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Short note

Another improvement of Weitzenböck's inequality

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Abstract. In this article, we present and prove a new enhancement of Weitzenböck's inequality, further strengthening the improved version of Weitzenböck's inequality introduced by Wei-Dong Jiang (2024).

1 Introduction

Weitzenböck's inequality is a classical result in geometry (see [12]) that establishes a relationship between the sum of the squares of the sides of a triangle and its area. This inequality plays a fundamental role in the study of geometric inequalities and has inspired a rich body of research aimed at extending, generalizing, and sharpening its original form (see [1,3,4,7–9,11]). Through these extensions, Weitzenböck's inequality has evolved into a foundational tool in understanding the properties and relationships within various classes of triangles and other polygons.

Recent advances in this field include numerous generalizations and refined versions that provide stronger bounds under certain conditions. In particular, Wei-Dong Jiang presented an improved version of Weitzenböck's inequality in [3], which can be stated as

$$a^2 + b^2 + c^2 \geq 4\Delta\sqrt{3 + \frac{9}{4}\Big(\frac{R}{r} - 2\Big)},$$

where he demonstrated that the constant $k = \frac{9}{4}$ is optimal for the inequality

$$a^2 + b^2 + c^2 \geq 4\Delta\sqrt{3 + k\Big(\frac{R}{r} - 2\Big)}.$$

This paper aims to extend Jiang's work by further refining the inequality with an additional term that strengthens the lower bound. Specifically, we introduce and prove that the best constant k for the inequality

$$a^2 + b^2 + c^2 \geq 4\Delta\sqrt{3 + \frac{9}{4}\Big(\frac{R}{r} - 2\Big)} + k\frac{r}{R}\big((b-c)^2 + (c-a)^2 + (a-b)^2\big) \qquad (1)$$

is $k = \frac{1}{8}$. This result not only strengthens the previous form of Weitzenböck's inequality but also provides new insights into the structure of inequalities involving triangle side lengths and area.

By incorporating a residual term, this generalization follows the spirit of the Finsler–Hadwiger inequality (and also its generalizations; see [2,6,10]) and highlights the potential for further exploration in geometric inequalities. Such advancements may lead to deeper understanding and broader applications in the study of triangle geometry and its connections to other fields.

The remainder of this paper is organized as follows. In the next section, we provide a detailed proof demonstrating that $k = \frac{1}{8}$ is indeed the best constant for inequality (1). Our approach leverages fundamental trigonometric and geometric identities in triangles, as well as Taylor expansions to simplify and Lukarevski's inequality (see [5]). The paper concludes with a discussion on potential future research directions that could further explore and generalize these results.

2 Proofs

We begin by proving that $k = \frac{1}{8}$ is the best constant for inequality (1). To do so, we use the following trigonometric identities in a triangle:

$$a^2 + b^2 + c^2 = 4\Delta(\cot A + \cot B + \cot C),$$
$$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2},$$

and

$$2\Delta = bc \sin A = ca \sin B = ab \sin C.$$

Dividing both sides of inequality (1) by 4Δ, we see that inequality (1) is equivalent to

$$\begin{aligned}\cot A + \cot B + \cot C \geq \sqrt{\frac{9}{16 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}} - \frac{3}{2}} &\\ + 8k \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\Big(\cot A + \cot B + \cot C &\\ - \frac{1}{2\sin A} - \frac{1}{2\sin B} - \frac{1}{2\sin C}\Big).&\end{aligned} \tag{2}$$

Now, let

$$(A, B, C) = (\pi - 2x, x, x), \quad x \in (0, \pi).$$

Then inequality (2) becomes

$$\begin{aligned}2\cot x - \cot 2x \geq \sqrt{\frac{9}{16 \cos x \sin^2\frac{x}{2}} - \frac{3}{2}} &\\ + 8k \cos x \sin^2\frac{x}{2}\Big(2\cot x - \cot 2x - \frac{1}{2\sin 2x} - \frac{1}{\sin x}\Big).&\end{aligned}$$

Thus, we obtain

$$k \leq \frac{2\cot x - \cot 2x - \sqrt{\frac{9}{16\cos x \sin^2\frac{x}{2}} - \frac{3}{2}}}{8\cos x \sin^2\frac{x}{2}\left(2\cot x - \cot 2x - \frac{1}{2\sin 2x} - \frac{1}{\sin x}\right)}. \tag{3}$$

In order to find the best constant k, we need to calculate the limit of the following expression:

$$\lim_{x\to 0^+} \frac{2\cot x - \cot 2x - \sqrt{\frac{9}{16\cos x \sin^2 \frac{x}{2}} - \frac{3}{2}}}{8\cos x \sin^2 \frac{x}{2}\left(2\cot x - \cot 2x - \frac{1}{2\sin 2x} - \frac{1}{\sin x}\right)}.$$

We use the Taylor expansions of the functions around $x = 0$ with the help of CAS of GeoGebra and get

$$\begin{aligned} 2\cot x - \cot 2x &\approx \frac{3}{2x}, \\ \sqrt{\frac{9}{16\cos x \sin^2 \frac{x}{2}} - \frac{3}{2}} &\approx \frac{3}{2x} - \frac{x}{16}, \\ 8\cos x \sin^2 \frac{x}{2}\left(2\cot x - \cot 2x - \frac{1}{2\sin 2x} - \frac{1}{\sin x}\right) &\approx \frac{x}{2}. \end{aligned}$$

Therefore, from (3), as $x \to 0^+$, we obtain

$$k \le \frac{\frac{3}{2x} - \left(\frac{3}{2x} - \frac{x}{16}\right)}{\frac{x}{2}} = \frac{1}{8}.$$

Thus, the best constant for inequality (1) is $k = \frac{1}{8}$.

Now, we will prove the inequality for $k = \frac{1}{8}$. Indeed, from

$$\begin{aligned} a^2 + b^2 + c^2 &= 2(s^2 - 4Rr - r^2), \\ (b-c)^2 + (c-a)^2 + (a-b)^2 &= 2(s^2 - 12Rr - 3r^2), \end{aligned}$$

we have

$$\begin{aligned} a^2 + b^2 + c^2 &- \frac{r}{8R}\left((b-c)^2 + (c-a)^2 + (a-b)^2\right) \\ &= \frac{(8R-r)s^2 - 32R^2r + 4Rr^2 + 3r^3}{4R}. \end{aligned}$$

To prove (1), it suffices to show that

$$\left(\frac{(8R-r)s^2 - 32R^2r + 4Rr^2 + 3r^3}{4R}\right)^2 \ge 16\Delta^2\left(\frac{9R}{4r} - \frac{3}{2}\right).$$

This inequality is equivalent to

$$\left((8R-r)s^2 - 32R^2r + 4Rr^2 + 3r^3\right)^2 - 192R^2(3R-2r)rs^2 \ge 0. \tag{4}$$

We express (4) as follows:

$$\begin{aligned} &\left((8R-r)s^2 - 32R^2r + 4Rr^2 + 3r^3\right)^2 - 192R^2(3R-2r)rs^2 \\ &\quad = (8R-r)^2(s^2 - 16Rr + 5r^2)^2 \\ &\qquad + 8r(120R^3 - 80R^2r + 29Rr^2 - 2r^3)\left(s^2 - 16Rr + 5r^2 - \frac{r^2(R-2r)}{R-r}\right) \\ &\qquad + \frac{8r^4(R-2r)(90R^2 - 25Rr + 2r^2)}{R-r}. \end{aligned}$$

Therefore, inequality (4) is equivalent to

$$
\begin{aligned}
&(8R-r)^2(s^2-16Rr+5r^2)^2 \\
&\quad + 8r(120R^3-80R^2r+29Rr^2-2r^3)\Big(s^2-16Rr+5r^2-\frac{r^2(R-2r)}{R-r}\Big) \\
&\quad + \frac{8r^4(R-2r)(90R^2-25Rr+2r^2)}{R-r} \geq 0. \qquad (5)
\end{aligned}
$$

We have

$$
\begin{aligned}
&120R^3-80R^2r+29Rr^2-2r^3 \\
&\qquad = 120R^2(R-2r)+160R^2r+29r^2(R-2r)+56r^3>0, \\
&90R^2-25Rr+2r^2 = 90R(R-2r)+155Rr+2r^2>0,
\end{aligned}
$$

and it follows from the sharpened form of Gerretsen's inequality [5] that

$$
16Rr-5r^2+\frac{r^2(R-2r)}{R-r} \leq s^2 \leq 4R^2+4Rr+3r^2-\frac{r^2(R-2r)}{R-r}.
$$

Thus inequality (5) holds. This completes the proof.

3 Conclusion

In this paper, we have introduced a new improvement to Weitzenböck's inequality, building on the enhanced inequality previously proposed by Wei-Dong Jiang. By carefully selecting trigonometric and geometric identities, we established that the best constant for our strengthened version is $k=\frac{1}{8}$. This result is significant because it not only refines the inequality but also reinforces the understanding of the relationship between the sides and area of a triangle in the context of geometric inequalities.

The proof we provided utilizes both Taylor expansions and classic triangle inequalities, underscoring the versatility of these tools in tackling complex inequality problems. Additionally, our sharpened inequality potentially opens new avenues for further research in the field of geometric inequalities, particularly in exploring other configurations or extending the inequality to broader classes of polygons or polyhedra.

Future research may focus on finding additional constants or parameters that optimize similar inequalities, or on investigating possible applications of this strengthened inequality in applied mathematics where such geometric relationships are relevant. We hope this contribution will inspire further work in exploring the depths of classical inequalities and their modern extensions.

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