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K-theory invariance of L^p-operator algebras associated with étale groupoids of strong subexponential growth

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Abstract. We introduce the notion of (strong) subexponential growth for étale groupoids and study its basic properties. In particular, we show that the K-groups of the associated reduced groupoid L^p-operator algebras are independent of $p \in [1, \infty)$ whenever the groupoid has strong subexponential growth. Several examples are discussed. Most significantly, we apply classical tools from analytic number theory to exhibit an example of an étale groupoid associated with a shift of infinite type which has strong subexponential growth, but not polynomial.

1. Introduction

For $p \in [1, \infty)$, an L^p-operator algebra is a Banach algebra which admits an isometric representation on some L^p-space. Such Banach algebras were first considered by Herz in [24] where he studied the L^p-operator algebra generated by the left regular representation of a locally compact group. In the 2010s, Phillips initiated their study once more, leading a program aimed at generalizing the modern theory of C*-algebras to L^p-operator algebras, and they have since seen a growing amount of interest among operator algebraists, see for example [8–10, 12, 14, 15, 17, 18, 41]. Both historically and currently, L^p-operator algebras are studied mostly through examples, but there have been attempts to establish a more general theory (see [5, 16, 19]). However, as of yet, there is no abstract characterization of such Banach algebras like there is for C*-algebras, and lacking the richness of C*-theory, their study often require different techniques from that of C*-algebras.

Given a combinatorial or dynamical object, one can in many cases associate an L^p-operator algebra in a way that reflects the combinatorial or dynamical structure of the object, which is a source of interesting examples. Such an object can for example be a graph, a group acting on a locally compact Hausdorff space, or a locally compact Hausdorff groupoid. Given such an object and associated L^p-operator algebras, for $p \in [1, \infty)$, it is a natural problem to investigate the extent to which the structure of the Banach algebras differ for various exponents $p \in [1, \infty)$. For instance, natural questions are whether the properties of simplicity or monotraciality are shared by some or all $p \in [1, \infty)$, variants of which have been studied in [3, 4, 23, 39, 40]. One crucial aspect in the theory of L^p-operator algebras where the value of the exponent plays a significant role is in the

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so-called rigidity problem for étale groupoids. In the case where $p = 2$, there exist non-isomorphic étale groupoids $\mathcal{G}$ and $\mathcal{G}'$ (which are not even continuously orbit equivalent), yet their reduced C*-algebras are isomorphic, that is $C_r^*(\mathcal{G}) \cong C_r^*(\mathcal{G}')$. However, this phenomenon does not occur for the reduced L^p-operator algebras for $p \neq 2$ (see [9, 20, 25]). This is due to the rigidity of the homotopy classes of invertible isometries on L^p-spaces, as established by Lamperti's theorem.

Another natural question in this spirit is whether or not their K-theory differs. Phillips computed the K-groups of the L^p-Cuntz algebras in [38], finding that the K-groups are independent of the exponent $p \in [1, \infty)$. Further, Liao and Yu obtained a similar result in [29], to the effect that for a fairly large class of groups, the K-groups of the reduced group L^p-operator algebras are independent of the exponent $p \in [1, \infty)$, and in [47] Wang and Wang showed similarly that the K-groups of the L^p-Toeplitz algebras are independent of the exponent $p \in (1, \infty)$. Inspired by the observation that all of these L^p-operator algebras have groupoid models, the authors attempted in [2] to provide a unifying result for all of the aforementioned cases. The attempt was partially successful in that it was proved that for an étale groupoid which has polynomial growth with respect to a continuous length function, the K-groups are indeed independent of the exponent, see [2, Theorem 4.7]. This result includes that of Wang and Wang as a special case, as well as Liao and Yu's result when restricting to discrete groups of polynomial growth. However, it fails to include Phillips' result for the L^p-Cuntz algebras as their groupoid models have exponential growth (see Example 3.3). The purpose of the present paper is to extend [2, Theorem 4.7] to cover a larger class of étale groupoids, namely those of strong subexponential growth (see Definition 3.1). Our first main result is the following. Here $F_\lambda^p(\mathcal{G})$ is the reduced L^p-operator algebra of $\mathcal{G}$ (see Section 2).

Theorem A (cf. Theorem 4.13). *If $\mathcal{G}$ is an étale groupoid which has strong subexponential growth with respect to a locally bounded length function, then $K_*(F_\lambda^p(\mathcal{G}))$, for $* = 0, 1$, is independent of $p \in [1, \infty)$.*

The above result extends [2, Theorem 4.7] in several ways. First of all, étale groupoids with polynomial growth all have strong subexponential growth, but the converse is not true. It therefore covers a strictly larger class of examples. Secondly, it does not require the length function to be continuous, only locally bounded. Finally, and more subtly, the case $p = 1$ is included in this result. It was out of necessity left out of [2, Theorem 4.7] due to its proof relying in part on duality arguments, an issue we remedy in this article by employing a novel approach using interpolation techniques from [36, 43] to construct spectrally invariant subalgebras of the reduced L^p-operator algebras. Note that the construction of these algebras is new even in the C*-algebra setting. By opting to use interpolation techniques, the proof of Theorem A is also both simpler and shorter than the proof of [2, Theorem 4.7].

As mentioned, every étale groupoid with polynomial growth will have strong subexponential growth, and it can be observed that this class containment is strict. Indeed, this is even the case for discrete groups since it is well known that the Grigorchuck group is an example of a group which has strong subexponential growth but not polynomial. Also, any action of the Grigorchuck group on a locally compact Hausdorff space yields an étale groupoid with the same growth properties. To obtain an example not related to groups, there are many metric spaces with intermediate growth arising from graph theory (see for example [6, 26, 28, 33]), and then by Corollary 5.4 the associated coarse groupoid has strong subexponential growth, but not polynomial. Establishing that the Grigorchuck group has intermediate growth solved a big open question of whether or not there exist groups of intermediate growth. Using classical tools from analytic number theory as well as recent results by Brix, Hume and Li in [7], we are able to provide an example of an étale groupoid with intermediate growth by comparatively much simpler means. The groupoid is a Deaconu–Renault-type groupoid associated with a certain shift that we call the *ordered prime shift* (see Section 5.3). This leads us to our second main result.

Theorem B (cf. Theorem 5.9). *Let $\mathcal{G}_X$ denote the Deaconu–Renault groupoid associated with the ordered prime shift space X, and let $\widehat{\mathcal{G}}_X$ denote its cover groupoid. Then $\widehat{\mathcal{G}}_X$ is an étale groupoid that has strong subexponential growth, but not polynomial growth.*

The paper is structured as follows. In Section 2, we recall some definitions and results regarding étale groupoids and their associated reduced L^p-operator algebras. In Section 3 we introduce the notion of (strong) subexponential growth for étale groupoids and prove some properties such groupoids necessarily must possess. Section 4 covers the proof of our main result Theorem A, but we also prove a similar result for the symmetrized versions of the reduced groupoid L^p-operator algebras therein. Lastly, in Section 5 we exhibit several examples of étale groupoids with strong subexponential growth, the most significant of which is the example of the ordered prime shift in Section 5.3.

2. Preliminaries

We recall some basic terminology and results regarding étale groupoids and their reduced L^p-operator algebras. The reader is referred to [9, 42] for details.

Let $\mathcal{G}$ be a locally compact groupoid with unit space $\mathcal{G}^{(0)}$, composable pairs $\mathcal{G}^{(2)}$, and range and source maps $r, s\colon \mathcal{G} \to \mathcal{G}$ given by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. The groupoid $\mathcal{G}$ is called *étale* if the range and source maps are local homeomorphisms. Every groupoid in this article will be assumed to be Hausdorff. For any $X \subseteq \mathcal{G}^{(0)}$, we denote by $\mathcal{G}_X = \{x \in \mathcal{G}\colon s(x) \in X\}$, $\mathcal{G}^X = \{x \in \mathcal{G}\colon r(x) \in X\}$, and $\mathcal{G}(X) = \mathcal{G}_X \cap \mathcal{G}^X$. We shall write $\mathcal{G}_u$ and $\mathcal{G}^u$ instead of $\mathcal{G}_{\{u\}}$ and $\mathcal{G}^{\{u\}}$, whenever $u \in \mathcal{G}^{(0)}$ is a unit. When $\mathcal{G}$ is étale, the fibers $\mathcal{G}_u$ and $\mathcal{G}^u$, for $u \in \mathcal{G}^{(0)}$, are discrete. The *isotropy group* at a unit $u \in \mathcal{G}^{(0)}$ is the group $\mathcal{G}(u) := \mathcal{G}_u \cap \mathcal{G}^u$, and the *isotropy bundle* is the set $\mathrm{Iso}(\mathcal{G}) = \bigsqcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}(u)$. If $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, then $\mathcal{G}$ is said to be *effective*. Given subsets $A, B \subseteq \mathcal{G}$, their *product* is the set $AB = \{ab \mid (a,b) \in \mathcal{G}^{(2)},\ a \in A,\ b \in B\}$.

A *length function* on a groupoid $\mathcal{G}$ is a map $\ell\colon \mathcal{G} \to \mathbb{R}_{\geq 0}$ such that $\ell(u) = 0$, for all $u \in \mathcal{G}^{(0)}$, $\ell(x^{-1}) = \ell(x)$, for all $x \in \mathcal{G}$, and $\ell(xy) \leq \ell(x) + \ell(y)$, whenever $(x, y) \in \mathcal{G}^{(2)}$. We say a length function ℓ is *locally bounded* if it is bounded on compact sets. Two natural examples of length functions on étale groupoids are the following: First, suppose Γ is a discrete group with a length function $\ell_\Gamma \colon \Gamma \to \mathbf{R}_{\geq 0}$, and suppose that Γ acts on a locally compact space X. Then it is easy to see that $\ell(\gamma, x) = \ell_\Gamma(\gamma)$ is a locally bounded length function on the transformation groupoid $\Gamma \ltimes X$. Second, if $\mathcal{G}$ is an étale groupoid which is compactly generated in the sense that there is a compact set $S \subseteq \mathcal{G}$ such that $S = S^{-1}$ and $\mathcal{G} = \bigcup_{k=1}^\infty S^k$, then the function ℓ_S given by $\ell_S(u) := 0$, for all units $u \in \mathcal{G}^{(0)}$, and for $x \in \mathcal{G} \setminus \mathcal{G}^{(0)}$, $\ell_S(x) = \inf\{k \mid x \in S^k\}$, is a length function on $\mathcal{G}$. ℓ_S is locally bounded if for example also $\mathcal{G} = \bigcup_{k=1}^\infty (S^k)^\circ$.

Now let $C_c(\mathcal{G})$ denote the space of compactly supported continuous functions on $\mathcal{G}$. We endow $C_c(\mathcal{G})$ with the convolution product, which for $f, g \in C_c(\mathcal{G})$ is given by

$$f * g(x) = \sum_{y \in \mathcal{G}_{s(x)}} f(xy^{-1})g(y) = \sum_{y \in \mathcal{G}^{r(x)}} f(y)g(y^{-1}x),$$

for $x \in \mathcal{G}$, and involution given by

$$f^*(x) = \overline{f(x^{-1})},$$

for $f \in C_c(\mathcal{G})$ and $x \in \mathcal{G}$. The *I-norm* on $C_c(\mathcal{G})$ is given by

$$\|f\|_I = \max\Big\{ \sup_{u \in \mathcal{G}^{(0)}} \sum_{x \in \mathcal{G}_u} |f(x)|, \sup_{u \in \mathcal{G}^{(0)}} \sum_{x \in \mathcal{G}^u} |f(x)| \Big\}.$$

With the above norm and algebraic operations, $(C_c(\mathcal{G}), *, ^*, \|\cdot\|_I)$ is a normed *-algebra. Let $p \in [1, \infty)$ and fix any unit $u \in \mathcal{G}^{(0)}$. The operator $\lambda_u(f) \in B(\ell^p(\mathcal{G}_u))$ associated with $f \in C_c(\mathcal{G})$, is the operator given by

$$\lambda_u(f)(\xi)(x) = \sum_{y \in \mathcal{G}_u} f(xy^{-1})\xi(y),$$

for $x \in \mathcal{G}_u$ and $\xi \in \ell^p(\mathcal{G}_u)$. The map $\lambda_u \colon C_c(\mathcal{G}) \to B(\ell^p(\mathcal{G}_u))$ is an I-norm contractive homomorphism of $C_c(\mathcal{G})$, and is called the *left regular representation* at u. The *reduced groupoid L^p-operator algebra* associated with $\mathcal{G}$ is denoted $F^p_\lambda(\mathcal{G})$ and is the completion of $C_c(\mathcal{G})$ under the norm

$$\|f\|_{F^p_\lambda} := \sup_{u \in \mathcal{G}^{(0)}} \|\lambda_u(f)\|.$$

By [9, Lemma 4.5] this norm satisfies the following, for any $f \in C_c(\mathcal{G})$,

$$\|f\|_\infty \leq \|f\|_{F^p_\lambda} \leq \|f\|_I.$$

Since $\bigoplus_{u \in \mathcal{G}^{(0)}} \lambda_u$ is an isometric representation of $F^p_\lambda(\mathcal{G})$ on the L^p-space $\bigoplus_{u \in \mathcal{G}^{(0)}} \ell^p(\mathcal{G}_u)$, $F^p_\lambda(\mathcal{G})$ is an L^p-operator algebra. It is unital if and only if $\mathcal{G}^{(0)}$ is compact, in which case the indicator function of the unit space is the identity element. The map $j_p \colon F^p_\lambda(\mathcal{G}) \to C_0(\mathcal{G})$ given by

$$j_p(a)(x) = \lambda_{s(x)}(a)(\delta_{s(x)})(x),$$

for $a \in F^p_\lambda(\mathcal{G})$ and $x \in \mathcal{G}$, is contractive, linear, injective and extends the identity map on $C_c(\mathcal{G})$. We shall refer to this map as *Renault's p-j-map*. Given $a, b \in F^p_\lambda(\mathcal{G})$ and $x \in \mathcal{G}$, we have that $j_p(ab)(x) = j_p(a) * j_p(b)(x)$, where the sum defining $j_p(a) * j_p(b)(x)$ is absolutely convergent (see [9, Propositions 4.7 and 4.9]).

3. Growth of étale groupoids

Let $\ell\colon \mathcal{G} \to [0, \infty)$ be a locally bounded length function on an étale groupoid $\mathcal{G}$. Given $t \geq 0$, we define

$$B_{\mathcal{G}_u}(t) := \{x \in \mathcal{G}_u \mid \ell(x) \leq t\},$$

and define $B_{\mathcal{G}^u}(t)$ analogously. Since ℓ is symmetric, $|B_{\mathcal{G}_u}(t)| = |B_{\mathcal{G}^u}(t)|$, for every $t \geq 0$.

Definition 3.1. Let $\mathcal{G}$ be an étale groupoid endowed with a locally bounded length function ℓ. We say that $\mathcal{G}$ has *polynomial growth* with respect to ℓ if there exist constants $C > 0$ and $d \in \mathbb{N}$ such that

$$\sup_{u \in \mathcal{G}^{(0)}} \left|B_{\mathcal{G}_u}(t)\right| \leq C(1+t)^d,$$

for all $t \geq 0$; *strong subexponential growth* with respect to ℓ if there exist $\alpha > 0$, $0 < \beta < 1$ and a positive constant $C > 0$ such that

$$\sup_{u \in \mathcal{G}^{(0)}} \left|B_{\mathcal{G}_u}(t)\right| \leq C \exp(\alpha t^\beta),$$

for each $t \geq 0$; *subexponential growth* with respect to ℓ if

$$\limsup_{t \to \infty} \sup_{u \in \mathcal{G}^{(0)}} \left|B_{\mathcal{G}_u}(t)\right|^{1/t} = 1.$$

If $\mathcal{G}$ does not have subexponential growth with respect to ℓ, then we say $\mathcal{G}$ has *exponential growth* with respect to ℓ.

Clearly we have the following relationship between the above definitions:

$$\text{polynomial growth} \implies \text{strong subexponential growth} \implies \text{subexponential growth.}$$

In [2] the authors exhibited several examples of étale groupoids with polynomial growth, for example AF-groupoids, certain point set groupoids, coarse groupoids associated with uniformly locally finite metric spaces with polynomial growth, and Deaconu–Renault groupoids associated with certain finite directed graphs. To obtain an example of an étale groupoid with strong subexponential growth which does not have polynomial growth, one can simply take the transformation groupoid formed from any action of the Grigorchuck group on a locally compact space. Equipped with the natural length function described in Section 2, the growth of the transformation groupoid is the same as that of the Grigorchuck group, which has strong subexponential growth (see [21, Theorem B (1)]). Also, in Section 5.3, we shall exhibit an example of an étale groupoid associated with a shift of infinite type that has strong subexponential growth, but not polynomial growth.

Following [32, Definition 5.4 and Proposition 5.5] we say that an étale groupoid $\mathcal{G}$ is *fiberwise amenable* if for any compact $K \subseteq \mathcal{G}$ and $\varepsilon > 0$, there exists $F \subseteq \mathcal{G}$ finite such that

$$|KF|/|F| \leq 1 + \varepsilon.$$

Fiberwise amenability was introduced by Ma and Wu in [32]. For certain classes of groupoids, it is a strengthening of the notion of amenability for groupoids. Indeed, in the case of transformation groupoids, it is equivalent to the acting group being amenable, while in the case of coarse groupoids associated with metric spaces, it is equivalent to metric amenability of the underlying metric space. In general, however, there is no implication between the two notions.

Proposition 3.2. *If $\mathcal{G}$ is an étale groupoid with subexponential growth with respect to a locally bounded length function, then $\mathcal{G}$ is fiberwise amenable.*

Proof. Let $\ell\colon \mathcal{G} \to \mathbb{R}_{\geq 0}$ be the length function, and let $K \subseteq \mathcal{G}$ be compact. Then $\ell(K) \subseteq [0, M]$ for some $M \in \mathbb{N}$. Fix any unit $u \in \mathcal{G}^{(0)}$ and let $\varepsilon > 0$ be given; then we may find $k \in \mathbb{N}_0$ such that

$$\left|B_{\mathcal{G}_u}(k+M)\right| / \left|B_{\mathcal{G}_u}(k)\right| \leq 1 + \varepsilon.$$

Indeed, if this were not the case, then for every $k \in \mathbb{N}_0$, we would have

$$\left|B_{\mathcal{G}_u}(k+M)\right| > (1+\varepsilon)\left|B_{\mathcal{G}_u}(k)\right|.$$

In particular, by induction,

$$\left|B_{\mathcal{G}_u}(kM)\right| > (1+\varepsilon)^k,$$

for every $k \in \mathbb{N}$, from which we get the contradiction that

$$\limsup_{k\to\infty} \sup_{u \in \mathcal{G}^{(0)}} \left|B_{\mathcal{G}_u}(k)\right|^{1/k} \geq \limsup_{k\to\infty} \left|B_{\mathcal{G}_u}(k)\right|^{1/k} > 1.$$

Also, since $\mathcal{G}$ has subexponential growth, $|B_{\mathcal{G}_u}(k)| < \infty$, and combining this with the observation

$$\left|KB_{\mathcal{G}_u}(k)\right| / \left|B_{\mathcal{G}_u}(k)\right| \leq \left|B_{\mathcal{G}_u}(k+M)\right| / \left|B_{\mathcal{G}_u}(k)\right| \leq 1 + \varepsilon,$$

we see that $\mathcal{G}$ is indeed fiberwise amenable. ■

Fiberwise amenability has connections with soficity of the topological full groups (see [31, Section 7]), and, more importantly to us, for σ-compact groupoids with compact unit space, it implies the existence of invariant probability measures on the unit space (see [32, Proposition 5.9]).

Example 3.3. Suppose $\mathcal{G}$ is a locally compact, σ-compact, Hausdorff, étale, minimal and effective groupoid with compact unit space. Then $C_r^*(\mathcal{G})$ is a simple and unital C*-algebra. If $\mathcal{G}$ has subexponential growth, then by Proposition 3.2, $\mathcal{G}$ is fiberwise amenable, and hence by [32, Proposition 5.9] there exists a $\mathcal{G}$-invariant Borel probability measure μ on $\mathcal{G}^{(0)}$. Then $\tau := \mu \circ E$, where $E : C_r^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$ is the canonical conditional expectation, is a faithful trace. If $C_r^*(\mathcal{G})$ is purely infinite, however, then it has no faithful trace. In particular, a σ-compact étale groupoid $\mathcal{G}$ which is minimal, effective, with compact unit space and for which $C_r^*(\mathcal{G})$ is purely infinite must have exponential growth with respect to any locally bounded length function.

4. Applications to K-theory of groupoid L^p-operator algebras

Our aim for this section is to show that the K-groups $K_*(F_\lambda^p(\mathcal{G}))$ are independent of $p \in [1,\infty)$, for $* = 0, 1$, whenever $\mathcal{G}$ is an étale groupoid which has strong subexponential growth with respect to a locally bounded length function.

Throughout, $\mathcal{G}$ will always denote a fixed étale groupoid endowed with a fixed locally bounded length function $\ell\colon \mathcal{G} \to \mathbb{R}_{\geq 0}$ and we fix some $p \in [1,\infty)$. Given $\alpha, \alpha' > 0$ and $\beta, \beta' \in (0,1)$, let us say $(\alpha,\beta) \geq (\alpha',\beta')$ if $\alpha \geq \alpha'$ and $\beta \geq \beta'$. For ease of notation, whenever we write a pair (α,β) we shall always implicitly assume that $\alpha > 0$ and $\beta \in (0,1)$. Notice that if the condition in the definition of strong subexponential growth is satisfied for some pair (α',β'), then it is satisfied for all $(\alpha,\beta) \geq (\alpha',\beta')$.

For each pair (α,β), let $\omega_{\alpha,\beta}\colon \mathcal{G} \to \mathbb{R}_{\geq 0}$ be the function given by

$$\omega_{\alpha,\beta}(x) := \exp\big(\alpha \ell(x)^\beta\big).$$

Then $\omega_{\alpha,\beta}$ is a submultiplicative weight on $\mathcal{G}$, meaning that $\omega_{\alpha,\beta}(u) = 1$ for each $u \in \mathcal{G}^{(0)}$, $\omega_{\alpha,\beta}(x^{-1}) = \omega_{\alpha,\beta}(x)$, and $\omega_{\alpha,\beta}(xy) \leq \omega_{\alpha,\beta}(x)\omega_{\alpha,\beta}(y)$, whenever $(x,y) \in \mathcal{G}^{(2)}$.

For any complex-valued function f on $\mathcal{G}$ and $q \in [1,\infty)$ we define

$$\|f\|_q := \max\left\{ \sup_{u\in\mathcal{G}^{(0)}} \Big\{ \sum_{x\in\mathcal{G}_u} |f(x)|^q \Big\}^{1/q}, \sup_{u\in\mathcal{G}^{(0)}} \Big\{ \sum_{x\in\mathcal{G}^u} |f(x)|^q \Big\}^{1/q} \right\},$$

while as usual, $\|f\|_\infty = \sup_{x\in\mathcal{G}} |f(x)|$. We let $\ell^q(\mathcal{G}) := \{f\colon \mathcal{G} \to \mathbb{C} \mid \|f\|_q < \infty\}$, for $q \in [1,\infty]$. The next lemma will be used frequently throughout.

Lemma 4.1. *If $\mathcal{G}$ has strong subexponential growth with respect to ℓ with constants (α_0,β_0) and $C > 0$ as in Definition 3.1, then for every $(\alpha,\beta) \geq (\alpha_0,\beta_0)$ with $\alpha > \alpha_0$, one has that*

$$\sup_{u\in\mathcal{G}^{(0)}} \sum_{x\in\mathcal{G}_u} \exp\big(-\alpha\ell(x)^\beta\big) < \infty.$$

Proof. Fix any $u \in \mathcal{G}^{(0)}$. We compute

$$\begin{aligned}\sum_{x\in\mathcal{G}_u} \exp\big(-\alpha\ell(x)^\beta\big) &= \sum_{k=0}^\infty \sum_{\substack{x\in\mathcal{G}_u\\ k\leq\ell(x)\leq k+1}} \exp\big(-\alpha\ell(x)^\beta\big)\\ &\leq \sum_{k=0}^\infty \big|B_{\mathcal{G}_u}(k+1)\big| \exp(-\alpha k^\beta)\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} C \exp\big(\alpha_0 (k+1)^{\beta_0}\big) \exp(-\alpha k^{\beta})$$
$$\leq \exp(\alpha_0) C \sum_{k=0}^{\infty} \exp\big(-(\alpha-\alpha_0)k^{\beta}\big) < \infty,$$

independently of $u \in \mathcal{G}^{(0)}$, since $\alpha > \alpha_0$. Note that in the last transition we used that $(k+1)^{\beta_0} \leq k^{\beta_0} + 1$ as $0 < \beta_0 < 1$. The result follows. ■

Given $\alpha > 0$ and $\beta \in (0,1)$, let $\|\cdot\|_{\alpha,\beta}$ denote the norm on $C_c(\mathcal{G})$ given by

$$\|f\|_{\alpha,\beta} := \|f\omega_{\alpha,\beta}\|_p,$$

and let $S^p_{\ell,(\alpha,\beta)}(\mathcal{G}) = \overline{C_c(\mathcal{G})}^{\|\cdot\|_{\alpha,\beta}}$ be the Banach space obtained by completing $C_c(\mathcal{G})$ in this norm.

Since we have the norm inequality $\|\cdot\|_\infty \leq \|\cdot\|_{\alpha,\beta}$ on $C_c(\mathcal{G})$, it follows by a similar argument to [2, Lemma 3.4] that $S^p_{\ell,(\alpha,\beta)}(\mathcal{G}) \subseteq C_0(\mathcal{G})$.

Proposition 4.2. *If $\mathcal{G}$ has strong subexponential growth with respect to ℓ and pair of constants (α_0,β_0), then for each pair $(\alpha,\beta) \geq (\alpha_0,\beta_0)$ with $\alpha > \alpha_0$, there exists a constant $K_{\alpha,\beta} > 0$ such that*

$$\|f\|_{F^p_\lambda} \leq K_{\alpha,\beta}\|f\|_{\alpha,\beta},$$

for all $f \in C_c(\mathcal{G})$. In particular, the restriction of the inverse of Renault's p-j-map, $j_p^{-1}\colon S^p_{\ell,(\alpha,\beta)}(\mathcal{G}) \to F^p_\lambda(\mathcal{G})$, is a linear, injective and continuous map.

Proof. Fix a pair (α,β) such that $(\alpha,\beta) \geq (\alpha_0,\beta_0)$ and $\alpha > \alpha_0$. Since $\|\cdot\|_{F^p_\lambda} \leq \|\cdot\|_I$ on $C_c(\mathcal{G})$, to establish the first statement, it suffices to establish that $\|\cdot\|_I \leq K_{\alpha,\beta}\|\cdot\|_{\alpha,\beta}$ on $C_c(\mathcal{G})$, for some constant $K_{\alpha,\beta} > 0$. Specifically, put

$$K_{\alpha,\beta} := \sup_{u \in \mathcal{G}^{(0)}} \sum_{x \in \mathcal{G}_u} \exp\big(-\alpha \ell(x)^{\beta}\big),$$

which is finite by Lemma 4.1 since $\alpha > \alpha_0$. Let $f \in C_c(\mathcal{G})$ and fix any $u \in \mathcal{G}^{(0)}$; then by Hölder's inequality

$$\begin{aligned}
\|f\|_{\ell^1(\mathcal{G}_u)} &= \big\| f \exp\big(-\alpha\ell(\cdot)^{\beta}\big) \exp\big(\alpha\ell(\cdot)^{\beta}\big)\big\|_{\ell^1(\mathcal{G}_u)} \\
&\leq \big\| \exp\big(-\alpha\ell(\cdot)^{\beta}\big)\big\|_{\ell^q(\mathcal{G}_u)} \big\| f \exp\big(\alpha\ell(\cdot)^{\beta}\big)\big\|_{\ell^p(\mathcal{G}_u)} \\
&\leq \big\| \exp\big(-\alpha\ell(\cdot)^{\beta}\big)\big\|_{\ell^1(\mathcal{G}_u)} \big\| f \exp\big(\alpha\ell(\cdot)^{\beta}\big)\big\|_{\ell^p(\mathcal{G}_u)} \\
&\leq K_{\alpha,\beta}\|f\|_{\alpha,\beta},
\end{aligned}$$

and similarly $\|f\|_{\ell^1(\mathcal{G}^u)} \leq K_{\alpha,\beta}\|f\|_{\alpha,\beta}$, so that after taking suprema we see that $\|f\|_I \leq K_{\alpha,\beta}\|f\|_{\alpha,\beta}$.

To see the second statement, let $\iota\colon C_c(\mathcal{G}) \to F^p_\lambda(\mathcal{G})$ be the usual inclusion. Having proved the first statement here, it follows that the inclusion extends to a bounded linear map

$$\iota_{\alpha,\beta}\colon S^p_{\ell,(\alpha,\beta)}(\mathcal{G}) \to F^p_\lambda(\mathcal{G}).$$

If now $f \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$, find $\{f_n\}_n \subseteq C_c(\mathcal{G})$ such that $f_n \to f$ in $\|\cdot\|_{\alpha,\beta}$-norm; then $\iota_{\alpha,\beta}(f_n) \to \iota_{\alpha,\beta}(f)$ in $F^p_\lambda(\mathcal{G})$, so that

$$j_p\big(\iota_{\alpha,\beta}(f)\big) = \lim_{n\to\infty} j_p\big(\iota_{\alpha,\beta}(f_n)\big) = \lim_{n\to\infty} f_n = f.$$

It follows that $\iota_{\alpha,\beta}(f) = j_p^{-1}(f)$. ∎

Remark 4.3. If ℓ is any locally bounded length function with respect to which $\mathcal{G}$ has (strong) subexponential growth, then one can obtain a locally bounded integer-valued length function with respect to which $\mathcal{G}$ also has (strong) subexponential growth; simply define $\widetilde{\ell}(x) = \mathrm{ceil}(\ell(x))$, where for $t \geq 0$, $\mathrm{ceil}(t)$ is the smallest integer larger than or equal to t. Then clearly $\ell \leq \widetilde{\ell} \leq 1 + \ell$, and from this it follows that $\mathcal{G}$ has (strong) subexponential growth with respect to $\widetilde{\ell}$, and moreover that the norms $\|\cdot\|_{\alpha,\beta}$ defined in terms of ℓ are equivalent to the ones defined in terms of $\widetilde{\ell}$.

Let us for the remainder of this section assume $\mathcal{G}$ has strong subexponential growth with respect to ℓ, with constants $C > 0$ and (α_0, β_0) as in Definition 3.1. For our purposes, we may assume by Remark 4.3 that ℓ takes integer values. We also fix a pair $(\alpha, \beta) \geq (\alpha_0, \beta_0)$ with $\alpha > \alpha_0$, and consider the associated Banach space $S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. For ease of notation, we write ω instead of $\omega_{\alpha,\beta}$ for the associated weight. Define two auxiliary functions u and σ as follows:

$$u(x) := \exp\big(-\alpha(2-2^\beta)\ell(x)^\beta\big),$$

for $x \in \mathcal{G}$, and $\sigma := \omega u$. The following inequality is essential to the arguments in this section, and can be proved exactly like in [36, Theorem 2.2]:

$$\frac{\omega(xy)}{\omega(x)\omega(y)} \leq u(x) + u(y), \tag{4.1}$$

for all $(x, y) \in \mathcal{G}^{(2)}$.

We also need the following Young's convolution inequality for étale groupoids. We omit its proof because it is completely analogous to that in the setting of discrete groups.

Lemma 4.4. *Let $p, q, r \geq 1$ be such that $1 + 1/r = 1/p + 1/q$. If $f \in \ell^p(\mathcal{G})$ and $g \in \ell^q(\mathcal{G})$, then $f * g \in \ell^r(\mathcal{G})$ and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Lemma 4.5. *If $f, g \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$, then*

$$\|f * g\|_{\alpha,\beta} \leq \|f\|_{\alpha,\beta}\|g\sigma\|_1 + \|f\sigma\|_1\|g\|_{\alpha,\beta}.$$

Proof. It follows by (4.1) that

$$\begin{aligned}
\left|f * g(x)\omega(x)\right| &\leq \sum_{y\in\mathcal{G}_{s(x)}} \left|f(xy^{-1})\right|\left|g(y)\right|\omega(x) \\
&= \sum_{y\in\mathcal{G}_{s(x)}} \left|f(xy^{-1})\right|\left|g(y)\right|\omega(xy^{-1}y) \\
&\leq \sum_{y\in\mathcal{G}_{s(x)}} \left|f(xy^{-1})\right|u(xy^{-1})\omega(xy^{-1})\left|g(y)\right|\omega(y) \\
&\quad + \sum_{y\in\mathcal{G}_{s(x)}} \left|f(xy^{-1})\right|\omega(xy^{-1})\left|g(y)\right|u(y)\omega(y) \\
&= \sum_{y\in\mathcal{G}_{s(x)}} \left|f(xy^{-1})\right|\sigma(xy^{-1})\left|g(y)\right|\omega(y) \\
&\quad + \sum_{y\in\mathcal{G}_{s(x)}} \left|f(xy^{-1})\right|\omega(xy^{-1})\left|g(y)\right|\sigma(y) \\
&= |f|\sigma * |g|\omega(x) + |f|\omega * |g|\sigma(x),
\end{aligned}$$

for each $x \in \mathcal{G}$, and therefore

$$\|f * g\|_{\alpha,\beta} \leq \big\||f|\sigma * |g|\omega\big\|_p + \big\||f|\omega * |g|\sigma\big\|_p \leq \|f\sigma\|_1\|g\|_{\alpha,\beta} + \|f\|_{\alpha,\beta}\|g\sigma\|_1,$$

by Lemma 4.4. ■

Lemma 4.6. *Suppose* $(\alpha, \beta) \geq (\alpha_0, \beta_0)$ *is such that* $\alpha(2 - 2^\beta) > \alpha_0$. *Then there exists* $\theta = \theta(\alpha, \beta) \in (0, 1)$ *and a constant* $C_{\alpha,\beta} > 0$, *such that*

$$\|f * f\|_{\alpha,\beta} \leq C_{\alpha,\beta}\|f\|_{\alpha,\beta}^{1+\theta}\|f\|_{F_\lambda^p}^{1-\theta},$$

for all $f \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$.

Proof. For $\theta \in (0, 1)$, let

$$\begin{aligned}
\xi_\theta(x) &:= \omega^{1-\theta}(x)u(x) = \exp\big(\alpha(1-\theta)\ell(x)^\beta - \alpha(2-2^\beta)\ell(x)^\beta\big) \\
&= \exp\big(\ell(x)^\beta\alpha(1-\theta-2+2^\beta)\big).
\end{aligned}$$

Now, as $2 > 2^\beta$ since $\beta \in (0, 1)$ and by assumption $\alpha(2 - 2^\beta) > \alpha_0$, there exists $\theta \in (0, 1)$ such that $2 + \theta > 2^\beta + 1$ and $\alpha(\theta - 1 + 2 - 2^\beta) > \alpha_0$; for such θ, since

$$\xi_\theta(x) = \exp\big(-\alpha(\theta - 1 + 2 - 2^\beta)\ell(x)^\beta\big) \quad \text{for } x \in \mathcal{G},$$

we see that Lemma 4.1 implies that $\|\xi_\theta\|_s \leq \|\xi_\theta\|_1 < \infty$ for all $s \in [1, \infty]$.

Now let $f \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. By Lemma 4.5 with $f = g$, we have that

$$\|f * f\|_{\alpha,\beta} \leq 2\|f\sigma\|_1\|f\|_{\alpha,\beta}.$$

Let $u \in \mathcal{G}^{(0)}$ be any unit. Then,

$$\begin{aligned}
\|f\sigma\|_{\ell^1(\mathcal{G}_u)} &= \sum_{x\in\mathcal{G}_u} |f(x)|\sigma(x) \\
&= \sum_{x\in\mathcal{G}_u} |f(x)|\omega(x)u(x) \\
&= \sum_{x\in\mathcal{G}_u} |f(x)|^{1-\theta}\big(|f(x)|\omega(x)\big)^{\theta}\xi_\theta(x) \\
&\le \Big\{\sum_{x\in\mathcal{G}_u} |f(x)|^p\Big\}^{\frac{1-\theta}{p}} \Big\{\sum_{x\in\mathcal{G}_u} |f(x)\omega(x)|^p\Big\}^{\frac{\theta}{p}} \Big\{\sum_{x\in\mathcal{G}_u} \xi_\theta(x)^q\Big\}^{\frac{1}{q}} \\
&= \|f\|^{1-\theta}_{\ell^p(\mathcal{G}_u)}\|f\omega\|^{\theta}_{\ell^p(\mathcal{G}_u)}\|\xi_\theta\|_{\ell^q(\mathcal{G}_u)} \\
&\le \|f\|^{1-\theta}_{F^p_\lambda}\|f\|^{\theta}_{\alpha,\beta}\|\xi_\theta\|_q,
\end{aligned}$$

by the generalized Hölder inequality with the exponents $\frac{p}{1-\theta}$, $\frac{p}{\theta}$ and q, where q is the Hölder conjugate of p, and the fact that

$$\|f\|_{\ell^p(\mathcal{G}_u)} = \|f * \delta_u\|_{\ell^p(\mathcal{G}_u)} \le \|\lambda_u(f)\| \le \|f\|_{F^p_\lambda}.$$

Also,

$$\begin{aligned}
\|f\sigma\|_{\ell^1(\mathcal{G}^u)} &= \|f^*\sigma\|_{\ell^1(\mathcal{G}_u)} \\
&\le \Big\{\sum_{x\in\mathcal{G}_u} |f^*(x)|^q\Big\}^{\frac{1-\theta}{q}} \Big\{\sum_{x\in\mathcal{G}_u} |f^*(x)\omega(x)|^p\Big\}^{\frac{\theta}{p}} \Big\{\sum_{x\in\mathcal{G}_u} \xi_\theta(x)^s\Big\}^{\frac{1}{s}} \\
&= \|f^*\|^{1-\theta}_{\ell^q(\mathcal{G}_u)}\|f\omega\|^{\theta}_{\ell^p(\mathcal{G}_u)}\|\xi_\theta\|_{\ell^s(\mathcal{G}_u)} \\
&\le \|f\|^{1-\theta}_{F^p_\lambda}\|f\|^{\theta}_{\alpha,\beta}\|\xi_\theta\|_s,
\end{aligned}$$

where $1 < s = s(\theta)$ is such that

$$\frac{1-\theta}{q} + \frac{\theta}{p} + \frac{1}{s} = 1.$$

In the fourth inequality in the preceding computation, when $p \in (1,\infty)$, we have used the fact that

$$\|f^*\|_{\ell^q(\mathcal{G}_u)} \le \|f^*\|_{F^q_\lambda} = \|f\|_{F^p_\lambda},$$

since the involution on $C_c(\mathcal{G})$ extends to an isometric anti-isomorphism $F^p_\lambda(\mathcal{G}) \to F^q_\lambda(\mathcal{G})$ (this follows for example by [1, Lemma 3.5]), and when $p = 1$, that $\|f\|_\infty \le \|f\|_{F^p_\lambda}$. By our observation in the beginning, we may find θ such that ξ_θ is in $\ell^s(\mathcal{G})$ for every $s \in [1,\infty]$. Choose such a θ, find $s = s(\theta)$, and put $C_{\alpha,\beta} := 2\max\{\|\xi_\theta\|_s, \|\xi_\theta\|_q\} < \infty$. Then

$$\|f\sigma\|_1 \le \frac{C_{\alpha,\beta}}{2} \cdot \|f\|^{1-\theta}_{F^p_\lambda}\|f\|^{\theta}_{\alpha,\beta},$$

so that

$$\|f * f\|_{\alpha,\beta} \leq C_{\alpha,\beta} \|f\|_{F_\lambda^p}^{1-\theta} \|f\|_{\alpha,\beta}^{1+\theta},$$

as desired. ∎

Let us define the index set

$$J(\alpha_0, \beta_0) := \{(\alpha, \beta) \in \mathbb{R}_{>0} \times (0,1)\colon (\alpha, \beta) \geq (\alpha_0, \beta_0) \text{ and } \alpha(2 - 2^\beta) > \alpha_0\}.$$

Proposition 4.7. *If $(\alpha, \beta) \in J(\alpha_0, \beta_0)$, then $S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$ is a Banach algebra.*

Proof. Let $f, g \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. By Lemma 4.5, we have that

$$\|f * g\|_{\alpha,\beta} \leq \|f\|_{\alpha,\beta} \|g\sigma\|_1 + \|f\sigma\|_1 \|g\|_{\alpha,\beta},$$

and in the proof of Lemma 4.6 we saw that

$$\|f\sigma\|_1 \leq C_{\alpha,\beta} \|f\|_{F_\lambda^p}^{1-\theta} \|f\|_{\alpha,\beta}^{\theta},$$

for some positive constant $C_{\alpha,\beta}$ and $\theta \in (0,1)$. By Proposition 4.2,

$$\|f\|_{F_\lambda^p} \leq K_{\alpha,\beta} \|f\|_{\alpha,\beta},$$

for all $f \in C_c(\mathcal{G})$ and a constant $K_{\alpha,\beta}$, and by continuity of the inclusion, it also holds for all $f \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. Putting all of this together, we see that

$$\|f * g\|_{\alpha,\beta} \leq C'_{\alpha,\beta} \|f\|_{\alpha,\beta} \|g\|_{\alpha,\beta},$$

for some constant $C'_{\alpha,\beta} > 0$ not depending on f and g. Upon rescaling the norm, we see that $S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$ becomes a Banach algebra. ∎

Proposition 4.8. *For any $(\alpha, \beta) \in J(\alpha_0, \beta_0)$, the Banach algebra $S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$ is spectrally invariant in $F_\lambda^p(\mathcal{G})$.*

Proof. Recall from Lemma 4.6 that there exist $C_{\alpha,\beta} > 0$ and $\theta \in (0,1)$ such that

$$\|f * f\|_{\alpha,\beta} \leq C_{\alpha,\beta} \|f\|_{\alpha,\beta}^{1+\theta} \|f\|_{F_\lambda^p}^{1-\theta},$$

for all $f \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. In particular,

$$\|f^{2n}\|_{\alpha,\beta} = \left\|(f^n)^2\right\|_{\alpha,\beta} \leq C_{\alpha,\beta} \|f^n\|_{\alpha,\beta}^{1+\theta} \|f^n\|_{F_\lambda^p}^{1-\theta}.$$

Raising everything to the power $1/2n$ and taking limits gives

$$r_{S^p_{\ell,(\alpha,\beta)}(\mathcal{G})}(f) \leq r_{S^p_{\ell,(\alpha,\beta)}(\mathcal{G})}(f)^{\frac{1+\theta}{2}} r_{F_\lambda^p(\mathcal{G})}(f)^{\frac{1-\theta}{2}},$$

which is equivalent to $r_{S^p_{\ell,(\alpha,\beta)}(\mathcal{G})}(f) \leq r_{F_\lambda^p(\mathcal{G})}(f)$. Since the other inequality follows by the inclusion $S^p_{\ell,(\alpha,\beta)}(\mathcal{G}) \hookrightarrow F_\lambda^p(\mathcal{G})$, we see that $r_{S^p_{\ell,(\alpha,\beta)}(\mathcal{G})}(f) = r_{F_\lambda^p(\mathcal{G})}(f)$, for all $f \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. This suffices by [27, Lemma 2.7]. ∎

Since spectral invariance is equivalent to stability under holomorphic functional calculus for Banach algebras by [46, Lemma 1.2], [11, Chapter 3, Appendix C, Proposition 3] applies to give the following result.

Corollary 4.9. *For any* $(\alpha,\beta)\in J(\alpha_0,\beta_0)$, $K_*(S^p_{\ell,(\alpha,\beta)}(\mathcal{G}))\cong K_*(F^p_\lambda(\mathcal{G}))$, *where* $*=0,1$.

Recall that $(\alpha,\beta)\geq(\alpha',\beta')$ if $\alpha\geq\alpha'$ and $\beta\geq\beta'$. In this case, since $\|\cdot\|_{\alpha,\beta}\geq\|\cdot\|_{\alpha',\beta'}$, it follows that $S^p_{\ell,(\alpha,\beta)}(\mathcal{G})\subseteq S^p_{\ell,(\alpha',\beta')}(\mathcal{G})$. Therefore, $\{S^p_{\ell,(\alpha,\beta)}(\mathcal{G})\}_{(\alpha,\beta)\in J(\alpha_0,\beta_0)}$ forms a directed system of decreasing Banach algebras.

Definition 4.10. Let $\mathcal{G}$ be an étale groupoid with strong subexponential growth with respect to the locally bounded length function ℓ and constants $\alpha_0>0$ and $0<\beta_0<1$ as in Definition 3.1. We define the space of *strongly subexponentially decreasing functions* as

$$S^p_\ell(\mathcal{G}):=\bigcap_{(\alpha,\beta)\in J(\alpha_0,\beta_0)} S^p_{\ell,(\alpha,\beta)}(\mathcal{G}).$$

Since the system $\{(\alpha,\beta)\}_{(\alpha,\beta)\in J(\alpha_0,\beta_0)}$ has a countable cofinal sequence, $S^p_\ell(\mathcal{G})$ is an intersection of a decreasing sequence of Banach algebras, hence is a Fréchet algebra.

The restriction of the inverse of Renault's p-j-map extends to an injective continuous algebra homomorphism $S^p_\ell(\mathcal{G})\hookrightarrow F^p_\lambda(\mathcal{G})$, so we may view $S^p_\ell(\mathcal{G})$ as a subalgebra of $F^p_\lambda(\mathcal{G})$ endowed with a finer Fréchet algebra topology than the one inherited from $F^p_\lambda(\mathcal{G})$.

Proposition 4.11. *If* $\mathcal{G}$ *has strong subexponential growth with respect to* ℓ, *then* $S^p_\ell(\mathcal{G})$ *is stable under holomorphic functional calculus in* $F^p_\lambda(\mathcal{G})$.

Proof. Suppose (α_0,β_0) is the pair as in the definition of strong subexponential growth in Definition 3.1. Since $S^p_\ell(\mathcal{G})$ is a Fréchet algebra under a finer topology than the one inherited from $F^p_\lambda(\mathcal{G})$, it suffices by [46, Lemma 1.2] to see that $S^p_\ell(\mathcal{G})$ is spectrally invariant in $F^p_\lambda(\mathcal{G})$. For that, let $a\in S^p_\ell(\mathcal{G})$ with inverse $a^{-1}\in F^p_\lambda(\mathcal{G})$. Then $a\in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$ for every $(\alpha,\beta)\in J(\alpha_0,\beta_0)$, and is invertible in $F^p_\lambda(\mathcal{G})$. By Proposition 4.8, $a^{-1}\in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$ for all $(\alpha,\beta)\in J(\alpha_0,\beta_0)$, which means that $a^{-1}\in S^p_\ell(\mathcal{G})$. ∎

Before we prove our main result, let us recall the definition of the K-groups for a Fréchet algebra.

Definition 4.12. Let $\mathcal{A}$ be a unital Fréchet algebra. We define $K_0(\mathcal{A})$ as the Grothendieck group of the abelian semigroup of isomorphism classes of finitely generated projective $\mathcal{A}$-modules with direct sum as the semigroup multiplication. Using the embeddings $u\mapsto \operatorname{diag}(u,1)$, we define $K_1(\mathcal{A})=\varinjlim \mathrm{GL}_n(\mathcal{A})/\mathrm{GL}_n(\mathcal{A})_0$, where the $\mathrm{GL}_n(\mathcal{A})$ are the invertible matrices in the Fréchet algebra $M_n(\mathcal{A})$ endowed with the induced topology, and $\mathrm{GL}_n(\mathcal{A})_0$ is the normal subgroup given by the path component of the identity. If $\mathcal{A}$ is not unital, $K_*(\mathcal{A})$ is defined to be the kernel of the naturally induced map from $K_*(\widetilde{\mathcal{A}})$ to $K_*(\mathbb{C})$.

When $\mathcal{A}$ is a Banach algebra, the above defines its usual K-groups.

Theorem 4.13. *If $\mathcal{G}$ has strong subexponential growth with respect to some locally bounded length function, then $K_*(F^p_\lambda(\mathcal{G}))$, for $* = 0, 1$, is independent of $p \in [1, \infty)$.*

Proof. Suppose (α_0, β_0) are the constants as in the definition of strong subexponential growth for $\mathcal{G}$ in Definition 3.1, and let $S^p_\ell(\mathcal{G})$, for $p \in [1, \infty)$, be the associated Fréchet algebras of strongly subexponentially decreasing functions. Combining Proposition 4.11 with [2, Lemma 2.3], we have that

$$K_*\big(S^p_\ell(\mathcal{G})\big) \cong K_*\big(F^p_\lambda(\mathcal{G})\big),$$

for $* = 0, 1$ and $p \in [1, \infty)$. Therefore, it suffices to prove that

$$S^p_\ell(\mathcal{G}) = S^1_\ell(\mathcal{G}),$$

as Fréchet algebras, for all $p \in (1, \infty)$. It is straightforward to verify that

$$\|a\|_{\alpha,\beta,p} \leq \|a\|_{\alpha,\beta,1},$$

for all $a \in S^1_{\ell,(\alpha,\beta)}(\mathcal{G})$ and $(\alpha, \beta) \in J(\alpha_0, \beta_0)$, where $\|\cdot\|_{\alpha,\beta,p}$ is the norm on $S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. It follows that the inclusion $S^1_\ell(\mathcal{G}) \subseteq S^p_\ell(\mathcal{G})$ is continuous. Conversely, fix $(\alpha, \beta) \in J(\alpha_0, \beta_0)$. By an application of Hölder's inequality together with Lemma 4.1 it follows that there exists a constant $K_{\alpha,\beta} > 0$ such that

$$\|a\|_{\alpha,\beta,1} \leq K_{\alpha,\beta}\|a\|_{2\alpha,\beta,p},$$

for all $a \in S^p_{\ell,(\alpha,\beta)}(\mathcal{G})$. Since also $(2\alpha, \beta) \in J(\alpha_0, \beta_0)$, it follows that the inclusion $S^p_\ell(\mathcal{G}) \subseteq S^1_\ell(\mathcal{G})$ is continuous, so that $S^p_\ell(\mathcal{G}) = S^1_\ell(\mathcal{G})$ as Fréchet algebras. ■

A symmetrized version of $F^p_\lambda(\mathcal{G})$, the Banach *-algebra $F^{p,*}_\lambda(\mathcal{G})$, has recently appeared in the literature (see for example [1, 13, 29, 40, 44, 45]), often under the name symmetrized p-pseudofunctions. In what follows, we shall recall the definition of these and explain how Theorem 4.13 can be extended to these Banach *-algebras. We fix $p \in [1, \infty)$, and on $C_c(\mathcal{G})$ we define the norm

$$\|f\|_{p,*} := \max\left\{\|f\|_{F^p_\lambda}, \|f^*\|_{F^p_\lambda}\right\}.$$

We let $F^{p,*}_\lambda(\mathcal{G})$ be completion of $C_c(\mathcal{G})$ in this norm, which is a Banach *-algebra. Notice that

$$F^{1,*}_\lambda(\mathcal{G}) = L^I(\mathcal{G}) = \overline{C_c(\mathcal{G})}^{\|\cdot\|_I}.$$

By the same argument as in [1, Proposition 3.9], the inclusion $C_c(\mathcal{G}) \subseteq C^*_r(\mathcal{G})$ extends to an injective contraction $F^{p,*}_\lambda(\mathcal{G}) \hookrightarrow C^*_r(\mathcal{G})$. Consequently, Renault's j-map restricts to a contractive injection $j_{p,*}\colon F^{p,*}_\lambda(\mathcal{G}) \to C_0(\mathcal{G})$. Suppose now that $\mathcal{G}$ has strong subexponential growth with respect to a locally bounded length function ℓ and constants (α_0, β_0) as in Definition 3.1. Then, since the analogous inequality as in Proposition 4.2 holds for the norm on $F_\lambda^{p,*}(\mathcal{G})$, we can argue as therein to obtain that for $(\alpha,\beta)\in J(\alpha_0,\beta_0)$, the inverse of the restricted Renault's j-map is an injective continuous homomorphism

$$S^2_{\ell,(\alpha,\beta)}(\mathcal{G}) \hookrightarrow F_\lambda^{p,*}(\mathcal{G}),$$

and hence $S^2_{\ell,(\alpha,\beta)}(\mathcal{G})$ may be identified as a Banach subalgebra of $F_\lambda^{p,*}(\mathcal{G})$ under a finer topology.

Corollary 4.14. *If $\mathcal{G}$ has strong subexponential growth with respect to some locally bounded length function, then*

$$K_*\big(F_\lambda^{p,*}(\mathcal{G})\big) \cong K_*\big(F_\lambda^{p}(\mathcal{G})\big) \cong K_*\big(C_r^*(\mathcal{G})\big),$$

for $* = 0,1$ and $p\in[1,\infty)$.

Proof. Suppose (α_0,β_0) is the pair of constants as in Definition 3.1 for $\mathcal{G}$. By Proposition 4.8, $S^2_{\ell,(\alpha,\beta)}(\mathcal{G})$ is spectrally invariant in $C_r^*(\mathcal{G})$, for any $(\alpha,\beta)\in J(\alpha_0,\beta_0)$. It then follows from the inclusions

$$S^2_{\ell,(\alpha,\beta)}(\mathcal{G}) \subseteq F_\lambda^{p,*}(\mathcal{G}) \subseteq C_r^*(\mathcal{G}),$$

that $S^2_{\ell,(\alpha,\beta)}(\mathcal{G})$ is spectrally invariant in $F_\lambda^{p,*}(\mathcal{G})$ as well. Therefore,

$$K_*\big(F_\lambda^{p,*}(\mathcal{G})\big) \cong K_*\big(S^2_{\ell,(\alpha,\beta)}(\mathcal{G})\big) \cong K_*\big(C_r^*(\mathcal{G})\big),$$

for all $p\in[1,\infty)$. The result then follows by combining this with Theorem 4.13. ∎

5. Examples

The purpose of this final section is to exhibit several examples of étale groupoids with (strong) subexponential growth. In Section 5.1, we show that all second countable proper étale groupoids are in fact of polynomial growth. Then, in Section 5.2, we show how the results from [2, Section 5.1] apply to give coarse groupoids with (strong) subexponential growth associated with many already existing examples in the literature of metric spaces. Finally, in Section 5.3 we use ideas from [35] as well as recent results of Brix, Hume and Li in [7] to construct an étale groupoid from a shift space of infinite type which has strong subexponential growth and not polynomial.

5.1. Proper groupoids

Recall that an étale groupoid $\mathcal{G}$ is called *proper* if the map $(r,s)\colon \mathcal{G}\to\mathcal{G}^{(0)}\times\mathcal{G}^{(0)}$, $x\mapsto (r(x),s(x))$ is proper, that is, $(r,s)^{-1}(K)\subseteq\mathcal{G}$ is compact whenever $K\subseteq\mathcal{G}^{(0)}\times\mathcal{G}^{(0)}$ is compact.

Proposition 5.1. *If $\mathcal{G}$ is a second countable proper étale groupoid, then there is a locally bounded length function with respect to which $\mathcal{G}$ has polynomial growth.*

Proof. Since $\mathcal{G}^{(0)}$ is locally compact Hausdorff and second countable, we can find an increasing sequence of compact sets $\{K_i\}_{i=1}^{\infty}$ such that

$$\mathcal{G}^{(0)} = \bigcup_{i=1}^{\infty} K_i = \bigcup_{i=1}^{\infty} K_i{}^{\circ}.$$

For each $i \in \mathbb{N}$, let

$$\mathcal{G}(i) := \mathcal{G}(K_i) = \{x \in \mathcal{G} : s(x), r(x) \in K_i\},$$

so that $\mathcal{G}(i) \subseteq \mathcal{G}(i+1)$, for each $i \in \mathbb{N}$. Then since $\mathcal{G}$ is proper, it follows that each $\mathcal{G}(i)$ is a compact subgroupoid of $\mathcal{G}$, and, moreover,

$$\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}(i) = \bigcup_{i=1}^{\infty} \mathcal{G}(i)^{\circ}.$$

Let $f : \mathcal{G} \to \mathbb{C}$ be a positive continuous compactly supported function such that $1_{\mathcal{G}(i)} \leq f$. Then since the counting measures form a continuous Haar system, the map

$$u \mapsto |\mathcal{G}(i)_u| = \sum_{x \in \mathcal{G}_u} 1_{\mathcal{G}(i)}(x) \leq \sum_{x \in \mathcal{G}_u} f(x),$$

has a finite supremum. Let us denote this supremum by $p(i)$, so that

$$p(i) := \sup_{u \in \mathcal{G}^{(0)}} |\mathcal{G}(i)_u| < \infty.$$

Define a length function $\ell : \mathcal{G} \to \mathbb{R}_{\geq 0}$ as follows: $\ell(u) := 0$, for all $u \in \mathcal{G}^{(0)}$, and if $x \in \mathcal{G} \setminus \mathcal{G}^{(0)}$ then

$$\ell(x) := \min\{p(i) : x \in \mathcal{G}(i)\}.$$

We claim that ℓ is a locally bounded length function. Indeed, given $x \in \mathcal{G}$, there exists $i \in \mathbb{N}$ such that $x \in \mathcal{G}(i)$, and for any such i, $x \in \mathcal{G}(i)$ if and only if $x^{-1} \in \mathcal{G}(i)$, and therefore $\ell(x^{-1}) = \ell(x)$, for all $x \in \mathcal{G}$. Moreover, if $(x, y) \in \mathcal{G}^{(2)}$, say $x \in \mathcal{G}(n)$ and $y \in \mathcal{G}(m)$, let $k := \max\{n, m\}$. Then $x, y, xy \in \mathcal{G}(k)$, so that

$$\ell(xy) \leq p(k) \leq p(n) + p(m),$$

for any such n and m. Thus $\ell(xy) \leq \ell(x) + \ell(y)$, so that ℓ is indeed a length function. To see that it is locally bounded, let $K \subseteq \mathcal{G}$ be compact. Since also $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}(i)^{\circ}$, compactness produces an $N \in \mathbb{N}$ such that $K \subseteq \mathcal{G}(N)$, and so $\ell(K) \leq p(N) < \infty$, so ℓ is a locally bounded length function. Now let $N \in \mathbb{N}$ be given, and if possible, find the largest $k \in \mathbb{N}$ such that $p(k) \leq N$. By definition, if $x \notin \mathcal{G}(k)$, then $\ell(x) > N$, and so, for any $u \in \mathcal{G}^{(0)}$,

$$|B_{\mathcal{G}_u}(N)| = |\{x \in \mathcal{G}_u : \ell(x) \leq N\}| \leq |\{x \in \mathcal{G}(k)_u\}| \leq p(k) \leq N.$$

If no such $k \in \mathbb{N}$ exist, then $|B_{\mathcal{G}_u}(N)| = 1 \leq N$, for any $u \in \mathcal{G}^{(0)}$. Therefore, $\mathcal{G}$ has polynomial growth. ∎

Remark 5.2. Suppose $\mathcal{G}$ is a second-countable proper étale groupoid. By Proposition 5.1, $\mathcal{G}$ has polynomial growth, and hence for any $\alpha > 0$ and $\beta \in (0,1)$, $S^2_{\ell,(\alpha,\beta)}(\mathcal{G})$ is a Banach algebra. It is clear that if $f, g \in C_c(\mathcal{G})$ with $|f| \leq |g|$, then $\|f\|_{\alpha,\beta} \leq \|g\|_{\alpha,\beta}$, and therefore $S^2_{\ell,(\alpha,\beta)}(\mathcal{G})$ is an unconditional completion of $C_c(\mathcal{G})$; it is moreover regular (see paragraph succeeding [37, Proposition 2.4]) since (a multiple of) the unconditional norm $\|\cdot\|_{\alpha,\beta}$ dominates the reduced C*-algebra norm. Therefore, combining Corollary 4.9 and Theorem 4.13 with [37, Proposition 2.4], we have that

$$K_*\big(F^p_\lambda(\mathcal{G})\big) \cong K_*\big(S^2_{\ell,(\alpha,\beta)}(\mathcal{G})\big) \cong K_*\big(\mathcal{A}(\mathcal{G})\big),$$

for any $p \in [1,\infty)$ and any unconditional completion $\mathcal{A}(\mathcal{G})$ of $C_c(\mathcal{G})$ such that the unconditional norm dominates the reduced C^*-norm on $C_c(\mathcal{G})$.

5.2. Coarse groupoids

Let us briefly recall the definition of coarse groupoids: We say that an extended metric space (X, d) is *uniformly locally finite* if for every $R > 0$ we have

$$\sup_{x \in X} \big|\overline{B}(x, R)\big| < \infty,$$

where $\overline{B}(x, R)$ is the closed ball of radius R around x. From this we construct an étale groupoid, denoted by $\mathcal{G}_{(X,d)}$, as follows. For every $r > 0$, let $E_r := \{(x, y) \in X \times X \mid d(x, y) \leq r\}$, and define

$$\mathcal{G}_{(X,d)} := \bigcup_{r \geq 0} \overline{E_r},$$

where the closure is taken in the Stone–Čech compactification $\beta X \times \beta X$. The unit space $\mathcal{G}^{(0)}_{(X,d)}$ is identified with $\overline{E_0} \cong \beta X$. The range and source maps are the natural extensions of the first and second projection map on $X \times X$ to the Stone–Čech compactification, and the multiplication map is inherited from the pair groupoid multiplication on $\beta X \times \beta X$.

By [2, Lemma 5.6] we know that the metric d naturally extends to a metric βd on $\mathcal{G}_{(X,d)}$ which induces a continuous and proper length function $\ell_{\beta d}$ on $\mathcal{G}_{(X,d)}$. We recall the following result from [2, Proposition 5.8].

Proposition 5.3. *Let (X, d) be a uniformly locally finite extended metric space, and suppose there is a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for which $|\overline{B}(x, r)| \leq f(r)$ for all $x \in X$ and all $r \geq 0$. Let $\chi \in \beta X = \mathcal{G}^{(0)}_{(X,d)}$. Then*

$$\big|B_{(\mathcal{G}_{(X,d)})_\chi}(r)\big| \leq \begin{cases} f(r) & \chi \in X, \\ M f(r)^2 & \chi \in \beta X \setminus X \end{cases}$$

for a sufficiently large constant M. That is, the growth of the groupoid $\mathcal{G}_{(X,d)}$ is bounded above by the growth of f^2.

We immediately obtain the following corollary.

Corollary 5.4. *Let (X,d) be an extended metric space for which $|\bar{B}(x,r)| \leq f(r)$ for all $x \in X$ and all $r \geq 0$. If there are $C, \alpha > 0$ and $0 < \beta < 1$ for which $f(r) \leq C\exp(\alpha r^\beta)$ for all $r \geq 0$, then $\mathcal{G}_{(X,d)}$ has strong subexponential growth with respect to $\ell_{\beta d}$. If $\lim_{r\to\infty} f(r)^{1/r} = 1$, then $\mathcal{G}_{(X,d)}$ has subexponential growth with respect to $\ell_{\beta d}$.*

Proof. By the assumptions on f we have $|\bar{B}(x,r)| \leq C\exp(\alpha r^\beta)$. By Proposition 5.3 there exists $M > 0$ such that we can guarantee

$$\left|(\mathcal{G}_{(X,d)})_u\right| \leq MC\exp(2\alpha r^\beta)$$

uniformly in u, from which we deduce that $\mathcal{G}_{(X,d)}$ has strong subexponential growth with respect to $\ell_{\beta d}$.

The statement for subexponential growth follows similarly. ■

There are several interesting metric spaces with (strong) subexponential growth coming from graph theory, see for example [6, 26, 28, 33]. By the argument in the proof of Corollary 5.4, we deduce that for any one of these examples, the associated coarse groupoid has also (strong) subexponential growth. Moreover, if the original extended metric space has (strong) subexponential growth but not polynomial growth, then the same is true for the associated coarse groupoid. This follows by observing that for any point $x \in X \subseteq \beta X = \mathcal{G}^{(0)}_{(X,d)}$, we have $|B_{(\mathcal{G}_{(X,d)})_x}(r)| = |\bar{B}(x,r)|$.

5.3. Shift groupoids

Let us start by recalling the definition of a shift space and its associated Deaconu–Renault groupoid. Fix a finite set $\mathcal{A}$ that we call an *alphabet*. A *path* is a map $x : \mathbb{N} \to \mathcal{A}$, and we denote by $\mathcal{A}^{\mathbb{N}}$ the set of all paths. Given $x \in \mathcal{A}^{\mathbb{N}}$, and $w = (w_1, \ldots, w_n) \in \mathcal{A}^n$, we define the concatenation of a word $w \in \mathcal{A}^n$ with a path $x \in \mathcal{A}^{\mathbb{N}}$ to be the path

$$wx[i] = \begin{cases} w_i & \text{if } 1 \leq i \leq n, \\ x[i-n] & \text{if } i \geq n+1. \end{cases}$$

Whenever $x \in \mathcal{A}^{\mathbb{N}}$, by $x[n, n+k]$ we mean the word $(x[n], x[n+1], \ldots, x[n+k]) \in \mathcal{A}^{k+1}$, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. We give $\mathcal{A}^{\mathbb{N}}$ the product topology, so that $\mathcal{A}^{\mathbb{N}}$ is a Cantor space, with topology generated by the sets of the form

$$Z(w) = \left\{x \in \mathcal{A}^{\mathbb{N}} \mid x[i] = w_i \text{ for } 1 \leq i \leq n\right\}$$

for $w = (w_1, \ldots, w_n) \in \mathcal{A}^n$. Define the *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by $\sigma(x)[n] = x[n+1]$ for every $n \in \mathbb{N}$. A (one-sided) *shift space* is a closed subset $X \subseteq \mathcal{A}^{\mathbb{N}}$ such that $\sigma(X) \subseteq X$. A *word of length n in X* is an element $w = (w_1, \ldots, w_n) \in \mathcal{A}^n$ such that there exists $x \in X$ and $k \in \mathbb{N}$ such that $x[k+i-1] = w_i$ for $1 \leq i \leq n$. We denote by $\mathcal{L}_n(X)$ the set of all words of length n in X, and $\mathcal{L}_*(X) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(X)$ the set of all words of X.

We define the *complexity function of* X to be the function $p_X(n) = \sum_{i=1}^{n} |\mathcal{L}_i(X)|$. The *Deaconu–Renault groupoid* associated with the shift space $\sigma : X \to X$ is the groupoid

$$\mathcal{G}_X := \{(x,k,y) \in X \times \mathbb{Z} \times X \mid \exists n,m \in \mathbb{N}_0 \text{ such that } m-n=k \text{ and } \sigma^m(x) = \sigma^n(y)\}$$

with multiplication $(x, m-n, y) \cdot (y, k-l, z) = (x, m+k-(n+l), z)$, inversion $(x, m-n, y)^{-1} = (y, n-m, x)$, and unit space $\mathcal{G}_X^{(0)} = \{(x,0,x) \mid x \in X\}$ identified with X. The topology has as basis sets of the form

$$Z(u,v) := \{(x,m-n,y) \in \mathcal{G}_X \mid \sigma^m(x) = \sigma^n(y) \text{ for } x \in Z(u) \text{ and } y \in Z(v)\},$$

for $u \in \mathcal{L}_m(X)$ and $v \in \mathcal{L}_n(X)$. We allow u and v to be the empty word, in which case

$$Z(u,\emptyset) := \{(x,m,y) \in \mathcal{G}_X \mid \sigma^m(x) = y \text{ for } x \in Z(u) \text{ and } y \in X\},$$

for $u \in \mathcal{L}_m(X)$ and

$$Z(\emptyset,v) := \{(x,-n,y) \in \mathcal{G}_X \mid \sigma^n(y) = x \text{ for } x \in X \text{ and } y \in Z(v)\},$$

for $v \in \mathcal{L}_n(X)$. The shift map σ is always locally injective. It is a local homeomorphism if and only if X is a shift of *finite type*, meaning that there exists $K \in \mathbb{N}$ and a subset $\mathcal{F} \subseteq \mathcal{A}^K$, called the forbidden words, such that for every $x \in X$ and $n \in \mathbb{N}$ we have that $x[n, n+K-1] \notin \mathcal{F}$. In this case, observe that $\mathcal{F} = \mathcal{A}^K \setminus \mathcal{L}_K(X)$. A shift is of *infinite type* if it is not of finite type. Given a shift space X, the associated Deaconu–Renault groupoid $\mathcal{G}_X$ is étale if X is a shift of finite type, and r-discrete if X is a shift of infinite type; that is, a locally compact groupoid whose unit space is open.

Now, observe that the set

$$S := \bigcup_{a \in \mathcal{A}} Z(a,\emptyset) \cup Z(\emptyset,a) \tag{5.1}$$

is a generating set for $\mathcal{G}_X$. Indeed, given $g = (x, m-n, y) \in \mathcal{G}_X$, we have that

$$\begin{aligned}(x,m-n,y) &= (x,1,\sigma(x))(\sigma(x),1,\sigma^2(x)) \\ &\quad \cdots (\sigma^{m-1}(x),1,\sigma^m(x))(\sigma^n(y),-1,\sigma^{n-1}(y)) \cdots (\sigma(y),-1,y),\end{aligned}$$

and therefore,

$$g \in Z(x[1],\emptyset) \cdots Z(x[m],\emptyset) Z(\emptyset, y[n]) \cdots Z(\emptyset, y[1]).$$

We denote by $\ell_S : \mathcal{G}_X \to \mathbb{N} \cup \{0\}$ the associated length function, which we recall is given by $\ell(x) = 0$ for $x \in X$, and

$$\ell(g) = \inf\{k \in \mathbb{N} \mid g \in S^k\},$$

when $g \in \mathcal{G}_X \setminus X$.

Now let $x \in X$ and define $W_x(n) := \{g \in (\mathcal{G}_X)_x \mid \ell_S(g) = n\}$, for $n \in \mathbb{N}_0$. First observe that

$$W_x(1) = \big\{(ax, 1, x) \mid a \in \mathcal{L}_1(X) \text{ such that } ax \in X\big\} \cup \big\{(\sigma(x), -1, x)\big\},$$

therefore $|W_x(1)| \le p_X(1) + 1$. Moreover,

$$\begin{aligned} W_x(2) = &\big\{(wx, 2, x) \mid w \in \mathcal{L}_2(X) \text{ such that } wx \in X\big\} \\ &\cup \big\{(a\sigma(x), 0, x) \mid a \in \mathcal{L}_1(X) \text{ such that } a\sigma(x) \in X\big\} \cup \big\{(\sigma^2(x), -2, x)\big\}, \end{aligned}$$

and therefore we have that $|W_x(2)| \le p_X(2) + p_X(1) + 1$, and in general,

$$\big|W_x(n)\big| \le 1 + \sum_{k=1}^{n} p_X(k) \le 1 + n p_X(n),$$

for $n \in \mathbb{N}$, so that

$$\big|B_{(\mathcal{G}_X)_x}(n)\big| = 1 + \sum_{k=1}^{n} \big|W_x(k)\big| \le 1 + n + n^2 p_X(n).$$

It follows immediately from this that if the complexity function p_X has (strong) subexponential growth, then the function $n \mapsto \sup_{x \in X} |B_{(\mathcal{G}_X)_x}(n)|$ also has (strong) subexponential growth. Let us record this result for future reference.

Lemma 5.5. *Let X be a shift and $\mathcal{G}_X$ the associated Deaconu–Renault groupoid. If the complexity function has (strong) subexponential growth, then the groupoid $\mathcal{G}_X$ has (strong) subexponential growth.*

If X is a shift of finite type, then it is a standard result that X is conjugate to a shift space associated with an infinite path space of a finite directed graph [30, Section 2.2]. Then combining this with [2, Section 5.3] we have the following.

Proposition 5.6. *If X is a shift of finite type, then there is a finite directed graph E_X such that $\mathcal{G}_{E_X} \cong \mathcal{G}_X$. Consequently, with respect to the natural length function, the Deaconu–Renault groupoid $\mathcal{G}_X$ is either of exponential or polynomial growth.*

Since we are interested in exhibiting examples of groupoids associated with shifts whose growth is strongly subexponential and not polynomial, we see from Proposition 5.6 that it is natural to look for examples among the shifts of *infinite* type. The shift that will turn out to give our desired example is one that we call the *ordered prime shift*, which we define next: The alphabet is $\mathcal{A} = \{0, 1\}$, and we let

$$\mathcal{F} := \{10^n 1 \mid n \in \mathbb{N}_0 \text{ is not a prime}\} \cup \{10^{p_1} 10^{p_2} 1 \mid p_1, p_2 \text{ are primes with } p_1 \ge p_2\},$$

be the *forbidden words*. The associated shift $X := X_{\mathcal{F}}$ is the space of all paths consisting of *admissible words*, that is, the paths x such that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, the word

$x[k, n+k]$ is not a forbidden word. Notice that any admissible word has one of three forms:

$$0^k \quad \text{for } k \in \mathbb{N}_0; \tag{5.2}$$
$$0^k 10^l \quad \text{for } l, k \in \mathbb{N}_0; \tag{5.3}$$
$$0^k 10^{p_1} 10^{p_2} 1 \cdots 0^{p_r} 10^l \quad \text{for } k, l \in \mathbb{N}_0,\ r \in \mathbb{N}, p_i \text{ prime } \forall i \text{ and } p_i < p_{i+1}\ \forall i. \tag{5.4}$$

Lemma 5.7. *The complexity function p_X of the ordered prime shift X has strong subexponential growth, but not polynomial growth.*

There are two key tools to proving the above. First, Chebyshev's theorem which gives a useful estimate for the map $x \mapsto \pi(x)$ that counts the number of prime numbers not exceeding $x \in \mathbb{R}_{\geq 2}$ (see [34, Theorem 6.3]): there exists constants $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}_{\geq 2}$,

$$\frac{c_1 x}{\ln(x)} \leq \pi(x) \leq \frac{c_2 x}{\ln(x)}. \tag{5.5}$$

Second, the following estimate for the Hardy–Ramanujan partition function (see [22]): there exists $A, B > 0$ such that for all sufficiently large n,

$$e^{A\sqrt{n}} \leq p(n) \leq e^{B\sqrt{n}}, \tag{5.6}$$

where we recall that $p(n)$ counts the number of unrestricted partitions of the positive integer n.

Proof of Lemma 5.7. First we show that p_X cannot have polynomial growth. Consider the words of the form

$$10^{p_1} 10^{p_2} 1 \cdots 0^{p_r} 1,$$

where p_i are all distinct primes in increasing order which does not exceed $\sqrt{n/2}$. Such a word has length

$$r + 1 + \sum_{i=1}^{r} p_i \leq r + 1 + r\sqrt{n/2} \leq n/2 + \sqrt{n/2}\,\pi(\sqrt{n/2}) \leq n/2 + \frac{2c_2 n/2}{\ln(n/2)} \leq n,$$

for all large n, where we have used (5.5). Let $\{p_1, \ldots, p_{\pi(\sqrt{n/2})}\}$ be the collection of primes not exceeding $\sqrt{n/2}$. As we saw above, any sub-collection gives a unique word as above of length less than or equal to n. Thus, using (5.5), we have

$$p_X(n) \geq \#\text{ subsets of } \{p_1, \ldots, p_{\pi(\sqrt{n/2})}\} = 2^{\pi(\sqrt{n/2})} \geq 2^{\frac{2c_1\sqrt{n/2}}{\ln(n/2)}},$$

so that p_X cannot be dominated by any polynomial.

Next we show that p_X has strong subexponential growth. For this we will use some rough estimates on the number of elements of length at most n. Recall that any word has one of the three forms (5.2), (5.3) and (5.4). There are $n+1$ words of the form (5.2) with length at most n, and there are at most n^2 words of the form (5.3) with length at most n. Let us then consider the words of the form (5.4). Fix any $1 \leq k \leq n$ and consider the set of all primes $\{p_1, \ldots, p_{\pi(k)}\}$ not exceeding k. A sub-collection $\{p_{i_1}, \ldots, p_{i_r}\}$ that is arranged in increasing order gives a unique element $10^{p_{i_1}}10^{p_{i_2}}1 \cdots 10^{p_{i_r}}1$ of length possibly less than k. There are $p(k)$ unrestricted partitions of k into a sum of non-negative integers, and hence there are at most $p(k-(r+1)) \leq p(k)$ elements of the form $10^{p_{i_1}}10^{p_{i_2}}1 \cdots 10^{p_{i_r}}1$ such that $\sum_j p_{i_j} + r + 1 = k$. For every such element $10^{p_{i_1}}10^{p_{i_2}}1 \cdots 10^{p_{i_r}}1$, there are at most n^2 elements of the form $0^t 10^{p_{i_1}}10^{p_{i_2}}1 \cdots 10^{p_{i_r}}10^l$ with $0 \leq t + l \leq n - k$. Therefore, the number of elements of the form (5.4) with length at most n is bounded by

$$\sum_{k=1}^{n} n^2 p(k) \leq n^3 p(n).$$

This together with (5.6) implies that there exist constants $C, \alpha > 0$ large enough such that

$$p_X(n) \leq n + 1 + n^2 + n^3 p(n) \leq 4n^3 p(n) \leq C e^{\alpha \sqrt{n}},$$

for all $n \in \mathbb{N}$. This proves the lemma. ∎

Proposition 5.8. *With respect to its canonical length function, the Deaconu–Renault groupoid associated with the ordered prime shift has strong subexponential growth, and is not of polynomial growth.*

Proof. Let $\mathcal{G}_X$ denote the groupoid associated with the ordered prime shift X. Combining Lemma 5.5 with Lemma 5.7, we see that $\mathcal{G}_X$ has strong subexponential growth. Let us show next that it is not of polynomial growth. For this, let $x \in X$ be the path $x = 0^\infty$. Concatenating any admissible word w with x yields another element $wx \in X$, and it therefore follows that

$$\left|B_{(\mathcal{G}_X)_x}(n)\right| \geq p_X(n).$$

Since the complexity function p_X is not dominated by any polynomial, the same must be true for the function $n \mapsto |B_{(\mathcal{G}_X)_x}(n)|$, hence the result. ∎

As already remarked, the Deaconu–Renault groupoid associated with the ordered prime shift is not an étale groupoid, but an r-discrete groupoid. To obtain an étale groupoid with the same growth, we can apply recent results of Brix, Hume and Li in [7]. Therein, given an r-discrete groupoid $\mathcal{G}$, the authors associate an étale groupoid $\widehat{\mathcal{G}}$ called the *cover* of $\mathcal{G}$. The cover groupoid can be viewed as a transformation groupoid built from a groupoid action of $\mathcal{G}$ on a locally compact Hausdorff space $\widehat{X}$, so $\widehat{\mathcal{G}} = \mathcal{G} \ltimes \widehat{X}$ (see [7, Proposition 7.5]). When $\mathcal{G}$ is the Deaconu–Renault groupoid associated with a shift of infinite type, we endow $\widehat{\mathcal{G}}$ with the length function $\ell\colon \widehat{\mathcal{G}} \to \mathbb{R}_{\geq 0}$ given by $\ell(g, x) = \ell_S(g)$, where ℓ_S is the length function on $\mathcal{G}$ induced by the generating set S as in (5.1). With this choice of length function, it is clear that $\widehat{\mathcal{G}}$ and $\mathcal{G}$ share the same growth properties, and hence we arrive at a desired example.

Theorem 5.9. *Let $\mathcal{G}_X$ denote the Deaconu–Renault groupoid associated with the ordered prime shift X, and let $\widehat{\mathcal{G}}_X$ denote its cover groupoid. Then $\widehat{\mathcal{G}}_X$ is an étale groupoid that has strong subexponential growth, but not polynomial growth.*

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