\mathbf{f} \sim_r \mathbf{g}$, if there is an automorphism φ of $\Bbbk$-algebra of R such that $g_i = \varphi(f_i)$. They are called *contact equivalent*, denoted by $\mathbf{f} \sim_c \mathbf{g}$, if $R/(\mathbf{f}) \cong R/(\mathbf{g})$ as $\Bbbk$-algebras.

Remark 2.8. Note that if there are units $u_1, \ldots, u_s \in R$ and an automorphism of $\Bbbk$-algebras $\varphi\colon R \to R$ such that $g_i = u_i\varphi(f_i)$, then $\mathbf{f}$ is contact equivalent to $\mathbf{g}$, since φ induces an isomorphism of $\Bbbk$-algebras $R/(\mathbf{f}) \cong R/(\mathbf{g})$. Conversely, if R has the lifting property (see Lemma 3.5) and f and g are two elements in R, which are contact equivalent, then there are a unit $u \in R$ and an automorphism of $\Bbbk$-algebras $\varphi\colon R \to R$ such that $g = u\varphi(f)$. In general, the contact equivalence of $\mathbf{f}$ and $\mathbf{g}$ does not imply the existence of u_i and φ, as the following example shows.

Let $R = \Bbbk[\![x, y]\!]$. Let $\mathbf{f} = (f_1, f_2)$ and $\mathbf{g} = (g_1, g_2)$ be two systems of elements in R with

$$f_1(x, y) = x^2,\ f_2(x, y) = y^3 \quad \text{and} \quad g_1(x, y) = x^2,\ g_2(x, y) = x^2 + y^3.$$

Then $\mathbf{f} \sim_c \mathbf{g}$, but there are not units $u_1, u_2 \in R$ and automorphisms of $\Bbbk$-algebras $\varphi\colon R \to R$ such that $g_i = u_i\varphi(f_i)$.

In their paper [11], Hussain, Ma, Yau, and Zuo proposed the following conjecture, and proved it for $d = n = 2$, see Theorem A in [11].

Conjecture 2.9 (Conjecture 1.5 in [11]). *Let* f *and* g *be in* $\mathbb{C}\{\mathbf{x}\}$, *with* $f(\mathbf{0}) = g(\mathbf{0}) = 0$. *If* f *is contact equivalent to* g *at* $\mathbf{0}$, *then, for any* $n \in \mathbb{N}^*$, $\mathbb{C}\{\mathbf{x}\}/(f) + \mathcal{J}_n(f)$ *is isomorphic to* $\mathbb{C}\{\mathbf{x}\}/(g) + \mathcal{J}_n(g)$ *as* $\mathbb{C}$*-algebras.*

This conjecture was recently proved by Le and Yasuda in Theorem 2.26 of [13] without the assumption that f and g have an isolated singularity at the origin. An elementary proof of the conjecture for the second Nash blowup algebras ($n = 2$) is given in [18]. In the following, we prove the conjecture in a very general sense. Namely, the ground field is arbitrary, the ring R is analytic or geometric (see Section 2.1), and f and g are two systems of elements in R, not just hypersurfaces. The converse is shown to hold true for hypersurface singularities in the next section (Theorem 3.2).

Theorem 2.10. *Assume that R is analytic or geometric. Let $\mathbf{f}$ and $\mathbf{g}$ be two systems of s nonzero elements in R.*

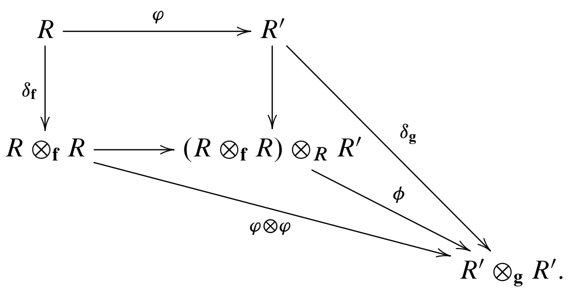
(i) *If $\mathbf{f}$ and $\mathbf{g}$ are right equivalent, then $\mathcal{M}_n(\mathbf{f}) \cong \mathcal{M}_n(\mathbf{g})$ for all $n \geq 1$.*

(ii) *If $\mathbf{f}$ and $\mathbf{g}$ are contact equivalent, then $\mathcal{T}_n(\mathbf{f}) \cong \mathcal{T}_n(\mathbf{g})$ for all $n \geq 1$.*

Proof. (i) By definition, there exists an isomorphism $\varphi\colon R \to R' = R$ defining the R-module structure of R', such that $g_i = \varphi(f_i)$ in R' for all $i = 1, \dots, s$. Let

$$\varphi \otimes \varphi : R \otimes_{\mathbf{f}} R \to R' \otimes_{\mathbf{g}} R'$$

denote the isomorphism mapping $\alpha \otimes \beta$ to $\varphi(\alpha) \otimes \varphi(\beta)$. Consider the tensor product $(R \otimes_{\mathbf{f}} R) \otimes_R R'$ of two R-algebras $(R \otimes_{\mathbf{f}} R)$ and R'. The morphisms $\varphi \otimes \varphi$ and δ_g give rise, by the universal property of tensor product, to a morphism $\phi\colon (R \otimes_{\mathbf{f}} R) \otimes_R R \to R' \otimes_{\mathbf{g}} R'$ and the following commutative diagram:



Note that φ is isomorphic and so are the morphisms $R \otimes_{\mathbf{f}} R \to (R \otimes_{\mathbf{f}} R) \otimes_R R'$ and $\varphi \otimes \varphi$. Therefore, ϕ is an isomorphism. We obtain the following isomorphism of R'-modules induced by ϕ:

$$\mathcal{I}_{\mathbf{f}} \otimes_R R' \cong \mathcal{I}_{\mathbf{g}},$$

and therefore

$$\Omega_{\mathbf{f}}^{(n)} \otimes_R R' \cong \Omega_{\mathbf{g}}^{(n)}.$$

Since taking Fitting ideals commutes with base change, this yields that

$$\varphi\big(\mathbf{Fitt}_r(\Omega_{\mathbf{f}}^{(n)})\big) = \mathbf{Fitt}_r(\Omega_{\mathbf{f}}^{(n)} \otimes_R R') = \mathbf{Fitt}_r(\Omega_{\mathbf{g}}^{(n)})$$

as ideals in $R' = R$. Hence $\varphi(\mathcal{J}_n(\mathbf{f})) = \mathcal{J}_n(\mathbf{g})$ and (i) follows.

(ii) Since $\mathbf{f} \sim_c \mathbf{g}$, there exists an isomorphism $\varphi\colon R/(\mathbf{f}) \to R/(\mathbf{g})$. Note that

$$R \otimes_{\mathbf{f}} R \cong R \otimes_{\Bbbk} R/(\mathbf{f} \otimes 1 - 1 \otimes \mathbf{f}), \quad \alpha \otimes \beta \mapsto \alpha \otimes \beta.$$

and hence

$$(R \otimes_{\mathbf{f}} R) \otimes_R R/(\mathbf{f}) \cong R \otimes_{\Bbbk} R/(\mathbf{f} \otimes 1, 1 \otimes \mathbf{f}) \cong R/(\mathbf{f}) \otimes_{\Bbbk} R/(\mathbf{f})$$

as $\Bbbk$-algebras. Similarly, one has the following isomorphism of $\Bbbk$-algebras:

$$(R \otimes_{\mathbf{g}} R) \otimes_R R/(\mathbf{g}) \cong R/(\mathbf{g}) \otimes_{\Bbbk} R/(\mathbf{g}).$$

Then φ induces an isomorphism

$$\tilde{\varphi} : (R \otimes_{\mathbf{f}} R) \otimes_R R/(\mathbf{f}) \cong (R \otimes_{\mathbf{g}} R) \otimes_R R/(\mathbf{g})$$

which makes the following diagram commutative:

$$\begin{array}{ccc} (R \otimes_{\mathbf{f}} R) \otimes_R R/(\mathbf{f}) & \xrightarrow{\tilde{\varphi}} & (R \otimes_{\mathbf{g}} R) \otimes_R R/(\mathbf{g}) \\ \downarrow{\scriptstyle \tilde{\rho}_{\mathbf{f}}} & & \downarrow{\scriptstyle \tilde{\rho}_{\mathbf{g}}} \\ R/(\mathbf{f}) & \xrightarrow[\varphi]{} & R/(\mathbf{g}), \end{array}$$

where the morphism $\tilde{\rho}_{\mathbf{f}}$ is defined as

$$\tilde{\rho}_{\mathbf{f}} : (R \otimes_{\mathbf{f}} R) \otimes_R R/(\mathbf{f}) \xrightarrow{\rho_{\mathbf{f}} \otimes \mathrm{id}} R \otimes_R R/(\mathbf{f}) \xrightarrow{\cong} R/(\mathbf{f}).$$

The morphism $\tilde{\rho}_{\mathbf{g}}$ is defined similarly. Note that

$$\mathcal{I}_{\mathbf{f}} \otimes_R R/(\mathbf{f}) = \ker \tilde{\rho}_{\mathbf{f}} \quad \text{and} \quad \mathcal{I}_{\mathbf{g}} \otimes_R R/(\mathbf{g}) = \ker \tilde{\rho}_{\mathbf{g}}.$$

Hence

$$(\mathcal{I}_{\mathbf{f}} \otimes_R R/(\mathbf{f})) \otimes_{R/(\mathbf{f})} R/(\mathbf{g}) \cong \tilde{\varphi}(\mathcal{I}_{\mathbf{f}} \otimes_R R/(\mathbf{f})) = \mathcal{I}_{\mathbf{g}} \otimes_R R/(\mathbf{g}),$$

as $R/(\mathbf{g})$-modules. This yields the following isomorphism of $R/(\mathbf{g})$-modules:

$$(\Omega_{\mathbf{f}}^{(n)} \otimes_R R/(\mathbf{f})) \otimes_{R/(\mathbf{f})} R/(\mathbf{g}) \cong \Omega_{\mathbf{g}}^{(n)} \otimes_R R/(\mathbf{g}).$$

Considering the natural morphisms $\varphi_{\mathbf{f}}\colon R \to R/(\mathbf{f})$ and $\varphi_{\mathbf{g}}\colon R \to R/(\mathbf{g})$, we obtain the following identities:

$$\varphi_{\mathbf{f}}(\mathcal{J}_n(\mathbf{f})) = \varphi_{\mathbf{f}}(\mathbf{Fitt}_r(\Omega_{\mathbf{f}}^{(n)})) = \mathbf{Fitt}_r(\Omega_{\mathbf{f}}^{(n)} \otimes_R R/(\mathbf{f})),$$

and similarly,

$$\varphi_{\mathbf{g}}(\mathcal{J}_n(\mathbf{g})) = \mathbf{Fitt}_r(\Omega_{\mathbf{g}}^{(n)} \otimes_R R/(\mathbf{g})).$$

It follows that

$$\begin{aligned} \varphi(\varphi_{\mathbf{f}}(\mathcal{J}_n(\mathbf{f}))) &= \varphi(\mathbf{Fitt}_r(\Omega_{\mathbf{f}}^{(n)} \otimes_R R/(\mathbf{f}))) = \mathbf{Fitt}_r((\Omega_{\mathbf{f}}^{(n)} \otimes_R R/(\mathbf{f})) \otimes_{R/(\mathbf{f})} R/(\mathbf{g})) \\ &= \mathbf{Fitt}_r(\Omega_{\mathbf{g}}^{(n)} \otimes_R R/(\mathbf{g})) = \varphi_{\mathbf{g}}(\mathcal{J}_n(\mathbf{g})). \end{aligned}$$

Hence

$$\mathcal{T}_n(\mathbf{f}) \cong \mathcal{T}_n(\mathbf{g}). \qquad \blacksquare$$

3. Mather–Yau theorem for higher Nash blowup algebras

The well-known Mather–Yau theorem [14] says that the isomorphism type of the local ring of an isolated complex hypersurface singularity is determined by its Tjurina algebra. It was generalized to the case of complex-analytic varieties of isolated singularity type[1] by Gaffney and Hauser [7]. In the case of positive characteristic, this was achieved by Greuel and Pham [10] and Kerner [12] by considering higher Tjurina algebras, see Theorem 3.1. In this section, we give a characterization of the isomorphism type of the local ring of a hypersurface singularity f without any assumption on f by using higher Nash blowup algebras (Theorem 3.2). Recall that a hypersurface singularity is an element in the maximal ideal $\mathfrak{m} = (\mathbf{x})$ of the ring $R = \Bbbk[\![\mathbf{x}]\!]$ or $\Bbbk\{\mathbf{x}\}$. The *multiplicity* of f, denoted by $\operatorname{mt}(f)$, is the maximal k such that $f \in \mathfrak{m}^k$. The *k-th Tjurina algebra* of f is defined as

$$T_n(f) = R/(f) + \mathfrak{m}^k \operatorname{j}(f),$$

where $\operatorname{j}(f)$ denotes the Jacobian ideal of f. Notice that $\operatorname{j}(f) = \mathcal{J}_1(f)$ by Proposition 2.6. The dimension $\dim_{\Bbbk} T_0(f)$ is called the Tjurina number of f, and denoted by $\tau(f)$.

Theorem 3.1 ([7, 10, 12, 14]). *Let $f, g \in \Bbbk[\![\mathbf{x}]\!]$ be such that $\operatorname{mt}(f) \geq 2$ and $\tau(f) < \infty$. Let*

$$k = \begin{cases} 2\tau(f) - 2\operatorname{mt}(f) + 4, & \text{if } \operatorname{char}(\Bbbk) > 0, \\ 1 & \text{if } \operatorname{char}(\Bbbk) = 0, \\ 0 & \text{if } \operatorname{char}(\Bbbk) = 0 \text{ and } \Bbbk = \bar{\Bbbk}. \end{cases}$$

Then the following are equivalent:

(i) $f \sim_c g$.

(ii) *$T_n(f) \cong T_n(g)$ as $\Bbbk$-algebras for all $n \geq k$.*

(iii) *$T_n(f) \cong T_n(g)$ as $\Bbbk$-algebras for some $n \geq k$.*

The following result says that for f and any $\Bbbk$, the isomorphism type of the local ring $R/(f)$ is determined by the n-th Nash blowup algebras with $n \geq 2$. Notice that if $\Bbbk$ is an algebraically closed field of characteristic zero, then it can be determined by the first Nash blowup algebra [10, 14]. This is not true, in general, if the field $\Bbbk$ is either of positive characteristic or not algebraically closed, as Example 3.7 shows.

Theorem 3.2. *Let $R = \Bbbk[\![\mathbf{x}]\!]$ or $R = \Bbbk\{\mathbf{x}\}$. Let f and g be two elements in R. The following are equivalent*:

(i) $f \sim_c g$.

(ii) *$\mathcal{T}_n(f) \cong \mathcal{T}_n(g)$ for all $n \geq 2$.*

(iii) *$\mathcal{T}_n(f) \cong \mathcal{T}_n(g)$ for some $n \geq 2$.*

For the proof of this theorem, we shall need the following well-known results of Samuel [15], Artin [2], André [1] and Schemmel [16], as well as a couple of technical results (the lifting Lemma 1.1.23 in [8] and Lemma 3.6).

[1] By definition, $(X, 0)$ has isolated singularity type if $\operatorname{Sing}(X, 0) \neq \operatorname{Sing}(X, a)$ for $a \neq 0$ varying in a sufficiently small neighborhood of $0 \in \mathbb{C}^n$.

Theorem 3.3 ([15]). *Let $R = \Bbbk[\![\mathbf{x}]\!]$ be a power series ring over a field $\Bbbk$, and let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the maximal ideal of R. Suppose that f and g are elements in R such that $\mathrm{j}(f) \neq R$. If $g \equiv f \mod \mathfrak{m}\,\mathrm{j}(f)^2$, then there exists an automorphism s of R such that $s(f) = g$.*

Theorem 3.4 ([1, 2, 16]). *Let $\Bbbk$ be a valued field of characteristic zero and let $\mathbf{f}(\mathbf{x}, \mathbf{y})$ be a vector of convergent power series in two sets of variables $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_l)$. Assume given a formal power series solution $\widehat{\mathbf{y}}(\mathbf{x})$ such that $\hat{y}_i \in \mathfrak{m}$ for all $i = 1, \ldots, l$, and*

$$\mathbf{f}(\mathbf{x}, \widehat{\mathbf{y}}(\mathbf{x})) = 0.$$

Then there exists, for any $c \in \mathbb{N}$, a convergent power series solution $\widetilde{\mathbf{y}}(\mathbf{x})$,

$$\mathbf{f}(\mathbf{x}, \widetilde{\mathbf{y}}(\mathbf{x})) = 0,$$

which coincides with $\widehat{\mathbf{y}}(\mathbf{x})$ up to degree c,

$$\tilde{y}_i(\mathbf{x}) \equiv \hat{y}_i(\mathbf{x}) \mod \mathfrak{m}^c, \quad \forall i = 1, \ldots, l.$$

Lemma 3.5. *The rings $\Bbbk[\![\mathbf{x}]\!]$ has the following lifting property: let φ be a morphism of $\Bbbk$-algebras,*

$$\varphi\colon A = \Bbbk[\![\mathbf{x}]\!]/I \to B = \Bbbk[\![\mathbf{x}]\!]/J.$$

Then, φ possesses a lifting $\tilde{\varphi}\colon \Bbbk[\![\mathbf{x}]\!] \to \Bbbk[\![\mathbf{x}]\!]$, that is, the following diagram commutes:

$$\begin{array}{ccc} \Bbbk[\![\mathbf{x}]\!] & \xrightarrow{\tilde{\varphi}} & \Bbbk[\![\mathbf{x}]\!] \\ \downarrow & & \downarrow \\ \Bbbk[\![\mathbf{x}]\!]/I & \xrightarrow[\varphi]{} & \Bbbk[\![\mathbf{x}]\!]/J. \end{array}$$

Moreover, the morphism $\tilde{\varphi}$ can be selected as an isomorphism if φ is an isomorphism.

Lemma 3.6. *Let $R = \Bbbk\{\mathbf{x}\}$ or $\Bbbk[\![\mathbf{x}]\!]$ and let $\mathfrak{m} = (\mathbf{x})$. Assume that $\mathrm{mt}(f) \geq 2$ and $n \geq 2$. Then*

(i) $\mathcal{J}_n(f) \subseteq \mathcal{J}_{n-1}(f)$;
(ii) $\mathcal{J}_n(f) \subseteq \mathfrak{m}\mathcal{J}_1(f)^2$ *if either $d \geq 3$ or $n \geq 3$ or $\mathrm{mt}(f) \geq 3$;*
(iii) $(f) + \mathcal{J}_n(f) \subseteq (f) + \mathfrak{m}\mathcal{J}_1(f)^2$.

Proof. Part (i) follows from Proposition 2.6 and a direct calculation. Part (ii) is due to Proposition 2.7 in case either $d \geq 3$ or $n \geq 3$. Assume that $\mathrm{mt}(f) \geq 3$, $d = n = 2$. Considering f as an element in $\Bbbk[\![x, y]\!]$, one can see that

$$\mathcal{J}_2(f) = (f_x^3, f_x^2 f_y, f_x f_y^2, f_y^3, f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_x^2 f_{yy}).$$

Hence

$$\mathcal{J}_n(f) \subseteq \mathcal{J}_2(f) \subseteq \mathfrak{m}\mathcal{J}_1(f)^2,$$

which completes the proof of (ii).

To prove (iii), it remains to consider the case when $d \leq 2$ and $\mathrm{mt}(f) = n = 2$. If $d = 1$, then as $\mathrm{mt}(f) = 2$, one has $\mathcal{J}_2(f) = (f_x^2) \subseteq (f)$, and hence

$$(f) + \mathcal{J}_2(f) = (f) \subseteq (f) + \mathfrak{m}\mathcal{J}_1(f)^2.$$

Suppose that $d = 2$ and $f \in \Bbbk[\![x, y]\!]$. Since $\mathrm{mt}(f) = 2$, we can show that f is right equivalent to a plane curve singularities in one of the following forms, whereas $a \in \Bbbk$ (see, for instance, Proposition 3.6 in [9]):

(a) $f_1 = xy,\ f_2 = x^2 + ay^k,\ k \geq 3$ if $\mathrm{char}(\Bbbk) \neq 2$.

(b) $f_1 = xy,\ f_3 = x^2 + h(x, y),\ \mathrm{mt}(h) \geq 3$ if $\mathrm{char}(\Bbbk) = 2$.

We may compute the ideal $(f) + \mathcal{J}_2(f)$ as follows:

$$(f_1) + \mathcal{J}_2(f_1) = (xy, x^3, y^3),$$
$$(f_2) + \mathcal{J}_2(f_2) = (x^2 + ay^k, x^3, axy^{2k-2}, ax^2y^{k-1}, ay^{3(k-1)})$$

and

$$(f_3) + \mathcal{J}_2(f_3) = (f_3) + \mathcal{J}_2(h) \subseteq (f_3) + \mathfrak{m}\mathcal{J}_1(h)^2 = (f_3) + \mathfrak{m}\mathcal{J}_1(f_3)^2,$$

using (ii) in the inclusion. This completes the lemma. ■

Proof of Theorem 3.2. The implication (i) ⇒ (ii) is due to Theorem 2.10, while (ii) ⇒ (iii) is trivial. It remains to prove the implication (iii) ⇒ (i). We first establish this implication for $R = \Bbbk[\![\mathbf{x}]\!]$. Assume that $\mathcal{T}_n(f) \cong \mathcal{T}_n(g)$ for some $n \geq 2$. Then by the lifting lemma (see for instance Lemma I.1.23 in [8]), there exists an automorphism ϕ of R such that

$$(g) + \mathcal{J}_n(g) = \phi((f) + \mathcal{J}_n(f)) = (\phi(f)) + \mathcal{J}_n(\phi(f)),$$

where the last identity is according to the proof of Theorem 2.10 (i). Replacing f by $\phi(f)$, we may assume that $(f) + \mathcal{J}_n(f) = (g) + \mathcal{J}_n(g)$. We may assume, without loss of generality, that $\mathrm{mt}(f) \geq \mathrm{mt}(g)$. Considering the first case that $\mathrm{mt}(f) = 1$, one obtains $\mathrm{mt}(g) = 1$ and hence $f \sim_c g$. Assume that $\mathrm{mt}(f) \geq 2$; then, by Lemma 3.6, one has

$$(f) + \mathcal{J}_n(f) \subseteq (f) + \mathfrak{m}\mathcal{J}_1(f)^2,$$

where $\mathfrak{m} = (\mathbf{x})$ is the maximal ideal of R. Since

$$g \in (f) + \mathcal{J}_n(f) \subseteq (f) + \mathfrak{m}\mathcal{J}_1(f)^2 = (f) + \mathfrak{m}\,\mathrm{j}(f)^2,$$

there exist $u, h \in R$ such that

$$g = uf + h, h \in \mathfrak{m}\,\mathrm{j}(f)^2.$$

As $h \in \mathfrak{m}\,\mathrm{j}(f)^2$, it follows that

$$\mathrm{mt}(h) \geq 1 + 2(\mathrm{mt}(f) - 1) > \mathrm{mt}(f).$$

Then u must be a unit in R since $\mathrm{mt}(g) \leq \mathrm{mt}(f) < \mathrm{mt}(h)$. This yields that

$$u^{-1}g - f = u^{-1}h \in \mathfrak{m}\,\mathrm{j}(f)^2.$$

Applying Samuel's theorem (Theorem 3.3), one obtains an automorphism s of R such that

$$s(f) = u^{-1}g, \tag{3.1}$$

which gives $f \sim_c g$.

We now consider the case that $R = \Bbbk\{\mathbf{x}\}$. Let $F(\mathbf{x}, \mathbf{y}) \in \Bbbk\{\mathbf{x}, \mathbf{y}\}$ be a convergent series in $\Bbbk\{\mathbf{x}, \mathbf{y}\}$ defined as

$$F(\mathbf{x}, \mathbf{y}) = y_{d+1} f(y_1, \ldots, y_d) - g(x_1, \ldots, x_d),$$

where $\mathbf{y} = (y_1, \ldots, y_{d+1})$. By (3.1), F admits a formal solution $\hat{\mathbf{y}}(\mathbf{x}) \in \Bbbk[\![\mathbf{x}]\!]^{d+1}$ with

$$\hat{y}_i = s(x_i), \quad \forall i = 1, \ldots, d, \qquad \text{and} \qquad \hat{y}_{d+1} = u(\mathbf{x}).$$

Applying Artin's approximation theorem (Theorem 3.4), we obtain a convergent solution $\tilde{\mathbf{y}} \in \Bbbk\{\mathbf{x}\}^{d+1}$ of F such that

$$\hat{y}_i - \tilde{y}_i \in \mathfrak{m}^2, \quad \forall i = 1, \ldots, d+1. \tag{3.2}$$

Let $\tilde{s}$ be an endomorphism of $\Bbbk\{\mathbf{x}\}$ defined as $\tilde{s}(x_i) = \tilde{y}_i$, for all $i = 1, \ldots, d$. It follows from(3.2) that s and $\tilde{s}$ induce the same morphism,

$$\dot{s} = \dot{\tilde{s}} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2.$$

Since s is an isomorphism, $\tilde{s}$ is also an isomorphism, according to the inverse function theorem (see Theorem I.1.21 in [8]). Moreover, since $\hat{y}_{d+1}$ is a unit, it follows from (3.2) that $\tilde{y}_{d+1}$ is a unit in $\Bbbk\{\mathbf{x}\}$. As $\tilde{\mathbf{y}}$ is a solution of F, we have

$$\tilde{y}_{d+1}\tilde{s}(f) = \tilde{y}_{d+1} f(\tilde{y}_1, \ldots, \tilde{y}_d) = g,$$

and therefore $f \sim_c g$. ■

Example 3.7 (Mather–Yau, Gaffney and Hauser). (a) Let $p = \operatorname{char}(\Bbbk) > 0$. Take

$$f(x, y) = x^{p+1} + y^{p+1} \quad \text{and} \quad g = f + x^p.$$

Then

$$f_x = g_x = x^p, f_y = g_y = y^p$$

and therefore $\mathcal{T}_1(f) = \mathcal{T}_1(g)$ but $f \not\sim_c g$.

(b) Let $\Bbbk = \mathbb{R}$. Take

$$f(x, y) = x^2 + y^2 \quad \text{and} \quad g = x^2 - y^2$$

in $\mathbb{R}[\![x, y]\!]$. Then $\mathrm{j}(f) = \mathrm{j}(g) = (x, y)$, and therefore

$$\mathcal{T}_1(f) = \mathcal{T}_1(g) \cong \mathbb{R},$$

but it is obvious that $f \not\sim_c g$.

Acknowledgements. We would like to thank the referees for their comments and suggestions which prompted us to improve the paper. A part of this work was done while the author were visiting Vietnam Institute for Advanced Study in Mathematics (VIASM) and Hanoi Institute of Mathematics, VAST in 2024. The author would like to thank these institutes for hospitality and support.

Funding. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2024.09.

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Received April 3, 2025; revised October 21, 2025.

Hong Duc Nguyen
TIMAS, Thang Long University
Nghiem Xuan Yem, 10000 Hanoi, Vietnam;
duc.nh@thanglong.edu.vn
Author IDs: zbMATH nguyen-hong-duc. MR 842994 ORCID 0000-0003-2807-7336