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Near invariance of quasi-energy spectrum of Floquet Hamiltonians

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Abstract. The spectral analysis of the unitary monodromy operator, associated with a time-periodically (parametrically) forced Schrödinger equation, is a question of longstanding interest. Here, we consider this question for Hamiltonians of the form

$$H^{\varepsilon}(t) = H^0 + \varepsilon^a W(\varepsilon^a t, -i\nabla),$$

where H^0 is an unperturbed autonomous Hamiltonian, $a \geq 1$, and $W(T,\cdot)$ has a period of $T_{\mathrm{per}} > 0$. In particular, in the small $\varepsilon > 0$ regime, we seek a comparison between the spectral properties of the monodromy operator, the one-period flow map associated with the $H^{\varepsilon}(t)$ dynamics, and that of the autonomous (unforced) flow, $\exp[-iH^0 T_{\mathrm{per}}\varepsilon^{-a}]$. We consider H^0 which is spatially periodic on $\mathbb{R}^n$ with respect to a lattice. Using the decomposition of H^0 and $H^{\varepsilon}(t)$ into their actions on spaces (Floquet–Bloch fibers) of pseudo-periodic functions, we establish a spectral near-invariance property for the monodromy operator, when acting on data which are ε-localized in energy and quasi-momentum. Our analysis requires the following steps: (i) spectrally-localized data are approximated by -emphband-limited (Floquet–Bloch) wavepackets; (iii) the envelope dynamics of such wavepackets is well approximated by an effective (homogenized) PDE; and (iii) an exact invariance property for band-limited Floquet–Bloch wavepackets, which follows from the effective dynamics. We apply our general results to a number of periodic Hamiltonians, H^0, of interest in the study of photonic and quantum materials.

1. Introduction

We consider a class of n-dimensional Schrödinger equations with time-periodic forcing, governing $\psi = \psi(t,\mathbf{x})$, a complex-valued function of $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$i\partial_t \psi = H^{\varepsilon}(t)\psi, \quad H^{\varepsilon}(t) \equiv H^0 + \varepsilon^a W(\varepsilon^a t, -i\nabla), \tag{1.1}$$

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where $a \geq 1$, H^0 is self adjoint, and $W(T, -i\nabla)$ and $H^\varepsilon(t)$ are self adjoint for all $t \geq 0$ and sufficiently smooth as functions of t. Furthermore, $T \mapsto W(T, \cdot)$ is periodic of period $T_{\rm per} > 0$, i.e., $W(T, \cdot) = W(T + T_{\rm per}, \cdot)$ for all $T \in \mathbb{R}$. Hence,

$$t \mapsto H^\varepsilon(t) \text{ is periodic of period } T^\varepsilon_{\rm per} \equiv T_{\rm per}\varepsilon^{-a},\ a \geq 1. \tag{1.2}$$

We consider (1.1) for $\varepsilon > 0$ and small: the regime of small and slowly varying time-periodic forcing. Very briefly, wave-packet initial-data will deform on a time-scale which depends on its spectral localization. The parameter $a \geq 1$ is therefore chosen so that this time-scale and the forcing period $T^\varepsilon_{\rm per} \sim \varepsilon^{-a}$ are matched; see Section 1.1.

Since $H^\varepsilon(t)$ is time-dependent (non-autonomous), the spectra of the family of operators $\{H^\varepsilon(t)\}_{t\in\mathbb{R}}$ does not determine the time-dynamics (1.1). Instead, one must study one-period evolution map associated with (1.1): for initial data $\psi(t)|_{t=0} = \psi_0 \in L^2(\mathbb{R}^n)$, the solution $\psi(t) \in L^2(\mathbb{R}^n)$ of the initial value problem (1.1) is defined by the unitary operator

$$\psi^\varepsilon(t) = U^\varepsilon(t)\psi_0. \tag{1.3a}$$

The one-period evolution map, or *monodromy operator,* is the unitary operator

$$M^\varepsilon \equiv U^\varepsilon(T^\varepsilon_{\rm per}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{1.3b}$$

The driven and undriven problems can be compared by viewing the autonomous case $W = 0$, i.e., the dynamics of $i\partial_t\psi = H^0\psi$, as having (trivial) $T^\varepsilon_{\rm per}$ periodicity. The associated solution operator is thus $e^{-iH^0 t}$ and the monodromy operator is

$$M^\varepsilon_0 = e^{-iH^0 T^\varepsilon_{\rm per}}.$$

The spectrum of M^ε_0 acting in $L^2(\mathbb{R}^n)$ has a simple relation to the spectrum of H^0:

$$\mathrm{Spec}_{L^2(\mathbb{R}^n)}(M^\varepsilon_0) = \{e^{-iET^\varepsilon_{\rm per}} \mid E \in \mathrm{Spec}_{L^2(\mathbb{R}^n)}(H^0)\}.$$

This relation motivates the notion of *quasi-energy.* A point on the spectrum of the monodromy operator $z \in \mathrm{Spec}_{L^2(\mathbb{R}^n)}(M^\varepsilon) \subseteq S^1$ can be written as $z = e^{-iT^\varepsilon_{\rm per}\nu}$. The phase $T^\varepsilon_{\rm per}\nu$ is called a *Floquet exponent*, and $\nu \in \mathbb{R}/2\pi\mathbb{Z}$ is called a *quasi-energy*. We ask the following.

Question 1. *What is the relation between the spectrum of the monodromy operator M^ε and that of M^ε_0, arising from the non-trivial time-periodic forcing?*

In general, time-periodic settings, beyond being unitary, very little is known about the spectrum of the monodromy operator. Note, in particular, that, even though the parametric forcing is slow in time, the study of Question 1 is not covered by standard adiabatic theory, to the best of our knowledge; see discussion of Section 1.2.

In this paper, we gain insight on this question for the class of operators M^ε, where the unperturbed Hamiltonian, H^0, is periodic with respect to spatial translations in a lattice $\Lambda \subset \mathbb{R}^n_{\mathbf{x}}$, i.e., $V(\mathbf{x}+\mathbf{v}) = V(\mathbf{x})$ for all $\mathbf{v} \in \Lambda$ and all $\mathbf{x} \in \mathbb{R}^n$. Since H^0 commutes with Λ-translations, it may be decomposed into its action on distinct spaces of $\mathbf{k}$-pseudo-periodic functions, where $\mathbf{k}$ varies over the *Brillouin zone*; see Section 2.1. This decomposition allows us to formulate and address a spectrally-local version of Question 1.

Question 2. *What is the relation between the spectrum of the monodromy operator* M^ε *and that of* M_0^ε*, when restricted to* $L^2(\mathbb{R}^n)$ *data concentrated near an energy* $E_\star$ *and quasi-momentum* $\mathbf{k}_\star$*?*

1.1. Discussion of main results

We begin by discussing more specifically the type of spectral localization we have in mind. First, fix a general quasi-momentum $\mathbf{k}_\star$ and energy $E_\star$, and consider initial data which are "ε-spectrally localized" with respect to H^0 – a Bloch wave-packet of band-width ε (see Proposition 4.7). The (unforced) evolution of such data by $U^0(t) = e^{-iH^0 t}$, on large finite time-scales, has the structure of a slowly-varying spatial and temporal modulation of Bloch modes with energy $E_\star$ and quasi-momentum $\mathbf{k}_\star$. The slow evolution is described by an *effective Hamiltonian,* which is determined by the local character of the band structure (energy dispersion curves and eigenspaces) near $(\mathbf{k}_\star, E_\star)$. The effective Hamiltonian captures the transport and spreading dynamics of such data. This work utilizes the existence of an effective Hamiltonian (and indeed, many known results, see Section 5) to answer Question 2.

We choose the forcing time-scale of our time-dependent Hamiltonian, $H^\varepsilon(t)$*, so that there is a non-trivial interplay between the unforced transport dynamics and the effect of time-dependent forcing.*[1] The parameter $a \geq 1$ in (1.1) is chosen to achieve this balance of effects. For example, the choice $a = 1$ in (1.1) is appropriate for the dynamics where our Bloch wavepacket is concentrated near a point where the dispersion is locally linear, thus allowing for simple transport or conical diffraction, for example. In Section 5 we discuss a number of examples.

A key to our analysis is the space BL_ε of band-limited wavepackets (Definition 4.1): the space of Fourier band-limited envelope modulations of Bloch modes

[1] Such balancing corresponds to what is typically done in experiments. For example, a wavepacket excitation is designed by a choice of a laser frequency and pulse band-width, and balanced with forcing to measure effects on experimentally accessible spatial and temporal scales.

($\mathbf{k}_\star$-pseudo-periodic eigenstates) of H^0 with energy $E_\star$. BL_ε states are good approximations to ε-spectrally localized a Floquet–Bloch wavepackets of band-width ε (Proposition 4.7).

Let Π^ε denote the spectral projection-valued measure associated with the unitary operator M^ε, whose spectrum is on the unit circle S^1 (see Section 2); let $\mathcal{P}_0^\varepsilon$ be the projection onto the subspace of $L^2(\mathbb{R}^n)$, which is the space of functions which are ε-localized, with respect to H^0, in quasi-momentum and energy about $(\mathbf{k}_\star, E_\star)$. In particular, note that by applying the spectral theorem to H_0, then $\mathcal{P}_0^\varepsilon M_0^\varepsilon = M_0^\varepsilon \mathcal{P}_0^\varepsilon$. Our main *near-invariance* results are the following.

(1) **Theorem 4.4** (Near invariance on range($\mathcal{P}_0^\varepsilon$)). *Let $\mathcal{I} \subset S^1$ denote any arc such that the spectrum of $M_0^\varepsilon \circ \mathcal{P}_0^\varepsilon$ is contained in $\mathcal{I}$.*[2] *Then, for $\varepsilon > 0$ sufficiently small,*

$$\Pi^\varepsilon[\mathcal{I}] \circ \mathcal{P}_0^\varepsilon = \mathcal{P}_0^\varepsilon + \mathcal{O}_{B(L^2(\mathbb{R}^n))}(\varepsilon^{n+1})$$

or equivalently $\Pi^\varepsilon[S^1 \setminus \mathcal{I}] \circ \mathcal{P}_0^\varepsilon = \mathcal{O}_{B(L^2(\mathbb{R}^n))}(\varepsilon^{n+1})$.

This provides an answer to Question 2. When states in the range of $\mathcal{P}_0^\varepsilon$ evolve under the forced monodromy operator, M^ε, the resulting state has very small projection onto quasi-energies far from their quasi-energies under $M_0^\varepsilon = \exp(-iH_0 T_{\rm per}^\varepsilon)$. Equivalently, denoting the spectral measure of M_0^ε by Π_0^ε,

$$(\Pi^\varepsilon[\mathcal{I}] - \Pi_0^\varepsilon[\mathcal{I}]) \circ \mathcal{P}_0^\varepsilon = \mathcal{O}_{B(L^2(\mathbb{R}^n))}(\varepsilon^{n+1}).$$

Underlying the proof of Theorem 4.4 is the following strict invariance result on BL_ε, the space of band-limited wave-packets.

(2) **Theorem 4.6** (Strict invariance on BL_ε). *Let $\mathcal{I}$ be as in Theorem 4.4. Then, for all $\varepsilon > 0$ and sufficiently small, we have the following* strict *invariance property for the evolution of* BL_ε *under the flow of M^ε:*

$$\Pi^\varepsilon[\mathcal{I}] \circ \mathrm{Proj}_{\mathrm{BL}_\varepsilon} = \mathrm{Proj}_{\mathrm{BL}_\varepsilon},$$

or equivalently, $\Pi^\varepsilon[S^1 \setminus \mathcal{I}] \circ \mathrm{Proj}_{\mathrm{BL}_\varepsilon} = 0$.

Remark 1.1. The results above are not captured by naive perturbative expansions of the propagator. Although the forcing term in (1.1) has magnitude ε^a, the monodromy M^ε is defined by evolution over one period $T_{\rm per}^\varepsilon \sim \varepsilon^{-a}$. Thus, an a priori estimate based solely on coefficient size and period does not guarantee a small correction to M^ε. The relevant control arises from the existence of an effective Hamiltonian

[2] For linear operators with compatible domains, we write composition as $A \circ B$, so $(A \circ B)u := A(Bu)$.

approximating the slow (envelope) dynamics. Our main theorem shows that the consequence such an approximation, on the spectral window of interest, is that the associated remainder is small in the sense pertinent to quasi-energies.

1.2. Relevant analytical work on temporally-forced Hamiltonian PDEs

In the present article, we give a novel perspective on the $L^2(\mathbb{R}^n)$ spectrum of parametrically forced Schrödinger equations. Our approach is part of an ongoing interest in the wave-packet dynamics of non-autonomous (in time) and periodic (in space) Schrödinger equations [19, 42] and their associated non-autonomous homogenized dynamics [3, 30]. This mathematical effort track the rapid development in the experimental study temporally-driven material, a category known as "Floquet media." The techniques to implement this paradigm has been demonstrated in condensed-matter physics [31, 38, 45], photonics [37, 40], acoustics [48], and more. Here, we draw connections and distinctions between our result and other approaches to two types of similar problems: reducibility and adiabatic theory.

Reducibility. By translation invariance with respect to a lattice (see Section 2.1), the dynamics of (1.1) can reduce to the spectral study of the *Floquet Hamiltonian*

$$\mathcal{K} \equiv i\partial_t - H^\varepsilon(t),$$

over the family of spaces $\{L^2(S^1; L^2_{\mathbf{k}})\}_{\mathbf{k}\in\mathcal{B}}$. For each fixed $\mathbf{k}$, spectral problems of this latter type correspond to time-periodically forced wave equations on the spatial torus. By constructing a change of variables which approximately maps the original Hamiltonian to an autonomous Hamiltonian (reducibility), it is shown that $\mathcal{K}(t)$ has pure point spectra, with quantitative control on the effect of the forcing on the unperturbed point spectrum [4, 5, 10, 14, 24, 34]. For these results to hold, one needs strong assumptions regarding the growth of point eigenvalues of H^0. For Schrödinger operators, the Weyl asymptotics imply that these growth assumptions are only satisfied for spatial dimension $n = 1$. An exception to that is [10], which establish analogous results for $n \geq 2$ for the case where $|V(x)|$ is assumed to be sufficiently small. To the best of our knowledge, beyond these works, the nature of the spectrum of $\mathcal{K}$ remains an open problem [34].

Adiabatic theory. Adiabatic theory studies slowly varying Hamiltonians of the form

$$i\partial_t\psi^\varepsilon(t) = H(\varepsilon t)\psi^\varepsilon(t), \quad 0 < \varepsilon \ll 1.$$

Here $s \mapsto H(s)$ is a sufficiently regular curve of self-adjoint operators on a Hilbert space $\mathcal{H}$. A common hypothesis is the existence, for each s, of an isolated spectral subspace of $H(s)$ with associated projector $P(s)$, where the map $s \mapsto P(s)$ is smooth in an appropriate sense. Under such assumptions, if $\psi^\varepsilon(0) \in \operatorname{Ran} P(0)$, then for times $t = O(\varepsilon^{-1})$ the solution remains close to $\operatorname{Ran} P(\varepsilon t)$; the error order depends on the regularity of H (see, e.g., [2, 9, 15, 17, 21, 25, 27, 35, 36, 41]).

The conclusions of adiabatic theorems concern transport along moving spectral subspaces for $H(\varepsilon t)$. They do not, in general, yield statements about the *spectrum of the unitary monodromy* associated with a time–periodic evolution. Our results address this different question (Question 2), and use a different mechanism – the existence of an effective Hamiltonian H_{eff} governing the slow dynamics.

1.3. Outline of the paper

In Section 2 we provide the necessary background on Floquet–Bloch theory of periodic Hamiltonians, the spectral theorem for unitary operators, and introduce relevant notation. Section 3 presents a brief intuitive introduction to effective dynamics and homogenization of periodic Hamiltonians. The main results of this article are presented in Section 4. In Section 5 we demonstrate how these results apply to a number of specific periodic Hamiltonians, H^0 and the associated H^ε, of physical interest. The proofs of the main results are presented in Section 6. Finally, a formal derivation of an effective transport equation (see Section 5.1) is presented in Section 7.

2. Mathematical preliminaries

2.1. Floquet–Bloch theory

We consider Hamiltonians $H^0 = -\Delta + V$, where V is periodic with respect to a lattice $\Lambda = \mathbb{Z}\mathbf{v}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{v}_n$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$.[3] Since H^0 commutes with lattice translations, the Hamiltonian H^0 admits a fiber-decomposition $H^0 = \int_{\mathcal{B}}^{\oplus} H^0_{\mathbf{k}}\, d\mathbf{k}$, where $H^0_{\mathbf{k}}$ denotes the operator H^0 acting in the space $L^2_{\mathbf{k}}$, consisting of L^2_{loc} functions which are $\mathbf{k}$-pseudo-periodic, i.e., $\psi \in L^2_{\mathbf{k}}$ if $\psi(\mathbf{x} + \mathbf{v}) = e^{i\mathbf{k}\cdot\mathbf{v}}\psi(\mathbf{x})$ a.e. in $\mathbf{x} \in \mathbb{R}^2$. The set $\mathcal{B}$ is the Brillouin zone, a choice of fundamental cell in $(\mathbb{R}^n_{\mathbf{x}})^* = \mathbb{R}^n_{\mathbf{k}}$. For each quasi-momentum, $\mathbf{k} \in \mathcal{B}$, $H^0_{\mathbf{k}}$ is self adjoint and has a compact resolvent. Hence, for each $\mathbf{k}$, $H^0_{\mathbf{k}}$ has a real sequence of discrete eigenvalues of finite multiplicity

$$E_1(\mathbf{k}) \le E_2(\mathbf{k}) \le \cdots \le E_b(\mathbf{k}) \le \cdots$$

[3] We believe our analysis can be extended to general classes of elliptic operators (scalar Hamiltonians and systems) H^0, whose coefficients are periodic with respect to a lattice; see [33].

tending to infinity. The corresponding $L^2_{\mathbf{k}}$ eigenfunctions, denoted by $\Phi_b(\mathbf{x};\mathbf{k})$, satisfy

$$H^0\Phi_b(\mathbf{x};\mathbf{k}) = E_b(\mathbf{k})\Phi_b(\mathbf{x};\mathbf{k}), \quad \mathbf{x} \mapsto \Phi_b(\mathbf{x},\mathbf{k}) \in L^2_{\mathbf{k}}.$$

Equivalently, $\mathbf{x} \mapsto e^{-i\mathbf{k}\cdot\mathbf{x}}\Phi_b(\mathbf{x},\mathbf{k}) \in L^2(\mathbb{R}^n/\Lambda)$. These eigenfunctions may be chosen to form an orthonormal basis for $L^2_{\mathbf{k}}$. Each function $\mathbf{k} \mapsto E_b(\mathbf{k})$ is continuous and piecewise analytic [32, Theorem 5.5], and hence Lipschitz continuous.

Each image, $E_b(\mathcal{B})$, is a subinterval of $\mathbb{R}$ called the *b-th spectral band.* The graphs of $\mathbf{k} \mapsto E_b(\mathbf{k})$ are called *dispersion surfaces*. The collection of all pairs $(\mathbf{k}, E_b(\mathbf{k}))$ and corresponding normalized $L^2_{\mathbf{k}}$ eigenfunctions, $\Phi_b(\mathbf{x};\mathbf{k})$, is called the *band structure* of H^0. Finally, the family of Floquet–Bloch modes $\bigcup_{\mathbf{k}\in\mathcal{B}}\{\Phi_b(\cdot,\mathbf{k})\}_{b\geq 1}$ is complete in $L^2(\mathbb{R}^n)$; for any $f \in L^2(\mathbb{R}^n)$,

$$f(\mathbf{x}) = \frac{1}{\mathrm{vol}(\mathcal{B})}\sum_{b\geq 1}\int_{\mathcal{B}} \langle \Phi_b(\cdot,\mathbf{k}), f\rangle_{L^2(\mathbb{R}^n)}\Phi_b(\mathbf{x},\mathbf{k})\, d\mathbf{k},$$

where the sum is interpreted as a convergence of partial sums in $L^2(\mathbb{R}^n)$. For simplicity, we will assume henceforward that $\mathrm{vol}(\mathcal{B}) = 1$.

2.2. $U^\varepsilon(t)$, M^ε and its associated spectral measure

In (1.3a) we introduced the (unitary in $L^2(\mathbb{R}^n)$) evolution $U^\varepsilon(t)$ associated with the dynamics (1.1). In this section we give a brief outline of a construction of $U^\varepsilon(t)$ and then discuss the spectral measure of the associated monodromy operator. Under very general assumptions on H^0 and the operators $\{W(t,\cdot)\}$, a unitary propagator can be shown to exist, see e.g., [23, 49].

From $U^0(t) = e^{-iH^0 t}$ and $U^\varepsilon(t)$, we obtain the unitary monodromy operators M_0^ε and M^ε which, by the spectral theorem, are equipped with associated spectral (projection-valued) measures Π_0^ε and Π^ε, respectively. For completeness, we review the definition and properties of a spectral measure and the spectral theorem for unitary operators. We refer the reader to [8, 18, 39, 44] for details.

Let $\mathcal{H}$ be a Hilbert space, let X be a set, and Σ a σ-algebra in X. A map $\Pi\colon \Sigma \to B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the Banach space of bounded linear operators on $\mathcal{H}$ is called a *projection-valued measure* if the following properties hold:

(1) $\Pi(I)$ is an orthogonal projection for every $I \in \Sigma$;

(2) $\Pi(\emptyset) = 0$ and $\Pi(X) = \mathrm{Id}$;

(3) If $\{I_j\}_{j\geq 1} \subset \Sigma$ are disjoint, then

$$\Pi\Big(\bigcup_{j\geq 1} I_j\Big)v = \sum_{j\geq 1}\Pi(I_j)v, \quad v \in \mathcal{H};$$

(4) $\Pi(I_1 \cap I_2) = \Pi(I_1)\Pi(I_2)$ for all $I_1, I_2 \in \Sigma$.

Theorem 2.1. *Let U be a unitary operator on $\mathcal{H}$. There exists a unique projection-valued measure $\Pi = \Pi_U$ on the Borel σ-algebra of S^1, which contains the spectrum of U, such that for every $f \in \mathcal{H}$,*

$$\int_{S^1} z\, d\Pi(z) f = Uf.$$

2.3. Notation and conventions

(1) We adopt the convention of considering all $\mathbb{C}^N$ vectors as column vectors. If $a, b \in \mathbb{C}^N$, $a^\top b = a \cdot b$.

(2) Fourier transform on $L^2(\mathbb{R}^n)$. For a function, f, defined on $\mathbb{R}^n$, define its *Fourier transform* as

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \hat{f}(\xi) \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x}\, dx.$$

Furthermore, introduce

$$\check{g}(x) \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(\xi) e^{ix\cdot\xi}\, d\xi.$$

The mappings $f \mapsto \hat{f}$ and $g \mapsto \check{g}$ map the Schwartz class, $\mathcal{S}(\mathbb{R}^n)$, to itself and for all $f \in \mathcal{S}(\mathbb{R}^n)$, we have the Plancherel identity $\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}$ and the inversion formula $(\hat{f})\check{} = f$. Hence, $\check{f} = \mathcal{F}^{-1} f$ on $\mathcal{S}(\mathbb{R}^n)$. By density, both mappings extend to bounded linear transformations on $L^2(\mathbb{R}^n)$ which satisfy the Plancherel identity, and the inversion formula.

(3) Pauli matrices are given by $\sigma_0 = \mathrm{I}$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(4) For a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$, we write $(v_1, v_2) \cdot (\sigma_1, \sigma_2) = v_1\sigma_1 + v_2\sigma_2$.

3. Wavepackets, the geometry of dispersion surfaces, and periodic homogenization

Our spectrally local formulation concerning the quasi-energy spectrum $H^\varepsilon(t)$, Question 2, is a natural relaxation of Question 1. In physical settings, a crystalline structure is experimentally probed in a narrow spectral range, e.g., a bulk material is externally excited (e.g., electrically, optically, elastically, acoustically). Such settings induce the propagation of spectrally localized wavepackets (quasi-particles), whose envelope dynamics are given by a simplified effective Hamiltonian.

To illustrate this last point and how effective Hamiltonians emerge, consider the following "toy model" of continuously translation-invariant and time-periodically forced Hamiltonian dynamics governing a wave-field $\psi = \psi(t, \mathbf{x})$:

$$\begin{cases} i\partial_t \psi(t, \mathbf{x}) = E(-i\nabla)\psi + \varepsilon^a \mathcal{Q}(\varepsilon^a t)\psi \\ \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}), \end{cases} \tag{3.1}$$

where ψ_0 is sufficiently smooth and localized on $\mathbb{R}^n$ and $\mathcal{Q}$ is a real integrable function. The real-valued *dispersion relation* $\xi \mapsto E(\xi)$ is, for simplicity, taken to be smooth. Clearly, an explicit solution can be given in terms of the Fourier transform, but our goal here will be to discuss the notion of effective dynamics.

Consider initial data whose Fourier transform is concentrated near $\xi_\star \in \mathbb{R}^n$:

$$\widehat{\psi_0^\varepsilon}(\xi) = \varepsilon^{-n}\widehat{\Psi}_0(\varepsilon^{-1}(\xi - \xi_\star)), \quad \widehat{\Psi}_0 \in \mathcal{S}(\mathbb{R}^n), \quad 0 < \varepsilon \ll 1.$$

The solution of the initial value problem (3.1) may be written as

$$\psi^\varepsilon(t, \mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{i(\xi_\star \cdot \mathbf{x} - E(\xi_\star)t)} \int e^{i[[E(\xi_\star + \varepsilon\tilde{\xi}) - E(\xi_\star)]t + \varepsilon\tilde{\xi}\cdot\mathbf{x} + \Theta(\varepsilon^a t)]}\widehat{\Psi}_0(\tilde{\xi})d\tilde{\xi},$$

where $\Theta(T) \equiv \int_0^T \mathcal{Q}(s)ds$.

If $\nabla E(\xi_\star) \neq 0$, then by Taylor expansion of $E(\xi)$ about $\xi_\star$, we obtain the following approximation of the solution $\psi_\varepsilon(t, \mathbf{x})$ of (3.1) with $a = 1$, which is valid on the time scale: $0 \le t \lesssim \varepsilon^{-1}$:

$$\psi_\varepsilon(t, \mathbf{x}) \approx e^{i(\xi_\star \cdot \mathbf{x} - E(\xi_\star)t)} \cdot B(\varepsilon t, \varepsilon\mathbf{x}),$$

where the envelope $B(T, X)$ is governed by a driven transport equation

$$i\partial_T B(T, X) = [i\nabla_\xi E(\xi_\star) \cdot \nabla_X + \mathcal{Q}(T)]B(T, X).$$

If, on the other hand, $\nabla E(\xi_\star) = 0$ and $D^2_{\mathbf{k}} E(\xi_\star)$, the $n \times n$ Hessian matrix, is non-singular, then we obtain the following approximate solution $\psi_\varepsilon(t, \mathbf{x})$ of (3.1) with $a = 2$, which is valid on the time scale: $0 \le t \lesssim \varepsilon^{-2}$:

$$\psi_\varepsilon(t, \mathbf{x}) \approx e^{i(\xi_\star \cdot \mathbf{x} - E(\xi_\star)t)} \cdot B(\varepsilon^2 t, \varepsilon\mathbf{x}),$$

where $B(T, X)$ satisfies an (generally anisotropic) effective Schrödinger equation:

$$i\partial_T B(T, X) = \Big[\nabla_X \cdot \frac{1}{2} D^2_\xi E(\xi_\star)\nabla_X + \mathcal{Q}(T)\Big]B(T, X).$$

In each case, the function $B(T, X)$, which provides the slow envelope evolution, is governed by a time-dependent effective Hamiltonian:

$$i\partial_T B(T, X) = H_{\text{eff}}(-i\nabla, T)B(T, X),$$

in which both the effects of deformation under H^0 and temporal forcing are captured. Note also that $H_{\text{eff}}(-i\nabla, T)$ commutes with continuous spatial translations and therefore can be analysed using the Fourier transform.

In general, for spatially homogeneous media and for the case of crystalline (lattice periodic) media described by H^0, which is invariant under discrete translations in a lattice, the dispersion relation eigenvalue-branches may be degenerate. At such degeneracies the dispersion relations $\mathbf{k} \mapsto E_b(\mathbf{k})$ may not be smooth, although they are Lipschitz continuous if H^0 is self adjoint. Furthermore, in such cases, the eigenvector maps $\mathbf{k} \mapsto \Phi_b(\mathbf{x}; \mathbf{k})$ may even be multivalued.[4] Nevertheless, Fourier-type analysis (based on Floquet–Bloch modes) and multiple-scale/homogenization methods can be used to rigorously derive, with accompanying error bounds, effective envelope dynamics. Examples are

(i) effective mass Schrödinger equations [1,22] when $E_\star$ corresponding is at an isolated band edge, at which the dispersion surface is generically quadratic,

(ii) effective Dirac equations (with time-independent and time-dependent Hamiltonians) for dispersion surfaces touching conically (Dirac points) [13, 19, 42],

(iii) effective matrix-Schrödinger equations, for quadratically degenerate dispersion surfaces [28], and

(iv) effective Dirac operators of magnetic type for non-uniform spatial deformations of honeycomb media [16].

4. Main results

4.1. Hypotheses and definitions

Our first assumption concerns the character of the energy band structure near $(\mathbf{k}_\star, E_\star)$; in particular if $(\mathbf{k}_\star, E_\star)$ is a degeneracy, then this degeneracy is isolated.

[4] In this paper, we discuss only isolated point degeneracies. Other types of band degeneracies may arise. Examples are (i) the touching of two bands along a submanifold of quasi-momenta due the underlying symmetries and (ii) degeneracies of infinite multiplicity such as “flat bands,” as in e.g., the Landau Hamiltonian [20]. We do not treat these situations in the present work.

Hypothesis 1 (Spectral separation). Let $(\mathbf{k}_\star, E_\star)$ be such that $H^0_{\mathbf{k}_\star}$ has an eigenvalue $E_\star$ of multiplicity $N \geq 1$, i.e., for some $b_\star \geq 1$,

$$E_{b_\star - 1}(\mathbf{k}_\star) < E_\star = E_{b_\star}(\mathbf{k}_\star) = E_{b_\star+1}(\mathbf{k}_\star) = \cdots = E_{b_\star+N-1}(\mathbf{k}_\star) < E_{b_\star+N}(\mathbf{k}_\star).$$

Furthermore, $(\mathbf{k}_\star, E_\star)$ is isolated in the band structure in the sense that

$$E_{b_\star-1}(\mathbf{k}) < E_\star < E_{b_\star+N}(\mathbf{k})$$

for all $\mathbf{k}$ in an open neighbourhood of the quasi-momentum $\mathbf{k}_\star$. Introduce an orthonormal basis for the degenerate eigenspace:

$$\{\Phi_b(\mathbf{x}, \mathbf{k}_\star) \mid b_\star \leq b \leq b_\star + N - 1\}.$$

With Question 2 in mind and assuming spectral separation as defined in Hypothesis 1, we define a projection, $\mathcal{P}^\varepsilon_0$, associated with a subspace of $L^2(\mathbb{R}^n)$ consisting of states, which are superpositions of modes whose quasi-momenta and energy are near $(\mathbf{k}_\star, E_\star)$:

$$\mathcal{P}^\varepsilon_0 \equiv \int\limits_{|\mathbf{k}-\mathbf{k}_\star|<\varepsilon} \mathrm{Proj}(|H^0_{\mathbf{k}} - E_\star| < L\varepsilon)\, d\mathbf{k}, \tag{4.1}$$

where $L > 0$ is fixed. Here and henceforward, such integrals over k are to be understood in the sense of the Floquet–Bloch deomposition of $L^2(\mathbb{R}^n)$.

We next present two additional assumptions concerning the underlying wave-packet dynamics. Let $(\mathbf{k}_\star, E_\star)$ satisfy the spectral separation Hypothesis 1 with parameters $N \geq 1$ and $b_\star \geq 1$. Denote the vector of degenerate Floquet–Bloch modes

$$\Phi_\star(\mathbf{x}) \equiv \begin{pmatrix} \Phi_{b_\star}(\mathbf{x}; \mathbf{k}_\star) \\ \vdots \\ \Phi_{b_\star+N-1}(\mathbf{x}; \mathbf{k}_\star) \end{pmatrix}.$$

We next introduce the subspace of $L^2(\mathbb{R}^n)$, consisting of Fourier band-limited wave-packets, which are modulations $\Phi_\star$.

Definition 4.1 (Band-limited wave-packets). For fixed parameters $\varepsilon, d_0 > 0$, we define

$$\mathrm{BL}_\varepsilon \equiv \{u = \alpha(\varepsilon\mathbf{x})^\top \Phi_\star(\mathbf{x}) \mid \mathrm{supp}(\hat{\alpha}) \subseteq B_{d_0}(0) \text{ and } \alpha \in L^2(\mathbb{R}^n; \mathbb{C}^N)\}. \tag{4.2}$$

Hypothesis 2 (Translation invariant effective dynamics). There is a one-parameter family of unitary operators on BL_ε, $U^\varepsilon_{\mathrm{eff}}(t)$, with the following properties.

(1) *Spatially translation invariant effective dynamics.* For $\psi_0 = \alpha_0^\top(\varepsilon\mathbf{x})\Phi_\star(\mathbf{x}) \in \mathrm{BL}_\varepsilon$, $U^\varepsilon_{\mathrm{eff}}(t)$ is defined by

$$U^\varepsilon_{\mathrm{eff}}(t)\psi_0 = \frac{e^{-iE_\star t}}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{i\xi\cdot\varepsilon\mathbf{x}} \widehat{\mathfrak{U}}_{\mathrm{eff}}(\varepsilon^a t; \xi)\hat{\alpha}_0(\xi)\, d\xi \cdot \Phi_\star(\mathbf{x}),$$

where $(T,\xi)\mapsto \widehat{\mathfrak{U}}_{\rm eff}(T;\xi)$ is a smooth mapping from $\mathbb{R}_T\times\mathbb{R}^n_\xi$ into the space of unitary $N\times N$ matrices.

(2) *Approximation by effective dynamics.* Let $U^\varepsilon_{\rm eff}(t)$ be defined as in (1). If $\psi_0\in \mathrm{BL}_\varepsilon$, then

$$\lim_{\varepsilon\to 0+}\sup_{0\le t\le T^\varepsilon_{\rm per}}(U^\varepsilon_{\rm eff}(t)-U^\varepsilon(t))\psi_0\|_{L^2(\mathbb{R}^n)}=0,$$

where $T^\varepsilon_{\rm per}$ is given in (1.2).

Remark 4.2. For $\psi_0=\alpha_0^\top(\varepsilon\mathbf{x})\Phi_\star(\mathbf{x})\in\mathrm{BL}_\varepsilon$, Hypothesis 2 implies *slow envelope effective dynamics.* Indeed, let

$$\alpha(T,\cdot)=\mathfrak{U}_{\rm eff}(T;-i\nabla)[\alpha_0].$$

Then, using the space-time scaling $\mathcal{S}_\varepsilon[f](\mathbf{x},t)=f(\varepsilon\mathbf{x},\varepsilon^a t)$, we may write

$$U^\varepsilon_{\rm eff}(t)\psi_0=\mathcal{S}_\varepsilon\circ\mathfrak{U}_{\rm eff}(t;-i\nabla)[\alpha_0]\cdot\mathcal{S}_\varepsilon^{-1}\Phi_\star(\mathbf{x})=\alpha(\varepsilon^a t,\varepsilon\mathbf{x})\cdot\Phi_\star(\mathbf{x}).$$

Equivalently, $\alpha(T,X)$ evolves under the effective Hamiltonian $H_{\rm eff}(T,-i\nabla)$, which generates the unitary flow $\mathcal{U}_{\rm eff}$:

$$i\partial_T\alpha=H_{\rm eff}(T,-i\nabla)\alpha,\quad \alpha(0,X)=\alpha_0(X).$$

The effective evolution operator, $U^\varepsilon_{\rm eff}(t)$, naturally gives rise to an *Effective monodromy operator defined on* BL_ε:

$$M^\varepsilon_{\rm eff}\equiv U^\varepsilon_{\rm eff}(T^\varepsilon_{\rm per});$$

for $\psi_0=\alpha_0^\top(\varepsilon\mathbf{x})\Phi_\star(\mathbf{x})\in\mathrm{BL}_\varepsilon$,

$$(M^\varepsilon_{\rm eff}\psi_0)(\mathbf{x})=(\mathfrak{U}_{\rm eff}(T_{\rm per};-i\nabla)[\alpha_0])(\varepsilon\mathbf{x})\cdot\Phi_\star(\mathbf{x}).$$

Hypothesis 3 (Spectrum of the effective monodromy operator). For every $d_0>0$ sufficiently small, there exists $g_0\in[0,\pi)$ such that

$$\mathrm{Spec}_{\mathrm{BL}_\varepsilon}(M^\varepsilon_{\rm eff})\subseteq\{e^{-i\nu}\mid \nu\in(E_\star T^\varepsilon_{\rm per}-g_0,E_\star T^\varepsilon_{\rm per}+g_0)\}.\tag{4.3}$$

Remark 4.3 (Notational assumption; $E_\star=0$ from here on). In the proofs of our results below we shall, without loss of generality, by replacing H^0 by $H^0-E_\star$, set $E_\star=0$. Under this convention, (4.3) in Hypothesis 3 simply reads as

$$\mathrm{Spec}_{\mathrm{BL}_\varepsilon}(M^\varepsilon_{\rm eff})\subseteq\{e^{-i\nu}\mid \nu\in(-g_0,g_0)\}.$$

4.2. A theorem on near-invariance of quasi-energy spectrum

Since the monodromy operator M^{ε} is unitary (see (1.3)), M^{ε} has a spectral representation as an integral with respect to a projection-valued spectral measure, Π^{ε}, which is supported on the unit circle; see Section 2.2. We now state our main theorem, which addresses Question 2.

Denote by (a, b) the arc $\{e^{-iy} \mid y \in (a, b)\} \subseteq S^1$.

Theorem 4.4 (Near invariance). *Consider the periodically forced Schrödinger equation (1.1). Assume that for some quasi-momentum/energy pair $(\mathbf{k}_\star, E_\star) = (\mathbf{k}_\star, 0)$ (see Remark 4.3) Hypotheses 1–3 are satisfied. Let $\mathcal{P}_0^{\varepsilon}$, defined in (4.1), denote the $L^2(\mathbb{R}^n)$ projection onto Bloch modes of H^0 of energy and quasi-momentum in an ε-neighbourhood of $(\mathbf{k}_\star, E_\star) = (\mathbf{k}_\star, 0)$.*

Then, for every $g \in (g_0, \pi)$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\Pi^{\varepsilon}[(-g, g)] \circ \mathcal{P}_0^{\varepsilon} = \mathcal{P}_0^{\varepsilon} + \mathcal{O}_{\mathcal{B}(L^2)}(\varepsilon^{n+1}).$$

Theorem 4.4 is a near-invariance (or stability) result for a spectral subspace associated with H^0, the range of $\mathcal{P}_0^{\varepsilon}$, under the perturbed dynamics $H^{\varepsilon}(t)$. Indeed, let the parametric forcing term be zero, i.e., $W = 0$; Then, under Hypothesis 1, the non-driven monodromy operator M_0^{ε}, restricted to the range of $\mathcal{P}_0^{\varepsilon}$, is given by, for any $u \in L^2(\mathbb{R}^n)$:

$$M_0^{\varepsilon} \mathcal{P}_0^{\varepsilon} u(\mathbf{x}) = \sum_{b=b_\star}^{b_\star+N-1} \int_{|\mathbf{k}-\mathbf{k}_\star|<\varepsilon} \langle \Phi_b(\cdot; \mathbf{k}), u \rangle \Phi_b(\mathbf{x}; \mathbf{k}) e^{-iE_b(\mathbf{k}) T_{\mathrm{per}}^{\varepsilon}} \, d\mathbf{k}. \tag{4.4}$$

Suppose that the relevant dispersion surfaces near $\mathbf{k}_\star$ are bounded by a polynomial of degree a, i.e., there is a constant $C > 0$ such that for $|\mathbf{k} - \mathbf{k}_\star| \lesssim \varepsilon$ and for all $b_\star \leq b \leq b_\star + N - 1$, we have

$$|E_b(\mathbf{k})| \leq C |\mathbf{k} - \mathbf{k}_\star|^a + \mathcal{O}(\varepsilon^{a+1}) = \mathcal{O}(\varepsilon^a).$$

Denoting the spectral measure of M_0^{ε} by Π_0^{ε}, we have that for a fixed g and sufficiently small $\varepsilon > 0$, by inspecting the exponents in (4.4),

$$\Pi_0^{\varepsilon}[(-g, g)] \circ \mathcal{P}_0^{\varepsilon} = \mathcal{P}_0^{\varepsilon} \quad \text{and} \quad \Pi_0^{\varepsilon}[S^1 \setminus (-g, g)] \circ \mathcal{P}_0^{\varepsilon} = 0. \tag{4.5}$$

As discussed in Remark 1.1, it is non-trivial that a form of (4.5) persists for time-periodic forcing $W \neq 0$ in (1.1), due to the formally order-one cumulative effect of a perturbation of size ε^a on the time-scale $T_{\mathrm{per}}^{\varepsilon} \sim \varepsilon^{-a}$.

4.3. The main result for the space of band limited wavepackets BL_ε

As a step toward the proof of Theorem 4.4, we first prove its analog, Theorem 4.6, a strict invariance property for functions in BL_ε (see (4.2)), a closed subspace of $L^2(\mathbb{R}^n)$. Since BL_ε approximates the range of $\mathcal{P}_0^\varepsilon$ (Proposition 4.7), we can then use Theorem 4.6 to prove Theorem 4.4, which concerns the range of $\mathcal{P}_0^\varepsilon$.

Lemma 4.5. *There exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, BL_ε defined in (4.2) is a closed subspace of $L^2(\mathbb{R}^n)$. Hence, $L^2(\mathbb{R}^n)$ has the decomposition*

$$L^2(\mathbb{R}^n) = \mathrm{BL}_\varepsilon \oplus \mathrm{BL}_\varepsilon^\perp,$$

with corresponding orthogonal projections on $L^2(\mathbb{R}^n)$ denoted

$$\mathrm{Proj}_{\mathrm{BL}_\varepsilon} \quad \text{and} \quad \mathrm{Proj}^\perp_{\mathrm{BL}_\varepsilon} = \mathrm{I} - \mathrm{Proj}_{\mathrm{BL}_\varepsilon}.$$

For a proof, see Appendix B.1.

BL_ε is a very natural space with which to study the effects of time-dependent forcing. In fact, the proof of Theorem 4.4, follows from its analog for the space BL_ε.

Theorem 4.6 (Invariance on BL_ε). *Consider (1.1) and suppose it satisfies Hypotheses 1–3 at some quasi-momentum energy pair $(\mathbf{k}_\star, E_\star = 0)$. Fix $d_0 \in (0, \pi)$ and $g > 0$ such that $g \in (g_0, \pi)$. Then, for every $g \in (g_0, \pi)$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\Pi^\varepsilon[(-g, g)] \circ \mathrm{Proj}_{\mathrm{BL}_\varepsilon} = \mathrm{Proj}_{\mathrm{BL}_\varepsilon}.$$

Equivalently,

$$\Pi^\varepsilon[S^1 \setminus (-g, g)] \circ \mathrm{Proj}_{\mathrm{BL}_\varepsilon} = 0. \tag{4.6}$$

Theorem 4.6 is proved in Section 6. Here, we first use it to give a proof of the main result, Theorem 4.4 (concerning $\mathcal{P}_0^\varepsilon$).

Proof of the main result, Theorem 4.4. To prove Theorem 4.4 we shall use Theorem 4.6 above and the following proposition, which is proved in Section 6.1.

Proposition 4.7 (BL_ε approximates $\mathrm{ran}(\mathcal{P}_0^\varepsilon)$). *There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, the following holds: for every $f \in L^2(\mathbb{R}^n)$, there is a $u_\varepsilon[f] \in \mathrm{BL}_\varepsilon$ with $d_0 = 1$ (see (4.2)) such that*

$$\mathcal{P}_0^\varepsilon f = u_\varepsilon[f] + \mathcal{O}(\varepsilon^{n+1} \|f\|_{L^2(\mathbb{R}^n)}). \tag{4.7}$$

Conversely, there exists $C > 0$ such that for every $u \in \mathrm{BL}_\varepsilon$ with d_0 sufficiently small, then

$$\|(I - \mathcal{P}_0^\varepsilon)u\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{n+1} \|u\|_{L^2(\mathbb{R}^n)}. \tag{4.8}$$

By Proposition 4.7, $\mathcal{P}_0^\varepsilon u = u_{\mathrm{bl}} + r$, where $u_{\mathrm{bl}} \in \mathrm{BL}_\varepsilon$ and $\|r\|_{L^2} = \mathcal{O}(\varepsilon^{n+1}\|v\|_{L^2})$. Now,

$$\begin{aligned}\Pi^\varepsilon[S^1 \setminus (-g,g)] \circ \mathcal{P}_0^\varepsilon u &= \Pi^\varepsilon[S^1 \setminus (-g,g)]u_{\mathrm{bl}} + \Pi^\varepsilon[S^1 \setminus (-g,g)]r \\ &= 0 + \Pi^\varepsilon[S^1 \setminus (-g,g)]r,\end{aligned}$$

where, since $u_{\mathrm{bl}} \in \mathrm{BL}_\varepsilon$, the last equality is the result of Theorem 4.6. Finally, since $\Pi^\varepsilon[S^1 \setminus (-g,g)]$ is a projection,

$$\|\Pi^\varepsilon[S^1 \setminus (-g,g)]r\|_{L^2(\mathbb{R}^n)} \le \|r\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{n+1}),$$

which completes the proof. ∎

5. Applications of the main result, Theorem 4.4

In this section we apply Theorem 4.4 to time-periodically forced (Floquet) Hamiltonians of the form:

$$H^\varepsilon(t) = H^0 + 2i\varepsilon^a \underline{A}(\varepsilon^a t)\cdot\nabla. \tag{5.1}$$

Here, $\underline{A}\colon \mathbb{R} \to \mathbb{R}^n$ is T_{per}-periodic with zero mean, i.e., $\int_0^{T_{\mathrm{per}}} \underline{A}(T)\,dT = 0$. A discussion, with references, of how this class of models arises in condensed matter physics and photonics is presented in Appendix A.

The setting of Theorem 4.4 is a Floquet Hamiltonian, here (5.1), and a neighbourhood of an energy quasi-momentum pair $(E_\star, \mathbf{k}_\star)$ in the band structure of H^0, in which the class of wave-packet initial data are spectrally localized. Here, we characterize the local character of the band structure at $(E_\star, \mathbf{k}_\star)$ by a number of parameters. As in Hypothesis 1, we denote by N the multiplicity of $E_\star$. The parameter a in (5.1) is chosen to match the rate at which to energy, E, deviates from $E_\star$ for $|\mathbf{k}-\mathbf{k}_\star|$ small. Table 1 summarizes four cases of physical interest, which are discussed in the following subsections.

Sect.	dispersion rate a	N, degeneracy order	dimension n	effective equation
5.1	1	1	$n \ge 1$	Transport (5.3)
5.2	1	2 (+conical touching)	$n = 2$	Dirac system (5.5)
5.3	2	1	$n \ge 1$	Schrödinger (5.6)
5.4	2	2 (+quadratic touching)	$n = 2$	Schrödinger system (5.7)

Table 1. Summary of examples discussed in Sections 5.1–5.4. Parameters N and a are defined in (1.1) and Hypothesis 1, respectively.

In what follows, BL_ε wavepackets are always denoted by $u(\mathbf{x}) = \alpha(\varepsilon\mathbf{x})^\top\Phi(\mathbf{x})$, where the dimension of α and Φ is N, the degree of the degeneracy at $(\mathbf{k}_\star, E_\star)$.

In the non-driven case, i.e., when $\underline{A}=0$, the effective/homogenized models, which govern the large time dynamics of wave-packet envelopes, are continuously translation-invariant PDEs of the form $i\partial_T\alpha = H_{\mathrm{eff}}(-i\nabla)\alpha$; see references below. Our analysis shows that, for (5.1) with $\underline{A}\neq 0$, the dynamics of wave-packet envelopes is governed by

$$i\partial_T\alpha = H_{\mathrm{eff}}(-i\nabla, T)\alpha,$$

where the non-autonomous Hamiltonian $H_{\mathrm{eff}}(-i\nabla, T)$ is obtained from arising from $H_{\mathrm{eff}}(-i\nabla)$ via the formal replacement $-i\nabla_X \mapsto P_{\underline{A}}(T) \equiv -i\nabla_X + \underline{A}(T)$. In each example below, we display $H_{\mathrm{eff}}(-i\nabla, T)$.

5.1. $E_\star$ simple and $(\mathbf{k}_\star, E_\star)$ a non-critical point – ballistic transport

For a given Hamiltonian H^0, let $(\mathbf{k}_\star, b_\star)$ be a pair of a quasi-momentum and index $b_\star\in\mathbb{N}$ such that $E_{b_\star}(\mathbf{k}_\star)=0$ is a simple eigenvalue of H^0 in $L^2_{\mathbf{k}_\star}$ with a linear dispersion relation, i.e.,

$$E_{b_\star-1}(\mathbf{k}_\star) < E_\star < E_{b_\star+1}(\mathbf{k}_\star), \tag{5.2a}$$

and

$$\mathbf{c} \equiv -\vec{\nabla}_k E_{b_\star}(\mathbf{k})|_{\mathbf{k}=\mathbf{k}_\star} \neq 0, \tag{5.2b}$$

where $\mathbf{c}\in\mathbb{R}^n$ (since the dispersion surfaces are real-valued). By continuity of the energy bands, Hypothesis 1 holds. The effective Hamiltonian, governing the BL_ε-wave-packet envelope $\alpha(X,T)$, is given by

$$H_{\mathrm{eff}}(-i\nabla, T) = \mathbf{c}\cdot P_{\underline{A}(T)} = \mathbf{c}(-i\nabla_X + \underline{A}(T)). \tag{5.3}$$

In this case, following the notations of Hypothesis 2, $\|U^\varepsilon(t) - U^\varepsilon_{\mathrm{eff}}(t)\|_{L^2} \lesssim \varepsilon$ for $t\lesssim\varepsilon^{-1}$. The proof of this statement follows closely that of of the case of a double conical degeneracy (Dirac point; see Section 5.2), which is presented in detail in [42]. We include a formal derivation of (5.3) in Section 7.

To verify Hypothesis 3, we apply the Fourier transform (in the X variable)

$$\mathcal{F}[\alpha(T,X)](\xi) = \hat{\alpha}(T;\xi),$$

and get the family of ODE initial value problems, parametrized by $\xi\in\mathbb{R}^n$:

$$i\partial_T\hat{\alpha}(T;\xi) = \mathbf{c}\cdot(\xi + \underline{A}(T))\hat{\alpha}(T;\xi), \quad \hat{\alpha}(0;\xi) = \hat{\alpha}_0(\xi),$$

for which the solution is

$$\hat{\alpha}(T;\xi) = \exp[-i\mathbf{c}\cdot(\xi T + \mathbf{h}(T))]\hat{\alpha}_0(\xi), \quad \mathbf{h}(T) \equiv \int_0^T \underline{A}(T')\, dT'.$$

Hence, for a fixed $d_0 > 0$ and $u \in \mathrm{BL}_\varepsilon$,

$$M^\varepsilon_{\mathrm{eff}} u \equiv U^\varepsilon_{\mathrm{eff}}(T_{\mathrm{per}}\varepsilon^{-1})u = (2\pi)^{-r/2}\varepsilon^{n/2} \int_{|\xi|\le d_0} e^{i\xi\cdot\varepsilon\mathbf{x}}\hat{\alpha}_0(\xi)e^{-i\mathbf{c}\cdot\xi T_{\mathrm{per}}}\Phi(x)\, d\xi,$$

where we recall that, by assumption, $\mathbf{h}(T_{\mathrm{per}}) = 0$. And so, by choosing wavepackets supported on the ball $|\xi| < d_0$,

$$\sigma(M^\varepsilon_{\mathrm{eff}}) \text{ on } \mathrm{BL}_\varepsilon = \{e^{iy} \mid y \in [-d_0 T_{\mathrm{per}}|\mathbf{c}|, d_0 T_{\mathrm{per}}|\mathbf{c}|]\},$$

which verifies Hypothesis 3.

5.2. $E_\star$ of multiplicity two; Conical touching of dispersion surfaces at $(\mathbf{k}_\star, E_\star)$, a Dirac point

An example which plays an important role in the modelling of two-dimensional materials such as graphene is the case where $H^0 = -\Delta + V(\mathbf{x})$, where V is a honeycomb lattice potential, i.e., V has the symmetries of a honeycomb tiling of $\mathbb{R}^2$. (A one-dimensional variant of such potentials, *dimer potentials*, was studied in [11] and the following discussion can adapted to this setting as well.) For generic honeycomb lattice potentials, conical degeneracies (Dirac points) occur in the band structure at pairs $(\mathbf{k}_\star, E_\star)$, where $\mathbf{k}_\star$ is any vertex (high symmetry quasi-momentum) of the hexagonal Brillouin zone [12]. In a neighbourhood of a Dirac point, one has two consecutive dispersion surfaces, $E_-(\mathbf{k}_\star) \le E_+(\mathbf{k}_\star)$, satisfying

$$E_\pm(\mathbf{k}) = E_\star \pm v_{\mathrm{D}}|\mathbf{k}-\mathbf{k}_\star| + \mathcal{O}(|\mathbf{k}-\mathbf{k}_\star|^2), \quad v_{\mathrm{D}} > 0. \tag{5.4}$$

The slope of the cone, v_{D}, is referred to as the *Dirac* or *Fermi* velocity. From (5.2b), we see that Hypothesis 1 is satisfied.

Hypothesis 2 is also satisfied for this class of equations. Indeed, [42, Theoem 3.2] shows that, with scaling parameter $a = 1$ (see (5.4)), the effective envelope dynamics of (5.1) for data in BL_ε are governed by a driven Dirac Hamiltonian:

$$H_{\mathrm{eff}}(T, -i\nabla_X) = v_{\mathrm{D}}(\sigma_1, \sigma_2)\cdot(P_{\underline{A}_1(T)}, P_{\underline{A}_2(T)}), \tag{5.5}$$

and that $\|(U_{\mathrm{eff}}(T^\varepsilon_{\mathrm{per}}) - U^\varepsilon(T^\varepsilon_{\mathrm{per}}))f\|_{L^2(\mathbb{R}^n)} \lesssim \varepsilon\|f\|_{L^2(\mathbb{R}^n)}$ for data $f \in \mathrm{BL}_\varepsilon$.

Finally, Hypothesis 3 is satisfied by [42, Proposition 3.5]. There, we apply the Fourier transform to the effective Dirac equation above and find the eigenvalues of its monodromy operator at Fourier momentum: $\xi = (0,0)^\top$. By continuity with respect to ξ, one can find g_0 for a sufficiently small d_0 to satisfy Hypothesis 3. In the semi-classical regime (which is different from the present paper), wave-packet dynamics near a conical singularity have been studied extensively, see e.g., [7, 46, 47].

5.3. $E_\star$ simple, $(\mathbf{k}_\star, E_\star)$ a non-degenerate critical (quadratic) point of a band

Suppose $(\mathbf{k}_\star, E_\star)$ is such that $E_\star$ is a simple $L^2_{\mathbf{k}_\star}$-eigenvalue of H^0 and $\mathbf{k}_\star$ is a non-degenerate critical point of the band dispersion function E_b: $\vec{\nabla}_{\mathbf{k}} E_b(\mathbf{k}_\star) = \vec{0}$ and $\det D^2_{\mathbf{k}} E_b(\mathbf{k}_\star) \neq 0$. Then, $a = 2$, and the envelope dynamics for are given by an driven effective Schrödinger-type Hamiltonian:

$$H_{\text{eff}}(T, -i\nabla_X) = P_{\underline{A}(T)} \cdot \frac{1}{2} D^2_{\mathbf{k}} E(\mathbf{k}_\star) P_{\underline{A}(T)}. \tag{5.6}$$

The validity on time scales is of order ε^{-2}, and therefore Hypothesis 2 follows along the lines of [1] or [42].

Note that such quadratic points may occur at spectral band edges, in which case the Hessian $D^2_{\mathbf{k}} E(\mathbf{k}_\star)$ is positive or negative definite or at $(\mathbf{k}_\star, E_\star)$; or where $E_\star$ is interior to a spectral band, in which case the Hessian $D^2_{\mathbf{k}} E(\mathbf{k}_\star)$ might have an indefinite signature. Finally, a similar homogenization argument can be carried in the case where $\vec{\nabla}_{\mathbf{k}} E_b(\mathbf{k}_\star) \neq \vec{0}$ and $D^2_{\mathbf{k}} E(\mathbf{k}_\star)$ is non-degenerate. In this case, one gets a Schrödinger equation on the time-scales of ε^{-2}, with a drift term on the time-scale of ε^{-1}; see, for example, [1].

5.4. $E_\star$ of multiplicity two; Quadratic touching of two dispersion surfaces at $(\mathbf{k}_\star, E_\star)$

Consider a two-dimensional Hamiltonian $H^0 = -\Delta + V(\mathbf{x})$ where the potential V which is periodic with respect to the lattice $\Lambda = \mathbb{Z}^2$, real-valued, even, and invariant under a $\pi/2$-rotation. We can take the Brillouin zone, $\mathcal{B}$, to be a square, centered at the origin in $\mathbb{R}^2_{\mathbf{k}}$. The vertices of $\mathcal{B}$ are *high-symmetry quasi-momenta*. In [28] it is proved that the band structure of H^0 has consecutive band dispersion surfaces which touch quadratically over the vertices of $\mathcal{B}$ at an eigenvalue with a two-fold degenerate eigenvalue.

Hence, we consider (5.1) with $a = 2$ for BL_ε data near these high-symmetry $\mathbf{k}_\star$-points. The effective envelope dynamics of BL_ε data can be shown, in a manner analogous to the derivation in [29], to be governed by the matrix-Schrödinger effective Hamiltonian:

$$\begin{aligned} &H_{\text{eff}}(-i\nabla, T) \\ &\quad = \alpha(P^2_{A_1(T)} + P^2_{A_2(T)})\sigma_0 + \tilde{\gamma}(P^2_{A_1(T)} - P^2_{A,2})\sigma_2 + 2\beta P_{A_1(T)} P_{A_2(T)}\sigma_1, \end{aligned} \tag{5.7a}$$

where

$$P_{A_j(T)} \equiv -i\partial_{X_j} + A_j(T), \quad j = 1, 2. \tag{5.7b}$$

Here, the coefficients $\alpha, \tilde{\gamma}, \beta \in \mathbb{R}$ can be expressed as $L^2(\mathbb{R}/\mathbb{Z}^2)$ inner products involving a basis for the 2-dimensional $L^2_{\mathbf{k}_\star}$-kernel of $(H^0 - E_\star)$ (see [29]), and $\sigma_0, \sigma_1, \sigma_2$ are Pauli matrices. As in the case of the effective Dirac equation (Section 5.2), Hypothesis 3 is verified as follows. (i) Fourier-transforming (5.7) yields a system of two linear and time-periodic system of (Floquet) ODES, which is parametrized by ξ. (ii) Since this matrix defining this system of ODEs has trace equal to 0 and is continuous in ξ, the Floquet multipliers $e^{\pm i\mu(\xi)}$ are continuous functions of ξ on unit circle. (iii) Hence, for BL_ε data with a fixed band-width $d_0 > 0$ (see Definition 4.1), there exists a continuous function $g_0(d_0)$ such that the BL_ε data of $M^\varepsilon_{\mathrm{eff}}$ are contained in the arc $(-g_0(d_0), g_0(d_0))$.

6. Proof of Theorem 4.6

Let us first recall the following centering lemma for unitary operators. Intuitively, it says that if a unitary operator acts on a function which is spectrally localized, it is approximately the same as acting as a multiplication operator. We proved a weaker version of this lemma in [42], and include the proof here for completeness.

Lemma 6.1. *Let $\mathcal{I} \subset S^1$ such that $\Pi^\varepsilon(\mathcal{I})u = u$ and let $e^{-i\nu_0} \in \mathcal{I}$ be the mid-point of the arch $\mathcal{I}$. Then,*

$$M^\varepsilon u = e^{-i\nu_0}u + \eta, \quad \textit{where } \|\eta\|_{L^2(\mathbb{R}^n)} \le 2\sin\Big(\frac{|\mathcal{I}|_{S^1}}{4}\Big)\cdot\|u\|,$$

where $|\mathcal{I}|_{S^1}$ is the arclength of $\mathcal{I}$.

Proof. Let $z_0 = e^{-i\nu_0}$. Then,

$$M^\varepsilon u = \int_{\mathcal{I}} z\, d\Pi^\varepsilon(z)u = \int_{\mathcal{I}} (z_0 - z_0 - z)\, d\Pi^\varepsilon(z)u = z_0 u + \eta,$$

where

$$\eta \equiv \int_{\mathcal{I}} (z - z_0)\, d\Pi^\varepsilon(z)u.$$

Since $z_0 = e^{-i\nu_0}$, we only need to bound $\|\eta\|_{L^2}$. By the spectral theorem (see Section 2.2), we have that

$$\|\eta\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathcal{I}} |z - z_0|^2 \langle d\Pi^\varepsilon(z)u, u\rangle_{L^2(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\le \max_{z\in\mathcal{I}} |z-z_0|^2 \cdot \int_{S^1} \langle d\Pi^\varepsilon(z)u,u\rangle_{L^2(\mathbb{R}^n)} \\
&= \max_{e^{-i\nu}\in\mathcal{I}} |e^{-i\nu}-e^{-i\nu_0}|^2\cdot \|u\|^2 \\
&\le 4\sin^2\Big(\frac{|\mathcal{I}|_{S^1}}{4}\Big)\cdot\|u\|^2,
\end{aligned}
$$

where we have used the expression for the arc length

$$
|e^{i\beta}-e^{i\beta'}|_{S^1} = 2\sin\Big(\frac{|\beta-\beta'|}{2}\Big)
$$

for any $\beta,\beta'\in[0,2\pi)$ with $|\beta-\beta'|\le\pi$, combined with the fact that ν_0 is the midpoint of $\mathcal{I}$. ■

Proof of Theorem 4.6 . To prove (4.6), let $v\in \mathrm{BL}_\varepsilon$ and let

$$
v' \equiv \Pi^\varepsilon[S^1\setminus(-g,g)]v,
$$

for $g\in(g_0,\pi)$, where g_0 is defined in Hypothesis 3. We will now show that $v'=0$. Lemma 6.1 implies that, since π is the midpoint of the arch $\mathcal{I}=S^1\setminus(-g,g)$,

$$
M^\varepsilon v' = e^{-i\pi}v'+\eta_{v'} = -v'+\eta_{v'},
$$

where

$$
\|\eta_{v'}\|_{L^2(\mathbb{R}^n)} \le 2\sin\Big(\frac{\pi-g}{2}\Big)\cdot\|v'\|_{L^2(\mathbb{R}^n)}.
$$

Hence,

$$
\begin{aligned}
&\|(M^\varepsilon - M^\varepsilon_{\mathrm{eff}})v'\|_{L^2(\mathbb{R}^n)} \\
&\quad = \|(-\mathrm{Id} - M^\varepsilon_{\mathrm{eff}})v' + \eta_{v'}\|_{L^2(\mathbb{R}^n)} \\
&\quad \ge \|(-\mathrm{Id} - M^\varepsilon_{\mathrm{eff}})v'\|_{L^2(\mathbb{R}^n)} - \|\eta_{v'}\|_{L^2(\mathbb{R}^n)} \\
&\quad \ge \|(-\mathrm{Id} - M^\varepsilon_{\mathrm{eff}})v'\|_{L^2(\mathbb{R}^n)} - 2\sin\Big(\frac{\pi-g}{2}\Big)\cdot\|v'\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
\tag{6.1}
$$

To bound $\|(-\mathrm{Id} - M^\varepsilon_{\mathrm{eff}})v'\|_{L^2(\mathbb{R}^n)}$ from below, we will prove the following lemma.

Lemma 6.2. *For any $g\in(g_0,\pi)$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, $\nu_0\in(g_0,\pi]$,*[5] *and any $f\in\mathrm{BL}_\varepsilon$,*

$$
\|(e^{-i\nu_0}-M^\varepsilon_{\mathrm{eff}})f\|_{L^2(\mathbb{R}^n)} \ge 2\sin\Big(\frac{\nu_0-g_0}{2}\Big)\|f\|_{L^2(\mathbb{R}^n)}.
$$

[5] An analogous formula holds if $\nu_0\in[\pi,2\pi-g_0)$.

Let us first use Lemma 6.2 with $\nu_0 = \pi$ to prove the main result, Theorem 4.6, and then return to its proof. Combined with (6.1), we have that

$$\begin{aligned}
\|(M^{\varepsilon} - M^{\varepsilon}_{\text{eff}})v'\|_{L^2(\mathbb{R}^n)} &\geq \cdots \\
&\geq \frac{2}{\text{vol}(\Omega)^{1/2}} \sin\Big(\frac{\pi - g_0}{2}\Big)\|v'\|_{L^2(\mathbb{R}^n)} \\
&\quad - 2\sin\Big(\frac{\pi - g}{2}\Big) \cdot \|v'\|_{L^2(\mathbb{R}^n)} \\
&\geq 2\Big[\frac{1}{\text{vol}(\Omega)^{1/2}} \sin\Big(\frac{\pi - g_0}{2}\Big) - \sin\Big(\frac{\pi - g}{2}\Big)\Big] \cdot \|v'\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, since $v' \in \text{BL}_{\varepsilon}$, Hypothesis 2 regarding the effective dynamics provides an upper bound on $\|(M^{\varepsilon} - M^{\varepsilon}_{\text{eff}})v'\|_{L^2(\mathbb{R}^n)}$. When combined this yields that

$$\begin{aligned}
&2\Big[\frac{1}{\text{vol}(\Omega)^{1/2}} \sin\Big(\frac{\pi - g_0}{2}\Big) - \sin\Big(\frac{\pi - g}{2}\Big)\Big] \cdot \|v'\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq \|(M^{\varepsilon} - M^{\varepsilon}_{\text{eff}})v'\|_{L^2(\mathbb{R}^n)} \leq o(\varepsilon) \cdot \|v'\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Since $\pi > g > g_0$, the difference on the left-hand side above is always positive. Therefore, for sufficiently small $\varepsilon > 0$, the above inequality is only possible if $v' = 0$. ■

Proof of Lemma 6.2 . By the explicit form of $M^{\varepsilon}_{\text{eff}}$ given in Hypothesis 2, we can write for every $f \in \text{BL}_{\varepsilon}$,

$$\begin{aligned}
(e^{-i\nu_0} - M^{\varepsilon}_{\text{eff}})f &= (2\pi)^{-n/2}\varepsilon^{n/2} \int\limits_{|\xi|\leq d_0} e^{-i\xi\cdot\varepsilon\mathbf{x}}[(e^{-i\nu_0}\,\text{Id} - \widehat{M}^{\varepsilon}_{\text{eff}}(\xi))\hat{\alpha}_0(\xi)]^{\top}\Phi(\mathbf{x})\,d\xi \\
&= \varepsilon^{n/2}\gamma(\varepsilon\mathbf{x})^{\top}\Phi(\mathbf{x}),
\end{aligned} \tag{6.2}$$

where

$$\gamma(X) \equiv (2\pi)^{-n/2} \int\limits_{|\xi|\leq d_0} e^{-i\xi\cdot X}[(e^{-i\nu_0}\,\text{Id} - M^{\varepsilon}_{\text{eff}}(\xi))\hat{\alpha}_0(\xi)]\,d\xi$$

Next, we recall the following averaging lemma.

Lemma 6.3. *Let $q \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\text{supp}(\hat{q}) \subseteq B(0,d)$ for some $d > 0$, and let $p \in L^2(\Omega)$ be Λ-periodic. Then, there exists $\varepsilon_0 > 0$ which depends on d, such that for any fixed $0 < \varepsilon < \varepsilon_0$,*

$$\int\limits_{\mathbb{R}^n} p(\mathbf{x})q(\varepsilon\mathbf{x})\,d\mathbf{x} = \frac{\varepsilon^{-n}}{\text{vol}(\Omega)}\Big(\int\limits_{\Omega} p(\mathbf{x})\,d\mathbf{x}\Big)\cdot\Big(\int\limits_{\mathbb{R}^n} q(X)\,dX\Big).$$

For proof, see Appendix B.2. Applying Lemma 6.3 to (6.2) yields, using the orthonormality of $\Phi_b, \dots, \Phi_{b+N-1}$ (for brevity, set $b_\star = 1$ without loss of generality)

$$\begin{aligned}
\|\varepsilon^{n/2}\gamma(\varepsilon\mathbf{x})\Phi(\mathbf{x})\|^2_{L^2(\mathbb{R}^n)} &= \varepsilon^n \int\limits_{\mathbb{R}^n} |\gamma(\varepsilon\mathbf{x})^\top \Phi(\mathbf{x})|^2 \, d\mathbf{x} \\
&= \varepsilon^n \int\limits_{\mathbb{R}^n} \sum_{j,m=1}^{N} \gamma_j(\varepsilon\mathbf{x})\Phi_j(\mathbf{x})\bar{\gamma}_m(\varepsilon\mathbf{x})\bar{\Phi}_m(\mathbf{x}) \, d\mathbf{x} \\
&= \varepsilon^n \frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)} \sum_{j,m=1}^{N} \langle \gamma_m, \gamma_j \rangle_{L^2(\mathbb{R}^N)} \cdot \langle \Phi_m, \Phi_j \rangle_{L^2_{\mathbf{k}_\star}} \\
&= \frac{1}{\mathrm{vol}(\Omega)} \sum_{j=1}^{N} \|\gamma_j\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{\mathrm{vol}(\Omega)} \|\gamma\|^2_{L^2(\mathbb{R}^n;\mathbb{C}^N)},
\end{aligned}$$

where in applying Lemma 6.3, we used the fact that, while the support of the Fourier transform of $\gamma_j\bar{\gamma}_m$ might not be $B(0, d_0)$, it is still compact, since it is included in $B(0, 2d_0)$.

Hence, to prove Lemma 6.2, we need to bound the norm of $\|\gamma\|_{L^2(\mathbb{R}^n;\mathbb{C}^N)}$ from below. We now note that for every $\xi \in \mathbb{R}^n$, the Fourier-transformed monodromy $\widehat{M}^\varepsilon_{\text{eff}}(\xi)$ is an $N \times N$ unitary matrix (where N is the degree of the degeneracy in Hypothesis 1). Let $P(\xi)$ be the unitary matrix which diagonalizes the monodromy, i.e.,

$$\widehat{M}^\varepsilon_{\text{eff}}(\xi) = P(\xi)D(\xi)P^*(\xi), \quad D(\xi)_{\ell,j} = e^{-i\nu_j(\xi)}\delta_{j,\ell}, \quad 1 \le j, \ell \le N.$$

Hence, using Plancherel theorem and the orthogonality of $P(\xi)$, we have that

$$\begin{aligned}
&\|\gamma\|^2_{L^2(\mathbb{R}^n)} \\
&= \left\| (2\pi)^{-n/2} \int\limits_{|\xi| \le d_0} e^{-i\xi\cdot X} [P(\xi)(e^{-i\nu_0}\,\mathrm{Id} - D(\xi))P^*(\xi)\hat{\alpha}_0(\xi)] \, d\xi \right\|^2_{L^2(\mathbb{R}^n_X;\mathbb{C}^N)} \\
&= \|P(\xi)(e^{-i\nu_0}\,\mathrm{Id} - D(\xi))P^*(\xi)\hat{\alpha}_0(\xi)\|^2_{L^2(\mathbb{R}^n_\xi;\mathbb{C}^N)} \\
&= \|(e^{-i\nu_0}\,\mathrm{Id} - D(\xi))P^*(\xi)\hat{\alpha}_0(\xi)\|^2_{L^2(\mathbb{R}^n_\xi;\mathbb{C}^N)} \\
&= \sum_{j=1}^{N} \|(e^{-i\nu_0} - e^{-i\nu_j(\xi)})(P^*(\xi)\hat{\alpha}_0(\xi))_j\|^2_{L^2(\mathbb{R}^n_\xi)} \\
&= \sum_{j=1}^{N} \int\limits_{|\xi| \le d_0} |e^{-i\nu_0} - e^{-i\nu_j(\xi)}|^2 |(P^*(\xi)\hat{\alpha}_0(\xi))_j|^2 \, d\xi
\end{aligned}$$

$$\geq \sum_{j=1}^{N} \min_{|\xi'|\leq d_0} |e^{-i\nu_0} - e^{-i\nu_j(\xi')}|^2 \cdot \|(P^*(\xi)\hat{\alpha}_0(\xi))_j\|^2_{L^2(\mathbb{R}^n_\xi)}$$
$$\geq \min_{|\xi|\leq d_0} \min_{1\leq j\leq N} |e^{-i\nu_0} - e^{-i\nu_j(\xi)T_{\text{per}}}|^2 \cdot \|\alpha_0\|^2_{L^2(\mathbb{R}^n;\mathbb{C}^N)}$$
$$\geq 4\sin^2\Big(\frac{\nu_0 - g_0}{2}\Big) \cdot \|\alpha_0\|^2_{L^2(\mathbb{R}^n;\mathbb{C}^N)},$$

where the last inequality is derived from the arc-length formula between two angles, as well as from Hypothesis 3 on the spectrum of $M^{\varepsilon}_{\text{eff}}$. ∎

6.1. Proof of Proposition 4.7

We note here that the proof of Proposition 4.7 is very similar to that which appears in [42]. However, due to many changes in the notation and change in dimensionality, we include it here for completeness.

6.1.1. From projections to wavepackets; Proof of (4.7). Let $\varepsilon > 0$ be taken sufficiently small, and let $f \in L^2(\mathbb{R}^n)$. Express H^0 acting in $L^2(\mathbb{R}^2)$ as a direct integral $H^0 = \int_{\mathcal{B}}^{\oplus} H^0_k \, d\mathbf{k}$, where $H^0_{\mathbf{k}}$ denotes the operator $H = -\Delta + V$ acting in $L^2_{\mathbf{k}}$. Then, taking $E_b(\mathbf{k}_\star) = 0$ without loss of generality, we can rewrite (4.1)

$$\mathcal{P}^{\varepsilon}_0 = \int_{\mathcal{B}} d\mathbf{k}\, \chi\Big(\frac{|\mathbf{k}-\mathbf{k}_\star|}{\varepsilon} < 1\Big) \operatorname{Proj}_{L^2_{\mathbf{k}}}(|H^0_{\mathbf{k}}| < L\varepsilon) f$$
$$= \int_{\mathcal{B}} d\mathbf{k}\, \chi\Big(\frac{|\mathbf{k}-\mathbf{k}_\star|}{\varepsilon} < 1\Big) \Big[\frac{1}{2\pi i} \oint_{|\zeta|=2L\varepsilon} (\zeta I - H_{\mathbf{k}})^{-1} \, d\zeta\Big] f, \tag{6.3}$$

where the factor 2 in the $2L\varepsilon$ radius in the contour integral is not necessarily sharp. In order to expand for $\mathbf{k}$ near $\mathbf{k}_\star$, we next express the operators $H_{\mathbf{k}}$ in terms of operators which acts in the fixed space $L^2_{\mathbf{k}_\star}$. Note that $H_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{x}} H(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$, where $H(\mathbf{k}) \equiv -(\nabla + i\mathbf{k})^2 + V$ acts in $L^2(\mathbb{R}^n/\Lambda)$. Furthermore, $(\zeta I - H_{\mathbf{k}})^{-1} = e^{i\mathbf{k}\cdot\mathbf{x}}(\zeta I - H(\mathbf{k}))^{-1} e^{-i\mathbf{k}\cdot\mathbf{x}}$.

Substitution into (6.3) yields

$$\cdots = \int_{\mathcal{B}} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \chi\Big(\frac{|\mathbf{k}-\mathbf{k}_\star|}{\varepsilon} < 1\Big) \Big[\frac{1}{2\pi i} \oint_{|\zeta|=2L\varepsilon} (\zeta I - H(\mathbf{k}))^{-1} \, d\zeta\Big] e^{-i\mathbf{k}\cdot\mathbf{x}} f$$
$$= \int_{\mathcal{B}} d\kappa e^{i(\mathbf{k}_\star+\kappa)\cdot\mathbf{x}} \chi\Big(\frac{\kappa}{\varepsilon} < 1\Big) \Big[\frac{1}{2\pi i} \oint_{|\zeta|=2L\varepsilon} (\zeta I - H(\mathbf{k}_\star+\kappa))^{-1} \, d\zeta\Big] e^{-i(\mathbf{k}_\star+\kappa)\cdot\mathbf{x}} f.$$

The contour integral inside the square brackets is smooth $L^2(\mathbb{R}^2/\Lambda)$-valued function of κ, and so by Taylor expansion:

$$\cdots = \int\limits_{\mathcal{B}} d\kappa e^{i(\mathbf{k}_\star+\kappa)\cdot\mathbf{x}}\chi\Big(\frac{|\kappa|}{\varepsilon}<1\Big)\Big[\frac{1}{2\pi i}\oint\limits_{|\zeta|=2L\varepsilon}(\zeta I - H(\mathbf{k}_\star))^{-1}\,d\zeta\Big]e^{-i(\mathbf{k}_\star+\kappa)\cdot\mathbf{x}} f$$
$$+\int\limits_{\mathcal{B}}\chi\Big(\frac{\kappa}{\varepsilon}<1\Big)\kappa\,\mathrm{Error}[f;\kappa]\,d\kappa. \tag{6.4}$$

The last term in (6.4) is linear in f and easily seen to be bounded in $L^2(\mathbb{R}^2)$ by $\varepsilon^{n+1}\|f\|_{L^2}$ since the domain of integration is over a disc of radius ε.

The dominant term in (6.4) may be re-expressed as

$$\begin{aligned}
&\int\limits_{\mathcal{B}} d\kappa\chi\Big(\frac{|\kappa|}{\varepsilon}<1\Big)e^{i\kappa\cdot\mathbf{x}}\Big[\frac{1}{2\pi i}\oint\limits_{|\zeta|=2L\varepsilon}e^{i\mathbf{K}\cdot\mathbf{x}}(\zeta I - H(\mathbf{k}_\star))^{-1}e^{-i\mathbf{K}\cdot\mathbf{x}}\,d\zeta\Big]e^{-i\kappa\cdot\mathbf{x}}f(\mathbf{x})\\
&\quad=\int\limits_{\mathcal{B}} d\kappa\chi\Big(\frac{|\kappa|}{\varepsilon}<1\Big)e^{i\kappa\cdot\mathbf{x}}\Big[\frac{1}{2\pi i}\oint\limits_{|\zeta-E_D|=2L\varepsilon}(\zeta I - H_{\mathbf{k}_\star})^{-1}\,d\zeta\Big]e^{-i\kappa\cdot\mathbf{x}}f(\mathbf{x})\\
&\quad=\int\limits_{\mathcal{B}} d\kappa\chi\Big(\frac{|\kappa|}{\varepsilon}<1\Big)e^{i\kappa\cdot\mathbf{x}}\ \mathrm{Proj}_{L^2_{\mathbf{k}_\star}}(|H_{\mathbf{k}_\star}|<2L\varepsilon)e^{-i\kappa\cdot\mathbf{x}}f(\mathbf{x})\\
&\quad=\int\limits_{\mathcal{B}} d\kappa\chi\Big(\frac{|\kappa|}{\varepsilon}<a\Big)e^{i\kappa\cdot\mathbf{x}}\Phi_\star^\top(\mathbf{x};\mathbf{k}_\star)\Big[\int\limits_{\mathbb{R}^n}d\mathbf{y}\,\overline{\Phi_\star(\mathbf{y};\mathbf{k}_\star)}f(\mathbf{y})e^{-i\kappa\cdot\mathbf{y}}\Big]\\
&\quad=\Phi_\star^\top(\mathbf{x};\mathbf{k}_\star)\int\limits_{\mathbb{R}^n}d\mathbf{y}\,\overline{\Phi_\star(\mathbf{y};\mathbf{k}_\star)}(\mathbf{y})f(\mathbf{y})\Big[\int\limits_{\mathcal{B}}d\kappa\chi\Big(\frac{|\kappa|}{\varepsilon}<1\Big)e^{i\kappa\cdot(\mathbf{x}-\mathbf{y})}\Big]\\
&\quad=\Phi_\star^\top(\mathbf{x};\mathbf{k}_\star)\int\limits_{\mathbb{R}^n}d\mathbf{y}\,\overline{\Phi_\star(\mathbf{y};\mathbf{K})}f(\mathbf{y})\Big[\int\limits_{\mathbb{R}^n}d\kappa\chi\Big(\frac{|\kappa|}{\varepsilon}<1\Big)e^{i\kappa\cdot(\mathbf{x}-\mathbf{y})}\Big].
\end{aligned}$$

In all, we have that

$$\mathcal{P}_0^\varepsilon f = u_\varepsilon[f] + \mathcal{O}_{L^2(\mathbb{R}^n)}(\varepsilon^{n+1}\|f\|_{L^2}), \tag{6.5}$$

where

$$u_\varepsilon[f](\mathbf{x}) \equiv \Phi^\top(\mathbf{x};\mathbf{k}_\star)\beta_\varepsilon[f](\mathbf{x}) \tag{6.6a}$$

and

$$\beta_\varepsilon[f](\mathbf{x}) \equiv \Big[\big(\overline{\Phi_\star(\cdot;\mathbf{k}_\star)}f\big) * \mathcal{F}_\xi^{-1}\Big[\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big)\Big]\Big](\mathbf{x}), \tag{6.6b}$$

where $\mathcal{F}^{-1}[g](\xi)$ denotes the inverse Fourier transform and $*$ denotes convolution.

We next show that $u_\varepsilon \in \mathrm{BL}_\varepsilon$ with $d_0 = 1$ by showing that

$$\mathcal{F}[\beta_\varepsilon[f]] \in \chi(|\xi| < \varepsilon) L^2(\mathbb{R}^n).$$

Indeed, by the convolution rule,

$$\begin{aligned}\mathcal{F}[\beta_\varepsilon[f]](\xi) &= \mathcal{F}\Big[(\overline{\Phi(\cdot;\mathbf{k}_\star)} f) * \mathcal{F}^{-1}\Big[\chi\Big(\frac{|\xi|}{\varepsilon} < 1\Big)\Big]\Big] \\ &= \mathcal{F}[\overline{\Phi_\star(\cdot;\mathbf{k}_\star)} f](\xi)\chi\Big(\frac{|\xi|}{\varepsilon} < 1\Big),\end{aligned}$$

which is supported in $\{|\xi| < \varepsilon a\}$. This completes the proof of (4.7).

Remark 6.4. This proof shows that, more generally, if the definition of $\mathcal{P}_0^\varepsilon$ would have been changed to a projection onto the disc $|\mathbf{k} - \mathbf{k}_\star| < a\varepsilon$ with $a \neq 1$, then the proposition would have carried through with a different value of d_0.

6.1.2. From wavepackets to projections; Proof of (4.8). Consider $u(\mathbf{x}) \in \mathrm{BL}_\varepsilon$ for some $d_0 \in (0, 1)$ and $\varepsilon > 0$ sufficiently small, then by definition of (4.2), there exists $\alpha_\varepsilon \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ such that

$$u(\mathbf{x}) = \Phi_\star^\top(\mathbf{x};\mathbf{k}_\star)\alpha_\varepsilon(\mathbf{x}), \quad \text{where } \mathcal{F}[\alpha_\varepsilon](\xi) = \chi\Big(\frac{|\xi|}{\varepsilon} < d_0\Big)\mathcal{F}[\alpha_\varepsilon](\xi).$$

On the other hand, by (6.5), for any BL_ε function and $\varepsilon > 0$ sufficiently small, there exists a function γ_ε such that

$$\mathcal{P}_0^\varepsilon u = \Phi_\star^\top(\mathbf{x},\mathbf{k}_\star)\gamma_\varepsilon[u](\mathbf{x}) + \mathcal{O}(\varepsilon^{n+1}\|u\|_{L^2(\mathbb{R}^n)}). \tag{6.7}$$

To prove (4.8) it suffices to show that $\gamma_\varepsilon(\mathbf{x}) = \alpha_\varepsilon(\mathbf{x})$. Substitution of

$$u = \Phi_\star^\top(\mathbf{x};\mathbf{k}_\star)\alpha_\varepsilon(\mathbf{x})$$

into (6.6) yields

$$\gamma_\varepsilon[u] = (\overline{\Phi_\star(\cdot;\mathbf{k}_\star)}\Phi_\star^\top(\cdot;\mathbf{k}_\star)\alpha_\varepsilon) * \mathcal{F}^{-1}\Big[\chi\Big(\frac{|\xi|}{\varepsilon} < 1\Big)\Big].$$

We next compute the Fourier transform of $\gamma_\varepsilon[u]$. For $b_\star \le j < b + N$,

$$\begin{aligned}\mathcal{F}[\gamma_\varepsilon[g]]_j &= \mathcal{F}[\overline{\Phi_\star(\mathbf{x};\mathbf{k}_\star)}\Phi_\star^\top(\mathbf{x};\mathbf{k}_\star)\alpha_\varepsilon(\mathbf{x})]_j\,\chi\Big(\frac{|\xi|}{\varepsilon} < 1\Big) \\ &= \sum_{\ell=b}^{N+b-1} \mathcal{F}[\overline{\Phi_j(\mathbf{x};\mathbf{k}_\star)}\Phi_\ell(\mathbf{x};\mathbf{k}_\star)\alpha_{\varepsilon,\ell}(\mathbf{x})]\chi\Big(\frac{|\xi|}{\varepsilon} < 1\Big)\end{aligned} \tag{6.8}$$

Consider the expression being summed in (6.8). Since $\Phi_j(\mathbf{x};\mathbf{k}_\star) = e^{i\mathbf{k}_\star\cdot\mathbf{x}}\phi_j(\mathbf{x};\mathbf{k}_\star)$ with $\phi_j(\mathbf{x};\mathbf{k}_\star) \in L^2(\mathbb{R}^n/\Lambda)$ periodic, we have

$$p_{j,\ell}(\mathbf{x}) \equiv \overline{\Phi_j(\mathbf{x},\mathbf{k}_\star)}\Phi_\ell(\mathbf{x},\mathbf{k}_\star) = \overline{\phi_j(\mathbf{x},\mathbf{k}_\star)}\phi_\ell(\mathbf{x},\mathbf{k}_\star) \in L^2(\mathbb{R}^n/\Lambda).$$

We expand $p_{j,\ell}(\mathbf{x})$ for each $j,\ell = 1,2$ in a Fourier series with respect to the lattice Λ: for every $g \in L^2(\mathbb{R}^n/\Lambda)$,

$$g(\mathbf{x}) = \sum_{\mathbf{n}\in\Lambda^*}\hat{g}(\mathbf{n})e^{i\mathbf{n}\cdot\mathbf{x}}, \quad \hat{g}(\mathbf{n}) \equiv \int_\Omega e^{-i\mathbf{n}\cdot\mathbf{x}}g(\mathbf{x})\,d\mathbf{x}.$$

Substituting the Fourier series into (6.8) yields

$$\begin{aligned}
\mathcal{F}[\gamma_\varepsilon[u]]_j &= \sum_{\ell=n}^{N+b-1}\mathcal{F}\Big[\sum_{\mathbf{n}\in\Lambda^*}\hat{p}_{j,\ell}(\mathbf{n})e^{i\mathbf{n}\cdot x}\alpha_{\varepsilon,\ell}(\mathbf{x})\Big]\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big)\\
&= \sum_{\ell=n}^{N+b-1}\sum_{\mathbf{n}\in\Lambda^*}\hat{p}_{j,\ell}(\mathbf{n})\mathcal{F}[e^{i\mathbf{n}\cdot x}\alpha_{\varepsilon,\ell}(\mathbf{x})]\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big)\\
&= \sum_{\ell=n}^{N+b-1}\sum_{\mathbf{n}\in\Lambda^*}\hat{p}_{j,\ell}(\mathbf{n})\hat{\alpha}_{\varepsilon,\ell}(\xi-\mathbf{n})\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big).
\end{aligned} \tag{6.9}$$

Note that by definition, $\hat{\alpha}_\varepsilon$ has compact support in the disc of radius εa around the origin. In the expansion above, (6.9), for $\varepsilon > 0$ sufficiently small, a term does not vanish if two conditions are met: (i) $|\xi - \mathbf{n}| < \varepsilon$ (where $\mathbf{n} \in \Lambda^*$) and (ii) $|\xi| < \varepsilon$. Hence, the only non-zero term in (6.9) arises from the lattice point $\mathbf{n} = \vec{0}$. Then, by definition of the Fourier coefficient $\hat{p}_{j,\ell}(\vec{0})$ and the orthogonality of the different Φ_j's, we have that

$$\begin{aligned}
\mathcal{F}[\gamma_\varepsilon[u]]_j &= \sum_{\ell=b}^{N+b-1}\hat{p}_{j,\ell}(\vec{0})\hat{\alpha}_{\varepsilon,\ell}(\xi)\cdot\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big)\\
&= \sum_{\ell=n}^{N+b-1}\int_\Omega\overline{\Phi_j(\mathbf{y};\mathbf{k}_\star)}\Phi_l(\mathbf{y};\mathbf{k}_\star)\,d\mathbf{y}\ \hat{\alpha}_{\varepsilon,\ell}(\xi)\cdot\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big)\\
&= \hat{\alpha}_{\varepsilon,j}(\xi)\cdot\chi\Big(\frac{|\xi|}{\varepsilon}<1\Big) = \mathrm{vol}(\Omega)^{-1}\hat{\alpha}_{\varepsilon,j}(\xi).
\end{aligned}$$

Summarizing, we have $\gamma_\varepsilon[u] = \alpha_\varepsilon$ and so substitution into (6.7) yields

$$\mathrm{Proj}_{L^2(\mathbb{R}^2)}(|H^0 - E_\star| \le \varepsilon)u = u(\mathbf{x}) + \mathcal{O}(\varepsilon^{n+1}\|u\|_{L^2(\mathbb{R}^2)}).$$

This is equivalent to (4.8). The proof of Proposition 4.7 is now complete.

7. Effective transport dynamics

Consider (5.1) with $a = 1$ and initial data of the form

$$\psi_0(\mathbf{x}) = \varepsilon^{n/2}\alpha_0(\varepsilon\mathbf{x})\Phi_b(\mathbf{x};\mathbf{k}_\star), \quad \alpha_0 \in H^s(\mathbb{R}^n), \tag{7.1}$$

for sufficiently high $s > 0$, and where the $\varepsilon^{n/2}$ factor keeps the overall norm of ψ_0 independent of ε. Initial data in BL_ε are then a sub-class of (7.1). In this subsection, we formally derive the effective transport equation and its propagator $U^\varepsilon_{\mathrm{eff}}$, as given in (5.3). The proof of its validity follows closely that of [42, Theorem 3.2] and [19, Theorem 5.1], and is therefore omitted from this manuscript.

To construct a solution, we assume separation of scales, with slow time variables

$$T \equiv \varepsilon t, \quad X \equiv \varepsilon \mathbf{x},$$

and introduce the expansion

$$\psi(t,x) = \psi^0(t,x) + \varepsilon\psi^1(t,x) + \cdots$$

where for every $j \geq 0$,

$$\psi^j(t,x) = \Psi^j(t,x,T,X)|_{T=\varepsilon t, X=\varepsilon x}.$$

By expanding

$$\partial_t \mapsto \partial_t + \varepsilon T, \quad \vec{\nabla} \mapsto \vec{\nabla}_{\mathbf{x}} + \varepsilon\vec{\nabla}_X, \quad \Delta \mapsto \Delta_{\mathbf{x}} + 2\varepsilon\vec{\nabla}_{\mathbf{x}}\cdot\vec{\nabla}_X + \Delta_X$$

and substituting into (1.1), we solve for each power of ε.

Order $\boldsymbol{\varepsilon^0}$. We have

$$(i\partial_t - H^0)\Psi^0 = 0, \quad \Psi(t=0, T=0, x, X) = \alpha_0(X)\Phi(\mathbf{x}),$$

and so $\Psi^0(t,T,\mathbf{x},X) = \alpha_0(X)\Phi(\mathbf{x})$.

Order $\boldsymbol{\varepsilon^1}$. We have

$$(i\partial_t - H^0)\Psi^1 = (-i\partial_T + 2\vec{\nabla}_{\mathbf{x}}\cdot\vec{\nabla}_X + 2i\underline{A}(T)\cdot\vec{\nabla}_{\mathbf{x}})\Psi^0.$$

To invert $(i\partial_t - H^0)$ in $L^2_{\mathbf{k}_\star}$ and solve for Ψ^1, we need to verify that the right-hand side is $L^2_{\mathbf{k}}$ orthogonal to the kernel, i.e., to $\Phi = \Phi_b(\cdot;\mathbf{k})$ (from here on, we suppress the $\mathbf{k}_\star$ and b dependence for brevity).

Here, it is useful to note that (5.2) is equivalent to a statement on the Bloch mode $\Phi_b(x;\mathbf{k}_\star)$.

Lemma 7.1. *Given* (5.2), *then*

$$\langle \Phi_b(\cdot;\mathbf{k}_\star), 2\vec{\nabla}\Phi_b(\cdot;\mathbf{k}_\star)\rangle_{L^2_{\mathbf{k}_\star}} = i\mathbf{c} \neq \vec{0}.$$

Combining Lemma 7.1 and normalizing $\langle \Phi, \Phi\rangle_{L^2_{\mathbf{k}_\star}} = 1$, we get the desired result

$$i\partial_T \alpha(T,X) = i\mathbf{c}\cdot(\vec{\nabla}_X + i\underline{A}(T))\alpha.$$

Proof of Lemma 7.1. By definition, Φ satisfies

$$H\Phi(\mathbf{x};\mathbf{k}) = E(\mathbf{k})\Phi, \quad \Phi \in L^2_{\mathbf{k}}.$$

Write

$$\Phi(\mathbf{x};\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{x}} p_{\mathbf{k}}(\mathbf{x}), \quad p_{\mathbf{k}} \in L^2_{\mathrm{per}}(\Omega), \tag{7.2}$$

which transforms the TISE to

$$\widehat{H}(\mathbf{k})p_{\mathbf{k}} = E(\mathbf{k})p_{\mathbf{k}}, \quad \widehat{H}(\mathbf{k}) \equiv H - 2i\mathbf{k}\cdot\vec{\nabla}_x + |\mathbf{k}|^2. \tag{7.3}$$

We now take $\vec{\nabla}_k$ on both sides of (7.3). By noting that

$$\vec{\nabla}_k \widehat{H}(\mathbf{k}) = -2i\vec{\nabla}_x + 2\mathbf{k},$$

we get that

$$\widehat{H}(\mathbf{k})\vec{\nabla}_k p_{\mathbf{k}} - 2i\vec{\nabla}_x p_{\mathbf{k}} + 2\mathbf{k}p_{\mathbf{k}} = \vec{\nabla}_k E(\mathbf{k})p_{\mathbf{k}} + E(\mathbf{k})\vec{\nabla}_k p_{\mathbf{k}}.$$

We rearrange some of the terms and take the inner product from the left with $p_{\mathbf{k}}$

$$\langle p_{\mathbf{k}}, p_{\mathbf{k}}\vec{\nabla}_k E(\mathbf{k})\rangle = \langle p_{\mathbf{k}}, (\widehat{H}(\mathbf{k}) - E(\mathbf{k}))\vec{\nabla}_k p_{\mathbf{k}}\rangle - \langle p_{\mathbf{k}}, 2i\vec{\nabla}_x p_{\mathbf{k}}\rangle + \langle p_{\mathbf{k}}, 2\mathbf{k}p_{\mathbf{k}}\rangle.$$

Here we note that, by definition, $\|\phi(\cdot;\mathbf{k})\|_2 = \|p_{\mathbf{k}}\|_2 = 1$. Moreover, since $\widehat{H}(\mathbf{k}) - E(\mathbf{k})$ is self adjoint, combined with (7.3), the first inner product on the right-hand side vanishes. By differentiating (7.2), we get

$$\begin{aligned}\vec{\nabla}_k E(\mathbf{k}) &= \langle p_{\mathbf{k}}, 2(-i\vec{\nabla}_x + \mathbf{k})p_{\mathbf{k}}\rangle \\ &= \langle \Phi(\mathbf{x};\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}, -2i\vec{\nabla}_x(\Phi(\mathbf{x};\mathbf{k}))e^{-i\mathbf{k}\cdot\mathbf{x}}\rangle \\ &= -\langle \Phi(\cdot;\mathbf{k}), 2i\vec{\nabla}_x\Phi(\cdot;\mathbf{k})\rangle.\end{aligned}$$

■

A. Physical interpretations of the model

An example of physical interest is the case of (5.1), i.e., (1.1) with $W(T, -i\nabla) = -2i\underline{A}(T)\cdot\nabla$. Note here that (5.1) can be transformed to an equivalent "magnetic" form

$$i\partial_t\psi = (i\vec{\nabla} + \varepsilon^a\underline{A}(\varepsilon^a t))^2\psi + V\psi,$$

where $\underline{A}$ is a vector potential. This class of PDEs arises in physical settings, such as those described below.

(a) The modelling of time periodic conductors (e.g., graphene), excited by a time-varying electric field [38,45]. Here, $H^0 = -\Delta + V$ is a single-electron Hamiltonian for graphene and the time dependence in $H^\varepsilon(t)$ models the excitation of the graphene sheet by an external electric field with no magnetic field (by Maxwell's equations, since $\underline{A}$ is constant in space, see e.g., [31]).

(b) For $n = 1, 2$, the propagation of light in a periodic array of helically coiled optical fiber waveguides [6, 26, 37, 40]. Here, the Schrödinger equation describes the propagation in the time-like longitudinal direction of a continuous-wave (CW) laser beam propagating through a hexagonal or triangular transverse array of optical fiber waveguides. Beginning with Maxwell's equations, under the nearly monochromatic and paraxial approximations, one obtains $i\psi_z(z, \mathbf{x}) = H^0\psi$ for the longitudinal evolution of the slowly varying envelope of the classical electric field. Suppose the fibers are longitudinally coiled. Then, in a rotating coordinate frame, we obtain (1.1) where the time-periodic perturbation, $\underline{A}$, captures effect of periodic coiling.

B. Auxilary proofs

B.1. Proof of Lemma 4.5

It suffices to prove that BL_ε is a closed subspace of $L^2(\mathbb{R}^n)$. Let $(\alpha_m^\top(\varepsilon\cdot)\Phi)_{m=1}^\infty \subset L^2(\mathbb{R}^n)$ be a sequence converging to some $u \in L^2(\mathbb{R}^n)$, and then let us show that $u \in \mathrm{BL}_\varepsilon$, too. To do so, we first prove that $(\alpha_m)_{m=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R}^n; \mathbb{C}^N)$.[6] This is a consequence of the averaging lemma, Lemma 6.3. Indeed, choose for any $1 \le j, l, \le N$, and every two indexes $m_1, m_2 \ge 1$,

$$q_{j,l}(X) = \overline{(\alpha_{j,m_1} - \alpha_{j,m_2})(X)}(\alpha_{l,m_1} - \alpha_{l,m_2})(X), \quad p_{j,l}(\mathbf{x}) = \overline{\Phi_j(\mathbf{x})}\Phi_l(\mathbf{x}).$$

[6] Recall that $N \ge 1$ is the degree of the degeneracy at $(\mathbf{k}_\star, E_\star)$, see Hypothesis 1.

Since $\{\Phi_1, \ldots, \Phi_N\}$ is an orthonormal set in $L^2(\Omega)$, for all $\varepsilon > 0$ sufficiently small, and every two indexes m_1, m_2,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} |(\alpha_{m_1}(\varepsilon\mathbf{x}) - \alpha_{m_2}(\varepsilon\mathbf{x}))^\top \Phi(\mathbf{x})|^2 \, d\mathbf{x} \\
&\quad = \frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)} \sum_{j,l=1}^{N} \Big(\int\limits_{\Omega} p_{j,l}(\mathbf{x}) \, d\mathbf{x} \Big) \cdot \Big(\int\limits_{\mathbb{R}^n} q_{j,l}(X) \, dX \Big) \\
&\quad = \frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)} \sum_{j,l=1}^{N} \delta_{j,l} \cdot \Big(\int\limits_{\mathbb{R}^n} q_{j,l}(X) \, dX \Big) \\
&\quad = \frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)} \|\alpha_{m_1} - \alpha_{m_2}\|_2^2.
\end{aligned}$$

Since the left-hand side tends to zero as m_1 and m_2 tend to infinity, so does the right-hand side. Therefore, $(\alpha_m)_{m=1}^{\infty}$ is a Cauchy sequence and converges in $L^2(\mathbb{R}^n; \mathbb{C}^N)$ to some $\alpha_\star$. Hence, the sequence $\{\varepsilon\alpha_m^\top(\varepsilon\cdot)\Phi\}$ converges to $\varepsilon\alpha_\star^\top(\varepsilon\cdot)\Phi$.

Finally, while the limit does have the correct form, for u to be in BL_ε we still need to show that $\mathrm{supp}(\hat{\alpha}_\star) \subset \{|\xi| \leq d_0\}$ to complete the proof. Indeed, since $(\alpha_m)_{m=1}^{\infty}$ converges in L^2, by the Plancherel identity. Since all $\hat{\alpha}_m$ are complactly supported on the same compact set, it mus be that the sequence $\hat{\alpha}_m(\xi)$ converges almost everywhere in ξ, up to a subsequence. Furthermore, for all m we have $\hat{\alpha}_m(\xi) \equiv 0$ for $|\xi| > d_0$, so we conclude $\hat{\alpha}_\star(\xi) \equiv 0$ for $|\xi| > d_0$.

B.2. Proof of Lemma 6.3

Since the fundamental cell of the lattice Ω tiles the plane, we partition $\mathbb{R}^n$ with respect to the lattice, i.e., $\mathbb{R}^n = \bigcup_{\mathbf{m}\in\Lambda}(\Omega + \mathbf{m})$. Therefore,

$$\begin{aligned}
\int\limits_{\mathbb{R}^2} p(\mathbf{x})q(\varepsilon\mathbf{x}) \, d\mathbf{x} &= \sum_{\mathbf{m}\in\Lambda} \int\limits_{\Omega+\mathbf{m}} p(\mathbf{x})q(\varepsilon\mathbf{x}) \, d\mathbf{x} \quad \text{(change of variables } \mathbf{x} = \mathbf{y} + \mathbf{m}) \\
&= \sum_{\mathbf{m}\in\Lambda} \int\limits_{\Omega} p(\mathbf{y})q(\varepsilon(\mathbf{y} + \mathbf{m})) \, d\mathbf{y} \\
&= \int\limits_{\Omega} p(\mathbf{y}) \Big[\sum_{\mathbf{m}\in\Lambda} q(\varepsilon(\mathbf{y} + \mathbf{m})) \Big] \, d\mathbf{y}.
\end{aligned} \tag{B.1}$$

Using a generalization of Poisson summation formula for general lattices [43], then

$$\sum_{\mathbf{m}\in\Lambda} q(\mathbf{y} + \mathbf{m}) = \frac{1}{\mathrm{vol}(\Omega)} \sum_{\mathbf{n}\in\Lambda^*} \hat{q}(\mathbf{n}) e^{2\pi i \mathbf{y}\cdot\mathbf{n}},$$

where $\hat{q}$ is the Fourier transform of q, and Λ^* is the dual lattice to Λ. Since in $\mathbb{R}^n$ we have that $\widehat{q(\varepsilon\cdot)}(\xi) = \varepsilon^{-n}\hat{q}(\varepsilon^{-1}\xi)$, then

$$\sum_{\mathbf{m}\in\Lambda} q(\varepsilon(\mathbf{y}+\mathbf{m})) = \frac{1}{\mathrm{vol}(\Omega)}\sum_{\mathbf{n}\in\Lambda^*}\varepsilon^{-n}\hat{q}\Big(\frac{\mathbf{n}}{\varepsilon}\Big)e^{2\pi i\mathbf{y}\cdot\mathbf{n}}.$$

Now, since q is band-limited, then for sufficiently small ε all of the terms in the last sum vanish but $\mathbf{n} = (0,\dots,0)$. Therefore, and since q is integrable,

$$\sum_{\mathbf{m}\in\Lambda} q(\varepsilon(\mathbf{y}+\mathbf{m})) = \frac{1}{\mathrm{vol}(\Omega)}\varepsilon^{-n}\hat{q}(0) = \frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)}\int_{\mathbb{R}^n} q(\mathbf{x})\,d\mathbf{x},$$

which when substituted into (B.1), yields

$$\begin{aligned}\int_{\mathbb{R}^2} p(\mathbf{x})q(\varepsilon\mathbf{x})\,d\mathbf{x} = \dots &= \int_{\Omega} p(\mathbf{y})\Big[\frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)}\int_{\mathbb{R}^n} q(\mathbf{x})\,d\mathbf{x}\Big]\,d\mathbf{y}\\ &= \frac{\varepsilon^{-n}}{\mathrm{vol}(\Omega)}\Big(\int_{\Omega} p(\mathbf{y})\,d\mathbf{y}\Big)\cdot\Big(\int_{\mathbb{R}^n} q(\mathbf{x})\,d\mathbf{x}\Big).\end{aligned}$$

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