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Short note Pythagorean triples and Catalan's equation

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Abstract. We prove that there are only six primitive Pythagorean triangles whose area has exactly three prime divisors.

1 Introduction

Innumerable properties of Pythagorean triangles have been described by Dickson [3, Chapter IV, pp. 165–191], Sierpiński [4] and Beiler [1, pp. 104–134]. Curiously, the property below seems to have gone completely unnoticed. It relies on a combination of basic properties of Pythagorean triples and Catalan's elementary equation, that we recall below from [5].

(C) Integer solutions of $|3^\beta - 2^\alpha| = 1$ are $\beta = 1 = \alpha$, $\beta = 2, \alpha = 3$ and $\beta = 1, \alpha = 2$.

Our aim is to prove the following theorem.

Theorem. *The only primitive Pythagorean triangles whose area has exactly three prime divisors are*

$$[5, 12, 13],\ [8, 15, 17],\ [9, 40, 41],\ [7, 24, 25],\ [16, 63, 65],\ [17, 144, 145].$$

2 Proof of the theorem

Let integers a, b, c be the sides and hypotenuse, respectively, of a primitive Pythagorean triangle of area K having three prime divisors. As is well known [2], there exist coprime positive integers m, n of different parity, i.e. one odd and one even, such that $a = 2mn$, $b = m^2 - n^2$ (or vice versa) and $c = m^2 + n^2$. Observing $K = mn(m-n)(m+n)$, first modulo 2 and then modulo 3, implies that 2 and 3 divide K. Let p be the third prime divisor of K. Since m, n are coprime by definition, $m, n, m+n$ and $m, n, m-n$ are pairwise coprime. Furthermore, as m, n have different parity, $m+n, m-n$ are coprime. Hence, since K only has 3 prime divisors, at least one of the integers $n, m, m-n, m+n$ is 1, and so, as $n < m$, either $n = 1$ or $m - n = 1$. Notice, if $n = 1 = n - m$, then $m = 2$ and $n = 1$ producing the $[3, 4, 5]$ triangle, whose area has only two prime factors. Consequently, exactly one factor equals 1 and the other three coprime factors must therefore be $2^\alpha, 3^\beta, p^\gamma$ for certain strictly positive integers α, β, γ.

If $n = 1$, then m being even, we have $m = 2^\alpha$ and so either $2^\alpha - 1 = 3^\beta$ or $2^\alpha + 1 = 3^\beta$. In the first case, (C) implies $\alpha = 2$, $\beta = 1$, $\gamma = 1$, giving $[8, 15, 17]$, and in the second, either $\alpha = 3$, $\beta = 2$, giving $p^\gamma = 7$ and hence $[16, 63, 65]$, or $\alpha = \beta = 1$, giving the contradiction $p^\gamma = 1$.

If $m - n = 1$, then we solve the following four systems of equations:

$$m, n = 2^\alpha, \quad n, m = 3^\beta \text{ and } m + n = p^\gamma,$$
$$m, n = 2^\alpha, \quad n, m = p^\gamma \text{ and } m + n = 3^\beta.$$

In the first case, $2^\alpha - 3^\beta = 1$ implies $\alpha = 2$, $\beta = 1$ and so $p^\gamma = 7$, i.e. $[24, 7, 25]$. In the second case, $3^\beta - 2^\alpha = 1$, either we obtain $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and so $[12, 5, 13]$, or $\alpha = 3$, $\beta = 2$, $\gamma = 1$, meaning $[144, 17, 145]$.

In the last two cases, we first get $2^\alpha - p^\gamma = 1$, $2^\alpha + p^\gamma = 3^\beta$, giving $2^{\alpha+1} = 3^\beta + 1$. Hence, (C) implies $\alpha = 1 = \beta$ and so $\gamma = 0$, an undesired solution, since K would have only two prime divisors.

Finally, $p^\gamma - 2^\alpha = 1$, $p^\gamma + 2^\alpha = 3^\beta$ implies, once again by (C), $\alpha = 2$, $\beta = 2$, $\gamma = 1$, meaning $[40, 9, 41]$, and the undesired solution $\alpha = 0$, $\beta = 1$ and $p = 2$, i.e. the $[3, 4, 5]$ triangle of area 6. ∎

Remark. By applying the same method of proof as above, one easily sees that the preceding $[3, 4, 5]$ is the only primitive triangle whose area has exactly two prime divisors. However, the question on whether the number of solutions is finite or infinite for four prime divisors of the area of primitive Pythagorean triangles eludes the author completely.

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References

[1] A. H. Beiler, *Recreations in the theory of numbers – The queen of mathematics entertain*. 2nd Edn., Dover Publications, New York 1966 Zbl 0154.04001

[2] J. H. Silverman, *A friendly introduction to number theory*. Prentice Hall, Upper Saddle River 1997 Zbl 0973.11003

[3] L. E. Dickson, *History of the theory of numbers. Vol. II: Diophantine analysis*. Chelsea Publishing Co., New York, 1966 Zbl 1214.11002 MR 245500

[4] W. Sierpiński, *Pythagorean triangles*. Scripta Math. Stud. 9, Yeshiva University, Graduate School of Science, New York, 1962 MR 191870

[5] W. Sierpiński, *250 problems in elementary number theory*. Mod. Anal. Comput. Meth. Sci. Math. 26, American Elsevier Publishing Co., Inc., New York; PWN—Polish Scientific Publishers, Warsaw, 1970 Zbl 0211.37201 MR 269580

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