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Classification of solutions to $-\Delta u = e^{-2u}$ in the half-space

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Abstract. We classify all positive solutions to $-\Delta u = e^{-2u}$ in the half space $\mathbb{R}^N_+$, under Dirichlet boundary condition.

1. Introduction

In this paper, we classify positive weak solutions to the following Dirichlet problem with exponentially decreasing nonlinearity:

$$(\mathscr{P}) \qquad \begin{cases} -\Delta u = e^{-2u} & \text{in } \mathbb{R}^N_+ \\ u > 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

where $N \geq 2$, $x = (x', x_N) \in \mathbb{R}^N_+$ with $x' \in \mathbb{R}^{N-1}$ and $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_N > 0\}$.

We say $u\colon \overline{\mathbb{R}^N_+} \to \overline{\mathbb{R}_+}$ is a weak solution to ($\mathscr{P}$) in the sense that, $u|_{B_R^+} \in H^1(B_R^+)$ with $B_R^+ := B_R(0) \cap \mathbb{R}^N_+$ for any $R > 0$, $u = 0$ on $\partial\mathbb{R}^N_+$ in the sense of trace, and

$$(1.1) \qquad \int_{\mathbb{R}^N_+} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N_+} e^{-2u} \varphi \, \mathrm{d}x, \quad \forall \varphi \in C_c^1(\mathbb{R}^N_+).$$

Equation ($\mathscr{P}$) can be regarded as the exponential decreasing nonlinear counterpart of the second order Liouville equation with exponential increasing nonlinearity:

$$-\Delta u = e^{2u}.$$

For classification results on second order, higher order and higher dimensional Liouville equations with exponential increasing nonlinearities, we refer to [8–12, 14–17, 20, 23, 25, 26, 31] and the references therein, in which the finite total mass condition $\int e^{2u} \mathrm{d}x < +\infty$ is usually necessary. The Dirichlet and Navier problems for Liouville equations in half space does not possess any positive classical solution in half space (see, e.g., [15–17]). The exponential decay at infinity of the nonlinearity causes that Problem ($\mathscr{P}$) can also

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be regarded as a natural counterpart of the singular equation $-\Delta u = u^{-\gamma}$ with negative exponent ($\gamma > 0$) investigated in [27, 28] as $\gamma \to +\infty$, which arises in the blow-up analysis near the boundary of bounded domain for solutions to $-\Delta u = 1/u^{\gamma} + f(u)$ (see, e.g., [7, 19]). The classification results in [27, 28] conclude the whole picture for the classification of solutions to semi-linear elliptic equations $-\Delta u = u^p$ ($p \in (-\infty, \frac{2N}{N-2}]$) in half space involving (positive/negative) power nonlinearities. For more literature on the study of singular elliptic problems with negative exponents, refer to the seminal paper [13], see also [1, 3–6, 18, 21, 24, 29] and the references therein.

Throughout this paper, we shall use the following notation.

Definition 1.1. For any given $h > 0$, we define the strip Σ_h^+ in the half space $\mathbb{R}_+^N$ by

$$\Sigma_h^+ := \{x \in \mathbb{R}_+^N \mid 0 < x_N < h\}. \tag{1.2}$$

When $N \geq 3$, the following hypothesis will be needed when we prove the $1 - D$ symmetry of solution u.

(A) There exists $\Lambda > 0$ such that u is bounded from above in the strip Σ_Λ^+. We define

$$\Theta := \sup_{\Sigma_\Lambda^+} u(x) < +\infty.$$

Our main result is the following theorem.

Theorem 1.2. *Assume that $N \geq 2$ and let u be a weak solution to ($\mathcal{P}$). Then, we have that $u \in C^2(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N})$ and there exists $H = H(N) > 0$ such that*

$$u(x) \geq \ln\left(1 + H^{-1}x_N\right) - \frac{1}{2}\ln\ln\left(1 + H^{-1}x_N\right) \quad \textit{in } \mathbb{R}_+^N \setminus \Sigma_{2H}^+, \tag{1.3}$$

and hence

$$\liminf_{x_N \to +\infty} \frac{u(x)}{\ln(1 + x_N)} \geq 1.$$

Assume further that u satisfies the assumption (**A**) *if $N \geq 3$ (no further assumption if $N = 2$). Then u has 1-D symmetry, that is,*

$$u(x) = u(x', x_N) = u(x_N).$$

Moreover, we have, either

$$u = u(x_N) = \ln(1 + x_N),$$

or

$$u = u(x_N) = u_L(x_N), \quad \forall L > 0,$$

where

$$u_L(t) = \ln[\sinh(L(t + t_0(L)))] - \ln L, \quad \textit{with} \quad t_0(L) = \frac{1}{L}\ln(L + \sqrt{L^2 + 1}),$$

is the unique solution to

$$\begin{cases} -u_L''(t) = e^{-2u_L(t)}, & \forall t > 0, \\ u_L(t) > 0, & \forall t > 0, \\ u_L(0) = 0, \quad \lim_{t \to +\infty} u_L'(t) = L. \end{cases} \tag{1.4}$$

Remark 1.3. Our classification result in Theorem 1.2 can be compared with the classification results for $-\Delta u = u^{-\gamma}$ with $\gamma > 0$ in [27, 28]. Comparing with semi-linear elliptic equations involving (positive/negative) power nonlinearities, there are some key difficulties and ingredients in our proof of Theorem 1.2. First, it is necessary but very difficult for us to show the a priori lower bound estimate $u(x) \geq u_*(x_N)$, where $u_*(x_N) := \ln(1 + x_N)$ is the special solution to $(\mathcal{P})$. Fortunately, we are able to show the “ln-ln ln" type almost sharp estimate (1.3) in Theorem 2.1, which is exactly enough for our proof and plays a key role in the proof. Second, there is no scaling that can keep both the equation and the boundary value in problem $(\mathcal{P})$ invariant at the same time. Consequently, we need to show some uniform estimates and uniform results with respect to the family of boundary value problems caused by scalings. For instance, the lower bound estimates in Theorem 2.1 including (1.3) are valid for general boundary value problem (2.8) (see Remark 2.2 and the proof of Theorem 2.1).

We would like to summarize the outline of our proof and the rest of our paper as follows. In Section 2, we prove the accurate lower bound estimates and $C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$-regularity without the assumption (**A**), and the precise upper bound estimates on both u and $|\nabla u|$ under the assumption (**A**). Actually, we show in (i) of Theorem 2.1 that

$$u(x) \geq \ln(1 + H^{-1}x_N) - \frac{1}{2} \ln\ln(1 + H^{-1}x_N)$$

far from the boundary $\partial\mathbb{R}^N_+$, which is an almost sharp estimate and key ingredient in our proof. In Section 3, thanks to these accurate asymptotic estimates established in Section 2, we are able to exploit a celebrated result from Berestycki, Caffarelli and Nirenberg [2] to derive the 1-D symmetry of solutions for $N \geq 3$ under the assumption (**A**). One should note that, when $N = 2$, we can also show the 1-D symmetry of solution u by using Kelvin type transforms and the method of moving planes without the assumption (**A**). Section 4 is devoted to carrying out the ODE analysis and completing our proof of Theorem 1.2.

In the following, we will use C to denote a general positive constant that may depend on N, Λ and Θ, and whose value may differ from line to line.

2. Regularity of weak solutions and precise asymptotic estimates

In this section, we assume u is a positive weak solution to $(\mathcal{P})$, and we will show regularity and precise asymptotic behaviors of u.

2.1. The precise lower bound estimates and regularity without assumption (A)

First, without the boundedness assumption (**A**), we will prove the precise lower bound estimates and the $C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$-regularity for u. In fact, the standard elliptic estimates (see, for instance, [22] and [30]) imply that, the weak solution u to $(\mathcal{P})$ belongs to $C^2(\mathbb{R}^N_+) \cap L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^N_+})$, because the nonlinear term e^{-2u} is bounded. Then, it follows from the standard local estimates up to the boundary (cf., e.g., Theorem 8.29 in [22]) that $u \in C(\overline{\mathbb{R}^N_+})$. That is, we have $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$.

Theorem 2.1. *There exist constants* $H = H(N) > 0$ *and* $c = c(N) > 0$ *such that*

$$\begin{array}{lll} \text{(i)} & u(x) \geq \ln(1 + H^{-1}x_N) - \dfrac{1}{2}\ln\ln(1 + H^{-1}x_N) & \textit{in } \mathbb{R}^N_+ \setminus \Sigma^+_{2H}, \\ \text{(ii)} & u(x) \geq c\ln(1 + x_N) & \textit{in } \mathbb{R}^N_+ \setminus \Sigma^+_1, \\ \text{(iii)} & u(x) \geq c x_N^2 & \textit{in } \Sigma^+_1. \end{array}$$

Proof. Let $\phi_1 \in C^2(\overline{B_1(0)})$ be the first eigenfunction that solves

$$\begin{cases} -\Delta\phi_1 = \lambda_1\phi_1 & \text{in } B_1(0), \\ \phi_1 > 0 & \text{in } B_1(0), \\ \phi_1 = 0 & \text{on } \partial B_1(0), \end{cases} \tag{2.1}$$

with $\phi_1(0) = \|\phi_1\|_{L^\infty(B_1(0))} = 1$. Let us define

$$w(x) := c\ln(1 + \phi_1(x))$$

with constant $0 < c \leq 1/\ln 2$ to be determined later. One can see that $0 < w \leq c\ln 2 \leq 1$ in $B_1(0)$. Through straightforward computations, one has

$$\Delta w = \frac{c\Delta\phi_1}{1+\phi_1} - \frac{c|\nabla\phi_1|^2}{(1+\phi_1)^2}.$$

By (2.1), we obtain

$$\begin{aligned} -\Delta w &= \frac{c|\nabla\phi_1|^2}{(1+\phi_1)^2} + \frac{c\lambda_1\phi_1}{1+\phi_1} = \frac{c}{(1+\phi_1)^2}\big(|\nabla\phi_1|^2 + \lambda_1\phi_1(1+\phi_1)\big) \\ &=: \eta(x)\, e^{-2w} \quad \text{in } B_1(0), \end{aligned}$$

where

$$\eta(x) := c[(|\nabla\phi_1|^2 + \lambda_1\phi_1^2) + \lambda_1\phi_1](1+\phi_1)^{2c-2}.$$

First, we choose $c = 1/\ln 2$, then

$$0 < \eta(x) = \frac{1}{\ln 2}[(|\nabla\phi_1|^2 + \lambda_1\phi_1^2) + \lambda_1\phi_1](1+\phi_1)^{2c-2} \leq H^2 \quad \text{in } B_1(0),$$

where

$$H = H(N) := \frac{e}{2\sqrt{\ln 2}}\sqrt{\sup_{B_1(0)} |\nabla\phi_1|^2 + 2\lambda_1} > 0$$

depends only on N. Thus w solves

$$\begin{cases} -\Delta w \leq H^2 e^{-2w} & \text{in } B_1(0), \\ 0 < w \leq 1 & \text{in } B_1(0), \\ w = 0 & \text{on } \partial B_1(0). \end{cases} \tag{2.2}$$

For arbitrarily fixed $x_0 = (x_0', x_{0,N}) \in \mathbb{R}^N_+ \setminus \Sigma^+_2$, let

$$\widehat{w}_{x_0,R} := \Big[\ln(1+R) - \frac{1}{2}\ln\ln(1+R)\Big]\, w\big(\frac{x - x_0}{R}\big) \quad \text{in } B_R(x_0),$$

where $R = x_{0,N} \geq 2$. Then, we infer from (2.5) that

$$
\begin{aligned}
(2.3)\qquad -\Delta \widehat{w}_{x_0,R} &= -\frac{\ln(1+R) - \frac{1}{2}\ln\ln(1+R)}{R^2}\,\Delta w\Big(\frac{x-x_0}{R}\Big) \\
&\leq H^2\,\frac{\ln(1+R) - \frac{1}{2}\ln\ln(1+R)}{R^2\,e^{2w((x-x_0)/R)}} \leq H^2\,\frac{\ln(1+R)}{R^2\,e^{2w((x-x_0)/R)}} \\
&\leq \frac{H^2}{e^{2(\ln R - \frac{1}{2}\ln\ln(1+R)+1)\,w((x-x_0)/R)}} \\
&\leq \frac{H^2}{e^{2[\ln(1+R) - \frac{1}{2}\ln\ln(1+R)]w((x-x_0)/R)}} = H^2\,e^{-2\widehat{w}_{x_0,R}} \quad \text{in } B_R(x_0).
\end{aligned}
$$

Recalling that u is a solution to ($\mathscr{P}$), and letting $u_H(x) := u(Hx)$, then we have

$$
(2.4)\qquad \begin{cases} -\Delta u_H = H^2\,e^{-2u_H} & \text{in } B_R(x_0), \\ -\Delta \widehat{w}_{x_0,R} \leq H^2\,e^{-2\widehat{w}_{x_0,R}} & \text{in } B_R(x_0), \end{cases}
$$

with $u_H(x) \geq 0$ and $\widehat{w}_{x_0,R} = 0$ on $\partial B_R(x_0)$. Since

$$
e^{-2\widehat{w}_{x_0,R}} - e^{-2u_H} = c(x)(\widehat{w}_{x_0,R} - u_H), \quad c(x) = \frac{e^{-2\widehat{w}_{x_0,R}} - e^{-2u_H}}{\widehat{w}_{x_0,R} - u_H} \leq 0,
$$

we infer from the maximum principle that $u_H \geq \widehat{w}_{x_0,R}$ in $B_R(x_0)$, and hence

$$
u(Hx_0) \geq \Big[\ln(1+R) - \frac{1}{2}\ln\ln(1+R)\Big]w(0) = \ln(1+R) - \frac{1}{2}\ln\ln(1+R).
$$

Since $x_0 \in \mathbb{R}^N_+ \setminus \Sigma_2^+$ is arbitrary and $R = x_{0,N} \geq 2$, we derive

$$
u(Hx) \geq \Big[\ln(1+x_N) - \frac{1}{2}\ln\ln(1+x_N)\Big]
$$

for any $x = (x', x_N) \in \mathbb{R}^N_+$ with $x_N \geq 2$. Consequently, we get

$$
u(x) \geq \Big[\ln(1+H^{-1}x_N) - \frac{1}{2}\ln\ln(1+H^{-1}x_N)\Big]
$$

for any $x = (x', x_N) \in \mathbb{R}^N_+$ with $x_N \geq 2H$. This proves (i) in Theorem 2.1.

Next, we choose $c = c(N) \in \big(0, \frac{1}{2\ln 2}\big)$ small enough such that

$$
\eta(x) = c\big[(|\nabla\phi_1|^2 + \lambda_1\phi_1^2) + \lambda_1\phi_1(1+\phi_1)^{2c-2}\big] \leq 1 \quad \text{in } B_1(0),
$$

and hence w solves

$$
(2.5)\qquad \begin{cases} -\Delta w \leq e^{-2w} & \text{in } B_1(0), \\ 0 < w \leq c\ln 2 < 1/2 & \text{in } B_1(0), \\ w = 0 & \text{on } \partial B_1(0). \end{cases}
$$

For arbitrarily fixed $x_0=(x_0',x_{0,N})\in\mathbb{R}^N_+\setminus\Sigma_1^+$, let

$$w_{x_0,R}:=\ln(1+R)\,w\Big(\frac{x-x_0}{R}\Big)\quad\text{in } B_R(x_0),$$

where $R=x_{0,N}\geq 1$. Then, we infer from (2.5) that

$$\begin{aligned}-\Delta w_{x_0,R}&=-\frac{\ln(1+R)}{R^2}\,\Delta w\Big(\frac{x-x_0}{R}\Big)\\&\leq\frac{\ln(1+R)}{R}\,\frac{1}{Re^{2w((x-x_0)/R)}}\leq\frac{\ln(1+R)}{R}\,\frac{1}{R^{2c\ln 2}e^{2w((x-x_0)/R)}}\\&\leq\frac{1}{e^{2(\ln R+1)w((x-x_0)/R)}}\leq\frac{1}{e^{2\ln(1+R)w((x-x_0)/R)}}=e^{-2w_{x_0,R}}\quad\text{in } B_R(x_0).\end{aligned}$$

Recalling that u is a solution to ($\mathcal{P}$), we have

$$\begin{cases}-\Delta u=e^{-2u} & \text{in } B_R(x_0),\\ -\Delta w_{x_0,R}\leq e^{-2w_{x_0,R}} & \text{in } B_R(x_0),\end{cases}\tag{2.6}$$

with $u(x)\geq 0$ and $w_{x_0,R}=0$ on $\partial B_R(x_0)$. Then we infer from the maximum principle that $u\geq w_{x_0,R}$ in $B_R(x_0)$, and hence

$$u(x_0)\geq\ln(1+R)w(0)=c\ln 2\ln(1+R).$$

Since $x_0\in\mathbb{R}^N_+\setminus\Sigma_1^+$ is arbitrary and $R=x_{0,N}\geq 1$, we derive $u\geq c\ln 2\ln(1+x_N)$ for any $x=(x',x_N)\in\mathbb{R}^N_+$ with $x_N\geq 1$. This proves (ii) in Theorem 2.1.

For arbitrarily fixed $x_0=(x_0',x_{0,N})\in\Sigma_1^+$, let

$$\widetilde{w}_{x_0,R}:=R^2w\Big(\frac{x-x_0}{R}\Big)\quad\text{in } B_R(x_0),$$

where $R=x_{0,N}<1$. Then, we infer from (2.5) that

$$\begin{aligned}-\Delta\widetilde{w}_{x_0,R}=-\Delta w\Big(\frac{x-x_0}{R}\Big)&\leq e^{-2w((x-x_0)/R)}\\&\leq e^{-2R^2w((x-x_0)/R)}=e^{-2\widetilde{w}_{x_0,R}}\quad\text{in } B_R(x_0).\end{aligned}$$

Recalling that u is a solution to ($\mathcal{P}$), we have

$$\begin{cases}-\Delta u=e^{-2u} & \text{in } B_R(x_0),\\ -\Delta\widetilde{w}_{x_0,R}\leq e^{-2\widetilde{w}_{x_0,R}} & \text{in } B_R(x_0),\end{cases}\tag{2.7}$$

with $u(x)\geq 0$ and $\widetilde{w}_{x_0,R}=0$ on $\partial B_R(x_0)$. Then it follows from the maximum principle that $u\geq\widetilde{w}_{x_0,R}$ in $B_R(x_0)$, and hence

$$u(x_0)\geq w(0)R^2=(c\ln 2)R^2.$$

Recalling that $R=x_{0,N}\leq 1$, since $x_0\in\Sigma_1^+$ is arbitrary, one has $u(x)\geq(c\ln 2)x_N^2$ in Σ_1^+. This proves (iii) in Theorem 2.1, and hence we have finished our proof of Theorem 2.1. ■

Remark 2.2. It is clear from the proof of Theorem 2.1 that the lower bound estimates (i), (ii) and (iii) in Theorem 2.1 hold for all weak solutions u to the general problem

$$\begin{cases} -\Delta u = e^{-2u} & \text{in } \mathbb{R}^N_+, \\ u > 0 & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{2.8}$$

which generalizes the Dirichlet problem ($\mathscr{P}$).

2.2. The precise upper bound estimates with assumption (A)

In this subsection, we need the boundedness assumption (**A**) to derive the precise upper bound estimates on both u and $|\nabla u|$.

Lemma 2.3. *Under the assumption* (**A**), *we have the following sharp upper bound in the strip* Σ^+_Λ:

$$u(x) \leq C \ln(1 + x_N) \quad \text{in } \Sigma^+_\Lambda$$

with constant $C = C(\Lambda, \Theta) > 0$.

Proof. Let $\widehat{u}(x_N) = \ln(1 + x_N)$ be the 1-D solution to

$$\begin{cases} -\widehat{u}'' = e^{-2\widehat{u}} & \text{in } \mathbb{R}^+, \\ \widehat{u}(0) = 0, & \\ \widehat{u} > 0 & \text{in } \mathbb{R}^+. \end{cases}$$

Therefore, for $\rho > 1$, $\widehat{u}_\rho := \rho \widehat{u}$ solves

$$\begin{cases} -\Delta \widehat{u}_\rho = \rho e^{-2\widehat{u}} > e^{-2\widehat{u}_\rho} & \text{in } \mathbb{R}^N_+, \\ \widehat{u}_\rho > 0 & \text{in } \mathbb{R}^N_+, \\ \widehat{u}_\rho = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{2.9}$$

Now, we take $\rho := \Theta/\ln(1 + \Lambda) + 1$. It follows that

$$\widehat{u}_\rho(x) = \rho \widehat{u}(\Lambda) = \rho \ln(1 + \Lambda) \geq \Theta \quad \text{on } \{x \in \mathbb{R}^N_+ : x_N = \Lambda\}. \tag{2.10}$$

Thus we have

$$u \leq \widehat{u}_\rho \quad \text{on } \partial\Sigma^+_\Lambda. \tag{2.11}$$

Let $\zeta := u - \widehat{u}_\rho$. Then $\zeta \in C(\overline{\Sigma^+_\Lambda})$, $\|\zeta\|_{L^\infty(\Sigma^+_\Lambda)} \leq 2\Theta + \ln(1 + \Lambda)$, and ζ satisfies

$$\begin{cases} \Delta\zeta + c(x)\zeta > 0 & \text{in } \Sigma^+_\Lambda, \\ \zeta \leq 0 & \text{on } \partial\Sigma^+_\Lambda, \end{cases} \tag{2.12}$$

where $c(x) := (e^{-2u} - e^{-2\widehat{u}_\rho})/(u - \widehat{u}_\rho) \leq 0$. Therefore, by applying a celebrated result from Berestycki, Caffarelli and Nirenberg (see Lemma 2.1 in [2]) to (2.12), we derive that $u \leq \widehat{u}_\rho$ in the strip Σ^+_Λ. This finishes our proof of Lemma 2.3. ■

Lemma 2.4. *Under the assumption* (**A**), *there exists a positive constant* $C = C(\Theta, \Lambda, N)$ *such that*

$$u(x) \leq Cx_N \quad \textit{in } \mathbb{R}^N_+ \setminus \Sigma^+_\Lambda.$$

Proof. If $\Lambda < 4H$, by applying the invariant scaling for the equation $-\Delta u = e^{-2u}$,

$$u_{\Lambda/(4H)}(x) = u\Big(\frac{\Lambda}{4H}x\Big) - \ln\Big(\frac{\Lambda}{4H}\Big), \tag{2.13}$$

one can verify that $u_{\Lambda/(4H)}$ still solves $-\Delta u_{\Lambda/(4H)} = e^{-2u_{\Lambda/(4H)}}$ in $\mathbb{R}^N_+$, with $u_{\Lambda/(4H)} > 0$ in $\mathbb{R}^N_+$ and $u_{\Lambda/(4H)} = -\ln(\Lambda/(4H)) > 0$ on $\partial\mathbb{R}^N_+$. It follows from Theorem 2.1 and Remark 2.2 that the precise lower bound estimates (i), (ii) and (iii) hold for $u_{\Lambda/(4H)}$. In particular, we have $u_{\Lambda/(4H)}(x) \geq \ln(1 + H^{-1}x_N) - \frac{1}{2}\ln\ln(1 + H^{-1}x_N)$ in $\mathbb{R}^N_+ \setminus \Sigma^+_{2H}$. Consequently, by replacing u with $u_{\Lambda/(4H)}$ if $\Lambda < 4H$, in what follows, we may assume, without loss of generalities, that $\Lambda = 4H$ in the hypothesis (**A**), i.e., the solution u is bounded in the strip Σ^+_{4H}, where $H = H(N)$ is given by Lemma 2.1. Note that, if $\Lambda < 4H$, the new function $u_{\Lambda/(4H)}$ (still denoted by u) is positive on $\partial\mathbb{R}^N_+$.

For any $x_0 = (x_0', x_{0,N}) \in \mathbb{R}^+_N$ with $x_{0,N} \geq \Lambda = 4H$, let $R := x_{0,N}/4 \geq H$ and $u_R(x) := u(x_0 + R(x - x_0))$. Then, we deduce from equation ($\mathcal{P}$) that

$$-\Delta u_R = R^2 \frac{1}{u_R e^{2u_R}} u_R =: c(x)u_R \quad \text{in } B_4(x_0), \tag{2.14}$$

and $u_R > 0$ in $B_4(x_0)$. Since (i) in Lemma 2.1 implies that

$$u(x) \geq \ln(1 + H^{-1}x_N) - \frac{1}{2}\ln\ln(1 + H^{-1}x_N)$$

for any $x = (x', x_N) \in \mathbb{R}^+_N$ with $x_N \geq 2H$, noting that $2R = x_{0,N}/2 \geq 2H$, we deduce that

$$\begin{aligned} u^R(x)e^{2u_R(x)} &\geq \frac{(1 + 2H^{-1}R)^2\left[\ln(1 + 2H^{-1}R) - \frac{1}{2}\ln\ln(1 + 2H^{-1}R)\right]}{\ln(1 + 2H^{-1}R)} \\ &\geq 2H^{-2}R^2 \quad \text{in } B_2(x_0). \end{aligned}$$

Therefore, we have

$$0 < c(x) := \frac{R^2}{u_R\, e^{2u_R}} \leq \frac{R^2}{2H^{-2}R^2} = \frac{H^2}{2} \quad \text{in } B_2(x_0),$$

where the constant $H = H(N)$ is the same as (i) in Theorem 2.1. Consequently, by applying the Harnack inequality (cf., e.g., Theorem 8.20 in [22]) to

$$-\Delta u_R = c(x)u_R \quad \text{in } B_2(x_0),$$

we get

$$\sup_{B_1(x_0)} u_R \leq C_H \inf_{B_1(x_0)} u_R, \tag{2.15}$$

where $C_H = C_H(N)$. Thus we can deduce that

$$u(x_0) \leq \sup_{B_R(x_0)} u = \sup_{B_1(x_0)} u_R \leq C_H \inf_{B_1(x_0)} u_R = C_H \inf_{B_R(x_0)} u \leq C_H u(x) \quad \text{in } B_R(x_0),$$

and hence

$$u(x) \geq C_H^{-1} u(x_0) \quad \text{on } \partial B_R(x_0).$$

For any $\sigma, \tau \in \mathbb{R}$, let us define the auxiliary function

$$\Gamma_{\sigma,\tau} := \begin{cases} \sigma\big(\frac{1}{|x-x_0|^{N-2}} + \tau\big) & \text{if } N \geq 3, \\ \sigma\big(\ln \frac{1}{|x-x_0|} + \tau\big) & \text{if } N = 2, \end{cases}$$

which satisfies

$$\Delta \Gamma_{\sigma,\tau} = 0 \quad \text{in } \mathbb{R}^N \setminus \{x_0\}.$$

Now we choose σ and τ such that

$$\begin{cases} \Gamma_{\sigma,\tau} = C_H^{-1} u(x_0) & \text{on } \partial B_R(x_0), \\ \Gamma_{\sigma,\tau} = 0 & \text{on } \partial B_{4R}(x_0). \end{cases} \tag{2.16}$$

Through direct computations, we can see that the system (2.16) holds as long as we take

$$\sigma = \frac{C_H^{-1}\, u(x_0)(4R)^{N-2}}{4^{N-2}-1} =: \tilde{c}_N\, u(x_0) R^{N-2} \quad \text{and} \quad \tau = -\frac{1}{(4R)^{N-2}} \quad \text{if } N \geq 3, \tag{2.17}$$

$$\sigma = \frac{C_H^{-1}\, u(x_0)}{2 \ln 2} =: \tilde{c}_2 u(x_0) \quad \text{and} \quad \tau = \ln(4R) \quad \text{if } N = 2, \tag{2.18}$$

where

$$\tilde{c}_N = \frac{C_H^{-1}\, 4^{N-2}}{4^{N-2}-1} \quad \text{for } N \geq 3, \text{ and} \quad \tilde{c}_2 = \frac{C_H^{-1}}{2 \ln 2}.$$

With such choice of σ and τ, we have

$$\begin{cases} -\Delta u = e^{-2u} > 0 & \text{in } B_{4R}(x_0) \setminus B_R(x_0), \\ -\Delta \Gamma_{\sigma,\tau} = 0 & \text{in } B_{4R}(x_0) \setminus B_R(x_0), \\ u, \Gamma_{\sigma,\tau} > 0 & \text{in } B_{4R}(x_0) \setminus B_R(x_0), \end{cases}$$

and $u \geq \Gamma_{\sigma,\tau}$ on $\partial(B_{4R}(x_0) \setminus B_R(x_0))$. Then we deduce from the maximum principle that

$$u(x) \geq \Gamma_{\sigma,\tau}(x) \quad \text{in } B_{4R}(x_0) \setminus B_R(x_0). \tag{2.19}$$

Thus, by choosing $(x_0', H) \in B_{4R}(x_0) \setminus B_R(x_0)$, it follows from (2.17), (2.18) and (2.19) that

$$\begin{aligned} u(x_0', H) \geq \Gamma_{\sigma,\tau}(x_0', H) &= \sigma\Big(\frac{1}{|(x_0', H)-(x_0', x_{0,N})|^{N-2}} + \tau\Big) = c\Big(\frac{1}{|4R-H|^{N-2}} + \tau\Big) \\ &= \tilde{c}_N\, u(x_0)\, R^{N-2}\Big(\frac{1}{(4R-H)^{N-2}} - \frac{1}{(4R)^{N-2}}\Big) \quad \text{if } N \geq 3, \\ u(x_0', H) \geq \Gamma_{\sigma,\tau}(x_0', H) &= \sigma\Big(\ln \frac{1}{|(x_0', H)-(x_0', x_{0,N})|} + \tau\Big) = c\Big(\ln \frac{1}{|4R-H|} + \tau\Big) \\ &= \tilde{c}_2\, u(x_0)\, [\ln(4R) - \ln(4R-H)] \quad \text{if } N = 2. \end{aligned}$$

Then, by Lagrange's mean value theorem, we have

$$u(x_0', H) \geq \begin{cases} \dfrac{\tilde{c}_N(N-2)Hu(x_0)}{4^{N-1}R} & \text{if } N \geq 3, \\ \dfrac{\tilde{c}_2\, H\, u(x_0)}{4R} & \text{if } N = 2. \end{cases}$$

Since $\|u\|_{L^\infty(\Sigma_{4H}^+)} = \Theta$ due to assumption (**A**), it follows immediately that

$$u(x_0) \leq CR = \frac{C}{4}\, x_{0,N}$$

for some constant $C = C(\Theta, N)$ independent of R. Since $x_0 \in \mathbb{R}_+^N \setminus \Sigma_{4H}^+$ is arbitrary, we obtain

$$u(x) \leq Cx_N \quad \text{in } \{x \in \mathbb{R}_+^N : x_N \geq 4H\}.$$

Re-scaling by using the invariant scaling (2.13) if $\Lambda < 4H$, we finally derive $u(x) \leq Cx_N$ in $\mathbb{R}_+^N \setminus \Sigma_\Lambda^+$ for some positive constant $C = C(\Lambda, \Theta, N)$. This completes our proof of Lemma 2.4. ■

As an immediate consequence of Theorem 2.1 and Lemma 2.4, we have the following.

Corollary 2.5. *Under the assumption* (**A**)*, there exists* $C = C(\Lambda, \Theta, N)$ *such that*

$$u(x) \leq Cx_N \quad \textit{in } \mathbb{R}_+^N.$$

Furthermore, we can show that actually $\nabla u \in L^\infty(\mathbb{R}_+^N)$.

Lemma 2.6. *Under the assumption* (**A**)*, there exists* $C = C(\Lambda, \Theta, N)$ *such that*

$$|\nabla u| \leq C \quad \textit{in } \mathbb{R}_+^N.$$

Proof. For arbitrary $x_0 = (x', x_{0,N}) \in \mathbb{R}_+^N$, we set $R = x_{0,N}$ and define

$$u_R(x) = \frac{u(Rx)}{R} \quad \text{in } B_{1/2}\Big(\frac{x_0}{R}\Big).$$

Then, by equation ($\mathcal{P}$), we know u_R satisfies

$$-\Delta u_R = \frac{R}{e^{2u(Rx)}} =: f(x) \quad \text{in } B_{1/2}\Big(\frac{x_0}{R}\Big).$$

Exploiting Corollary 2.5, we deduce that

$$u_R(x) = \frac{u(Rx)}{R} \leq C\,\frac{Rx_N}{R} = Cx_N \leq \frac{3}{2}C \quad \text{in } B_{1/2}\Big(\frac{x_0}{R}\Big),$$

where the constant $C = C(\Theta, \Lambda, N)$ is independent of R. Note that $Rx \in B_{R/2}(x_0)$ for any $x \in B_{1/2}(x_0/R)$. If $R = x_{0,N} \leq 4H$, then

$$f(x) = \frac{R}{e^{2u(Rx)}} \leq R \leq 4H \quad \text{in } B_{1/2}\Big(\frac{x_0}{R}\Big),$$

where the constant $H = H(N)$ comes from (i) in Theorem 2.1. If $R = x_{0,N} > 4H$, then it follows from (i) in Theorem 2.1 that

$$u(Rx) \geq \ln(1 + H^{-1}Rx_N) - \frac{1}{2}\ln\ln(1 + H^{-1}Rx_N) \quad \text{in } B_{1/2}\Big(\frac{x_0}{R}\Big),$$

and hence

$$f(x) \leq \frac{R\ln(1 + H^{-1}Rx_N)}{(1 + H^{-1}Rx_N)^2} \leq \frac{H^{-1}R^2x_N}{H^{-2}R^2x_N^2} \leq \frac{H}{x_N} \leq 2H \quad \text{in } B_{1/2}\Big(\frac{x_0}{R}\Big).$$

In a word, we have $f(x) = R/e^{2u(Rx)} \leq 4H$ in $B_{1/2}(x_0/R)$.

As a consequence, by regularity estimates (see, e.g., Theorem 3.9 in [22]), we derive

$$|\nabla u_R(x)| = |\nabla u(Rx)| \leq C(\Theta, \Lambda, N) \quad \text{in } B_{1/4}\Big(\frac{x_0}{R}\Big),$$

and hence

$$|\nabla u(x)| \leq C(\Theta, \Lambda, N) \quad \text{in } B_{R/4}(x_0).$$

Due to the arbitrariness of $x_0 \in \mathbb{R}^N_+$, we conclude that $\|\nabla u\|_{L^\infty(\mathbb{R}^N_+)} \leq C(\Theta, \Lambda, N)$. This finishes the proof of Lemma 2.6. ■

3. 1-D symmetry of solutions

Let u be a weak solution to $(\mathcal{P})$. We are now ready to derive the 1-D symmetry result.

3.1. 1-D symmetry under the assumption (A) for general $N \geq 2$

Theorem 3.1. *Suppose u is a weak solution to $(\mathcal{P})$. Under the assumption* (**A**), *it holds*

$$u(x) = u(x', x_N) = u(x_N) \quad \textit{in } \mathbb{R}^N_+.$$

Proof. Let $\mu \in \mathbb{R}$ and $\theta > 0$ to be chosen later. For any unit vector $\nu \in \partial B_1'(0) := \{x \in \partial\mathbb{R}^N_+ : |x| = 1\}$, define

$$u_{\mu,\theta} := u(x + \mu\nu + \theta e_N).$$

One can easily verify that $-\Delta u_{\mu,\theta} = e^{-2u_{\mu,\theta}}$ in $\mathbb{R}^N_+$. Setting $\omega := u - u_{\mu,\theta}$, we get

$$-\Delta\omega = e^{-2u} - e^{-2u_{\mu,\theta}} = g(x)\omega \quad \text{in } \mathbb{R}^N_+, \tag{3.1}$$

where $g(x) := (e^{-2u} - e^{-2u_{\mu,\theta}})/(u - u_{\mu,\theta}) \in C(\overline{\mathbb{R}^N_+})$ and $g(x) \leq 0$. From Theorem 2.1 and Lemma 2.3, we infer that, there exist constants $C_1, C_2 > 0$ such that

$$C_1x_N^2 \leq u(x) \leq C_2\ln(1 + x_N) \quad \text{in } \Sigma^+_{\min\{1,\Lambda\}}. \tag{3.2}$$

For $\theta > 0$, by (3.2), there exist $\gamma = \gamma(\theta) > 0$ and $\widetilde{\lambda} = \widetilde{\lambda}(\theta) < \min\{1, \Lambda\}$ (with $\gamma, \widetilde{\lambda} \sim \theta^2$ as $\theta \to 0^+$) such that $u < \gamma$ in $\Sigma_{\widetilde{\lambda}}$ and $u_{\mu,\theta} > 2\gamma$ in $\Sigma_{\widetilde{\lambda}}$ for any $\mu \in \mathbb{R}$. Thus

$$\omega = u - u_{\mu,\theta} < -\gamma < 0 \quad \text{in } \Sigma_{\widetilde{\lambda}}.$$

Define $D := \mathbb{R}^N_+ \setminus \Sigma_{\tilde{\lambda}}$. Then $\omega \leq -\gamma < 0$ on ∂D. Moreover, it follows from Lemma 2.6 that

$$\|\omega\|_{L^\infty(D)} \leq \|\nabla u\|_{L^\infty(\mathbb{R}^N_+)}(|\mu| + \theta) \leq C(|\mu| + \theta).$$

As a consequence, by applying Lemma 2.1 in [2] from Berestycki, Caffarelli and Nirenberg to the problem

$$\begin{cases} \Delta\omega + g(x)\omega = 0 & \text{in } D, \\ \omega \leq 0 & \text{on } \partial D, \end{cases}$$

we obtain $\omega := u - u_{\mu,\theta} \leq 0$ in D. Thus $\omega \leq 0$ in $\mathbb{R}^N_+$, i.e., $u \leq u_{\mu,\theta}$ in $\mathbb{R}^N_+$. Now, letting $\theta \to 0^+$, we obtain

$$u \leq u_\mu, \quad \forall \mu \in \mathbb{R},$$

and hence $u(x) = u(x + \mu\nu)$ for any $\mu \in \mathbb{R}$. By the arbitrariness of $\mu \in \mathbb{R}$ and $\nu \in \partial B_1'(0)$, we deduce that $u(x) = u(x_N)$. This finishes our proof of Theorem 3.1. ■

3.2. 1-D symmetry without the assumption (A) for $N = 2$

When the spatial dimension $N = 2$, we can prove the 1-D symmetry of weak solution u to $(\mathcal{P})$ by using Kelvin type transforms, without the boundedness assumption (**A**).

Theorem 3.2. *Assume* $N = 2$. *If* u *is a weak solution to* $(\mathcal{P})$*, then* u *has* 1-*D symmetry, namely,*

$$u(x) = u(x', x_N) = u(x_N).$$

Proof. For $N = 2$, we define the Kelvin type transform of the solution u by

$$\bar{u}(x) := u\left(\frac{x}{|x|^2}\right), \quad x \neq 0.$$

It follows from equation $(\mathcal{P})$ that $\bar{u}$ solves the equation

$$-\Delta\bar{u} = \frac{1}{|x|^4} e^{-2\bar{u}}, \quad x \neq 0,$$

in the weak sense:

$$\int_{\mathbb{R}^2_+} \nabla\bar{u} \cdot \nabla\varphi \,\mathrm{d}x = \int_{\mathbb{R}^2_+} \frac{1}{|x|^4} e^{-2\bar{u}} \varphi \,\mathrm{d}x, \quad \forall \varphi \in C^1_c(\mathbb{R}^2_+). \tag{3.3}$$

Note that, by Theorem 2.1, $\bar{u}$ has singularity at the origin 0. By the regularity result in Section 2, we know that $u \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$, thus $\bar{u} \in C^2(\mathbb{R}^2_+)$ and $\bar{u}$ is continuous up to the boundary $\partial\mathbb{R}^2_+$ except the origin 0.

We need now to introduce some standard notations. For any $\lambda < 0$, we define

$$\Sigma_\lambda := \{x = (x_1, x') = (x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N_+ \mid x_1 < \lambda\},$$

and

$$x_\lambda = (2\lambda - x_1, x'), \quad \bar{u}_\lambda(x) = \bar{u}(x_\lambda) = \bar{u}(2\lambda - x_1, x') \quad \text{in } \Sigma_\lambda.$$

Note that $\bar{u}_\lambda$ has singularity at $0_\lambda = (-2\lambda, x')$.

By the reflection invariance and (3.3), we deduce that

$$\int_{\mathbb{R}^N_+} \nabla \bar{u}_\lambda \cdot \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N_+} \frac{1}{|x_\lambda|^4} e^{-2\bar{u}_\lambda} \varphi \, \mathrm{d}x, \quad \forall \varphi \in C^1_c(\mathbb{R}^N_+). \tag{3.4}$$

Let logarithmic cut-off functions $0 \le \psi_R, \phi_\varepsilon \le 1$, be defined by

$$\psi_R(x) = \begin{cases} 1 & \text{for } |x| \le \sqrt{R}, \\ 2\log(R/|x|)/\log R & \text{for } \sqrt{R} \le |x| \le R, \\ 0 & \text{for } |x| \ge R, \end{cases}$$

and

$$\phi_\varepsilon(x) = \begin{cases} 0 & \text{for } |x - 0_\lambda| \le \varepsilon, \\ 2\log(\varepsilon/|x|)/\log \varepsilon & \text{for } \varepsilon \le |x - 0_\lambda| \le \sqrt{\varepsilon}, \\ 1 & \text{for } |x - 0_\lambda| \ge \sqrt{\varepsilon}. \end{cases}$$

Then, ψ_R and ϕ_ε satisfy

$$\begin{cases} \psi_R = 1 & \text{in } B_{\sqrt{R}}(0), \\ \psi_R = 0 & \text{in } \mathbb{R}^N \setminus B_R(0), \\ |\nabla \psi_R| = \dfrac{2}{|x| \log R} & \text{in } B_R(0) \setminus B_{\sqrt{R}}(0), \end{cases} \tag{3.5}$$

and

$$\begin{cases} \phi_\varepsilon = 0 & \text{in } B_\varepsilon(0_\lambda), \\ \phi_\varepsilon = 1 & \text{in } \mathbb{R}^N \setminus B_{\sqrt{\varepsilon}}(0_\lambda), \\ |\nabla \phi_\varepsilon| = \dfrac{2}{|x| |\log \varepsilon|} & \text{in } B_{\sqrt{\varepsilon}}(0_\lambda) \setminus B_\varepsilon(0_\lambda), \end{cases} \tag{3.6}$$

and moreover,

$$\int_{B_R(0) \setminus B_{\sqrt{R}}(0)} |\nabla \psi_R|^2 \le \frac{4\pi}{\log R} \xrightarrow[R \to \infty]{} 0 \tag{3.7}$$

and

$$\int_{B_{\sqrt{\varepsilon}}(0_\lambda) \setminus B_\varepsilon(0_\lambda)} |\nabla \phi_\varepsilon|^2 \le \frac{4\pi}{|\log \varepsilon|} \xrightarrow[\varepsilon \to 0]{} 0. \tag{3.8}$$

For any given $\delta > 0$, we define

$$w = w_{\varepsilon,\delta,R,\lambda} := (\bar{u} - \bar{u}_\lambda - \delta)_+ \phi_\varepsilon^2 \psi_R^2 \chi_{\Sigma_\lambda},$$

where χ_A denotes the characteristic function of the set A. One can see that $w \in H^1_0(\mathbb{R}^N_+)$ and w is compactly supported in $\mathbb{R}^N_+$. Therefore, by density, we can use w as test function in (3.3) and (3.4) and derive

$$\int_{\Sigma_\lambda} \nabla \bar{u} \cdot \nabla w \, \mathrm{d}x = \int_{\Sigma_\lambda} \frac{1}{|x|^4} e^{-2\bar{u}} w \, \mathrm{d}x,$$
$$\int_{\Sigma_\lambda} \nabla \bar{u}_\lambda \cdot \nabla w \, \mathrm{d}x = \int_{\Sigma_\lambda} \frac{1}{|x_\lambda|^4} e^{-2\bar{u}_\lambda} w \, \mathrm{d}x.$$

Note that $\bar{u} \geq \bar{u}_\lambda$ on $\operatorname{supp}(w)$ and $|x_\lambda| \leq |x|$ in Σ_λ, by subtracting the above two integral formulae, we get

$$\int_{\Sigma_\lambda} \nabla(\bar{u}-\bar{u}_\lambda)\cdot\nabla w \, \mathrm{d}x = \int_{\Sigma_\lambda} \Big(\frac{1}{|x|^4} e^{-2\bar{u}} - \frac{1}{|x_\lambda|^4} e^{-2\bar{u}_\lambda}\Big) w \, \mathrm{d}x \leq 0. \tag{3.9}$$

Define

$$h_\delta := (\bar{u}-\bar{u}_\lambda-\delta)_+.$$

Note that

$$\nabla w = \nabla h_\delta \phi_\varepsilon^2 \psi_R^2 + 2h_\delta \phi_\varepsilon \nabla\phi_\varepsilon \psi_R^2 + 2h_\delta \phi_\varepsilon^2 \psi_R \nabla\psi_R \quad \text{in } \Sigma_\lambda.$$

We infer from (3.9) that

$$\int_{\Sigma_\lambda} |\nabla h_\delta|^2 \phi_\varepsilon^2 \psi_R^2 \, \mathrm{d}x \leq 2\int_{\Sigma_\lambda} h_\delta |\nabla h_\delta| \phi_\varepsilon^2 \psi_R |\nabla \phi_R| \, \mathrm{d}x + 2\int_{\Sigma_\lambda} h_\delta |\nabla h_\delta| \phi_\varepsilon \psi_R^2 |\nabla\phi_\varepsilon| \, \mathrm{d}x.$$

By using Young's inequality, we get

$$\begin{aligned} &\int_{\Sigma_\lambda} |\nabla h_\delta|^2 \phi_\varepsilon^2 \psi_R^2 \\ &\leq 8\int_{\Sigma_\lambda\cap(B_R(0)\setminus B_{\sqrt{R}}(0))} h_\delta^2 \phi_\varepsilon^2 |\nabla\psi_R|^2 + 8\int_{\Sigma_\lambda\cap(B_{\sqrt{\varepsilon}}(0_\lambda)\setminus B_\varepsilon(0_\lambda))} h_\delta^2 \psi_R^2 |\nabla\phi_\varepsilon|^2 \\ &=: I_1(R) + I_2(\varepsilon). \end{aligned} \tag{3.10}$$

Since

$$h_\delta \leq \bar{u} \leq C \quad \text{in } B_R(0)\setminus B_{\sqrt{R}}(0)$$

for R large, it follows from (3.7) that

$$\lim_{R\to+\infty} I_1(R) = 0.$$

On the other hand, note that $h_\delta \leq \bar{u} \leq C$ in $\Sigma_\lambda \cap (B_{\sqrt{\varepsilon}}(0_\lambda)\setminus B_\varepsilon(0_\lambda))$, it follows from (3.8) that

$$\int_{\Sigma_\lambda\cap(B_{\sqrt{\varepsilon}}(0_\lambda)\setminus B_\varepsilon(0_\lambda))} h_\delta^2 \psi_R^2 |\nabla\phi_\varepsilon|^2 \leq C\int_{\Sigma_\lambda\cap(B_{\sqrt{\varepsilon}}(0_\lambda)\setminus B_\varepsilon(0_\lambda))} |\nabla\phi_\varepsilon|^2 = o_\varepsilon(1),$$

that is,

$$\lim_{\varepsilon\to 0} I_2(\varepsilon) = 0.$$

Consequently, passing to the limit as $R \to +\infty$ and $\varepsilon \to 0^+$ in (3.10) yields that

$$\int_{\Sigma_\lambda} |\nabla h_\delta|^2 \mathrm{d}x = 0.$$

This implies that $\bar{u} \leq \bar{u}_\lambda + \delta$ in Σ_λ. Then, letting $\delta \to 0^+$, we get

$$\bar{u} \leq \bar{u}_\lambda \quad \text{in } \Sigma_\lambda \quad \text{for any } \lambda < 0.$$

Repeating the argument in the opposite $(-x_1)$-direction, we conclude that

$$\bar{u}(x_1, x') = \bar{u}(-x_1, x'),$$

and hence

$$u(x_1, x') = u(-x_1, x'),$$

i.e., u is symmetric with respect to the hyperplane $\{x_1 = 0\}$, i.e., the x_N-axis for $N = 2$. Taking the translation invariance of the problem $(\mathscr{P})$ into account, we can see that u has to be rotationally symmetric with respect to any line parallel to the x_N-axis, which implies that u must be constant on any hyperplane $\{x_N = h \geq 0\}$. Therefore, we have

$$u(x) = u(x', x_N) = u(x_N).$$

This completes our proof of Theorem 3.2. ■

4. Completion of the proof of Theorem 1.2: ODE analysis

4.1. ODE analysis

Since we have proved that $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ and $u(x) = u(x_N)$, it follows that $u \in C^2(\mathbb{R}_+) \cap C(\overline{\mathbb{R}_+})$ and u solves the following ODE:

$$\begin{cases} -u'' = e^{-2u} & \text{in } \mathbb{R}_+, \\ u(t) > 0 & \text{in } \mathbb{R}_+, \\ u(0) = 0. \end{cases} \tag{4.1}$$

One can easily verify that

$$u_*(t) = \ln(1+t) \tag{4.2}$$

is a special solution to the ODE (4.1), and for any $L > 0$,

$$u_L(t) := \ln[\sinh(L(t + t_0(L)))] - \ln L, \quad \text{with} \quad t_0(L) = \frac{1}{L}\ln\left(L + \sqrt{L^2+1}\right)$$

is also a special solution to the ODE (4.1) such that $\lim_{t\to\infty} u'_L(t) = L$.

Lemma 4.1. *Suppose $u(t) \not\equiv u_*(t)$ is another solution to the ODE (4.1), then there exist $t_0 > 0$ and $C > 0$ such that $u(t_0) > Ct_0 + u_*(t_0)$ and $u'(t_0) > (Ct + u_*(t))'(t_0)$.*

Proof. Suppose $u(t) \not\equiv u_*(t)$ is another solution to the ODE (4.1), then there exists $t_0 > 0$ such that either $u(t_0) < u_*(t_0)$ and $u'(t_0) < u'_*(t_0)$, or $u(t_0) > u_*(t_0)$ and $u'(t_0) > u'_*(t_0)$.

If $u(t_0) < u_*(t_0)$ and $u'(t_0) < u'_*(t_0)$, suppose there exists $\tau > 0$ such that $u(t) < u_*(t)$ for any $t \in [t_0, t_0 + \tau]$ and $u(t_0+\tau) = u_*(t_0+\tau)$. Then it follows from (4.1) that $(u' - u'_*)'(t) = -(e^{-2u} - e^{-2u_*}) < 0$ for any $t \in [t_0, t_0+\tau]$, and hence $u'(t) - u'_*(t) < u'(t_0) - u'_*(t_0) < 0$ for any $t \in [t_0, t_0+\tau]$, which implies

$$u(t_0+\tau) - u_*(t_0+\tau) < u(t_0) - u_*(t_0) + (u'(t_0) - u'_*(t_0))\tau < u(t_0) - u_*(t_0) < 0.$$

This contradicts $u(t_0+\tau) = u_*(t_0+\tau)$. Therefore, we must have $u(t) < u_*(t)$ for all $t \geq t_0$. The ODE (4.1) implies again that $u'(t) - u'_*(t) < u'(t_0) - u'_*(t_0) < 0$ for all $t > t_0$.

Thus

$$\begin{aligned} u(t) &< u_*(t) + (u(t_0) - u_*(t_0)) + (u'(t_0) - u'_*(t_0))(t - t_0) \\ &< \ln(1+t) + (u'(t_0) - u'_*(t_0))(t - t_0) \to -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

which contradicts $u(t) > 0$.

As a consequence, if $u(t) \not\equiv u_*(t)$, then there must exist $t_0 > 0$ such that $u(t_0) > u_*(t_0)$ and $u'(t_0) > u'_*(t_0)$. Furthermore, $u(t_0) > Ct_0 + u_*(t_0)$ and $u'(t_0) > C + u'_*(t_0)$ for $C > 0$ small enough. ■

Next, we will show that if $u(t) \not\equiv u_*(t)$ is a solution to (4.1), then u must have linear growth, i.e., $\lim_{t\to+\infty} u'(t) = L$ for some $L > 0$.

To this end, for any $C > 0$, we define the following auxiliary function:

$$z_C(t) := Ct + \ln(1+t) = Ct + u_*(t).$$

Since $u_*(t) < z_C(t)$ for $t > 0$, one has

$$-z''_C(t) = -u''_*(t) = e^{-2u_*(t)} > e^{-2z_C(t)}, \quad \forall t > 0.$$

Note that $z_C(0) = u(0) = 0$, $z_C(t) > 0$ for $t > 0$, thus z_C is a super-solution for the ODE (4.1). Moreover, $z'_C(t) \to C$ as $t \to +\infty$.

Now, for arbitrarily fixed $t_0 > 0$ and $C > 0$, we consider the following problem:

$$\begin{cases} -w''(t) = e^{-2w(t)}, & \forall t > t_0, \\ w(t_0) > z_C(t_0), \\ w'(t_0) > z'_C(t_0). \end{cases} \tag{4.3}$$

Note that Lemma 4.1 implies that any solution $u(t) \not\equiv u_*(t)$ to the ODE (4.1) satisfies (4.3) for some $t_0 > 0$ and $C > 0$. We have the following lemma.

Lemma 4.2. *For any fixed $t_0 > 0$ and $C > 0$, there exists a solution w to the problem* (4.3). *Moreover, we have $w(t) > z_C(t)$ for $t \geq t_0$ and there exists a finite constant $L > C > 0$ such that* $\lim_{t\to\infty} w'(t) = L < +\infty$.

Proof. Since the nonlinearity e^{-2w} is locally Lipschitz, there exists a unique local solution w for problem (4.3). Next, we will show that $w(t) > z_C(t)$ for $t \geq t_0$ and hence the solution w is global.

Suppose on the contrary that there exists $T > t_0$ such that $w(t) > z_C(t)$ for $t_0 < t < T$ but $w(T) = z_C(T)$. Then we have

$$(w'(t) - z'_C(t))' > -e^{-2w(t)} + e^{-2z_C(t)} \geq 0, \quad \forall t \in [t_0, T].$$

This yields that $w'(t) - z'_C(t) > w'(t_0) - z'_C(t_0) > 0$ for each $t \in [t_0, T]$, thus

$$w(T) > z_C(T) + (w'(t_0) - z'_C(t_0))(T - t_0) > z_C(T),$$

which contradicts $w(T) = z_C(T)$. Therefore, it must hold that $w(t) > z_C(t) > 0$ for $t \geq t_0$, thus the nonlinear term e^{-2w} is globally Lipschitz and hence the solution w actually exists on the whole $[t_0, +\infty)$.

Moreover, we deduce from

$$w(t) > z_C(t) \quad \text{and} \quad (w'(t) - z'_C(t))' > -e^{-2w(t)} + e^{-2z_C(t)} > 0 \quad \text{on } [t_0, +\infty)$$

that

$$w'(t) - z_C'(t) > w'(t_0) - z_C'(t_0) > 0 \quad \text{for each } t \in [t_0, +\infty).$$

Furthermore, $w''(t) = -e^{-2w} < 0$ on $[t_0, +\infty)$ implies that $w'(t)$ is monotone decreasing and

$$w'(t) > z_C'(t) + (w'(t_0) - z_C'(t_0)) > C + (w'(t_0) - z_C'(t_0)) > C \quad \text{on } [t_0, +\infty).$$

Thus there exists a constant $L > C$ such that $\lim_{t\to+\infty} w'(t) = L > \lim_{t\to\infty} z_C'(t) = C$. This finishes our proof of Lemma 4.2. ∎

As a direct consequence of Lemma 4.2, we have the following corollary.

Corollary 4.3. *Suppose $u(t) \not\equiv u_*(t)$ is a solution to (4.1). Then u must have linear growth, i.e., there exists a constant $L > 0$ such that* $\lim_{t\to+\infty} u'(t) = L > 0$.

Finally, we will show that u_L is the unique solution to the ODE (4.1) with arbitrary asymptotic linear growth rate $L > 0$ at ∞, and hence classify all solutions $u(t) \not\equiv u_*(t)$ to (4.1).

Theorem 4.4. *For arbitrarily fixed $L > 0$, u_L is the unique solution to*

$$\begin{cases} -u_L''(t) = e^{-2u_L(t)}, & \forall t > 0, \\ u_L(t) > 0, & \forall t > 0, \\ u_L(0) = 0, \quad \lim_{t\to+\infty} u_L'(t) = L. \end{cases} \tag{4.4}$$

Proof. Assume by contradiction that u_L^1, u_L^2 are two different solutions to (4.4) with the same asymptotic linear growth rate $L > 0$ at ∞. First, we will show that $u_L^1(t) \neq u_L^2(t)$ for all $t > 0$. If not, suppose that there exist $0 \leq t_* < t^*$, which are a pair of neighboring roots of the equation $u_L^1(t) = u_L^2(t)$. Without loss of generalities, we may assume that $u_L^1(t) < u_L^2(t)$ for any $t \in (t_*, t^*)$ but $u_L^1(t_*) = u_L^2(t_*)$ and $u_L^1(t^*) = u_L^2(t^*)$. For any $\eta > 0$, taking $\varphi := (u_L^2 - u_L^1 - \eta)_+$ as a test function for problem (4.4), we derive

$$\int_{t_*}^{t^*} [(u_L^2 - u_L^1 - \eta)_+']^2 \, \mathrm{d}t = \int_{t_*}^{t^*} (e^{-2u_L^2} - e^{-2u_L^1})(u_L^2 - u_L^1 - \eta)_+ \, \mathrm{d}t \leq 0.$$

Thus $u_L^2 \leq u_L^1 + \eta$ in $[t_*, t^*]$ for all $\eta > 0$, and hence $u_L^1 = u_L^2$ in $[t_*, t^*]$. It follows that $u_L^1(t^*) = u_L^2(t^*)$ and $(u_L^1)'(t^*) = (u_L^2)'(t^*)$. Consequently, by uniqueness of solution to ODE $-u_L'' = e^{-2u_L}$, we must have $u_L^1 \equiv u_L^2$ in $[t_*, +\infty)$. Note that, if we cannot take $t_* = 0$, then there must exist a sequence of pairs of neighboring roots $0 < (t_*)_k < (t^*)_k$ of the equation $u_L^1(t) = u_L^2(t)$ such that $(t_*)_k \to 0$ as $k \to +\infty$. Therefore, we finally deduce that $u_L^1 \equiv u_L^2$ in the whole $[0, +\infty)$, which contradicts $u_L^1 \not\equiv u_L^2$.

Therefore, without loss of generalities, we may assume that $u_L^1(t) < u_L^2(t)$ for all $t \in \mathbb{R}^+$. It follows from

$$({u_L^1}' - {u_L^2}')'(t) = e^{-2u_L^2(t)} - e^{-2u_L^1(t)} < 0 \quad \text{in } \mathbb{R}^+$$

and ${u_L^1}'(t), {u_L^2}'(t) \to L$ as $t \to +\infty$ that

$${u_L^1}'(t) - {u_L^2}'(t) > 0 \quad \text{for all } t \in \mathbb{R}^+.$$

Thus $u_L^1(t) - u_L^2(t)$ is strictly increasing in $\mathbb{R}^+$. Recalling that $u_L^1(0) = u_L^2(0) = 0$, we derive $u_L^1(t) > u_L^2(t)$ in $\mathbb{R}^+$, which is absurd. Therefore, we must have $u_L^1 \equiv u_L^2$ for any $t \geq 0$, and hence the uniqueness holds true. This concludes our proof of Theorem 4.4. ∎

4.2. Conclusion: proof of Theorem 1.2

From the regularity result in Section 2 and from Theorems 3.1 and 3.2, we know that $u \in C^2(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N})$ and $u(x) = u(x_N)$, and hence $u \in C^2(\mathbb{R}_+) \cap C(\overline{\mathbb{R}_+})$ and solves the ODE (4.1). Consequently, we deduce from the ODE analysis in Section 4 (see Lemmas 4.1 and 4.2, Corollary 4.3, and Theorem 4.4) that, either $u = u_*(x_N) = \ln(1 + x_N)$, or u has asymptotic linear growth rate L at ∞ and hence $u = u_L(x_N)$ $(\forall L > 0)$, where

$$u_L(t) = \ln[\sinh(L(t + t_0(L)))] - \ln L, \quad \text{with} \quad t_0(L) = \frac{1}{L}\ln\left(L + \sqrt{L^2 + 1}\right),$$

is the unique solution to the ODE (4.1) such that $\lim_{x_N \to +\infty} u_L'(x_N) = L > 0$.

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