Elemente der Mathematik 81 (2026), 45–53
DOI 10.4171/EM/561

Expansions of positive integers in terms of Fibonacci terms with equally spaced indices

Sungkon Chang

Sungkon Chang is professor of mathematics at the Georgia Southern University. His research areas are elementary number theory and the arithmetic of elliptic curves. He plays the classical guitar in his spare time.

1 Introduction

Let $\{Q_k\}_{k=1}^{\infty}$ be a sequence of real numbers in the interval $[0,1]$. In [6], Chung and Graham considered the following measure of irregularity in the sequence suggested by a question of D. J. Newman (see [11]):

$$C = \inf_n \liminf_{m\to\infty} n|Q_{m+n} - Q_m|,$$

which is a more refined version of the measure introduced by Bruijn and Erdös in [9]. Chung and Graham proved that

$$C \le \alpha := \left(1 + \sum_{k=1}^{\infty} \frac{1}{F_{2k}}\right)^{-1} \approx 0.39441967,$$

Im Jahr 1972 erschien ein Artikel von Edouard Zeckendorf mit dem heute nach ihm benannten Satz, dass jede positive ganze Zahl n in eindeutiger Weise als Summe verschiedener, nicht benachbarter Fibonacci-Zahlen F_k geschrieben werden kann. Tatsächlich wurde dieses Theorem bereits 20 Jahre früher von Cornelis Gerrit Lekkerkerker gefunden. Chung und Graham zeigten 1984, dass jede positive ganze Zahl n auch eine eindeutige Darstellung der Form

$$n = \sum_{k=1}^{\infty} \varepsilon_k^{(n)} F_{2k}$$

besitzt, mit Fibonacci-Zahlen F_{2k} mit geradem Index und Koeffizienten $\varepsilon_k^{(n)}$, die einer gewissen Regel gehorchen. Dabei beginnt man die Fibonacci-Folge mit $F_1 = F_2 = 1$. Wie verhält es sich nun aber bei der Verwendung von Fibonacci-Zahlen $F_{2+d(k-1)}$ deren Indizes aus einer anderen arithmetischen Folge stammen? Der vorliegende Artikel geht dieser Frage auf den Grund.

where F_k are the terms of the Fibonacci sequence, i.e., $(F_1, F_2) = (1, 1)$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 1$. What a surprising appearance of the Fibonacci sequence in the seemingly unrelated problem! Moreover, for each positive integer n, they construct a sequence $\{\varepsilon_k^{(n)}\}_{k=1}^{\infty}$ in $\{0, 1, 2\}$, and prove that if

$$Q_n = \alpha \sum_{k=1}^{\infty} \varepsilon_k^{(n)} / F_{2k},$$

then the value of C for $\{Q_n\}_{n=1}^{\infty}$ is equal to α, which implies that α is the least upper bound of C.

In [6], Chung and Graham show that, for each positive integer n, there is a unique sequence $\{\varepsilon_k^{(n)}\}_{k=1}^{\infty}$ such that

$$n = \sum_{k=1}^{\infty} \varepsilon_k^{(n)} F_{2k} \tag{1}$$

and the terms $\varepsilon_k^{(n)}$ satisfy Rule 1.1 given below. They use these sequences $\{\varepsilon_k^{(n)}\}_{k=1}^{\infty}$ to reach the least upper bound of C.

Rule 1.1. (1) $\varepsilon_k^{(n)} \in \{0, 1, 2\}$ for all $k \in \mathbb{N}$.

(2) If $\varepsilon_k^{(n)} = \varepsilon_j^{(n)} = 2$, where $1 \leq k < j$, then $\varepsilon_i^{(k)} = 0$ for some $k < i < j$. In particular, $2 = \varepsilon_k^{(n)} = \varepsilon_{k+1}^{(n)}$ is not allowed.

We call the summation (1) the *Chung–Graham expansion of* n, and the conditions described in Rule 1.1 the *Chung–Graham rule of expansion*.

This reminds us of Zeckendorf's Theorem [20], which states that every positive integer can be uniquely written as a sum of non-adjacent distinct Fibonacci terms. In other words, if n is a positive integer, then there is a unique sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ such that $n = \sum_{k=1}^{\infty} \varepsilon_k F_k$ and

$$\varepsilon_1 = 0,\ \varepsilon_k \in \{0, 1\},\ \varepsilon_k = 1 \implies \varepsilon_{k+1} = 0 \text{ for all } k \in \mathbb{N}. \tag{2}$$

The values of F_1 and F_2 are equal to each other, and by requiring $\varepsilon_1 = 0$, we achieve the uniqueness of the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$. These examples lead us to the broader question of expanding positive integers by a sequence, a topic with a rich body of literature; see [1–4, 7, 8, 10, 12–14, 16–18].

Let us introduce terminology to discuss the topic more precisely. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of non-negative integers, respectively. We denote a sequence $\{G_k\}_{k=1}^{\infty}$ simply by G, and identify the sequence with the infinite tuple consisting of its terms, i.e., $G = (G_1, G_2, G_3, \ldots)$.

Definition 1.2. Let ε be a sequence in $\mathbb{N}_0$, and let G be a sequence in $\mathbb{N}$. If

$$n = \sum_{k=1}^{\infty} \varepsilon_k G_k \in \mathbb{N}_0,$$

then we call the summation an *expansion of* n *by* G and the sequence ε a *coefficient sequence*. By definition, coefficient sequences have only finitely many non-zero entries. If there is a set of conditions that ε is required to satisfy, we call the set a *rule of expansion*.

For the Chung–Graham expansions, the rule of expansion is as stated in Rule 1.1. For the Zeckendorf expansions, the rule of expansion is the conditions listed in (2).

The natural question for us is how to generalize the Chung–Graham expansions using other terms with equally spaced indices. For example, if we use F_{4k-2} for $k \geq 1$, i.e.,

$$F_2, F_6, F_{10}, F_{14}, \ldots,$$

what should be a rule of expansion under which each positive integer can be uniquely expanded by these terms? We provide an answer to this question in this note. In fact, we introduce a rule of expansion for the sequence $\{F_{2+d(k-1)}\}_{k=1}^{\infty}$, where $d \in \mathbb{N}$ is a fixed even integer. Our approach can be applied when d is odd or when the sequence is $\{F_{1+d(k-1)}\}_{k=1}^{\infty}$ for all integers $d \geq 2$. The work is nearly identical to the case where $d > 0$ is even, which is treated here, with the terms $F_{2+d(k-1)}$, and we leave the details to the reader for simplicity in the presentation of this note.

2 The rule of expansion

Let d be a positive even integer. We introduce the rule of expansion for the sequence $\{F_{2+d(k-1)}\}_{k=1}^{\infty}$. Let K be the sequence given by $(K_1, K_2) = (1, 3)$, $K_{k+2} = K_{k+1} + K_k$ for all $k \in \mathbb{N}$. Then

$$K = (1, 3, 4, 7, 11, 18, 29, \ldots).$$

Definition 2.1. Let $A := K_d - 1$ and $B := F_{2+d} - 1$. A coefficient sequence ε, which has only finitely many non-zero entries by definition, is said to satisfy the *Chung–Graham rule of expansion for even interval d* if the following are satisfied.

(1) $0 \leq \varepsilon_1 \leq B$ and $0 \leq \varepsilon_k \leq A$ for all $k \geq 2$.

(2) If there is an index $m \geq 2$ such that $\varepsilon_m = A$ and $\varepsilon_k = A - 1$ for all $2 \leq k \leq m - 1$, then $\varepsilon_1 < B$.

(3) If $\varepsilon_k = \varepsilon_j = A$ for $2 \leq k < j$, then there is an index c such that $k < c < j$ and $\varepsilon_c \leq A - 2$. In particular, $\varepsilon_k = \varepsilon_{k+1} = A$ for $k \geq 2$ is not allowed.

Let $d = 2$. Then $A = B = 2$, and it is equivalent to the Chung–Graham rule of expansion introduced in Section 1. Item (1) states $\varepsilon_k \in \{0, 1, 2\}$ for all $k \in \mathbb{N}$. Item (3) is the Chung–Graham rule of expansion applied to indices higher than 1, and item (2) is related to the Chung–Graham rule of expansion that involves the index 1, e.g., $(1, 1, 1, 1, 2)$ is allowed while $(2, 1, 1, 1, 2)$ is not.

Let $d = 4$. Then $A = 6$ and $B = 7$. Listed below are examples of coefficient sequences that satisfy the Chung–Graham rule of expansion for interval 4. If relevant to item (2), the first entries are written in boldface:

$$(\mathbf{6}, \mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{6}, 0, 2, 5, 6),\ (7, 4, 5, 5, 6, 5, 5),\ (\mathbf{6}, \mathbf{6}, 3, 5, 5, 0, 4).$$

The Chung–Graham expansions are generalized as follows.

Theorem 2.2. *Let d be a positive even integer. Then, for each $n \in \mathbb{N}$, there is a unique coefficient sequence ε satisfying the Chung–Graham rule of expansion for even interval d such that $n = \sum_{k=1}^{\infty} \varepsilon_k F_{2+d(k-1)}$.*

A majority of research works in the literature focus on finding a rule of expansion given a sequence, and a relatively small number of works concern the converse of this task. Namely, given a rule of expansion, what are the sequences by which each $n \in \mathbb{N}$ is uniquely expanded under the rule? If such a sequence exists, we call it a *base sequence* for the rule of expansion. This direction of research may begin by imposing some reasonable requirements on the rule of expansion. Introduced in [5] by the author is a general approach to the rule of expansion, and if a rule of expansion satisfies the general principle, by [5, Theorem 16], there is a unique increasing base sequence for the rule of expansion. It follows from this theorem that, for the Chung–Graham rule of expansion for even interval d, there is a unique increasing base sequence, namely, $\{F_{2+d(k-1)}\}_{k=1}^{\infty}$. For the Zeckendorf rule of expansion described in (2), Daykin proved in [8] that there is a unique base sequence among all sequences in $\mathbb{N}$, namely, the Fibonacci sequence. Arguably, this is the most intriguing definition of the Fibonacci sequence.

Given a coefficient sequence ε, we denote by $\operatorname{ord}(\varepsilon)$ the largest index ℓ such that $\varepsilon_\ell \neq 0$ if ε is not the zero sequence. If ε is the zero sequence, then we define $\operatorname{ord}(\varepsilon) := 1$. If $\ell = \operatorname{ord}(\varepsilon)$, then we also identify ε with the finite tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell)$.

3 The recurrence and the maximal coefficient sequences

The Chung–Graham rule of expansion for even interval d is derived from a linear recurrence that $\{F_{2+d(k-1)}\}_{k=1}^{\infty}$ satisfies. We discuss in this section the recurrence and its relationship with the rule of expansion.

Let $\varphi = \frac{1}{2}(1+\sqrt{5})$ be the golden ratio, and let $\tilde{\varphi}$ be its Galois conjugate $\frac{1}{2}(1-\sqrt{5})$. By Binet's Formula,

$$F_k = \frac{1}{\sqrt{5}}(\varphi^k - \tilde{\varphi}^k), \quad K_k = \varphi^k + \tilde{\varphi}^k, \tag{3}$$

where K is the sequence defined in Section 2. Let H be the sequence given by $H_k = F_{2+d(k-1)}$. Then the formula in (3) implies $H_k = a(\varphi^d)^k + b(\tilde{\varphi}^d)^k$, where $a, b \in \mathbb{Q}[\varphi]$. Notice that $\varphi^d = \frac{1}{2}(K_d + F_d\sqrt{5})$ for all $d \in \mathbb{N}$, which follows trivially from (3). Then the minimal polynomial for φ^d is $x^2 - K_d x + \frac{1}{4}(K_d^2 - 5F_d^2)$.

Lemma 3.1. *For $d \in \mathbb{N}$, we have $K_d^2 - 5F_d^2 = (-1)^d \cdot 4$.*

Proof. Notice that $\varphi\tilde{\varphi} = -1$, and hence, $\varphi^d\tilde{\varphi}^d = (-1)^d$. The assertion follows from

$$\varphi^d\tilde{\varphi}^d = \frac{1}{2}(K_d - F_d\sqrt{5}) \cdot \frac{1}{2}(K_d + F_d\sqrt{5}) = \frac{1}{4}(K_d^2 - 5F_d^2). \qquad \blacksquare$$

Thus, the two numbers $\varphi^d, \tilde{\varphi}^d$ satisfy the characteristic equation $x^2 = K_d x - (-1)^d$. If d is even, then the sequence H satisfies the recurrence

$$H_{k+2} = K_d H_{k+1} - H_k. \tag{4}$$

For the sequences satisfying the linear recurrence with positive constant coefficients, the standard rule of expansion and its base sequence associated with this recurrence are well known; see [15, 19]. For the linear recurrence with some negative constant coefficients, a rule of expansion and its base sequence are introduced by the author in [5]. However, the base sequence introduced in [5] is not equal to H for each $d \geq 4$. For these cases, we tweak the rule so that, under a new one, H is a base sequence. The rule of expansion introduced in Definition 2.1 is a variation of the rule introduced in [5].

We conclude this section explaining a key component of the rule of expansion introduced in [5]. It is called the *maximal coefficient sequence of order n*, denoted by $\beta^{(n)}$, and it satisfies the property

$$1 + \sum_{k=1}^{\infty} \beta_k^{(n)} H_k = H_{n+1}. \tag{5}$$

This can be viewed as a generalization of consecutive 9s in the last digits of a decimal expansion, e.g., $1 + 12345\mathbf{9999} = 12346\mathbf{0}000$, where the block of 9s is carried over to the next digit.

For the sequence H, we may use the recurrence (4) to find $\beta^{(n)}$. Notice that if $m \geq 3$, then

$$H_m - H_{m-1} = (K_d - 2)H_{m-1} + (H_{m-1} - H_{m-2}). \tag{6}$$

Then, for $n \geq 3$, we have

$$\begin{aligned} H_{n+1} &= K_d H_n - H_{n-1} = (K_d - 1)H_n + (H_n - H_{n-1}) \\ &= (K_d - 1)H_n + (K_d - 2)H_{n-1} + (H_{n-1} - H_{n-2}) \quad \text{by (6)} \\ &= (K_d - 1)H_n + (K_d - 2)H_{n-1} + \cdots + (K_d - 2)H_2 + (H_2 - 1) \\ &= (K_d - 1)H_n + (K_d - 2)H_{n-1} + \cdots + (K_d - 2)H_2 + (H_2 - 2) + 1. \end{aligned} \tag{7}$$

Thus, for property (5), we may define the entries of $\beta^{(n)}$ to be the coefficients of expansion (7).

Definition 3.2. Define the coefficient sequence $\beta^{(n)}$ which is of order n as follows:

$$\beta^{(1)} = (B), \quad \beta^{(2)} = (B - 1, A),$$

and for every $n \geq 3$,

$$\beta^{(n)} := (B - 1, A - 1, A - 1, \ldots, A - 1, A - 1, A),$$

which contains n terms.

The standard rule of expansion introduced in [5] chooses the value of H_2 to be equal to K_d, i.e., $H_2 - 2 = K_d - 2$, so that the entries of $\beta^{(n)}$ described in (7) have preperiodic structure and H becomes its base sequence. However, for our sequence H, we have $H_2 = F_{2+d} \neq K_d$ if $d \geq 4$, and we may keep $H_2 - 2$ as the first entry of $\beta^{(n)}$, which will break the preperiodic structure at the first entry.

Lemma 3.3. *Identity* (5) *is satisfied for all* $n \geq 1$.

Proof. Recall that H satisfies the recurrence in (4). If $n = 1$, then

$$1 + \sum_{k=1}^{\infty} \beta_k^{(1)} H_k = 1 + BH_1 = 1 + F_{2+d} - 1 = H_2.$$

If $n = 2$, then

$$1 + \sum_{k=1}^{\infty} \beta_k^{(2)} H_k = 1 + (B-1)H_1 + AH_2 = 1 + (H_2 - 2) + AH_2 \\ = -1 + (A+1)H_2 = H_3.$$

For $n \geq 3$, identity (7) proves the assertion. ∎

4 Proof of Theorem 2.2

Let $\mathcal{F}$ be the collection of coefficient sequences that satisfy the Chung–Graham rule of expansion for even interval d.

Lemma 4.1. *Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\ell) \in \mathcal{F}$. Then the coefficient sequence $\varepsilon' := (\varepsilon_1, \ldots, \varepsilon_m)$ and $\tilde{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\ell, a)$ are members of $\mathcal{F}$ for each $1 \leq m < \ell$ and $0 \leq a \leq A - 1$.*

Proof. Notice that if ε' does not satisfy an item of Definition 2.1, then ε does not satisfy the item, either. Since $\varepsilon \in \mathcal{F}$, it follows that ε' must satisfy the items of the rule. Since $\tilde{\varepsilon}_{\ell+1} \leq A - 1$ and $\ell + 1 \geq 2$, the only item of the rule that concerns the $(\ell + 1)$th entry is the first item, which is clearly true. ∎

4.1 Existence

We use induction on $n \in \mathbb{N}$ to prove the existence of $\varepsilon \in \mathcal{F}$ such that $n = \sum_{k=1}^{\infty} \varepsilon_k H_k$.

Lemma 4.2. *Let $n > B$ be an integer, and let ℓ be the maximal index such that $H_\ell \leq n$. Then $\ell \geq 2$ and $(A + 1)H_\ell > n$.*

Proof. Since $n \geq B + 1 = F_{2+d} = H_2$, we have $\ell \geq 2$. By the maximality of ℓ, we have

$$n < H_{\ell+1} = (A+1)H_\ell - H_{\ell-1} < (A+1)H_\ell.$$ ∎

If $1 \leq n \leq B$, then $n = \sum_{k=1}^{\infty} \varepsilon_k H_k$, where $\varepsilon_1 = n$ and $\varepsilon_k = 0$ for all $k \geq 2$. Since $\varepsilon \in \mathcal{F}$, this proves the existence for $n \leq B$. Let $n \geq B + 1$. Let ℓ be the maximal index such that $H_\ell \leq n$, and let a be the largest positive integer a such that $aH_\ell \leq n$. Then $\ell \geq 2$ and $a \leq A$ by Lemma 4.2.

Let $n' := n - aH_\ell$. By the induction hypothesis,

$$n' = \sum_{k=1}^{\infty} \varepsilon'_k H_k \quad \text{for some } \varepsilon' \in \mathcal{F}.$$

The maximality of a implies $\operatorname{ord}(\varepsilon') < \ell$. If $a \leq A - 1$, then $\varepsilon := (\varepsilon'_1, \ldots, \varepsilon'_{\ell-1}, a) \in \mathcal{F}$ by Lemma 4.1 and $n = \sum_{k=1}^{\infty} \varepsilon_k H_k$. Suppose $a = A$. If $\ell = 2$ and $n' \geq B$, then Lemma 3.3 implies $n = AH_2 + n' \geq AH_2 + B = H_3$, which contradicts the maximality of ℓ. Thus, if $\ell = 2$, then $n' < B$, $\varepsilon := (n', A) \in \mathcal{F}$, and $n = \sum_{k=1}^{\infty} \varepsilon_k H_k$.

Suppose $\ell \geq 3$, and let $m \geq 0$ be the maximal index such that

$$\tilde{n} := \sum_{k=\ell-m}^{\ell-1} (A-1)H_k + AH_\ell \leq n.$$

Consider the case $\ell - m \le 2$. Then $n - (\sum_{k=2}^{\ell-1}(A-1)H_k + AH_\ell) < B$ since

$$n - \left(\sum_{k=2}^{\ell-1}(A-1)H_k + AH_\ell\right) \ge B \implies n \ge 1 + \sum_{k=1}^{\infty}\beta_k^{(\ell)}H_k = H_{\ell+1},$$

which contradicts the maximality of ℓ. Thus,

$$n = \varepsilon_1 H_1 + \sum_{k=2}^{\ell-1}(A-1)H_k + AH_\ell,$$

where $\varepsilon_1 < B$, which satisfies the Chung–Graham rule of expansion for the interval d.

Let $\ell - m \ge 3$, and let b be the maximal coefficient such that $bH_{\ell-m-1} \le n - \tilde{n}$. The maximality of m implies $b \le A-2$. By the induction hypothesis, $n - \tilde{n} - bH_{\ell-m-1} = \sum_{k=1}^{\infty}\varepsilon_k H_k$ for some $\varepsilon \in \mathcal{F}$. The maximality of b implies $q := \operatorname{ord}(\varepsilon) < \ell - m - 1$. Thus, we have the following expansion:

$$n = \sum_{k=1}^{q}\varepsilon_k H_k + bH_{\ell-m-1} + \sum_{k=\ell-m}^{\ell-1}(A-1)H_k + AH_\ell,$$

which satisfies the Chung–Graham rule of expansion for the interval d. This concludes the proof of the existence.

4.2 Uniqueness

Notice that

$$A = K_d - 1 \le B = F_{2+d} - 1 \tag{8}$$

for all $d \ge 1$, which can be proved by induction. The equality in (8) holds only when $d = 2$.

Lemma 4.3. *Let $\varepsilon \in \mathcal{F}$ and $\ell := \operatorname{ord}(\varepsilon)$. Then $\sum_{k=1}^{\infty}\varepsilon_k H_k < H_{\ell+1}$.*

Proof. We use induction on ℓ. If $\ell = 1$, then $n \le BH_1 < B + 1 = H_2$. Assume that there is $\ell \ge 1$ such that the statement is true for all $1 \le \ell' \le \ell$. Let $\varepsilon \in \mathcal{F}$ such that $\operatorname{ord}(\varepsilon) = \ell + 1$. Suppose that $\varepsilon_{\ell+1} \le A - 1$. Then, by Lemma 4.1, $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_\ell)$ is a member of $\mathcal{F}$. By the induction hypothesis, $\sum_{k=1}^{\infty}\varepsilon'_k H_k < H_{\ell+1}$, and hence,

$$\sum_{k=1}^{\infty}\varepsilon_k H_k < H_{\ell+1} + \varepsilon_{\ell+1}H_{\ell+1} \le AH_{\ell+1} < \sum_{k=1}^{\infty}\beta_k^{(\ell+1)}H_k < H_{\ell+2}.$$

Suppose that $\varepsilon_{\ell+1} = A$, and let $m \ge 0$ be the maximal index such that $\varepsilon_k = A - 1$ for all k in the interval $[\ell + 1 - m, \ell]$. If $\ell + 1 - m \le 2$, then item (2) of Definition 2.1 and (8) imply

$$\sum_{k=1}^{\infty}\varepsilon_k H_k \le \sum_{k=1}^{\infty}\beta_k^{(\ell+1)}H_k < H_{\ell+2}.$$

Let $\ell + 1 - m \ge 3$. Then item (3) of Definition 2.1 and the maximality of m imply that $\varepsilon_{\ell-m} \le A - 2$. Notice that $\varepsilon' := (\varepsilon_1, \dots, \varepsilon_{\ell-m-1}) \in \mathcal{F}$ by Lemma 4.1. By the induction hypothesis, $\sum_{k=1}^{\infty} \varepsilon'_k H_k < H_{\ell-m}$, which implies

$$\sum_{k=1}^{\infty} \varepsilon_k H_k < H_{\ell-m} + \sum_{k=\ell-m}^{\infty} \varepsilon_k H_k \le \sum_{k=\ell-m}^{\ell} (A-1)H_k + AH_{\ell+1}$$
$$< \sum_{k=1}^{\infty} \beta_k^{(\ell+1)} H_k < H_{\ell+2}. \qquad \blacksquare$$

Let ε and δ be members of $\mathcal{F}$ such that

$$\sum_{k=1}^{\infty} \varepsilon_k H_k = \sum_{k=1}^{\infty} \delta_k H_k.$$

Suppose that there is a largest index q such that $\varepsilon_q \neq \delta_q$, and without loss of generality, assume $\varepsilon_q < \delta_q$. Then

$$H_q \le \sum_{k=1}^{q-1} \delta_k H_k + (\delta_q - \varepsilon_q) H_q = \sum_{k=1}^{q-1} \varepsilon_k H_k.$$

It follows from Lemmas 4.1 and 4.3 that $H_q \le \sum_{k=1}^{q-1} \varepsilon_k H_k < H_q$, which is a contradiction. Thus, we prove $\varepsilon_k = \delta_k$ for all $k \in \mathbb{N}$.

Acknowledgments. The author would like to thank the referee for carefully reading the paper and providing suggestions that significantly improved its readability, especially for recommending a direct proof of existence and uniqueness rather than one based on lexicographical order.

References

[1] V. Berthé, C. Frougny, M. Rigo, and J. Sakarovitch, The carry propagation of the successor function. *Adv. in Appl. Math.* **120** (2020), article no. 102062, 55 Zbl 1484.11012 MR 4107002

[2] J. L. Brown, Jr., Note on complete sequences of integers. *Amer. Math. Monthly* **68** (1961), 557–560 Zbl 0115.04305 MR 130207

[3] P. S. Bruckman, The generalized Zeckendorf theorems. *Fibonacci Quart.* **27** (1989), no. 4, 338–347 Zbl 0679.10011 MR 1006980

[4] M. Catral, P. L. Ford, P. E. Harris, S. J. Miller, and D. Nelson, Legal decomposition arising from non-positive linear recurrences. *Fibonacci Quart.* **54** (2016), no. 4, 348–365 Zbl 1400.11037 MR 3584005

[5] S. Chang, The weak converse of Zeckendorf's theorem. *Res. Number Theory* **7** (2021), no. 3, article no. 48, 39 Zbl 1484.11024 MR 4292843

[6] F. R. K. Chung and R. L. Graham, On irregularities of distribution. In *Finite and infinite sets, Vol. I, II (Eger, 1981)*, pp. 181–222, Colloq. Math. Soc. János Bolyai 37, North-Holland, Amsterdam, 1984 Zbl 0566.10042 MR 818235

[7] K. Cordwell, M. Hlavacek, C. Huynh, S. J. Miller, C. Peterson, and Y. N. T. Vu, Summand minimality and asymptotic convergence of generalized Zeckendorf decompositions. *Res. Number Theory* **4** (2018), no. 4, article no. 43, 27 Zbl 1441.11028 MR 3862851

[8] D. E. Daykin, Representation of natural numbers as sums of generalised Fibonacci numbers. *J. London Math. Soc.* **35** (1960), 143–160 Zbl 0215.06801 MR 112863

[9] N. G. de Bruijn and P. Erdös, Sequences of points on a circle. *Indagationes Math.* **11** (1949), 46–49 Zbl 0031.34803 MR 33331

[10] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller, D. Moon, and U. Varma, Generalizing Zeckendorf's Theorem to f-decompositions. *J. Number Theory* **141** (2014), 136–158 Zbl 1309.11013 MR 3195393

[11] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*. Monogr. Enseign. Math. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980 Zbl 0434.10001 MR 592420

[12] A. S. Fraenkel, Systems of numeration. *Amer. Math. Monthly* **92** (1985), no. 2, 105–114 Zbl 0568.10005 MR 777556

[13] A. Frigeri, A note on Fibonacci numbers of even index. *Integers* **18** (2018), article no. A87, 7 Zbl 1444.11031 MR 3874978

[14] P. J. Grabner, R. F. Tichy, I. Nemes, and A. Pethő, Generalized Zeckendorf expansions. *Appl. Math. Lett.* **7** (1994), no. 2, 25–28 Zbl 0792.11002 MR 1350140

[15] N. Hamlin and W. A. Webb, Representing positive integers as a sum of linear recurrence sequences. *Fibonacci Quart.* **50** (2012), no. 2, 99–105 Zbl 1273.11025 MR 2927927

[16] V. E. Hoggatt, Jr., Generalized Zeckendorf theorem. *Fibonacci Quart.* **10** (1972), no. 1, 89–93 Zbl 0239.10006 MR 300968

[17] C. G. Lekkerkerker, Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. *Simon Stevin* **29** (1952–53), 190-195

[18] T. C. Martinez, S. J. Miller, C. Mizgerd, J. Murphy, and C. Sun, Generalizing Zeckendorf's theorem to homogeneous linear recurrences, II. *Fibonacci Quart.* **60** (2022), no. 5, 231–254 Zbl 1520.11011 MR 4539724

[19] S. J. Miller and Y. Wang, From Fibonacci numbers to central limit type theorems. *J. Combin. Theory Ser. A* **119** (2012), no. 7, 1398–1413 Zbl 1245.05006 MR 2925932

[20] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. *Bull. Soc. Roy. Sci. Liège* **41** (1972), 179–182 Zbl 0252.10011 MR 308032

Sungkon Chang
Department of Mathematical Sciences
Georgia Southern University
1332 Southern Dr.
Statesboro, GA 30458, USA
schang@georgiasouthern.edu