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Propagation of Zariski dense orbits

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Abstract. Let X/K be a smooth projective variety defined over a number field, and let $f : X \to X$ be a morphism defined over K. We formulate a number of statements of varying strengths asserting, roughly, that if there is at least one point $P_0 \in X(K)$ whose f-orbit $\mathcal{O}_f(P_0) := \{f^n(P) : n \in \mathbb{N}\}$ is Zariski dense, then after replacing K by a finite extension, there are many f-orbits in $X(K)$. For example, a weak conclusion would be that $X(K)$ is not the union of finitely many (grand) f-orbits, while a strong conclusion would be that any set of representatives for the Zariski dense grand f-orbits in $X(K)$ is itself Zariski dense. We prove statements of this sort for various classes of varieties and maps, including projective spaces, abelian varieties, and surfaces.

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1. Introduction

Let K be a number field, let X/K be a smooth projective variety, and let

$$f : X \to X$$

be an endomorphism of defined over K.

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The theme of this article is that if there is even a single point $P_0 \in X(K)$ whose forward f-orbit

$$\mathcal{O}_f(P_0) := \{f^n(P_0) : n \geq 0\}$$

is Zariski dense in X, then after replacing K by a finite extension, the set $X(K)$ contains lots of distinct large orbits whose points are widely distributed. There are many ways to turn this vague idea, which we call an *orbit propagation principle*, into a precise statement. We will describe four orbit propagation principles here, and we refer the reader to Section 2, and especially Table 1, for many others. In order to state these principles, we begin with some useful definitions and notation.

Definition 1.1. The set of points with Zariski dense f-orbit is denoted

$$X_f^{\text{dense}} := \{P \in X : \overline{\mathcal{O}_f(P)} = X\}.$$

Definition 1.2. The *grand orbit* of a point $P \in X$ is the set of points whose orbits eventually merge with the orbit of P,

$$\mathcal{O}_f^{\text{grand}}(P) := \{Q \in X : \mathcal{O}_f(P) \cap \mathcal{O}_f(Q) \neq \emptyset\}.$$

We say that P and Q are *grand f-orbit equivalent*, and we write $P \equiv_f Q$, if

$$\mathcal{O}_f(P) \cap \mathcal{O}_f(Q) \neq \emptyset,$$

or equivalently, if

$$\mathcal{O}_f^{\text{grand}}(P) = \mathcal{O}_f^{\text{grand}}(Q).$$

It is an exercise to show that grand f-orbit equivalence is an equivalence relation; see Lemma B.1(a).

Conjecture 1.3. *Assume that $X(K)$ has at least one Zariski dense f-orbit. Then there is a finite extension K'/K such that*:

(a) (Orbit propagation principle (B1)) *For all $P_1, \dots, P_r \in X(K')$, the set*

$$X(K') \smallsetminus (\mathcal{O}_f(P_1) \cup \cdots \cup \mathcal{O}_f(P_r))$$

is Zariski dense in X.

(b) (Orbit propagation principle (C1)) *For all $P_1, \dots, P_r \in X(K')$, the set*

$$X(K') \smallsetminus \big(\mathcal{O}_f^{\text{grand}}(P_1) \cup \cdots \cup \mathcal{O}_f^{\text{grand}}(P_r)\big)$$

is Zariski dense in X.

(b′) (Orbit propagation principle (C2∃)) *The set $X(K')$ contains a Zariski dense set of coset representatives for $X(K')/{\equiv_f}$.*

(c) (Orbit propagation principle (C3∀)) *Every set of coset representatives in $X(K')$ for the quotient*

$$X_f^{\text{dense}}(K')/\equiv_f \quad \textit{is Zariski dense in } X.$$

Intuition. Conjectures 1.3(a) and (b) say that a finite union of (grand) orbits can cover only a small portion of the K'-rational points. Conjectures 1.3(b′) and (c) say that taking one (any) K'-rational point from each (Zariski dense) grand orbit in $X(K')$ always results in a Zariski dense set.

To what extent do propagation principles of varying strengths imply one another? There are some implications that are universally true and (mostly) easy to prove. For example, in the notation of Conjecture 1.3, we always have

$$(\mathrm{C3}\forall) \implies (\mathrm{C2}\exists) \iff (\mathrm{C1}) \implies (\mathrm{B1}).$$

See Table 2 for a more extensive diagram of universal implications, and Appendix B.2 for proofs.

The bulk of this article is devoted to proving non-trivial orbit propagation properties of varying strengths from the initial weak assumption that there is a single Zariski dense f-orbit. We have no general statement, but we prove results for various classes of varieties and maps. We state here some exemplary results. We refer the reader to Theorem 2.7 for a complete description of the results in this paper and to Sections 4–8 for the proofs.

Theorem 1.4.

(a) *The orbit propagation principle* (B1) (*Conjecture* 1.3(a)) *is true for endomorphisms of smooth projective surfaces.*

(b) *The orbit propagation principles* (C1) and (C2∃) (*Conjectures* 1.3(b, b′))*, which are equivalent, are true for endomorphisms of projective space* $\mathbb{P}^N$.

(c) *The orbit propagation principle* (C3∀) (*Conjecture* 1.3(c)) *is true for linear endomorphisms of* $\mathbb{P}^N$ *and for endomorphisms of geometrically simple abelian varieties.*

Proof. (a) See Theorem 8.1.

(b) See Theorem 4.1(a).

(c) See Theorem 4.1(c) for $\mathbb{P}^N$, and Theorem 6.1 for abelian varieties. ■

Remark 1.5. We note that although we are able to prove our strongest orbit propagation principle (C3∀) (Conjecture 1.3(c)) in only a limited number of cases, we know of no examples for which it fails to be true.

We now describe the structure of this paper. We start in Section 2 with definitions, notation, the description of a number of different orbit propagation principles, and a statement of our main results. Section 3 describes in more detail the motivation that led to the idea of orbit propagation. Then Sections 4–8 contain the proofs of our main results, using a variety of techniques and tools that include canonical heights and height counting functions, p-adic methods, algebro-geometric techniques, and deep theorems of Faltings et al. on the intersection of subvarieties of abelian varieties with subgroups of finite type. We include two appendices. Appendix A briefly discusses various versions of the related Zariski density conjecture and gives pointers to the literature. Appendix B contains a number of auxiliary results, including elementary properties of orbits (Lemma B.1) in Section B.1, elementary implications relating the various orbit propagation properties (Proposition B.2 and Lemma B.3) in Section B.2, and a weak height counting estimate for $\mathbb{P}^N(K)$ (Lemma B.7) in Section B.3.

We conclude this introduction with a few additional remarks.

Remark 1.6 (The Zariski density conjecture). There is a large literature on the following *Zariski density conjecture*. We briefly describe the relation of the Zariski density conjecture to the present paper, with further discussion and historical details in Section A.

Conjecture 1.7 (The Zariski density conjecture). *Let $X/\overline{\mathbb{Q}}$ be a smooth variety, and let $f\colon X \dashrightarrow X$ be a dominant rational map. Then one of the following is true*:

(a) *There is a point $P \in X(\overline{\mathbb{Q}})$ whose orbit $\mathcal{O}_f(P)$ is well-defined and Zariski dense in X.*

(b) *There is a dominant rational map $\varphi\colon X \dashrightarrow \mathbb{P}^1$ such that $\varphi \circ f = \varphi$. (Note that in this case, the orbits of f are restricted to lie in the fibers of φ, and thus f cannot have any Zariski dense orbits.)*

Xie and others have formulated stronger versions of Conjecture 1.7, and in particular, Xie (see Section 1.3 of [2]) notes that "if we have one Zariski dense orbit, we expect many such orbits." However, both the original conjecture and its various generalizations assert only that $X(\overline{\mathbb{Q}})$ contains "many" points with Zariski dense orbit. They do not appear to consider the problem of whether there is a finite extension $K/\mathbb{Q}$ such that $X(K)$ contains many such points; see Section A for details. Thus the Zariski density conjectures and our orbit propagation conjectures are complementary. The implication relationship, at least for endomorphisms of non-singular projective varieties, is summarized in Figure 1.

$$\begin{aligned}\left(\begin{array}{l}\text{There are no dominant}\\ \text{fibrations } \varphi\colon X\to\mathbb{P}^1\\ \text{satisfying } \varphi\circ f=\varphi\end{array}\right) &\xRightarrow[\text{density conjecture}]{\text{(Strong) Zariski}} \left(\begin{array}{l}X(\overline{\mathbb{Q}})\text{ has one (many) points}\\ \text{with Zariski dense orbits}\end{array}\right)\\ &\xRightarrow[\text{tion conjecture(s)}]{\text{Orbit propaga-}} \left(\begin{array}{l}\text{There is a number field } K/\mathbb{Q}\text{ such}\\ \text{that } X(K)\text{ has many widely spaced}\\ \text{points with Zariski dense orbits}\end{array}\right).\end{aligned}$$

Figure 1. Relation between the Zariski density conjectures and the orbit propagation conjectures.

Remark 1.8 (Motivation). Our motivation for an orbit propagation principle arose from a 35-year old conjecture [39] of the first author. Very roughly, the conjecture says that if $X(K)$ has infinitely many points, then ignoring "error terms," the height counting function of $X(K)$ should grow like a power of T, a power of $\log(T)$, or be bounded. (See Section 3 for a precise statement.) On the other hand, a conjecture of Kawaguchi and the second author [23] suggests that if f is sufficiently dynamically complicated (formally, if the dynamical degree of f satisfies $\delta(f) > 1$), then the height counting function of $\mathcal{O}_f(P)$ of the orbit of a point grows no faster than $\log\log(T)$. Hence $X(K)$ should have lots of different orbits. This vague idea led first to various weak conjectures such as (A) implies (B1) and (C1) in Table 1, and eventually to stronger statements, culminating in the strong orbit propagation principle described in Conjecture 1.3(c).

Remark 1.9. Rather than starting with the assumption that there is at least one Zariski dense orbit, we may instead assume only that there is a finite extension K'/K such that $X(K')$ is Zariski dense. This is often called *potential density* of rational points. Then, for example, in the fibered case described in Conjecture 1.7(b) for which there are no Zariski dense orbits, potential density easily implies the orbit propagation principle (C1) (Conjecture 1.3(b)), which in turn is equivalent to the orbit propagation principle (C2∃) (Conjecture 1.3(b′)). It is then natural to ask whether the stronger orbit propagation principle (C2∀) in Table 1 is true under the potential density assumption.

Remark 1.10 (Rational self-maps of singular and/or quasi-projective varieties?). In this paper, we consider orbit propagation principles only for self-morphisms f of smooth projective varieties X. One might ask whether similar statements hold for rational maps and/or for non-smooth quasi-projective varieties, as is allowed for example in the Zariski density conjecture (Conjecture 1.7). We leave such questions for the future.

2. Notation, definitions, conjectures and main results

Definition 2.1. Throughout this article, we fix the following notation:

- X is a smooth projective variety with $\dim(X) \geq 1$,
- f is an endomorphism $f\colon X \to X$.

Let $P \in X$ be a geometric point of X. We define[1] various sorts of orbits of P.

- (Forward) f-orbit:

$$\mathcal{O}_f(P) = \mathcal{O}_f^+(P) := \{f^n(P) : n \in \mathbb{N}\}.$$

- Backward f-orbit:

$$\mathcal{O}_f^-(P) := \{Q \in X : P \in \mathcal{O}_f(Q)\}.$$

- Full f-orbit:

$$\mathcal{O}_f^\pm(P) := \mathcal{O}_f^+(P) \cup \mathcal{O}_f^-(P).$$

- Grand f-orbit:

$$\mathcal{O}_f^{\text{grand}}(P) = \{Q \in X : \mathcal{O}_f(P) \cap \mathcal{O}_f(Q) \neq \emptyset\}.$$

Definition 2.2. Let $P, Q \in X$. If

$$\mathcal{O}_f(P) \cap \mathcal{O}_f(Q) \neq \emptyset,$$

then we say that P and Q are *grand f-orbit equivalent*, and we write

$$P \equiv_f Q.$$

Definition 2.3. We denote the set of points with Zariski dense f-orbit by

$$X_f^{\text{dense}} := \{P \in X : \overline{\mathcal{O}_f(P)} = X\}.$$

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[1] Our set of natural numbers $\mathbb{N}$ includes 0, so $\mathcal{O}_f(P)$ includes the point $P = f^0(P)$.

We refer the reader to Section B for various elementary properties of different types of orbits a proof, including in particular the fact that grand f-orbit equivalence is an equivalence relation on the set of points of X.

Conjecture 2.4 (Orbit propagation). *Let $\mathcal{X}$ be a set of (smooth projective connected) varieties defined over $\overline{\mathbb{Q}}$, and for each $X \in \mathcal{X}$, let $\mathcal{F}_X$ be a collection of $\overline{\mathbb{Q}}$-morphisms $X \to X$. We say that $(\mathcal{X}, \mathcal{F})$ has an* orbit propagation property *if for every $X \in \mathcal{X}$ and every $f \in \mathcal{F}_X$, the following implication holds:*

$$\left(\begin{array}{l}X(\overline{\mathbb{Q}})\textit{ contains at}\\ \textit{least one Zariski}\\ \textit{dense } f\textit{-orbit}\end{array}\right) \Longrightarrow \left(\begin{array}{l}\textit{there is a number field } K/\mathbb{Q} \textit{ that is a field of}\\ \textit{definition for } X \textit{ and } f \textit{ such that } X(K)\\ \textit{contains ``many large'' } f\textit{-orbits}\end{array}\right).$$

The term "many large" may be quantified using the various orbit statements in Table 1. So we say that $(\mathcal{X}, \mathcal{F})$ has an orbit propagation property of a specified type *if for every $X \in \mathcal{X}$ and every $f \in \mathcal{F}_X$, it satisfies*

$$X_f^{\text{dense}}(\overline{\mathbb{Q}}) \neq \emptyset \Longrightarrow \left(\begin{array}{l}\textit{there is a number field } K/\mathbb{Q} \textit{ that is a field of}\\ \textit{definition for } X \textit{ and } f \textit{ such that a specified } (\mathrm{B})\\ \textit{or } (\mathrm{C}) \textit{ statement in Table 1 is valid for } X(K)\end{array}\right).$$

Remark 2.5. We note that since we assume that X is a smooth projective variety and that $f : X \to X$ is a morphism, assumption (A) that there exists at least one Zariski dense orbit forces f to be surjective, and hence finite.

Remark 2.6. When we refer to Table 1 and make an assertion such as

$$(\mathrm{A}) \Longrightarrow (\mathrm{C1}),$$

what we always mean is that if there is a number field K such that (A) is true, then possibly after replacing K with a finite extension, the statement (C1) is also true.

The many orbit propagation statements in Table 1 are related by a number of straightforward implications,[2] which we describe pictorially in Table 2 and prove in Proposition B.2. In particular, we note that

$$(\mathrm{C3}\forall) = \text{"The one orbit propagation property to rule them all!"}$$

So one might ask whether

$$(\mathrm{A}) \overset{???}{\Longrightarrow} (\mathrm{C3}\forall). \tag{2.1}$$

We do not know of any examples for which (2.1) fails to be true, and although we are only able to prove it in certain cases, there are many classes of varieties and maps for which we can prove weaker orbit propagation implications. Theorem 2.7 summarizes our results.

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[2] Although to be strictly accurate, the fact that (C1) implies (C2∃) is not completely straightforward. In particular, our proof depends on the fact that we are working over a countable fields.

(A)-Statement: One forward orbit

(A) There is at least one Zariski dense f-orbit in $X(K)$, i.e.,

$$X_f^{\text{dense}}(K) \neq \emptyset.$$

(B)-Statements: Many forward orbits

(B1) For any finite collection of f-orbits $\Gamma_1, \dots, \Gamma_r$, the set

$$X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r) \quad \text{is Zariski dense in } X.$$

(B1∞) For every proper Zariski closed set $Y \subsetneq X$, the set

$$X(K) \smallsetminus \bigcup_{P \in Y(K)} \mathcal{O}_f(P) \quad \text{is Zariski dense in } X.$$

(C)-Statements: Many grand forward orbits

(C1) For any finite collection of grand f-orbits $\Gamma_1, \dots, \Gamma_r$, the set

$$X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r) \quad \text{is Zariski dense in } X.$$

(C1∞) For every proper Zariski closed set $Y \subsetneq X$, the set

$$X(K) \smallsetminus \bigcup_{P \in Y(K)} \mathcal{O}_f^{\text{grand}}(P) \quad \text{is Zariski dense in } X.$$

(C2$\exists$) There exists a Zariski dense set of representatives in $X(K)$ for $X(K)/\equiv_f$.
(C2$\forall$) Every complete set of representatives for

$$X(K)/{\equiv_f} \quad \text{is Zariski dense in } X.$$

(C3$\exists$) There exists a Zariski dense set of representatives in $X_f^{\text{dense}}(K)$ for $X_f^{\text{dense}}(K)/\equiv_f$.
(C3$\forall$) $X_f^{\text{dense}}(K) \neq \emptyset$ and every complete set of representatives for

$$X_f^{\text{dense}}(K)/{\equiv_f} \quad \text{is Zariski dense in } X.$$

Table 1. Orbit propagation statements.

$$\begin{tikzcd}
(\mathrm{C3}\forall) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\mathrm{C2}\forall) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (\mathrm{C1}\infty) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\mathrm{B1}\infty) \arrow[d, Rightarrow] \\
(\mathrm{C3}\exists) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\mathrm{C2}\exists) \arrow[r, Leftrightarrow] & (\mathrm{C1}) \arrow[r, Rightarrow] & (\mathrm{B1}) \\
(\mathrm{A}) & & &
\end{tikzcd}$$

Table 2. Universal implications relating the orbit propagation properties in Table 1. See Proposition B.2.

Theorem 2.7. *The following orbit propagation implications hold for the indicated varieties and maps, with the proviso from Remark 2.6 that one may need to replace the field K with a finite extension.*

(1) *Projective space,* $\deg(f) \geq 2$:

$$\text{(A)} \implies \text{(C1)} \iff \text{(C2}\exists\text{)} \quad \text{and} \quad \text{(A)} \implies \text{(B1}\infty\text{)} \quad (\textit{Theorem } 4.1\,\text{(a,b)}).$$

(2) *Projective space,* $\deg(f) = 1$:

$$\text{(A)} \implies \text{(C3}\forall\text{)} \quad (\textit{Theorem } 4.1\,\text{(c)}).$$

(3) *$K3$ surfaces*:

$$\text{(A)} \implies \text{(C1)} \iff \text{(C2}\exists\text{)} \quad (\textit{Theorem } 5.1).$$

(4) *Geometrically simple abelian varieties*:

$$\text{(A)} \implies \text{(C3}\forall\text{)} \quad (\textit{Theorem } 6.1).$$

(5) *Smooth rational varieties and étale f*:

$$\text{(A)} \implies \text{(B1)} \quad (\textit{Corollary } 7.5).$$

(6) *Étale quotients of abelian varieties*:

$$\text{(A)} \implies \text{(B1)} \quad (\textit{Corollary } 7.6).$$

(7) *Smooth projective surfaces*:

$$\text{(A)} \implies \text{(B1)} \quad (\textit{Theorem } 8.1).$$

3. Motivation for the propagation conjecture

Our motivation for formulating some sort of propagation conjecture rests on an older, highly speculative, conjecture of the second author. That conjecture says roughly that, up to lower order terms, the height counting function for the integral points on an algebraic variety can have only one of three possible growth rates. The precise formulation requires some care balancing extending the field and discarding Zariski closed sets having fast growth rates. We give a precise statement for projective varieties and K-rational points; see the cited reference for a general formulation for quasi-projective varieties and S-integral points.

Definition 3.1. Let X/K be a smooth projective variety, and let

$$H : X(\bar{K}) \to [1, \infty)$$

be a Weil height function associated to an ample divisor. We say that X has *arithmetic order* $\mathsf{A}(X)$ if there are an integer $m \geq 2$ and a non-empty Zariski open subset $U_0 \subseteq X$

such that for every non-empty Zariski open subset $U \subseteq U_0 \subseteq X$, there is a finite extension L_0/K such that for every finite extension L/L_0,

$$\lim_{T\to\infty} \frac{\log^{(m)} \#\{P \in U(L) : H(P) \le T\}}{\log^{(m+\mathsf{A}(X))} T} = 1. \tag{3.1}$$

The notation $\log^{(m)}$ denotes the m-fold iterate, and by convention we set $\mathsf{A}(X) = \infty$ if the limit is 0 for all U and all L.

Since the chain of logic in Definition 3.1 is somewhat complicated, we note that it may be written succinctly using logical notation as

$$\exists m \ge 2,\ \exists U_0 \subseteq X,\ \forall U \subseteq U_0,\ \exists L_0/K,\ \forall L/L_0,\ (3.1) \text{ is true.}$$

Conjecture 3.2 (Silverman, 1987, see [39]). *Let X/K be a smooth projective variety defined over a number field. Then the arithmetic order $\mathsf{A}(X)$ exists and satisfies*

$$\mathsf{A}(X) = 0 \quad \textit{or} \quad \mathsf{A}(X) = 1 \quad \textit{or} \quad \mathsf{A}(X) = \infty.$$

Remark 3.3. We note that it is tempting to simply set $m = 2$ in Definition 3.1, so we refer the reader to [39] for an example that suggests why it may be necessary in some cases to take $m \ge 3$. In any case, the conclusion of Conjecture 3.2 says roughly that one of the following is true, where we are being very coarse about ignoring error terms:

$$\#\{P \in X(K) : H(P) \le T\} \begin{cases} \text{grows like a power of } T, \\ \text{grows like a power of } \log T, \\ \text{is bounded at } T \to \infty. \end{cases} \tag{3.2}$$

We next observe that in many situations, if $f : X \to X$ is an endomorphism of a smooth projective variety defined over a number field K, and if $P \in X(P)$, then the logarithmic height of the points in f-orbit of P tend to grow exponentially. Indeed, if the dynamical degree of f satisfies $\delta(f) > 1$, and if $\mathcal{O}_f(P)$ is Zariski dense, then it is conjectured [23] that

$$\lim_{n\to\infty} \sqrt[n]{\log H(f^n(P))} = \delta(f).$$

If this is true, then the height counting function for the points in the orbit satisfies

$$\#\{Q \in \mathcal{O}_f(P) : H(Q) \le T\} \ll \log\log T, \tag{3.3}$$

where the implied constant depends on f and P, but is independent of T. Thus if $X(K)$ were to be the union of a finite number of f-orbits, at least one of which is Zariski dense, then its height counting function would not be bounded, yet would increase too slowly to satisfy the other growth conditions in Conjecture 3.2. (This can be seen more clearly, albeit less precisely, by comparing (3.2) and (3.3).) Hence Conjecture 3.2 suggests that if $X(K)$ contains at least one Zariski dense f-orbit, then (possibly after extending K) it must contain infinitely many non-overlapping f-orbits.

We acknowledge that making a new conjecture on the basis of an older, not widely known, conjecture is somewhat dubious. Further, using Conjecture 3.2 as a starting point, a natural conjecture would be a relatively weak statement such as the following.

Proto-Conjecture 3.4. *Let K be a number field, let X/K be a smooth projective variety, and let $f: X \to X$ be an endomorphism defined over K with dynamical degree $\delta(f) > 1$. If $\#X(K) = \infty$, then $X(K)$ is not the union of finitely many f-orbits.*

But as we explored this proto-conjecture, we realized that we were unable to rule out even much stronger statements, including dropping the $\delta(f) > 1$ requirement, looking at grand orbits, requiring orbits to be Zariski dense, and changing the "not a union of finitely many f-orbits" to a statement that complete sets of representatives for the f-orbits must be Zariski dense. This led us to the plethora of orbit propagation properties listed in Table 1. The remainder of this paper is devoted to proving, for certain classes of varieties and maps, various versions of the statement that one Zariski dense f-orbit leads to many such orbits.

Remark 3.5. We thank De-Qi Zhang for the following remarks and questions.

(a) We have assumed that X is a smooth projective variety, but Conjecture 1.3 makes sense even if X is singular.

(b) Suppose that $f: X \to X$ descends or lifts to a surjective endomorphism $Y \to Y$ of a projective variety of the same dimension via a generically finite rational map or finite morphism $X \to Y$ or $Y \to X$. Is the validity of Conjecture 1.3 for X equivalent to the validity of Conjecture 1.3 for Y? If so, then the case of singular surfaces with $\deg(f) \ge 2$ can mostly be reduced to the case of smooth surfaces except possibly in some cases of polarized f.

(c) To what extent can one prove Conjecture 1.3 or various other implications among the orbit propagation statements in Table 1 if one assumes that $f: X \to X$ is a *polarized endomorphism*, i.e., if there exist an ample line bundle $\mathcal{L} \in \operatorname{Pic}(X) \otimes \mathbb{R}$ and a real number $d > 1$ such that $f^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$? We remark that examples of such morphisms include maps of degree $\deg(f) \ge 2$ on $\mathbb{P}^N$ and infinite order self-isogenies of abelian varieties, for which we have proven orbit propagation results in Theorems 4.1 (a,b) and 6.1. We also observe that if f is polarizable, then Zhang's original conjecture [43] says that there is a Zariski-dense orbit defined over $\overline{\mathbb{Q}}$, and in any case, a polarizable f does not admit a rational fibration to $\mathbb{P}^1$; see Section 1 of [42].

Remark 3.6. There are many papers in arithmetic dynamics that study the heights of points in orbits. We cannot survey the entire literature, but we mention as examples the Kawaguchi–Silverman conjecture that relates the height-defined arithmetic degree to the geometrically defined dynamical degree [24, 27], and results and conjectures on height gaps in dynamics such as [5, 8].

4. Projective space

Theorem 4.1. *Let $N \ge 1$, let $f: \mathbb{P}^N \to \mathbb{P}^N$ be an endomorphism defined over $\overline{\mathbb{Q}}$, and assume that there is a point $P_0 \in \mathbb{P}^N(\overline{\mathbb{Q}})$ whose orbit $\mathcal{O}_f(P_0)$ is Zariski dense in $\mathbb{P}^N$. There exists a number field K that is a common field of definition for f and P_0 so that the following hold*:

(a) (*For* $\deg(f) \ge 2$). *For every finite collection of grand f-orbits*

$$\Gamma_1, \dots, \Gamma_r \subset \mathbb{P}^N(\bar{\mathbb{Q}}),$$

the set

$$\mathbb{P}^N(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r) \quad \text{is Zariski dense in } \mathbb{P}^N.$$

Equivalently, from Proposition B.2, there exists a Zariski dense set in $\mathbb{P}^N(K)$ that contains one point in each grand f-orbit generated by the points in $\mathbb{P}^N(K)$. In the terminology of Table 1, non-linear endomorphisms of $\mathbb{P}^N$ satisfy the orbit propagation statement

$$\text{(A)} \implies \text{(C1)} \iff \text{(C2}\exists\text{)}.$$

(b) (*For* $\deg(f) \ge 2$). *For every proper Zariski closed set $Y \subsetneq \mathbb{P}^N$, the set*

$$\mathbb{P}^N(K) \smallsetminus \bigcup_{P \in Y(K)} \mathcal{O}_f(P) \quad \text{is Zariski dense in } \mathbb{P}^N.$$

In the terminology of Table 1, non-linear endomorphisms of $\mathbb{P}^N$ satisfy the orbit propagation statement

$$\text{(A)} \implies \text{(B1}\infty\text{)}.$$

(c) (*For* $\deg(f) = 1$). *Every complete set of representatives in $(\mathbb{P}^N)^{\text{dense}}_f(K)$ for the set of equivalence classes $(\mathbb{P}^N)^{\text{dense}}_f(K)/{\equiv_f}$ is Zariski dense in $\mathbb{P}^N$. In the terminology of Table 1, linear endomorphisms of $\mathbb{P}^N$ satisfy the orbit propagation statement*

$$\text{(A)} \implies \text{(C3}\forall\text{)}.$$

Proof. We remark that the proof of Theorem 4.1 uses three lemmas whose statements and proofs we defer until the end of this section.

(a,b) The assumption that $\deg(f) \ge 2$ means, see [9], that there is a canonical height function

$$\hat{h}_f : \mathbb{P}^N(\bar{K}) \longrightarrow \mathbb{R}_{\ge 0}$$

satisfying

$$\hat{h}_f \circ f = d \cdot \hat{h}_f, \tag{4.1}$$

$$|\hat{h}_f - h| \le C_1(f), \tag{4.2}$$

$$\hat{h}_f(P) = 0 \iff \#\mathcal{O}_f(P) < \infty. \tag{4.3}$$

Combining (4.2) and (4.3) with the fact that $\mathbb{P}^N(K)$ has only finitely many points of bounded height, we see that $\hat{h}_f$ takes on a minimal positive value, which we denote by

$$\hat{h}^{\min}_f(\mathbb{P}^N, K) := \inf_{\substack{P \in \mathbb{P}^N(K) \\ \#\mathcal{O}_f(P) = \infty}} \hat{h}_f(P) > 0. \tag{4.4}$$

Let Γ be an f-orbit or a grand f-orbit such that $\Gamma \cap \mathbb{P}^N(K)$ is non-empty. We define

$$\hat{h}^{\min}_f(\Gamma, K) = \inf\{\hat{h}_f(Q) : Q \in \Gamma \cap \mathbb{P}^N(K)\}.$$

Using (4.2) and the fact that there are only finitely many points in $\mathbb{P}^N(K)$ of bounded Weil height, we see that there exists a point

$$Q_{\Gamma,K} \in \Gamma \cap \mathbb{P}^N(K)$$

(not necessarily unique if Γ is a grand orbit) such that

$$\hat{h}_f(Q_{\Gamma,K}) = \hat{h}_f^{\min}(\Gamma, K).$$

Further, we see that

$$\hat{h}_f^{\min}(\Gamma, K) > 0 \iff \Gamma \text{ contains an } f\text{-wandering point.} \tag{4.5}$$

(This is equivalent to every point in Γ being f-wandering.)

(a) Let Γ be a grand f-orbit. Then

$$\begin{aligned}
&Q \in \Gamma \cap \mathbb{P}^N(K) \\
&\quad\Longrightarrow\quad f^i(Q) = f^j(Q_{\Gamma,K}) && \text{for some } i, j \in \mathbb{N}, \\
&\quad\Longrightarrow\quad d^i \cdot \hat{h}_f(Q) = d^j \cdot \hat{h}_f(Q_{\Gamma,K}) && \text{for some } i, j \in \mathbb{N}, \\
&\quad\Longrightarrow\quad d^i \cdot \hat{h}_f(Q) = d^j \cdot \hat{h}_f(Q_{\Gamma,K}) && \text{with } j \ge i, \text{ since } \hat{h}_f(Q_{\Gamma,K}) \text{ is the smallest value of } \hat{h}_f \text{ for the points in } \Gamma, \\
&\quad\Longrightarrow\quad \hat{h}_f(Q) \in d^{\mathbb{N}} \cdot \hat{h}_f^{\min}(\Gamma, K) && \text{since } \hat{h}_f(Q_{\Gamma,K}) = \hat{h}_f^{\min}(\Gamma, K), \\
&\quad\Longrightarrow\quad h(Q) \in \bigcup_{n\in\mathbb{N}} \bigl[d^n \hat{h}_f^{\min}(\Gamma, K) - C_1(f), d^n \hat{h}_f^{\min}(\Gamma, K) + C_1(f)\bigr] \\
& && \text{since } |\hat{h}_f - h| \le C_1(f).
\end{aligned}$$

We now suppose that $\Gamma_1, \dots, \Gamma_r$ are grand orbits, and to ease notation, we let

$$C_2(i) := C_2(f, \Gamma_i, K) = \hat{h}_f^{\min}(\Gamma_i, K).$$

The above calculation shows that

$$\bigcup_{i=1}^r \Gamma_i \cap \mathbb{P}^N(K) \subseteq \bigcup_{i=1}^r \bigcup_{n\in\mathbb{N}} \bigl\{Q \in \mathbb{P}^N(K) : |h(Q) - d^n C_2(i)| \le C_1(f)\bigr\}.$$

Hence taking heights, we find that the set of heights of the points in the union of the $\Gamma_i \cap \mathbb{P}^N(K)$ is contained in a union of intervals,

$$\bigcup_{i=1}^r \{h(Q) : Q \in \Gamma_i \cap \mathbb{P}^N(K)\} \subseteq \bigcup_{i=1}^r \bigcup_{n\in\mathbb{N}} [d^n C_2(i) - C_1,\ d^n C_2(i) + C_1].$$

An elementary estimate shows that the double union on the right-hand side omits intervals in $\mathbb{R}_{\ge 0}$ of arbitrarily large length; see Lemma 4.3 for a more precise result. In particular, we can find infinitely many intervals of length 1 that are omitted, say $0 \le t_1 < t_2 < t_3 < \cdots$, satisfying

$$t_{i+1} > t_i + 1 \quad \text{and} \quad \Bigl(\bigcup_{i\ge 1}[t_i, t_i + 1]\Bigr) \cap \Bigl(\bigcup_{i=1}^{r}\{h(Q) : Q \in \Gamma_i \cap \mathbb{P}^N(K)\}\Bigr) = \emptyset.$$

Every interval of length 1 contains a number of the form $\log(a)$ with $a, \in \mathbb{N}$, so we can find a sequence of distinct positive integers $a_1, a_2, \dots$ satisfying

$$\log(a_j) \notin \bigcup_{i=1}^{r}\{h(Q) : Q \in \Gamma_i \cap \mathbb{P}^N(K)\} \quad \text{for all } j \ge 1.$$

We consider the set of points

$$\mathcal{A} := \bigcup_{j\ge 1}\bigl\{[a_j, b_1, \dots, b_N] \in \mathbb{P}^N(\mathbb{Q}) : b_1, \dots, b_N, \in \mathbb{Z},\ 0 \le |b_1|, \dots, |b_N| \le a_j\bigr\}.$$

The heights of the points in $\mathcal{A}$ are all of the form $\log(a_j)$, so they are not heights of K-rational points in any of the grand orbits $\Gamma_1, \dots, \Gamma_r$. This proves that

$$\mathcal{A} \ \cap\ \bigcup_{i=1}^{r}(\Gamma_i \cap \mathbb{P}^N(K)) = \emptyset.$$

On the other hand, it is clear that $\mathcal{A}$ is Zariski dense in $\mathbb{P}^N$. Hence $\mathbb{P}^N(K)$ contains a Zariski dense set of points not lying in any of the grand orbits $\Gamma_1, \dots, \Gamma_r$, which completes the proof of the orbit propagation property (C1).

(b) To simplify formulas, we are going to use Weil and canonical heights relative to the field K. For any subset $\mathcal{P} \subseteq \mathbb{P}^N(K)$, we define a counting function

$$\mathsf{N}(\mathcal{P}, T) := \#\{P \in \mathcal{P} : H(P) \le T\}. \tag{4.6}$$

With our height normalization, Lemma B.7(a) tells us that[3]

$$C_3(K, N)T^{N+1} \le \mathsf{N}(\mathbb{P}^N(K), T) \le C_4(K, N)T^{N+1}. \tag{4.7}$$

Let $P \in X(K)$ be an f-wandering point. Then

$$\begin{aligned} \mathsf{N}(\mathcal{O}_f(P), T) &= \#\{n \ge 0 : H(f^n(P)) \le T\} \\ &\le \#\{n \ge 0 : \hat{h}_f(f^n(P)) \le \log(T) + C_1(f)\} && \text{from (4.2)}, \\ &\le \#\{n \ge 0 : d^n\hat{h}_f(P) \le \log(T) + C_1(f)\} \\ &\le 1 + \log_d\Bigl(\frac{\log T + C_1(f)}{\hat{h}_f(P)}\Bigr) \\ &\le 1 + \log_d\Bigl(\frac{\log T + C_1(f)}{\hat{h}_f^{\min}(\mathbb{P}^N, K)}\Bigr) && \text{from (4.4)}, \\ &\le C_5(K, N, f)\cdot\log\log(T). && (4.8) \end{aligned}$$

---

[3] Schanuel [36] gives a formula, with error term, for $\mathsf{N}(\mathbb{P}^N(K), T)$, but we will not need anything that precise.

We also note that

$$\begin{aligned}
&h(P) > \log(T) + 2C_1(f) \\
&\qquad \Longrightarrow \quad \hat{h}_f(P) > \log(T) + C_1(f) && \text{from (4.2)}, \\
&\qquad \Longrightarrow \quad d^n \hat{h}_f(P) > d^n \log(T) + d^n C_1(f) && \text{for all } n \in \mathbb{N}, \\
&\qquad \Longrightarrow \quad \hat{h}_f(f^n(P)) > d^n \log(T) + d^n C_1(f) && \text{for all } n \in \mathbb{N}, \text{ from (4.1)}, \\
&\qquad \Longrightarrow \quad h(f^n(P)) > d^n \log(T) + (d^n - 1) C_1(f) && \text{for all } n \in \mathbb{N}, \text{ from (4.2)}, \\
&\qquad \Longrightarrow \quad h(f^n(P)) > \log(T) && \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Hence

$$h(P) > \log(T) + 2C_1(f) \quad \Longrightarrow \quad \mathsf{N}(\mathcal{O}_f(P), T) = 0. \tag{4.9}$$

The Zariski closed set Y consists of a finite number of irreducible subvarieties of $\mathbb{P}^N$ of dimension at most $N-1$. It follows Lemma B.7 (b) that

$$\mathsf{N}(Y(K), T) \le C_6(K, Y) \cdot T^N. \tag{4.10}$$

We estimate

$$\begin{aligned}
\mathsf{N}\Bigl(\bigcup_{P \in Y(K)} \mathcal{O}_f(P), T \Bigr) &\le \sum_{P \in Y(K)} \mathsf{N}(\mathcal{O}_f(P), T) \\
&= \sum_{\substack{P \in Y(K) \\ h(P) \le \log(T) + 2C_1(f)}} \mathsf{N}(\mathcal{O}_f(P), T) && \text{from (4.9)}, \\
&\le \sum_{\substack{P \in Y(K) \\ h(P) \le \log(T) + 2C_1(f)}} C_5(K, N, f) \cdot \log\log(T) && \text{from (4.8)}, \\
&= \mathsf{N}(Y(K), C_7(f) \cdot T) \cdot C_5(K, N, f) \cdot \log\log(T) && \text{where } C_7 = e^{2C_1(f)}, \\
&\le C_6(K, Y) \cdot (C_7(f) \cdot T)^N \cdot C_5(K, N, f) \cdot \log\log(T) && \text{from (4.10)}, \\
&\le C_8(K, Y, N, f) \cdot T^N \cdot \log\log(T). && (4.11)
\end{aligned}$$

Combining (4.7) and (4.11) yields

$$\mathsf{N}\Bigl(\mathbb{P}^N(K) \smallsetminus \bigcup_{P \in Y(K)} \mathcal{O}_f(P), T \Bigr) \ge C_3(K, N) T^{N+1} - C_8(K, Y, N, f) \cdot T^N \cdot \log\log(T).$$

This function grows faster than any multiple of T^N, so (4.10) tells us that

$$\mathbb{P}^N(K) \smallsetminus \bigcup_{P \in Y(K)} \mathcal{O}_f(P) \quad \text{is not contained in a Zariski closed subset of } \mathbb{P}^N.$$

This completes the proof of (b).

(c) The assumption that $\deg(f) = 1$ tells us that $f \in \mathrm{PGL}_{N+1}(\bar{\mathbb{Q}})$. Making a change of coordinates, we may assume that f is represented by a matrix $A_f \in \mathrm{GL}_{N+1}(\bar{\mathbb{Q}})$ in Jordan normal form, say

$$A_f = \begin{pmatrix} \lambda_0 & * & 0 & \cdots & 0 \\ 0 & \lambda_1 & * & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_N \end{pmatrix} = \Lambda + \Theta, \tag{4.12}$$

where the stars are 0 or 1. As indicated in (4.12), we write Λ to denote the diagonal matrix with entries $\lambda_0, \dots, \lambda_N$, and we let $\Theta = A_f - \Lambda$ be the nilpotent matrix containing the off-diagonal entries of A_f. In particular, we have

$$\Lambda\Theta = \Theta\Lambda \quad \text{and} \quad \Theta^N = 0.$$

It follows that for all $n \in \mathbb{Z}$,

$$A_f^n = \sum_{j=0}^{N-1} \binom{n}{j} \Lambda^{n-j} \Theta^j. \tag{4.13}$$

We note that (4.13) holds for all integers n, using the usual definition of $\binom{n}{j}$ for $n < 0$, and that it holds for $0 \le n < N$, since $\binom{n}{j} = 0$ for $j > n$.

We claim that our assumption that f has the propagation property (A) implies in particular that the eigenvalues are non-zero, i.e., none of the diagonal entries are 0. To see this, suppose the contrary. Then due to the configuration of Jordan normal form, one of the rows of A_f is 0, say the kth row, where k is some value between 0 and N. It follows that for any point $P \in \mathbb{P}^N$, the orbit of P is not Zariski dense; more precisely,

$$\mathcal{O}_f(P) \subset \{P\} \cup \{x_k = 0\} \subsetneq \mathbb{P}^N.$$

This contradicts our assumption that there is at least one Zariski dense orbit, which completes the proof of the claim that the λ_i are all non-zero.

We set the following notation:

- $K/\mathbb{Q}$ is a number field containing all of the λ_i;
- S is a set of places of K, including all archimedean places, such that $\lambda_i \in R_S^*$ for all i;
- R_S is the ring of S-integers of K.

For $n \in \mathbb{Z}$, we write the corresponding power of A_f as

$$A_f^n = \begin{pmatrix} \mu_{00}^{(n)} & \mu_{01}^{(n)} & \cdots & \mu_{0N}^{(n)} \\ 0 & \mu_{11}^{(n)} & \cdots & \mu_{1N}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{NN}^{(n)} \end{pmatrix}.$$

We note that

$$\mu_{ii}^{(n)} = \lambda_i^n \in R_S^* \text{ for all } i, \quad \text{and} \quad \mu_{ij}^{(n)} \in R_S \text{ for all } i, j.$$

We are first going to prove that f has the propagation property (C2$\forall$). We let $\mathcal{Q} \subset \mathbb{P}^N(K)$ be a complete set of representatives for $\mathbb{P}^N(K)/{\equiv_f}$, and we need to show that $\mathcal{Q}$ is Zariski dense in $\mathbb{P}^N$. So we let $\varphi(\boldsymbol{x})$ be a homogeneous polynomial in $K[\boldsymbol{x}]$ satisfying

$$\varphi(Q) = 0 \quad \text{for all } Q \in \mathcal{Q}, \tag{4.14}$$

and our goal is to prove that $\varphi = 0$.

We let $d = \deg(\varphi)$, and we write φ explicitly as

$$\varphi(\boldsymbol{x}) = \sum_{k_0+k_1+\cdots+k_N=d} a(k_0,k_1,\ldots,k_N)x_0^{k_0}x_1^{k_1}\cdots x_N^{k_N} = \sum_{|\boldsymbol{k}|=d} a(\boldsymbol{k})\boldsymbol{x}^{\boldsymbol{k}},$$

where as indicated, we adopt the notation

$$\boldsymbol{k} = (k_0,\ldots,k_N) \quad \text{and} \quad |\boldsymbol{k}| = k_0 + k_1 + \cdots + k_N. \tag{4.15}$$

We adjoin finitely many additional primes to the set S so that the non-zero coefficients of φ are S-units, i.e.,

$$a(\boldsymbol{k}) \ne 0 \implies a(\boldsymbol{k}) \in R_S^*.$$

By assumption, the set of K-rational points $\mathbb{P}^N(K)$ is a disjoint union

$$\mathbb{P}^N(K) = \bigsqcup_{Q\in\mathcal{Q}} \mathcal{O}_f^{\text{grand}}(Q).$$

Hence for all $P \in \mathbb{P}^N(K)$, there exist a point $Q_P \in \mathbb{P}^N(K)$ and an integer $n_P \in \mathbb{Z}$ such that[4]

$$\varphi(Q_P) = 0 \quad \text{and} \quad P = f^{-n_P}(Q_P).$$

Note that we have complete freedom in our choice of $P \in \mathbb{P}^N(K)$.

We fix a prime $\mathfrak{p}$ of R_K with $\mathfrak{p} \notin S$, and we let $\pi \in R_K$ be a uniformizer for $\mathfrak{p}$. Then we set

$$P(\boldsymbol{r}) = [\pi^{-r_0}, \pi^{-r_1}, \ldots, \pi^{-r_N}] \quad \text{for each } \boldsymbol{r} = (r_0, r_1, \ldots, r_N) \in \mathbb{N}^{N+1}.$$

For the moment, we assume that the r_i are positive and decreasing,

$$r_0 > r_1 > \cdots > r_N \ge 1. \tag{4.16}$$

Later we will specify them more precisely.

We let $0 \le i \le N$, and for $n \in \mathbb{Z}$ we consider the $\mathfrak{p}$-adic valuation of[5]

$$f^n(P(\boldsymbol{r}))[i] := \text{the } i\text{th coordinate of } f^n(P(\boldsymbol{r})).$$

Thus

$$\operatorname{ord}_{\mathfrak{p}}\left(f^n(P(\boldsymbol{r}))[i]\right) = \operatorname{ord}_{\mathfrak{p}}\Bigl(\sum_{j=i}^{N} \mu_{ij}^{(n)} \cdot \pi^{-r_j}\Bigr) = \operatorname{ord}_{\mathfrak{p}}\Bigl(\lambda_i^n \pi^{-r_i} + \sum_{j=i+1}^{N} \mu_{ij}^{(n)} \cdot \pi^{-r_j}\Bigr).$$

---

[4] Since $n_P \in \mathbb{Z}$, it is notationally convenient to insert a negative sign here.

[5] More formally, we lift $P(\boldsymbol{r})$ to $(\pi^{-r_0}, \ldots, \pi^{-r_N}) \in \mathbb{A}^{N+1}$, apply A_f^n, and take the ith coordinate; but for ease of exposition, we will simply refer to the ith coordinate in $\mathbb{P}^N$.

The facts that $\mathfrak{p} \notin S$ and $\lambda_i \in R_S^*$ and $\mu_{ij}^{(n)} \in R_S$ yield

$$\operatorname{ord}_{\mathfrak{p}}(\lambda_i^n \pi^{-r_i}) = -r_i \quad \text{and} \quad \operatorname{ord}_{\mathfrak{p}}(\mu_{ij}^{(n)} \cdot \pi^{-r_j}) \geq -r_j.$$

It follows from (4.16) that we have a strict inequality

$$\operatorname{ord}_{\mathfrak{p}}(\lambda_i^n \pi^{-r_i}) < \operatorname{ord}_{\mathfrak{p}}(\mu_{ij}^{(n)} \cdot \pi^{-r_j}) \quad \text{for all } i < j,$$

and hence the non-archimedean triangle inequality yields

$$\operatorname{ord}_{\mathfrak{p}}\big(f^n(P(\boldsymbol{r}))[i]\big) = \operatorname{ord}_{\mathfrak{p}}(\lambda_i^n \pi^{-r_i}) = -r_i. \tag{4.17}$$

We compute

$$0 = \varphi(Q_{P(\boldsymbol{r})}) = \varphi(f^{n_{P(\boldsymbol{r})}}(P(\boldsymbol{r}))) = \sum_{|\boldsymbol{k}|=d} a(\boldsymbol{k})(f^{n_{P(\boldsymbol{r})}}(P(\boldsymbol{r})))^{\boldsymbol{k}}$$

$$= \sum_{|\boldsymbol{k}|=d} a(\boldsymbol{k}) \prod_{i=0}^{N} f^{n_{P(\boldsymbol{r})}}(P(\boldsymbol{r}))[i]^{k_i}. \tag{4.18}$$

We use (4.17) to compute the valuation of the non-zero monomials appearing in (4.18). Thus if $a(\boldsymbol{k}) \neq 0$, then

$$\operatorname{ord}_{\mathfrak{p}}\big(a(\boldsymbol{k})(f^{n_{P(\boldsymbol{r})}}(P(\boldsymbol{r})))^{\boldsymbol{k}}\big) = \operatorname{ord}_{\mathfrak{p}}(a(\boldsymbol{k})) + \sum_{i=0}^{N} \operatorname{ord}_{\mathfrak{p}}\big(f^{n_{P(\boldsymbol{r})}}(P(\boldsymbol{r}))[i]^{k_i}\big)$$

$$= \sum_{i=0}^{N} -r_i k_i. \tag{4.19}$$

The last equality follows from (4.17) and the fact that the non-zero $a(\boldsymbol{k})$ are S-units.

We now set the r_i to equal

$$r_i = (d+1)^{N-i} \quad \text{for } 0 \leq i \leq N.$$

It then follows from (4.19) and Lemma 4.4 that the non-zero monomials appearing in the expansion (4.18) have distinct negative $\mathfrak{p}$-adic valuations. Thus if $\varphi(\boldsymbol{x})$ has any non-zero monomials, then the non-archimedean triangle inequality implies that the sum cannot equal 0. Since $\varphi(Q_{P(\boldsymbol{r})}) = 0$ by construction, it follows that the polynomial φ has no non-zero monomials, i.e., we have proven that $\varphi = 0$, which completes the proof that $\mathcal{Q}$ is Zariski dense in $\mathbb{P}^N$, and thus the proof that the map f has property (C2∀).

We next invoke Lemma 4.2, which tells us that the set of dense-orbit points $(\mathbb{P}^N)_f^{\text{dense}}$ contains a non-empty Zariski open subset of $\mathbb{P}^N$. This fact, combined with the assumed propagation property (A) and the proven propagation property (C2∀) allows us to apply Lemma B.3 (a) and conclude that f has the propagation property (C3∀). ■

The following result, which is used in the proof of Theorem 4.1 (c), is essentially proven by Ghioca and Hu in [15]. We indicate how to modify their proof to obtain the desired result.

Lemma 4.2. *Let $f\colon \mathbb{P}^N \to \mathbb{P}^N$ be a linear map, and assume that there is at least one Zariski-dense f-orbit. Then X_f^{dense} contains a non-empty Zariski open subset of $\mathbb{P}^N$.*

Proof. By assumption, there is a point $P_0 \in \mathbb{P}^N$ whose f-orbit is Zariski dense. We change coordinates so that f is represented by a matrix in Jordan normal form. Then the hyperplane $H := \{X_N = 0\}$ is f-invariant, so letting $\mathbb{A}^N := \mathbb{P}^N \smallsetminus H_N$ and dehomogenizing, we see that f induces a linear map on

$$A : \mathbb{A}^N \longrightarrow \mathbb{A}^N \quad \text{given by a matrix } A \in \mathrm{GL}_N(K).$$

Further, we must have $P_0 \in \mathbb{A}^N(K)$, since $f(H) \subseteq H$, so points in H_N do not have Zariski dense orbits.

We now follow the proof of Theorem 2.1 in [15]. The assumption that there is a Zariski-dense orbit implies in particular that there are no non-constant rational functions $\varphi \colon \mathbb{A}^N \dashrightarrow \mathbb{P}^1$ satisfying $\varphi \circ A = \varphi$. (Otherwise A-orbits would lie in the fibers of φ.) The conclusion of Theorem 2.1 in [15] is that there exists at least one Zariski-dense A-orbit in $\mathbb{A}^N$, but the proof actually shows that the A-orbit of the specific point $(1, \dots, 1)$ is Zariski dense in $\mathbb{A}^N$. (Remember that we are assuming that A is in Jordan normal form.) We briefly indicate how to modify the proof of Theorem 2.1 in [15] to show that every point

$$\boldsymbol{\beta} = (\beta_1, \dots, \beta_N) \in \mathbb{T}^N := \{[x_0, x_1, \dots, x_N] \in \mathbb{P}^N : x_0 \cdots x_N \neq 0\}$$

has Zariski-dense A-orbit.

The proof in Theorem 2.1 of [15] that $(1, 1, \dots, 1)$ has Zariski-dense A-orbit is reduced to two cases, labeled Cases 4 and 5 in [15]. We start with Case 4, which is the case that A is a diagonal matrix with multiplicatively independent diagonal entries.[6] Let

$$F(x_1, \dots, x_N) = \sum_{i_1,\dots,i_N} c_{i_1,\dots,i_N} \prod_{j=1}^{N} x_j^{i_j} \tag{4.20}$$

be a polynomial such that $F(A^n \boldsymbol{\beta}) = 0$ for all $n \geq 0$. Our goal is to show that $F = 0$. Letting $\Lambda_{i_1,\dots,i_N} = \lambda_1^{i_1} \cdots \lambda_N^{i_N}$, we have

$$F(A^n \boldsymbol{\beta}) = \sum_{i_1,\dots,i_N} (c_{i_1,\dots,i_N} \beta_1^{i_1} \cdots \beta_N^{i_N}) \cdot \Lambda_{i_1,\dots,i_N}. \tag{4.21}$$

As explained in [15], the sequence (4.21) is a non-degenerate linear recurrence as a function of n, so it can vanish for only finitely many values of n unless all of the coefficients $c_{i_1,\dots,i_N} \beta_1^{i_1} \cdots \beta_N^{i_N}$ vanish. But the β_j are non-zero by assumption, so this forces the $c_{i_1,\dots,i_N}$ to vanish, which completes the proof that $F = 0$.

Similarly, in Case 5 of Theorem 2.1 in [15], the matrix A is diagonal except for one 2-dimensional Jordan block. So letting $\lambda_1 = \lambda_2$ be the eigenvalue in the Jordan block,

$$A^n(\boldsymbol{\beta}) = (\beta_1 \lambda_1^n + \beta_2 n \lambda_1^{n-1}, \beta_2 \lambda_1^n, \beta_3 \lambda_3^n, \dots, \beta_N \lambda_N^n).$$

---

[6] We remark that in Case 4, there is an easier proof that avoids the use of deep results on linear recurrences and shows that $(\mathbb{P}^N)_f^{\text{dense}} = \mathbb{T}^N$. One simply starts with one point with dense orbit and translates it to all of $\mathbb{T}^N$ using diagonal matrices, while noting that diagonal matrices commute with A and thus preserve the Zariski density property of the orbit. However, it is unclear whether this translation proof can be adapted to handle the non-diagonal matrices in Case 5.

We define a linear transformation

$$B(x_1,\ldots,x_N)=(\lambda_1(\beta_2x_1-\beta_1x_2),x_2,\ldots,x_N).$$

The fact that $\beta_2\neq 0$ ensures that B is invertible, so it suffices to show that $B\mathcal{O}_A(\boldsymbol{\beta})$ is Zariski dense. We have

$$BA^n(\boldsymbol{\beta})=(\beta_2^2n\lambda_1^n,\beta_2\lambda_1^n,\beta_3\lambda_3^n,\ldots,\beta_N\lambda_N^n).$$

Let $F(\boldsymbol{x})$ be a polynomial as in (4.20) vanishing on $B\mathcal{O}_A(\boldsymbol{\beta})$. Then

$$\begin{aligned}F(BA^n(\boldsymbol{\beta}))&=\sum_{i_1,\ldots,i_N}c_{i_1,\ldots,i_N}(\beta_2^2n\lambda_1^n)^{i_1}\cdot(\beta_2\lambda_1^n)^{i_2}\cdot\prod_{j=3}^{N}(\beta_j\lambda_j^n)^{i_j}\\&=\sum_{k,i_3,\ldots,i_N}\Bigl(\sum_{i_1+i_2=k}c_{i_1,\ldots,i_N}\,\beta_2^{2i_1+i_2}\beta_3^{i_3}\cdots\beta_N^{i_N}n^{i_1}\Bigr)\lambda_1^k\lambda_3^{i_3}\cdots\lambda_N^{i_N}.\end{aligned}$$

As in the proof of Case 5 of Theorem 2.1 in [15], the sequence $F(BA^n(\boldsymbol{\beta}))$ is a non-degenerate linear recurrence, so if it vanishes for infinitely many $n\geq 0$ (much less for all $n\geq 0$), every coefficient must vanish, i.e.,

$$c_{i_1,\ldots,i_N}\beta_2^{2i_1+i_2}\beta_3^{i_3}\cdots\beta_N^{i_N}=0\quad\text{for all }i_1,\ldots,i_N.$$

Then using the fact that $\beta_2\cdots\beta_N\neq 0$, we conclude that $F=0$.

This concludes the proof of the fact that, after an appropriate change of coordinates, the torus $\mathbb{T}^N\subset\mathbb{A}^N\subset\mathbb{P}^N$ is contained in the set $(\mathbb{P}^N)_f^{\text{dense}}$ of points with Zariski-dense f-orbits. ■

We conclude this section with two elementary results. The first describes intervals contained within other intervals that was used in the proof of Theorem 4.1 (a). The second is an injectivity result for dot products of sequences.

Lemma 4.3. *Let $\alpha_1,\ldots,\alpha_r>0$ and $\beta\geq 0$ and $d>1$. For $1\leq i\leq r$ and $n\in\mathbb{N}$, define real intervals*

$$I_{i,n}:=[\alpha_id^n-\beta,\alpha_id^n+\beta]\quad\textit{and}\quad I:=\bigcup_{1\leq i\leq r}\bigcup_{n\in\mathbb{N}}I_{i,n}.$$

There are constants C_9 and $C_{10}>0$, depending only on $\alpha_1,\ldots,\alpha_r$, β and d, so that

$$T\geq C_9\implies[0,T]\setminus I\textit{ contains an interval of length at least }C_{10}T/\log(T).$$

Proof. Fix T. In the following calculation, all constants are independent of T. The number of intervals $I_{i,n}$ that have a point in common with the interval $[0,T]$ is bounded by

$$\begin{aligned}\#\bigl\{(i,n)\in[r]\times\mathbb{N}:I_{i,n}\cap[0,T]\neq\emptyset\bigr\}&\leq\sum_{i=1}^{r}\#\{n\in\mathbb{N}:\alpha_id^n-\beta\leq T\}\\&\leq\sum_{i=1}^{r}\Bigl(\log_d\Bigl(\frac{T+\beta}{\alpha_i}\Bigr)+2\Bigr)\leq C_{11}\log_d(T)\quad\text{for }T\geq C_{12}.\end{aligned}$$

Suppose that $[0,T] \smallsetminus I$ contains no intervals of length at least B. This implies that if we lengthen each $I_{i,n}$ by $B/2$ on each side, then $\{I_{i,n}\}$ will cover all of $[0,T]$. In other words, if we define

$$I_{i,n}(B) := [\alpha_i d^n - \beta - B/2, \alpha_i d^n + \beta + B/2],$$

then the assumption that $[0,T] \smallsetminus I$ contains no intervals of length at least B implies that

$$[0,T] \subseteq \bigcup_{\substack{i\in[r],\, n\in\mathbb{N}\\ I_{i,n}\cap[0,T]\neq\emptyset}} I_{i,n}(B).$$

Hence

$$T \le \sum_{\substack{i\in[r],\, n\in\mathbb{N}\\ I_{i,n}\cap[0,T]\neq\emptyset}} \operatorname{Length}(I_{i,n}(B)) \le \sum_{\substack{i\in[r],\, n\in\mathbb{N}\\ I_{i,n}\cap[0,T]\neq\emptyset}} (2\beta + B) \le (2\beta + B)\cdot(C_{11}\log_d(T)),$$

where the last inequality holds for $T \ge C_{12}$. Rearranging this inequality, we have proven that if $T \ge C_{12}$ and if $[0,T] \smallsetminus I$ contains no intervals of length at least B, then

$$B \ge \frac{T}{C_{11}\log_d(T)} - 2\beta.$$

Adjusting constants, it follows that if $T \ge C_{16}$, then $[0,T] \smallsetminus I$ will contain an interval of length $C_{17}T/\log(T)$. ■

Lemma 4.4. *Define a list of integers*

(4.22) $$r_i = (d+1)^{N-i} \quad \text{for } 0 \le i \le N.$$

Then with notation as in (4.15)*, the map*

(4.23) $$\{\boldsymbol{k} \in \mathbb{N}^{N+1} : |\boldsymbol{k}| = d\} \longrightarrow \mathbb{N}, \quad \boldsymbol{k} \longmapsto \boldsymbol{r}\cdot\boldsymbol{k} = \sum_{i=0}^{N} r_i k_i$$

is injective.

Proof. Writing $|\boldsymbol{a}|_\infty = \max\{a_i\}$ for the sup norm, we have the stronger statement that there are injections

$$\{\boldsymbol{a} \in \mathbb{N}^{N+1} : |\boldsymbol{a}| = d\} \hookrightarrow \{\boldsymbol{a} \in \mathbb{N}^{N+1} : |\boldsymbol{a}|_\infty \le d\} \xhookrightarrow{R_d} \mathbb{N},$$

where R_d is the map

$$R_d(\boldsymbol{a}) = \sum_{i=0}^{N} a_i (d+1)^{N-i}.$$

The injectivity of R_d is simply the statement that every non-negative integer has a unique expression in base $d+1$. (We thank the referee for this simplified argument.) ■

5. K3 surfaces

Theorem 5.1. *Let $X/\bar{\mathbb{Q}}$ be a smooth projective K3 surface, let $f\colon X \to X$ be an endomorphism, and assume that there is a point $P_0 \in X(\bar{\mathbb{Q}})$ whose f-orbit $\mathcal{O}_f(P_0)$ is Zariski dense in X. Then there is a common field of definition K for X, f and P_0 such that for any finite collection of grand f-orbits $\Gamma_1, \dots, \Gamma_r \subset X(\bar{\mathbb{Q}})$, we have*

$$\overline{X(K) \smallsetminus (\Gamma_1 \cup \dots \cup \Gamma_r)} = X. \tag{5.1}$$

In the terminology of Table 1, endomorphisms of X satisfy the orbit propagation statement

$$\text{(A)} \implies \text{(C1)}.$$

Proof. We first recall the well-known fact that all dominant endomorphisms of a smooth algebraic K3 surface X are automorphisms; cf. Section 1.2 of [10]. To see this, we start with the general formula

$$f^*K_X = K_X + (\text{Ramification divisor of } f).$$

Since K3 surfaces have trivial canonical bundle $K_X = 0$, we conclude that f is unramified. But $X(\mathbb{C})$ is simply connected, and hence f is an automorphism.

We note that the existence of a point with Zariski dense f-orbit implies in particular that f, and hence also f^{-1}, have infinite order. Further, as noted in Lemma B.1, the fact that f is an automorphism implies that the grand f-orbit of a point P is the union of the f-orbit of P and the f^{-1}-orbit of P. Hence it suffices to prove (5.1) for f under the assumption that the Γ_i are forward f-orbits, since we can then apply the same reasoning to f^{-1}.

We say that a curve $C \subset X$ is *f-periodic*[7] if $f^n(C) = C$ for some $n \geq 1$. Otherwise we say that C is *f-wandering*. Our first step is to produce an f-wandering rational curve on X. To do this, we use the following two results.

Lemma 5.2 (Theorem 2.1 in [26]). *Let D be a non-trivial effective divisor on X. Then D is linearly equivalent to a sum of rational curves.*

Lemma 5.3 (Corollary 3.5 in Chapter 5 of [21]). *Let D be an ample divisor on X, and let $g\colon X \to X$ be an automorphism such that $g_*D \sim D$. Then g is of finite order.*

Resuming the proof of Theorem 5.1, we suppose that all rational curves in X are f-periodic and derive a contradiction. Let D be an ample effective divisor on X. By Lemma 5.2, there are rational curves $Y_1, \dots, Y_m$ and integers $a_1, \dots, a_m$ such that

$$D \sim a_1Y_1 + \dots + a_mY_m.$$

The rational curves Y_j are assumed to be f-periodic. Let $p \geq 1$ be a positive integer such that $f^p(Y_j) = Y_j$ for all $1 \leq j \leq m$. Since f is an automorphism, this implies that $f_*^pY_j = Y_j$ as divisors. Hence

$$f_*^pD \sim a_1f_*^pY_1 + \dots + a_mf_*^pY_m = a_1Y_1 + \dots + a_mY_m \sim D.$$

Since D is ample by assumption, Lemma 5.3 implies that f^p has finite order, which contradicts the assumption that there exists a Zariski dense f-orbit.

---

[7] We note that since f is an automorphism, a curve is f-periodic if and only if it is f-preperiodic.

We let Y be an f-wandering rational curve on X. Replacing K with a finite extension, which by abuse of notation we continue to call K, we may assume that Y is defined over K and that $Y(K)$ is Zariski dense in Y.

Let $\mathcal{F} \subset X(K)$ be a finite set of points, and let

$$\mathcal{O}_f^{\text{grand}}(\mathcal{F}) := \bigcup_{Q\in\mathcal{F}} \mathcal{O}_f^{\text{grand}}(Q) = \bigcup_{Q\in\mathcal{F}} \bigl(\mathcal{O}_f(Q) \cup \mathcal{O}_{f^{-1}}(Q)\bigr)$$

be the union of the f-orbits of the points in $\mathcal{F}$. Our goal is to prove that $X(K)$ has a Zariski dense set that is disjoint from $\mathcal{O}_f^{\text{grand}}(\mathcal{F})$.

We know from earlier that $Y(K)$ is Zariski dense in Y. Further, the dynamical Mordell–Lang conjecture is true for étale maps [6], so in particular for automorphisms such as f and f^{-1}. Hence

$$\mathcal{O}_f^{\text{grand}}(Q) \cap C \text{ is finite for every point } Q \in X(K) \text{ and every curve } C \subset X. \tag{5.2}$$

In particular, since $f^n(Y)$ is a curve for every $n \geq 0$, and since $\mathcal{O}_f^{\text{grand}}(\mathcal{F})$ is a finite union of f-orbits and f^{-1}-orbits, it follows from (5.2) that

$$\underbrace{f^n(Y(K)) \smallsetminus \mathcal{O}_f^{\text{grand}}(\mathcal{F})}_{\text{this is an infinite subset of the curve } f^n(Y).} \quad \text{is Zariski dense in } f^n(Y). \tag{5.3}$$

But the fact that Y is wandering implies that the union of curves

$$\bigcup_{n\geq 0} f^n(Y) \quad \text{is Zariski dense in } X, \tag{5.4}$$

and combining (5.3) and (5.4), we conclude that

$$\bigcup_{n\geq 0} \bigl(f^n(Y(K)) \smallsetminus \mathcal{O}_f^{\text{grand}}(\mathcal{F})\bigr) \quad \text{is Zariski dense in } X.$$

Hence

$$X(K) \smallsetminus \mathcal{O}_f^{\text{grand}}(\mathcal{F}) \supseteq \Bigl(\bigcup_{n\geq 0} f^n(Y(K))\Bigr) \smallsetminus \mathcal{O}_f^{\text{grand}}(\mathcal{F}),$$

is also Zariski dense in X, which completes the proof that (A) implies (C1) for K3 surfaces. ■

6. Abelian varieties

In this section, we prove that our strongest propagation principle is true for geometrically simple abelian varieties.

Theorem 6.1. *Let $X/\overline{\mathbb{Q}}$ be a geometrically simple abelian variety, let $f\colon X \to X$ be an endomorphism of X (as an abstract variety), and let $P_0 \in X(\overline{\mathbb{Q}})$ be a point such* *that $\mathcal{O}_f(P_0)$ is Zariski dense in X. Then there is a number field $K/\mathbb{Q}$ such that X and f are defined over K and such that every complete set of representatives for*

$$X_f^{\mathrm{dense}}(K)/{\equiv_f} \quad \textit{is Zariski dense in } X.$$

In the terminology of Table 1, endomorphisms of geometrically simple abelian varieties satisfy the orbit propagation statement

$$\text{(A)} \implies \text{(C3}\forall\text{)}.$$

Remark 6.2. The methods used by Ghioca and Scanlon [18] to prove the Zariski density conjecture (Conjecture 1.7) for abelian varieties might be adaptable to removing the simplicity assumption in Theorem 6.1, but we will not pursue this in the present article.

Remark 6.3. More generally, let X be a split semi-abelian variety, and let $f: X \to X$ be an endomorphism, all defined over $\overline{\mathbb{Q}}$. There is an explicit construction in [16] that produces points in $X(\overline{\mathbb{Q}})$ having Zariski dense f-orbits. The paper [16] does not appear to discuss whether one can create many such points defined over a fixed number field, but the construction is sufficiently explicit that it might give an alternative approach to proving orbit propagation results for (semi)-abelian varieties.

Proof of Theorem 6.1. We start by fixing a field of definition K for X, f, and P_0, but we may at times replace K with a finite extension. Every map between abelian varieties is the composition of a homomorphism and a translation (Corollary 1 in Section 4 of [32]), say

$$f(x) = \varphi(x) + Q_0, \quad \text{with } \varphi: X \to X \text{ a homomorphism and with } Q_0 \in X(K).$$

The assumption that $P_0 \in X(K)$ satisfies

$$\overline{\mathcal{O}_f(P_0)} = X$$

implies, in particular, that f is dominant, and thus that φ is an isogeny.

The assumption that X is geometrically simple implies that the kernel of $\varphi - 1$ is either a finite subgroup of X or all of X. We consider these two case in turn.

Case 1. $\operatorname{Ker}(\varphi - 1)$ *is finite.*

In this case $\varphi - 1$ is an isogeny, i.e., the map $\varphi - 1: X \to X$ is a finite surjective map. Hence we can find a point $R_0 \in X(\bar{K})$ satisfying

$$(\varphi - 1)(R_0) = Q_0.$$

Writing T_P in general for the translation-by-P map, we have

$$\begin{aligned} T_{R_0} \circ f \circ T_{-R_0}(x) &= f(x - R_0) + R_0 = \varphi(x - R_0) + Q_0 + R_0 \\ &= \varphi(x) - \varphi(R_0) + Q_0 + R_0 = \varphi(x). \end{aligned}$$

Thus f and φ are conjugate via the automorphism $T_{R_0} \in \operatorname{Aut}(X)$, so they have the same dynamics. Hence in Case 1, it suffices to consider the case that f itself is an isogeny.

Let $\mathcal{M}$ be an infinite set of positive integers with the following properties:

$$\gcd(\deg(f), m) = 1 \quad \text{for all } m \in \mathcal{M}. \tag{6.1}$$

$$\gcd(m_1, m_2) = 1 \quad \text{for all distinct } m_1, m_2 \in \mathcal{M}. \tag{6.2}$$

For example, we could take $\mathcal{M}$ to be the set of all primes not dividing $\deg(f)$. Then for $m_1, m_2 \in \mathcal{M}$, we compute

$$\begin{aligned}
&\mathcal{O}_f(m_1P_0) \cap \mathcal{O}_f(m_2P_0) \neq \emptyset \\
&\iff f^{n_1}(m_1P_0) = f^{n_2}(m_2P_0) && \text{for some } n_1, n_2 \in \mathbb{N}, \\
&\implies (f^{n_1} \circ m_1 - f^{n_2} \circ m_2)(P_0) = 0 && \text{for some } n_1, n_2 \in \mathbb{N}, \\
&\implies f^{n_1} \circ m_1 - f^{n_2} \circ m_2 = 0 && \text{in } \mathbb{Z}[f] \subseteq \operatorname{Isog}(X) \text{ for some } n_1, n_2 \in \mathbb{N}, \\
& && \text{since } P_0 \text{ is non-torsion because its } f\text{-orbit} \\
& && \text{is Zariski dense,} \\
&\implies (\deg f)^{n_1} \cdot m_1^{2g} = \deg(f)^{n_2} \cdot m_2^{2g} && \text{for some } n_1, n_2 \in \mathbb{N}, \\
&\implies m_1 = m_2 && \text{from (6.1) and (6.2).}
\end{aligned}$$

Taking the contrapositive, we have proven that

$$m_1, m_2 \in \mathcal{M}_f \text{ and } m_1 \neq m_2 \implies \mathcal{O}_f(m_1P_0) \cap \mathcal{O}_f(m_2P_0) = \emptyset.$$

Hence the grand f-orbits generated by the points in $\{mP_0 : m \in \mathcal{M}_f\}$ are distinct.

We claim that the f-orbits of these points are also Zariski dense. To prove this, we note that since f is an isogeny, it commutes with the multiplication-by-m, and since f is a finite map of a proper variety, it sends closed sets to closed sets. Hence

$$\begin{aligned}
\overline{\mathcal{O}_f(mP_0)} &= \overline{m \cdot \mathcal{O}_f(P_0)} && \text{since } fm = mf, \\
&= m \cdot \overline{\mathcal{O}_f(P_0)} && \text{since } m \text{ is finite and } X \text{ is proper.}
\end{aligned} \tag{6.3}$$

It follows immediately from (6.3) that[8]

$$\overline{\mathcal{O}_f(P_0)} = X \implies \overline{m\mathcal{O}_f(P_0)} = X.$$

Since the f-orbit of P_0 is Zariski dense by assumption, the same is true of the f-orbit of mP_0 for all positive integers m.

We let

$$\mathcal{M} \cdot P_0 := \{mP_0 : m \in \mathcal{M}\},$$

and we consider the following statements:

(1) Every point in $\mathcal{M} \cdot P_0$ has a Zariski dense f-orbit.

(2) The points in $\mathcal{M} \cdot P_0$ generate distinct grand f-orbits.

(3) The set $\mathcal{M} \cdot P_0$ is Zariski dense.

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[8] The converse also holds, since if $\mathcal{O}_f(P_0)$ is not Zariski dense, then its Zariski closure is a subvariety whose dimension is strictly smaller than $\dim(X)$, and multiplication-by-m preserves the dimension, so $\mathcal{O}_f(mP_0)$ is also not Zariski dense.

We have proven (1) and (2), and we now prove (3). We first show that $\mathcal{M} \cdot P_0$ is an infinite set. Since $\mathcal{M}$ is an infinite set of integers, it suffices to show that P_0 is a non-torsion point. To see this, we assume that $mP_0 = 0$ for some non-zero integer and derive a contradiction. The fact that f is an isogeny implies that f commutes with the multiplication-by-m map, so we have

$$\mathcal{O}_f(P_0) \subseteq \mathbb{Z}[f] \cdot P_0 = (\mathbb{Z}[f]/m\mathbb{Z}[f]) \cdot P_0.$$

The ring $\mathbb{Z}[f]/m\mathbb{Z}[f]$ is finite, so the f-orbit $\mathcal{O}_f(P_0)$ is finite, i.e., the point P_0 is f-preperiodic. This contradicts the assumption that $\mathcal{O}_f(P_0)$ is Zariski density, and completes the proof that $\mathcal{M} \cdot P_0$ is an infinite set of points.

The Zariski closure $\overline{\mathcal{M} \cdot P_0}$ contains the infinite set $\mathcal{M} \cdot P_0$, which in turn is a subset of the finitely generated (indeed, rank 1) subgroup $\mathbb{Z} \cdot P_0$ of X. It follows from Faltings' theorem [12] (originally the Mordell–Lang conjecture) that $\overline{\mathcal{M} \cdot P_0}$ contains a translate of an abelian subvariety of X, necessarily positive dimensional since $\#(\mathcal{M} \cdot P_0) = \infty$. The assumed simplicity of X tells us that the only such abelian subvariety is X itself. Hence $\overline{\mathcal{M} \cdot P_0} = X$, which completes the proof of (3).

We can restate (1) as the inclusion $\mathcal{M} \cdot P_0 \subseteq X_f^{\text{dense}}$, and then (3) tells us that $X_f^{\text{dense}}(K)$ contains a Zariski dense set of points. We have thus proven the following useful fact:

$$\overline{X_f^{\text{dense}}(K)} = X. \tag{6.4}$$

Our next goal is to prove that f has the propagation property (C2∀). So we let $\mathcal{Q} \subset X(K)$ be a set of representatives for the grand f-orbits, i.e., a set satisfying

$$\mathcal{Q} \overset{\text{bijective}}{\longleftrightarrow} X(K)/{\equiv_f}, \tag{6.5}$$

and we need to show that $\mathcal{Q}$ is Zariski dense. We let

$$Y = \bar{\mathcal{Q}} \subseteq X,$$

and our goal is to prove that $Y = X$.

We proved earlier that the set $\mathcal{M} \cdot P_0$ is Zariski dense and that each point in $\mathcal{M} \cdot P_0$ is in a distinct grand f-orbit. The choice (6.5) of $\mathcal{Q}$ says that every grand f-orbit containing a K-rational point will contain a unique point of $\mathcal{Q}$. This allows us to define an injection

$$\psi : \mathcal{M} \cdot P_0 \hookrightarrow \mathcal{Q}, \qquad \left(\begin{array}{l} \psi(mP_0) \text{ is the unique point } Q \in \mathcal{Q} \\ \text{such that } \mathcal{O}_f(Q)^{\text{grand}} = \mathcal{O}_f(mP_0)^{\text{grand}} \end{array} \right).$$

The existence of Zariski dense f-orbit $\mathcal{O}_f(P_0)$ tells us, in particular, that $f \neq 0$, so there exists an isogeny $g \colon X \to X$ satisfying

$$f \circ g = g \circ f = q \in \mathbb{Z}_{>0}.$$

The condition that $\mathcal{O}_f(Q)^{\text{grand}} = \mathcal{O}_f(mP_0)^{\text{grand}}$ with $Q = \psi(mP_0)$ says that there are integers $n_1, n_2 \geq 0$ satisfying

$$f^{n_1}(Q) = f^{n_2}(mP_0). \tag{6.6}$$

Applying g^{n_1} to both sides of (6.6), we find that for $mP_0 \in \mathcal{M}$, there are non-negative integers n_1 and n_2 satisfying

$$q\psi(mP_0) = g^{n_1} \circ f^{n_1}(\psi(mP_0)) = g^{n_1} \circ f^{n_2}(mP_0) \in \operatorname{Isog}(X) \cdot P_0.$$

Hence[9]

$$\psi(mP_0) \in (\operatorname{Isog}(X) \cdot P_0)^{\mathrm{div}} \quad \text{for all } m \in \mathcal{M}.$$

To recapitulate, we have proven that

$$\psi(\mathcal{M} \cdot P_0) \subseteq \mathcal{Q} \cap (\operatorname{Isog}(X) \cdot P_0)^{\mathrm{div}} \subseteq Y \cap (\operatorname{Isog}(X) \cdot P_0)^{\mathrm{div}}. \tag{6.7}$$

The ring of isogenies $\operatorname{Isog}(X)$ is a finitely generated $\mathbb{Z}$-module. Indeed, it is a classical result that $\operatorname{Isog}(X)$ is an order in a central simple algebra of rank at most $2\dim(X)$; see, for example, Section 19 of [32]. Hence the subgroup $\operatorname{Isog}(X) \cdot P$ is a finitely generated subgroup of X. We now apply [29], which represents a culmination of fundamental work of Faltings, Vojta, Raynaud, Hindry, and others on the intersection of a subvariety of a semi-abelian variety with a subgroup of finite type. This result tells us that

$$\overline{Y \cap (\operatorname{Isog}(X) \cdot P_0)^{\mathrm{div}}} = \left(\begin{array}{l}\text{a union of finitely many translates of abelian} \\ \text{subvarieties of } X\end{array}\right). \tag{6.8}$$

It follows from (6.7) that the set (6.8) contains $\psi(\mathcal{M} \cdot P_0)$. We know that ψ is injective, and we proved earlier that $\mathcal{M} \cdot P_0$ is an infinite set. It follows that at least one of the abelian subvarieties appearing in the set (6.8) is positive dimensional. On the other hand, since Y is Zariski closed, it contains the set (6.8), so we conclude that Y contains a translate of a positive dimensional abelian subvariety of X. Our assumption that X is geometrically simple implies X has no non-trival positive dimensional abelian subvarieties. Hence $Y = X$, which concludes our proof that f has the propagation property (C2∀).

Our ultimate goal is to prove property (C3∀). Since we have already proven (C2∀), Lemma B.3 (b) tells us that it suffices to prove that

$$\operatorname{Image}\bigl((X \smallsetminus X_f^{\mathrm{dense}})(K) \longrightarrow X(K)/\equiv_f\bigr) \quad \text{is finite,} \tag{6.9}$$

i.e., to prove that the points in $X(K)$ whose f-orbit is not Zariski dense lie in finitely many grand f-orbits. To see why (6.9) is true, let $\Gamma \subset X$ be a non-Zariski dense grand orbit that contains a K-rational point $Q \in X(K)$. Then we have the following facts:

- The forward orbit $\mathcal{O}_f(Q)$ is not Zariski dense in X.
- The subgroup $\mathbb{Z}[f]Q$ of X generated by the points in $\mathcal{O}_f(Q)$ is a finitely generated subgroup of $X(K)$, since $\mathbb{Z}[f]$ is a finitely-generated $\mathbb{Z}$-module.
- The intersection

  $$\overline{\mathcal{O}_f(Q)} \cap \mathbb{Z}[f]Q \quad \text{is Zariski dense in} \quad \overline{\mathcal{O}_f(Q)},$$

  since $\mathcal{O}_f(Q) \subseteq \mathbb{Z}[f]Q$.

---

[9] We use that notation that for any subgroup $Z \subset X(\bar{K})$, the division subgroup associated to Z is $Z^{\mathrm{div}} := \{P \in X(\bar{K}) : nP \in Z \text{ for some integer } n \geq 1\}$. For example, $\{0\}^{\mathrm{div}} = X(\bar{K})_{\mathrm{tors}}$.

These three observations and Faltings' theorem [12] allow us to conclude that $\overline{\mathcal{O}_f(Q)}$ is a finite union of torsion-point translates of proper abelian subvarieties of X. The assumption that X is geometrically simple implies that the only proper abelian subvariety of X is $\{0\}$. Hence $\mathcal{O}_f(Q)$ consists entirely of torsion points, and in particular, the point Q itself is a torsion point.

We have thus shown that if Γ is a grand orbit such that $\Gamma \cap X(K)$ is non-empty and not Zariski dense, then

$$\Gamma \cap X(K)_{\text{tors}} \neq \emptyset,$$

i.e., Γ contains a K-rational torsion point of X. The set $X(K)_{\text{tors}}$ is finite, which proves that there are only finitely many such Γ.

Case 2. $\operatorname{Ker}(\varphi - 1) = X$.

In this case $\varphi = 1$, so f is a pure translation

$$f(x) = x + Q_0.$$

Hence for all $n \in \mathbb{N}$, we have $f^n(x) = x + nQ_0$, which shows that

$$\mathcal{O}_f(x) = x + \mathbb{N} \cdot Q_0. \tag{6.10}$$

It follows that for any $P \in X$, we have

$$\mathcal{O}_f(P) = P + \mathbb{N} \cdot Q_0 = (P - P_0) + (P_0 + \mathbb{N} \cdot Q_0) = (P - P_0) + \mathcal{O}_f(P_0),$$

i.e., all f-orbits are translates of one another. Hence the fact that $\mathcal{O}_f(P_0)$ is Zariski dense in X implies that every f-orbit is Zariski dense in X, i.e.,

$$X_f^{\text{dense}} = X. \tag{6.11}$$

It follows from (6.11) that

$$X_f^{\text{dense}}(K)/{\equiv_f} \xleftrightarrow{\text{bijective}} X(K)/{\equiv_f}.$$

Thus (C2∀) and (C3∀) are equivalent statements, so it suffices to prove the former.

We replace K with a finite extension, which by abuse of notation we again denote by K, such that

$$\operatorname{rank} X(K) \geq 2.$$

This is always possible. More generally it is known that $\operatorname{rank} X(\bar{K}) = \infty$; see for example Theorem 10.1 in [13]. We let $Q_1 \in X(K)$ be a point such that Q_0 and Q_1 are $\mathbb{Z}$-linearly independent.

We claim that distinct multiples of Q_1 have distinct grand f-orbits, i.e., the map

$$\begin{pmatrix} \mathbb{Z} \longrightarrow \{\text{grand } f\text{-orbits}\} \\ m \longmapsto \mathcal{O}_f^{\text{grand}}(mQ_1) \end{pmatrix} \quad \text{is injective.} \tag{6.12}$$

To prove (6.12), we let $m, m' \in \mathbb{Z}$ and compute

$$\begin{aligned}
&\mathcal{O}_f(mQ_1)^{\text{grand}} \cap \mathcal{O}_f(m'Q_1)^{\text{grand}} \ne \emptyset \\
&\iff \quad f^n(mQ_1) = f^{n'}(m'Q_1) && \text{for some } n, n' \in \mathbb{N}, \\
&\iff \quad mQ_1 + nQ_0 = m'Q_1 + n'Q_0 && \text{for some } n, n' \in \mathbb{N}, \\
&\iff \quad (m-m')Q_1 = (n'-n)Q_0 && \text{for some } n, n' \in \mathbb{N}, \\
&\implies \quad m = m' && \text{since } Q_0 \text{ and } Q_1 \text{ are } \mathbb{Z}\text{-linearly independent.}
\end{aligned}$$

This completes the proof that distinct points in $\mathbb{Z} \cdot Q_1$ have distinct grand f-orbits.

Let $\mathcal{Q} \subset X(K)$ be a set of representatives for $X(K)/{\equiv_f}$. To prove that (C2∀) is true, we need to show that $\overline{\mathcal{Q}} = X$, i.e., that $\mathcal{Q}$ is Zariski dense in X.

For each $m \in \mathbb{Z}$, the grand orbit

$$\mathcal{O}_f^{\text{grand}}(mQ_1) = f^{\mathbb{Z}}(mQ_1) = mQ_1 + \mathbb{Z}Q_0$$

contains a unique point in $\mathcal{Q}$, say

$$f^{\nu(m)}(mQ_1) \in \mathcal{Q}, \quad \text{where } \nu(m) \in \mathbb{Z}.$$

We thus have

$$mQ_1 + \nu(m)Q_0 \in \mathcal{Q} \quad \text{for all } m \in \mathbb{Z}. \tag{6.13}$$

The points in (6.13) are distinct, since they lie in distinct grand orbits from (6.12), so we see that

$$\mathcal{Q} \cap (\mathbb{Z}Q_1 + \mathbb{Z}Q_0) \quad \text{is an infinite set.}$$

It follows that $\overline{\mathcal{Q}}$ is a Zariski closed subset of X that has infinite intersection with the infinite finitely generated subgroup $\mathbb{Z}Q_1 + \mathbb{Z}Q_0$ of X. It follows from Faltings' theorem [12] that $\overline{\mathcal{Q}}$ contains a translate of a positive dimensional abelian subvariety of X. The assumed geometric simplicity of X says that the only such subvariety is X itself, which completes the proof that $\overline{\mathcal{Q}} = X$. ∎

7. Étale maps and p-adic dimension

In this section, we use p-adic arguments to prove orbit propagation results for étale maps.

Definition 7.1. We need to explain what we mean by the *p-adic dimension* of an algebraic variety. For a smooth variety, we use the same analytic definition as for $\mathbb{R}$ and $\mathbb{C}$, namely we use local charts and locally convergent power series, and we declare that $\mathbb{A}^n(\mathbb{Q}_p)$ has p-adic dimension n. For a singular variety, we write the variety as a union of smooth varieties by inductively removing singular loci, and then the p-adic dimension of $X(\mathbb{Q}_p)$ is the maximum of the p-adic dimensions of the smooth varieties appearing in the decomposition. And since we will be using two notions of dimension in the section, we will write $\dim_{p\text{-adic}}$ for the p-adic dimension of a p-adic set, and we will write $\dim_{\text{Krull}}$ for the usual Krull, or algebraic, dimension of an algebraic variety.

Lemma 7.2. (a) *Let $X/\mathbb{Q}_p$ be a smooth algebraic variety. Then*

$$X(\mathbb{Q}_p) \neq \emptyset \implies \dim_{p\text{-adic}} X(\mathbb{Q}_p) = \dim_{\mathrm{Krull}} X.$$

(b) *Let K be a number field, and let X/K be a smooth algebraic variety. Then for all but finitely many primes v of K satisfying $K_v = \mathbb{Q}_v$, we have*

$$\dim_{p\text{-adic}} X(K_v) = \dim_{\mathrm{Krull}} X.$$

(c) *Let $X/\mathbb{Q}_p$ be a smooth algebraic variety, and let $Z \subsetneq X$ be a proper* (*not necessarily smooth*) *algebraic subset of Z defined over $\mathbb{Q}_p$. Then*

$$\dim_{p\text{-adic}} Z(\mathbb{Q}_p) < \dim_{\mathrm{Krull}} X.$$

(d) *Let K be a number field, let X/K be a smooth projective variety with $X(K) \neq \emptyset$, and suppose that there are infinitely many degree 1 primes v of K such that $X(K)$ is v-adically dense in $X(K_v)$. Then $X(K)$ is Zariski dense in X.*

Proof. (a) Let $n = \dim_{\mathrm{Krull}} X$ and $Q \in X(\mathbb{Q}_v)$. The smoothness of X and the v-adic implicit function theorem (see [38], p. 73) say that there is neighborhood of Q that is v-adic analytically isomorphic to a neighborhood of the origin in $\mathbb{A}^n(\mathbb{Q}_v)$.

(b) We choose a model $\mathcal{X}$ for X over the ring of integers of K. The smoothness of X tells us that the reduction of $\mathcal{X}$ modulo v is smooth for all but finitely many primes v. If $\mathrm{char}(v)$ is sufficiently large, then the standard Lang–Weil estimate (see [25]) shows that $X(\mathbb{F}_v) \neq \emptyset$, and then smoothness and Hensel's lemma show that $X(K_v) \neq \emptyset$. Restricting to primes with $K_v = \mathbb{Q}_v$, the desired result follows from (a).

(c) We can check the assertion on Zariski open sets, and by Noetherian induction on the dimension, we are reduced to proving that if $V/\mathbb{Q}_p$ is a smooth (affine) variety, then

$$\dim_{p\text{-adic}} V(\mathbb{Q}_p) \leq \dim_{\mathrm{Krull}} V.$$

If $V(\mathbb{Q}_p) = \emptyset$, this is vacuously true. Otherwise we may apply (a).

(d) We write Cl_v for the v-adic closure of a set and Cl_Z for the Zariski closure. So our assumption is that there are infinitely many degree 1 primes v with $\mathsf{Cl}_v(X(K)) = X(K_v)$. We use this assumption and (b) to deduce that there is a degree 1 prime v, say of characteristic $p \geq 3$, such that

$$\dim_{p\text{-adic}} \mathsf{Cl}_v(X(K)) = \dim_{p\text{-adic}} X(K_v) = \dim_{\mathrm{Krull}} X. \tag{7.1}$$

On the other hand, if $\mathsf{Cl}_Z(X(K))$ is a proper subset of X, then (c) says that

$$\dim_{p\text{-adic}}(\mathsf{Cl}_Z(X(K)))(K_v) < \dim_{\mathrm{Krull}} X \quad \text{(strict inequality)}. \tag{7.2}$$

Since (7.1) and (7.2) are incompatible, we conclude that $\mathsf{Cl}_Z(X(K))$ is equal to X. ■

Proposition 7.3. *Let $p \geq 3$ be prime, let $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathbb{Z}_p$ be a smooth projective morphism with geometrically irreducible fibers, and let $F\colon \mathcal{X} \to \mathcal{X}$ be a finite étale $\mathbb{Z}_p$-morphism. Let $X = \mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Q}_p$ be the generic fiber of π, and let $f\colon X \to X$ be the restriction of F to the generic fiber. Let $x \in X(\mathbb{Q}_p)$. Then the p-adic closure of $\mathcal{O}_f(x)$ in $X(\mathbb{Q}_p)$ consists of finitely many p-adic arcs, so in particular it has p-adic dimension* 1.

Proof. We let $X' = \mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p$ be the special fiber of π, and we let n be the relative dimension n of π, so X and X' are smooth projective varieties of dimension n. By properness, the point $x \in X(\mathbb{Q}_p)$ extends to a section $\sigma_x\colon \operatorname{Spec} \mathbb{Z}_p \to \mathcal{X}$ of π, and the reduction of σ_x to the special fiber gives a point $x' \in X'(\Phi_p)$. Since $X(\Phi_p)$ is finite, there is no loss of generality if we assume that $f(x') = x'$.

Expanding f locally on the residue disk of x' centered along σ_x, we see that f is given by an n-tuple of convergent power series $g_1, \dots, g_n \in \mathbb{Z}_p[\![t_1, \dots, t_n]\!]$, where the t_j are local parameters along σ. As in Proposition 2.2 of [6], we may choose the t_j so that the g_j are congruent to linear polynomials modulo p. Since f is étale, the Jacobian of the g_j reduces modulo p to a matrix in $\mathrm{GL}_n(\mathbb{F}_p)$ that is necessarily of finite order. Replacing f by a suitable iterate, we get that

$$(g_1, \dots, g_n) \equiv (t_1, \dots, t_n) \pmod{p}.$$

It now follows from Theorem 7 in [1] (see also [35]) that the map

$$\mathbb{N} \longrightarrow X(\mathbb{Q}_p), \quad n \longmapsto f^n(x),$$

extends to a locally p-adic analytic map

$$\mathbb{Z}_p \longrightarrow X(\mathbb{Q}_p). \tag{7.3}$$

Hence the p-adic closure of $\mathcal{O}_f(x)$ is contained in the image of (7.3). ■

Theorem 7.4. *Let K be a number field, let X/K be a smooth projective variety of dimension at least* 2, *and let $f\colon X \to X$ be a finite étale morphism. Suppose that there are infinitely many degree* 1 *primes v of K, i.e., primes satisfying $K_v = \mathbb{Q}_v$, such that $X(K)$ is v-adically dense in $X(\mathbb{Q}_v)$. Then* (B1) *in Table* 1 *is true for* (X, f), *i.e., if $\Gamma_1, \dots, \Gamma_r \subset X(K)$ are f-orbits, then*

$$X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r) \quad \textit{is Zariski dense in } X.$$

Before proving Theorem 7.4, we state and prove two corollaries, one of which will require a preliminary lemma.

Corollary 7.5. *Let X/K be a smooth rational variety defined over a number field, and let $f\colon X \to X$ be a finite étale morphism defined over K. Then* (B1) *in Table* 1 *is true for* (X, f). *In particular,* (B1) *is true for automorphisms of rational varieties.*

Corollary 7.6. *Let X be an étale quotient of an abelian variety,*[10] *and let $f\colon X \to X$ a morphism. Then* (B1) *in Table* 1 *is true for* (X, f). *In particular,* (B1) *is true for abelian varieties and for bielliptic surfaces.*

Proof of Corollary 7.5. Possibly after replacing K with a finite extension, we can find a birational map $\varphi\colon \mathbb{P}^N \dashrightarrow X$ defined over K, and we let $U \subseteq \mathbb{P}^N$ be a non-empty Zariski open subset defined over K such that $\varphi|_U$ is an isomorphism onto its image. The fact

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[10] In other words, there is an abelian variety A and a finite group of automorphisms $G \subset \mathrm{Aut}(A)$ (not necessarily isogenies) so that $X \cong A/G$ and $A \to X$ is étale.

that U is an open subset of $\mathbb{P}^N$ implies that $U(K)$ is v-adically dense in $U(K_v)$ for all places of K. (This follows simply from the fact that K is v-adically dense in K_v.) It follows that $\varphi(U(K)) = \varphi(U)(K)$ is v-adically dense in $\varphi(U)(K_v)$, and then since $\varphi(U)$ is a Zariski open subset of the irreducible variety X, we see that $\varphi(U)(K)$ is v-adically dense in $X(K_v)$. Hence the large set $X(K)$ is v-adically dense in $X(K_v)$, and the desired conclusion follows from Theorem 7.4. ■

Lemma 7.7. *Let A be abelian variety, let $\pi\colon A \to B$ be an étale map, and let $f\colon B \to B$ be a surjective morphism. Then f is étale.*

Proof. We let X be the fiber product of A and B relative to the map π and f, so we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "p"'] & A \arrow[d, "\pi"'] \\
B \arrow[r, "f"] & B.
\end{tikzcd}$$

The assumption that π is étale implies that X is smooth and p is étale.

The fact that p is étale tells us that $\kappa(X) = 0$ and that Ω^1_X is semiample, since B has these properties, so [14] allows us to conclude that X is the quotient of an abelian variety C. Considering the composite ma $C \to X \to A$, we conclude that $X \to A$ is étale, and thus that $f\colon B \to B$ is also étale. ■

Proof of Corollary 7.6. Lemma 7.7 says that it is enough to check the case that X is an abelian variety. Since every map between dominant map $X \to X$ of an abelian variety is étale, Theorem 7.4 says that it suffices to check that for all but finitely many places v of K, the set of K-rational points $X(K)$ is v-adically dense in $X(K_v)$. This will be true provided that the rank of the Mordell-Weil group of each simple isogeny factor of $X(K)$ is sufficiently large with respect to the dimension of X; see [40]. And we can make these ranks sufficiently large by taking a finite extension of K. ■

Proof of Theorem 7.4. We choose a model $\mathcal{X} \to \operatorname{Spec}(\mathcal{O}_K)$ of X over the ring of integers $\mathcal{O}_K$ of K, and then f extends to a rational map $F : \mathcal{X} \dashrightarrow \mathcal{X}$. For any finite set of places S of K that includes the archimedean places, we denote the ring of S-integers by $\mathcal{O}_{K,S}$. We choose such a set S having the property that the scheme $\mathcal{X}_S = \mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{K,S}$ is smooth and proper over $\mathcal{O}_{K,S}$ with geometrically irreducible fibers and such that the map $F_S\colon \mathcal{X}_S \to \mathcal{X}_S$ is a dominant étale morphism. Since the degree 1 primes have positive density, the assumption of Theorem 7.4 and the finiteness of S imply that we can find a degree 1 prime $v \notin S$ of characteristic $p \geq 3$ such that $X(K)$ is p-adically dense in $X(\mathbb{Q}_p)$. Using the smoothness of X, the Lang–Weil estimate [25], and Hensel's lemma, we may also assume that $X(\mathbb{Q}_p) \neq \emptyset$.

Using the closure notation from the proof of Lemma 7.2(d), we have

$$\begin{aligned}
2 &\leq \dim_{\mathrm{Krull}} X && \text{by assumption,} \\
&= \dim_{p\text{-adic}} X(\mathbb{Q}_p) && \text{from Lemma 7.2(a),} \qquad (7.4)\\
&= \dim_{p\text{-adic}} \mathsf{Cl}_p(X(K)) && \text{since } \mathsf{Cl}_p(X(K)) = X(\mathbb{Q}_p) \text{ by assumption.} \qquad (7.5)
\end{aligned}$$

On the other hand, for $x \in X(K)$, it follows from Proposition 7.3 that

$$\dim_{p\text{-adic}} \mathsf{Cl}_p(\mathcal{O}_f(x)) \le 1.$$

To ease notation, we let

$$\Gamma := \Gamma_1 \cup \cdots \cup \Gamma_r$$

be the finite union of orbits in the statement of Theorem 7.4. Then, since the dimension of a finite union of sets is the maximum of the dimensions of the components, we have

$$\dim_{p\text{-adic}} \mathsf{Cl}_p(\Gamma) \le 1. \tag{7.6}$$

Thus (7.4) says that $\mathsf{Cl}_p(X(K))$ has p-adic dimension at least 2, while (7.6) says that $\mathsf{Cl}_p(\Gamma)$ has p-adic dimension at most 1, so the complement satisfies

$$\dim_{p\text{-adic}}(\mathsf{Cl}_p(X(K)) \smallsetminus \mathsf{Cl}_p(\Gamma))) = \dim_{p\text{-adic}} \mathsf{Cl}_p(X(K)).$$

Applying (7.5) gives

$$\dim_{p\text{-adic}}(\mathsf{Cl}_p(X(K)) \smallsetminus \mathsf{Cl}_p(\Gamma)) = \dim_{\text{Krull}} X. \tag{7.7}$$

Elementary topology[11] tells us that

$$\mathsf{Cl}_p(X(K) \smallsetminus \Gamma) \supseteq \mathsf{Cl}_p(X(K)) \smallsetminus \mathsf{Cl}_p(\Gamma),$$

so taking p-adic dimension and using (7.7) yields

$$\dim_{p\text{-adic}}(\mathsf{Cl}_p(X(K) \smallsetminus \Gamma)) \ge \dim_{\text{Krull}} X. \tag{7.8}$$

Let

$$Z := \mathsf{Cl}_Z(X(K) \smallsetminus \Gamma).$$

We note that

$$\begin{aligned} \dim_{p\text{-adic}} Z(\mathbb{Q}_p) &\ge \dim_{p\text{-adic}} \mathsf{Cl}_p(X(K) \smallsetminus \Gamma) && \text{since } Z \text{ is a Zariski closed} \\ & && \text{set that contains } X(K) \smallsetminus \Gamma, \\ &\ge \dim_{\text{Krull}} X && \text{from (7.8).} \end{aligned}$$

On the other hand, if $Z \subsetneq X$ is a proper subset of X, then Lemma 7.2(c) says that

$$\dim_{p\text{-adic}} Z(\mathbb{Q}_p) < \dim_{\text{Krull}} X.$$

Hence $Z = X$, which is the desired conclusion. ■

8. Surfaces

Our goal in this section is to prove an orbit propagation result for all smooth projective surfaces. We note that we have already proven a number of cases that apply in particular to surfaces. These results are summarized in Table 3.

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[11] It is an exercise to check that if U and V are subsets of a topological space, then their closures satisfy $\mathsf{Cl}(U \smallsetminus V) \supseteq \mathsf{Cl}(U) \smallsetminus \mathsf{Cl}(V)$. See for example [37].

$\mathbb{P}^2$, $\deg(f) \geq 2$	(A) $\Longrightarrow$ (C1) and (B1$\exists$)	Theorem 4.1 (a,b)
$\mathbb{P}^2$, $\deg(f) = 1$	(A) $\Longrightarrow$ (C3$\forall$)	Theorem 4.1 (c)
Geometrically simple abelian surfaces	(A) $\Longrightarrow$ (C3$\forall$)	Theorem 6.1
K3 surfaces	(A) $\Longrightarrow$ (C1)	Theorem 5.1
Rational surface f étale	(A) $\Longrightarrow$ (B1)	Corollary 7.5
étale quotient of an abelian surface	(A) $\Longrightarrow$ (B1)	Corollary 7.6

Table 3. Orbit propagation results that apply to surfaces.

Theorem 8.1. *Let K be a number field and let X/K be a smooth projective surface. Then, possibly after replacing K by a finite extension, for every finite collection of f-orbits $\Gamma_1, \dots, \Gamma_r \subset X(K)$, we have*

$$\overline{X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r)} = X.$$

In the terminology of Table 1, smooth projective surfaces satisfy the orbit propagation statement

$$\text{(A)} \implies \text{(B1)}.$$

Before starting the proof of Theorem 8.1, we give some lemmas that will be used in the proof.

Lemma 8.2. *Let Y be a relatively minimal surface with $\kappa(Y) = -\infty$. Let X be obtained from Y by a sequence of blow-ups. If $f: X \to X$ is a non-invertible surjective morphism, then some iterate of f is induced from an endomorphism of Y. In particular, if $\kappa(X) = -\infty$, then Theorem 8.1 for all relatively minimal surfaces Y and all non-invertible maps implies Theorem 8.1 for X with a non-invertible map f.*

Proof. See Lemma 4.1 in [28]. ■

Lemma 8.3. *Let X be a smooth projective complex surface that is not birational to $\mathbb{P}^2$. Then X contains at most finitely many (-1)-curves.*

Proof. This is well known. See for example [34]. ■

We next consider a version of Lemma 8.2 for non-rational surfaces.

Lemma 8.4. *Let Y be a minimal surface that is not birational to $\mathbb{P}^2$. Let X be obtained from Y by a sequence of blow-ups. If $f: X \to X$ is a surjective morphism, then some iterate of f is induced from an endomorphism of Y. In particular, Theorem 8.1 for Y implies Theorem 8.1 for X.*

Proof. If f is an automorphism, then it maps exceptional curves into exceptional curves and we are done. So we assume that f is non-invertible and consider two cases. First, if $\kappa(X) \geq 0$, then Lemma 4.2 in [28] tells us that $X = Y$. Second, if $\kappa(X) = -\infty$, then Lemma 8.2 gives the desired result. ■

Proof of Theorem 8.1. We note from Lemma 8.4 that Theorem 8.1 for minimal surfaces implies Theorem 8.1 for all surfaces, except possibly in the case that X is a rational surface and $f\colon X \to X$ is an automorphism. But that case is covered by Corollary 7.5. So we have reduced the proof of Theorem 8.1 to the case of minimal surfaces. The proof of Theorem 8.1 is now a case-by-case analysis via the classical classification of surfaces (see Section V.6 of [19]).

We remark that all of the implications in Table 3 are at least as strong as the assertion of Theorem 8.1, since each of the properties (C1), (C3∃), and (C3∀) imply (B1); cf. Table 2. Hence Theorem 8.1 is true for the surfaces listed in Table 3.

Case 1. *$\kappa(X) = -\infty$, X is rational.*

Since we are assuming that X is minimal, this implies that $X \cong \mathbb{P}^2$. Therefore, Theorem 4.1 (a,b,c) give something stronger than the desired result.

Case 2. *$\kappa(X) = -\infty$, X is ruled.*

Let $\pi\colon X \to B$ be a surjective map from X to a curve B whose fibers are $\bar{K}$-isomorphic to $\mathbb{P}^1$. If $g(B) \geq 2$, then $B(K)$ is finite, so there are no dense orbits. We may thus assume that $g(B)$ is 0 or 1.

Possibly after replacing f by f^2 (cf. Lemma 5.4 in [28]), we may assume that f is semi-conjugate to π, i.e., there are a map $h\colon B \to B$ and a commutative diagram

(8.1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"'] \\ B \arrow[r, "h"] & B. \end{tikzcd}$$

Furthermore, π has a section $\sigma\colon B \to X$; see Lemma 5.1 in [28]. Replacing K by a finite extension, we may assume that everything is defined over K. Then for every finite extension L/K, we have

$$\pi(X(L)) \supseteq \pi(\sigma(B(L))) = B(L).$$

Therefore, $\pi(X(L)) = B(L)$, where B is either $\mathbb{P}^1$ or an elliptic curve. The implication (A) ⇒ (B1) is true for the curve B by the dimension 1 cases of Theorem 4.1 and 6.1, where for the latter we note that an elliptic curve is always geometrically simple.

We want to prove that (A) implies (B1) for X, so we assume there is a point $x_0 \in X(K)$ whose orbit $\mathcal{O}_f(x_0)$ is Zariski dense in X. It follows from the commutative diagram (8.1) that $\mathcal{O}_h(\pi(x_0))$ is Zariski dense in B, so as noted in the previous paragraph, we know that (B1) is true for B.

Let $\Gamma_1, \dots, \Gamma_r$ be f-orbits in $X(K)$. Then $\pi(\Gamma_1), \dots, \pi(\Gamma_r)$ are h orbits in B, so the validity of (B1) for B tells us that

$$B^\circ(K) := B(K) \smallsetminus (\pi(\Gamma_1), \dots, \pi(\Gamma_r)) \quad \text{is Zariski dense in } B.$$

Note that the fibers of π over the points in $B^\circ(K)$ are K-isomorphism to $\mathbb{P}^1$, since the fiber over $b \in B^\circ(K)$ is a rational curve $X_b := \pi^{-1}(b)$ that contains the K-rational point $\sigma(b)$. Hence $X_b(K)$ is Zariski dense in X, so the density of $B^\circ(K)$ in B implies that

$$\bigcup_{b \in B^\circ(K)} X_b(K) \quad \text{is Zariski dense in } X.$$

This set is disjoint from $\Gamma_1 \cup \cdots \cup \Gamma_r$ be construction, which completes the proof that $X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r)$ is Zariski dense in X, and thus the proof that (B1) holds for X.

Case 3. *$\kappa(X) = 0$, X is a $K3$ surface.*

Theorem 5.1 proves (A) $\Rightarrow$ (C1) for K3 surfaces, which is stronger than the desired result.

Case 4. *$\kappa(X) = 0$, X is an Enriques surface.*

There is an étale quotient map $\pi\colon X' \to X$ with X' a $K3$ surface, and the endomorphism $f\colon X \to X$ lifts to X' because K3 surfaces are simply connected. Hence the desired result for X follows from the fact that it is true the K3 surface X'.

Case 5. *$\kappa(X) = 0$, X is an abelian surface.*

This is covered by Corollary 7.6.

Case 6. *$\kappa(X) = 0$, X is a bielliptic surface.*

This is covered by Corollary 7.6.

Case 7. *$\kappa(X) = 1$, X is an elliptic surface*; *$\kappa(X) = 2$, X is of general type.*

In general, if $\kappa(X) > 0$, then we can use Theorem A in [33] by Nakayama and Zhang. They prove that a dominant rational self-map of a smooth projective variety of positive Kodaira dimension factors over a positive dimensional base, and hence does not have any Zariski dense orbits. See also Corollary 8.2 in [28] for a more elementary proof of the case that we need. Thus all versions of orbit propagation are vacuously true, since there are no Zariski dense orbits. (We mention that varieties of general type cannot even have endomorphisms of infinite order, so orbits are necessarily finite in that case, making orbit propagation even more vacuous.) ■

A. The Zariski dense orbit conjecture

In this section, for the convenience of the reader, we discuss various versions of the Zariski density conjecture. As noted in the introduction and in Figure 1, these conjectures are complementary to our orbit propagation conjectures, since the former says that a geometric property implies the existence of Zariski dense orbits over an algebraically closed field, while the latter says that the existence of one such orbit implies that there are many Zariski dense orbits defined over a finitely generated field. To do full justice to the Zariski density conjecture, we work in somewhat greater generality than in the other parts of this paper, using the notation described in Table 4.

Definition A.1. A map $f\colon X \dashrightarrow X$ is *fibered* if there is a dominant rational map $\varphi\colon X \to \mathbb{P}^1$ satisfying $\varphi \circ f = \varphi$; or equivalently, if $\boldsymbol{k}(X)^f \neq \boldsymbol{k}$.

Clearly, the existence of a point $x \in X(\boldsymbol{k})$ whose f-orbit is well-defined and Zariski dense implies that f cannot be fibered. Building on a conjecture of Zhang [43], Amerik–Campana [3] and Medvedev–Scanlon [30] independently suggested the converse should hold, with further refinements by Amerik–Bogomolov–Rovinsky [2] and a stronger formulation by Xie [42]. (See also Amerik [1].)

$\boldsymbol{k}$	an algebraically closed field of characteristic 0 (the case that $\operatorname{trdeg}_{\mathbb{Q}}(\boldsymbol{k}) < \infty$ is generally more difficult),
$X/\boldsymbol{k}$	an irreducible (quasi-projective) variety defined over $\boldsymbol{k}$,
f	a dominant rational map $f : X \dashrightarrow X$ defined over $\boldsymbol{k}$,
$\boldsymbol{k}(X)$	the function field of X,
$\boldsymbol{k}(X)^f$	the subfield of f-invariant functions, i.e., the subfield $\boldsymbol{k}(X)^f = \{\varphi \in \boldsymbol{k}(X) : \varphi \circ f = \varphi\}$.

Table 4. Notation for the Zariski density conjecture.

Conjecture A.2. *With notation as in Table 4:*

(a) (The Zariski-density (ZD) orbit conjecture [3], Conjecture 1.2 in [2] and Conjecture 5.10 in [30])

$$X_f^{\text{dense}}(\boldsymbol{k}) = \emptyset \iff f \textit{ is fibered.}$$

(b) (The strong Zariski-density (SZD) orbit conjecture, Conjecture 1.4 in [42]) *Assume that the dominant rational map f is not fibered. Then for every non-empty Zariski open subset $U \subset X$, the set*[12]

$$X_f^{\text{dense}}(\boldsymbol{k}) \cap U \qquad \textit{is Zariski dense in } U.$$

Remark A.3. Xie (see Section 1.3 of [42]) notes that “if we have one Zariski dense orbit, we expect many such orbits. However, the Zariski topology is too weak to describe such phenomena; in particular, one cannot expect to have a nonempty Zariski open set of points having Zariski dense orbits.” He goes on to develop a very interesting *adelic topology*[13] on $X(\boldsymbol{k})$. Xie’s adelic topology has many agreeable properties; see Section 1.2 of [42]. In particular, it is stronger than the Zariski topology, and thus has more open sets, which allows him to make the following conjecture.

Conjecture A.4. (The adelic Zariski-density (AZD) orbit conjecture, Conjecture 1.10 in [2]) *With notation as in Table 4, if the dominant rational map f is not fibered, then there exists a non-empty adelic open subset $U \subseteq X(\boldsymbol{k})$ such that*

$$U \subseteq X_f^{\text{dense}}(\boldsymbol{k}).$$

In other words, there is an adelic open set U such that every point in U has a well-defined Zariski dense forward orbit.

Remark A.5. We thank Junyi Xie for the following remarks (private communication): Let $U \subset X(\boldsymbol{k})$ be a non-empty adelic open set as in Conjecture A.4. Then U is automatically Zariski dense in X, but if C is defined over a finitely generated subfield of $\boldsymbol{k}$,

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[12] Xie further conjectures that there is a point $x \in X_f^{\text{dense}}(\boldsymbol{k}) \cap U$ satisfying $\mathcal{O}_f(x) \subset U$.

[13] We note that Xie’s adelic topology is not the restriction to $X(\boldsymbol{k})$ of the natural topology on the adelic points of X. In particular, Xie’s adelic topology is T_1 (Fréchet), but not T_2 (Hausdorff), while the restriction topology is T_2.

it need not be true that there is a finitely generated field of definition $K \subset \boldsymbol{k}$ of X such that $U \cap X(K)$ is Zariski dense in X. For example, if $X/\mathbb{Q}$ is a field of genus at least 2, any non-empty adelic set $U \subset X(\overline{\mathbb{Q}})$ is Zariski dense in X, but $X(K)$ is finite for every number field $K/\mathbb{Q}$.

Xie notes that it might be possible to modify the definition of the adelic topology so that Conjecture A.4 for $\mathbb{P}^N$ is reasonable with the added condition that $U \cap \mathbb{P}^N(L)$ is Zariski dense for some finitely generated subfield L of $\boldsymbol{k}$. However, this alternative topology would probably not be open under flat morphisms.

We describe some of the cases for which the ZD conjecture and its various generalizations have been proven.

Theorem A.6. *The ZD conjecture is true for the following classes of varieties and maps, and in many cases, the SZD conjecture and/or the AZD conjecture are true*:

- (a) *The field* $\boldsymbol{k}$ *is uncountable, e.g.,* $\boldsymbol{k} = \mathbb{C}$ *or* $\mathbb{C}_p$ (*see* [3] *and Corollary* 9 *in* [1]).
- (b) $X/\boldsymbol{k}$ *is an irreducible* (*non-singular*) *projective surface, and* $f: X \to X$ *is a dominant endomorphism* (*see Theorem* 1.11 *in* [22] *and Theorem* 1.15 *in* [42]).
- (c) $X/\boldsymbol{k}$ *is a* (*semi*)*-abelian variety, and* $f: X \to X$ *is a dominant endomorphism* (*see* [16–18, 42]).
- (d) $X = \mathbb{A}^N$ *and* $f: \mathbb{A}^N \to \mathbb{A}^N$ *is a dominant endomorphism of the form* $f(x_1, \dots, x_N) = (f_1(x_1), \dots, f_N(x_N))$ (*see Theorem* 1.13 *in* [42]).
- (e) $X = (\mathbb{P}^1)^N$ *and* $f: X \to X$ *is a dominant endomorphism* (*see Theorem* 1.16 *in* [42], *joint with T. Tucker*).
- (f) $X = \mathbb{A}^2$ *and* $f: \mathbb{A}^2 \to \mathbb{A}^2$ *is a dominant* (*regular*) *endomorphism* (*see* [41]).
- (g) X *is a connected commutative linear algebraic groups and* $f: X \to X$ *is a group endomorphism* (*see* [15]).
- (h) *The field* $\boldsymbol{k}$ *is algebraically closed, uncountable, and has characteristic* $p > 0$ (*see Corollary* 6.1 *in* [4]).

See Remark 1.13 in [22] and Section 11 of [31] for further details.

B. Additional results

This section contains additional material that illuminates or is used in the text.

B.1. Elementary properties of orbits

The following lemma describes some elementary properties of orbits.

Lemma B.1. (a) *Grand* f*-orbit equivalence is an equivalence relation.*

(b) *Let* Γ_1 *and* Γ_2 *be* f*-grand orbits. Then either*

$$\Gamma_1 = \Gamma_2 \quad \textit{or} \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

Hence for $P, Q \in X$, *we have*

$$P \equiv_f Q \iff \mathcal{O}_f^{\text{grand}}(P) = \mathcal{O}_f^{\text{grand}}(Q).$$

(c) *If* Γ *is a grand orbit, then either all of its points are preperiodic, or all of its points are wandering.*

(d) *Let* Γ *be a grand orbit. If* Γ *contains a point with Zariski dense forward orbit, then every point in* Γ *has Zariski dense forward orbit.*

(e) *If* f *is an automorphism, then the* f*-grand orbit of* Q *is the union of orbits of* f *and its inverse,*

$$\mathcal{O}_f(Q)^{\text{grand}} = \mathcal{O}_f(Q) \cup \mathcal{O}_{f^{-1}}(Q).$$

(f) *If* f *is an automorphism, then* P *is* f*-wandering if and only if* P *is* f^{-1}*-wandering.*

(g) *Let* $P, Q \in X$. *Then*

$$P \equiv_f Q \implies \left(P \in X_f^{\text{dense}} \iff Q \in X_f^{\text{dense}}\right).$$

Proof. The elementary proof is left to the reader. ■

B.2. Implications between propagation statements

We prove the implications that hold universally for the various propagation statements in Table 1.

Proposition B.2. *The following implications hold for the various orbit propagation statements in Table 1, with the convention described in Remark 2.6. These implications are illustrated in Table 2.*

$$\text{(C1)} \not\Longrightarrow \text{(A)}, \tag{B.1}$$

$$\text{(B1}\infty) \Longrightarrow \text{(B1)}, \tag{B.2}$$

$$\text{(C1}\infty) \Longrightarrow \text{(C1)}, \tag{B.3}$$

$$(\text{C2}\forall) \Longrightarrow (\text{C1}\infty), \tag{B.4}$$

$$(\text{C2}\forall) \Longrightarrow (\text{C2}\exists), \tag{B.5}$$

$$\text{(C1)} \Longrightarrow \text{(B1)}, \tag{B.6}$$

$$(\text{C2}\exists) \Longrightarrow \text{(C1)}, \tag{B.7}$$

$$\text{(C1)} \Longrightarrow (\text{C2}\exists), \tag{B.8}$$

$$(\text{C1}\infty) \Longrightarrow (\text{B1}\infty), \tag{B.9}$$

$$(\text{C3}\exists) \Longrightarrow \text{(A)}, \tag{B.10}$$

$$(\text{C3}\forall) \Longrightarrow (\text{C3}\exists), \tag{B.11}$$

$$(\text{C3}\forall) \Longrightarrow (\text{C2}\forall), \tag{B.12}$$

$$(\text{C3}\exists) \Longrightarrow (\text{C2}\exists), \tag{B.13}$$

$$(\text{C1}\infty) \Longrightarrow (\text{C2}\forall). \tag{B.14}$$

Proof. (B.1) (C1) $\not\Longrightarrow$ (A)

If the map f admits a non-trivial fibration, then (C1) is true, since any finite set of grand f-orbits will be contained in a finite number of fibers, but (A) is not true, since every grand orbits is contained in a fiber, so there are no Zariski dense grand orbits. We note that it is not clear if (B1∞) or (C1∞) implies (A), but neither do we see an obvious counterexample.

(B.2) (B1∞) $\Longrightarrow$ (B1)

Let $\Gamma_1, \dots, \Gamma_r$ be a collection of f-orbits, say $\Gamma_i = \mathcal{O}_f(P_i)$. Then $Y = \{P_1, \dots, P_r\}$ is a proper Zariski closed subset of X, so applying (B1∞) to Y gives (B1).

(B.3) (C1∞) $\Longrightarrow$ (C1)

Let $\Gamma_1, \dots, \Gamma_r$ be a collection of grand f-orbits, say $\Gamma_i = \mathcal{O}_f^{\text{grand}}(P_i)$. Then $Y = \{P_1, \dots, P_r\}$ is a proper Zariski closed subset of X, so applying (C1∞) to Y gives (C1).

(B.4) (C2$\forall$) $\Longrightarrow$ (C1∞)

Let $Y \subsetneq X$ be a proper Zariski closed set, and let $\mathcal{Q} \subset X(K)$ be any complete set of representatives for $X(K)/{\equiv_f}$. Thus every grand f-orbit Γ that contains a point of $X(K)$ has the property that $\#(\Gamma \cap \mathcal{Q}) = 1$. We create a modified version of $\mathcal{Q}$, which we denote by $\mathcal{Q}'$, as follows: for each $Q \in \mathcal{Q}$, if

$$\mathcal{O}_f^{\text{grand}}(Q) \cap Y(K) \neq \emptyset,$$

then we replace Q with one of the points in $\mathcal{O}_f^{\text{grand}}(Q) \cap Y(K)$. This modified set $\mathcal{Q}'$ is still a complete set of representatives for $X(K)/{\equiv_f}$. Our assumption that (C2$\forall$) is true tells us that $\mathcal{Q}'$ is Zariski dense in X. Since Y is not Zariski dense in X, it follows that

$$\mathcal{Q}' \smallsetminus Y \quad \text{is Zariski dense in } X. \tag{B.15}$$

On the other hand, the construction of $\mathcal{Q}'$ ensures that

$$Q \in \mathcal{Q}' \cap \bigcup_{P \in Y(K)} \mathcal{O}_f^{\text{grand}}(P) \implies Q \in Y. \tag{B.16}$$

Combining (B.15) and (B.16), we deduce that

$$\mathcal{Q}' \smallsetminus \Bigl(\bigcup_{P \in Y(K)} \mathcal{O}_f^{\text{grand}}(P) \Bigr) \supset Q \smallsetminus Y \quad \text{is Zariski dense in } X,$$

and since $\mathcal{Q}'$ is a subset of $X(K)$, we conclude that (C1∞) is true.

We remark that if we use the same argument to try to prove that (C2$\exists$) implies (C1∞), we run into the problem that the replacement procedure used to create $\mathcal{Q}'$ could potentially replace every point of $\mathcal{Q}$ with a point in Y, which prevents us from concluding that $\mathcal{Q}'$ is Zariski dense in X.

(B.5) (C2$\forall$) $\Longrightarrow$ (C2$\exists$)

There certainly exists at least one set of representatives $\mathcal{Q} \subset X(K)$ for

$$X(K)/ \equiv_f .$$

Our assumption that (C2$\forall$) holds tells us that $\mathcal{Q}$ is Zariski dense. Hence there exists at least one Zariski dense set of representatives, so (C2$\exists$) is true.

(B.6) (C1) $\Longrightarrow$ (B1)

Let $\Gamma_1, \dots, \Gamma_r$ be f-orbits, and let $\Lambda_1, \dots, \Lambda_s$ be the grand f-orbits generated by the points in $\Gamma_1, \dots, \Gamma_r$. Then each Γ_i is contained in a (unique) Λ_j, so we have the following inclusion:

$$\overbrace{X(K) \smallsetminus (\Lambda_1 \cup \cdots \cup \Lambda_s)}^{\text{(C1) tells us that this set is Zariski dense,}} \subseteq \underbrace{X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r)}_{\text{hence this set is also Zariski dense}} .$$

(B.7) (C2∃) $\implies$ (C1)

We start by extending K so that $X(K) \neq \emptyset$. The assumption that (C2∃) is true says that, after replacing K by a further finite extension, there is a set of points $\mathcal{Q} \subseteq X(K)$ such that $\mathcal{Q}$ is Zariski dense in X and such that $\mathcal{Q}$ is a complete set of representatives for $X_f(K)/{\equiv_f}$.

Let $\Gamma_1, \dots, \Gamma_r$ be grand f-orbits. Then for each i, there is a unique point $Q_i \in \mathcal{Q} \cap \Gamma_i$. It follows that

$$\mathcal{Q} \smallsetminus \{Q_1, \dots, Q_r\} \subseteq X(K) \smallsetminus (\Gamma_1 \cup \dots \cup \Gamma_r).$$

By assumption, the set $\mathcal{Q}$ is Zariski dense in X, and thus the same is true for the slightly smaller set $\mathcal{Q} \smallsetminus \{Q_1, \dots, Q_r\}$, since $\dim(X) \geq 1$. Hence $X(K) \smallsetminus (\Gamma_1 \cup \dots \cup \Gamma_r)$ contains a Zariski dense set, which completes the proof that (C1) is true.

(B.8) (C1) $\implies$ (C2∃)

We start by enumerating the countably many[14] proper Zariski closed subsets of X, say they are $Z_1, Z_2, Z_3, \dots$. We select a sequence of points $P_1, P_2, \dots$ in $X(K)$ using the following algorithm:

- The assumption that (C1) is true tells us in particular that $X(K)$ is Zariski dense in X, so in any case we know that $X(K) \neq \emptyset$. So we can choose a point $P_1 \in X(K)$. Let $\Gamma_1 = \mathcal{O}_f^{\text{grand}}(P_1)$.
- LOOP $i = 1, 2, 3, 4, \dots$
- The assumption that (C1) is true tells us that $X(K) \smallsetminus (\Gamma_1 \cup \dots \cup \Gamma_i)$ is Zariski dense in X. The union of the proper Zariski closed subsets $Z_1, \dots, Z_i$ is not Zariski dense in X, so we can choose a point P_{i+1} satisfying

  $$P_{i+1} \in X(K) \smallsetminus \bigl((\Gamma_1 \cup \dots \cup \Gamma_i) \cup (Z_1 \cup \dots \cup Z_i)\bigr).$$

  We let $\Gamma_{i+1} = \mathcal{O}_f^{\text{grand}}(P_{i+1})$.
- END LOOP

The output from this algorithm is a set of points

$$\mathcal{P} = \{P_1, P_2, P_3, \dots\} \subseteq X(K)$$

whose grand orbits are distinct. We claim that $\mathcal{P}$ is Zariski dense. If not, then $\bar{\mathcal{P}}$ is a proper Zariski closed subset of X, so by our enumeration $Z_1, Z_2, \dots$ of all of the proper Zariski closed subsets of X, there is an index j such that $\mathcal{P} \subseteq Z_j$. But $P_{j+1} \in \mathcal{P}$ and, by construction, we have $P_{j+1} \notin Z_j$. This contradiction shows that $\mathcal{P}$ is a Zariski dense set of points in $X(K)$ that represent distinct elements of $X(K)/{\equiv_f}$. Extending $\mathcal{P}$ to a full set of representatives shows that (C2∃) is true.

(B.9) (C1∞) $\implies$ (B1∞)

We prove these simultaneously. Every f-orbit is contained in a (unique) grand f-orbit, since for every point P we have

$$\mathcal{O}_f(P) \subseteq \mathcal{O}_f^{\text{grand}}(P).$$

---

[14] The fact that a projective variety defined over a countable field has countably many Zariski closed subsets may be reduced to the same assertion for $\mathbb{P}^N$, and then it follows from the fact that there are only countably many ideals in the Noetherian ring $K[X_0, \dots, X_n]$.

This implies that if $\mathcal{P} \subset X(K)$ is any set of points, then

$$\Bigl(X(K) \smallsetminus \bigcup_{P\in\mathcal{P}} \mathcal{O}_f(P)\Bigr) \supseteq \Bigl(X(K) \smallsetminus \bigcup_{P\in\mathcal{P}} \mathcal{O}_f^{\text{grand}}(P)\Bigr).$$

Taking $\mathcal{P}$ to be a finite set of points gives (C1) $\longrightarrow$ (B1), and taking $\mathcal{P} = Y(K)$ gives (C1∞) $\longrightarrow$ (B1∞).

(B.10) (C3$\exists$) $\Longrightarrow$ (A)

We start by extending K so that $X(K) \ne \emptyset$. The assumption that (C3$\exists$) is true says that, after replacing K by a further finite extension, there is a set of points $\mathcal{Q} \subseteq X(K)$ such that $\mathcal{Q}$ is Zariski dense in X and such that $\mathcal{Q}$ is a complete set of representatives for $X_f^{\text{dense}}(K)/{\equiv_f}$. Let $Q \in \mathcal{Q}$. Lemma B.1(g) tells us that $Q \in X_f^{\text{dense}}$, and we know $Q \in X(K)$, so we have found a point $Q \in X_f^{\text{dense}}(K)$. By definition, the orbit $\mathcal{O}_f(Q)$ is Zariski dense in X, which proves that (A) is true.

(B.11) (C3$\forall$) $\Longrightarrow$ (C3$\exists$)

By assumption, there is a point $Q \in X_f^{\text{dense}}(K)$. It follows that $X_f^{\text{dense}}(K)/{\equiv_f}$ is nonempty, so there exists a set $\mathcal{Q} \subset X_f^{\text{dense}}(K)$ of representatives for $X_f^{\text{dense}}(K)/{\equiv_f}$. The assumption (C3$\forall$) tells us that $\mathcal{Q}$ is Zariski dense in X, so the existence of $\mathcal{Q}$ implies that (C3$\exists$) is true.

(B.12) (C3$\forall$) $\Longrightarrow$ (C2$\forall$)

Let $\mathcal{Q} \subset X(K)$ be a complete set of representatives for the grand orbits $X(K)/{\equiv_f}$. Then part (g) of Lemma B.1 tells us that $\mathcal{Q} \cap X_f^{\text{dense}}(K)$ is a complete set of representative for $X_f^{\text{dense}}(K)/{\equiv_f}$. The assumption that (C3$\forall$) is true tells us that $\mathcal{Q} \cap X_f^{\text{dense}}(K)$ is Zariski dense in X, from which it is clear that $\mathcal{Q}$ is Zariski dense in X.

(B.13) (C3$\exists$) $\Longrightarrow$ (C2$\exists$)

We are given that there exists a set $\mathcal{Q} \subset X_f^{\text{dense}}(K)$ such that $\mathcal{Q}$ is Zariski dense in X and such that $\mathcal{Q}$ is a complete set of representatives for $X_f^{\text{dense}}(K)/{\equiv_f}$. Let $\mathcal{Q}' \subset (X \smallsetminus X_f^{\text{dense}})(K)$ be a complete set of representatives for

$$(X \smallsetminus X_f^{\text{dense}})(K)/{\equiv_f}.$$

Lemma B.1(g) tells us that $\mathcal{Q} \cup \mathcal{Q}'$ is a complete set of representatives for the grand orbits $X(K)/{\equiv_f}$, and that $\mathcal{Q} \cup \mathcal{Q}'$ is Zariski dense in X, since it contains $\mathcal{Q}$. Hence (C2$\exists$) is true.

(B.14) (C1∞) $\Longrightarrow$ (C2$\forall$)

Let $\mathcal{Q} \subset X(K)$ be a complete set of representatives for the grand orbits $X(K)/{\equiv_f}$, and let

$$Y = \overline{\mathcal{Q}} = \text{the Zariski closure of } \mathcal{Q}.$$

We suppose that $Y \ne X$, and our goal is to show that (C1∞) is false for this Y. The choice of $\mathcal{Q}$ tells us that

$$X(K) = \bigcup_{Q\in\mathcal{Q}} \bigl(\mathcal{O}_f^{\text{grand}}(Q) \cap X(K)\bigr). \tag{B.17}$$

(Indeed, it even tells us that the union is a disjoint union, but we will not need this fact.) The facts $Y = \overline{\mathcal{Q}}$ and $\mathcal{Q} \subseteq X(K)$ imply that

$$Y(K) \supseteq \mathcal{Q}. \tag{B.18}$$

Hence

$$\begin{aligned} X(K) \smallsetminus \Bigl(\bigcup_{P \in Y(K)} \mathcal{O}_f^{\text{grand}}(P) \Bigr) &\subseteq X(K) \smallsetminus \Bigl(\bigcup_{Q \in \mathcal{Q}} \mathcal{O}_f^{\text{grand}}(Q) \Bigr) && \text{from (B.18)}, \\ &= \emptyset && \text{from (B.17)}. \end{aligned} \tag{B.19}$$

This certainly contradicts (C1∞), since (C1∞) would imply that the set (B.19) is Zariski dense in X. ∎

The next result describes two ways to deduce (C3$\forall$) from (C2$\forall$), which gives it a slightly different flavor from the results in Proposition B.2. We note that the applicability of Lemma B.3 is probably limited, e.g., since the periodic points of polarizable endomorphisms of projective varieties are Zariski dense (Theorem 5.1 in [11]). But it will be useful for analyzing linear maps of $\mathbb{P}^N$ and maps of (geometrically simple) abelian varieties.

Lemma B.3. *Let $f : X \to X$ such that* (A) *and* (C2$\forall$) *are true, and suppose further that f has one of the following properties*:

(a) *X_f^{dense} contains a non-empty Zariski open set.*

(b) Image$((X \smallsetminus X_f^{\text{dense}})(K) \longrightarrow X(K)/{\equiv_f})$ *is finite.*

Then (C3$\forall$) *is true.*

Proof. Let

$$\mathcal{Q} \subset X_f^{\text{dense}}(K) \quad \text{such that} \quad \mathcal{Q} \xleftrightarrow{\text{bijective}} X_f^{\text{dense}}(K)/\equiv_f.$$

Our goal is to prove that $\mathcal{Q}$ is Zariski dense in X. To do this, we consider the complement of X_f^{dense}, and we choose a set

$$\mathcal{Q}' \subset (X \smallsetminus X_f^{\text{dense}})(K) \quad \text{such that} \quad \mathcal{Q}' \xleftrightarrow{\text{bijective}} (X \smallsetminus X_f^{\text{dense}})(K)/\equiv_f.$$

Then the union satisfies

$$\mathcal{Q} \cup \mathcal{Q}' \xleftrightarrow{\text{bijective}} X(K)/{\equiv_f},$$

so the assumption that (C2$\forall$) is true tells us that $\mathcal{Q} \cup \mathcal{Q}'$ is Zariski dense in X.

(a) The assumption in (a) implies that the set $\mathcal{Q}'$ is not Zariski dense in X, since

$$\mathcal{Q}' \subset X \smallsetminus X_f^{\text{dense}},$$

and the assumption that X_f^{dense} contains a non-empty Zariski open set implies that the complement of X_f^{dense} is not Zariski dense. Hence $\mathcal{Q}$ is Zariski dense in X, which completes the proof that f has the propagation property (C3$\forall$).

(b) The assumption in (b) implies that $\mathcal{Q}'$ is a finite set, so in particular it is not Zariski dense (since $\dim(X) \ge 1$). The rest of the argument is as in the proof of (a). ∎

We will apply the next lemma, which is undoubtedly well known, to the case of an automorphism. But since it is no harder to handle the case of arbitrary backward branches, we formulate it in that generality.

Lemma B.4. *Let $P_0 \in X$ be a point whose forward orbit $\mathcal{O}_f(P_0)$ is Zariski dense in X, and let*

$$\mathcal{B} = \{P_0, P_1, P_2, \dots\} \subset X(\bar{K})$$

be a complete backward f-branch of P_0, i.e., a sequence of points in $X(\bar{K})$ satisfying

$$f(P_{i+1}) = P_i \quad \text{for all } i \geq 0.$$

Then $\mathcal{B}$ is Zariski dense in X.

Proof. Our first observation is that the points $P_0, P_1, \dots$ are distinct. To see why, suppose that $P_i = P_j$ for some $i > j$. Then

$$f^{i-j}(P_0) = f^{i-j}(f^j(P_j)) = \underbrace{f^i(P_j) = f^i(P_i)}_{\text{since } P_i = P_j} = P_0,$$

which implies that P_0 is periodic; this contradicts the assumed Zariski density of $\mathcal{O}_f(P_0)$. (Note we always assume that $\dim(X) \geq 1$.)

Our goal is to prove that the Zariski closure $\bar{\mathcal{B}}$ of $\mathcal{B}$ is equal to X. For each integer $m \geq 0$, we let

$$\mathcal{B}_m := \{P_m, P_{m+1}, P_{m+2}, \dots\}$$

denote the branch with its first m points omitted. The actions of f on these sets and their closures satisfy

$$\text{(B.20)} \qquad f : \mathcal{B}_{m+1} \longrightarrow \mathcal{B}_m \quad \text{and} \quad f : \overline{\mathcal{B}_{m+1}} \longrightarrow \overline{\mathcal{B}_m}.$$

We now suppose that

$$\text{(B.21)} \qquad P_m \notin \overline{\mathcal{B}_{m+1}} \quad \text{for all } m \geq 0,$$

and derive a contradiction. Since $\mathcal{B}_m = \{P_m\} \cup \mathcal{B}_{m+1}$, it would follow from (B.21) that the single point $\{P_m\}$ is an irreducible component of $\overline{\mathcal{B}_m}$; and hence, since $\mathcal{B} \smallsetminus \mathcal{B}_m$ is a finite set, it would follow that $\{P_m\}$ is an irreducible component of $\bar{\mathcal{B}}$. The Zariski closed set $\bar{\mathcal{B}}$ has only finitely many irreducible components, and we proved earlier that the P_i are distinct, which completes our proof that (B.21) is false.

Therefore, there exists an index $\ell \geq 0$ such that

$$\text{(B.22)} \qquad P_\ell \in \overline{\mathcal{B}_{\ell+1}}.$$

This implies that

$$\overline{\mathcal{B}_\ell} = \overline{\{P_\ell\} \cup \mathcal{B}_{\ell+1}} = \{P_\ell\} \cup \overline{\mathcal{B}_{\ell+1}} = \overline{\mathcal{B}_{\ell+1}},$$

and then (B.20) tells us that f induces a map

$$\text{(B.23)} \qquad f : \overline{\mathcal{B}_{\ell+1}} \longrightarrow \overline{\mathcal{B}_{\ell+1}}.$$

We know from (B.22) that the point P_ℓ is in $\mathcal{B}_{\ell+1}$, so (B.23) implies that

$$\mathcal{O}_f(P_\ell) \subset \overline{\mathcal{B}_{\ell+1}}. \tag{B.24}$$

We compute

$$\begin{aligned}
X &= \overline{\mathcal{O}_f(P_0)} && \text{by assumption,}\\
&\subseteq \overline{\mathcal{O}_f(P_\ell)} && \text{since } f^\ell(P_\ell) = P_0,\\
&\subseteq \overline{\mathcal{B}_{\ell+1}} && \text{from (B.24),}\\
&= \overline{\mathcal{B} \smallsetminus \{P_0, \dots, P_\ell\}} && \text{by definition of } \mathcal{B}_m,\\
&\subseteq \bar{\mathcal{B}}.
\end{aligned}$$

Hence $\bar{\mathcal{B}} = X$, which completes the proof that $\mathcal{B}$ is Zariski dense in X. ■

If $f : X \to X$ is an automorphism, then the existence of a Zariski dense orbit automatically implies a weak propagation property, as in the following result. The proof follows from the fact that $\mathcal{O}_f(P)$ is Zariski dense if and only if $\mathcal{O}_{f^{-1}}(P)$ is Zariski dense. This allows us to use points in the backward orbit to prove that (A) implies propagation property (B1); but this is a bit of a cheat, and in any case the idea cannot be used to prove a grand orbit propagation property such as (C1).

Proposition B.5. *Let $f : X \to X$ be an automorphism. Then*

$$\text{(A)} \implies \text{(B1)}.$$

Proof. We are given that there is a point $P_0 \in X(K)$ whose f-orbit $\mathcal{O}_f(P_0)$ is Zariski dense. The forward f^{-1}-orbit of P_0 is a backward f-branch of P_0 as in Lemma B.4, so Lemma B.4 tells us that $\mathcal{O}_{f^{-1}(P_0)}$ is Zariski dense in X.

We now commence the proof of the propagation property (B1). Let $\Gamma_1, \dots, \Gamma_r$ be a collection of f-orbits. The fact that the Γ_i are forward f-orbits implies[15] that they have only finitely many points in common with the f^{-1}-orbit $\mathcal{O}_{f^{-1}}(P_0)$. Hence

$$X(K) \smallsetminus (\Gamma_1 \cup \dots \cup \Gamma_r) \supseteq \mathcal{O}_{f^{-1}}(P_0) \smallsetminus (\Gamma_1 \cup \dots \cup \Gamma_r) = \underbrace{\mathcal{O}_{f^{-1}}(P_0) \smallsetminus \{\text{finite set}\}}_{\text{Zariski dense in } X.}.$$

This completes the proof that f has the propagation property (B1). ■

Remark B.6. The dynamical Mordell–Lang conjecture describes intersection properties of forward orbits with subvarieties; see [7] for a detailed discussion. If we assume the dynamical Mordell–Lang conjecture for f^{-1}, then we can strengthen Proposition B.5 to the statement that (A) implies that f has the propagation property (B1∞). Briefly, if a Zariski closed subset $Y \subseteq X$ contains $f^{-n}(P_0)$ for infinitely many $n \ge 0$, then Mordell–Lang says that Y contains an f^{-1}-invariant subvariety Z that contains those points. In particular, there is a point $f^{-m}(P_0) \in Z$. But then Z is also f-invariant, so it contains $f^m(f^{-m}(P_0)) = P_0$. Hence $\mathcal{O}_f(P_0) \subset Z$, and then the Zariski density of $\mathcal{O}_f(P_0)$ forces $Z = X$, and thus also $Y = X$.

---

[15] Suppose that $\mathcal{O}_f(Q) \cap \mathcal{O}_{f^{-1}}(P_0) \ne \emptyset$. Then $Q = f^{-m}(P_0)$ for some $m \ge 0$, so $\mathcal{O}_f(Q) \cap \mathcal{O}_{f^{-1}}(P_0) = \{f^n(P_0) : -m \le n \le 0\}$ is a finite set.

B.3. A coarse height counting estimate

There are many stronger results that imply the following height counting lemma, but we include it to indicate that the estimate that we need can be obtained in an elementary manner.

Lemma B.7. *Let K be a number field, and let $N \geq 1$, and compute counting functions using the multiplicative K-height on $\mathbb{P}^N(K)$; see Section* B.2 *of* [20].

(a) *There are constants $C_{18}(K,N) > 0$ and $C_{19}(K,N) > 0$ such that*

$$C_{18}(K,N)T^{N+1} \leq \mathsf{N}(\mathbb{P}^N(K),T) \leq C_{19}(K,N)T^{N+1}.$$

(b) *Let $Y \subsetneq \mathbb{P}^N$ be a Zariski closed set. There is a constant $C_{20}(K,Y)$ such that*

$$\mathsf{N}(Y(K),T) \leq C_{20}(K,Y)\,T^{1+\dim(Y)}, \tag{B.25}$$

where the dimension of an algebraic subset of $\mathbb{P}^N$ is defined to be the maximum of the dimensions of its geometrically irreducible components. In particular, if $Y \subsetneq \mathbb{P}^N$ is a proper subset, then

$$\mathsf{N}(Y(K),T) \leq C_{20}(K,Y)\,T^N.$$

Proof. (a) Schanuel's formula [36] gives a precise asymptotic formula for $\mathsf{N}(\mathbb{P}^N(K),T)$, but the weak estimate that we have cited in (a) follows from the standard proofs that $\mathbb{P}^N(K)$ has only finitely many points of bounded height.

(b) We prove (B.25) by induction on $\dim(Y)$. If $\dim(Y) = 0$, then Y is a finite set of points, so its counting function is bounded and (B.25) is trivially true.

Suppose now that $m \geq 1$, that we know (B.25) for all algebraic sets of dimension at most $m-1$, and that $Y \subseteq \mathbb{P}^N$ is a Zariski closed subset with $\dim(Y) = m$. (Of course, we must have $m \leq N$.) Let $Y_1,\dots,Y_r$ be the irreducible components of Y of dimension m. By the induction hypothesis, it suffices to prove (B.25) for the union of the Y_i, which reduces us to the case that Y is irreducible and of dimension m. A generic projection of Y onto a linear subspace of dimension m yields a quasi-finite rational map

$$\varphi : Y \dashrightarrow \mathbb{P}^m.$$

Let $Y^\circ \subseteq Y$ be a non-empty Zariski open subset on which φ is a quasi-finite morphism of degree d. Then

$$\begin{aligned} \mathsf{N}(Y^\circ(K),T) &\leq (\deg\varphi)\cdot\mathsf{N}(\mathbb{P}^m(K),T) \\ &\leq (\deg\varphi)\cdot C_{21}(m,K)\,T^{1+m} \qquad \text{from (a)} \\ &= C_{21}(K,Y)\,T^{1+\dim(Y)}. \end{aligned} \tag{B.26}$$

We next observe that $Y \smallsetminus Y^\circ$ is a Zariski closed subset of $\mathbb{P}^N$ of dimension at most $m-1$, so our induction hypothesis gives

$$\mathsf{N}((Y \smallsetminus Y^\circ)(K),T) \leq C_{22}(K,Y)\,T^{1+\dim(Y\smallsetminus Y^\circ)} \leq C_{22}(K,Y)\,T^{\dim(Y)}. \tag{B.27}$$

Combining (B.26) and (B.27) yields (B.25) for Y, which completes our induction proof that (B.25) is true for all algebraic subsets of $\mathbb{P}^N$. In particular, if $Y \subsetneq \mathbb{P}^N$ is a proper Zariski closed subset, then its dimension is at most $N - 1$. This completes the proof of Lemma B.7 (b). ■

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